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# A (very short) introduction to buildings[☆]

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**Abstract**

This is an informal elementary introduction to buildings – what they are and where they come from.

This is an informal elementary introduction to buildings – written for, and by, a non-expert. The aim is to get to the definition of a building and feel that it is an entirely natural thing. To maintain the lecture style examples have replaced proofs. The notes at the end indicate where these proofs can be found.

Most of what we say has its origins in the work of Jacques Tits, and our account borrows heavily from the books of Abramenko and Brown [1] and Ronan [15]. Section 1 illustrates all the essential features of a building in the context of an example, but without mentioning any building terminology. In principle anyone could read this. Sections 2-4 firm-up and generalize these specifics: Coxeter groups appear in §2, chambers systems in §3 and the definition of a building in §4. Section 5 addresses where buildings come from by describing the first important example: the spherical building of an algebraic group.

## 1. The flag complex of a vector space

Let $V$ be a three dimensional vector space over a field $k$. Let $\Delta$ be the graph with vertices the non-trivial proper subspaces of $V$, and an edge connecting the vertices $V_i$ and $V_j$ whenever $V_i$ is a subspace of $V_j$:

$$V_i \circ\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\bullet\; V_j \iff V_i \subset V_j.$$

Figure 1 shows the graph $\Delta$ when $k$ is the field of orders $q = 2$ and 3. There are $1+q+q^2$ one dimensional subspaces – illustrated by the white vertices – and $1 + q + q^2$ two dimensional subspaces, illustrated by the black vertices. Each one dimensional space is contained in $1+q$ two dimensional spaces and each two dimensional space contains $1 + q$ one dimensional spaces. The duality here might remind the reader of projective geometry. Call the edges $V_i \subset V_j$ of $\Delta$ *chambers*.

Some more structure can be wrung out of this picture: there is an "$\mathfrak{S}_3$-valued metric", with $\mathfrak{S}_3$ the symmetric group, that gives the shortest route(s) through $\Delta$ between any two chambers. To see how, suppose $c, c'$ are chambers and we want a shortest route of edges connecting them:

$$c = V_1 \subset V_2 \overset{\text{shortest route}}{\rightsquigarrow} c' = V'_1 \subset V'_2.$$

Make $c$ and $c'$ as different as possible by assuming that $V_1 \neq V'_1$, $V_2 \neq V'_2$ and $V_2 \cap V'_2$ is a line different from $V_1, V'_1$. Changing notation, let $L_1, L_2, L_3$ be lines with $L_1 = V_1$, $L_3 = V'_1$ and $L_2 = V_2 \cap V'_2$. One then gets $V_2 = L_1 + L_2$ and $V'_2 = L_2 + L_3$.

☆Based on a short series of lectures given at the Université de Fribourg, Switzerland in June 2011. The author is grateful for the department's hospitality both then and over the years, and particularly to Ruth Kellerhals for her

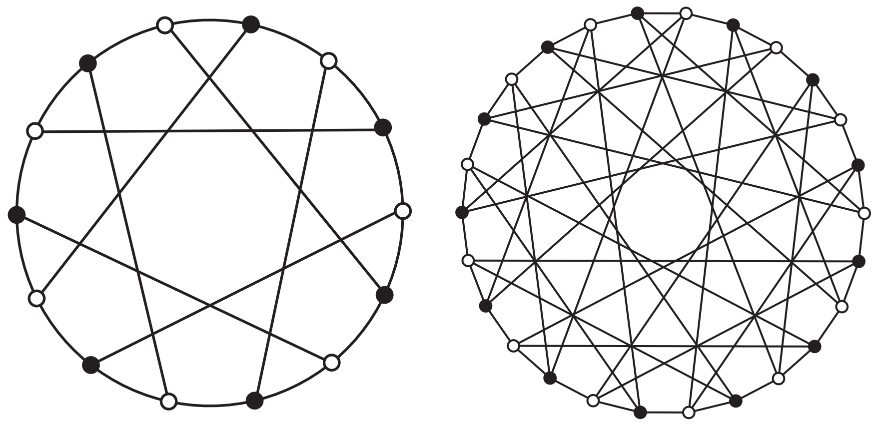

Figure 1: The flag complex $\Delta$ of the three dimensional vector space over the fields of order 2 *(left)* and 3 *(right)*.

We get a small piece of $\Delta$, a local picture containing $c, c'$, as in Figure 2. The field $k$ wasn't mentioned at all in the previous paragraph, so this is the local picture for $\Delta$ over any field. The *global* picture gets more complicated however as the field $k$ gets bigger as Figure 1 illustrates.

Say that chambers are *i-adjacent* if any difference between them occurs only in the $i$-th position, so $V_1 \subset V_2 \supset V_1'$, $(V_1 \neq V_1')$ are a pair of 1-adjacent chambers and $V_2 \supset V_1 \subset V_2'$, $(V_2 \neq V_2')$ a pair of 2-adjacent chambers (a chamber is also $i$-adjacent to itself for any $i$). Place the label $i$ on a vertex of the local picture in Figure 2 if the two chambers meeting at the vertex are $i$-adjacent.

The shortest routes from $c$ to $c'$ *in the local picture* are given by

$$c \xrightarrow[s_1 s_2 s_1]{s_2 s_1 s_2} c'$$

where the route $s_1 s_2 s_1$ means cross a 1-labeled vertex, then a 2-labeled vertex and then a 1-labeled vertex. Routes are read from left to right, although it obviously doesn't matter with the two above. These routes then take values in the symmetric group $\mathfrak{S}_3$ by letting $s_1 = (1, 2)$ and $s_2 = (2, 3)$, so that both $s_1 s_2 s_1$ and $s_2 s_1 s_2$ give the permutation $(1, 3) \in \mathfrak{S}_3$. Our actions will always be on the left, so in particular permutations in $\mathfrak{S}_3$ are composed from right to left. Define the $\mathfrak{S}_3$-distance between $c, c'$ to be $\delta(c, c') = (1, 3)$.

For an arbitrary pair of chambers define $\delta(c, c')$ to be the element of $\mathfrak{S}_3$ obtained by situating the chambers $c, c'$ in some local picture and taking the shortest route(s) as in Figure 2. The resulting map $\delta : \Delta \times \Delta \to \mathfrak{S}_3$ can be thought of as a metric on $\Delta$ taking values in $\mathfrak{S}_3$.

We will see in §4 why this map is well defined and doesn't depend on which local picture we choose containing $c, c'$, although an ad-hoc argument shows that an element of $\mathfrak{S}_3$ can be associated in a canonical fashion to any pair of chambers. Take the $c, c'$ above and write

$$c = 0 \subset L_1 \subset L_1 + L_2 \subset V = V_0 \subset V_1 \subset V_2 \subset V_3$$

and $c' = V_0' \subset \cdots \subset V_3'$ similarly. For each $i$ the filtration $V_0 \subset V_1 \subset V_2 \subset V_3$ of $V$ induces a filtration of the one dimensional quotient $V_i'/V_{i-1}'$:

$$(V_i' \cap V_0)/V_{i-1}' \subset \cdots \subset (V_i' \cap V_3)/V_{i-1}' \tag{1}$$

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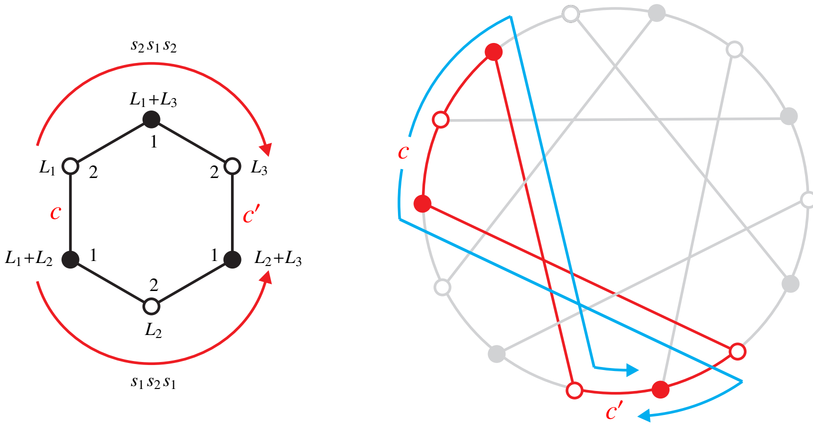



Figure 2: Local picture of $\Delta$ containing the pair of chambers $c, c'$ and the shortest routes between them *(left)*; situating the pair $c, c'$ in a local picture with the shortest routes $s_1s_2s_1 = s_2s_1s_2 = (1,3)$ *(right)*.

(by $(V'_i \cap V_0)/V'_{i-1}$, etc, we mean the image of $V'_i \cap V_0$ under the quotient map $V \to V/V'_{i-1}$). Any filtration of a one dimensional space must start with a sequence of trivial subspaces and end with a sequence of $V'_i/V'_{i-1}$'s. At some point in the middle the filtration jumps from being zero dimensional to one dimensional; for the $c, c'$ above:

| $i$ | $V'_i/V'_{i-1}$ | filtration (1) | "jump index" $j$ |
|---|---|---|---|
| 1 | $L_3$ | $0 \subset 0 \subset 0 \subset L_3$ | 3 |
| 2 | $(L_2+L_3)/L_3$ | $0 \subset 0 \subset (L_2+L_3)/L_3 \subset (L_2+L_3)/L_3$ | 2 |
| 3 | $V/(L_2+L_3)$ | $0 \subset V/(L_2+L_3) \subset V/(L_2+L_3) \subset V/(L_2+L_3)$ | 1 |

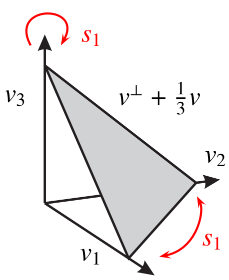
Defining $\pi(i) = j$ gives $\pi = (1,3) \in \mathfrak{S}_3$. Summarizing:

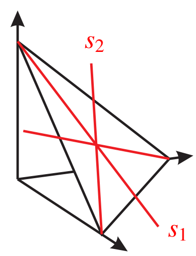
*First rough definition of a building* A building is a set of *chambers* with $i$-adjacency between them, the $i$ coming from some set $S$, together with a "$W$-valued metric" for $W$ some group.

Returning to the running example, the symmetric group $\mathfrak{S}_3$ is a reflection group, with Figure 2 and the resulting metric $\delta$ coming from the geometry of these reflections. To see why suppose we have a three dimensional Euclidean space – a real vector space with an inner product. Let $v_1, v_2, v_3$ be an orthonormal basis and let $\mathfrak{S}_3$ act on the space by permuting coordinates: $\pi \cdot v_i := v_{\pi(i)}$ for $\pi \in \mathfrak{S}_3$ (and extend linearly). This action is not essential as the vector $v = v_1 + v_2 + v_3$ is fixed by all $\pi \in \mathfrak{S}_3$. This can be gotten around by passing to the perp space

$$v^\perp = \{\sum \lambda_i v_i \mid \sum \lambda_i = 0\}.$$

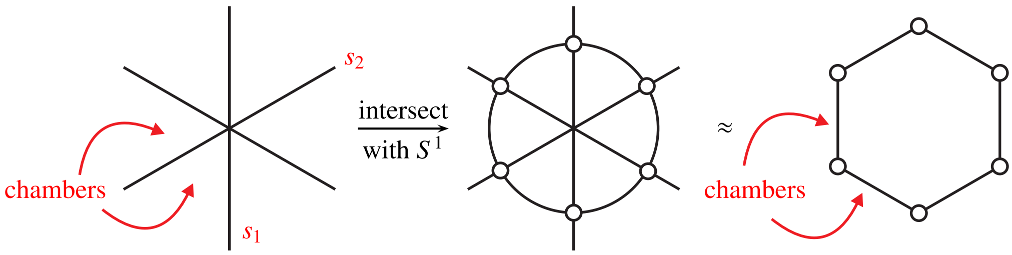
The picture to keep in mind is the following, where $v^\perp$ is translated off the origin to make it easier to see:



The element $s_1 = (1,2)$ acts as on the left – as the reflection in the plane with equation $x_1 - x_2 = 0$. Similarly $s_2 = (2,3)$ and $(1,3)$ are reflections in the planes $x_2 - x_3 = 0$ and $x_1 - x_3 = 0$. These three planes chop the intersection of $v^\perp + \frac{1}{3}v$ with the positive quadrant into a triangle with its boundary barycentrically subdivided (or hexagon). So we start to see the local picture of Figure 2 coming from the geometry of these reflecting hyperplanes.

