ANASOV DIFFEOMORPHISM OF FLAT MANIFOLDS

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Introduction

Let M be a compact differentiable manifold without boundary. A Riemannian structure on M is called <u>flat</u> if all sectional curvatures vanish at each point; then M is called a <u>flat manifold</u>. A diffeomorphism $f : M \to M$ is called an <u>Anosov diffeomorphism</u> if for some (and hence any) Riemannian metric on M there exist constants $c > 0$, $\lambda < 1$ such that at any point m of M the tangent space TM_m decomposes as the direct sum of a contracting part and an expanding part; more precisely $TM_m = E^s \oplus E^u$, where $\|Tf^r v\| \leqslant c\lambda^r \|v\|$ for all $v \in E^s$ and all integers $r > 0$ and $\|Tf^{-r} w\| \leqslant c\lambda^r \|w\|$ for all $w \in E^u$ and all integers $r > 0$ (the letters s and u stand, as usual, for stable and unstable; they are also used for the dimensions of the spaces involved).

<u>Example</u> If we write $\underset{\sim}{T}^2 = \underset{\sim}{R}^2/\underset{\sim}{Z}^2$ for the flat torus, then the automorphism of $\underset{\sim}{R}^2$ given by the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ induces an Anosov diffeomorphism on $\underset{\sim}{T}^2$. On the Klein bottle, however, it is impossible to construct an Anosov diffeomorphism.

This raises the obvious question : On which manifolds can we construct Anosov diffeomorphisms ? c.f. Smale [14] p.760. Smale gives examples of Anosov diffeomorphisms on nilmanifolds (p.761). Shub [13] gives examples on a four-dimensional flat manifold which is not a torus, and on a six-dimensional infranil manifold.

We give below a complete algebraic characterization of those flat manifolds which support Anosov diffeomorphisms (see Theorem 2.3.1). Each flat manifold comes prepacked with its own finite group F (the linear holonomy group) and a representation T of this group into $GL(n, \underset{\sim}{Z})$, where n is the dimension of the manifold.

In chapter 1 we find necessary and also sufficient conditions for M to support an Anosov diffeomorphism, and show that these depend only on the representation T .

In chapter 2 we examine these conditions, as a problem in abstract representation theory and arrive at the surprising conclusion that the conditions are equivalent. They depend on the manner in which T decomposes as we enlarge the coefficient domain first from $\underset{\sim}{Z}$ to $\underset{\sim}{Q}$, and then to $\underset{\sim}{R}$.

What we do is this : first we decompose T over $\underset{\sim}{Q}$. If any pieces occur more than once in the decomposition we ignore them. We now take those pieces which occur precisely once and attempt to decompose them over $\underset{\sim}{R}$. If we are successful every time, the manifold will support an Anosov diffeomorphism, but if any of them is irreducible over $\underset{\sim}{R}$, then the manifold will not support an Anosov diffeomorphism.

In chapter 3 we apply our results to specific problems, generate lots of examples and finally use one of the examples to illustrate a formula of Williams [15] on zeta functions of diffeomorphisms.

To reduce the weight of the proofs in chapters 1 and 2, we have assembled those parts of the proofs which have nothing to do with Anosov diffeomorphisms into a chapter 0 which we call "Prerequisites". It is used heavily for reference, and to establish notation.

I should like to take this opportunity to extend my thanks to the many people who gave me help and encouragement, especially to my supervisors David Epstein and Mike Shub, who have shown more patience with me than I deserve. I should also like to thank the management of I.H.E.S. for their wonderful hospitality during Easter 1970, when a significant portion of this work was accomplished.

Chapter 0 Prerequisites

§0.1 Cohomology of Groups

The easiest reference for this section is Eilenberg and MacLane [7].

Let A be an abelian group, written additively, and Γ be an arbitrary group, written multiplicatively, which acts on A on the left, the action being written multiplicatively, but with a dot, to distinguish it from the multiplication in Γ. We define $H^1(\Gamma, A)$ to be the group of all maps $\psi : \Gamma \to A$ which satisfy the equation $\psi(x_1x_2) = x_1 . \psi(x_2) + \psi(x_1)$, (these are usually called crossed homomorphisms, although they are not homomorphisms) modulo the group of maps ψ of the form $\psi(x) = x . a - a$ for some $a \in A$ (usually called the principal crossed homomorphisms). $H^2(\Gamma, A)$ is the group of all maps $\psi : \Gamma \times \Gamma \to A$ which satisfy the equation $\psi(x_1x_2, x_3) - \psi(x_1, x_2x_3) = x_1 . \psi(x_2, x_3) - \psi(x_1, x_2)$ modulo the group of those of the form $x_1 . g(x_2) - g(x_1x_2) + g(x_1)$ where $g : \Gamma \to A$ is an arbitrary map.

We shall be considering the special case where A is an abelian normal subgroup of Γ, and Γ acts on A on the left by conjugation. Then since the action of A on itself is trivial, the crossed homomorphisms $\psi : A \to A$ are just the usual homomorphisms and the only principal crossed homomorphism is zero. So $H^1(A, A) = \text{Hom}(A, A)$. We define $H^1(A, A)^\Gamma$ to be the subgroup of $H^1(A, A)$ given by the restriction $\psi(x . a) = x . \psi(a)$ for all $x \in \Gamma$, $a \in A$. Then $H^1(A, A)^\Gamma = \text{Hom}^\Gamma(A, A)$ the Γ-module homomorphisms $: A \to A$.

If $F = \Gamma/A$ then the action of Γ on A induces a natural action of F on A. We shall need the following

<u>Theorem 0.1.1</u> There exists an exact sequence

$$0 \to H^1(F, \Lambda) \to H^1(\Gamma, \Lambda) \to \mathrm{Hom}^\Gamma(\Lambda, \Lambda) \to H^2(F, \Lambda) \to H^2(\Gamma, \Lambda)$$

where the map $H^1(\Gamma, \Lambda) \to \mathrm{Hom}^\Gamma(\Lambda, \Lambda)$ is the restriction map.

<u>Proof</u> see Hochschild - Serre [10] □

<u>Lemma 0.1.2</u> If F is a finite group, and $\alpha \in H^2(F, \Lambda)$ then $|F| \alpha = 0$

<u>Proof</u> Let f be a representative cocycle for α.

Then $f : F \times F \to \Lambda$ satisfies $f(x_1 x_2, x_3) - f(x_1, x_2 x_3) = x_1 \cdot f(x_2 x_3) - f(x_1, x_2)$

Now let $g : F \to \Lambda$ be defined by

$$g(x_1) = \sum_{x_3 \in F} f(x_1, x_3)$$

then $x_1 \cdot g(x_2) - g(x_1 x_2) + g(x_1)$

$$= x_1 \cdot \sum_{x_3 \in F} f(x_2, x_3) - \sum_{x_3 \in F} f(x_1 x_2, x_3) + \sum_{x_3 \in F} f(x_1, x_3)$$

$$= \sum_{x_3 \in F} x_1 \cdot f(x_2, x_3) - \sum_{x_3 \in F} f(x_1 x_2, x_3) + \sum_{x_3 \in F} f(x_1, x_2 x_3)$$

$$= \sum_{x_3 \in F} f(x_1, x_2)$$

$$= |F| \, f(x_1, x_2)$$

so $|F| f$ is a coboundary and $|F| \alpha = 0 \in H^2(F, \Lambda)$ □

§0.2 Linear Algebra

In this section we collect together several unrelated results which we shall need, but which are not well known.

Notation By $g\ell(n, R)$ we shall mean the $n \times n$ matrices with coefficients in a commutative ring R .

By $GL(n, R)$ we shall mean that subset of $g\ell(n, R)$ of matrices whose inverses are also in $g\ell(n, R)$.

Lemma 0.2.1 If $H \in GL(n, \mathbb{Z})$ and $\ell \in \mathbb{Z}$ then $\exists\, k \in \mathbb{Z}_+^* = \{k \in \mathbb{Z} : k > 0\}$ such that $H^k - I \in g\ell(n, \ell\mathbb{Z})$.

Proof There is a natural map $q : GL(n, \mathbb{Z}) \to GL(n, \mathbb{Z}_\ell)$, which latter is a finite group. So $\exists\, k \in \mathbb{Z}_+^*$ such that $(qH)^k = id$. □

Lemma 0.2.2 If $A, B \in g\ell(n, F)$ where F is a field of characteristic prime to n , and $AB - BA = \mu I$, then $\mu = 0$.

Proof The trace of the left-hand-side is 0, of the right-hand-side is $n\mu$. □

Lemma 0.2.3 If $A, B \in g\ell(n, F)$ F any field ,

$$\text{then } \det\begin{pmatrix} \lambda I & A \\ B & \mu I \end{pmatrix} = \det(\lambda\mu I - AB)$$

Proof If $\lambda = 0$ this is trivial.

