SOLUBLE LIE RINGS

by

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ABSTRACT

This work has as its object the study of a rather neglected object, the Lie ring. The general method and type of problem tackled are suggested by analogy with the theory of infinite groups.

A recurring theme is the study of residual properties (mainly residual finiteness) of Lie rings, with particular emphasis on soluble rings. However this by no means presents the whole picture. Related problems in the field of Lie algebras are tackled in the first few chapters, chapters 3, 6, and 7 are not concerned with residual properties at all, and throughout many results are presented for Lie rings which are not necessarily soluble. Many of the results (mainly in the second half) will also hold in general nonassociative rings with suitable restrictions imposed, but presentation in this form would make many results which appear natural in the present context seem technical and obscure. Occasional reference is made to general nonassociative rings however.

Chapter 1 sets up the notation and a few of the most useful technical tools that are used in the sequel.

Chapters 2 and 3 are concerned with certain classes of finitely generated soluble Lie ring (and Lie algebras). The approach is through associative ring theory using the universal enveloping ring. Chapter 2 looks at maximal conditions and residual finiteness while chapter 3 examines the Frattini theory of these Lie rings.

Chapter 4 examines the residual properties of certain classes of Lie rings, notably nilpotent Lie rings and Lie

rings of matrices over integral domains.

Chapter 5 considers Lie rings whose underlying abelian groups satisfy certain rank restrictions. Necessary and sufficient conditions for residual finiteness are established for these rings.

In chapter 6 we examine which properties when shared by all the abelian subrings of a soluble Lie ring are inherited by the ring itself.

Chapter 7 gives a characterization of certain Lie rings which have the subideal intersection property (i.e. an arbitrary intersection of subideals is once again a subideal).

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CONTENTS

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NOTATION

The following abbreviations and notation are used throughout without further definition :

f.g.	finitely generated
iff	if and only if
$\lvert L \rvert$	the cardinality of L
$\mathbb{Z}$	the ring of integers
$\mathbb{Q}$	the field of rationals
C_n	the cyclic group of order n
C_{p^∞}	the Prufer p^∞ - group

If R is a commutative ring and A and B are R - modules :

$\mathrm{Hom}_R(A , B)$	the ring of R - homomorphisms from A to B
$\mathrm{End}_R(A)$	the ring of R - endomorphisms of A
$M_n(R)$	the full ring of $n \times n$ matrices over R

If $R = \mathbb{Z}$ then we write $\mathrm{Hom}(A , B)$ and $\mathrm{End}(A)$ instead of $\mathrm{Hom}_{\mathbb{Z}}(A , B)$ and $\mathrm{End}_{\mathbb{Z}}(A)$.

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CHAPTER 1 PRELIMINARIES

The notation of this thesis is nonstandard, but is influenced by that used by Stewart [37] and Amayo [1] for Lie algebras, and also by analogy with infinite group theory. Since no suitable reference exists basic definitions have been included for completeness.

§ 1.1 BASIC DEFINITIONS

A Lie ring is an abelian group $(L, +)$ with a bilinear multiplication $[\ ,\ ] : L \times L \longrightarrow L$ satisfying

(a) $[a, a] = 0$

(b) $[a, [b, c]] + [c, [a, b]] + [b, [c, a]] = 0$

for all $a, b, c \in L$.

Note that (a) implies that $[a, b] = -[b, a]$.

A Lie subring of L is an additive subgroup of L, which is closed under multiplication. We write $H \leqslant L$ if H is a Lie subring (not necessarily proper) of L. If $X \subseteq L$ then $\langle X \rangle$ is the subring generated by X. If $A, B \leqslant L$ we define $[A, B]$ to be the ~~subring~~ additive subgroup generated by all products $[a, b]$, $a \in A$, $b \in B$.

A subring H of L is an ideal of L, denoted by $H \lhd L$, if $[L, H] \subseteq H$.