Putting $v^\perp$ into the plane of the page decomposes the plane into six infinite wedge-shaped regions:

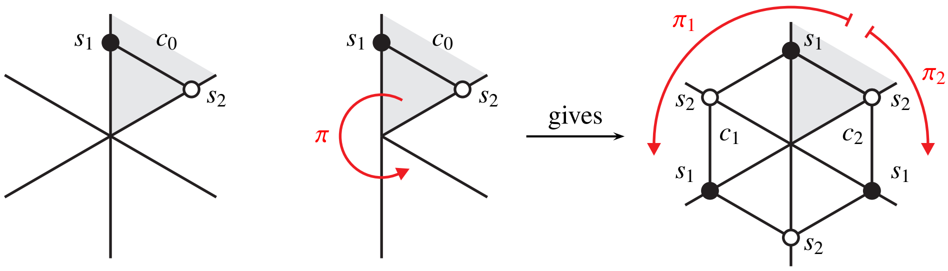



In the theory of reflection groups (§2) these regions are also called chambers. The chambers of our local picture are gotten back by intersecting these regions with the sphere $S^1$.

(In the next dimension up we can still draw pictures of some of these objects. Let $V$ be four dimensional over $k$ and $\Delta$ the two dimensional simplicial complex with vertices the non-trivial subspaces of $V$, edges (or 1-simplicies) the pairs $V_i \subset V_j$ and 2-simplicies the triples $V_i \subset V_j \subset V_k$. We can get the local picture by working backwards from a symmetric group action like we did above. If we have a four dimensional Euclidean space with orthonormal basis $v_1, v_2, v_3, v_4$, then the convex hull of the $v_i$ is a tetrahedron lying in the hyperplane $v^\perp + \frac{1}{4}v$ where $v = v_1 + v_2 + v_3 + v_4$. The six reflecting hyperplanes of the $\mathfrak{S}_4$-action have equations $x_i - x_j = 0$ and slice the boundary of the tetrahedron barycentrically. Identifying the hyperplane with three dimensions and intersecting the whole picture with the sphere $S^2$, we end up with Figure 3 (left). Flattening it out, we can retrospectively label the simplicies by lines $L_1, L_2, L_3, L_4 \in V$ and the spaces they generate.)

Returning to the hexagon, the $\mathfrak{S}_3$-action turns out to be regular on the chambers, i.e. given chambers $c, c'$ there is a unique $\pi \in \mathfrak{S}_3$ with $\pi c = c'$. This is most easily seen by brute force: fix a "fundamental" chamber $c_0$ and show that the six elements of $\mathfrak{S}_3$ send it to the six chambers in the decomposition above. In particular there is a one-one correspondence between the chambers and the elements of $\mathfrak{S}_3$ given by $\pi \in \mathfrak{S}_3 \leftrightarrow$ chamber $\pi c_0$.

This correspondence gives the adjacency labelings of the hexagonal local picture of Figure 2: choose the fixed chamber $c_0$ to be one of the two regions bounded by the reflecting lines for $s_1$ and $s_2$. Starting with the edge of the hexagon contained in $c_0$, label its vertices by the corresponding reflections as below left:




Now transfer this labeled edge to the other chambers using the $\mathfrak{S}_3$-action as in the picture above middle; the result is shown above right, where the $i$'s have become $s_i$'s. Vertices on opposite ends of the same line have different labels because the antipodal map $x \mapsto -x$ is not in the action of $\mathfrak{S}_3$ on the plane $v^\perp$.

Finally, to get the metric $\delta$ observe that if $c$ is some chamber of the local picture in Figure 2 and $\pi \in \mathfrak{S}_3$ sends $c_0$ to $c$, then $\pi = s_{i_1} \ldots s_{i_k}$ where $s_{i_k}, \ldots, s_{i_1}$ are the labels (read

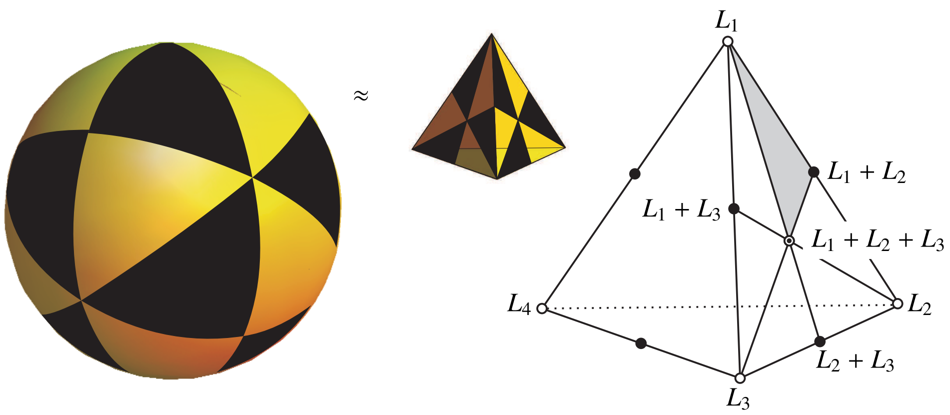



Figure 3: The local picture for the flag complex of a four dimensional space: the result of intersecting the hyperplanes $x_i - x_j = 0$ with $S^2$ *(left)*, flattened out *(middle)*, and the picture corresponding to the labelled hexagon of Figure 2 *(right)*. The shaded 2-simplex corresponds to the triple $L_1 \subset L_1 + L_2 \subset L_1 + L_2 + L_3$.

from left to right) on the vertices crossed in a path in the hexagon from $c_0$ to $c$. So for chambers $c_1, c_2$ we have $\delta(c_1, c_2) = \pi_1^{-1}\pi_2$ where $c_i = \pi_i c_0$. For our original $c_1, c_2$ we have $\pi_1 = s_1 s_2, \pi_2 = s_2$, hence $\delta(c_1, c_2) = s_2 s_1 s_2$ as shown in the picture above.

*Second rough definition of building* A building is a set of chambers with $i$-adjacency, the $i$ coming from some set $S$, together with a $W$-valued metric $\delta$, for $W$ a reflection group generated by $S$ and $\delta$ arising from the geometry of $W$.

In the next sections we will make precise and general the ideas in this rough definition, but working in the reverse order: we start with reflection groups (§2), then an abstract version of chambers and adjacency (§3) and finally $W$-valued metrics (§4).

## 2. Reflection Groups and Coxeter Groups

Reflection groups arise as the symmetries of familiar geometric objects; Coxeter groups are an abstraction of them. This section covers the basics. All vector spaces and linear maps here are over the reals $\mathbb{R}$.

A *reflection* of a finite dimensional vector space $V$ is a linear map $s : V \to V$ for which there is a decomposition

$$V = H_s \oplus L_s \tag{2}$$

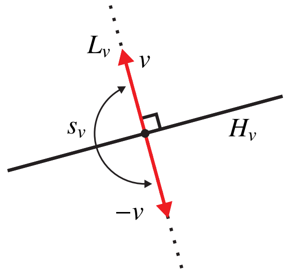
where $H_s$ is a hyperplane (a codimension one subspace); $L_s$ is one dimensional; the restriction of $s$ to $H_s$ is the identity; and the restriction to $L_s$ is the map $x \mapsto -x$. Thus a reflection fixes pointwise a mirror $H_s$, the reflecting hyperplane of $s$, and acts as multiplication by $-1$ in some direction (the reflecting line) not lying in the mirror. In particular $s$ is invertible and an involution[1].

A *reflection group* $W$ is a subgroup of $GL(V)$ generated by finitely many reflections.

**Example 2.1 (orthogonal reflections).** The most familiar kind of reflections are the orthogonal ones for which we further assume that $V$ is a Euclidean space, i.e. is equipped with an inner product. Then $s$ is orthogonal if in the decomposition (2) the line $L_s = H_s^\perp$, the orthogonal complement. In particular $L_s$, and hence the reflection, is determined by the

[1] We will have no need for them in these notes, but one can reflect a vector space over an arbitrary field $k$: the definition is identical except that the restriction of $s$ to the reflecting line $L_s$ is the map $x \mapsto \zeta x$, where $\zeta$ is a primititve root of unity in $k$. The only such $\zeta$ in $\mathbb{R}$ is $-1$, hence the definition we have given of *real* reflections. By contrast a complex reflection can have any finite order.

reflecting hyperplane, unlike a general reflection where both the hyperplane and the line are needed.




If $s$ is orthogonal then for any vector $v$ in $L_s$ we have $s : v \mapsto -v$ with $v^\perp$ fixed pointwise. Thus an orthogonal reflection $s$ can be specified by just a non-zero vector $v$, as the reflection with $H_s = v^\perp$ and $L_s$ spanned by $v$. We write $s = s_v$, $H_s = H_v$, $L_s = L_v$, and by choosing a sensible basis one gets that an orthogonal reflection is an orthogonal map of the Euclidean space.

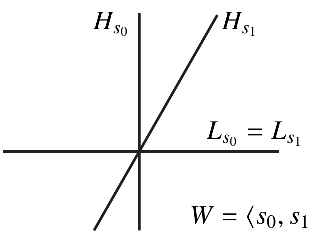
Let $\mathcal{H} = \{H_{v_1}, \dots, H_{v_m}\}$ be hyperplanes in Euclidean $V$ and $W$ the reflection group generated by the orthogonal reflections $s_{v_1}, \dots, s_{v_m}$. As an exercise the reader can show that if $W\mathcal{H} = \mathcal{H}$, i.e. $gH_{v_i} = H_{v_j}$ for all $g \in W$ and all $v_i$, then $W$ is finite (*hint*: $|W| \leq (2m)!$). It turns out (although this is harder) that $\mathcal{H}$ then consists of *all* the reflecting hyperplanes of $W$.

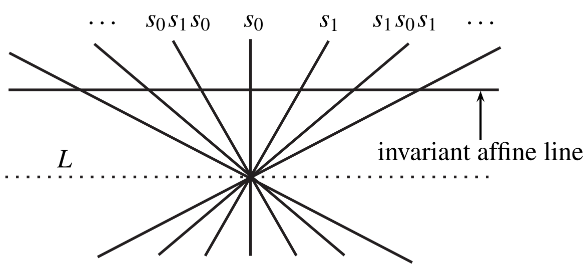
**Example 2.2 (a finite reflection group).** Let $V$ be Euclidean with an orthonormal basis $v_1, \dots, v_{n+1}$ and $\mathcal{H}$ the hyperplanes $H_{ij} := (v_i - v_j)^\perp$ for $1 \leq i \neq j \leq n+1$ (in other words, $H_{ij}$ is the hyperplane with equation $x_i - x_j = 0$). The reflection $s_{v_i - v_j}$ sends $v_i - v_j$ to $v_j - v_i$, thus swapping the vectors $v_i$ and $v_j$. Any other basis vector is orthogonal to $v_i - v_j$, so lies in $H_{ij}$, and is fixed. Thus if $\pi = (i, j) \in \mathfrak{S}_{n+1}$ then $s_{v_i - v_j} H_{k\ell} = H_{\pi(k),\pi(\ell)}$.

Now let $W$ be the group generated by the reflections $s_{v_i - v_j}$. We have just shown that $W\mathcal{H} = \mathcal{H}$, so $W$ is a finite reflection group by the exercise above. Indeed, $W$ is the symmetric group $\mathfrak{S}_{n+1}$ acting by permuting coordinates as in §1. To make this identification we have already seen that each $s_{v_i - v_j}$, and so every element of $W$, permutes the basis vectors $v_1, \dots, v_{n+1}$. This gives a homomorphism $W \to \mathfrak{S}_{n+1}$. Injectivity of this homomorphism follows as the $v_i$ span $V$ and surjectivity as the transpositions $(i, j)$ generate $\mathfrak{S}_{n+1}$.

