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# Metrical Diophantine approximation for quaternions

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(*Received* ; *revised*)

Dedicated to J. W. S. Cassels.

*Abstract*

Analogues of the classical theorems of Khintchine, Jarník and Jarník-Besicovitch in the metrical theory of Diophantine approximation are established for quaternions by applying results on the measure of general 'lim sup' sets.

## 1. *Introduction*

Diophantine approximation begins with a more quantitative understanding of the density of the rationals $\mathbb{Q}$ in the reals $\mathbb{R}$. For any real number $\xi$, one considers rational solutions $p/q$ to the inequality

$$\left|\xi - \frac{p}{q}\right| < \varepsilon,$$

where $\varepsilon$ is a small positive number depending on $p/q$. Dirichlet's theorem [**41**, Chap. XI], where $\varepsilon = (qN)^{-1}$ for any $N \in \mathbb{N}$ and a suitable positive integer $q \leqslant N$, is fundamental to the theory. Holding for all real numbers, it is a *global* result in Sprindžuk's classification of Diophantine approximation [**70**, pg. x], in contrast with *individual* results, which hold for special numbers, such as the golden ratio $\phi$, $e$, $\pi$, *etc.*, and with the *metrical* theory. The last theory uses measure theoretic ideas to describe sets of number theoretic interest and is the setting of this paper.

Dirichlet's theorem underpins four major theorems – or the *Four Peaks* – in the metrical theory of Diophantine approximation for $\mathbb{R}$. These results are concerned with the measure (usually Lebesgue or Hausdorff) of real numbers that infinitely often are 'close' to rationals, and those which 'avoid' rationals; these are called *well* approximable and *badly* approximable numbers respectively (definitions are given below). The four results are Khintchine's theorem (Theorem 3·1), two theorems of Jarník (Theorems 3·2 and 3·4) the celebrated Jarník-Besicovitch theorem (Theorem 3·3). Three of the peaks concern well-approximable numbers and one badly-approximable numbers. The quantitative form

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of Khintchine's theorem (see [**63**, **70**]) certainly merits peak status as well but will not be considered here.

This basic setting can be generalised in a number of directions: one 'topological', where the reals are replaced by $\mathbb{R}^n$ or even submanifolds of $\mathbb{R}^n$ and the nature of the Diophantine approximation modified appropriately; another is 'geometrical' where the reals are replaced by limit points of a discrete group acting on hyperbolic space; while yet another is 'algebraic', where the field $\mathbb{R}$ is replaced by other fields, skew-fields or division algebras, and $\mathbb{Q}$ is replaced by the field of fractions of 'integral' subrings. This paper follows the third direction: the approximation of quaternions $\mathbb{H}$ by ratios of integer-like quaternions. For us, 'integer-like' will mean the Hurwitz integers $\mathcal{H}$: these turn out to be the simplest subring of $\mathbb{H}$ with sufficiently nice algebraic properties – such as a division algorithm – for an interesting quaternionic number theory (see §4·1 and [**18**, **44**]).

As far as we can determine, little has been published on quaternionic Diophantine approximation. A. Speiser obtained an approximation constant for irrational quaternions [**69**]; his work was extended and sharpened by A. L. Schmidt [**61**, **62**]. K. Mahler proved an inequality for the product of Hurwitzian integral linear forms [**53**] but we can find nothing explicitly on the metrical theory. This paper sets out to fill this gap by establishing quaternionic analogues of the Four Peaks. Limitations of space and complications arising from non-associativity prevent including the further extension to octonions and completing the picture for real division algebras.

After some basic measure theory in §2 and a brief survey of real, complex and more general Diophantine approximation in §3, we set the stage for the quaternionic theory in §4. The main result here is a quaternionic analogue of Dirichlet's theorem (Theorem 4·1). The badly approximable quaternions are then defined in §4·3. Section 5 extends the fundamental Dirichlet inequality to the notion of $\Psi$-approximability. The ideas of resonance and near-resonance are explained and the basic structure of the set of $\Psi$-approximable numbers is described.

We are finally ready for the quaternionic Four Peaks in §6. The First Peak is the quaternionic Khintchine theorem (Theorem 6·1). Each of the statement and proof falls into two cases: convergence and divergence. The convergence case is the easier of the two and is established in §8. The divergence case, proved in §9, is much harder and requires deeper ideas, such as ubiquity (§7) and the mass transference principle (§9·1). The quaternionic Dirichlet's theorem (Theorem 4·1) is used in §9 to show that the Hurwitz rationals $\mathcal{Q}$ are a ubiquitous system. This involves rather lengthy and delicate analysis but is a prerequisite to applying the powerful Beresnevich-Velani Theorem, established in [**10**]. This is adapted to our needs as Theorem 9·2 and used to yield the analogue of Khintchine's theorem. The extension to quaternions of the quantitative form of Khintchine's theorem is an interesting open question.

The proof of Theorem 6·2, the quaternionic analogue of the Jarník's extension of Khintchine's theorem to Hausdorff measure, follows similar lines and is sketched. Theorem 6·6, the analogue of the Jarník-Besicovitch theorem, is a corollary of Theorem 6·2. Finally, some related ideas are used in §10 to prove Theorem 6·7 on the Hausdorff measure and dimension of the set of badly approximable quaternions.

A knowledge of measure theory and particularly Lebesgue and Hausdorff measure in $\mathbb{R}^k$ will be assumed. For completeness and to fix notation the elements of the theory are sketched. The reader is referred to [**11**, **26**, **27**, **28**, **54**, **59**] for further details.

## 2. *Measure and dimension*

We consider points in subsets of general Euclidean space $\mathbb{R}^n$, our primary interest of course being in $\mathbb{R}^4$, the underlying set of $\mathbb{H}$. When defined, the Lebesgue measure of a set $E$ will be denoted by $|E|$. The set $E \subseteq F \subseteq \mathbb{R}^n$ is said to be *null* if $|E| = 0$ and *full in* $F$ if its complement $F \setminus E$ is null (reference to $F$ will be omitted when there is no risk of ambiguity). Hausdorff measure and Hausdorff dimension are much more general and can be assigned to any set. In particular they can be applied to different null sets (also referred to as *exceptional sets*), so offering a possible means of distinguishing between them.

A *dimension function* $f\colon [0,\infty) \to [0,\infty)$ is a generalisation of the usual notion of dimension; $m$-dimensional Lebesgue measure corresponds to $f(t) = t^m$. More generally, the function $f$ will be taken to be increasing on $[0,\infty)$, with $f(x) > 0$ for $x > 0$ and $f(x) \to 0$ as $x \to 0$. For convenience $f$ will be assumed to be continuous, so that $f(0) = 0$. The *Hausdorff $f$-measure* $\mathscr{H}^f$ (or generalised Hausdorff measure with dimension function $f$) is defined in terms of a $\varepsilon$-cover $\mathcal{C}_\varepsilon = \{C_i\}$ of a set $E$, so that $E \subseteq \bigcup_{i=1}^\infty C_i$, where $\operatorname{diam}(C_i) \leqslant \varepsilon$. The measure $\mathscr{H}^f(E)$, defined as

$$\mathscr{H}^f(E) := \lim_{\varepsilon\to 0} \inf\{\sum_i f(\operatorname{diam}(C_i))\colon C_i \in \mathcal{C}_\varepsilon\}, \tag{2·1}$$

is a Borel measure and regular on Borel sets [**26**, **54**]. Hausdorff $s$-measure $\mathscr{H}^s$ corresponds to the function $f$ being given by $f(t) = t^s$, where $0 \leqslant s < \infty$. When $s = m$ a non-negative integer, Hausdorff $s$-measure is comparable with Lebesgue's $m$-dimensional measure. Indeed

$$\mathscr{H}^m(E) = 2^m |B(0,1)|^{-1} |E|, \tag{2·2}$$

where $B(0,1)$ is the unit $m$-dimensional ball, and the two measures agree when $m = 1$ [**54**, pg. 56]. The 4-dimensional Lebesgue measure (4-volume) of the 4-ball $B^{(4)}(\xi, r)$ of radius $r$ (and diameter $2r$) centred at $\xi$ is given by

$$|B(\xi, r)| = \frac{\pi^2}{2} r^4 \asymp r^4. \tag{2·3}$$

For each set $E$ the *Hausdorff dimension* $\dim_{\mathrm{H}} E$ of $E$ is defined by

$$\dim_{\mathrm{H}} E := \inf\{s \in \mathbb{R}\colon \mathscr{H}^s(E) = 0\},$$

so that

$$\mathscr{H}^s(E) = \begin{cases} \infty, & s < \dim_{\mathrm{H}} E, \\ 0, & s > \dim_{\mathrm{H}} E. \end{cases}$$

Thus the dimension is that critical value of $s$ at which $\mathscr{H}^s(E)$ 'drops' discontinuously from infinity. Hausdorff dimension has the natural properties of dimension. For example, if $E \subseteq E'$, then $\dim_{\mathrm{H}} E \leqslant \dim_{\mathrm{H}} E'$; and an open set, or a set of positive Lebesgue measure in $\mathbb{R}^n$, has maximal or full Hausdorff dimension $n$. Different null sets can have different Hausdorff dimension and so can be distinguished (*e.g.*, Theorem 3·3).

The Hausdorff $s$-measure at the critical point can be 0, $\infty$ or any intermediate value. Methods for determining the Hausdorff dimension, such as the regular systems given in [**4**] or the ubiquitous systems of [**25**], do not specify the $s$-measure at the critical point in general and a deeper approach is usually needed (see Theorem 9·1). In the case of lim sup sets, such as the $\Psi$-approximable numbers defined below, the measure of a natural cover arising from the definition leads to a sum which determines the Hausdorff measure at the critical point.

## 3. *Real and complex metrical Diophantine approximation*

Some of the salient features of metrical Diophantine approximation for the real and complex numbers are set out to aid comparison with the quaternions.

### 3·1. *Metrical Diophantine approximation for real numbers*

Historically, metrical Diophantine approximation began with Borel's study of the set

$$W_v := \left\{\xi \in \mathbb{R} \colon \left|\xi - \frac{p}{q}\right| < q^{-v} \text{ for infinitely many } p \in \mathbb{Z},\, q \in \mathbb{N}\right\},$$

where $W_v = \mathbb{R}$ for $v \leqslant 2$ and is null for $v > 2$ [**15**]. More generally, the function $x \mapsto x^{-v}$ is replaced by an *approximation function* $\Psi$, defined here to be a function $\Psi\colon (0,\infty) \to (0,\infty)$ with $\Psi(x) \to 0$ as $x \to \infty$. One studies the Lebesgue measure $|W(\Psi)|$ of the set

$$W(\Psi) := \left\{\xi \in \mathbb{R} \colon \left|\xi - \frac{p}{q}\right| < \Psi(q) \text{ for infinitely many } p \in \mathbb{Z},\, q \in \mathbb{N}\right\}$$

of $\Psi$-*approximable numbers*. Unless otherwise stated, the approximation function $\Psi$ will be taken to be decreasing (which we will take to mean non-increasing).

For technical reasons, it is often better to work within a compact set and we choose the subset $V(\Psi) := W(\Psi) \cap [0,1]$. There is no loss in generality since $\mathbb{R}$ is the union of integer translates of $[0,1]$, the integers $\mathbb{Z}$ are a null set and Lebesgue measure is translation invariant, allowing the measure of $W(\Psi)$ to be deduced from that of $V(\Psi)$. In particular, $V(\Psi)$ is full (in $[0,1]$) iff $A(\Psi)$ is full (in $\mathbb{R}$). Four of the principal results – the Four Peaks – in the theory for $\mathbb{R}$ now follow.

### 3·2. *The Four Peaks in the theory of real metrical Diophantine approximation.*

*The First Peak: Khintchine's theorem for* $\mathbb{R}$. In 1924 Khintchine introduced a 'length' criterion that gave a strikingly simple and almost complete answer to the 'size' of $W(\Psi)$ [**48**], extended to $\mathbb{R}^n$ (simultaneous Diophantine approximation) in [**49**]. The conditions on $\Psi$ have been improved since (see for example [**16**, Ch. VII], [**70**, Ch. 1] and §9·8) to give the following result for $\mathbb{R}$:

THEOREM 3·1. *Let* $\Psi\colon (0,\infty) \to (0,\infty)$. *Then*

$$W(\Psi) \text{ and } V(\Psi) \text{ are } \begin{cases} \text{null when } \sum_{m=1}^{\infty} m\,\Psi(m) < \infty, \\ \text{full when } \Psi \text{ is decreasing and } \sum_{m=1}^{\infty} m\,\Psi(m) = \infty \end{cases}$$

Note that $W(\Psi)$ being full implies the weaker statement that $|W(\Psi)| = \infty$, while $|V(\Psi)| = 1$ is equivalent to $W(\Psi)$ being full. Other approximation functions can be used: *e.g.*, $\psi(x) = x\Psi(x)$, where $\|\xi\|$ is the distance of $\xi$ from the nearest integer, which allowing the inequality to be expressed in the concise form $\|q\xi\| < \psi(q)$ (*e.g.*, [**11**, **16**]), with the numerator $p$ suppressed, while Dennis Sullivan in [**71**] uses $a(x) = x^2\Psi(x)$ (he also uses an equivalent integral criterion instead of the sum $\sum_{m\in\mathbb{N}} a(m)/m$). The subset $W'(\Psi) \subset W(\Psi)$ of points $\xi$ approximated by rationals $p/q$ with $p, q$ coprime will not be considered.

It is evident that the value of the sum

$$\sum_{m=1}^{\infty} m\,\Psi(m)$$

in Theorem 3·1 determines the Lebesgue measure of $W(\Psi)$ and so will be called a *critical* sum for $W(\Psi)$. Note that if the above critical sum converges, $\Psi$ must converge to 0 and moreover there is no need in this case for $\Psi$ to be monotonic. Khintchine's theorem is related to the 'pair-wise' form of the Borel-Cantelli Lemma (see [**8**, **23**, **42**]) which also falls into two cases according as a certain sum of probabilities converges or diverges.

The interpretation of the rationals as the orbit of a point at infinity under the action of the modular group provides a powerful geometrical approach to Diophantine approximation in the reals (*e.g.*, [**58**, **68**]) and more generally [**1**, **5**, **55**, **56**]. It was the basis of Sullivan's proof [**71**, Th. 3] of a slightly stronger form of Khintchine's theorem and more (see §3·3 below).