A subring $H \leqslant L$ is an ascendant subring if there exists an ordinal number σ and a collection $\{H_\alpha \mid 0 \leqslant \alpha \leqslant \sigma\}$ of subrings of L such that $H_0 = H$, $H_\sigma = L$ and $H_\alpha \lhd H_{\alpha+1}$ for all $0 \leqslant \alpha \leqslant \sigma$ and $H_\lambda = \bigcup_{\alpha<\lambda} H_\alpha$ for limit ordinals λ. We write $H \lhd^{\sigma} L$.

If $H \lhd^{n} L$ for a finite ordinal n we say H is a <u>subideal</u> of L and write H si L. If it is wished to emphasise the role of the integer n we refer to H as an n-step subideal of L.

<u>EXAMPLES</u>

(1) An abelian group A with trivial multiplication is a Lie ring Such rings are said to be <u>abelian</u>. Clearly an abelian Lie ring with any given additive group exists.

(2) If R is an associative ring, then R can be made into a Lie ring by defining $[a , b] = ab - ba$. We denote R with the new multiplication by R_L. It should be noted that Lie subrings of R_L need not arise from associative subrings of R. In view of this example theorems about Lie rings may also be considered theorems about associative rings.

(3) Every Lie algebra (over any field) can be considered as a Lie ring by restricting scalar multiplication to the integers. Lie subalgebras of such a Lie ring are Lie subrings, but the converse need not be true.

A <u>Lie ring homomorphism</u> $\varphi : L \longrightarrow M$ is an abelian group homomorphism such that

$$\varphi([a , b]) = [\varphi(a) , \varphi(b)] \quad , a , b \in L$$

Standard facts about homomorphisms, quotients, and direct sums are valid (c.f Higgins [17]).

If L , M are Lie rings we denote their underlying abelian groups by L^* and M^*. A map $\Theta : L \longrightarrow M$ is called a <u>*-homomorphism</u> if it is an abelian group homomorphism $L^* \longrightarrow M^*$. Clearly if Θ is a Lie homomorphism it is a *-homomorphism, but the converse need not hold. If φ is a Lie homomorphism we write φ^* for φ considered as a *-homomorphism.

By abuse of language we do not distinguish L^* and the abelian Lie ring with underlying group L^*.

If $H \leqslant L$ then the <u>index of H in L</u> denoted $|L : H|$ is the index of H^* in L^*. Clearly if $H \lhd L$ then $|L : H| = |L / H|$.

We say L is a torsion (respectively torsion free, divisible reduced) ring according as L^* has these properties (c.f Fuchs [10] for definitions). Similarly L is a p-ring for some prime p if L^* is a p-group.

A Lie ring L is said to be of <u>finite exponent</u> if there exists $n \in \mathbb{Z}$ such that $nL = 0$. Clearly if L is of finite exponent then it is torsion.

If $X \subseteq L$ then the <u>centraliser of X in L</u> is

$$C_L(X) = \{ y \in L \mid [X , y] = 0 \}.$$

$C_L(X)$ is a subring of L and if $X \lhd L$ then $C_L(X) \lhd L$.

The <u>centre of L</u> is

$$Z_1(L) = \{ x \in L \mid [L , x] = 0 \}$$

(that is $Z_1(L) = C_L(L)$).

§ 1.2 <u>CLASSES OF LIE RINGS AND CLOSURE OPERATIONS</u>

By a <u>class of Lie rings</u> we shall mean a class $\mathfrak{X}$ in the usual sense whose elements are Lie rings with the further properties

(1) $0 \in \mathfrak{X}$

(2) $L \in \mathfrak{X}$ and $K \cong L$ implies $K \in \mathfrak{X}$.

where 0 denotes the trivial Lie ring.

The symbols $\mathfrak{X}$, $\mathfrak{Y}$ will be reserved for arbitrary classes of Lie rings. Lie rings belonging to a class $\mathfrak{X}$ will often be called $\mathfrak{X}$ -rings.