If $\lambda \neq 0$ we have

$$\det\begin{pmatrix} \lambda I & A \\ B & \mu I \end{pmatrix} = \det\begin{pmatrix} \lambda I & A \\ B & \mu I \end{pmatrix}\begin{pmatrix} I & 0 \\ -B/\mu & I \end{pmatrix}$$

$$= \det\begin{pmatrix} \lambda I - AB/\mu & A \\ 0 & \mu I \end{pmatrix} .$$

□

§0.3 Number Theory

We call $\underset{\sim}{Z}$ the rational integers. If $a_0 + \ldots + a_{n-1} x^{n-1} + x^n$ is a monic polynomial with coefficients in $\underset{\sim}{Z}$ we call the roots in $\underset{\sim}{C}$ algebraic integers. The set of all algebraic integers, $\underset{\sim}{A}$, forms a ring and $\underset{\sim}{A} \cap \underset{\sim}{Q} = \underset{\sim}{Z}$. The units in $\underset{\sim}{A}$, which we call $\underset{\sim}{U}$, the algebraic units are the roots of the polynomials $a_0 + \ldots + a_{n-1}x^{n-1} + x^n$ where $a_0 = \pm 1$.

Let $K = \underset{\sim}{Q}(\zeta)$ be an algebraic extension of $\underset{\sim}{Q}$, and suppose that ζ has r_1 conjugates in $\underset{\sim}{R}$ and $2r_2$ conjugates in $\underset{\sim}{C} \setminus \underset{\sim}{R}$ (in the Galois theory sense). We shall be interested in whether $K \cap \underset{\sim}{U}$ contains elements all of whose conjugates are different from 1 in absolute value. Now it is clear that such elements must be of infinite order, and this suggests that we use the famous

Theorem 0.3.1 (Dirichlet)

Under the above conditions, $K \cap \underset{\sim}{U}$ is an abelian group with rank $r_1 + r_2 - 1$.

For the proof, see Pollard [12] 11.4 or Artin [2] Chapter 13. □

This, however, is not quite strong enough, since, although it is true that, if an element of $K \cap \underset{\sim}{U}$ has all its conjugates equal to 1 in absolute value, then it must be a root of unity and so of finite order, it is possible for an element of infinite order to be of absolute value 1 . For example, the polynomial $x^4 - 3x^3 + 3x^2 - 3x + 1$ has two roots on the real line, and two on the unit circle. However, there is a stronger version of the theorem.

Theorem 0.3.2

If $r_1 + r_2 - 1 > 0$ in the above situation, then $K \cap \underset{\sim}{U}$ does contain elements none of whose conjugates have absolute value 1 .

To prove this, we look again at the proof of Theorem 0.3.1 and use the fact that a lattice in $\underset{\sim}{R}^n$ which does not lie in a hyperplane cannot be confined to the $\underline{n}$ hyperplanes $x_i = 0$. □

Corollary 0.3.3

The complex quadratic extensions of $\underset{\sim}{Q}$ are the only extensions of $\underset{\sim}{Q}$ which do not contain units all of whose conjugates have absolute value different from 1 . □

§0.4 Representation Theory

This section is a survey of the representation theory we need, mainly in Chapter 2. The principal reference for this section is Curtis and Reiner [6] .

Let G be a finite group. A representation of G is a homomorphism $T : G \to \mathrm{Aut}\, V$ or $GL(n, R)$ where V is a left R-module and R is a ring. We shall consider only the cases $R = \mathbb{Z}$, and $R = \mathbb{F}$ a subfield of $\mathbb{C}$, and $V = R^n$, for some n .

Let us now suppose, then, that $R = \mathbb{F}$. T is said to be irreducible if there is no subspace V_1 of V (except $\{0\}$ and V) such that $T(g) V_1 \subset V_1$ for all $g \in G$. If T is not irreducible, it is said to be reducible . By a theorem of Maschke [C & R 10.8], if $T : G \to \mathrm{Aut}\, V$ is a representation, we may write $V = V_1 \oplus \ldots \oplus V_k$ with $T(g) V_i = V_i$ for all $g \in G$, and $T_i = T|V_i$ irreducible for each i $1 \leqslant i \leqslant k$. We call T_i the irreducible components of T, and write $T = T_1 \oplus \ldots \oplus T_k$.

If now $\mathbb{F}'$ is a larger field than $\mathbb{F}$, we get a new representation, which we also call $T : G \to \mathrm{Aut}\, (V \otimes_{\mathbb{F}} \mathbb{F}')$, defined in the obvious manner. T may now become reducible, even if it was irreducible before. We say T is irreducible over $\mathbb{F}$, but reducible over $\mathbb{F}'$. In general, the larger the field, the more T will split. We call T absolutely irreducible if it is irreducible over $\mathbb{C}$. If K is a subfield of $\mathbb{C}$ such that every irreducible component of T over K is absolutely irreducible, we call K a splitting field for T or for G. By a theorem of Brauer [C & R 41.1] if n is the exponent of G , and ζ is a primitive nth root of unity, then $\mathbb{Q}(\zeta)$ is a splitting field for G . We restrict ourselves to subfields of $\mathbb{Q}(\zeta)$.

If T_i and T_j are two components of T, we say they are <u>equivalent</u>, written $T_i \sim T_j$, if there is an isomorphism $J : V_i \to V_j$ such that $J T_i(g) = T_j(g) J$ for all $g \in G$. We need not specify the field, since if $\underset{\sim}{F}' \supset \underset{\sim}{F}$ then two representations are equivalent over $\underset{\sim}{F}$ if and only if they are equivalent over $\underset{\sim}{F}'$ [C & R 29.7]. If T_i and T_j are equivalent, and we choose a basis $\{v_{ik}\}_k$ for V_i and the basis $\{J v_{ik}\}_k$ for V_j, then the matrices of $T_i(g)$ and $T_j(g)$ with respect to these bases will be identical.

If T is an irreducible representation over $\underset{\sim}{Q}$, and K is a minimal splitting field for T, and T_1 is an irreducible component of T over K (or indeed $\underset{\sim}{C}$), then if we choose a basis for V_1 in K^n, $\{v_1 \ldots v_\ell\}$ say, then $\{v_i^\sigma\}_i$, where $\sigma \in \Gamma(K/\underset{\sim}{Q})$ the Galois group of K over $\underset{\sim}{Q}$, span a space invariant under T. Since T is irreducible over $\underset{\sim}{Q}$ and K is minimal and $\Gamma(K/\underset{\sim}{Q})$ is abelian, the v_i^σ for all i, $1 \leqslant i \leqslant \ell$ and all $\sigma \in \Gamma(K/\underset{\sim}{Q})$ are distinct and are a basis for K^n. With respect to this basis the matrix of T will be a block matrix, with blocks equal to T_1^σ. If now T and T' are two inequivalent irreducible representations of G over $\underset{\sim}{Q}$, and if $\{T_i\}_i$ are the irreducible components of T over $\underset{\sim}{C}$ and $\{T_j'\}_j$ are irreducible components of T' over $\underset{\sim}{C}$, then T_i is inequivalent to T_j' for any i, j.

We shall be especially interested in those elements of $GL(n, \underset{\sim}{F})$ which commute with all the matrices in $\operatorname{im} T$. The key theorem here is <u>Schur's Lemma</u> [C & R 27.3]. If $\underset{\sim}{F}$ is algebraically closed (e.g. $\underset{\sim}{F} = \underset{\sim}{C}$) and T and U: $G \to GL(n, \underset{\sim}{F})$ are irreducible representations of G, and $S \in GL(n, \underset{\sim}{F})$ is such that $T(g) S = S U(g)$ for all $g \in G$, then $S = 0$ if $T \not\sim U$ and $S = \xi I$ for some $\xi \in \underset{\sim}{F}$ if $T = U$.

From this we deduce directly that if $T : G \to GL(n, \underset{\sim}{C})$ is a representation of G, and $\underset{\sim}{C}^n$ has been referred to a suitable basis so that T appears as the direct sum of irreducible representations in such a way that equivalent ones are adjacent and actually equal, then if $S \in GL(n, \underset{\sim}{C})$ commutes with all the elements in $\operatorname{im} T$ then S is a block matrix, all non-diagonal blocks being zero, one diagonal block corresponding to each equivalence class of equivalent components and each of these blocks being itself a block matrix, the size of the blocks being the size of the corresponding components and each block being a scalar matrix.

For example, if T is decomposed as $\begin{pmatrix} T_1 & & \\ & T_1 & \\ & & T_2 \end{pmatrix}$ with $T_1 \not\sim T_2$

then S must be of the form $\begin{pmatrix} aI & bI & 0 \\ cI & dI & 0 \\ 0 & 0 & eI \end{pmatrix}$ $a, b, c, d, e \in \underset{\sim}{C}$.

The result holds also if we replace $\underset{\sim}{C}$ by K, a splitting field for T.