*The Second Peak: Jarník's Hausdorff $f$-measure theorem for* $\mathbb{R}$. In 1931, Jarník obtained Hausdorff measure results for simultaneous Diophantine approximation in $\mathbb{R}^n$, providing a more general measure theoretic picture of the sets involved [**47**] (see also [**8**, pg. 3]). This did not include Lebesgue measure which is excluded by a growth condition on $f$ at 0. Although originally proved for $\mathbb{R}^n$, Jarník's result is again stated for the case $n = 1$, with some unnecessary monotonicity conditions omitted.

THEOREM 3·2. *Let $f$ be a dimension function such that $f(x)/x \to \infty$ as $x \to 0$ and $f(x)/x$ decreases as $x$ increases. Then*

$$\mathscr{H}^f(W(\Psi)) = \mathscr{H}^f(V(\Psi)) = \begin{cases} 0 & \textit{when } \sum_{m=1}^{\infty} m\, f(\Psi(m)) < \infty, \\ \infty & \textit{when } \Psi \textit{ is decreasing and } \sum_{m=1}^{\infty} m\, f(\Psi(m)) = \infty. \end{cases}$$

The condition $f(x)/x \to \infty$ as $x \to 0$ means that Jarník's theorem does not imply Khintchine's theorem since the dimension function $f$ for 1-dimensional Lebesgue measure is given by $f(x) = x$. However, using the idea of ubiquity (explained below in §7), V. V. Beresnevich and S. L. Velani have united Theorem 3·1 and Jarník's theorem into a single general 'Khintchine-Jarník' theorem [**10**, §2.3]. The sum $\sum_{m=1}^{\infty} m\, f(\Psi(m))$ is the corresponding critical sum.

*The Third Peak: the Jarník-Besicovitch theorem for* $\mathbb{R}$. In 1929 Jarník [**45**, **46**] obtained the Hausdorff dimension of the set $W_v$, proved by Besicovitch independently in 1934 [**13**]. This result is readily seen as a consequence of Jarník's result above by putting $f(x) = x^s$ and $\Psi(x) = x^{-v}$, $v > 0$.

THEOREM 3·3. *Let $v \geqslant 0$. Then the Hausdorff dimension of $W_v$ is given by*

$$\dim_{\mathrm{H}} W_v = \dim_{\mathrm{H}} V_v = \begin{cases} 1 & \textit{when } v \leqslant 2, \\ \dfrac{2}{v} & \textit{when } v \geqslant 2. \end{cases}$$

When $1/\Psi$ has lower order $\lambda(1/\Psi) := \liminf_{N\to\infty}(\log 1/\Psi(N))/(\log N)$, then

$$\dim_{\mathrm{H}} W(\Psi) = \dim_{\mathrm{H}} V(\Psi) = \begin{cases} 1 & \text{when } \lambda(1/\Psi) \leqslant 2, \\ \dfrac{2}{\lambda(1/\Psi)} & \text{when } \lambda(1/\Psi) \geqslant 2 \end{cases}$$

(see [**22**, **25**]).

*The Fourth Peak: Jarník's theorem for $\mathfrak{B}$, the set of badly approximable numbers.* A real number $\beta$ is said to be *badly approximable* if there exists a constant $c = c(\beta)$ such that

$$\left|\beta - \frac{p}{q}\right| \geqslant \frac{c}{q^2}$$

for all rationals $p/q$. The set of badly approximable numbers is denoted by $\mathfrak{B}$ and can be regarded as a 'lim inf' set [**26**, pg. 1]. In his pioneering paper of 1928, Jarník established the Lebesgue measure and Hausdorff dimension of $\mathfrak{B}$ [**45**].

Theorem 3·4. *The set $\mathfrak{B}$ is null with full Hausdorff dimension, i.e., $|\mathfrak{B}| = 0$ and $\dim_{\mathrm{H}}(\mathfrak{B}) = 1$.*

The strengthening of this result by W. M. Schmidt, who showed that $\mathfrak{B}$ was a 'winning set' in a certain game [**64**], will not be considered for quaternions.

3·3. *Metrical Diophantine approximation for the complex numbers*

Approximating complex numbers by ratios of Gaussian integers $\mathbb{Z}[i]$, a half way house to approximating quaternions by ratios of Lipschitz or Hurwitz integer quaternions, was studied by Hermite and Hurwitz in the 19th century [**50**, Chapter IV,§ 1]. Continued fractions for complex numbers, so simple and effective for real numbers, turn out to be much more difficult than the real case [**16**, **41**, **67**, **29**]. In the 1950s, Farey sections for complex numbers, analogous to Farey fractions for real numbers, were developed by Cassels, Ledermann and Mahler, who carried out a detailed study [**17**] of a programme sketched out by Hurwitz [**43**, §8]; their work was simplified and extended by LeVeque [**52**]. A. L. Schmidt developed a natural and effective approach in [**60**, **66**], subsequently extended to the even more difficult case of quaternions [**61**, **62**].

Each of the Four Peaks has an analogue in the complex numbers. That of Khintchine's theorem was by proved by LeVeque [**52**], who combined Khintchine's continued fraction approach with ideas from hyperbolic geometry. Later Patterson, Sullivan and others made full use of groups acting on hyperbolic space to prove Diophantine approximation results in more general settings. Sullivan established a Khintchine theorem for Diophantine approximation in the imaginary quadratic fields $\mathbb{Q}(\sqrt{-d})$, where $d$ is a positive non-square integer [**71**, Theorem 1], corresponding to the Bianchi groups. In the case $d = 1$, the field is the complex numbers, corresponding to the Picard group, and Theorem 1 in [**71**] reduces to the complex analogue of Khintchine's theorem.

The Mass Transference Principle (see §9·1) could be applied to the complex analogue of Khintchine's theorem to deduce the complex analogue of Theorem 3·2 (indeed more general analogues involving Bianchi groups could be deduced from the more general analogues of Khintchine's theorem). The complex Jarník-Besicovitch theorem and a stronger form of Jarník's Theorem for badly approximable complex numbers were also proved in [**24**], using respectively ubiquity (Theorem 6.1; see also [**8**, Cor. 7]) and the $(\alpha, \beta)$ games of W. M. Schmidt [**64**] (Theorem 5.2).

3·4. *Generalisations*

The above results, with appropriate modifications, hold for simultaneous Diophantine approximation and more generally for systems of linear forms (where the Khintchine-Groshev theorem takes the place of Khintchine's theorem) [**21**, **65**, **70**]. The Khintchine-Groshev theorem was extended to non-degenerate manifolds in the case of convergence by Beresnevich, D. Y. Kleinbock and G. A. Margulis in [**12**, **6**] and in the case of divergence by the preceding authors and V. I. Bernik in [**7**]. The idea of ubiquity [**25**], which is closely related to regular systems, has been extended by Beresnevich, H. Dickinson and Velani to lim sup sets in compact metric spaces supporting a suitable non-atomic measure to create a broad unifying theory [**8**, **9**, **10**]. In particular, the results in [**8**] imply that the measure in the Beresnevich-Velani theorem [**10**, Th. 3] covers both Lebesgue and Hausdorff measure and will be applied to establish the first three of the quaternionic Four Peaks.

The approach using discrete group actions on hyperbolic space for $\mathbb{R}$ and $\mathbb{C}$, already alluded to above, leads naturally to the more general setting of Kleinian group actions on hyperbolic space ([**14**] has a comprehensive list of references). Beresnevich, Dickinson & Velani have established metrical Diophantine approximation results for more general Kleinian group analogues of the first three of the Four Peaks [**8**]. These specialise to metrical Diophantine approximation results for real and complex numbers, corresponding to the modular and Picard group respectively. The quaternionic case would correspond to the group $\mathbf{PSp}_{2,1}(\mathcal{H})$ but different normalisations require reconciling and the proofs would also require a knowledge of the theory of discrete groups acting on (quaternionic) hyperbolic space. A more direct and less abstract approach is taken in this paper.

In a continuation of [**9**], S. Kristensen, R. Thorn & Velani [**51**] extend the definition of a badly approximable point to a metric space. This allows the metrical structure of $\mathfrak{B}_{\mathbb{H}}$, the quaternionic analogue of badly approximable points, to be read off once a few geometric conditions are verified (see §4·3). It could also be possible to use the equivalence of badly approximable points and 'bounded' orbits (for details see [**20**]). 'Divergent' orbits correspond to well-approximable points but the results are less precise [**19**].

## 4. *Quaternionic Diophantine approximation*

We begin our study of quaternionic Diophantine approximation by identifying the appropriate analogues of the classical case and then proving an analogue of Dirichlet's theorem.

### 4·1. *Preliminaries on quaternionic arithmetic*

The skew field $\mathbb{H}$ of quaternions consists of the set

$$\{\xi = a + bi + cj + dk\colon a, b, c, d \in \mathbb{R}\},$$

subject to $i^2 = j^2 = k^2 = ijk = -1$ and $i, j, k$ anticommuting: $ij = -ji, jk = -kj$ and $ik = -ki$. The norm of a quaternion $\xi$ is taken to be the usual Euclidean norm

$$|\xi|_2 := (\xi\overline{\xi})^{1/2} = (|a|^2 + \cdots + |d|^2)^{1/2},$$

where $\overline{\xi} = a - bi - cj - dk$. This norm is multiplicative, with $|\xi\xi'|_2 = |\xi|_2|\xi'|_2 = |\xi'|_2|\xi|_2 = |\xi'\xi|_2$ for $\xi, \xi' \in \mathbb{H}$ (in [**41**, §§20.6–20.8] and [**44**] 'norm' is used in a different sense, with $N(\xi) = |\xi|_2^2$). When convenient, we will write $\xi = a + bi + cj + dk = (a, b, c, d)$ and $a = \Re(\xi)$.

There are 24 multiplicative units in $\mathbb{H}$:

$$\pm 1, \pm i, \pm j, \pm k \text{ and } \pm\frac{1}{2} + \pm\frac{1}{2}i + \pm\frac{1}{2}j + \pm\frac{1}{2}k,$$

forming the vertices of a regular 24-cell in $\mathbb{R}^4$.

The simplest-minded notion of *integers* in $\mathbb{H}$ is that of the Lipschitz integers $\mathcal{L} =$

$\mathbb{Z}[i,j,k] = \mathbb{Z} + i\mathbb{Z} + j\mathbb{Z} + k\mathbb{Z} \cong \mathbb{Z}^4$. However, this choice has a number of shortcomings: it does not include all the $\mathbb{H}$-units and is not a Euclidean domain, as the centre of a 4-dimensional cube has Euclidean distance 1 from the closest integral points (to be an integral domain, the distance of a quaternion to the closest integral point should always be $< 1$). For these reasons, the usual choice for quaternionic integers is the set $\mathcal{H}$ consisting of the quaternions $a + bi + cj + dk$, where either all of $a, b, c, d \in \mathbb{Z}$ or all $a, b, c, d \in \mathbb{Z} + \frac{1}{2}$, *i.e.*,

$$\mathcal{H} = \mathcal{L} \ \cup \left(\mathcal{L} + \frac{1}{2}(1 + i + j + k)\right).$$

Thus $\mathcal{H}$ consists of $\mathbb{Z}^4$ together with the mid-points of the standard 4-dimensional unit cubes in $\mathbb{Z}^4$ and is an integral domain with division algorithm, *i.e.*, if $\mathbf{p}, \mathbf{q} \in \mathcal{H}$ with $\mathbf{q} \neq 0$, then there exist $\mathbf{s}, \mathbf{r} \in \mathcal{H}$ with $|\mathbf{r}|_2 < |\mathbf{q}|_2$ and

$$\mathbf{p} = \mathbf{s}\mathbf{q} + \mathbf{r},$$

(for example, see [**41**, Th. 373]). As a result, up to a multiplicative unit, any two Hurwitz integers have a unique greatest right (respectively left) common divisor up to a left (resp. right) unit, whence Hurwitz integers have essentially a unique factorisation [**18**]. Two Hurwitz integers $\mathbf{q}, \mathbf{q}'$ are said to be right (or left) coprime when their right (or left) greatest common divisor is a unit; we will write $(\mathbf{q}, \mathbf{q}')_r = 1$. Two coprime integers $\mathbf{q}, \mathbf{q}'$ generate $\mathcal{H}$ in the sense that $\mathcal{H}$ is a sum of the principle ideals they generate, *i.e.*, $\mathbf{q}\mathcal{H} + \mathbf{q}'\mathcal{H} = \mathcal{H}$. A *prime* quaternion $\xi$ is divisible only by a unit and an associate of $\xi$, *i.e.*, if in the factorisation $\xi = \mathbf{q}\,\mathbf{q}'$, either $\mathbf{q}$ or $\mathbf{q}'$ is a unit. Prime integer quaternions have a neat characterisation (modulo units) in terms of rational primes: an integer quaternion $\xi$ is prime if and only if $|\xi|_2^2$ is a rational prime [**41**, Th. 377].

As a subgroup of $\mathbb{R}^4$, the Hurwitz integers $\mathcal{H}$ are free abelian with generators $\{i, j, k, \frac{1}{2}(1 + i + j + k)\}$ and form a scaled copy of the lattice spanned by the root system of the simple Lie algebra $\mathfrak{f}_4$. A fundamental region for $\mathcal{H}$ is given by the half-closed region in $\mathbb{R}^4$ with vertices $0, 1, i, j$ and $\frac{1}{2}(1 + i + j + k)$. It has 4-dimensional Lebesgue measure, or 4-volume, $|\Delta| = 1/2$. For convenience we choose the simpler region

$$\Delta = \{\xi \in \mathbb{H} \colon 0 \leqslant a, b, c < 1, 0 \leqslant d < 1/2\} = [0,1)^3 \times [0, 1/2). \tag{4·1}$$

The Hurwitz rationals $\mathcal{Q}$ are defined to be

$$\mathcal{Q} := \{\mathbf{p}\mathbf{q}^{-1} \colon \mathbf{p}, \mathbf{q} \in \mathcal{H}, \mathbf{q} \neq 0\}.$$

A quaternion is said to be *irrational* if at least one of its (real) coordinates is irrational. Approximating quaternions by Hurwitz rationals $\mathbf{p}\mathbf{q}^{-1} \in \mathcal{Q}$, where the Hurwitz integer $\mathbf{q}$ can be regarded as a ‘denominator’ of the Hurwitz rational $\mathbf{p}\mathbf{q}^{-1}$, is an obvious analogue of approximating a real number by rationals $p/q \in \mathbb{Q}$. Distinct Hurwitz rationals enjoy essentially the same ‘separation’ property as distinct rationals.