If $\mathfrak{X}$ is a class of Lie rings we define a new class $\mathfrak{X}^*$ by ; $L \in \mathfrak{X}^*$ iff $L^* \in \mathfrak{X}$ (where L^* is here being considered as an abelian Lie ring).

A (noncommutative and nonassociative) binary operation on classes of Lie rings is defined as follows ; if $\mathfrak{X}$ and $\mathfrak{Y}$ are two classes, then let $\mathfrak{X}\mathfrak{Y}$ be the class consisting of Lie rings L having an ideal H such that $H \in \mathfrak{X}$ and $L / H \in \mathfrak{Y}$. We often refer to such rings as <u>$\mathfrak{X}$ by $\mathfrak{Y}$ rings</u>. The definition can be extended to products of n classes by defining $\mathfrak{X}_1 \ldots \mathfrak{X}_n = (\mathfrak{X}_1 \ldots \mathfrak{X}_{n-1}) \mathfrak{X}_n$. We may put all $\mathfrak{X}_i = \mathfrak{X}$ and denote the result by $\mathfrak{X}^n$.

(0) will denote the class of trivial Lie rings. Other frequently encountered classes are

$\mathfrak{A}$	abelian Lie rings
$\mathfrak{G}$	finitely generated Lie rings
$\mathfrak{T}$	torsion Lie rings
$\mathfrak{H}$	torsion free Lie rings
$\mathfrak{F}$	finite Lie rings
$\mathfrak{E}$	Lie rings of finite exponent
$\mathfrak{C}$	cyclic (i.e. one generator) Lie rings (note $\mathfrak{C} < \mathfrak{A}$).

A <u>closure operation</u> A assigns to each class another class $A\mathfrak{X}$ in such a way that for all classes $\mathfrak{X}$, $\mathfrak{Y}$ the following axioms hold

(a) $A(0) = (0)$

(b) $\mathfrak{X} \leq A\mathfrak{X}$

(c) $A(A\mathfrak{X}) = A\mathfrak{X}$

(d) $\mathfrak{X} \leq \mathfrak{Y}$ implies $A\mathfrak{X} \leq A\mathfrak{Y}$

(where $\leq$ denotes ordinary class inclusion).

$A\mathfrak{X}$ is called the <u>A-closure</u> of $\mathfrak{X}$ and $\mathfrak{X}$ is said to be <u>A-closed</u> if $\mathfrak{X} = A\mathfrak{X}$.

It is often easier to define a closure operation A by specifying which classes are A-closed. Suppose $\mathscr{S}$ is a collection of classes such that $(0) \in \mathscr{S}$ and $\mathscr{S}$ is closed under arbitrary intersections. Then we can define for each class $\mathfrak{X}$ the class

$$A\mathfrak{X} = \bigcap \{ \mathfrak{Y} \in \mathscr{S} \mid \mathfrak{X} \leq \mathfrak{Y} \}$$

(where the empty intersection is the universal class). It is easily seen that A is a closure operation and that $\mathfrak{X}$ is A-closed iff $\mathfrak{X} \in \mathscr{S}$. Conversely if A is a closure operation the set $\mathscr{S}$ of all A-closed classes contains (0), is closed under arbitrary intersections and determines A.

Standard examples of closure operations are S , I , Q , E , L and R which are defined as follows ; $\mathfrak{X}$ is S-closed (I-closed, Q-closed) if every subring (ideal, quotient) of an $\mathfrak{X}$ ring is always an $\mathfrak{X}$ ring ; $\mathfrak{X}$ is E-closed if every extension of an $\mathfrak{X}$ ring by an $\mathfrak{X}$ ring is an $\mathfrak{X}$ ring (equivalently if $\mathfrak{X} = \mathfrak{X}^2$) ; $L \in L\mathfrak{X}$ iff every finite subset of L is contained in an $\mathfrak{X}$ subring of L. $L\mathfrak{X}$ is the class of locally $\mathfrak{X}$ rings ; $L \in R\mathfrak{X}$ iff for each $x \in L$, $x \neq 0$, there exists $I \triangleleft L$ such that $x \notin I$ and $L / I \in \mathfrak{X}$. $R\mathfrak{X}$ is the class of residually $\mathfrak{X}$ rings.