The convex hull of the $v_i$ is the standard $n$-simplex, barycentrically subdivided by its $n(n-1)$ hyperplanes of reflectional symmetry (the $\mathcal{H}$), each of which is a reflecting hyperplane of $W$. This is the picture we had for $n = 2$ and $n = 3$ in §1. Finite reflection groups are often called *spherical* as the geometrical realisation of their Coxeter complexes (the boundary of the barycentrically divided $n$-simplex in this case; see Example 3.3 for the general definition) are spheres.

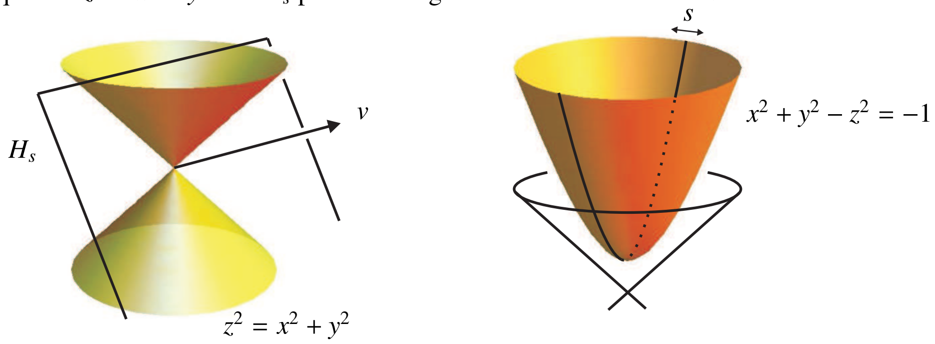
**Example 2.3 (an affine reflection group).** Let $V$ be 2-dimensional and consider reflections $s_0, s_1$ where the reflecting hyperplanes and lines are shown below left (there is no inner product this time). The reflecting hyperplanes are different but both have the same reflecting line: $L_{s_0} = L = L_{s_1}$. If $W$ is the group generated by $s_0, s_1$ then $W$ leaves invariant any affine line parallel to $L$ as the $s_i$ do. But if $\mathcal{H} = \{H_{s_0}, H_{s_1}\}$ then $W\mathcal{H} \neq \mathcal{H}$ as $s_0 H_{s_1} \notin \mathcal{H}$. Indeed, we must expand $\mathcal{H}$ to the infinite set shown below right before it becomes $W$-invariant:



In fact, by identifying the invariant affine line with the reals, $W$ is isomorphic to the group of "affine reflections" of $\mathbb{R}$ in the integers $\mathbb{Z}$, i.e. to the group of transformations of $\mathbb{R}$ generated by the maps $s_n : x \mapsto 2n - x$ for $n \in \mathbb{Z}$. The element $s_1 s_0$ acts on the affine line as the translation $x \mapsto x + 2$ so has infinite order. In particular $W$ is infinite. This also follows from $\mathcal{H}$ being infinite as the reflections in the hyperplanes in $\mathcal{H}$ are the $W$-conjugates of $s_0, s_1$.

**Example 2.4 (hyperbolic reflections).** Let $V$ be 3-dimensional and again there is no inner product. Let $a, b, c$ be real numbers such that $a^2 + b^2 > c^2$, and consider the reflection $s$ with reflecting hyperplane $H_s$ having the equation $ax + by - cz = 0$ and reflecting line $L_s$ spanned by the vector $v = (a, b, c)$. Then $v$ lies on the outside of the pair of cones with equation $z^2 = x^2 + y^2$ and $H_s$ passes through the interior of this cone:




One can check that $s$ leaves invariant each sheet of the two sheeted hyperboloid with equation $x^2 + y^2 - z^2 = -1$. Either sheet is a model for the hyperbolic plane. Intersecting $H_s$ with the top sheet gives a hyperbola – a straight line of hyperbolic geometry – and $s$ is the "hyperbolic reflection" of the plane in this line[2].

Returning to the finite orthogonal case, let $V$ be Euclidean, $\mathcal{H} = \{H_i\}_{i \in T}$ a finite set of hyperplanes and $W = \langle s_i \rangle_{i \in T}$ the group generated by the orthogonal reflections in the $H_i$. Suppose also that $W\mathcal{H} = \mathcal{H}$, so $W$ is finite and $\mathcal{H}$ is the set of all reflecting hyperplanes of $W$ as above.

For each $i \in T$ choose a linear functional $\alpha_i \in V^*$ with $H_i = \ker \alpha_i$. The choice of $\alpha_i$ is unique upto scalar multiple and $H_i$ consists of those $v \in V$ with $\alpha_i(v) = 0$. The two sides (or half-spaces) of the hyperplane consist of the $v$ with $\alpha_i(v) > 0$ or the $v$ with $\alpha_i(v) < 0$.

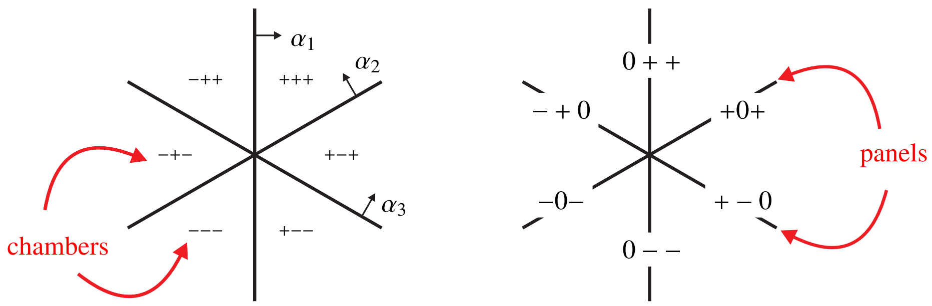
Fix an $T$-tuple $\varepsilon = (\varepsilon_i)_{i \in T}$, with $\varepsilon_i \in \{\pm 1\}$, and consider the set

$$c = c(\varepsilon) = \{v \in V \,|\, \varepsilon_i \alpha_i(v) > 0 \text{ for all } i\}. \tag{3}$$

So each $\alpha_i(v)$ is non-zero and $\alpha_i(v), \varepsilon_i$ have the same sign for all $i$. If this set is non-empty then call it a *chamber* of $W$. A non-empty set of the form

$$a = a(\varepsilon) = \{v \in V \,|\, \alpha_{i_0}(v) = 0 \text{ for some } i_0, \text{ and } \varepsilon_i \alpha_i(v) > 0 \text{ for all } i \neq i_0\} \tag{4}$$

is called a *panel*. Here is the example from §1:




[2]Although there is no inner product in Examples 2.3 and 2.4, it is possible to endow $V$ with a bilinear form so that the reflections are "orthogonal" with respect to this form.

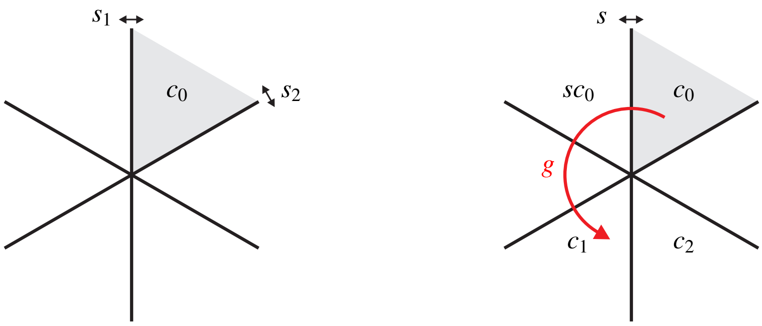
where there are three hyperplanes in $\mathcal{H}$ and the $\alpha_i$ are chosen so that $\alpha_i(v) > 0$ for those $v$ on the side indicated by the arrow. The chambers are marked by their $T$-tuples. There are $2^3$ $T$-tuples but only 6 chambers because the tuples $++-$ and $--+$ give empty sets in (3). Extend the notation to include panels (4) by placing a 0 in the $i_0$-th position. There are then $3.2^2$ such tuples but only 6 give non-empty panels, with two lying on each reflecting line.

There is an obvious notion of adjacency between chambers suggested by these pictures. Say that $a$ is a panel of the chamber $c$ if the corresponding $T$-tuples are identical except in one position where the tuple for $a$ has a 0. It turns out that this can also be defined topologically: $a$ is a panel of $c$ exactly when $\bar{a} \subset \bar{c}$, the closures of these sets with respect to the usual topology on $V$.

Chambers $c_1$ and $c_2$ are then *adjacent* if they share a common panel. In the Example from §1, chambers are adjacent when they share a common edge.

The adjacency relation can be refined by bringing the reflection group $W$ into the picture. In §1 we saw that $\mathfrak{S}_3$ acts regularly as a reflection group on the chambers. This turns out to be true in general for the $W$-action on the chambers: given chambers $c, c'$ there is a unique $g \in W$ with $gc = c'$. Fix one of the chambers $c_0$. Then the regular action gives the chambers are in one-one correspondence with the elements of $W$ via $g \in W \leftrightarrow$ chamber $gc_0$.

Now let $S = \{s_1, \dots, s_n\}$ be those reflections in $W$ whose hyperplanes $H_1, \dots, H_n$ are spanned by a panel of the fixed chamber $c_0$. Thus $S = \{s_1, s_2\}$ for the $c_0$ in the example from §1:




Suppose $c_1, c_2$ are a pair of adjacent chambers as above right. Then there is a $g \in W$ with $c_1 = gc_0$. Translating the picture back to $c_0$ we have $g^{-1}c_1 = c_0$ and $g^{-1}c_2$ are adjacent chambers, and the common panel of $c_1, c_2$ is sent by $g^{-1}$ to a common panel of $c_0$ and $g^{-1}c_2$ (these are most easily seen using the topological version of adjacency). If $s \in S$ is the reflection in the hyperplane spanned by the common panel of $c_0$ and $g^{-1}c_2$, then the chamber $g^{-1}c_2$ is the same as the chamber $sc_0$.

Thus $c_1 = gc_0$, $c_2 = (gs)c_0$, and we have the following more refined description of adjancey:

$$\text{the chambers adjacent to the chamber } gc_0 \text{ are the } (gs)c_0 \text{ for } s \in S. \tag{5}$$

When $S = \{s_1, \dots, s_n\}$ we say that chambers $gc_0$ and $gs_ic_0$ are $i$-adjacent. In our running example, the chambers adjacent to $gc_0$ are $gs_1c_0$ and $gs_2c_0$, and these are the two that were 1- and 2-adjacent to $gc_0$ in §1.

*Coxeter groups* We motivate the definition of Coxeter group by quoting two facts, staying with the assumptions above where $W$ is generated by orthogonal reflections in finitely many hyperplanes $\mathcal{H}$ with $W\mathcal{H} = \mathcal{H}$:

*Fact 1.* The group $W$ is generated by the reflections $s \in S$ in those hyperplanes spanned by a panel of the fixed chamber $c_0$.

In our running example we can see a how a proof might work using induction on the "distance" of a chamber from $c_0$. If $g$ is an element of $W$ then there is a chamber adjacent to the chamber $gc_0$ that is closer to $c_0$ than $gc_0$ is. If this closer chamber is $g'c_0$ say, then by (5) we have $g = g's$ for some $s \in S$. Repeat the process until $g$ completely decomposes as a word in the $s \in S$.

*Fact 2.* With respect to the generators $S$ the group $W$ admits a presentation

$$\langle s \in S \mid (s_i s_j)^{m_{ij}} = 1 \rangle \tag{6}$$

where the $m_{ij} \in \mathbb{Z}^{\geq 1}$ and are such that $m_{ij} = m_{ji}$, and $m_{ij} = 1$ if and only if $i = j$ (so in particular, $s_i^2 = 1$).

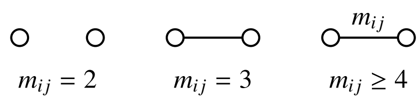
If $s_i$ and $s_j$ are reflections in $W$ finite, then the element $s_i s_j$ has finite order $m_{ij} \geq 2$. So the relations in the presentation (6) certainly hold. The content of Fact 2 is that these relations suffice. Geometrically, $s_i s_j$ is a rotation "about" the intersection $H_i \cap H_j$ of the corresponding hyperplanes.