If now T and T' are inequivalent irreducible representations of G over $\underset{\sim}{Q}$, then since their irreducible components over $\underset{\sim}{C}$ are also inequivalent, the only matrix S such that $T(g)S = ST'(g)$ for all $g \in G$ is the zero matrix. Consequently if T is any representation of G into $GL(n, \underset{\sim}{Q})$ and we choose a basis for $\underset{\sim}{Q}^n$ so that T appears as the direct sum $T = T_1 \oplus \ldots \oplus T_k$ of irreducible representations arranged so that equivalent ones are adjacent and equal, then S must be a block matrix, zero off the diagonal blocks, with one diagonal block corresponding to each class of equivalent components, this block being itself a block matrix, the blocks being of the size of the appropriate T_i and commuting with it.

For example, if T is decomposed as $\begin{pmatrix} T_1 & & \\ & T_1 & \\ & & T_2 \end{pmatrix}$

with $T_1 \not\sim T_2$ then S must be of the form

$$\begin{pmatrix} S_1 & S_2 & 0 \\ S_3 & S_4 & 0 \\ 0 & 0 & S_5 \end{pmatrix} \qquad \begin{matrix} S_i\ T_1(g) = T_1(g)\ S_i & \text{for } g \in G & 1 \leqslant i \leqslant 4 \\ S_5\ T_2(g) = T_2(g)\ S_5 & \text{for } g \in G\ . & \end{matrix}$$

We must now, therefore, consider what can commute with a representation T of G which is irreducible over $\underset{\sim}{Q}$. To do this, we extend our field to K, a minimal splitting field for T, and then express T as the direct sum $T_1 \oplus \dots \oplus T_k$ with $T_i = T_1^{\sigma_i}$ where $\sigma_i \in \Gamma(K/\underset{\sim}{Q})$. Let us suppose first that the T_i are all inequivalent. Then the only matrices which can commute with $\begin{pmatrix} T_1(g) & & \\ & \ddots & \\ & & T_k(g) \end{pmatrix}$ for all $g \in G$ are of the form $\begin{pmatrix} \lambda_1 I & & 0 \\ & \ddots & \\ 0 & & \lambda_k I \end{pmatrix}$ and this matrix comes from a rational one when referred to the original basis if and only if $\lambda_i = \lambda_1^{\sigma_i}$.

If the T_i are not all inequivalent they are equivalent in classes of μ components where μ is called the <u>Schur index</u> of T [C & R §70]. However, a matrix like the one above will still commute with T. If we arrange the T_i to put equivalent ones adjacent (to do this we must of course also rearrange $\Gamma(K/Q)$) then if S commutes with $T(g)$ for all $g \in G$, it must be a block matrix, zero off the diagonal blocks, of which there is one for each equivalence class. Each diagonal block is itself a block matrix, of size corresponding to the T_i, each diagonal block of which is a multiple of the identity and each off-diagonal block a matrix S_{ij} such that $S_{ij}\, T_i(g) = T_j(g)\, S_{ij}$ for each $g \in G$. Since T_i, T_j are irreducible, S_{ij} is unique up to constant multiples.

Let us look at the special case where $[K : \mathbb{Q}] = 2$, $K \not\subset \mathbb{R}$, and the Schur index is also 2. An example of this is $G = H_8$, which is given in C & R p.470, and worked out in detail below (§3.4). $\Gamma(K/\mathbb{Q})$ then has two elements, the non-trivial one corresponding to complex conjugation. We write the representation as $T = T_1 \oplus \bar{T}_1$. Then there is a matrix $J \in GL(n, K)$ such that $JT_1 = \bar{T}_1 J$. So $\bar{J}\bar{T}_1 = T_1 \bar{J}$, which gives $\bar{J}JT_1 = \bar{J}\bar{T}_1 J = T_1 \bar{J} J$. Since T_1 is irreducible over $\mathbb{C}$, we must have $\bar{J}J = \kappa I$ for some $\kappa \in K$. Now let $J = J_1 + iJ_2$, J_1, J_2 real $\kappa = \kappa_1 + i\kappa_2$, $\kappa_1, \kappa_2 \in \mathbb{R}$. Then $(J_1 - iJ_2)(J_1 + iJ_2) = \kappa_1 I + i\kappa_2 I$.

so $J_1^2 + J_2^2 = \kappa_1 I$ and $-J_2J_1 + J_1J_2 = \kappa_2 I$, so by Lemma 0.2.2 $\kappa_2 = 0$.

Now if T is irreducible over $\mathbb{R}$, then $\kappa_1 < 0$, as the following argument shows. Repeat the above construction, replacing $\mathbb{Q}$ by $\mathbb{R}$ and K by $\mathbb{C}$. Then by multiplying J by $1/\sqrt{|\kappa_1|}$ we may assume $\bar{J}J = \pm I$. We shall show $\bar{J}J = -I$.

Suppose $\bar{J}J = I$. Define $\phi : V \to V$ by $\phi|V_1 : V_1 \to V_2$ has matrix J, $\phi|V_2 : V_2 \to V_1$ has matrix $\bar{J}$, and extend linearly to V. If v has real coordinates, $v = v_1 + \bar{v}_1$, $v_1 \in V_1$ and $\phi(v) = Jv_1 + \bar{J}\bar{v}_1$, which also has real coordinates $\phi^2|V_1 = \bar{J}J = I$, $\phi^2|V_2 = J\bar{J} = \bar{I} = I$, so $\phi^2 = I$. Then ϕ has eigenvalues ± 1. Suppose now that ϕ has an eigenvalue $\epsilon (= \pm 1)$ corresponding to the eigenvector v , which can be chosen with real coordinates $\phi(v) = \epsilon v$. Now T is irreducible over $\underset{\sim}{R}$, so $T(g)v$ span V, and $\phi T(g) = T(g)\phi$ by definition of ϕ. So $\phi(v_i) = \epsilon v_i$ form a basis for $\underset{\sim}{R}^n$ and $\phi = \pm$ id. But this is impossible as $\phi(V_1) = V_2$ and $V_1 \cap V_2 = \{0\}$. This proves that $\bar{J}J = -I$. Of course, if we wish to keep $J \in GL(n, K)$ we must replace $\kappa_1 = \kappa$.

It follows that the only matrices which can commute with $T(g)$ are of the form $\begin{pmatrix} \lambda I & \nu\bar{J} \\ \xi J & \mu I \end{pmatrix}$ with $\bar{J}J = \kappa I$, and such a matrix will come from a rational one when referred to the original basis for $\underset{\sim}{Q}^n$ if and only if $\mu = \bar{\lambda}$, $\xi = \bar{\nu}$. Notice that by lemma 0.2.3

$$\det \begin{pmatrix} \lambda I & \nu\bar{J} \\ \bar{\nu}J & \bar{\lambda}I \end{pmatrix} = \det (\lambda\bar{\lambda}I - \nu\bar{\nu}\bar{J}J)$$

$$= \det (\lambda\bar{\lambda} - \kappa\nu\bar{\nu})I = (\lambda\bar{\lambda} - \kappa\nu\bar{\nu})^m \quad \text{where } n = 2m .$$

§0.5 Bieberbach Theorems

The principal reference for this section is Wolf [16] Chapter 3.

Let $E(n)$ be the Euclidean group for real n-dimensional space. Then $E(n)$ is the semi-direct product $O(n) \cdot T(n)$ where $O(n)$ is the orthogonal group and $T(n)$ the translation group for n-dimensional space. Of course $T(n) \cong \underset{\sim}{R}^n$. A closed subgroup Γ of $E(n)$ is called <u>uniform</u> if $E(n)/\Gamma$ is compact. If a closed subgroup Γ of $E(n)$ acts on $\underset{\sim}{R}^n$ in the usual way, the orbit space, $\underset{\sim}{R}^n/\Gamma$, with the quotient topology, is a compact manifold if and only if Γ is a discrete, uniform, torsion free subgroup of $E(n)$, and all compact flat manifolds are obtained in this way. The group Γ, which is the fundamental group of the quotient manifold, is often called the crystallographic or Bieberbach group of the manifold. We shall use the following three theorems of Bieberbach.