Suppose A , B are closure operations. Then the product AB defined by $AB\mathfrak{X} = A (B\mathfrak{X})$ need not be a closure operation (the third axiom need not hold). Define $\{A , B\}$ to be the closure operation whose closed classes are those classes $\mathfrak{X}$ which are both A-closed and B-closed. If we partially order operations on classesby writing $A \leq B$ iff

$A\mathfrak{X} \leq B\mathfrak{X}$ for any class $\mathfrak{X}$, then $\{A , B\}$ is the smallest closure operation greater than both A and B. It is easy to see that $AB = \{A , B\}$ (and hence is a closure operation) iff $BA \leq AB$.

§ 1.3 DERIVATIONS

A map $d : L \longrightarrow L$ is called a derivation of L if it is a *-homomorphism and for all $x , y \in L$

$$d([x , y]) = [d(x) , y] + [x , d(y)]$$

The set of all derivations of L forms a Lie ring under the usual map operations with Lie product defined by

$$[d_1 , d_2] = d_1 d_2 - d_2 d_1$$

We denote this Lie ring by $Der(L)$.

If $x \in L$ the map $ad_x : L \longrightarrow L$ (called the adjoint map) defined by $ad_x(y) = [y , x]$, $y \in L$, is a derivation of L. We call such derivations inner derivations and denote the collection of all of them by $Inn(L)$. $Inn(L) \lhd Der(L)$ and the map $L \longrightarrow Der(L)$ defined by $x \longmapsto ad_x$ is a Lie homomorphism with kernel $Z_1(L)$ and image $Inn(L)$ so we have

$$Inn(L) \cong L / Z_1(L)$$

The following innocuous looking lemmas prove very useful.

LEMMA 1.3.1

(a) $Der(L) \leq End(L^*)_L$

That is $Der(L)$ is a subring of the Lie ring formed from the associative ring of endomorphisms of its underlying additive group.

(b) If $L \in \mathfrak{U}$ then $Der(L) = End(L^*)_L$

PROOF

Derivations are $*$-homomorphisms. ∎

LEMMA 1.3.2

If $I \lhd L$ then $L / C_L(I) \leqslant \mathrm{Der}(I)$.

PROOF

For any $x \in L$ the map $\varphi_x : I \longrightarrow I$ defined by

$$\varphi_x(y) = [y, x] \qquad y \in I$$

is a derivation of I ($\varphi_x = \mathrm{ad}_x\big|_I$). The map $\varphi : L \longrightarrow \mathrm{Der}(I)$ given by $x \longmapsto \varphi_x$ is a Lie homomorphism with kernel $C_L(I)$ and the result follows. ∎

COROLLARY 1.3.3

If $I \lhd L$ and $I \in \mathfrak{F}$ then $L / C_L(I) \in \mathfrak{F}$ ∎

An ideal I of L is said to be characteristic if it is invariant under derivations of L. We write $I \text{ ch } L$.

Two important properties of characteristic ideals are that

$$I \text{ ch } K \lhd L \text{ implies } I \lhd L$$

and

$$I \text{ ch } K \text{ ch } L \text{ implies } I \text{ ch } L$$

Recall that a subgroup B of an abelian group A is called fully invariant if it is sent into itself by every endomorphism of A.

LEMMA 1.3.4

If $H \subseteq L$ and H^* is a fully invariant subgroup of L^* then $H \text{ ch } L$.

PROOF

If d is a derivation of L then d is a $*$-homomorphism and the result follows.

EXAMPLES

(1) Write $\tau(L)$ for the torsion subgroup of L^*, then
$\tau(L)$ ch L.

(2) Write $\partial(L)$ for the divisible subgroup of L^*, then
$\partial(L)$ ch L.

(3) nL ch L for all $n \in \mathbb{Z}$.