LEMMA 4·1. *If* $\mathbf{p}\mathbf{q}^{-1} \neq \mathbf{r}\mathbf{s}^{-1}$, *then*

$$|\mathbf{p}\mathbf{q}^{-1} - \mathbf{r}\mathbf{s}^{-1}|_2 \geqslant |\mathbf{q}|_2^{-1} |\mathbf{s}|_2^{-1}.$$

On expanding $|\mathbf{p}\mathbf{q}^{-1} - \mathbf{r}\mathbf{s}^{-1}|_2^2\, \mathbf{q}\,\overline{\mathbf{q}}\,\mathbf{s}\,\overline{\mathbf{s}}$ and multiplying out, one gets

$$0 < |\mathbf{p}\mathbf{q}^{-1} - \mathbf{r}\mathbf{s}^{-1}|_2^2\, |\mathbf{q}|_2^2\, |\mathbf{s}|_2^2 = |\mathbf{p}|_2^2\, |\mathbf{s}|_2^2 + |\mathbf{r}|_2^2\, |\mathbf{q}|_2^2 - 2\Re(\mathbf{p}\,\overline{\mathbf{q}}\,\mathbf{s}\,\overline{\mathbf{r}}) \in \mathbb{N}$$

and the lemma follows.

*For the rest of this paper,* $\mathbf{p}$ *and* $\mathbf{q}$ *will denote Hurwitz integers with* $\mathbf{q} \neq 0$ *unless otherwise stated.*

4·2. *Dirichlet's theorem for quaternions*

The quaternions $\mathbb{H} = \bigcup_{\mathbf{q}\in\mathcal{H}}(\Delta + \mathbf{q})$, the union of translates of the fundamental region $\Delta$. Hence for any $\xi \in \mathbb{H}$ and any non-zero $\mathbf{q} \in \mathcal{H}$, there exists a unique $\mathbf{p} = \mathbf{p}(\xi, \mathbf{q}) \in \mathcal{H}$ such that $\{\xi\}_\Delta$, the *Hurwitz fractional part* of $\xi$ (the analogue of the fractional part $\{\alpha\}$ of a real number $\alpha$), satisfies

$$\{\xi\}_\Delta := \xi - \mathbf{p} \in \Delta,$$

so that

$$|\{\xi\}_\Delta| = |\xi\mathbf{q} - \mathbf{p}|_2 \leqslant \frac{\sqrt{13}}{4} < 1.$$

This inequality can be strengthened by restricting the choice of $\mathbf{q}$ to give a quaternionic version of a uniform Dirichlet's theorem, where the approximation is by Hurwitz rationals $\mathcal{Q}$ with the Euclidean norm. A short geometry of numbers proof is given; it will be used in Lemma 9·3. The multiplicative constant 2 in (4·2) is chosen for convenience and could be replaced any number greater than $4/\pi$ 2 without affecting the results sought. Whether $4/\pi$ is best possible is an open question.

THEOREM 4·1. *Given any* $\xi \in \mathbb{H}$ *and any integer* $N > 1$*, there exist* $\mathbf{p}, \mathbf{q} \in \mathcal{H}$ *with* $1 \leqslant |\mathbf{q}|_2 \leqslant N$ *such that*

$$\left|\xi - \mathbf{p}\,\mathbf{q}^{-1}\right|_2 < \frac{2}{|\mathbf{q}|_2 N}. \tag{4·2}$$

*Moreover there are infinitely many* $\mathbf{p}, \mathbf{q} \in \mathcal{H}$ *such that*

$$\left|\xi - \mathbf{p}\,\mathbf{q}^{-1}\right|_2 < \frac{2}{|\mathbf{q}|_2^2}. \tag{4·3}$$

*Proof.* We seek non-zero $\mathbf{p}, \mathbf{q} \in \mathcal{H}$ as components for vectors in the set

$$K = \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \in \mathbb{H}^2 \colon |\xi\mathbf{y} - \mathbf{x}|_2 < \varepsilon, |\mathbf{y}|_2 \leqslant N \right\}.$$

Now the set $K$ is convex and

$$T(K) = \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \in \mathbb{H}^2 \colon |\mathbf{x}|_2 < \varepsilon, |\mathbf{y}|_2 \leqslant N \right\} = B(0,\varepsilon) \times B(0,N),$$

where the matrix $T = \begin{pmatrix} -1 & \xi \\ 0 & 1 \end{pmatrix}$ has determinant $\det T = -1$ and $|T(K)| = |\det T|\,|K| = |K|$, the 8-volume of $K$. Hence

$$|K| = |B(0,\varepsilon)| \times |B(0,N)| = \frac{\pi^2}{2}\varepsilon^4 \frac{\pi^2}{2} N^4 = \frac{\pi^4}{4}\varepsilon^4 N^4.$$

The 4-volume of a fundamental region $\Delta$ of the Hurwitz lattice is 1/2, so the 8-volume of $\Delta^2$ in $\mathbb{H}^2$ is 1/4. Hence by Minkowski's theorem [**41**, Theorem 447], if $|K| = \pi^4\varepsilon^4 N^4/4 > 2^8/4$, *i.e.*, if $\varepsilon > 4/(\pi N)$, then $K$ contains a non-zero lattice point $(\mathbf{p}, \mathbf{q})$ with $|\mathbf{q}|_2 \leqslant N$ and $|\xi\mathbf{q} - \mathbf{p}|_2 < \varepsilon$. Choosing $\varepsilon = 2/N > 4/(\pi N)$ gives

$$\left|\xi - \mathbf{p}\,\mathbf{q}^{-1}\right|_2 < \frac{\varepsilon}{|\mathbf{q}|_2} = \frac{2}{|\mathbf{q}|_2 N},$$

where $|\mathbf{q}|_2 \leqslant N$, which is (4·2).

To show that there are infinitely many pairs $\mathbf{p}$, $\mathbf{q}$ in $\mathcal{H}$ satisfying (4·2), observe that the quaternionic rationals are not required to be in lowest terms, so that when $\xi = \mathbf{ab}^{-1}$, $|\xi - \mathbf{ab}^{-1}|_2 = |\mathbf{ab}^{-1} - \mathbf{a}\,\mathbf{p}(\mathbf{b}\,\mathbf{p})^{-1}|_2 = 0$ for all non-zero $\mathbf{p} \in \mathcal{H}$. Thus the inequality (4·3) holds for infinitely many pairs $\mathbf{a}\,\mathbf{p}$, $\mathbf{b}\,\mathbf{p}$. Note that if $\mathbf{p}, \mathbf{q}$ are coprime and $\xi = \mathbf{ab}^{-1} \neq \mathbf{pq}^{-1}$, where $\mathbf{a}, \mathbf{b} \in \mathcal{H}$, then $|\mathbf{pq}^{-1} - \mathbf{ab}^{-1}|_2 \geqslant (|\mathbf{q}||\mathbf{b}|)^{-1}$ by Lemma 4·1, so that $|\mathbf{b}| < |\mathbf{q}|$ and there are only finitely many solutions for (4·3).

The case when $\xi$ is not a Hurwitz rational remains, *i.e.*, $\xi \neq \mathbf{ab}^{-1}$ for any $\mathbf{a}, \mathbf{b} \in \mathcal{H}$, so that for all $\mathbf{pq}^{-1}$, $|\xi - \mathbf{p}\,\mathbf{q}^{-1}|_2 > 0$. Suppose that the inequality (4·3) holds only for $\mathbf{pq}^{-1} = \mathbf{p}^{(m)}(\mathbf{q}^{(m)})^{-1}$, where $m = 1, \ldots, n$ and $|\mathbf{q}^{(m)}|_2 \leqslant N$. Then

$$0 < \min\left\{\frac{|\mathbf{q}^{(m)}|_2}{2}\left|\xi - \mathbf{p}^{(m)}(\mathbf{q}^{(m)})^{-1}\right|_2 : j = 1, \ldots, n\right\} = \eta$$

for some $\eta > 0$. Let $N = [1/\eta] + 1 > 1/\eta$, where $[x]$ is the integer part of the real number $x$. Then by (4·2), there exist $\mathbf{p}', \mathbf{q}' \in \mathcal{H}$ with $|\mathbf{q}'|_2 \leqslant N$ such that

$$\left|\xi - \mathbf{p}'(\mathbf{q}'^{-1})\right|_2 < \frac{2}{|\mathbf{q}'|_2\, N},$$

whence

$$\frac{|\mathbf{q}'|_2}{2}\left|\xi - \mathbf{p}'(\mathbf{q}'^{-1})\right|_2 < \frac{1}{N} < \eta,$$

and $\mathbf{p}'(\mathbf{q}')^{-1}$ cannot be one of the $\mathbf{p}^{(m)}(\mathbf{q}^{(m)})^{-1}$. This contradiction implies the result. $\square$

The smaller constant

$$c(\xi) = \liminf\{|\xi\mathbf{q} - \mathbf{p}|_2\, |\mathbf{q}| = |(\xi - \mathbf{pq}^{-1})\, \mathbf{q}^2|_2 : \mathbf{pq}^{-1} \in \mathcal{Q}\} \leqslant \sqrt{2/5} \qquad (4·4)$$

was established by Speiser [**69**] for asymptotic approximation, *i.e.*, for all $\xi \in \mathbb{H}$, there exist infinitely many pairs $\mathbf{p}, \mathbf{q} \in \mathcal{H}$ such that

$$\left|\xi - \mathbf{p}\,\mathbf{q}^{-1}\right|_2 \leqslant \sqrt{\frac{2}{5}}\frac{1}{|\mathbf{q}|_2^2} < \frac{1}{|\mathbf{q}|_2^2}. \qquad (4·5)$$

A. L. Schmidt [**61**] showed that $\sqrt{2/5}$ could not be reduced, so that it is analogous to Hurwitz's best possible rational approximation constant $1/\sqrt{5}$ for real numbers [**41**, §11.8] and Ford's $1/\sqrt{3}$ for complex numbers [**29**] (see also [**72**]). Since the approximating rational quaternions are not required to be in their lowest terms here, the inequality (4·5) holds for all $\xi \in \mathbb{H}$.

4·3. *Badly approximable quaternions*

In parallel with the classical case, Dirichlet's theorem for quaternions is best possible in the sense that the exponent 2 in (4·3) is best possible. Accordingly, a quaternion $\xi$ for which there exists a constant $c > 0$ such that

$$|\xi - \mathbf{pq}^{-1}| \geqslant \frac{c}{|\mathbf{q}|_2^2}$$

for all $\mathbf{pq}^{-1}$ is called *badly approximable*. By (4·5), $c \leqslant \sqrt{2/5}$. The set of badly approximable approximable quaternions will be denoted $\mathfrak{B}_{\mathbb{H}}$.

## 5. Ψ*-approximable quaternions*

The inequality (4·5) establishes that there are infinitely many $\mathbf{p}$, $\mathbf{q}^{-1}$ in $\mathcal{H}$ such that the approximants $\mathbf{pq}^{-1} \in \mathcal{Q}$ of Euclidean distance are at most $|\mathbf{q}|_2^{-2}$ from the quaternion $\xi \in \Delta$. As in the real case, it is natural to replace the error by a general approximation function $\Psi$, *i.e.*, a function $\Phi\colon (0,\infty) \to (0,\infty)$ such that $\Psi(x) \to 0$ as $x \to \infty$. We then consider the general inequality

$$|\xi - \mathbf{p}\,\mathbf{q}^{-1}|_2 < \Psi(|\mathbf{q}|_2) \tag{5·1}$$

for $\xi \in \mathbb{H}$, or without loss of generality, for $\xi$ in the compact set $\overline{\Delta}$ with $\Delta$ the $\mathcal{H}$-fundamental region from (4·1). The Euclidean norm $|\mathbf{q}|_2$ chosen for quaternions and the argument $|\mathbf{q}|_2$ of the approximation function $\Psi$ being defined on $\sqrt{\mathbb{N}} = \{\sqrt{k}\colon k \in \mathbb{N}\}$. (This minor complication would be avoided by working with the square of the norm but then the analogy with $\mathbb{R}$ would not be so close.) To make life simpler and to make comparison with other types of Diophantine approximation easier, we will take $\Psi$ to be a step function satisfying

$$\Psi(x) = \Psi([x]),$$

where $[x]$ is the integer part of $x$.

The main objective of this paper is to determine the metrical structure of the set

$$\mathcal{W}(\Psi) = \left\{\xi \in \mathbb{H}\colon |\xi - \mathbf{p}\,\mathbf{q}^{-1}|_2 < \Psi(|\mathbf{q}|_2) \;\text{ for infinitely many } \mathbf{p}, \mathbf{q} \in \mathcal{H}\right\}$$

and some related sets. Choosing the approximation function $\Psi$ as above is natural and fits in with a Duffin-Schaeffer conjecture [**70**, pg. 17] for quaternions that we will not address here. Nevertheless, the conjecture is still problematic as a more appropriate choice of argument for $\Psi$ would be the Hurwitz integer $\mathbf{q}$ rather than an integer $k$ (see [**42**]). Restricting the approximating Hurwitz rationals $\mathbf{pq}^{-1}$ to those with $\mathbf{p}, \mathbf{q}$ coprime, *i.e.*, to the subset

$$\mathcal{W}'(\Psi) = \left\{\xi \in \mathbb{H}\colon |\xi - \mathbf{p}\,\mathbf{q}^{-1}|_2 < \Psi(|\mathbf{q}|_2) \;\text{ for infinitely many } \mathbf{p}, \mathbf{q} \in \mathcal{H}, (\mathbf{p}, \mathbf{q})_r = 1\right\}$$

of $\mathcal{W}(\Psi)$, raises some minor technicalities and will not be considered.