<u>Theorem 0.5.1</u> If $\Gamma \subset E(n)$ is a crystallographic group then $A = \Gamma \cap T(n)$ is a normal subgroup of finite index in Γ, and any minimal set of generators of A is a vector space basis of $\underset{\sim}{R}^n$ relative to which the $O(n)$ components of the elements of Γ have all entries integral c.f. Wolf [16] 3.2.1. □

<u>Theorem 0.5.2</u> Any isomorphism $f : \Gamma \to \Sigma$ of crystallographic subgroups of $E(n)$ is of the form $\gamma \to B\gamma B^{-1}$ for some affine transformation $B : \underset{\sim}{R}^n \to \underset{\sim}{R}^n$. □

<u>Theorem 0.5.3</u> There are only finitely many isomorphism classes of n-dimensional Bieberbach groups. c.f. Wolf [16] 3.2.2. □

<u>Note</u> In the first theorem above we must be careful to notice that the $O(n)$ components of the elements of Γ need not be orthogonal when referred to the new basis. For example, one of the 3-dimensional flat manifolds given in Wolf [16] 3.5.5 is given by quotienting the 3-torus by

the group generated by the affine transformation

$$\underline{x} \leadsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \underline{x} + \begin{pmatrix} 1/3 \\ 0 \\ 0 \end{pmatrix}$$

, a group of order 3 , and the matrix

given is no longer orthogonal, nor can it be replaced by one which is. For essentially the only possibility for such a matrix is $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

and the transformation $\underline{x} \leadsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \underline{x} + \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is of period 3

only if $a + b + c = n \in \mathbb{Z}$ in which case $\begin{pmatrix} 0 \\ -a \\ -a-b \end{pmatrix}$ is mapped to

$$\begin{pmatrix} -n \\ -a \\ -a-b \end{pmatrix} = \begin{pmatrix} 0 \\ -a \\ -a-b \end{pmatrix} - \begin{pmatrix} n \\ 0 \\ 0 \end{pmatrix}$$

which is not permitted as the condition for Γ to be torsion free is equivalent to Γ/A acting on the torus without fixed points, c.f. Wolf [16] 3.1.3 (ii).

Were it not for this fact, much of this work would become trivial, as the only orthogonal matrices with integer coefficients are monomial matrices with all the non-zero elements equal to ± 1 .

We write A for $\Gamma \cap T(n)$, F for Γ/A, $\pi : \Gamma \to F$ for the natural projection and M for $\mathbb{R}^n/\Gamma = \mathbb{T}^n/F$. Note that F is isomorphic to the linear holonomy group of M (c.f. Wolf [16] 3.4.6). We then have an exact sequence

$$0 \to A \to \Gamma \to F \to 0$$

which we call the exact sequence associated with M.

If we refer $\underset{\sim}{R}^n$ to the basis given by Theorem 0.5.1 then the group A is simply translated by the elements of the integral lattice $\underset{\sim}{Z}^n$, and $\underset{\sim}{R}^n/A$ is the usual flat torus $\underset{\sim}{T}^n$. If β is an affine automorphism : $\underset{\sim}{R}^n \to \underset{\sim}{R}^n$ then we may write $\beta = B + \underline{b}$ where B is a linear map, $\underline{b}$ an n-vector, and the equation means $\beta\underline{x} = B\underline{x} + \underline{b}$ for all $\underline{x} \in \underset{\sim}{R}^n$. If $\alpha \in A$, $\gamma \in \Gamma$, then writing $\alpha = I + \underline{a}$, $\gamma = C + \underline{c}$ we have $\underline{a} \in \underset{\sim}{Z}^n$ and $C \in GL(n, \underset{\sim}{Z})$ when referred to the above basis. $\gamma\alpha\gamma^{-1} = (C + \underline{c})(I + \underline{a})(C^{-1} + C^{-1}\underline{c}) = I + C\underline{a} \in A$. $\phi \in F$ can be written as $C + \bar{c}$ where $\gamma \in \pi^{-1}(\phi)$ and $\bar{c} \in \underset{\sim}{R}^n/\underset{\sim}{Z}^n$ is the projection of $\underline{c}$. So the left-action of F on A induced by conjugation by Γ on A induces a representation of F, $T : F \to GL(n, \underset{\sim}{Z})$ given by $T\phi = C$. Write $\Phi = \text{im } T$.

Since the elements of F other than 1 act on $\underset{\sim}{T}^n$ without fixed points, an element $\phi \in F, \phi \neq 1$, such that $\gamma = C + \underline{c} \in \pi^{-1}(\phi)$, must have the property that $C\underline{x} + \underline{c} \neq \underline{x}$ for all $\underline{x} \in \underset{\sim}{R}^n$. So $(C - I)\underline{x} \neq -\underline{c}$. But this says that $C - I$ is singular, otherwise the equation $(C - I)\underline{x} = -\underline{c}$ would have a solution, and so 1 is an eigenvalue of C and -c is not in the image of $C - I$ for any choice of $\gamma \in \pi^{-1}(\phi)$.

The affine automorphism β above projects to an automorphism of M if the map $\gamma \rightsquigarrow \beta\gamma\beta^{-1}$, $\gamma \in \Gamma$, maps Γ onto itself; in other words if $\beta\Gamma\beta^{-1} = \Gamma$.

Since A is a characteristic subgroup of Γ we must have $\beta A \beta^{-1} = A$ which is true if and only if $B \in GL(n, \underset{\sim}{Z})$. Then we must have that $B C B^{-1} \in \Phi$ for each $C \in \Phi$, or $B \Phi B^{-1} = \Phi$.

Chapter 1 - Characterization Theorem - commenced

§1.1 First reduction

We can now proceed with our analysis of which flat manifolds support Anosov diffeomorphisms. Let M^n be the n-dimensional flat manifold associated with the exact sequence $0 \to A \to \Gamma \to F \to 0$ and let $T : F \to GL(n, \underset{\sim}{Z})$ be the representation of F described in §0.5. Let $\beta : \underset{\sim}{R}^n \to \underset{\sim}{R}^n$ be an affine automorphism of $\underset{\sim}{R}^n$ which projects to an automorphism of M. Write, as before, $\beta = B + \underline{b}$. This automorphism will be an Anosov diffeomorphism of M if the eigenvalues of B are all different from one in absolute value. An Anosov diffeomorphism obtained in this way will be called an Anosov automorphism of M. If β satisfies the weaker condition that none of the eigenvalues of B are roots of unity, we shall call β an ergodic automorphism of M. (c.f. Arnold and Avez [1]).

Theorem 1.1.1 If M is a flat manifold, then a necessary condition for M to support an Anosov diffeomorphism is that it supports an ergodic automorphism, a sufficient condition is that it support an Anosov automorphism.

Proof Let $\phi : M \to M$ be an Anosov diffeomorphism of M. Then $\phi_* : \pi_1 M \to \pi_1 M$ is an isomorphism. But $\pi_1 M = \Gamma$, and A is maximal abelian normal in Γ with finite index. So $\phi_* A = A$ (see Wolf [16] 3.2.9). Hence we can lift ϕ to $\tilde{\phi} : \underset{\sim}{T}^n \to \underset{\sim}{T}^n$ and consider the induced map on homology $\tilde{\phi}_* : H_1(\underset{\sim}{T}^n) \to H_1(\underset{\sim}{T}^n)$. But $H_1(\underset{\sim}{T}^n) \cong \underset{\sim}{Z}^n$; so $\tilde{\phi}_*$ may be represented by an element S of $GL(n, \underset{\sim}{Z})$. Now S defines an ergodic automorphism of $\underset{\sim}{T}^n$, c.f. Franks [9]. Since $\tilde{\phi}$ arises from a diffeomorphism of M, $\tilde{\phi} F \tilde{\phi}^{-1} = F$ and so, since F is finite, some power of S commutes with each element in $\operatorname{im} T$. Then by theorem 1.2.2 below, whose proof is independent of this, M supports an ergodic automorphism.

The second **part** of the theorem is trivial. ⊠

§1.2 Second reduction

We shall now show that the condition that M should support an Anosov automorphism depends only on the representation T . The proof follows closely that of the main theorem in Epstein and Shub [8] . The same proof, with trivial modifications gives us a corresponding theorem for ergodic automorphisms of M .

Theorem 1.2.1 A flat compact connected Riemannian manifold M^n of dimension n associated with the exact sequence $0 \to A \to \Gamma \to F \to 0$ and the representation $T : F \to GL(n, \mathbb{Z})$ supports an Anosov automorphism β if and only if there exists $H \in GL(n, \mathbb{Z})$ with no eigenvalues of absolute value one which commutes with all elements of $\Phi = \text{im } T$.

Note We are not saying that H = B .

Theorem 1.2.2 If M is as in the previous theorem then M supports an ergodic automorphism β' if and only if there exists $H' \in GL(n, \mathbb{Z})$ with no eigenvalues which are roots of unity which commutes with all elements of $\Phi = \text{im } T$.

We shall give a proof of Theorem 1.2.1 ; the proof of Theorem 1.2.2 is similar.

Proof Let us first assume that such a matrix exists, and construct an Anosov automorphism . We shall need

Lemma 1.2.3 If H is a matrix of the above type, then H^k is also of the above type for each integer $k \geq 1$, and $\exists$ integer k such that all entries of $H^k - I$ are divisible by $|F|$.

Proof This is a trivial consequence of Lemma 0.2.1. □

From now on k will be an integer with the above property.