(4) Let $L[n]$ be the set of all $x \in L$ such that $nx = 0$ where $n \in \mathbb{Z}$, then $L[n]$ ch L.

(5) Let L_p be the set of all $x \in L$ such that $p^k x = 0$ for some positive integer k, where p is a prime. Then L_p ch L. We call L_p the p - component of L. Note that

$$L_p = \bigcup_{k=1}^{\infty} L[p^k]$$

and $$\tau(L) = \bigoplus_p L_p$$ where p ranges over all primes.

If L is a Lie ring, $I \lhd L$, $K \leq L$ such that $I + K = L$, $I \cap K = 0$ then we say L is a split extension of I by K. As abelian groups we have $L^* = I^* \oplus K^*$ and each $k \in K$ induces a derivation $d(k) = ad_k\big|_I$ of I, and we obtain a Lie homomorphism $d : K \longrightarrow \mathrm{Der}(I)$ given by $k \longmapsto d(k)$. Further if $x, y \in I$ and $k, l \in K$

$$[x + k, y + l] = ([x, y] + xd(l) - yd(k)) + [k, l] \qquad \ldots\ldots(1)$$

Conversely given any Lie rings I, K and a Lie homomorphism $d : K \longrightarrow \mathrm{Der}(I)$, then (1) can be used to define a Lie product on $I^* \oplus K^*$ making it into a Lie ring. Consequently split extensions correspond to such homomorphisms and this provides us with a way of constructing split extensions.

§ 1.4 SERIES

Let Σ be a totally ordered set and L a Lie ring. A series of L of type Σ is a set

$$S = \{(\Lambda_\sigma, V_\sigma) \mid \sigma \in \Sigma ; \Lambda_\sigma, V_\sigma \leq L\}$$

such that

(a) $V_\sigma \lhd \Lambda_\sigma$

(b) $\tau < \sigma$ implies $\Lambda_\tau \leq \Lambda_\sigma$

(c) $L \setminus \{0\} = \bigcup_{\sigma \in \Sigma} (\Lambda_\sigma \setminus V_\sigma)$

The Lie rings $\Lambda_\sigma / V_\sigma$ are called the factors of the series. The sets $\Lambda_\sigma \setminus V_\sigma$ are called the layers of the series. A series with each factor in a class $\mathfrak{X}$ is called an $\mathfrak{X}$ series. Note that each $0 \neq x$ lies in a unique layer.

A series is said to be invariant (or an ideal series) if $\Lambda_\sigma, V_\sigma \lhd L$ for all σ ; and central if $[\Lambda_\sigma, L] \leq V_\sigma$ for all σ ; and characteristic if Λ_σ, V_σ ch L for all σ.

If $S = \{(\Lambda_\sigma, V_\sigma) \mid \sigma \in \Sigma\}$ where Σ is a well ordered set, then the Λ_σ's are superfluous. If we let $V_\rho = L$ then S may be written

$$0 = V_0 \lhd V_1 \lhd \ldots V_\rho = L$$

where if λ is a limit ordinal $\leq \rho$ then $V_\lambda = \bigcup_{\sigma < \lambda} V_\sigma$.

We say S is an ascending series. Dually we can define a descending series.

If Σ is a finite set then we have a finite series which we may write

$$0 = L_0 \lhd L_1 \lhd \ldots \lhd L_n = L$$

and which is both ascending and descending.

For more details concerning general series see Robinson [30], the methodology for Lie rings being the same as for groups.

L^n will denote the n^{th} term of the lower central series of L defined by

$$L^1 = L , \quad L^{n+1} = [L^n , L]$$

$L^{(\alpha)}$ (for all ordinals α) will denote the α^{th} term of the (transfinite) derived series of L defined by

$$L^{(0)} = L , \quad L^{(\alpha+1)} = [L^{(\alpha)} , L^{(\alpha)}]$$

and $L^{(\lambda)} = \bigcap_{\alpha<\lambda} L^{(\alpha)}$ for limit ordinals λ .