*Henceforth, unless otherwise stated,* $\Psi\colon \mathbb{N} \to (0,\infty)$ *will be a (monotonic) decreasing approximation function.*

Theorem 4·1 implies that if $\Psi(x)$ increases, then $\mathcal{W}(\Psi) = \mathbb{H}$. Removing monotonicity altogether turns out to be a difficult and subtle problem, associated with the Duffin-Schaeffer conjecture. Although we will be concerned mainly with monotonic decreasing approximation functions, we could, without loss of generality, take $\Psi$ to be simply monotonic in some general statements. The union of translates by Hurwitz integers of the compact subset

$$\mathcal{V}(\Psi) := \mathcal{W}(\Psi) \cap \overline{\Delta} = \left\{\xi \in \overline{\Delta}\colon |\xi - \mathbf{p}\,\mathbf{q}^{-1}|_2 < \Psi(|\mathbf{q}|_2) \;\text{ for infinitely many } \mathbf{p}, \mathbf{q} \in \mathcal{H}\right\},$$

of $\mathcal{W}(\Psi)$ yields

$$\mathcal{W}(\Psi) = \bigcup_{\mathbf{p} \in \mathcal{H}} (\mathcal{V}(\Psi) + \mathbf{p}). \tag{5·2}$$

Thus the measure of $\mathcal{W}(\Psi)$ can be obtained from that of $\mathcal{V}(\Psi)$. The same holds for the set $\mathcal{V}'(\Psi) := \mathcal{W}'(\Psi) \cap \overline{\Delta}$.

5·1. *Resonant points, resonant sets and near-resonant sets*

Diophantine equations and approximation can be associated with the physical phenomenon of resonance and for this reason the rationals $p/q$ are referred to as *resonant* points in $\mathbb{R}$ (the terminology is drawn from mechanics, see for example [**2**, §18]). From this point of view, the Hurwitz rationals $\mathbf{pq}^{-1} \in \mathcal{Q}$ are resonant points in $\mathbb{H}$. In view of (5·2), there is no loss of generality in considering quaternions restricted to $\overline{\Delta}$. For each non-zero $\mathbf{q} \in \mathcal{H}$, the lattice $\mathcal{R}_{\mathbf{q}}$ of Hurwitz rationals or resonant points $\mathbf{pq}^{-1}$ in $\overline{\Delta}$ given by

$$\mathcal{R}_{\mathbf{q}} = \left\{\mathbf{pq}^{-1} \colon \mathbf{p} \in \mathcal{H}\right\} \cap \overline{\Delta}$$

is useful in calculations. This resonant set is an analogue in $\mathbb{H}$ of the set of equally spaced points $\{p/q \colon 0 \leqslant p \leqslant q\}$ in $[0,1]$.

For each $\mathbf{q}$, the number $\#\mathcal{R}_{\mathbf{q}}$ of Hurwitz rationals $\mathbf{pq}^{-1}$ in $\overline{\Delta}$ is the number of $\mathbf{p}$ in $\mathbf{q}\overline{\Delta}$, *i.e.*,

$$\#\mathcal{R}_{\mathbf{q}} = \sum_{\mathbf{p}\in\mathcal{H}\colon\, \mathbf{pq}^{-1}\in\overline{\Delta}} 1 = |\mathbf{q}|_2^4 + O(|\mathbf{q}|_2^3) \asymp |\mathbf{q}|_2^4. \tag{5·3}$$

The set $\mathcal{R} := \{\mathcal{R}_{\mathbf{q}} \colon \mathbf{q} \in \mathcal{H} \setminus \{0\}\} = \mathcal{Q} \cap \overline{\Delta}$ consists of the Hurwitz rationals $\mathcal{Q}$ in $\overline{\Delta}$.

Let $B_0 := B(\xi_0; r) = \{\xi \in \mathbb{H} \colon |\xi - \xi_0|_2 < r\}$ be the quaternionic ball centred at $\xi_0$ with radius $r$ and 4-volume $|B_0| = \pi^2 r^4/2 \asymp r^4$ (2·3). The number of Hurwitz integers $\mathbf{p}$ in $N\overline{\Delta}$ is $N^4 + O(N^3)$. Thus by volume considerations, the number of resonant points $\mathbf{pq}^{-1}$ with $|\mathbf{p}|_2 < |\mathbf{q}|_2$ satisfies

$$\sum_{\mathbf{p}\in\mathcal{H}\colon\, |\mathbf{p}|_2<|\mathbf{q}|_2} 1 = 2\frac{\pi^2}{2}|\mathbf{q}|_2^4 + O(|\mathbf{q}|_2^3) = \pi^2|\mathbf{q}|_2^4 + O(|\mathbf{q}|_2^3) \asymp |\mathbf{q}|_2^4.$$

The number of resonant points $\mathbf{pq}^{-1}$ with $\mathbf{p}, \mathbf{q}$ coprime could be considered by using the quaternionic analogue of Euler's $\phi$ function but this raises some complicated technicalities and will not be pursued here.

For each non-zero $\mathbf{q} \in \mathcal{H}$, let

$$\mathcal{B}(\mathcal{R}_{\mathbf{q}}; \varepsilon) = \bigcup_{\mathbf{p}\in\mathcal{H}} B(\mathbf{pq}^{-1}, \varepsilon) \cap \overline{\Delta} = \left\{\xi \in \overline{\Delta} \colon \left|\xi - \mathbf{pq}^{-1}\right| < \varepsilon \text{ for some } \mathbf{p} \in \mathcal{H}\right\}$$

be the set of balls $B(\mathbf{pq}^{-1}, \varepsilon)$ in $\overline{\Delta}$. The points in $\mathcal{B}(\mathcal{R}_{\mathbf{q}}, \varepsilon)$ are within $\varepsilon$ of a resonant point and so will be called *near-resonant* points. The centres $\mathbf{pq}^{-1}$ lie in $\mathcal{R}_q$ and the number of such balls is $\asymp |\mathbf{q}|^4$. Clearly $\mathcal{B}(\mathcal{R}_{\mathbf{q}}, \varepsilon)$ is a finite lattice or array of quaternionic balls $B(\mathbf{pq}^{-1}, \varepsilon) \cap \overline{\Delta}$. By (2·3) and (5·3), we have $|B(\mathbf{pq}^{-1}, \varepsilon)| \ll \varepsilon^4$ and, provided $\varepsilon$ is small enough, the near-resonant set $\mathcal{B}(\mathcal{R}_{\mathbf{q}}, \varepsilon)$ has Lebesgue measure

$$|\mathcal{B}(\mathcal{R}_{\mathbf{q}}, \varepsilon)| \asymp |\mathbf{q}|_2^4\, \varepsilon^4. \tag{5·4}$$

5·2. *The structure of* $\mathcal{V}(\Psi)$

It is readily verified that the set $\mathcal{V}(\Psi) \subset \overline{\Delta}$ can be expressed in the form of a 'limsup set' involving unions of near-resonant sets as follows:

$$\begin{aligned} \mathcal{V}(\Psi) &= \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{[|\mathbf{q}|_2]=n} \mathcal{B}(\mathcal{R}_{\mathbf{q}}, \Psi(|\mathbf{q}|_2)) = \bigcap_{N=1}^{\infty} \bigcup_{|\mathbf{q}|_2\geqslant N} \mathcal{B}(\mathcal{R}_{\mathbf{q}}, \Psi(|\mathbf{q}|_2)) \\ &:= \limsup_{|\mathbf{q}|_2\to\infty} \mathcal{B}(\mathcal{R}_{\mathbf{q}}, \Psi(|\mathbf{q}|_2)). \end{aligned} \tag{5·5}$$

Similarly

$$\mathcal{W}(\Psi) = \bigcap_{N=1}^{\infty} \bigcup_{|\mathbf{q}|_2 \geqslant N} \bigcup_{\mathbf{p}\in\mathcal{H}} B(\mathbf{p}\mathbf{q}^{-1}, \Psi(|\mathbf{q}|_2)) = \limsup_{|\mathbf{q}|_2\to\infty} \bigcup_{\mathbf{p}\in\mathcal{H}} B(\mathbf{p}\mathbf{q}^{-1}, \Psi(|\mathbf{q}|_2)). \tag{5·6}$$

It follows that $\mathcal{V}(\Psi)$ has a natural cover

$$\mathcal{C}_N(\mathcal{V}(\Psi)) = \{\mathcal{B}(\mathcal{R}_{\mathbf{q}}, \Psi(|\mathbf{q}|_2))\colon |\mathbf{q}|_2 \geqslant N\} \tag{5·7}$$

for each $N = 1, 2, \ldots$. By (5·4), the Lebesgue measure of $\mathcal{B}(\mathcal{R}_{\mathbf{q}}, \Psi(|\mathbf{q}|_2))$ satisfies

$$|\mathcal{B}(\mathcal{R}_{\mathbf{q}}, \Psi(|\mathbf{q}|_2))| \asymp |\mathbf{q}|_2^4 \Psi(|\mathbf{q}|_2)^4.$$

5·3. *Approximation involving a power law*

In the special case that $\Psi(x) := x^{-v}$, $(v > 0)$, we write $\mathcal{V}(\Psi) := \mathcal{V}_v$ and $\mathcal{W}(\Psi) := \mathcal{W}_v$. When $v = 2$, it follows from their definitions ((5·5), (5·6)) and from (4·5) that

$$\mathcal{V}_2 = \limsup_{|\mathbf{q}|_2\to\infty} \mathcal{B}(\mathcal{R}_{\mathbf{q}}, |\mathbf{q}|_2^{-2}) = \overline{\Delta} \quad \text{and} \quad \mathcal{W}_2 = \limsup_{|\mathbf{q}|_2\to\infty} \bigcup_{\mathbf{p}\in\mathcal{H}} B(\mathbf{p}\mathbf{q}^{-1}, |\mathbf{q}|_2^{-2}) = \mathbb{H}. \tag{5·8}$$

It is evident that for $v' \geqslant v$, $\mathcal{W}_{v'} \subseteq \mathcal{W}_v$ and $\mathcal{V}_{v'} \subseteq \mathcal{V}_v$. For $v > 2$, $\mathcal{W}_v$ will be called the set of *very well approximable* quaternions. Analogous definitions can be made for $\mathbb{R}^n$ and other spaces.

## 6. *Metrical Diophantine approximation in* $\mathbb{H}$*: the quaternionic Four Peaks*

In order to provide a convenient comparison with the real case, the analogous results for quaternions are now set out in the same order as in §3·2.

*The First Peak: Khintchine's theorem for* $\mathbb{H}$. As in the real case, the quaternionic Khintchine's theorem relates the Lebesgue measure of the set $\mathcal{W}(\Psi)$ of $\Psi$-approximable quaternions to the convergence or divergence of a certain 'volume' sum while the analogue for Jarník's extension of Khintchine's theorem does the same for Hausdorff $f$-measure. The quaternionic version of Khintchine's theorem is now stated.

THEOREM 6·1. *Let* $\Psi\colon \mathbb{N} \to (0, \infty)$. *Then the sets*

$$\mathcal{W}(\Psi) \text{ and } \mathcal{V}(\Psi) \text{ are } \begin{cases} \text{null} & \text{when } \sum_{m=1}^{\infty} \Psi(m)^4\, m^7 < \infty, \\ \text{full} & \text{when } \Psi \text{ is decreasing and } \sum_{m=1}^{\infty} \Psi(m)^4\, m^7 = \infty. \end{cases}$$

Note that when $\mathcal{V}(\Psi)$ has full Lebesgue measure, $|\mathcal{V}(\Psi)| = |\Delta| = 1/2$. Again, it is evident that the value of the critical 'volume' or 'measure' sum

$$\sum_{m=1}^{\infty} \Psi(m)^4 m^7 \tag{6·1}$$

determines the Lebesgue measure of $\mathcal{W}(\Psi)$ and $\mathcal{V}(\Psi)$. Similar critical sums are associated with Hausdorff measures.

*The Second Peak: Jarník's Hausdorff measure theorem for* $\mathbb{H}$.

THEOREM 6·2. *Let $f$ be a dimension function with $f(x)/x^4$ decreasing and $f(x)/x^4 \to \infty$ as $x \to 0$. Then*

$$\mathscr{H}^f(\mathcal{W}(\Psi)) = \mathscr{H}^f(\mathcal{V}(\Psi)) = \begin{cases} 0 & \text{when } \sum_{m=1}^{\infty} m^7 f(\Psi(m)) < \infty, \\ \infty & \text{when } \Psi \text{ is decreasing and } \sum_{m=1}^{\infty} m^7 f(\Psi(m)) = \infty. \end{cases}$$

The sum

$$\sum_{m=1}^{\infty} m^7 f(\Psi(m)) \tag{6·2}$$

is the critical sum for Hausdorff $f$-measure. This $f$-measure version of Theorem 6·1 does not hold for Lebesgue measure but the two theorems can be combined into a single quaternionic 'Khintchine-Jarník' result (see [**10**, §2.3]).

Theorem 6·3. *Let $f$ be a dimension function with $f(x)/x^4$ decreasing. Then*

$$\mathscr{H}^f(\mathcal{V}(\Psi)) = \begin{cases} 0 & \text{when } \sum_{m=1}^{\infty} m^7 f(\Psi(m)) < \infty, \\ \mathscr{H}^f(\overline{\Delta}) & \text{when } \Psi \text{ is decreasing and } \sum_{m=1}^{\infty} m^7 f(\Psi(m)) = \infty. \end{cases}$$

The Mass Transference Principle (see §9·1 below) can also be used to deduce this theorem from Theorem 6·1. Specialising Theorem 6·2 to Hausdorff $s$-measure, where $f(x) = x^s$, gives

Theorem 6·4. *Suppose $0 \leqslant s < 4$. Then*

$$\mathscr{H}^s(\mathcal{W}(\Psi)) = \mathscr{H}^s(\mathcal{V}(\Psi)) = \begin{cases} 0, & \text{when } \sum_{m=1}^{\infty} m^7 \Psi(m)^s < \infty, \\ \infty, & \text{when } \Psi \text{ is decreasing and } \sum_{m=1}^{\infty} m^7 \Psi(m)^s = \infty. \end{cases}$$

Specialising further to the Hausdorff $s$-measure for a power law approximation function, *i.e.*, to $\Psi(m) = m^{-v}$, where $v > 0$, gives

Theorem 6·5. *Suppose $v > 2$. Then*

$$\mathscr{H}^s(\mathcal{W}_v) = \mathscr{H}^s(\mathcal{V}_v) = \begin{cases} 0 & \text{when } s > 8/v, \\ \infty & \text{when } s \leqslant 8/v. \end{cases}$$

*The Third Peak: the Jarník-Besicovitch theorem for $\mathbb{H}$.* The Hausdorff dimension of $\mathcal{W}_v$ is the point of discontiunuity of the Hausdorff measure $\mathscr{H}^s(\mathcal{W}_v)$ and so the quaternionic version of the Jarník-Besicovitch theorem follows by definition from the above result.