In Example 2.2 we have $S = \{s_1, \ldots, s_n\}$ where $s_i = s_{v_i - v_{i+1}}$. The $s_i s_{i+1}$ have order 3 and all other $s_i s_j$ have order 2. Moreover $W$ is isomorphic to $\mathfrak{S}_{n+1}$ via the map induced by $(i, i+1) \mapsto s_i$. Our running example of the action of $\mathfrak{S}_3$ on 3-dimensional $V$ is the $n = 2$ case of this.

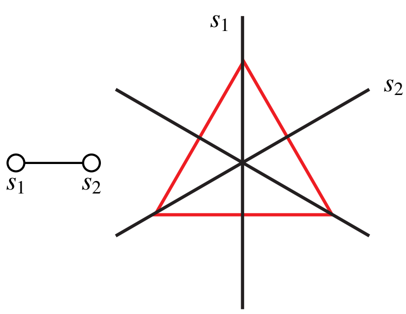
Here is the promised abstraction of reflection group: a group $W$ is called a *Coxeter group* if it admits a presentation (6) with respect to some finite $S$, where the $m_{ij} \in \mathbb{Z}^{\geq 1} \cup \{\infty\}$ satisfy the rules following (6). Sometimes the dependency on the relations $S$ is emphasized and $(W, S)$ is called a Coxeter *system*.

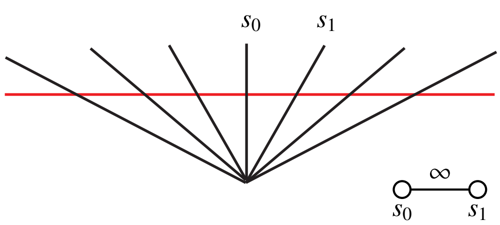
We want the new concept to cover all the examples we have seen so far in this section, including the affine group in Example 2.3 where the element $s_1 s_0$ had infinite order. This is why in the definition of Coxeter group the conditions on the $m_{ij}$ are relaxed to allow them to be infinite. A relation $(s_i s_j)^{m_{ij}} = 1$ is omitted from the presentation when $m_{ij} = \infty$.

There is a standard shorthand for a Coxeter presentation (6) called the Coxeter symbol. This is a graph with nodes the $s_i \in S$, and where nodes $s_i$ and $s_j$ are joined by an edge labeled $m_{ij}$ if $m_{ij} \geq 4$, an unlabeled edge if $m_{ij} = 3$ and no edge when $m_{ij} = 2$:




The examples from §1 and Example 2.3 are then:



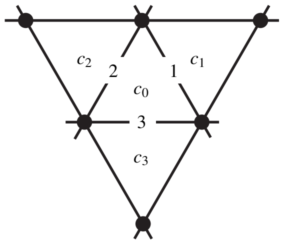

**Remark 2.1.** What is the relationship between the concrete reflection groups defined at the beginning of this section and the abstract Coxeter groups defined at the end? The answer is that the Coxeter groups are *discrete* reflection groups: for a Coxeter system $(W, S)$ one can construct a faithful representation $(W, S) \to GL(V)$ for some vector space $V$, where the $s \in S$ act on $V$ as reflections, and the image of $(W, S)$ is a discrete subgroup of $GL(V)$.

## 3. Chamber Systems and Coxeter Complexes

We have seen several examples of sets of chambers with different kinds of adjacency between them. This section introduces the formalization of this idea: chamber systems.

A *chamber system* over a finite set $I$ is a set $\Delta$ equipped with equivalence relations $\sim_i$, one for each $i \in I$. The $c \in \Delta$ are the *chambers* and two chambers are *$i$-adjacent* when $c \sim_i c'$.

The generic picture to keep in mind is below where chambers are $i$-adjacent if they share a common $i$-labeled edge. Thus, $c_0 \sim_1 c_1, c_0 \sim_2 c_2$, etc.




A *gallery* in a chamber system $\Delta$ is a sequence of chambers

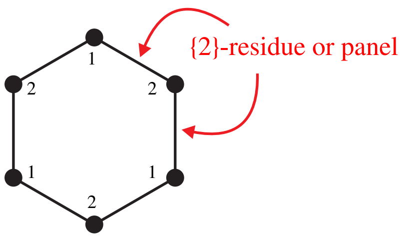
$$c_0 \sim_{i_1} c_1 \sim_{i_2} \cdots \sim_{i_k} c_k \tag{7}$$

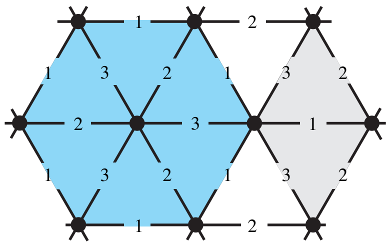
with $c_{j-1}$ and $c_j$ $i_j$-adjacent and $c_{j-1} \neq c_j$. The last condition is a technicality to help with the accounting. We say that the gallery (7) has type $i_1 i_2 \ldots i_k$, and write $c_0 \rightarrow_f c_k$ where $f = i_1 i_2 \ldots i_k$. If $J \subseteq I$ then a *$J$-gallery* is a gallery of type $i_1 i_2 \ldots i_k$ with the $i_j \in J$.

A subset $\Delta' \subseteq \Delta$ of chambers is *$J$-connected* when any two $c, c' \in \Delta'$ can be joined by a $J$-gallery that is contained in $\Delta'$. The *$J$-residues* of $\Delta$ are the $J$-connected components and they have *rank* $|J|$. Thus the chambers themselves are the rank 0 residues. The rank 1 residues are the equivalence classes of the equivalence relations $\sim_i$ as $i$ runs through $I$. Call these rank 1 residues the *panels* of $\Delta$. The chamber system itself has rank $|I|$.

A *morphism* $\alpha : \Delta \to \Delta'$ of chamber systems (both over the same set $I$) is a map of the chambers of $\Delta$ to the chambers of $\Delta'$ that preserves $i$-adjacence for all $i$: if $c \sim_i c'$ in $\Delta$ then $\alpha(c) \sim_i \alpha(c')$ in $\Delta'$. An *isomorphism* is a bijective morphism whose inverse is also a morphism.

**Example 3.1.** The local picture from §1 (below left) is a chamber system over $I = \{1, 2\}$, with chambers the edges, and two chambers $i$-adjacent when they share a common $i$-labeled vertex. The $\{i\}$-residues, or panels, are the pairs of edges having a $i$-labeled vertex in common; in particular each panel contains exactly two chambers and there is a one-one correspondence between the panels and the vertices:




The example above right has chambers the 2-simplicies, $I = \{1, 2, 3\}$, and two chambers $i$-adjacent when they share a common $i$-labeled edge. The six highlighted 2-simplicies are a $\{2, 3\}$-residue and the pair of 2-simplicies a $\{1\}$-residue or panel (so again, each panel contains two chambers). The six chambers in the rank 2 residue have a single common vertex at their center, and there is a one-one correspondence between the rank 2 residues and the vertices; similarly there is a one-one correspondence between the panels and the edges. So the chambers are the maximal dimensional simplicies and the residues correspond to the lower dimensional ones. We will return to this point below.

**Example 3.2 (flag complexes).** Generalizing the example of §1, let $V$ be an $n$-dimensional vector space over a field $k$. A *flag* is a sequence of subspaces $V_{i_0} \subset \cdots \subset V_{i_k}$ with $V_{i_j}$ a proper subspace of $V_{i_{j+1}}$. Let $\Delta$ be the chamber system over $I = \{1, \ldots, n-1\}$ whose chambers are the *maximal* flags $V_1 \subset \cdots \subset V_{n-1}$ with $\dim V_i = i$, and where

$$(V_1 \subset \cdots \subset V_{n-1}) \sim_i (V'_1 \subset \cdots \subset V'_{n-1})$$

when $V_j = V'_j$ for $j \neq i$, i.e. any difference between the maximal flags occurs only in the $i$-th position. The chambers in the panel (or $\{i\}$-residue) containing $V_1 \subset \cdots \subset V_{n-1}$ correspond to the 1-dimensional subspaces of the 2-dimensional space $V_{i+1}/V_{i-1}$. If $k$ is finite of order $q$ then each panel thus contains $q+1$ chambers; if $k$ is infinite then each panel contains infinitely many chambers.

**Example 3.3 (Coxeter complexes).** In §2 we defined chambers, panels and $i$-adjacence for a finite reflection group $W$ acting on a Euclidean space: the chambers were in one-one correspondence with the elements of $W$ via $g \leftrightarrow gc_0$ ($c_0$ a fixed fundamental chamber), and $gc_0$ and $g'c_0$ were $i$-adjacent when $g' = gs_i$.

Now let $(W, S)$ be a Coxeter system with $S = \{s_i\}_{i \in I}$. The *Coxeter complex* $\Delta_W$ is the chamber system over $I$ with chambers the elements of $W$ and

$$g \sim_i g' \text{ if and only if } g' = gs_i \text{ in } W. \tag{8}$$

Thus $g \sim_i gs_i$ and also $gs_i \sim_i gs_is_i = g$. The $\{i\}$-panel containing $g$ is thus $\{g, gs_i\}$, so each panel contains exactly two chambers (which can be thought of as lying on either side of the panel). This is the picture the geometry was giving us in §2. A gallery in $\Delta_W$ has the form

$$g \sim_{i_1} gs_{i_1} \sim_{i_2} gs_{i_1}s_{i_2} \sim \cdots \sim_{i_k} gs_{i_1}s_{i_2}\ldots s_{i_k}.$$

If $f = i_1 i_2 \ldots i_k$ and $s_f = s_{i_1}s_{i_2}\ldots s_{i_k}$, then there is a gallery $g \rightarrow_f g'$ in $\Delta_W$ exactly when $g' = gs_f$ in $W$.

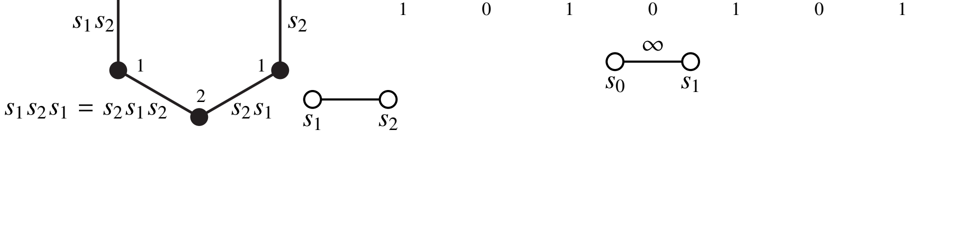
If $s_i, s_j \in S$ then starting at the chamber $g$ we can set off in the two directions given by the galleries:

$$g \sim_i gs_i \sim_j gs_is_j \sim_i gs_is_js_i \cdots \qquad \text{and} \qquad g \sim_j gs_j \sim_i gs_js_i \sim_j gs_js_is_j \cdots$$

If the order of $s_is_j$ is finite, then $(s_is_j)^{m_{ij}} = 1$ is equivalent to the relation

$$s_is_js_i \cdots = s_js_is_j \cdots ,$$

where there are $m_{ij}$ symbols on both sides, so the two galleries above, despite starting out in opposite directions, nevertheless end up at the same place: the chamber $gs_is_js_i \cdots = gs_js_is_j \cdots$. Thus the $\{i, j\}$-residues in $\Delta_W$ are circuits containing $2m_{ij}$ chambers when $s_is_j$ has finite order. If the order is not finite then the residue is an infinitely long line of chambers stretching in "both directions" from $g$. The two Coxeter groups from the end of §2 have Coxeter complexes illustrating both these phenomena:



*Aside.* In all our pictures of chamber systems, the chambers, panels and lower dimensional cells have been simplicies. It turns out that chamber systems are particularly nice examples of simplicial complexes where the chambers are the maximal dimensional simplicies. Moreover in all the chamber systems arising in these lectures there is a correspondence between the lower dimensional simplicies and the residues.