We now look for a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & A & \to & \Gamma & \to & F & \to & 0 \\ & & \downarrow L & & \downarrow f & & \downarrow I_F & & \\ 0 & \to & A & \to & \Gamma & \to & F & \to & 0 \end{array}$$

where $L = H^k|A$. For, if one exists, since L and I_F are isomorphisms, f is an isomorphism by the 5-lemma, i.e. an automorphism of Γ. Then by Theorem 0.5.2. there is an affine transformation $\beta : \underset{\sim}{R}^n \to \underset{\sim}{R}^n$ such that $f(\gamma) = \beta\gamma\beta^{-1}$ for $\gamma \in \Gamma$, and so $\beta\Gamma\beta^{-1} = \Gamma$. Thus β projects to an automorphism of M and we have only to prove that it is Anosov. We shall prove that $B = H^k$, which is of the correct form.

<u>Lemma 1.2.4</u> $B|\underset{\sim}{Z}^n = L$ where A is identified with $\underset{\sim}{Z}^n$.

<u>Proof</u> $\beta = B + \underline{b} \Rightarrow \beta\underline{x} = B\underline{x} + \underline{b}$ for $\underline{x} \in \underset{\sim}{R}^n$

$$\Rightarrow \underline{x} = B^{-1}(\beta\underline{x} - \underline{b})$$

$$= B^{-1}\beta\underline{x} - B^{-1}\underline{b}$$

so $\beta^{-1} = B^{-1} - B^{-1}\underline{b}$

if $\alpha \in A$, $\alpha = I + \underline{a}$ so $\alpha\underline{x} = \underline{x} + \underline{a}$

and $\beta\alpha\beta^{-1}\underline{x} = B(B^{-1}\underline{x} - B^{-1}\underline{b} + \underline{a}) + \underline{b}$

$$= \underline{x} + B\underline{a}$$

so $f(\alpha) = \beta\alpha\beta^{-1} = I + B\underline{a}$

or $f(\underline{a}) = B\underline{a}$ since $A = \underset{\sim}{Z}^n$.

But $f|A = L$. $\square$

Now, since $B|A = L = H^k|A$, and A contains a vector-space basis for $\underset{\sim}{R}^n$, we can deduce that $B = H^k$.

We must now show that such a commutative diagram exists.

Lemma 1.2.5 If $\psi : \Gamma \to A$ is a crossed homomorphism, then the following diagram is commutative, with all maps homomorphisms.

$$\begin{array}{ccccccccc} 0 & \to & A & \xrightarrow{\iota} & \Gamma & \xrightarrow{\pi} & F & \to & 0 \\ & & \downarrow g & & \downarrow f & & \downarrow I_F & & \\ 0 & \to & A & \xrightarrow{\iota} & \Gamma & \xrightarrow{\pi} & F & \to & 0 \end{array}$$

where $g(\alpha) = \psi(\alpha) + \underline{a}$ and $f(\gamma) = \psi(\gamma)\gamma$.

Proof We have to show that f is a homomorphism.

But $f(\gamma\delta) = \psi(\gamma\delta)\gamma\delta = \psi(\gamma)\ \gamma\ \ \psi(\delta)\ \gamma^{-1}\ \gamma\delta = \psi(\gamma)\ \gamma\ \psi(\delta)\ \delta = f(\gamma)f(\delta)$.

g is just $f|A$ written in additive notation; the left-hand square therefore commutes. To show that $\pi f = I_F\pi = \pi$, we show $\pi f(\gamma) = \pi(\gamma)$ or $\pi(f(\gamma)\gamma^{-1}) = I_F$. Since the sequence $0 \to A \to \Gamma \to F \to 0$ is exact, this condition is equivalent to $f(\gamma)\ \gamma^{-1} \in \operatorname{Im}\ \iota$ or $\psi(\gamma)\ \gamma\ \gamma^{-1} = \psi(\gamma) \in A$. But that is true by definition of ψ . □

We must now find such a crossed homomorphism . $(H^k - I)/|F|$ is the matrix of a Γ-module endomorphism of A , since H^k and I are, and it has integer entries by definition of k . Therefore $H^k - I \in \operatorname{Hom}^{\Gamma}(A, A)$ is sent to zero in $H^2(F, A)$ in the Hochschild - Serre exact sequence (0.1.1) by Lemma 0.1.2. By exactness, therefore, there is a crossed homomorphism ψ such that $\psi|A = H^k - I$; so if $f(\gamma) = \psi(\gamma)\gamma$ then $f|A = \psi|A + I = H^k$ as required.

To prove the converse, we consider the matrix B associated with the given Anosov automorphism β . This will satisfy all the requirements for H except possibly that it may not commute with all the elements of Φ. We have, however, $B \Phi B^{-1} = \Phi$ and so the action of B merely permutes the elements of Φ . But Φ is finite, and so there is an integer $r > 0$ such that B^r induces the identity permutation on Φ . Then we can take $H = B^r$. ⊠

§1.3 Applications

It is possible already at this stage to state some tangible consequences of our results. An easy consequence of theorem 1.2.1 is the following.

Proposition 1.3.1 If M is the flat manifold associated with the exact sequence $0 \to A \to \Gamma \to F \to 0$ and M' is the manifold which covers M and is associated with the exact sequence $0 \to A \to \Gamma' \to F' \to 0$, where F' is a subgroup of F, and if M supports an Anosov automorphism, then so does M'. □

Theorem 1.3.2 If M is a flat manifold of dimension n with linear holonomy group F, then there is a flat manifold M' of dimension $2n$ with linear holonomy group F which supports an Anosov automorphism.

Proof Let $\Delta(F \times F)$ be the diagonal subgroup of $F \times F$ and let $M' = \underset{\sim}{T}^n \times \underset{\sim}{T}^n / \Delta(F \times F)$ with the obvious action. Then M' supports an Anosov automorphism. For T' :

$\Delta(F \times F) \to GL(2n, \underset{\sim}{Z})$ maps $g \times g \rightsquigarrow \begin{pmatrix} T(g) & 0 \\ 0 & T(g) \end{pmatrix}$ so T' is just $T \oplus T$. This commutes with $\begin{pmatrix} I_n & I_n \\ I_n & 2I_n \end{pmatrix}$ which is of the correct form to apply theorem 1.2.1. □

Corollary 1.3.3 Any finite group F is the linear holonomy group of a flat manifold which supports an Anosov automorphism.

Proof This follows directly from theorem 1.3.2 and a theorem of Auslánder and Kuranishi [4] that any finite group F is the linear holonomy group of a flat manifold. □

Corollary 1.3.4 For each prime p there is a flat manifold of dimension $2p$ and linear holonomy group $\underset{\sim}{Z}_p$ which supports an Anosov automorphism.

Proof from theorem 1.3.2 and a theorem of Charlap [5] . □

Chapter 2 - Characterization Theorem - concluded

§2.1 Third reduction

The third reduction involves finding a criterion for the existence of the matrix H given by theorem 1.2.1 in terms of the $\underset{\sim}{Q}$ - irreducible components of the representation $T : F \to GL(n, \underset{\sim}{Z})$. We find that the only components that matter are those which occur with multiplicity one.

Theorem 2.1.1 M supports an Anosov automorphism if and only if each $\underset{\sim}{Q}$-irreducible component of T of multiplicity one has commuting with it an element $K \in GL(m, \underset{\sim}{Z})$ (m = dimension of component) with no eigenvalues of absolute value one.

Theorem 2.1.2 M supports an ergodic automorphism if and only if each $\underset{\sim}{Q}$-irreducible component of T of multiplicity one has commuting with it an element $K' \in GL(m, \underset{\sim}{Z})$ with no eigenvalue a root of unity.

Lemma 2.1.3 The existence of such a $K[K']$ is equivalent to the existence of $K_1\ [K_1'] \in GL(m, \underset{\sim}{Q})$ whose characteristic polynomial has integer coefficients, unit constant term and no zeros of absolute value one [a root of unity].

Proof We shall prove the Anosov case - the ergodic case is almost identical.

If K exists, it will do for K_1.

If K_1 exists, its rational canonical form $R \in GL(m, \underset{\sim}{Z})$. $R = P^{-1}K_1P$ for some $P \in GL(m, \underset{\sim}{Q})$. Let h be the product of the denominators of the elements in P and P^{-1}. Then by lemma 0.2.1 there is an integer $k \geq 1$ such that h divides all the entries of $R^k - I$. Then $K_1^k \in GL(m, \underset{\sim}{Z})$ and will do for K. □

Proof of theorem 2.1.1 (the proof of theorem 2.1.2 is similar)

Let us first suppose that M supports an Anosov automorphism, so that theorem 1.2.1 guarantees the existence of the matrix H. We now change the basis of $\underset{\sim}{Q}^n$ so that T splits up as $T_1 \oplus \ldots \oplus T_k$ with T_i $\underset{\sim}{Q}$-irreducible for each i and with equivalent T_i identical and adjacent. Then if some T_i occurs with multiplicity one, then the new matrix $H' = P^{-1}HP$, where P is the matrix of the new basis, will have a single block corresponding to this T_i and commuting with it. The characteristic polynomial of this block will divide the characteristic polynomial of H, so this block will do for K_1 in lemma 2.1.3.