Theorem 6·6. *Let $v \geqslant 0$. Then the Hausdorff dimension of $\mathcal{W}_v$ is given by*

$$\dim_{\mathrm{H}} \mathcal{W}_v = \dim_{\mathrm{H}} \mathcal{V}_v = \begin{cases} 4 & \text{when } v \leqslant 2, \\ \dfrac{8}{v} & \text{when } v > 2. \end{cases}$$

Note that $\mathscr{H}^s(\mathcal{W}_v) = \infty$ when $s = \dim_{\mathrm{H}} \mathcal{W}_v = 8/v$. A proof of this result will also be given in §9·2 below, using the Mass Transference Principle (see §9·1 below) and the quaternionic Dirichlet theorem (Theorem 4·1).

*The Fourth Peak: Jarník's theorem for $\mathfrak{B}_{\mathbb{H}}$.* The definition of $\mathfrak{B}_{\mathbb{H}}$, the set of badly approximable quaternions, is given in §4·3 above.

Theorem 6·7. *The set $\mathfrak{B}_{\mathbb{H}}$ is null with full Hausdorff dimension, i.e., $|\mathfrak{B}_{\mathbb{H}}| = 0$ and*

$$\dim_{\mathrm{H}}(\mathfrak{B}_{\mathbb{H}}) = 4.$$

## 7. *Ubiquitous systems*

As has been pointed out in §4, the metrical structure of lim sup sets which arise in number theory and elsewhere can be analysed very effectively using ubiquity. A ubiquitous system (or more simply ubiquity) is a more quantitative form of density underlying the classical Lebesgue and the more delicate Hausdorff measure results. Originally introduced to investigate lower bounds for Hausdorff dimension [**25**], ubiquitous systems have been extended considerably and now provide a way of determining the Lebesgue and Hausdorff measures of a very general class of 'limsup' sets [**8**, Theorems 1 & 2]. Indeed using the Mass Transference Principle, these two measures have been shown to be equivalent for this class of limsup sets, rather than Hausdorff measure being a refinement of Lebesgue [**10**].

7·1. *A metric space setting*

The definition of ubiquity given in [**8**] applies to a compact metric space $(\Omega, d)$ with a non-atomic finite measure $\mu$ (which includes $n$-dimensional Lebesgue measure). The resonant sets play the role of the approximants, which in the real line consist of the rationals. We will give a simplified version appropriate for Diophantine approximation in $\mathbb{H}$. The deep arguments in [**8**, **51**] are based on dyadic dissection suited to the Cantor-type constructions used in the proof. Thus the important ubiquity sum (9·7) is 2-adic, unlike the critical sum (6·1) which emerges from simpler standard estimates.

We start with a family $\mathcal{R}$ of *resonant sets* $R_j$ in $\Omega$, where $j$ lies in a countable discrete index set $J$ with each $j \in J$ having a weight $\lfloor j \rfloor$. The number of $j$ satisfying $\lfloor j \rfloor \leqslant N$ is assumed to be finite for each $N \in \mathbb{N}$. In $\mathbb{H}$ we take $j = \mathbf{q}$ , the index set $J = \{\mathbf{q} \in \mathcal{H} \colon \mathbf{q} \neq 0\}$, and the weight $\lfloor j \rfloor = \lfloor \mathbf{q} \rfloor := |\mathbf{q}|_2$. The resonant set $R_j = \mathcal{R}_{\mathbf{q}}$ corresponds to the lattice $\mathcal{R}_{\mathbf{q}}$ of resonant points $\mathbf{p}\mathbf{q}^{-1} \in \mathcal{H}$ or in $\mathcal{H} \cap \overline{\Delta}$. In the general formulation, the resonant sets $R_j$ can be lines, planes etc.

Let $B_0 := B(\xi_0, r) = \{\xi \in \Omega \colon d(\xi, \xi_0) < r\}$, for $r > 0$, be any fixed ball in $\Omega$ and let $\mathcal{R}$ be the family $\{R_j \colon j \in J\}$ of resonant points in $\Omega$. Further, let $\Psi$ be an approximation function, *i.e.*, $\Psi \colon (0, \infty) \to (0, \infty)$ converges to 0 at $\infty$. Let $\rho \colon \mathbb{N} \to (0, \infty)$ be a function with $\rho(m) = o(1)$. If for a given $B_0$,

$$\mu(B_0 \cap \bigcup_{1 \leqslant \lfloor j \rfloor \leqslant N} B(R_j, \rho(N))) \gg \mu(B_0), \tag{7·1}$$

where the implied constant in (7·1) is independent of $B_0$, then the family $\mathcal{R} = \{R_j \colon j \in J\}$ is said to be a *(strongly) ubiquitous system with respect to the function* $\rho$ and the weight $\lfloor \cdot \rfloor$. The idea here is that the family of near-resonant balls $B(R_j, \rho(N))$ meets the arbitrary ball $B_0$ in $\Omega$ substantially and covers it at least partially in measure. This can be regarded as a fairly general Dirichlet-type condition in which a 'significant' proportion of points is close to some resonant point $R_j$. It is evident that we want $\rho$ as small as possible. Note that in [**25**], $\rho$ was required to be decreasing; this condition is no longer required in the improved formulation in [**8**]. In applications, $\rho$ can often be chosen to be essentially a simple function, such as a power. In particular, for quaternionic Diophantine approximation, the choice of exponent is 2 (see §9·4). This is the same exponent as the ubiquity function for rational approximation on the real line $\mathbb{R}$ and is quite different from that for simultaneous rational approximation (see §9·8). The reason goes back to the similarity between the Dirichlet's theorems for the two spaces.

The set of points in $\Omega$ which are $\Psi$-approximable by the family $\mathcal{R} = \{\mathcal{R}_j \colon j \in J\}$ with respect to the weight $\lfloor \cdot \rfloor$ is defined by

$$\Lambda(\Psi) := \{\xi \in \Omega \colon \xi \in B(R_j, \Psi(\lfloor j \rfloor)) \text{ for infinitely many } j \in J\}. \tag{7·2}$$

If the family $\mathcal{R}$ is a ubiquitous system with respect to a suitable $\rho$ and weight, then the metrical structure of $\Lambda(\Psi)$ can be determined. Note that the set on the right hand side of (7·2) can be rewritten as a 'limsup' set (and hence falls into the ambit of the framework in [**8**]) as follows,

$$\Lambda(\Psi)=\bigcap_{N=1}^{\infty}\bigcup_{m=N}^{\infty}\bigcup_{\lfloor j\rfloor=m}B(R_j,\Psi(\lfloor j\rfloor))=\bigcup_{\lfloor j\rfloor\geqslant N}B(R_j,\Psi(\lfloor j\rfloor))=\limsup_{\lfloor j\rfloor\to\infty}B(R_j,\Psi(\lfloor j\rfloor)).$$

Thus for each $N=1,2,\ldots$, we have

$$\Lambda(\Psi)\subseteq\bigcup_{\lfloor j\rfloor\geqslant N}B(R_j,\Psi(\lfloor j\rfloor))=\mathcal{C}_N,$$

where $\mathcal{C}_N=\{B(R_j,\Psi(\lfloor j\rfloor))\colon\lfloor j\rfloor\geqslant N\}$ is the natural cover for $\Lambda(\Psi)$; the cover for $\mathcal{V}(\Psi)$ given in (5·7) is a special case.

## 8. *The proof of Khintchine's theorem for* $\mathbb{H}$ *when the critical sum converges*

The straightforward proof follows from (5·5) and the form of the natural cover $\mathcal{C}_N(\mathcal{V}(\Psi))$ for $\mathcal{V}(\Psi)$ (5·7). It follows by (5·4) that for each $N=1,2,\ldots$, the Lebesgue measure of $\mathcal{V}(\Psi)$ satisfies

$$|\mathcal{V}(\Psi)|\leqslant\sum_{m=N}^{\infty}\sum_{m\leqslant|\mathbf{q}|_2<m+1}|B(\mathcal{R}_{\mathbf{q}},\Psi(|\mathbf{q}|_2))|\ll\sum_{m=N}^{\infty}\sum_{m\leqslant|\mathbf{q}|_2<m+1}|\mathbf{q}|_2^4\,\Psi(|\mathbf{q}|_2)^4 \qquad (8·1)$$

By [**41**, Th. 386], the number $r_4(m)$ of Hurwitz integers $\mathbf{q}$ with $|\mathbf{q}|_2^2=m$ is given by

$$r_4(m)=8\sum_{d|m,\,4\nmid d}d$$

but for our purposes a simpler estimate suffices. By volume considerations,

$$\sum_{|\mathbf{q}|_2<m+1}1=\frac{\pi^2}{2}\,2\,m^4+O(m^3)\sim\pi^2m^4,$$

whence for each $m\in\mathbb{N}$,

$$\sum_{m\leqslant|\mathbf{q}|_2<m+1}1=\sum_{|\mathbf{q}|_2<m+1}1-\sum_{|\mathbf{q}|_2<m}1\ll m^3,$$

where in the sum on the left hand side, $|\mathbf{q}|_2$ ranges over the $2m+1$ values

$$m,\sqrt{m^2+1},\ldots,\sqrt{(m+1)^2-1}.$$

But $\Psi(|\mathbf{q}|_2):=\Psi([|\mathbf{q}|_2])=\Psi(m)$ when $m\leqslant|\mathbf{q}|_2<m+1$, so that

$$|\mathcal{V}(\Psi)|\ll\sum_{m=N}^{\infty}\Psi(m)^4(m+1)^4\sum_{m\leqslant|\mathbf{q}|_2<m+1}1\ll\sum_{m=N}^{\infty}\Psi(m)^4m^7. \qquad (8·2)$$

Thus for each $N=1,2,\ldots$, the measure of $\mathcal{V}(\Psi)$ satisfies

$$|\mathcal{V}(\Psi)|\ll\sum_{m=N}^{\infty}\Psi(m)^4m^7.$$

Since $N$ is arbitrary, if the critical sum (6·1) converges then the tail $\sum_{m=N}^{\infty}\Psi(m)^4m^7$ converges to 0 and $\mathcal{V}(\Psi)$ is a null set, *i.e.*,

$$|\mathcal{V}(\Psi)|=|\mathcal{W}(\Psi)|=0.$$

This is the convergence part of Theorem 6·1, the quaternionic analogue of Khintchine's theorem. Note that since $\mathcal{V}'(\Psi) \subset \mathcal{V}(\Psi)$, the convergence of the critical sum implies that $|\mathcal{V}'(\Psi)| = |\mathcal{W}'(\Psi)| = 0$ also.

## 9. *The proof when the critical sum diverges*

The case of divergence is much more difficult. The ideas involved, particularly ubiquity (see §7) and the remarkable Mass Transference Principle (see §9·1), require some further definitions and notation. First we explain the principle in a simple setting to clarify the ideas and give an application to indicate its power. Then we explain ubiquity.

### 9·1. *The Mass Transference Principle*

The Mass Transference Principle, introduced by Beresnevich and Velani in [**9**], is a remarkable technique which allows Lebesgue measure results for lim sup sets to be transferred to Hausdorff measures. A version adapted to our purposes is now given. Let $n$ be a non-negative integer and let $f$ be a dimension function (see §2) such that $x^{-n}f(x)$ is monotonic. For any ball $B = B(c, r)$ centred at $c$ and radius $r$, let

$$B^f := B(c, f(r)^{1/n}).$$

When for $s > 0$, $f(x) = x^{-s}$, write $B^f = B^s$, so that $B^s(c, r) = B(c, r^{s/n})$; note that $B^n = B$. Similarly for a family $\mathcal{B} = \{B(c_i, r_i)\}$ of balls in $\Omega$, let

$$\mathcal{B}^f := \{B(c_i, f(r_i)^{1/n})\},$$

so that $\mathcal{B}^n = \mathcal{B}$. Let $\{\mathcal{B}_i = \cup_j B(c_{i_j}, r_i) \colon i \in \mathbb{N}\}$ be a family of finite unions of balls $B(c_{i_j}, r_i)$ in $\mathbb{R}^n$ with the same radius $r_i \to 0$ as $i \to \infty$. Suppose that for any ball $B_0 \in \mathbb{R}^n$,

$$|(B_0 \cap \limsup_{i\to\infty} \mathcal{B}_i^f)| = |B_0|.$$

Then the Mass Transference Principle asserts that

$$\mathscr{H}^f(B_0 \cap \limsup_{i\to\infty} \mathcal{B}_i) = \mathscr{H}^f(B_0).$$

Thus the appropriate version of Khintchine's theorem would imply Jarník's $f$-measure theorem. This has not been proved for quaternions but (4·4) can be used with the mass transference principle to prove Theorem 6·6, the quaternionic analogue of the Jarník-Besicovitch theorem.

### 9·2. *An application to $\mathcal{V}_v$ : the quaternionic Jarník-Besicovitch theorem*

In §9·1, take $n = 4$, $f(x) = x^s$, $s < 4$ and $\Psi(x) = x^{-v}$, $v > 0$, $c_i = \mathbf{p}\mathbf{q}^{-1}$ (recall that $\mathbf{p}, \mathbf{q}$ are not necessarily coprime) and $r_i = \Psi(|\mathbf{q}_2|)$. Let the set $\mathcal{B}_{\mathbf{q}} := \mathcal{B}(\mathcal{R}_{\mathbf{q}}, |\mathbf{q}|_2^{-v})$ correspond to the set $\mathcal{B}_i$ in §9·1 and $\mathcal{B}_{\mathbf{q}}^{8/v} = \mathcal{B}(\mathcal{R}_{\mathbf{q}}, |\mathbf{q}|_2^{-2})$ correspond to $\mathcal{B}_i^{8/v}$.

Suppose $v \leqslant 2$. Then by (5·8),

$$\mathcal{V}_v = \limsup_{|\mathbf{q}_2|\to\infty} \mathcal{B}(\mathcal{R}_{\mathbf{q}}, |\mathbf{q}|_2^{-v}) = \overline{\Delta},$$

whence *a fortiori*, $|\mathcal{V}_v| = |\overline{\Delta}| = 1/2$ and $\dim \mathcal{V}_v = 4$.