$Z_\alpha(L)$ will denote the α^{th} term of the (transfinite) upper central series of L defined by

$Z_0(L) = 0$, $Z_1(L)$ is the centre of L ,

$Z_{\alpha+1}(L) / Z_\alpha(L) = Z_1(L / Z_\alpha(L))$

$$Z_\lambda(L) = \bigcup_{\alpha<\lambda} Z_\alpha(L) \quad \text{for limit ordinals } \lambda .$$

Note that L^n , $L^{(\alpha)}$, and $Z_\alpha(L)$ are all characteristic ideals of L.

We say that L is nilpotent of class $\leq n$ (or $L \in \mathfrak{N}_n$) if $L^{n+1} = 0$. If L is nilpotent of some class then $L \in \mathfrak{N}$.

We say that L is soluble (of derived length $\leq$ n) if $L^{(n)} = 0$. It is easy to see that L is soluble iff $L \in E\mathfrak{A}$ and that $\mathfrak{N} < E\mathfrak{A}$.

§ 1.5 REPRESENTATIONS AND MODULES

Let L be a Lie ring and A an abelian group. A representation of L is a Lie homomorphism $\rho : L \longrightarrow \mathrm{End}(A)_L$ such that

$$\rho (mx + ny) = m\rho(x) + n\rho(y)$$

$$\rho ([x , y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$$

for all $x , y \in L$, $m ,n \in \mathbb{Z}$.

We can define an L - action $A \times L \longrightarrow A$ by

$$ax = \rho(x)(a) \qquad a \in A, x \in L \; \dots (1)$$

Then
$$\left.\begin{aligned} (na+mb)x &= nax+mbx \\ a(nx+my) &= nax+may \\ a[x,y] &= (ax)y-(ay)x \end{aligned}\right\} \quad \ldots\ldots (2)$$
for all $a, b \in A$, $x, y \in L$, $n, m \in \mathbb{Z}$.

An abelian group with an L - action satisfying (2) is called an <u>L - module</u>. Using (1) we can pass back and forth from representations to modules.

Submodules, quotient modules etc. are defined in the obvious way and standard facts regarding them (e.g. the Noether isomorphism theorems) hold.

Modules arise naturally as follows : suppose $I \triangleleft L$

Now define an L - action on I by
$$xy = [x, y], \quad x \in I, \quad y \in L$$
Then I is an L - module. Similarly if $I, J \triangleleft L$ and $J \leq I$ then I / J is an L - module (and also an L / J - module).

Given an L - module A we let
$$C_L(A) = \{ x \in L \mid ax = 0 \text{ for all } a \in A \}$$
Then $C_L(A) \triangleleft L$. Thus $C_L(I / J)$ is defined if $I, J \triangleleft L$ and $J \leq I$.

An L - module A is <u>faithful</u> if $C_L(A) = 0$. In this case we say the associated representation ρ is faithful. Clearly $C_L(A)$ is the kernel of ρ.

A is <u>irreducible</u> if its only submodules are 0 and A and $A \neq 0$ (and ρ is irreducible iff A is).

§ 1.6 COMPLETIONS

Let L be a Lie ring, k a field of characteristic 0, then the <u>k - completion</u> of L is the Lie algebra $\mathcal{C}_k(L)$ with underlying abelian group $k \otimes_{\mathbb{Z}} L$ and Lie multiplication defined by

(i) $[s \otimes x + s' \otimes x' , t \otimes y + t' \otimes y']$

$$= st \otimes [x , y] + st' \otimes [x , y'] + s't \otimes [x' , y] + s't' \otimes [x' , y']$$

(ii) $s(t \otimes x) = st \otimes x$

for all $s , t , s' , t' \in k$ and $x , x' , y , y' \in L$.

<u>LEMMA 1.6.1</u> (Moran [29] p10ff)

Let L be a torsion free Lie ring then

(a) L is canonically isomorphic via the map $x \mapsto 1 \otimes x$ to a subring of $\mathcal{C}_k(L)$.