Now let us suppose that we have the various matrices K. We construct the matrix H'. If some T_i has multiplicity one, put the corresponding K in its appropriate place on the diagonal. If some T_i has multiplicity two, put $\begin{pmatrix} I & I \\ I & 2I \end{pmatrix}$ in the appropriate place. If some T_i has multiplicity three, put $\begin{pmatrix} I & I & I \\ I & 2I & 2I \\ I & 2I & 3I \end{pmatrix}$ in the appropriate place. If some T_i has even higher multiplicity, we can use a suitable combination of these. Then $H^1 \in GL(n, \underset{\sim}{Z})$, commutes with T referred to the basis P and has no eigenvalues of absolute value one. To show this last fact, we need only check that the polynomials $x^2 - 3x + 1$ and $x^3 - 6x^2 + 5x - 1$ have no zeros of absolute value one. The roots of the first are $(3 \pm \sqrt{5})/2$ and the sum of the coefficients of the second is odd which is sufficient on account of the entertaining

<u>Lemma 2.1.4</u> A cubic polynomial, monic, with integer coefficients and unit constant term has no zeros of absolute value one unless 1 or -1 is itself a zero. □

Then if we write $H'' = PH'P^{-1}$ then, as in the proof of lemma 2.1.3 some power of H'' will do for H and will guarantee the existence of an Anosov automorphism on M. ⊠

§2.2 Fourth reduction

We can now assume that T is $\underset{\sim}{Q}$-irreducible. In §0.4 we discussed which elements of $GL(n, \underset{\sim}{Q})$ commuted with a $\underset{\sim}{Q}$-irreducible representation $T : F \to GL(n, \underset{\sim}{Q})$. When we add the conditions that the matrix must be in $GL(n, \underset{\sim}{Z})$ and that its eigenvalues should not be of absolute value one (or alternatively should not be roots of unity) we obtain

Theorem 2.2.1 If $T : F \to GL(n, \underset{\sim}{Q})$ is irreducible over $\underset{\sim}{Q}$, then the following are equivalent

(i) there is in $GL(n, \underset{\sim}{Z})$ a matrix which commutes with im T and has no eigenvalues of absolute value one

(ii) there is in $GL(n, \underset{\sim}{Z})$ a matrix which commutes with im T and has no eigenvalues which are roots of unity

(iii) T is reducible over $\underset{\sim}{R}$.

Proof (i) => (ii) is trivial.

(ii) => (iii) Suppose that T is irreducible over $\underset{\sim}{R}$. We distinguish 3 cases.

Case 1 T is absolutely irreducible. Then, by Schur's lemma, the only matrices commuting with it are scalar matrices. But $\det(\lambda I_n) = \lambda^n$ which is ± 1 only if λ is a root of unity, and since λ is an eigenvalue of λI_n we cannot allow this.

Case 2 T decomposes over $\underset{\sim}{C}$ with Schur index one. If K is a minimal splitting field for T, then K is a complex quadratic extension of $\underset{\sim}{Q}$. Choosing a suitable basis for K^n, the matrix of T will be $\begin{pmatrix} T_1 & 0 \\ 0 & \bar{T}_1 \end{pmatrix}$ and, as the Schur index is one, $T_1 \not\sim \bar{T}_1$. By Schur's lemma, the only commuting matrices which came from $GL(n, \underset{\sim}{Q})$ are of the form $\begin{pmatrix} \lambda I & 0 \\ 0 & \bar{\lambda} I \end{pmatrix}$, with $\lambda \in K$. But $|\lambda| = |\bar{\lambda}|$ and if the characteristic polynomial is in $\mathbb{Z}[X]$, monic, with unit constant term then $|\lambda\bar{\lambda}| = 1$, so λ is on the unit circle. But $\lambda \in K$, a quadratic extension of $\mathbb{Q}$, so λ is a root of unity, but since it is an eigenvalue of the matrix, this is not permitted.

Case 3 T decomposes over $\mathbb{C}$ with Schur index two. Then if K is a minimal splitting field for T, then once again K is a complex quadratic extension of $\mathbb{Q}$. Choosing a suitable basis for K^n, the matrix of T will be $\begin{pmatrix} T_1 & 0 \\ 0 & \bar{T}_1 \end{pmatrix}$, with $T_1 \sim \bar{T}_1$. Then, as we found in §0.4, a commuting matrix which comes from $GL(n, \mathbb{Q})$ must be of the form $\begin{pmatrix} \lambda I & \nu\bar{J} \\ \bar{\nu}J & \bar{\lambda}I \end{pmatrix}$ with $J\bar{J} = \kappa I$, where $\kappa < 0$. If its characteristic equation is in $\mathbb{Z}[X]$, is monic, and has unit constant term, we must have $\det \begin{pmatrix} \lambda I & \nu\bar{J} \\ \bar{\nu}J & \bar{\lambda}I \end{pmatrix} = \pm 1$, so $(\lambda\bar{\lambda} - \kappa\nu\bar{\nu}) = 1$ (it cannot be -1 as it is positive). Then $|\lambda| \leq 1$, so $|\lambda + \bar{\lambda}| \leq 2$. The characteristic polynomial is $(x^2 - (\lambda + \bar{\lambda})x + 1)^m$, and since the zeros of $x^2 - 2x + 1$, $x^2 - x + 1$, $x^2 + 1$, $x^2 + x + 1$, and $x^2 + 2x + 1$ are all roots of unity, a suitable matrix is not available. An example of this case is given in §3.4.

(iii) => (i) If T is reducible over $\mathbb{R}$, let K be a minimal splitting field whose intersection with $\mathbb{R}$ is non-trivial. Then K is not a complex quadratic extension of $\mathbb{Q}$, and so, as we saw in §0.3, there are in K algebraic units none of whose conjugates (in the Galois sense) are on the unit circle. Let λ be such a unit and let $\{\sigma_i\}_i$ be the Galois group $\Gamma(K/\mathbb{Q})$. Then $\{\lambda^{\sigma_i}\}_i$ are the conjugates of λ. Write λ_i for λ^{σ_i}. If now we choose a basis for K^n so that T decomposes as the direct sum of T_i, where $T_i = T_1^{\sigma_i}$ then the block matrix with $\lambda_i I$ in the diagonal blocks, and zero elsewhere will commute with the image of T, it will come from $GL(n, \mathbb{Q})$, its characteristic polynomial will be in $\mathbb{Z}[x]$ and will have a unit for its constant term and none of its eigenvalues will be on the unit circle. It will thus satisfy the conditions of Lemma 2.1.3. ⊠

§2.3 Final Theorem

Theorem 2.3.1 Let M be a flat manifold associated with the exact sequence $0 \to A \to \Gamma \to F \to 0$ and the representation $T : F \to GL(n, \mathbb{Z})$. Then the following conditions on M are equivalent :

(i) M supports an Anosov diffeomorphism.

(ii) M supports an ergodic automorphism.

(iii) Each $\mathbb{Q}$-irreducible component of T which occurs with multiplicity one is reducible over $\mathbb{R}$.

(iv) M supports an Anosov automorphism.

Proof (i) => (ii) see Theorem 1.1.1.

(ii) => (iii) a direct consequence of theorems 2.1.2 and 2.2.1

(iii) => (iv) a direct consequence of theorems 2.1.1 and 2.2.1

(iv) => (i) trivial (see also theorem 1.1.1). ☒

Note Although we have proved here that every flat manifold which supports an Anosov diffeomorphism also supports an Anosov automorphism, there is not, as far as I know, any way of obtaining one directly from the other. It has been conjectured that if f is an Anosov diffeomorphism on a torus, then the induced map on homology is hyperbolic, but I have not heard of a proof. In order to construct a counterexample, it would be necessary to go into at least four dimensions, and attempt to construct an Anosov diffeomorphism whose induced map on homology was given by the matrix $\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 3 \end{pmatrix}$ whose characteristic polynomial, $x^4 - 3x^3 + 3x^2 - 3x + 1$, has two roots, not roots of unity, on the unit circle.

Chapter 3 - Applications and Examples

§3.1 Cyclic Linear Holonomy Group.