Suppose $v > 2$. By the definition of $\mathcal{B}_{\mathbf{q}}^{8/v}$ and by (4·5),

$$\limsup_{|\mathbf{q}|_2\to\infty} \mathcal{B}_{\mathbf{q}}^{8/v} = \limsup_{|\mathbf{q}|_2\to\infty} \mathcal{B}(\mathcal{R}_{\mathbf{q}}, |\mathbf{q}|_2^{-2}) = \overline{\Delta},$$

whence

$$\left|B_0 \cap \limsup_{|\mathbf{q}|_2\to\infty} \mathcal{B}_{\mathbf{q}}^{8/v}\right| = \left|B_0 \cap \overline{\Delta}\right| = |B_0|\,.$$

It follows by the Mass Transference Principle that

$$\mathscr{H}^{8/v}\left(B_0 \cap \limsup_{|\mathbf{q}|_2\to\infty} \mathcal{B}_{\mathbf{q}}\right) = \mathscr{H}^{8/v}\left(B_0 \cap \mathcal{V}_v\right) = \mathscr{H}^{8/v}(B_0) = \infty$$

since $B_0$ is open and $8/v < 4$. But $B_0 \cap \mathcal{V}_v \subset \mathcal{V}_v$, whence for $s \leqslant 8/v$,

$$\mathscr{H}^{s}(\mathcal{V}_v) = \mathscr{H}^{8/v}(\mathcal{V}_v) = \infty$$

and from the definition of Hausdorff dimension, $\dim_{\mathrm{H}} \mathcal{V}_v \geqslant 8/v$.

Next suppose $s > 8/v$. By (5·7), for each $N = 1, 2, \ldots$, the family of balls

$$\{B(\mathbf{p}\mathbf{q}^{-1}, |\mathbf{q}|_2^{-2}) \colon |\mathbf{p}|_2 \leqslant |\mathbf{q}|_2, |\mathbf{q}|_2 \geqslant N\}$$

is a cover for $\mathcal{V}_v$. Hence by (2·1), for each $N \in \mathbb{N}$,

$$\begin{aligned}
\mathscr{H}^{s}(\mathcal{V}_v) &\leqslant \sum_{m=N}^{\infty} \sum_{m\leqslant|\mathbf{q}|_2<m+1} \sum_{\mathbf{p}\colon\, \mathbf{p}\in\overline{\Delta}\,\mathbf{q}} \left(\operatorname{diam} B(\mathbf{p}\mathbf{q}^{-1}, |\mathbf{q}|_2^{-v})\right)^s \\
&\ll \sum_{m=N}^{\infty} \sum_{m\leqslant|\mathbf{q}|_2<m+1} \sum_{\mathbf{p}\colon\, \mathbf{p}\in\overline{\Delta}\,\mathbf{q}} |\mathbf{q}|_2^{-sv} \ll \sum_{m=N}^{\infty} \sum_{m\leqslant|\mathbf{q}|_2<m+1} |\mathbf{q}|_2^{4}|\mathbf{q}|_2^{-sv} \\
&\ll \sum_{m=N}^{\infty} m^{4-sv} \sum_{m\leqslant|\mathbf{q}|_2<m+1} 1 \ll \sum_{m=N}^{\infty} m^{7-vs} \to 0 \text{ as } N\to\infty,
\end{aligned}$$

since $s > 8/v$. Thus $\mathscr{H}^{s}(\mathcal{V}_v) = 0$ for $s > 8/v$ and $\dim_{\mathrm{H}} \mathcal{V}_v \leqslant 8/v$. Combining the values of $\mathscr{H}^{s}(\mathcal{V}_v)$ gives for $v > 2$,

$$\mathscr{H}^{s}(\mathcal{W}_v) = \mathscr{H}^{s}(\mathcal{V}_v) = \begin{cases} \infty & \text{when } s \leqslant \frac{8}{v}, \\ 0 & \text{when } s > \frac{8}{v}, \end{cases}$$

which is Theorem 6·2, which in turn implies Theorem 6·6, the quaternionic Jarník-Besicovitch theorem. Note that the Hausdorff $s$-measure is infinite at $s = \dim_{\mathrm{H}} \mathcal{V}_v$.

Since $\Psi$ is decreasing and $f$ increasing, the composition $f \circ \Psi$ is decreasing. Thus Khintchine's Theorem for quaternions (Theorem 6·1) implies that when the sum $\sum_m f(\Psi(m))m^7$ diverges,

$$|B_0 \cap (\limsup_{|\mathbf{q}|_2\to\infty} \mathcal{B}(\mathcal{R}_{\mathbf{q}}, f(\Psi(|\mathbf{q}|_2))))| = |B_0|.$$

Hence by the Mass Transfer Principle,

$$\mathscr{H}^{f}(B_0 \cap \mathcal{V}(\Psi)) = \mathscr{H}^{f}(\overline{\Delta}),$$

so that divergent case of Khintchine's theorem implies that of Jarník's $f$-measure theorem. However, this case of Khintchine's theorem needs to be proved and more ideas are needed to deal with the general decreasing approximation function $\Psi$.

9·3. *The quaternionic Khintchine theorem in the divergent case*

The objective here is to complete the determination of the Lebesgue and Hausdorff measures of $\Psi$-approximable quaternions when the critical sums diverge. We recall that the quaternions $\mathbb{H}$ form a 4-dimensional metric space which naturally carries Lebesgue measure. It is convenient to work with the compact set $[0,1]^3 \times [0,1/2] = \overline{\Delta}$, given in §5·1 above, for $\Omega$ and with the set of $\Psi$-approximable quaternions in $\overline{\Delta}$, *i.e.*, with $\mathcal{V}(\Psi) = \mathcal{W}(\Psi) \cap \overline{\Delta}$ instead of with $\mathcal{W}(\Psi)$.

We begin by stating a simplified version of the Beresnevich-Velani theorem [**10**, Th. 3] for the ubiquitous systems described in §7, and then deduce the analogue of Khintchine's theorem for $\mathbb{H}$ in the divergence case. The Beresnevich-Velani theorem holds for a compact metric space with a measure comparable to Lebesgue measure. The theorem can be regarded as a general Khintchine-Jarník result and illustrates the power of ubiquity and mass transfer (see §9·1 above). Note that in addition to converging to 0 at infinity, the ubiquity function $\rho$ must also satisfy the technical condition that for some positive constant $c < 1$,

$$\rho(2^{r+1}) \leqslant c\rho(2^r) \tag{9·1}$$

for $r$ sufficiently large. Such functions will be called *dyadically decaying*, a condition which is satisfied in the applications considered here. This condition is weaker than the requirement in earlier work (see for example [**25**]) that $\rho$ be decreasing. Note that the definition in [**8**] is more general: $\rho$ is '$u$-regular', a condition which involves a sequence $(u_n : n \in \mathbb{N})$. The very general Theorem 2 in [**8**] could also be used to first prove the analogue of Khintchine's theorem and then the analogue of the Khintchine-Jarník theorem deduced via mass transference §9·1. The dyadic decay condition can be imposed on $\Psi$ instead.

THEOREM 9·1 (Beresnevich-Velani). *Let $(\Omega, d)$ be a compact metric space equipped with a Borel measure $\mu$ which for some $\delta > 0$ satisfies*

$$\mu(B(\xi, r)) \asymp r^\delta \tag{9·2}$$

*for any sufficiently small ball $B(\xi, r)$ in $\Omega$. Suppose that the family $\mathcal{R}$ of resonant sets in $\Omega$ is a strongly $\mu$-ubiquitous system relative to the dyadically decaying function $\rho$ and that $\Psi$ is a decreasing approximation function. Let $f$ be a dimension function with $f(x)/x^\delta$ monotonic. If for some $\kappa > 1$, the ubiquity sum*

$$\sum_{m=1}^{\infty} \frac{f(\Psi(\kappa^m))}{\rho(\kappa^m)^\delta} \tag{9·3}$$

*diverges, then the Hausdorff $f$-measure $\mathscr{H}^f(\Lambda(\Psi))$ is given by*

$$\mathscr{H}^f(\Lambda(\Psi)) = \mathscr{H}^f(\Omega).$$

The hypotheses of Theorem 9·1 imply that $\mu$ is comparable to the $\delta$-dimensional Hausdorff measure $\mathscr{H}^\delta$ and that $\dim \Omega = \delta$. Note that in the Beresnevich-Velani theorem, the sum (9·3) is '$\kappa$-adic', whereas we have been working with 'standard' sums such as (6·1). By the choice of $\rho$ we will make and by Lemmas 9·1 and 9·2, the critical sum (6·1) will be comparable to the ubiquity sum (9·3).

9·4. *Perturbing divergent sums*

The following lemma, drawn from [**16**], is needed to construct the ubiquity function $\rho$.

LEMMA 9·1. *Let $F : \mathbb{N} \to (0, \infty)$ satisfy $\sum_{m=1}^{\infty} F(m) = \infty$. Then there exists a decreasing function $\eta : \mathbb{N} \to [0,1]$ with $\eta(m) = o(1)$, such that for any $\alpha > 0$, the sequence*

*$m\eta(m)^\alpha \to \infty$ as $m \to \infty$, $\eta(2^r) \leqslant 2\eta(2^{r+1})$ and such that $\sum_{m=1}^\infty F(m)\,\eta(m) = \infty$, $r = 1, 2, \ldots$.*

*Proof.* Since $\sum_{m=1}^\infty F(m) = \infty$, we can choose a strictly increasing sequence $(m_i \colon i = 1, 2, \ldots)$ with $m_1 = 1$ such that $m_{i+1} \geqslant 2m_i \geqslant \cdots \geqslant 2^i$ and

$$\sum_{m_i \leqslant m < m_{i+1}} F(m) > 1.$$

Define $\eta \colon \mathbb{N} \to [0, 1]$ by

$$\eta(m) = i^{-1}, \; m \in [m_i, m_{i+1}). \tag{9·4}$$

Evidently $\eta$ is decreasing, $o(1)$ and

$$m\,\eta(m)^\alpha \geqslant \frac{m_i}{i^\alpha} \geqslant 2^i\, i^{-\alpha} \to \infty$$

as $i$ and hence $m \to \infty$. In addition, if $\eta(2^r) = 1/i$, then by the choice of the intervals $[m_1, m_{i+1})$, $\eta(2^{r+1}) = 1/i$ or $1/(i+1)$, whence $\eta(2^{r+1}) \leqslant \eta(2^r) \leqslant 2\eta(2^{r+1})$.

Moreover

$$\sum_{m=1}^\infty F(m)\eta(m) = \sum_{i=1}^\infty \sum_{m_i \leqslant m < m_{i+1}} \eta(m)F(m) = \sum_{i=1}^\infty i^{-1} \sum_{m_i \leqslant m < m_{i+1}} F(m) > \sum_{i=1}^\infty i^{-1} = \infty.$$

□

Thus $F$ can be replaced by a smaller function $F\eta$ without affecting the divergence of the sum. Clearly $\eta$ depends on $F$.

9·5. *The functions $\eta$ and $\rho$*

Let $\eta = \eta(F)$ be the function in Lemma 9·1 corresponding to $F(m) = f(\Psi(m))\,m^7$; recall that in the divergent case $\sum_m f(\Psi(m))m^7 = \infty$ by hypothesis. Define the function $\rho \colon \mathbb{N} \to (0, 1]$ by

$$\rho(m) := \frac{2}{\eta(m)^{1/4} m^2}\,. \tag{9·5}$$

Then the function $\rho$ is the product of an inverse square and the slowly increasing function $1/\eta$. It turns out that $\eta$ decreases sufficiently slowly to ensure that $\rho(m') \leqslant 2^{1/4}\rho(m)$ for $m' \geqslant m$ (so that $\rho$ is decreasing modulo $2^{1/4}$).

LEMMA 9·2. *The function $\rho$ satisfies*

(i) *$\rho(m) = o(1)$,*
(ii) *$\rho(m)^{-1} = o(m^2)$,*
(iii) *$\rho(m') \ll \rho(m)$ for all $m' \geqslant m$,*
(iv) *$\rho$ decays dyadically.*

*Proof.*

(i) By Lemma 9·1, $\eta(m)^{1/4} m \to \infty$ as $m \to \infty$, so $\rho(m) := 2(\eta(m)^{1/4}\, m^2)^{-1} \to 0$.
(ii) Since $\rho(m) := 2(\eta(m)^{1/4}\, m^2)^{-1}$, we have that $\rho(m)^{-1} m^{-2} = \eta(m)^{1/4}/2 \to 0$ as $m \to \infty$.
(iii) Suppose $m \leqslant m'$. We consider cases; recall $i \in \mathbb{N}$ and that $m_{i+1} \geqslant 2m_i$. When $m \leqslant m'$ and $m, m' \in [m_i, m_{i+1})$,

$$\rho(m') = \frac{2}{\eta(m')^{1/4} m'^2} = \frac{2i^{1/4}}{m'^2} \leqslant \frac{2i^{1/4}}{m^2} = \rho(m).$$

If $m \in [m_i, m_{i+1})$ and $m' \in [m_{i+1}, m_{i+2})$, then

$$\rho(m') = \frac{2(i+1)^{1/4}}{m'^2} < \frac{2i^{1/4}}{m^2}\left(\frac{i+1}{i}\right)^{1/4} \leqslant 2^{1/4}\rho(m),$$

since $(1+i)/i \leqslant 2$. In the remaining case $m \in [m_i, m_{i+1})$ and $m' \in [m_{i'}, m_{i'+1})$, where $i' = i+j \geqslant i+2$, and $m$ and $m'$ satisfy

$$m' \geqslant m_{i+j} \geqslant 2^{j-1} m_{i+1} > 2^{j-1} m.$$

It follows that

$$\rho(m') = \frac{2i'^{1/4}}{m'^2} \leqslant \frac{2(i+j)^{1/4}}{2^{2j-2}\,m^2} = \frac{2i^{1/4}}{m^2}\,2^{-2j+2}\left(\frac{i+j}{i}\right)^{1/4} < \frac{3^{1/4}}{4}\,\rho(m) < \rho(m)$$

for $i \geqslant 1, j \geqslant 2$.