To see why, recall that an abstract simplicial complex $X$ with vertex set $V$ is a collection of subsets of $V$ such that

$$\text{(a). } \sigma \in X \text{ and } \tau \subset \sigma \Rightarrow \tau \in X \text{ and (b). } \{v\} \in X \text{ for all } v \in V.$$

A $\sigma = \{v_0, \ldots, v_k\}$ is a $k$-simplex of the simplicial complex $X$. The empty set $\varnothing$ is by convention the unique simplex of dimension $-1$.

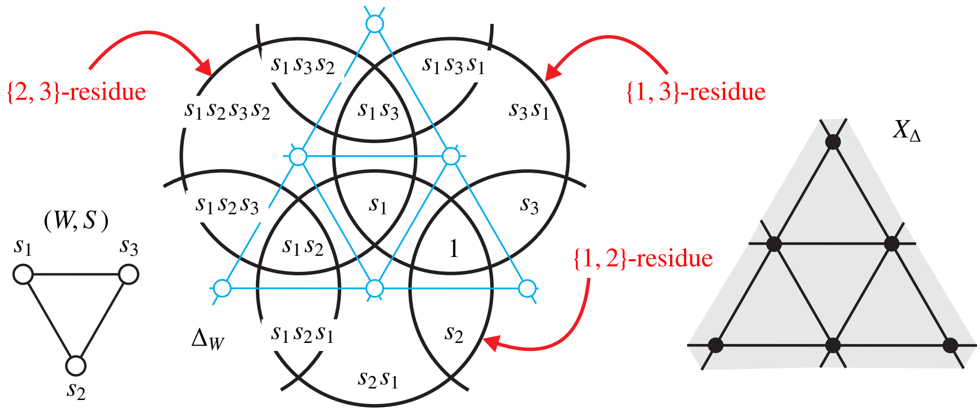
Now let $\Delta$ be a chamber system over $I$ and let $V$ be the set of residues of rank $|I|-1$ (recall that there is only one residue of rank $|I|$, namely $\Delta$ itself). Then let $X_\Delta$ be the simplicial complex with vertex set $V$ and such that if $R_0, \ldots, R_k$ are rank $|I|-1$ residues then

$$\sigma = \{R_0, \ldots, R_k\} \text{ is a } k\text{-simplex of } X_\Delta \Leftrightarrow \bigcap R_i \neq \varnothing.$$

In other words, $X_\Delta$ is the *nerve* of the covering of $\Delta$ by rank $|I|-1$ residues. Take the empty intersection to be the union $\bigcup_V R_i$, and observe that the maximum dimension a simplex can have is $|I|-1$.

If $\Delta$ is the flag complex chamber system of Example 3.2 with chambers the maximal flags, then the $k$-simplicies of $X_\Delta$ correspond to the flags $V_{i_0} \subset \cdots \subset V_{i_k}$ containing $k+1$ subspaces.

We illustrate with the Coxeter complex $\Delta_W$ of the Coxeter system $(W, S)$ with the symbol shown below left. Some elements of $W$ have been written down in a suggestive pattern, grouped into three rank 2 residues. The simplicial complex $X_\Delta$ acquires a 2-simplex from these residues as any two intersect in a residue of rank 1 and all three intersect in a residue of rank 0. In fact $X_W$ is the infinite tiling of the plane from Example 3.1:




It would seem from this example that if $R_0, \ldots, R_k$ are rank $|I|-1$ residues over $J_0, \ldots, J_k$ with $\bigcap R_i \neq \varnothing$, then $\bigcap R_i$ is a residue over $\bigcap J_i$. In fact this is always true for a Coxeter complex and indeed any building, although not for an arbitrary chamber system. As $\bigcap J_i$ has $|I|-(k+1)$ elements, there is a one-one correspondence between the simplicies of $X_\Delta$ and the residues of $\Delta$:

$$\text{codimension } \ell \text{ simplicies } \sigma = \{R_0, \ldots, R_m\} \leftrightarrow \text{ residues } \bigcap_{i=0}^{m} R_i \text{ of rank } \ell,$$

where $m = |I| - (\ell + 1)$. So for buildings the chambers of a chamber system $\Delta$ are the top dimensional simplicies of $X_\Delta$, with the lower dimensional simplicies given by the residues.

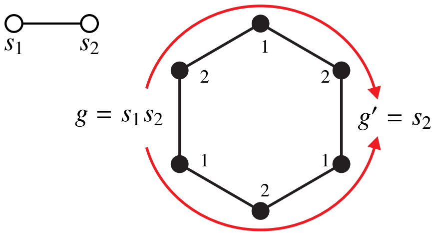
Returning to the general discussion, we now have all the properties of chamber systems that we need. We finish the section by defining a $W$-valued metric on a Coxeter complex $\Delta_W$.

If $(W, S)$ is a Coxeter system and $f = i_1 i_2 \dots i_k$ with $s_f = s_{i_1} s_{i_2} \dots s_{i_k}$, then we have seen that there is a gallery $g \to_f g'$ in $\Delta_W$ exactly when $g' = g s_f$ in $W$. Call such a gallery *minimal* if there is no gallery in $\Delta_W$ from $g$ to $g'$ that passes through fewer chambers. Call an expression $s_f = s_{i_1} s_{i_2} \dots s_{i_k}$ *reduced* if there is no expression in $W$ for $s_f$ involving fewer $s$'s (counted with multiplicity). Thus a gallery $g \to_f g'$ is minimal if and only if the expression $s_f$ is reduced.

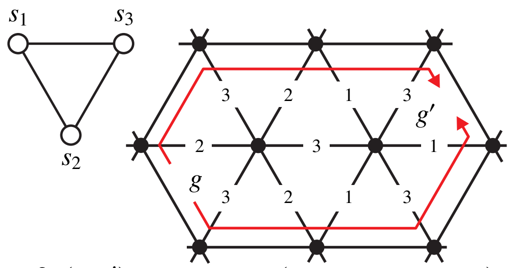
Define $\delta_W : \Delta_W \times \Delta_W \to W$ by $\delta_W(g, g') = g^{-1} g'$. Then

$$\delta_W(g, g') = s_f \Leftrightarrow g' = g s_f \Leftrightarrow \text{ there is a gallery } g \to_f g'. \tag{9}$$

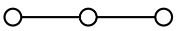
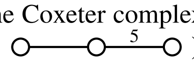
Moreover, $\delta_W(g, g')$ is reduced if and only if the gallery $g \to_f g'$ is minimal. A slight relaxation will define the metric on an arbitrary building. Here are two examples, one of which is our running one:




$\delta_W(g, g') = s_2 s_1 s_2$ $(= s_1 s_2 s_1)$




$\delta_W(g, g') = s_2 s_3 s_2 s_1 s_3$ $(= s_3 s_2 s_1 s_3 s_1 =$ etc$)$

*Another way to draw chamber systems.* A chamber system over $I$ can be drawn as a graph whose edges are "coloured" by $I$. The vertices of the graph are the chambers, and two vertices are joined by an edge labeled $i \in I$ iff the corresponding chambers are $i$-adjacent. These graphs are essentially the 1-skeletons of the duals of our simplicial complexes. If $\Delta_W$ is the Coxeter complex of the Coxeter system $(W, S)$ then this graph is the Cayley graph of $W$ with respect to the generating set $S$. Figure 4 (left) shows the graph for the local picture of the flag complex of a four dimensional space of Figure 3 (or the Coxeter complex of  ) and (right) the graph for the Coxeter complex of the group of symmetries of the dodecahedron (with Coxeter symbol  ).

## 4. Buildings and Apartments

Let $(W, S)$ be a Coxeter system with $S = \{s_i\}_{i \in I}$. A *building of type* $(W, S)$ is a chamber system $\Delta$ over $I$ such that:

**(B1).** every panel of $\Delta$ contains at least two chambers;

**(B2).** $\Delta$ has a $W$-valued metric $\delta : \Delta \times \Delta \to W$ such that if $s_f = s_{i_1} \dots s_{i_k}$ is a *reduced* expression in $W$ then

$$\delta(c, c') = s_f \Leftrightarrow \text{ there is a gallery } c \to_f c' \text{ in } \Delta.$$

**Example 4.1 (Coxeter complexes).** There is at least one building for every Coxeter system $(W, S)$, namely the Coxeter complex $\Delta_W$ with $\delta = \delta_W$ in (9), hence (B2). For (B1) we observed in Example 3.3 that the panels in $\Delta_W$ have the form $\{g, gs\}$ for $g \in W$ and $s \in S$. Such a building, where each panel has the minimum possible number of chambers, is said to be *thin*. It turns out that the thin buildings are precisely the Coxeter complexes.

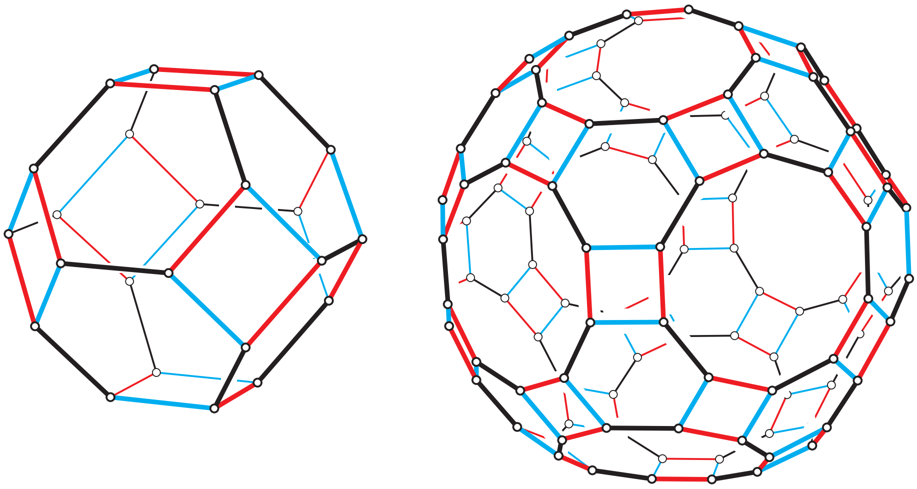


Figure 4: Chamber systems as edge coloured graphs. The local picture for the flag complex of a four dimensional space *(left)* and the Coxeter complex of the group of symmetries of a dodecahedron *(right)*. Both are chamber systems over $I = \{1, 2, 3\}$.

**Example 4.2 (a spherical building of type $\circ\!\!-\!\!-\!\!\circ\!\!-\cdots-\!\!\circ\!\!-\!\!-\!\!\circ$).** For $(W, S)$ having this symbol ($n-1$ vertices) we put a $W$-valued metric on the flag complex of Example 3.2. First identify $(W, S)$ with $\mathfrak{S}_n$ as in §2, with $s_i \mapsto (i, i+1)$ for $1 \leq i \leq n-1$. Let

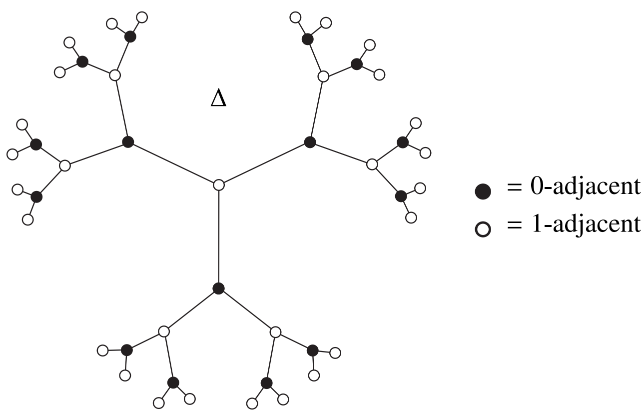
$$c = (V_1 \subset \cdots \subset V_{n-1}) \text{ and } c' = (V'_1 \subset \cdots \subset V'_{n-1})$$

be chambers and write $V_0 = V'_0 = 0$, $V_n = V'_n = V$. We can define $\delta(c, c') \in \mathfrak{S}_n$ using the filtration of $V'_i/V'_{i-1}$ of §1 in the obvious way. Alternatively, for $1 \leq i \leq n$, let

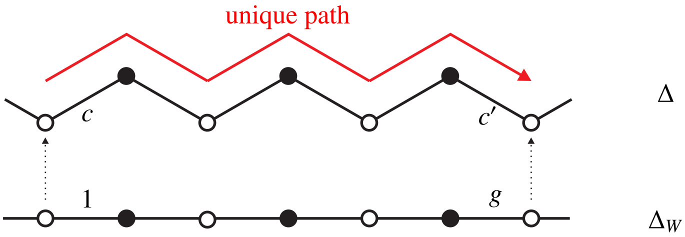
$$\pi(i) = \min\{j \,|\, V'_i \subset V'_{i-1} + V_j\}$$

and define $\delta(c, c') = \pi$. We show that we have a building (when $\dim V = 3$) at the end of this section.