Let us examine in greater detail the case where the linear holonomy group F is cyclic. The representation $T : F \to GL(n, \mathbb{Z})$ is completely determined once we specify the image of a generator of F ; let us call this matrix N. The matrix N is similar to an orthogonal matrix, and is therefore diagonalizable over $\mathbb{C}$; the decomposition of T depends only on the eigenvalues of N, corresponding, in fact, to the decomposition of N over the appropriate field. Thus the decomposition over $\mathbb{Q}$ will be indicated by grouping each eigenvalue θ of N with its conjugates under the action of $\Gamma(\mathbb{Q}(\theta)/\mathbb{Q})$. Since F is finite, each θ must be a primitive mth root of unity, for some $m \in \mathbb{Z}^+$, and its conjugates will be the other primitive mth roots of unity. A $\mathbb{Q}$-irreducible subrepresentation having multiplicity one corresponds to the appropriate root of unity being a simple eigenvalue of N, and irreducibility of this representation over $\mathbb{R}$ corresponds to the root of unity being in $\mathbb{Q}$ or a complex quadratic extension of $\mathbb{Q}$. So the condition given in Theorem 2.3.1 gives us the following

<u>Theorem 3.1.1</u> If M is a flat manifold whose linear holonomy group F is cyclic and $T : F \to GL(n, \mathbb{Z})$ is the natural representation, and if $N = T(g)$ where g is a generator of F, then M supports an Anosov diffeomorphism if and only if N has none of the following numbers as simple eigenvalues :
$1, -1, i, -i, \omega, \omega^2, -\omega, -\omega^2$ (where $\omega^3 = 1$). ⊠

It may be instructive to give an elementary proof of ii) => iii) in Theorem 2.3.1 in this special case. Let us suppose therefore that N is as above, θ is a simple eigenvalue of N and $H \in GL(n, \mathbb{Z})$ is a matrix which commutes with N and has no eigenvalues which are roots of unity. We change basis to the columns of a non-singular matrix P , so that if $N' = P^{-1}NP$, $n'_{11} = \theta$, $n'_{1i} = n'_{i1} = 0 \quad i \neq 1$. Since H commutes with N , we have $H' = P^{-1}HP$ commutes with N' and so $h'_{11} = \alpha$, $h'_{1i} = h'_{i1} = 0$ $i \neq 1$. So α is an eigenvalue of H . Writing now $PN' = NP$ and examining the first column of the two matrices we obtain the equations

$$p_{i1}\theta = \sum_{j=1}^{n} n_{ij}\, p_{j1}\ , \quad \text{or} \quad \sum_{j=1}^{n} (n_{ij} - \delta_{ij}\theta)\, p_{j1} = 0\ .$$

Since θ is a <u>simple</u> eigenvalue of N , there is a one-dimensional set of solutions for the p_{j1} , and any solution is therefore a multiple of one with entries in $\underset{\sim}{Q}(\theta)$. By changing P by a constant, we can ensure that the entries in the first column are themselves in $\underset{\sim}{Q}(\theta)$. Now $PH' = HP$, so $p_{i1}\alpha = \sum_{j=1}^{n} h_{ij}\, p_{j1}$, so choosing an i so that $p_{i1} \neq 0$ we obtain $\alpha = \sum_{j=1}^{n} h_{ij}p_{j1} / p_{i1} \in \underset{\sim}{Q}(\theta)$.

But since α is an eigenvalue of H , it is an algebraic unit which is not a root of unity and this is impossible if θ is any of $1, -1, i, -i, \omega, \omega^2, -\omega, -\omega^2$. □

§3.2 Dimension less than 6

Theorem 3.1.1 gives us an easy check on the known results about the existence of Anosov diffeomorphisms on flat manifolds of low dimension. If $n = 1$ the condition of the theorem is trivially not satisfied, and S^1 therefore does not support an Anosov diffeomorphism.

If $n = 2$ the matrix N must have 1 occurring as an eigenvalue at least once (see §0.5), and so twice, and N is the identity matrix. Thus no non-trivial cyclic groups are possible, and so no others either by Prop 1.3.1. So the torus $\underset{\sim}{T}^2$ is the only 2-dimensional flat manifold which supports an Anosov diffeomorphism. In particular, the Klein bottle does not.

If $n = 3$ the identity is still the only permissible matrix, and of the 10 three dimensional flat compact 3-manifolds listed in Wolf [16] p.122 only the torus will support an Anosov diffeomorphism.

If $n = 4$ we must still have 1's as eigenvalues of N, but it is now possible to have two -1's also. $F = \underset{\sim}{Z}_2$ is thus a possibility. No other cyclic group is possible. Any other F would have to have all its elements of period 2 and would therefore be abelian. The matrices of the representation could then be simultaneously diagonalized and would each then have two 1's and two -1's (or four 1's) on the diagonal, and since $(1, 1, -1, -1)$ with $(1, -1, 1, -1)$ would give four <u>distinct</u> subrepresentations and $(1, 1, -1, -1)$ with $(-1, -1, 1, 1)$ would necessitate having $(-1, -1, -1, -1)$ as well, no groups larger than $\underset{\sim}{Z}_2$ are possible. Although the group N must be similar to

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

there is in fact more than one possibility. The respective quotients of T^4 by the actions of the affine transformations

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & & 1 & \\ & 1 & 1 & \\ & & -1 & \\ & & & -1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

are not homeomorphic as the first has first homology group $H_1(M; \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ whereas the second has $H_1(M'; \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2$ (cf. Wolf [16] p.122) . The first of these was the example given by Shub [13].

If $n = 5$, similar reasoning gives us three possibilities. In addition to T^5, we may have $T^5/\mathbb{Z}_2$, where the generator of $\mathbb{Z}_2$ is represented by a matrix N which has either three 1's and two -1's or two 1's and three -1's. Again there is more than one possibility for M in each case. It should be noted that any manifold arising in the latter case will be non-orientable.

§3.3 Dimension 6

When the dimension reaches six, there arises suddenly a great wealth of examples. To classify them all would be a very long task - I shall give a representative sample. Notice first that Theorem 1.3.2 taken in conjunction with §3.5 of Wolf [16] gives already ten examples, all orientable. Of special interest is the one obtained from Wolf's $\mathcal{G}_6$ as it is the first known example of a flat manifold with first Betti number zero which supports an Anosov diffeomorphism.

A non-orientable example is formed by taking N with three 1's and three -1's for eigenvalues.

If F is cyclic its order must be 2, 3, 4, 5, 6, 8, 10 or 12. The appropriate generators for 5, 8, 10 and 12 are

$$\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & 1 & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & 1 & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & 1 & -1 \end{pmatrix} + \begin{pmatrix} 1/5 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}, \quad \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix} + \begin{pmatrix} 1/8 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix},$$

$$\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1/10 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \text{ and } \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix} + \begin{pmatrix} 1/12 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

respectively.

The only non-abelian example in dimension 6 which I have been able to find is $\mathcal{D}_8$, generated by

$$\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix} + \begin{pmatrix} 1/4 \\ \cdot \\ \frac{1}{2} \\ \cdot \\ \cdot \\ \frac{1}{2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix} + \begin{pmatrix} 1/4 \\ \frac{1}{2} \\ \frac{1}{2} \\ \cdot \\ \cdot \\ \frac{1}{2} \end{pmatrix}$$

Here again, the first Betti number, which is easily computed as the dimension of the intersection of the eigenspaces corresponding to eigenvalue 1, is zero.

<u>Note</u> The above remark gives a proof of the curious

<u>Proposition 3.3.1</u> No flat manifold with first Betti number one supports an Anosov diffeomorphism .

For the one-dimensional subspace would give rise to a one-dimensional sub-representation of multiplicity one defined over $\underset{\sim}{Q}$. $\square$

§3.4 Example of Schur index 2

We give here in detail the example of Schur index two from Curtis and Reiner [6] p.470 (see §0.4) and illustrate theorem 2.2.1 ii) => iii) Case 3 in this case.

Suppose $F = \underset{\sim}{H}_8$, the quaternion group and T has a sub-representation $T' : F \to GL(4, \underset{\sim}{Q})$ of multiplicity one given by $T'(1) = I$, $T'(-1) = -I$

$$T'(\underline{i}) = \begin{pmatrix} \cdot & -1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & -1 & \cdot \end{pmatrix} \qquad T'(\underline{j}) = \begin{pmatrix} \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{pmatrix}$$

$$T'(\underline{k}) = \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & -1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \end{pmatrix}$$

We decompose T' into its two absolutely irreducible components, $T_1 \oplus T_2$, by changing coordinates to the columns of

$$\begin{pmatrix} -1+i & 1-i & -1-i & 1+i \\ 1+i & 1+i & 1-i & 1-i \\ -1-i & 1+i & -1+i & 1-i \\ 1-i & 1-i & 1+i & 1+i \end{pmatrix} .$$

Then $T_1(\underline{i}) = \begin{pmatrix} i & \\ & -i \end{pmatrix}$ $\quad T_1(\underline{j}) = \begin{pmatrix} & i \\ i & \end{pmatrix}$ $\quad T_1(\underline{k}) = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$

$T_2(\underline{i}) = \begin{pmatrix} -i & \\ & i \end{pmatrix}$ $\quad T_2(\underline{j}) = \begin{pmatrix} & -i \\ -i & \end{pmatrix}$ $\quad T_2(\underline{k}) = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$

which are similar using the matrix $\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$.