(iv) To establish dyadic decay, first suppose $2^r, 2^{r+1} \in [m_i, m_{i+1})$. Then

$$\rho(2^{r+1}) = \frac{2i^{1/4}}{2^{2(r+1)}} = \frac{2}{4}\frac{i^{1/4}}{2^{2r}} = \frac{1}{4}\rho(2^r).$$

Next suppose $2^r \in [m_i, m_{i+1})$ and $2^{r+1} \notin [m_i, m_{i+1})$. Then since $m_{i+2} \geqslant 2m_{i+1}$, it follows that $2^{r+1} \in [m_{i+1}, m_{i+2})$ and

$$\rho(2^{r+1}) = \frac{2(i+1)^{1/4}}{2^{2(r+1)}} = \frac{2}{4}\frac{i^{1/4}}{2^{2r}}\left(\frac{i+1}{i}\right)^{1/4}.$$

But $1 < (1+i)/i \leqslant 2$ for $i \in \mathbb{N}$, whence for each $r \in \mathbb{N}$,

$$\frac{1}{4}\,\rho(2^r) < \rho(2^{r+1}) \leqslant \frac{2^{1/4}}{4}\,\rho(2^r) < \rho(2^r).$$

Thus $\rho$ decays dyadically (see (9·1)).

□

The main part of the proof is to use the quaternionic Dirichlet Theorem (Theorem 4·1) to establish that the Hurwitz rationals $\mathcal{Q}$ form a ubiquitous system.

LEMMA 9·3. *The Hurwitz rationals $\mathcal{Q}$ in $\overline{\Delta}$ are ubiquitous with respect to the function $\rho$ and the weight given by $\lfloor \mathbf{q} \rfloor = |\mathbf{q}|_2$.*

*Proof.* By the uniform Dirichlet theorem for $\mathbb{H}$ (Theorem 4·1), any point $\xi$ in $B_0$ in $\overline{\Delta}$ can be approximated with an error $2/(|\mathbf{q}|_2 N)$ for some $\mathbf{q}$ with $|\mathbf{q}|_2 \leqslant N$. Thus for each $N \in \mathbb{N}$,

$$B_0 \subseteq \bigcup_{1 \leqslant |\mathbf{q}|_2 \leqslant N} B\left(\mathcal{R}_{\mathbf{q}}; \frac{2}{|\mathbf{q}|_2 N}\right)$$

and so

$$B_0 = B_0 \cap \left(\bigcup_{1 \leqslant |\mathbf{q}|_2 \leqslant N} B\left(\mathcal{R}_{\mathbf{q}}; \frac{2}{|\mathbf{q}|_2 N}\right)\right),$$

where we recall $\mathcal{R}_{\mathbf{q}} = \{\mathbf{p}\mathbf{q}^{-1} \in \overline{\Delta}\}$.

To remove the dependence of the radius on the denominator $\mathbf{q}$, we select 'large' denominators $\mathbf{q}$ with $\varpi(N) \leqslant |\mathbf{q}|_2 \leqslant N$, where $\varpi \colon \mathbb{N} \to (0, \infty)$ is given by

$$\varpi(m) = \eta(m)^{1/4} m, \tag{9·6}$$

and where, by Lemma 9·1, $\varpi(m) \to \infty$ as $m \to \infty$. We remove Hurwitz rationals with 'small' denominators as follows. Let $E(N)$ be the set of $\xi \in B_0$ with 'small' denominator approximants $\mathbf{pq}^{-1}$, $1 \leqslant |\mathbf{q}|_2 < \varpi(N)$ with $|\xi - \mathbf{pq}^{-1}| < 2(|\mathbf{q}|_2 N)^{-1}$. Then $B_0 = E(N) \cup (B_0 \setminus E(N))$ and

$$E(N) \subseteq \bigcup_{1 \leqslant |\mathbf{q}|_2 < \varpi(N)} B\left(\mathcal{R}_{\mathbf{q}}, \frac{2}{N|\mathbf{q}|_2}\right).$$

By (5·4) and other estimates in §5·1, the Lebesgue measure of $E(N)$ satisfies

$$\begin{aligned}|E(N)| &\leqslant \left|\bigcup_{1 \leqslant |\mathbf{q}|_2 < \varpi(N)} B\left(\mathcal{R}_{\mathbf{q}}, \frac{2}{|\mathbf{q}|_2 N}\right)\right| \leqslant \sum_{1 \leqslant |\mathbf{q}|_2 < \varpi(N)} \left|B\left(\mathcal{R}_{\mathbf{q}}, \frac{2}{|\mathbf{q}|_2 N}\right))\right| \\ &\leqslant \sum_{1 \leqslant |\mathbf{q}|_2 < \varpi(N)} \frac{2^4}{|\mathbf{q}|_2^4 N^4} |\mathbf{q}|_2^4 = \frac{2^4}{N^4} \sum_{1 \leqslant |\mathbf{q}|_2 < \varpi(N)} 1 \\ &\ll N^{-4} \sum_{1 \leqslant m < \varpi(N)} m^3 \ll N^{-4} \varpi(N)^4.\end{aligned}$$

Since $\varpi(N) = \eta(N)^{1/4} N$, it follows that $\varpi(N)/N = o(1)$. Thus $|E(N)| \to 0$ and $|B_0 \setminus E(N)| \to |B_0|$ as $N \to \infty$. But by definition and by (9·6), for each $\xi \in B_0 \setminus E(N)$, there exist $\mathbf{p}, \mathbf{q} \in \mathcal{H}$ with $\varpi(N) \leqslant |\mathbf{q}|_2 \leqslant N$ such that

$$|\xi - \mathbf{pq}^{-1}| < \frac{2}{|\mathbf{q}|_2 N} \leqslant \frac{2}{\varpi(N) N} = \frac{2}{\eta(N)^{1/4} N^2} = \rho(N)$$

by (9·5) and (9·6). Moreover by Lemma 9·2, $\rho$ is dyadically decaying. Now

$$B_0 \setminus E(N) \subseteq B_0 \cap \left(\bigcup_{\varpi(N) \leqslant |\mathbf{q}|_2 \leqslant N} B(\mathcal{R}_{\mathbf{q}}, \rho(N))\right) \subseteq B_0 \cap \left(\bigcup_{1 \leq |\mathbf{q}|_2 \leq N} B(\mathcal{R}_{\mathbf{q}}, \rho(N))\right)$$

and it follows that for $N$ sufficiently large,

$$\left|B_0 \cap \bigcup_{1 \leq |\mathbf{q}|_2 \leq N} B(\mathcal{R}_{\mathbf{q}}, \rho(N))\right| \geqslant |B_0 \setminus E(N)| \geqslant \frac{1}{2}|B_0| \; (\gg r^4),$$

whence by (7·1) the Hurwitz rationals $\mathcal{Q}$ are ubiquitous with respect to the function $\rho$ given by (9·5) and the weight $|\cdot|_2$. $\square$

Note that the Hausdorff dimension of $\mathcal{V}(\Psi)$ in terms of the lower order of $\Psi$ can be obtained with less difficulty from this ubiquity result using the methods in [**11**, **25**]. To determine the measure requires the extra power of the Beresnevich-Velani Theorem.

We now state the specialisation of Theorem 9·1 to $\mathbb{H}$ and to Lebesgue and Hausdorff measure. This theorem unites the divergent cases of the quaternionic Khintchine and Jarník theorems.

THEOREM 9·2. *Let $\Omega = \overline{\Delta} \subset \mathbb{H}$ and $J = \mathcal{H} \setminus \{0\}$, so that $\delta = 4$, $\mathcal{R} = \mathcal{Q} \cap \overline{\Delta}$, $j = \mathbf{q}$, $R_j = \mathcal{R}_{\mathbf{q}}$ and $\Lambda(\Psi) = \mathcal{V}(\Psi)$. Let $f$ be a dimension function with $f(x)/x^4$ increasing and let $\rho$ be given by (9·5), so that $\mathcal{Q} \cap \overline{\Delta}$ is a ubiquitous system with respect to the weight $\lfloor \mathbf{q} \rfloor = |\mathbf{q}|_2$ and $\rho$. Suppose the ubiquity sum*

$$\sum_{r=1}^{\infty} \frac{f(\Psi(2^r))}{\rho(2^r)^4} \tag{9·7}$$

*diverges. If* $f(x) = x^4$, *then*

$$\mathscr{H}^4(\mathcal{V}(\Psi)) = \mathscr{H}^4(\overline{\Delta}) = 2^5\pi^{-2} \tag{9·8}$$

*and if* $f(x)/x^4 \to \infty$ *as* $x \to 0$, *then*

$$\mathscr{H}^f(\mathcal{V}(\Psi)) = \mathscr{H}^f(\overline{\Delta}) = \infty.$$

9·6. *The proof of Theorem* 6·1 *(Khintchine's theorem for* $\mathbb{H}$*)*

The proof when the critical sum (6·1) converges is given in §8. In the case of divergence, divergent dyadic and standard sums need to be compared.

LEMMA 9·4. *Let* $\Psi$ *be a decreasing approximation function and let* $f$ *be a dimension function. If the sum* (6·1) *diverges, then the ubiquity sum* (9·7) *also diverges.*

*Proof.* Take $F(m) = f(\Psi(m))\, m^7$ in Lemma 9·1. Then by (9·5), by the choice of $\eta$ in equation (9·4) and by Lemma 9·1, the divergence of the sum $\sum_{m=1}^{\infty} f(\Psi(m))\, m^7$ implies that the sum

$$\sum_{m=1}^{\infty} f(\Psi(m))\, m^7 \eta(m) = \sum_{m=1}^{\infty} f(\Psi(m)) m^7 \, \frac{1}{m^8\, \rho(m)^4} = \sum_{m=1}^{\infty} \frac{1}{m}\, \frac{f(\Psi(m))}{\rho(m)^4} \tag{9·9}$$

also diverges. Now since $f(\Psi(m))$ decreases as $m$ increases and since $\rho(m') \ll \rho(m)$ when $m' \geqslant m$ (Lemma 9·2),

$$\begin{aligned}
\sum_{m=1}^{\infty} \frac{1}{m}\, \frac{f(\Psi(m))}{\rho(m)^4} &= \sum_{r=0}^{\infty} \sum_{2^r \le m < 2^{r+1}} \frac{1}{m}\, \frac{f(\Psi(m))}{\rho(m)^4} \\
&\ll \sum_{r=0}^{\infty} 2^{-r} f(\Psi(2^r)) \rho(2^{r+1})^{-4} \sum_{2^r \le m < 2^{r+1}} 1 \\
&\ll \sum_{r=0}^{\infty} f(\Psi(2^r)) \rho(2^r)^{-4}
\end{aligned}$$

and the result follows. $\square$

Thus the divergence of the critical sum $\sum_{m=1}^{\infty} f(\Psi(m)) m^7$ (6·1) implies that the ubiquity sum (9·7) also diverges. When the dimension function $f$ is given by $f(x) = x^4$, it follows from (9·8) and (2·2) that $|\mathcal{V}(\Psi)| = |\overline{\Delta}| = 1/2$.

9·7. *Proofs of Jarník's Hausdorff measure theorem and the Jarník-Besicovitch Theorem for* $\mathbb{H}$

Theorem 9·2 and Lemma 9·4 can also be applied when $f(x)/x^4 \to \infty$ as $x \to 0$. Alternatively the Mass Transference Principle could be invoked (see §9·2).

*Jarník's Hausdorff measure theorem (Theorem* 6·2*).* Recall from (6·2) the definition of the critical sum:

$$\sum_{m=1}^{\infty} m^7 f(\Psi(m)).$$

*The case when the critical sum converges:* By (5·7), for each $N = 1, 2, \ldots$, the family of balls

$$\{B(\mathbf{p}\mathbf{q}^{-1}, \Psi(|\mathbf{q}|_2)) \colon |\mathbf{p}|_2 \le |\mathbf{q}|_2, |\mathbf{q}|_2 \geqslant N\}$$

is a cover for $\mathcal{V}(\Psi)$. Hence by (2·1), for each $N = 1, 2, \ldots$, the Hausdorff $f$ measure of $\mathcal{V}(\Psi)$ satisfies

$$\begin{aligned}\mathscr{H}^f(\mathcal{V}(\Psi)) &\leqslant \sum_{m=N}^{\infty} \sum_{m\leqslant|\mathbf{q}|_2<m+1} \sum_{|\mathbf{p}|_2\leq|\mathbf{q}|_2} f(\operatorname{diam} B(\mathbf{p}\mathbf{q}^{-1}, \Psi(|\mathbf{q}|_2))) \\ &\ll \sum_{m=N}^{\infty} \sum_{m\leqslant|\mathbf{q}|_2<m+1} |\mathbf{q}|_2^4\, f(2\Psi(|\mathbf{q}|_2)) \ll \sum_{m=N}^{\infty} m^4 f(2\Psi(m)) \sum_{m\leqslant|\mathbf{q}|_2<m+1} 1 \\ &\ll \sum_{m=N}^{\infty} m^7 f(2\Psi(m)).\end{aligned}$$

But by hypothesis, $f(x)/x^4$ decreases as $x$ increases and so

$$\begin{aligned}\mathscr{H}^f(\mathcal{V}(\Psi)) &\ll \sum_{m=N}^{\infty} m^7 f(2\Psi(m))\,(2\Psi(m))^{-4}\,(2\Psi(m))^4 \\ &\ll \sum_{m=N}^{\infty} m^7 f(\Psi(m))\,(\Psi(m))^{-4}\,2^4(\Psi(m))^4 \\ &\ll \sum_{m=N}^{\infty} m^7 f(\Psi(m)).\end{aligned}$$

Thus $\mathscr{H}^f(\mathcal{V}(\Psi)) = 0$ when $\sum_{m=1}^{\infty} m^7\ f(V(\Psi))$ converges.

*The case when the critical sum diverges:* Lemma 9·4 implies that the ubiquity sum (9·7) also diverges. Hence by Theorem 9·2,

$$\mathscr{H}^f(\mathcal{V}(\Psi)) = \mathscr{H}^f(\overline{\Delta}) = \infty$$

when $f(x)/x^4 \to \infty$ as $x \to 0$, which is Theorem 6·2.

*Theorem* 6·4 *and the Jarník-Besicovitch Theorem (Theorem* 6·6*).* The Hausdorff $s$-measure result follows by putting $f(x) = x^s$.

The Hausdorff dimension is the point of discontinuity of $\mathscr{H}^s(\mathcal{W}_v)$; this occurs at $s = 8/v$.

9·8. *Simultaneous Diophantine approximation in* $\mathbb{R}^4$

The theorems of Dirichlet, Khintchine, Jarník and Jarník-Besicovitch on simultaneous Diophantine approximation in 4-dimensional euclidean space $\mathbb{R}^4$ are stated for comparison with quaternions. First, Dirichlet's theorem in $\mathbb{R}^4$ [**41**] is stated.