**Example 4.3 (an affine building of type $\circ\!\overset{\infty}{-\!\!-}\!\circ$).** An affine building has type $(W, S)$ an affine reflection group as in Example 2.3. Taking this example, with $S = \{s_0, s_1\}$ and Coxeter symbol $\circ\!\overset{\infty}{-\!\!-}\!\circ$, let $\Delta$ be the chamber system over $I = \{0, 1\}$ shown below – an infinite 3-valent tree. The edges are the chambers, and two chambers are 0-adjacent when they share a common black vertex and 1-adjacent when they share a common white vertex. Each panel thus contains three chambers, hence (B1). The Coxeter complex $\Delta_W$ is in Example 3.3 (also a tree).

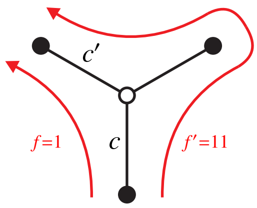


To define the $W$-metric on $\Delta$ recall that in a tree there is a unique path between chambers without "backtracking": a backtrack is a path that crosses an edge and then immediately comes back across the edge again. For chambers $c, c' \in \Delta$, match this unique path between $c$ and $c'$ with the same path starting at 1 in the Coxeter complex $\Delta_W$:



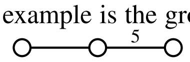

and define $\delta(c, c')$ to be the resulting $g$. To see (B2), let $\delta(c, c') = g \in W$ and suppose that $g = s_{j_1} \ldots s_{j_\ell}$. Then by (9) there is a gallery in $\Delta_W$ from 1 to $g$ of type $j_1 \ldots j_\ell$. As $\Delta_W$ is also a tree this gallery differs from the unique minimal one only by backtracks. First transfer this minimal gallery to $\Delta$ to get the minimal gallery from $c$ to $c'$, and then transfer the backtracks to obtain a gallery of type $j_1 \ldots j_\ell$ from $c$ to $c'$. Conversely if there is a gallery from $c$ to $c'$ of type $j_i \ldots j_\ell$ with $s_{j_1} \ldots s_{j_\ell}$ *reduced*, then in particular no two consecutive $s$'s are the same and so the gallery has no backtracks. Thus it is *the* unique minimal gallery from $c$ to $c'$ giving $\delta(c, c') = s_{j_1} \ldots s_{j_\ell}$ by definition.

In a Coxeter complex we have $\delta_W(c, c') = s_{i_1} \ldots s_{i_k}$ if and only if there is a gallery of type $i_1 \ldots i_k$ from $c$ to $c'$, but in an arbitrary building there is the extra condition that the word $s_{i_1} \ldots s_{i_k}$ be reduced. We can see why in the example above: if there is a gallery of type $i_1 \ldots i_k$ from $c$ to $c'$ with $s_{i_1} \ldots s_{i_k}$ not reduced, then $\delta(c, c')$ need not necessarily be $s_{i_1} \ldots s_{i_k}$. For example, if we have three adjacent chambers:




then there is a gallery of type 1 from $c$ to $c'$ with $s_1$ reduced, hence $\delta(c, c') = s_1$. The non-reduced gallery $c \to_{11} c'$ does not give $\delta(c, c') = s_1 s_1$, as $s_1 s_1 = 1 \neq s_1$.

Examples 4.2-4.3 are our first of *thick* buildings: one where every panel contains at least three chambers. "Thick" is generally taken to be synonymous with interesting.

It turns out that there are quite naturally arising Coxeter groups for which there are *no* thick buildings. One such example is the group of reflectional symmetries of a regular dodecahedron having symbol   .

In §1 (as well as Example 4.3) we defined the $W$-metric $\delta$ by situating a pair of chambers $c, c'$ inside a copy of the Coxeter complex $\Delta_W$ and transferring the metric $\delta_W$ defined in (9). We need to see that this process is well defined – although this is obvious in Example 4.3 – and that the resulting $\delta$ satisfies (B2). This leads to an alternative definition of building (Theorem 4.2 below) based on this idea of defining $\delta$ locally.

Let $(\Delta, \delta)$ and $(\Delta', \delta')$ be buildings of type $(W, S)$ and $X \subset (\Delta, \delta)$, $Y \subset (\Delta', \delta')$ be subsets. A morphism $\alpha : X \to Y$ is an *isometry* when it preserves the $W$-metrics: for all chambers $c, c'$ in $X$ we have $\delta'(\alpha(c), \alpha(c')) = \delta(c, c')$. A simple example is if $g_0 \in W$, then $g \mapsto g_0 g$ is an isometry $\Delta_W \to \Delta_W$.

The following result guarantees the existence of copies of the Coxeter complex in a building:

**Theorem 4.1.** *Let $\Delta$ be a building of type $(W, S)$ and $X$ a subset of the Coxeter complex $\Delta_W$. Then any isometry $X \to \Delta$ extends to an isometry $\Delta_W \to \Delta$.*

An *apartment* in a building $\Delta$ of type $(W, S)$ is an isometric image of the Coxeter complex $\Delta_W$, i.e. a subset of the form $\alpha(\Delta_W)$ for $\alpha : \Delta_W \to \Delta$ some isometry. Apartments are precisely the local pictures we saw in §1.

We are particularly interested in the following two consequences of Theorem 4.1:

$$\text{Any two chambers } c, c' \text{ lie in some apartment } A. \tag{10}$$

(If $\delta(c, c') = g \in W$, then $X = (1, g) \subset \Delta_W \mapsto (c, c') \subset \Delta$ is an isometry. It extends by Theorem 4.1 to an isometry $\Delta_W \to \Delta$ and hence an apartment containing $c, c'$.) So the $W$-metric on $\Delta$ can be recovered from the metric on the Coxeter complex; moreover, the metrics on overlapping Coxeter complexes agree on the overlaps:

$$\text{If chambers } c, c' \in A \text{ and } c, c' \in A' \text{ then there is an isometry } A \to A' \text{ fixing } A \cap A'. \tag{11}$$

(We leave this to the reader with the following hints: use the apartments to get an isometry $A \to A'$ fixing a chamber $c_0 \in A \cap A'$; then show that every chamber in the intersection is fixed by showing that in an apartment there is a unique chamber a given $W$-distance from $c_0$.)

It turns out that any chamber system covered by sufficiently many Coxeter complexes in a sufficiently nice way so that (10) and (11) hold can be made into a building by patching together the local metrics on the Coxeter complexes *ala* §1.

To formulate this properly we need to replace isometries by maps not involving metrics. Let $\Delta, \Delta'$ be chamber systems over the same set $I$. We leave it as an exercise to show that (i). $\alpha : (\Delta, \delta) \to (\Delta', \delta')$ is an isometry of buildings if and only if $\alpha : \Delta \to \Delta'$ is an injective morphism of chamber systems, and (ii). $\alpha$ is a surjective isometry of buildings if and only if $\alpha$ an isomorphism of chamber systems.

**Theorem 4.2.** *Let $(W, S)$ be a Coxeter system with $S = \{s_i\}_{i \in I}$ and $\Delta$ a chamber system over $I$. Suppose $\Delta$ contains a collection $\{A_\alpha\}$ of sub-chamber systems over $I$, called apartments, with each subsystem isomorphic (as a chamber system) to the Coxeter complex $\Delta_W$. Suppose also that*

**(B1′).** *any two chambers $c, c'$ of $\Delta$ are contained in some apartment $A$, and*

**(B2′).** *if chambers $c, c' \in A_\alpha$ and $\in A_\beta$, then there is an isomorphism $A_\alpha \to A_\beta$ fixing $A_\alpha \cap A_\beta$.*

*Define $\delta : \Delta \times \Delta \to W$ by $\delta(c, c') = \delta_W(\alpha(c), \alpha(c'))$ where $\alpha : \Delta_W \to A$ is an isomorphism with $c, c' \in A$. Then $(\Delta, \delta)$ is a building of type $(W, S)$.*

**Example 4.4 (the flag complex of §1 revisited).** The chamber system structure on the flag complex $\Delta$ of §1 was given there (and in Example 3.2, where we saw that $\Delta$ is thick). If $L_1, L_2, L_3$ are lines in $V$ spanned by independent vectors, then we get a hexagonal configuration as in §1. Let the apartments be all the hexagons obtained in this way. If $c = V_1 \subset V_2$ and $c' = V_1' \subset V_2'$ are chambers, then they can be situated in an apartment by extending $V_1, V_1'$ to a set $L_1, L_2, L_3$ of independent lines. If $V_1 \neq V_1'$, $V_2 \neq V_2'$ and $V_2 \cap V_2'$ is a line different from $V_1, V_1'$ as for the $c, c'$ of §1, then this extension is unique, so $c, c'$ lie in a unique apartment. Otherwise (e.g. if $V_2 \cap V_2'$ is one of $V_2$ or $V_2'$) there is some choice. In any case, if $L_1, L_2, L_3$ and $L_1', L_2', L_3'$ are two such extensions corresponding to apartments $A_\alpha, A_\beta$ containing $c, c'$, then any $g \in GL(V)$ with $g(L_i) = L_i'$ induces an isomorphism $A_\alpha \to A_\beta$ that fixes $A_\alpha \cap A_\beta$.

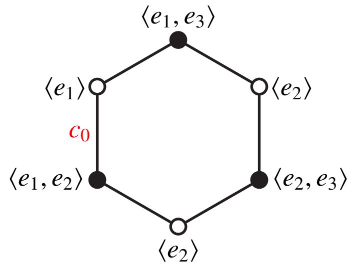



Figure 5: Apartment $A_0$

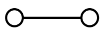
## 5. Spherical Buildings

So far our supply of *thick* buildings is a little disappointing: only the flag complex of §1 and the affine building of Example 4.3. In this section we considerably increase the library by extracting a building from the structure of a reductive algebraic group. These guys really are the motivating examples of buildings.

Call a building of type $(W, S)$ *spherical* when the Coxeter system $(W, S)$ is spherical (i.e. finite). It turns out that there is a uniform construction of a large class of thick spherical buildings. To motivate this we reconstruct the flag complex building $\Delta$ of §1 inside the general linear group $G = GL(V) \cong GL_3(k)$.

First, let $P \subset G$ be the subgroup of permutation matrices – those matrices with exactly one 1 in each row and column and all other entries 0; alternatively, the $a_\pi = \sum_j e_{\pi j,j}$, where $\pi \in \mathfrak{S}_3$ and $e_{ij}$ is the $3 \times 3$ matrix with a 1 in the $ij$-th position and 0's elsewhere. The map $\pi \mapsto a_\pi$ is an isomorphism $\mathfrak{S}_3 \to P$ with

$$s_1 = (1,2) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } s_2 = (2,3) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{12}$$

For the rest of this section we will blur the distinction between the symmetric group $\mathfrak{S}_3$, the group of permutation matrices $P$, and the Coxeter system $(W, S)$ with the symbol  .

Assume for the moment that:

**(G1).** The action of $G$ on the flag complex $\Delta$ given by $a : V_1 \subset V_2 \mapsto aV_1 \subset aV_2$ for $a \in G$, is by chamber system isomorphisms (hence via isometries by the comments immediately prior to Theorem 4.2).

**(G2).** Fix $g \in (W, S)$ and let $X(g) = \{(c, c') \in \Delta \times \Delta \,|\, \delta(c, c') = g\}$. Then for any $g$ the diagonal action $a : (c, c') \mapsto (ac, ac')$ of $G$ on $X(g)$ is transitive (thus $G$ acts transitively on the ordered pairs of chambers a fixed $W$-distance apart).