(b) $\mathcal{C}_k(L)$ satisfies an homogeneous identical relation $f(x_1,\ldots,x_n) = 0$ iff L satisfies the same relation. ∎

<u>COROLLARY 1.6.2</u> (Moran [29] p10ff)

If L is a torsion free Lie ring then

(a) $\mathcal{C}_k(L) \in \mathfrak{N}_c$ iff $L \in \mathfrak{N}_c$

(b) $\mathcal{C}_k(L) \in \mathfrak{A}^d$ iff $L \in \mathfrak{A}^d$

(c) If $H \triangleleft L$ then $\mathcal{C}_k(H) \triangleleft \mathcal{C}_k(L)$. ∎

Note in general that given $M \leq \mathcal{C}_k(L)$ there may not exist $H \leq L$ such that $M = k \otimes_{\mathbb{Z}} H$. However if $k = \mathbb{Q}$ then (by identifying x and $1 \otimes x$) given $y \in \mathcal{C}_k(L)$, there exists $n \in \mathbb{Z}$ such that $ny \in L$ and so given $M \leq \mathcal{C}_{\mathbb{Q}}(L)$ there exists $H \leq L$ such that $M = \mathbb{Q} \otimes_{\mathbb{Z}} H$. The $\mathbb{Q}$ - completion is also known as the rational completion. For details about completions see Moran [29] .

§ 1.7 CHAIN CONDITIONS

Let A be an abelian group, $\mathcal{S}$ a collection of subsets of A. We say that A has Max - $\mathcal{S}$ if $\mathcal{S}$ satisfies the maximal condition i.e. every ascending chain

$$S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots\cdots$$

of elements $S_i \in \mathcal{S}$ stops after a finite number of steps. That is $S_r = S_{r+1} = \ldots$ for some r. Dually we define Min - $\mathcal{S}$.

If L is a Lie ring and $\mathcal{S}$ is respectively the set of subrings, ideals, or subideals we write Max, Max - $\triangleleft$, Max - si for Max - $\mathcal{S}$ and Min, Min - $\triangleleft$, Min - si for Min - $\mathcal{S}$. These symbols also denote the corresponding classes of Lie rings.

If M is an L - module we say M has Max - L or Min - L according as M has the maximal or minimal condition for L - submodules.

The following result is standard.

LEMMA 1.7.1

(a) $L \in$ Max iff every subring of L is f.g.

(b) $L \in$ Max - $\triangleleft$ iff every ideal is f.g. as an ideal.

(c) $M \in$ Max - L iff every submodule of M is f.g. as an L - module. ∎

§ 1.8 LINEAR LIE RINGS

By analogy with group theory we say that a Lie ring L is R - linear of degree n , where R is a commutative ring with 1, and $n > 0$, if L has a faithful representation as a Lie ring of $n \times n$ matrices over R .

Let $\mathcal{R}_0$ denote the class of all linear Lie rings over integral domains of characteristic 0 . We are able to describe the structure of soluble rings in this class quite explicitly.

PROPOSITION 1.8.1

$$\mathcal{R}_0 \cap E\mathfrak{A} \leq \mathfrak{N}\mathfrak{A}$$

PROOF

Let $L \in \mathcal{R}_0 \cap E\mathfrak{A}$ say L is R - linear where R is an integral domain of characteristic 0 . Let k be the field of fractions of R . Consider the k - completion of L . L is torsion free and its completion is a finite dimensional soluble Lie algebra over k . Hence by Lie's theorem (Jacobson [19] p51) it is nilpotent by abelian. Corollary 1.6.2 then ensures that $L \in \mathfrak{N}\mathfrak{A}$. ∎

CHAPTER 2   FINITELY GENERATED SOLUBLE LIE RINGS

In [12] and [13] Hall studied finitely generated soluble groups using ring theoretic methods. He obtained the following results ;

(a) Finitely generated abelian by polycyclic groups satisfy the maximal condition for normal subgroups.