THEOREM 9·3. *For each* $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{R}^4$ *and* $N \in \mathbb{N}$, *there exists a* $\mathbf{p} = (p_1, p_2, p_3, p_4)$ *in* $\mathbb{Z}^4$, $q \in \mathbb{N}$ *such that*

$$\max_{1\leq m\leq 4}\left\{\left|\alpha_m - \frac{p_m}{q}\right|\right\} = \left|\alpha - \frac{\mathbf{p}}{q}\right|_\infty < \frac{1}{qN^{1/4}}.$$

*Moreover there are infinitely many* $\mathbf{p} \in \mathbb{Z}^4, q \in \mathbb{N}$ *such that*

$$\left|\alpha - \frac{\mathbf{p}}{q}\right|_\infty < \frac{1}{q^{5/4}}.$$

In the more general form of approximation, write $W^{(4)}(\Psi)$ for the set of $\Psi$-approximable points in $\mathbb{R}^4$, *i.e.*, points $\alpha$ such that

$$\left|\alpha - \frac{\mathbf{p}}{q}\right|_\infty < \Psi(q)$$

for infinitely many $\mathbf{p} \in \mathbb{Z}^4$ and $q \in \mathbb{N}$. Khintchine's theorem for the set $W^{(4)}(\Psi)$ takes the form

THEOREM 9·4. *The Lebesgue measure of $W^{(4)}(\Psi)$ is null or full according as the critical sum*

$$\sum_{m=1}^{\infty} m^4 \Psi(m)^4$$

*converges or diverges.*

Gallagher [**40**] showed that $\Psi$ need not be decreasing in dimensions $\geqslant 2$, and Pollington & Vaughan established that the Duffin-Schaeffer Conjecture also holds in this case [**57**]. Jarník's Hausdorff $f$-measure result [**8**, Theorem DV, pg. 66] is now stated for $W^{(4)}(\Psi)$.

THEOREM 9·5 (Jarník). *Let $f$ be a dimension function such that $f(x)/x^4$ decreases as $r$ increases and $f(x)/x^4 \to \infty$ as $x \to 0$. Then*

$$\mathscr{H}^f(W^{(4)}(\Psi)) = \begin{cases} 0 & \textit{when } \sum_{r=1}^{\infty} r^4 f(\Psi(r)) < \infty \\ \infty & \textit{when } \sum_{r=1}^{\infty} r^4 f(\Psi(r)) = \infty \ \textit{ and } \Psi \textit{ decreasing.} \end{cases}$$

As in the case for $\mathbb{R}$, the two results can be combined into a single 'Khintchine-Jarník' theorem.

Let $W_v$ denote the set of $\Psi$-approximable points in $\mathbb{R}^4$ when $\Psi(x) = x^{-v}$. The Jarník-Besicovitch theorem on simultaneous Diophantine approximation in $\mathbb{R}^4$ follows by taking the dimension function $f(x) = x^{-s}$, $s > 0$.

COROLLARY 9·6.

$$\dim_{\mathrm{H}}(W_v) = \begin{cases} \frac{5}{v} & \textit{when } v \le 5/4 \\ 4 & \textit{when } v \ge 5/4. \end{cases}$$

It is evident that exponents in the sums and the Hausdorff dimension are quite different. Note that the the identitity $\mathbf{p}\mathbf{q}^{-1} = \mathbf{p}\overline{\mathbf{q}}/n$, where $n = q_1^2 + \cdots + q_4^2$, gives a natural embedding of $\mathcal{W}(\Psi)$ into $W^{(4)}(\Psi \circ \sqrt{\ })$ (recall $\Psi(x) = \Psi([x])$). In particular $\mathcal{W}_v \hookrightarrow W^{(4)}_{v/2}$.

## 10. *Jarník's theorem for badly approximable quaternions*

The set $\mathfrak{B}_{\mathbb{H}}$ of badly approximable quaternions is defined analogously to the real case in §4·3 and are quaternions for which the exponent in Theorem 4·1 cannot be increased. As with ubiquitous systems in §7, this notion can be placed in a general setting of a metric space $(X, d)$ with a compact subspace $\Omega$ which contains the support of a non-atomic finite measure $\mu$ and a family $\mathcal{R} = \{R_j : j \in J\}$ of resonant sets, where $J$ is a countable discrete index set (see [**51**]). The Hausdorff dimension of the set $\mathfrak{B}_\Omega$ of badly approximable points in $\Omega$ can be determined if the following two conditions on $\mu$ and $\Psi$ hold.

First, for each ball $B(\xi, r)$, the measure $\mu$ satisfies

$$ar^\delta \le \mu(B(\xi, r)) \le br^\delta,$$

where $0 < a \leq 1 \leq b$. This condition is satisfied by Lebesgue measure and implies that the Hausdorff dimension of $\Omega$ is given by $\dim_{\mathrm{H}}(\Omega) = \delta$.

Secondly, for $\kappa > 1$ sufficently large, $\Psi$ satisfies the '$\kappa$-adic' decay condition

$$\ell(\kappa) \leq \frac{\Psi(\kappa^n)}{\Psi(\kappa^{n+1})} \leq u(\kappa), \; n \in \mathbb{N},$$

where $\ell(\kappa) \leq u(\kappa)$ and $\ell(\kappa) \to \infty$ as $\kappa \to \infty$ (*cf* (9·2) in Theorem 9·1). It is convenient to write for each $n \in \mathbb{N}$

$$\nu_n = \nu_n(\Psi, \kappa) := \left(\frac{\Psi(\kappa^n)}{\Psi(\kappa^{n+1})}\right)^{\delta}.$$

Recall from §4·3 that a point $\beta \in \Omega$ which for some constant $c(\beta) > 0$ satisfies

$$d(\xi, R_j) \geqslant c(\beta)\Psi(\lfloor j \rfloor) \text{ for all } j \in J$$

is called $\Psi$-*badly approximable*. The set of $\Psi$-badly approximable points in $X$ will be denoted by $\mathfrak{B}_X(\Psi)$. For each $n \in \mathbb{N}$, let $\xi \in \Omega$ and write for convenience

$$B^{(n)} := B(\xi, \Psi(\kappa^n)) = \{\xi' \in \Omega \colon d(\xi, \xi') \leq \Psi(\kappa^n)\}$$

and its scaling by $\theta \in (0, \infty)$ as

$$\theta B^{(n)} := B(\xi, \theta\Psi(\kappa^n)) = \{\xi' \in \Omega \colon d(\xi, \xi') \leq \theta\Psi(\kappa^n)\}.$$

Apart from some changes in notation, the following is Theorem 1 in [**51**] and gives conditions under which the Hausdorff dimension of the set of $\Psi$-badly approximable points in $\Omega$ can be obtained.

THEOREM 10·1. *Let $(X, d)$ be a metric space and $(\Omega, d, \mu)$ a compact subspace of $X$ with a measure $\mu$. Let the measure $\mu$ and the function $\Psi$ satisfy conditions (A) and (B) respectively. For $\kappa \geqslant \kappa_0 > 1$, suppose there exists some $\theta \in (0, \infty)$ so that for $n \in \mathbb{N}$ and any ball $B^{(n)}$, there exists a collection $\mathcal{C}^{(n+1)}$ of disjoint balls $2\theta B^{(n+1)} = B(\mathbf{c}, 2\theta\Psi(\kappa^{n+1}))$ in $\theta\, B^{(n)}$, satisfying*

$$\# \, \mathcal{C}^{(n+1)} \geqslant K_1 \nu_n \tag{10·1}$$

*and*

$$\# \left\{ 2\theta B^{(n+1)} \subset \theta\, B^{(n)} \colon \min_{\substack{j \in J, \\ \kappa^{n-1} \leqslant \lfloor j \rfloor < \kappa^n}} d(\mathbf{c}, R_j) \leqslant 2\theta\Psi(\kappa^{n+1}) \right\} \leqslant K_2 \nu_n, \tag{10·2}$$

*where $K_1, K_2$ are absolute constants, independent of $\kappa$ and $n$, with $K_1 > K_2 > 0$. Furthermore suppose that $\dim_{\mathrm{H}}(\cup_{j \in J} R_j) < \delta$. Then*

$$\dim_{\mathrm{H}} \mathfrak{B}_{\Omega}(\Psi) = \delta.$$

The general metric space setting is again specialised to $\mathbb{H}$ to give the analogue of Jarník's theorem for the Hausdorff measure and dimension of the set $\mathfrak{B}_{\mathbb{H}}$ of badly approximable quaternions. When $X = \mathbb{H}$ and $\Omega = \overline{\Delta}$, the measure $\mu$ is 4-dimensional Lebesgue measure, $\delta = 4$, the resonant set $R_j$ is the point $\mathbf{p}\mathbf{q}^{-1} \in \mathfrak{Q}$ and $\lfloor j \rfloor = |\mathbf{q}|_2$. In view of the exponent 2 in (4·5) being extremal, we can take

$$\Psi(|\mathbf{q}|_2) = |\mathbf{q}|_2^{-2},$$

so that $\mathfrak{B}_X(\Psi) = \mathfrak{B}_{\mathbb{H}}$. Thus in this case

$$\nu_n = \left(\frac{\Psi(\kappa^n)}{\Psi(\kappa^{n+1})}\right)^{\delta} = \left(\frac{\kappa^{-2n}}{\kappa^{-2(n+1)}}\right)^4 = \kappa^8,$$

whence $\nu_n$ is independent of $n$ and satisfied (10).

Let

$$\theta = 2^{-1}\kappa^{-2}$$

and let the 4-ball $B^{(n)} = B(\xi, \kappa^{-2n})$ lie in $\overline{\Delta}$. Then the shrunken ball $\theta B^{(n)} = B(\xi, \theta\kappa^{-2n})$ has radius $2^{-1}\kappa^{-2(n+1)}$. A collection $\mathcal{C}^{(n+1)}$ of closed disjoint balls in $\theta B^{(n)}$ is constructed. Divide the ball $\theta B^{(n)}$ into hypercubes $H^{(n+1)}$ of side length $\ell = 2^{5/4}\kappa^{-2(n+2)}$. The number of such hypercubes is at least

$$\frac{1}{2}\frac{|\theta B^{(n)}|}{\ell^4} = \frac{\pi^2}{4}\,2^{-4}\kappa^{-8(n+1)} \times 2^{-5}\,\kappa^{8(n+2)} = \frac{\pi^2}{2^{11}}\,\kappa^8.$$

Let $\mathcal{C}^{(n+1)}$ be the collection of balls $2\theta B^{(n+1)}$ of radius $\kappa^{-2(n+2)}$, centred at the centre $\mathbf{c}$ of a hypercube $H^{(n+1)}$. The number $\#\mathcal{C}^{(n+1)}$ of such balls satisfies

$$\#\,\mathcal{C}^{(n+1)} \geqslant 2^{-11}\pi^2\,\kappa^8$$

and we can choose $K_1 = \pi^2/2^{11}$ in (10·1).

The distance between two points in the ball $\theta B^{(n)} = B(\xi, \theta\kappa^{-2n})$ is at most $\kappa^{-2(n+1)}$. Consider two distinct Hurwitz rationals $\mathbf{pq}^{-1}, \mathbf{rs}^{-1}$, where $\kappa^n \leq |\mathbf{q}|_2, |\mathbf{s}|_2 < \kappa^{n+1}$ and $\kappa > 1$. By Lemma 4·1,

$$|\mathbf{pq}^{-1} - \mathbf{p}'\mathbf{q}'^{-1}| \geqslant |\mathbf{q}|_2^{-1}|\mathbf{s}|_2^{-1} > \kappa^{-2(n+1)},$$

so that $\theta B^{(n)}$ contains at most one Hurwitz rational $\mathbf{pq}^{-1}$ with $\kappa^n \leqslant |\mathbf{q}|_2 < \kappa^{n+1}$. Thus such a point $\mathbf{pq}^{-1}$ can be in at most one ball $2\theta B^{(n+1)} \in \mathcal{C}^{(n+1)}$. Hence for quaternions, the inequality (10·2) reduces to

$$\#\left\{2\theta B^{(n+1)} \subset \theta\, B^{(n)} \colon \mathbf{pq}^{-1} \in 2\theta B^{(n+1)},\ \kappa^n \leqslant |\mathbf{q}|_2 < \kappa^{n+1}\right\} \leqslant 1 < \frac{\pi^2}{2^{12}}\,\kappa^8$$

for $\kappa \geqslant 3$, so that we can choose $K_2 = \pi^2/2^{12} < K_1$. Finally, since the resonant sets $\mathbf{pq}^{-1}$ are points, $\dim_{\mathrm{H}}(\{\mathbf{pq}^{-1}\}) = 0$. It now follows from Theorem 10·1 that $\dim_{\mathrm{H}}(\mathfrak{B}_{\mathbb{H}}) = 4$, *i.e.*, $\mathfrak{B}_{\mathbb{H}} \subset \mathbb{H}$ has full Hausdorff dimension.

As in the classical case, Theorem 6·1 can be used to show that $\mathfrak{B}_{\mathbb{H}}$ is null. Indeed since $\Psi(m) = m^{-2}$, the sum $\sum_{m\in\mathbb{N}} \Psi(m)^4 m^7 = \sum_{m\in\mathbb{N}} m^{-1}$ diverges. Hence the set of $\beta \in \overline{\Delta}$ satisfying the inequality

$$|\beta - \mathbf{pq}^{-1}|_2 \geqslant \frac{1}{|\mathbf{q}|_2^2}$$

for all but finitely many $\mathbf{pq}^{-1}$, say $\mathbf{p}^{(m)}(\mathbf{q}^{(m)})^{-1}$, $m = 1, 2, \ldots, N = N(\xi)$, is null. Let

$$c(\beta) := \min\{1, |\beta - \mathbf{p}^{(m)}(\mathbf{q}^{(m)})^{-1}|_2|\mathbf{q}^{(m)}|_2^2 \colon m = 1, 2, \ldots N\}.$$

Then the set of $\beta \in \overline{\Delta}$ satisfying the inequality

$$|\beta - \mathbf{pq}^{-1}|_2 \geqslant \frac{c(\beta)}{|\mathbf{q}|_2^2}$$

for all $\mathbf{pq}^{-1}$ is null. This completes the proof of Theorem 6·7, the analogue of Jarník's theorem for the set $\mathfrak{B}_{\mathbb{H}}$ of badly approximable quaternions.

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