**(G3).** Let $A_0 \subset \Delta$ be the apartment given by the lines $L_i = \langle e_i\rangle$ with $\{e_1, e_2, e_3\}$ the usual basis for $V$, and $c_0$ the chamber $\langle e_1\rangle \subset \langle e_1, e_2\rangle$ – see Figure 5. Then $P$ acts on $A_0$. Moreover, the isometry $\Delta_W \to A_0$, $g \mapsto gc_0$ is equivariant with respect to the $(W, S)$-action $g \overset{g_0}{\mapsto} g_0 g$ on the Coxeter complex $\Delta_W$ and the $P$-action on the apartment $A_0$ (thus, the $(W, S)$-action on $\Delta_W$ is the same as the $P$-action on $A_0$).

These three allow us to reconstruct the chambers, adjacency and $\mathfrak{S}_3$-metric of $\Delta$ inside $G$:

*Reconstructing the chambers of* $\Delta$ *in* $G$. For $a \in G$ we have $ac_0 = c_0$ with $c_0 = \langle e_1\rangle \subset \langle e_1, e_2\rangle$, exactly when

$$a \in B := \left\{ \begin{pmatrix} \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet \\ 0 & 0 & \bullet \end{pmatrix} \in G \right\},$$

the subgroup of upper triangular matrices. It is easy to show that (G2) is equivalent to (G2a): the $G$-action on $\Delta$ is transitive on the chambers, and (G2b): for any $g \in (W, S)$ the action of the subgroup $B$ is transitive on the chambers $c$ such that $\delta(c_0, c) = g$.

Combining (G2a) with the fact that the chamber $c_0$ has stabilizer $B$, we get a 1-1 correspondence between the chambers of $\Delta$ and the left cosets $G/B$:

$$\text{chambers } ac_0 \in \Delta \xleftrightarrow{\text{1-1}} \text{cosets } aB \in G/B.$$

*Reconstructing the i-adjacency.* Let $c_1, c_2 \in \Delta$ be 1-adjacent chambers: $c_1 = V_1 \subset V_2$ and $c_2 = V_1' \subset V_2$, and let $c_i = a_i c_0$ with the $a_i \in G$. Then $a_1^{-1}a_2$ stabilizes the subspace $\langle e_1, e_2\rangle$, hence

$$a_1^{-1}a_2 \in \left\{ \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ 0 & 0 & \bullet \end{pmatrix} \in G \right\}. \tag{13}$$

The reader can show that for $s_1$ the permutation matrix in (12), the subgroup of matrices in (13) is the disjoint union $B\langle s_1\rangle B := B \cup Bs_1B$, where $BaB = \{bab' \mid b, b' \in B\}$ is a double coset. Thus, if we are to replace the chambers $c_1, c_2$ by the cosets $a_1B, a_2B$, then we need to replace $c_1 \sim_1 c_2$ by $a_1^{-1}a_2 \in B\langle s_1\rangle B$. Similarly

$$c_1 \sim_2 c_2 \text{ exactly when the } c_i = a_i c_0 \text{ with } a_1^{-1}a_2 \in \left\{ \begin{pmatrix} \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet \\ 0 & \bullet & \bullet \end{pmatrix} \in G \right\} = B\langle s_2\rangle B.$$

*Reconstructing the $\mathfrak{S}_3$-metric $\delta$.* Let $c_1, c_2 \in \Delta$ be chambers with $c_i = a_i c_0$. Suppose that $\delta(c_1, c_2) = g \in (W, S)$. As $G$ is acting by isometries (G1), we have $\delta(c_0, a_1^{-1}a_2c_0) = g$. In the Coxeter complex $\Delta_W$ we have by (9) that $\delta_W(1, g) = g$, so that by (G3), $\delta(c_0, gc_0) = g$ also. Thus by (G2b) there is a $b \in B$ with $(bc_0, bgc_0) = (c_0, a_1^{-1}a_2c_0)$, so in particular, $bgc_0 = a_1^{-1}a_2c_0$. As the elements of $G$ sending $c_0$ to $bgc_0$ are precisely the coset $bgB$, we get $a_1^{-1}a_2 \in bgB \subset BgB$.

Conversely, if $a_1^{-1}a_2 \in BgB$ then

$$\delta(c_1, c_2) = \delta(a_1c_0, a_2c_0) = \delta(c_0, a_1^{-1}a_2c_0) = \delta(c_0, bgb'c_0) = \delta(c_0, bgc_0),$$

for some $b \in B$, and so

$$\delta(c_0, bgc_0) = \delta(bc_0, bgc_0) = \delta(c_0, gc_0) = \delta_W(1, g) = g,$$

(the first as $B$ stabilizes $c_0$, the second by (G1) and the third by (G3)). We conclude that

$$\delta(c_1, c_2) = g \in (W, S) \text{ if and only if } a_1^{-1}a_2 \in BgB.$$

Summarizing, let the left cosets $G/B$ be a chamber system over $I = \{1, 2\}$ with adjacency defined by $a_1B \sim_i a_2B$ iff $a_1^{-1}a_2 \in B\langle s_i\rangle B$ and $\mathfrak{S}_3$-metric $\delta(a_1B, a_2B) = g$ iff $a_1^{-1}a_2 \in BgB$. Then $G/B$ is a building of type $\circ\!\!-\!\!\!-\!\!\!-\!\!\circ$, isomorphic to the flag complex of §1.

We leave it to the reader to show that the assumptions (G1)-(G3) hold (*hint*: for (G2) with $\delta(c_1, c_2) = \delta(c_1', c_2')$, situate $c_1, c_2$ in a hexagon as in §1 and $c_1', c_2'$ similarly. Then use the fact that $GL(V)$ acts transitively on ordered bases of $V$).

We are feeling our way towards a class of groups in which we can mimic this reconstruction of the flag complex. It turns out to be convenient to formulate the class abstractly first, and then bring in the natural examples later.

A *Tits system* or *BN-pair* for a group $G$ is a pair of subgroups $B$ and $N$ of $G$ satisfying the following axioms:

**(BN0).** $B$ and $N$ generate $G$;

**(BN1).** the subgroup $T = B \cap N$ is normal in $N$, and the quotient $N/T$ is a Coxeter system $(W, S)$ for some $S = \{s_i\}_{i \in I}$;

**(BN2).** for every $g \in W$ and $s \in S$ the product of double cosets[3] $BsB \cdot BgB \subset BgB \bigcup BsgB$;

**(BN3).** for every $s \in S$ we have $sBs \neq B$.

The group $W$ is called the *Weyl group* of $G$, and is in general not finite.

**Example 5.1.** $G = GL_n(k)$; $B =$ the upper triangular matrices in $G$; $N =$ the monomial matrices in $G$ (those having exactly one non-zero entry in each row and column),

$$T = \{\text{diag}(t_1, \ldots, t_n) \,|\, t_1 \ldots t_n \neq 0\},$$

and $W$ = the permutation matrices with

$$s_i = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \boxed{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}} & & \\ & & & & 1 & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

for $i \in \{1, \ldots, n-1\}$, where the number of 1's on the diagonal before the $2 \times 2$ block is $i - 1$. Let $e_i$ be the $n$-column vector $(0, \ldots, 1, \ldots, 0)^T$ with the 1 in the $i$-th position and $L_i = \{te_i \,|\, t \in k\}$. Then $N$ permutes the set of lines $\{L_1, \ldots, L_n\}$ and $W$ is isomorphic to the symmetric group on this set (hence $\cong \mathfrak{S}_n$). This example is misleadingly special in that the extension $1 \to T \to N \to W \to 1$ splits, so that the Weyl group $W$ can be realised, via the permutation matrices, as a subgroup of $G$. In general this doesn't happen.

**Theorem 5.1.** *Let $G$ be a group with a BN-pair and let $\Delta$ be a chamber system over $I$ with chambers the cosets $G/B$ and adjacency defined by $a_1B \sim_i a_2B$ iff $a_1^{-1}a_2 \in B\langle s_i \rangle B$. Define a $W$-metric by $\delta(a_1B, a_2B) = g \in W$ iff $a_1^{-1}a_2 \in BgB$. Then $(\Delta, \delta)$ is a thick building of type $(W, S)$.*

**Example 5.2.** $G$ = the symplectic group $Sp_{2n}(k) = \{g \in GL_{2n}(k) \,|\, g^T J g = J\}$ where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

with $I_n$ the $n \times n$ identity matrix; $B =$ the upper triangular matrices in $Sp_{2n}(k)$; $N =$ the monomial matrices in $Sp_{2n}(k)$, and

$$T = \{\text{diag}(t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}) \,|\, t_i \neq 0\}.$$

Let $\{e_1, \ldots, e_n, \overline{e}_1, \ldots, \overline{e}_n\}$ be $2n$-column vectors $(0, \ldots, 1, \ldots, 0)^T$ with the 1 in the $i$-th position for $e_i$ and the $(i+n)$-th position for $\overline{e}_i$. Let $L_i = \{te_i \,|\, t \in k\}$ and $\overline{L}_i = \{t\overline{e}_i \,|\, t \in k\}$, writing $\overline{\overline{L}} = L$. Then $N$ permutes the set $\{L_1, \ldots, L_n, \overline{L}_1, \ldots, \overline{L}_n\}$ and $W$ is isomorphic to the "signed" permutations $\mathfrak{S}_{\pm n} = \{\pi \in \mathfrak{S}_{2n} \,|\, \pi(\overline{L}_i) = \overline{\pi(L_i)}\}$.

[3] A $g \in W$ is not an element of $G$ but a coset $\overline{g}T$ for some representative in $\overline{g} \in N$ for $g$. As $T \subset B$, if $\overline{g}_1 T = \overline{g}_2 T$ then $B\overline{g}_1 B = B\overline{g}_2 B$, so we can unambiguously write $BgB$ to mean $B\overline{g}B$.

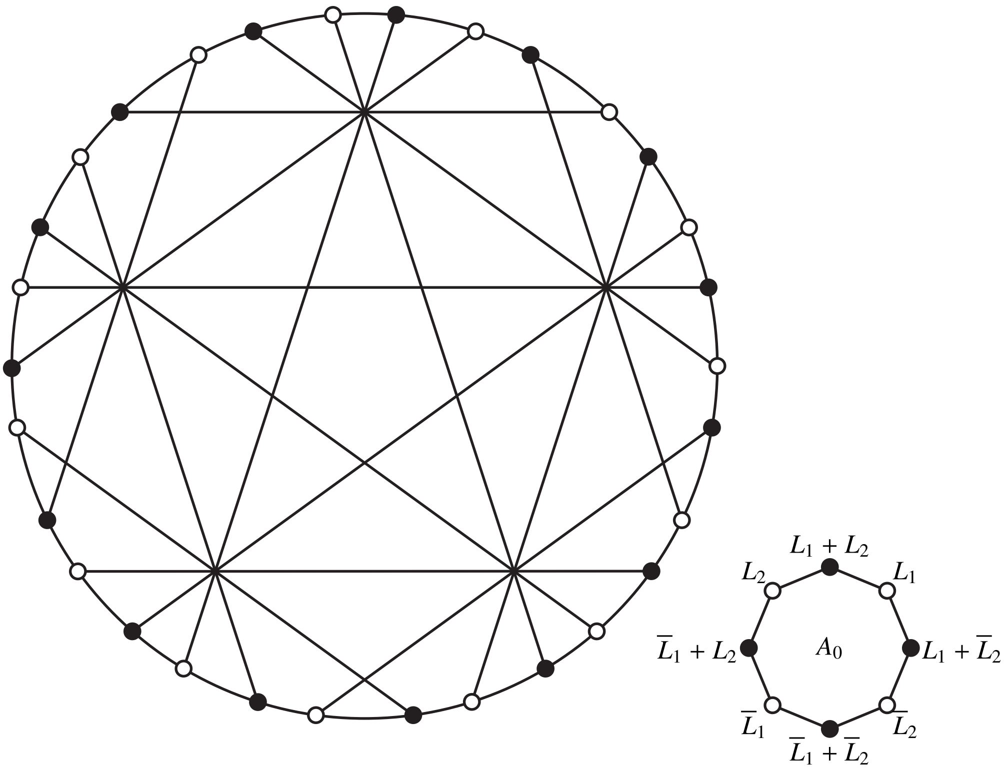



Figure 6: The spherical building of the symplectic group $Sp_4(\mathbb{F}_2)$ and apartment $A_0$.