This lets us write down the most general commuting matrix as, in the new coordinates

$$\begin{pmatrix} a + bi & \cdot & \cdot & c + di \\ \cdot & a + bi & -c - di & \cdot \\ \cdot & c - di & a - bi & \cdot \\ -c + di & \cdot & \cdot & a - bi \end{pmatrix}$$ which is

$$\begin{pmatrix} a & -c & b & -d \\ c & a & -d & -b \\ b & d & a & -c \\ d & -b & c & a \end{pmatrix}$$

in the old coordinates. Its determinant is $(a^2 + b^2 + c^2 + d^2)^2$, which is one if and only if only one of a, b, c, d is non-zero. But then it is monomial and so has eigenvalues which are roots of unity.

§3.5 Dimensions of Expanding and Contracting Manifolds

Our methods here give us a certain amount of information about the possible dimensions of the stable and unstable manifolds of an Anosov automorphism of a flat manifold M. The dimension of the unstable manifolds will simply be the number of eigenvalues of the matrix which are larger than one in absolute value. Although in our proofs it has been necessary on several occasions to take powers of the matrix in hand, this will not affect these dimensions. By examining the proof of the theorem 2.1.1 it is easy to see that the distinct subrepresentations over $\underset{\sim}{Q}$ behave independently and so, in particular, there must be at least one stable dimension and one unstable dimension for each of them. Also, if we have an absolutely irreducible subrepresentation of sise r, then the eigenvalues corresponding to it must all be equal, and so, in particular, must all be less than one, or all greater than one, in absolute value. If, therefore, the splitting is $1, n - 1$ then all subrepresentations must be of size one, and must all be equal, so that the group is abelian and each matrix in its image has all its eigenvalues equal. But each such matrix must have at least one 1 as eigenvalue, and so is the identity, and the manifold is a torus. In the non-toral four-dimensional examples, therefore, the splitting must be $2, 2$ and in the non-toral five-dimensional examples it must be $2, 3$ or $3, 2$. In the six-dimensional examples some of the manifolds with $F = \underset{\sim}{Z}_2$ will admit a $2, 4$ splitting, but all the larger groups demand a $3, 3$ splitting. The $\underset{\sim}{Z}_2 \times \underset{\sim}{Z}_2$ case, for example, has 3 distinct subrepresentations, the $\mathfrak{D}_8$ case has only two, but one of them is two-dimensional, and the various cyclic groups, although they have only two distinct subrepresentations over $\underset{\sim}{Q}$, must have a 2, 2 splitting on the non-trivial one because of the paucity of algebraic units in the appropriate field. This discussion, of course, only refers to Anosov automorphisms, but in view of the note in §2.3, we make the following

<u>Conjecture 3.5.1</u> The splittings which are possible for Anosov diffeomorphisms are the same as those for Anosov automorphisms. See Newhouse [11] for a detailed study of the case where the splitting is 1, n - 1, which he calls codimension one. □

§3.6 Zeta functions

The concept of the zeta function of a diffeomorphism, as developed by Artin and Mazur [3] is discussed in Smale [14]. If $f : M^n \to M^n$ is a diffeomorphism and all positive powers f^m of f have only finitely many fixed points (N_m, say) then the zeta function of f is defined as the formal power series $\zeta(t) = \exp \sum_{m=1}^{\infty} \frac{1}{m} N_m t^m$. Williams [15] has obtained a formula for the zeta function of an Anosov diffeomorphism. This formula has two distinct forms, depending on whether the unstable part, E^u, of the tangent bundle is orientable or not. Now the zeta function of f is the same as that of f^{-1}, as the formula above shows, and if M is non-orientable then E^u for f, and E^u for f^{-1} (which is just E^s for f), will have the property that one is orientable and the other is not. Williams suggested to me, therefore, that as I had an example of a non-orientable manifold which supported an Anosov diffeomorphism, I might calculate the zeta function using both formulae and discover what algebraic fact their equality was equivalent to.

The example I shall use then is a quotient of $\underset{\sim}{T}^5$ by $\underset{\sim}{Z}_2$, where the action of $\underset{\sim}{Z}_2$ on T^5 is via the affine transformation

$$\beta = \begin{pmatrix} -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

The Anosov automorphism I use is

$$\begin{pmatrix} 1 & 1 & 1 & \cdot & \cdot \\ 1 & 2 & 2 & \cdot & \cdot \\ 1 & 2 & 3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 2 \\ \cdot & \cdot & \cdot & 2 & 5 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

whose inverse is

$$\begin{pmatrix} 2 & -1 & 0 & \cdot & \cdot \\ -1 & 2 & -1 & \cdot & \cdot \\ 0 & -1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 5 & -2 \\ \cdot & \cdot & \cdot & -2 & 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} .$$

Williams' formulae are obtained as follows:

Let $f_{*i} : H_i(M, \underset{\sim}{R}) \to H_i(M, \underset{\sim}{R})$ be the map on homology induced by M . Now $H_i(M, \underset{\sim}{R}) = \underset{\sim}{R}^m$ for some m depending on i, and f_{*i} is therefore a linear map. Let χ_i be the characteristic polynomial of f_{*i}, and let $P_i(t) = (-t)^m \chi_i(1/t)$. Let $\widetilde{M}$ be the orientable double-cover of M. Then $H_i(\widetilde{M}, \underset{\sim}{R}) = H_i^+ \oplus H_i^-$ where β_{*i} has eigenvalue + 1 on H_i^+ and eigenvalue -1 on H_i^- . Let $\widetilde{\chi}_i(t)$ be the characteristic polynomial of $\widetilde{f}_{*i}|H_i^-$ and $\widetilde{P}_i(t) = (-t)^{m'} \chi_i(1/t)$ where m' is the dimension of H_i^- . Then if a) E^u is orientable and Df preserves orientation, b) E^u is orientable and Df reverses orientation or c) E^u is non-orientable then

$$\zeta_f = \quad a) \prod_{i=0}^{n} P_i^{\epsilon(i)} , \quad b) \prod_{i=0}^{n} P_i^{-\epsilon(i)} \qquad c) \prod_{i=0}^{n} \widetilde{P}_i^{\epsilon(i)}$$

where $\epsilon(i) = (-1)^{i+u+1}$.

In our example the zeta-function of f is obtained by using formula (c) and that if f^{-1} by using formula (a). The unstable dimension for f is 2 and that for f^{-1} is 3 (this is because $\lambda^3 - 6\lambda^2 + 5\lambda - 1$ has two zeros in $]0,1[$) so the formula for ζ_f is

$$\frac{P_1 P_3 P_5}{P_0 P_2 P_4}, \text{ that of } \zeta_{f^{-1}} \text{ is } \frac{\tilde{P}_0 \tilde{P}_2 \tilde{P}_4}{\tilde{P}_1 \tilde{P}_3 \tilde{P}_5} .$$

Now for f the induced maps on homology are given by the matrices

$$(1), \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 & 4 & 2 & 0 \\ 1 & 2 & 1 & 2 & 4 & 2 \\ 0 & 1 & 1 & 0 & 2 & 2 \\ 4 & 2 & 0 & 10 & 5 & 0 \\ 2 & 4 & 2 & 5 & 10 & 0 \\ 0 & 2 & 2 & 0 & 5 & 5 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \phi$$

and for f^{-1} the induced maps on H^- are given by the matrices

$$\phi, \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 10 & -5 & 0 & -4 & 2 & 0 \\ -5 & 10 & -5 & 2 & -4 & 2 \\ 0 & -5 & 5 & 0 & 2 & -2 \\ -4 & 2 & 0 & 2 & -1 & 0 \\ 2 & -4 & 2 & -1 & 2 & -1 \\ 0 & 2 & -2 & 0 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}, (1)$$

and the zeta-function works out in both cases to be

$$\frac{(\lambda^2 - 6\lambda + 1)(\lambda^6 - 36\lambda^5 + 206\lambda^4 - 378\lambda^3 + 229\lambda^2 - 30\lambda + 1)}{(\lambda - 1)(\lambda^4 - 7\lambda^3 + 11\lambda^2 - 6\lambda + 1)(\lambda^3 - 6\lambda^2 + 5\lambda - 1)}$$

or $$\frac{\lambda^8 - 42\lambda^7 + 423\lambda^6 - 1650\lambda^5 + 2703\lambda^4 - 1782\lambda^3 + 410\lambda^2 - 36\lambda + 1}{\lambda^8 - 14\lambda^7 + 71\lambda^6 - 166\lambda^5 + 207\lambda^4 - 146\lambda^3 + 58\lambda^2 - 12\lambda + 1}$$

The algebraic fact which causes these two to have the same formula is simply that the inverse of a matrix may be computed by evaluating the determinants of the minors.

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