This can be reformulated geometrically as follows. Let $V$ be a $2n$-dimensional space over $k$ and $(u, v)$ a symplectic form on $V$ – a non-degenerate alternating bilinear form[4]. Let $O(V)$ be those linear maps preserving the form, i.e. $O(V) = \{g \in GL(V) \,|\, (g(u), g(v)) = (u, v) \text{ for all } u, v \in V\}$. The form can be defined on a basis $\{e_1, \ldots, e_n, \overline{e}_1, \ldots, \overline{e}_n\}$ by

$$(e_i, e_j) = 0 = (\overline{e}_i, \overline{e}_j) \text{ and } (e_i, \overline{e}_j) = \delta_{ij} = -(\overline{e}_j, e_i),$$

so that $O(V) \cong Sp_{2n}(k)$. Call a subspace $U \subset V$ totally isotropic if $(u, v) = 0$ for all $u, v \in U$. It turns out that the maximal totally isotropic subspaces are $n$-dimensional. A (maximal) flag in $V$ is a sequence of totally isotropic subspaces $V_1 \subset \cdots \subset V_n$ with $\dim V_i = i$. Let $\Delta$ be the chamber system with chambers these flags and adjacencies over $I = \{1, \ldots, n\}$ as in the flag complex of Example 3.2: $(V_1 \subset \cdots \subset V_n) \sim_i (V'_1 \subset \cdots \subset V'_n)$ when $V_j = V'_j$ for $j \neq i$. Let $c_0$ be the chamber

$$\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, e_2, \ldots, e_n \rangle$$

and $A_0$ the set of images of $c_0$ under the signed permutations $\mathfrak{S}_{\pm n} = \{\pi \in \mathfrak{S}_{2n} \,|\, \pi(\overline{e}_i) = \overline{\pi(e_i)}\}$ (writing $\overline{\overline{e}} = e$). Finally, let $\{A_\alpha\}$ be the set of images of $A_0$ under $Sp_{2n}(k)$. Then this set of apartments $\Delta$ gives a building isomorphic to the spherical building of $Sp_{2n}(k)$ arising from Theorem 5.1 and Example 5.2.

We finish where we started by drawing a picture. Let $V$ be four dimensional over the field of order 2 and equipped with symplectic form $(u, v)$. Let $\Delta$ be the graph with vertices the proper non-trivial totally isotropic subspaces of $V$, with an edge connecting the (white) one dimensional vertex $V_i$ to the (black) two dimensional vertex $V_j$ whenever

[4]Alternating means $(u, u) = 0$ for all $u$, and non-degenerate that $V^\perp = \{0\}$.

$V_i$ is a subspace of $V_j$. Any one dimensional subspace (of which there are 15) is totally isotropic, and is contained in 3 two dimensional totally isotropic subspaces, each of which in turn contains 3 one dimensional subspaces. There are thus 15 two dimensional vertices. The local pictures/apartments are octagons (or barycentrically subdivided diamonds). The apartment $A_0$ above has white vertices $L_1, L_2, \overline{L}_1, \overline{L}_2$, using the notation of Example 5.2, and black vertices $L_1 + L_2, L_1 + \overline{L}_2, \overline{L}_1 + L_2$ and $\overline{L}_1 + \overline{L}_2$. See Figure 6.

**Remark 5.1.** Examples 5.1 and 5.2 are of classical groups of matrices. This can be generalized. Let $k = \overline{k}$ be algebraically closed and $G$ a connected algebraic group defined over $k$. Suppose also that $G$ is reductive, i.e. that its unipotent radical is trivial. Let $B$ be a Borel subgroup (a maximal closed connected soluble subgroup) and $T \subset B$ a maximal torus – a subgroup isomorphic to $(k^\times)^m$ for some $m$. Finally, let $W = N/T$ be the Weyl group of $G$, where $N$ is the normalizer in $G$ of $T$. This is isomorphic to a *finite* Coxeter group $(W, S)$ with $S = \{s_i\}_{i\in I}$. The result is a $BN$-pair for $G$. For a general non-algebraically closed $k$ a $BN$-pair can still be extracted from $G$, but one has to tread more carefully.

### Notes and References

As mentioned in the Introduction, most of what we have said has its origins in the work of Tits, and we start by listing his (many) original contributions. Coxeter groups as a notion first appeared in his 1961 mimeographed notes, *Groupes et géométries de Coxeter*. These were reproduced in [7, pages 740–754]. The name is a homage to [8]. The Bourbaki volume [2] dealing with Coxeter groups was produced after "numerous conversations" with Tits. Buildings as simplicial complexes go back to the very beginnings of the subject, but the first complete account can be found in [20]. Buildings as chamber systems with a $W$-metric have their origins in [23]. The earliest reference to $BN$-pairs that we could find in Tits's work is in [18]; they start to prove an essential tool in [19].

*Section 1.* This is mostly folklore. The reader is to be minded of projective geometry as $\Delta$ is the incidence graph of the standard projective plane over $k$. The ad-hoc argument (essentially the Jordan-Hölder Theorem) for associating the permutation $(1, 3)$ to the pair of chambers is from [1, §4.3].

*Section 2.* Standard references on reflection groups and Coxeter groups are [2] (still the only place you can find some things), [12] and [13]. The definition of reflection in (2) is from [2, V.2.2]. That $\mathcal{H}$ consists of all the reflecting hyperplanes of $W$ is [12, Proposition 1.14]. The general theory of finite reflection groups, including their classification, can be found in Chapters 1 and 2 of [12]. Example 2.3, although fairly standard, is taken from [1, §2.2.2]. The general theory of affine groups is in [12, Chapter 4]. For the hyperboloid or Minkowski model of hyperbolic space, hyperbolic lines, etc, see [14, Chapter 3]. The standard reference on hyperbolic reflection groups is [24]. The treatment of chambers, panels and adjacency is taken from [1, §1.1.4]. That $W$ acts regularly on the chambers is [12, Theorem 1.12]. Fact 1 is [12, Theorem 1.5] and Fact 2 is [12, Theorem 1.9]. For the general theory of Coxeter groups see [12, Chapter 5]. The representation $(W, S) \to GL(V)$ described in Remark 2.1 is called the geometric or reflectional or Tits representation, and is one of the crucial results of [7]. See [12, §5.3] for its definition; faithfulness is [12, Corollary 5.4] or [1, Theorem 2.59] (where it is also shown that the image in $GL(V)$ of $(W, S)$ is discrete).

*Section 3.* Apart from the aside, this section is based mainly on Chapters 1-2 of [15]; the initial chamber system notions and Example 3.2 are directly from [15, §1.1]. Chapter 2 of this book is entirely devoted to Coxeter complexes. A thorough exploration of the general connections between chambers systems and simplicial complexes is given in [1, Appendix A]. The building specific set-up is in [1, §5.6]. The construction of the simplicial complex

$X_\Delta$ as the nerve of the covering by rank $|I|-1$ residues is [1, Exercise 5.98]. The statement about the intersection of residues being a residue is [1, Exercise 5.32]. The edge coloured graph way of viewing chamber systems is a point of view adopted in [25].

*Section 4.* This section is based on Chapter 3 of [15] from which the definition of building is taken. That the Coxeter complexes comprise the thin buildings is from [15, §3.2]. The alternative definition of the permutation associated to a pair of chambers of a flag complex in Example 4.2 is taken from [25, Example 7.4]. The infinite 3-valent tree of Example 4.3 is an example of a building that does not have much structure as a combinatorial object. Nevertheless it can be constructed in an interesting way from a vector space over a field with a discrete valuation (and as such is an important special case of the Bruhat-Tits theory [6]) in the following way. Let $K$ be a non-archimedean local field with residue field $k$ and valuation ring $A$ (for example $K$ is the $p$-adics $\mathbb{Q}_p$ with $k=\mathbb{Z}/p\mathbb{Z}$ and $A$ the $p$-adic integers). If $V$ is a 2-dimensional vector space over $K$, then a lattice $L\subset V$ is a free $A$-module of rank 2. Consider the equivalence classes $\Lambda$ of lattices under the relation $L\sim Lx$ for $x\in K^\times$, and let $\Delta$ be the graph with vertices these classes and an edge joining $\Lambda,\Lambda'$ iff there are $L\in\Lambda, L'\in\Lambda'$ with $L'\subset L$ and $L/L'\cong k$. Then $\Delta$ is a tree, and Example 4.3 is the case where $k$ has two elements ($K=\mathbb{Q}_2$ for example). See [17, II.1.1] for details. In general there is a construction that extracts a $BN$-pair, and an affine building, from an algebraic group defined over such a $K$, and Example 4.3 is this affine building for $SL_2\mathbb{Q}_2$. For affine buildings in general see [26]. The fact that the affine building for $SL_2\mathbb{Q}_p$ is a tree was used by Serre to reprove a theorem of Ihara that a torsion free lattice in $SL_2\mathbb{Q}_p$ is a free group [17]. A theorem of Walter Feit and Graham Higman [10] has consequence that a finite thick building has type $(W,S)$ a finite reflection group where each irreducible component of $W$ is of type $A_n, B_n/C_n, D_n, E_6, E_7, E_8, F_4, G_2$ or $I_2(8)$ (see [1, Theorem 6.94]; see [12, Chapter 2] for a description of these types of finite reflection group). Hence there can be no finite thick buildings of type the symmetry group of the dodecahedron, for which $(W,S)$ has type $H_3$. That there are no *infinite* thick buildings of type $H_3$ is shown in [22]. Theorem 4.1 is [15, Theorem 3.6] and Theorem 4.2 is [15, Theorem 3.11]. Prior to [23] axioms (B1′) and (B2′) of Theorem 4.2 provided the standard definition of building.

*Section 5.* This section is based on [15, Chapter 5]. Properties (G1)-(G3) are the specialization to $GL_3$ of a strongly transitive group action [15, §5.1]. The argument that reconstructs the $W$-metric is taken from the proof of [15, Theorem 5.2]. The axioms for a $BN$-pair are from [15, §5.1]. A proof that Example 5.1 is a $BN$-pair using nothing but row and column operations can be found in [1, §6.5]. Theorem 5.1 is [15, Theorem 5.3]. The flag complex of a symplectic space is from [15, Chapter 1]. Figure 6 has several names: in graph theory circles it is called Tutte's eight-cage, and is the unique smallest cubic graph with girth 8 (where these minimal 8-circuits are, of course, the apartments). It is a pleasantly mindless exercise to label the vertices of the Figure with the totally isotropic subspaces (*hint:* start with the 8-circuit at the top as the apartment $A_0$). There is also a very simple construction that goes back to Sylvester (1844) – this (and much else) is engagingly described in [9]. There are 30 odd permutations of order 2 in $\mathfrak{S}_6$: 15 transpositions – like $(1,2)$ – and 15 products of three disjoint transpositions, like $(1,2)(3,4)(5,6)$. Let these be the vertices of the eight-cage, and join a vertex $\sigma$ in one of these two groups to the three $\tau_1,\tau_2,\tau_3$ in the other group for which $\sigma=\tau_1\tau_2\tau_3$. That the $B$ (Borel subgroup) and $N$ (normalizer of a maximal torus) extracted from a reductive group $G$ in Remark 5.1 are a $BN$-pair for $G$ is shown in [11, §29.1].

*Further reading.* Surely the shortest introduction to buildings is [5]; [4], [16] and [21] are slightly longer. The book [1] is a greatly expanded version of [3], while [15] is an updated version of the 1988 original. A nice introduction to spherical buildings, including an account of Tits's classification [20] of the thick spherical buildings of type $(W,S)$ for $|S|\geq 3$, is [25]; the sequel [26] treats affine buildings.

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