(b) Finitely generated abelian by nilpotent groups are residually finite.

Certain analogous results have been obtained for Lie algebras:

(a) Finitely generated abelian by finite (dimensional) Lie algebras satisfy the maximal condition for ideals (Amayo and Stewart [2])

(b) Finitely generated metabelian Lie algebras are residually finite and in characteristic 0 there exist f.g. abelian by nilpotent Lie algebras which are not residually finite (Amayo [1]).

In the first section of this chapter we give a basis free version of the proof in [2] thereby extending (a) to a class of generalised Lie algebras which includes Lie rings.

Using methods based upon Hall's f.g. abelian by nilpotent Lie rings are shown to be residually finite. The question of whether f.g. abelian by polycyclic Lie rings are residually finite is not answered but in this direction we prove that they are residually of finite exponent.

Finally the partial breakdown in the analogy for characteristic 0 Lie algebras in (b) is shown not to hold for fields of prime characteristic. Using results of Curtis [7] on the universal enveloping algebra it is shown that f.g. abelian by finite Lie algebras over fields of characteristic $p > 0$ are residually finite. A new proof is also provided for Amayo's result.

Hence for Lie rings and Lie algebras of characteristic $p > 0$ analogous or stronger results than for groups are obtained.

§ 2.1 PBW ALGEBRAS AND Max - $\triangleleft$

Let L denote a Lie algebra over a commutative ring with 1, say R. The universal enveloping algebra $U(L)$ of L is an associative unitary R - algebra and a map $\varepsilon : L \longrightarrow U(L)$ such that ε is a Lie homomorphim $L \longrightarrow U(L)_L$, and if A is any associative unitary R - algebra and $\alpha : L \longrightarrow A_L$ is any Lie homomorphism then there exists a unique associative algebra homomorphism $\varphi : U(L) \longrightarrow A$ such that

$$\begin{tikzcd} U(L) \arrow[r, dashed, "\varphi !"] & A \\ L \arrow[u, "\varepsilon"] \arrow[ur, "\alpha"'] & \end{tikzcd}$$

commutes.

For details regarding the existence and properties of $U(L)$ see Serre [35] and Bourbaki [4] .

We define a filtration of $U(L)$ as follows; let U_n be the submodule of $U(L)$ generated by the products $\varepsilon(x_1)\ldots\varepsilon(x_m)$, $m \leq n$ and $x_i \in L$, $i = 1,\ldots m$. Then we have $U_o = R$, $U = R \oplus \langle\varepsilon(L)\rangle$ (module direct sum) and

$$U_0 \subset U_1 \subset \ldots \subset U_n \subset U_{n+1} \subseteq \ldots$$

Now define

$$gr\ U(L) = \sum_{n=0}^{\infty} gr_n U$$

where $gr_n U = U_n / U_{n-1}$ and multiplication is defined componentwise by

$$(x_i + U_{i-1})(x_j + U_{j-1}) = (x_i x_j + U_{i+j-1})$$

gr U(L) is called the graded algebra associated to U(L).
It is associative, has a 1 and is commutative (Serre [35] LA 3.5)

Further the canonical map $L \longrightarrow$ gr U(L) extends to a homomorphism ξ : $S(L) \longrightarrow$ gr U(L) where S(L) is the symmetric algebra of L (i.e. $S(L) = U(L^*)$ cf Serre [35] LA 3.3)

We will call L a PBW - algebra (for Poincaré, Birkhoff, Witt) if the map ξ defined above is an isomrphism. for all homomorphic images of L.

The original Poincaré, Birkhoff, Witt theorem shows that L is a PBW - algebra if it is a free R - module (which is always true if R is a field) cf Serre [35] LA 3.5. Lazard [23] proves that L is a PBW - algebra if R is a principal ideal domain (and consequently that Lie rings are PBW - algebras). Not all Lie algebras are PBW - algebras (Širšov [36]).

It follows easily that if L is a PBW - algebra the map ε is an injection and in this case we will identify L with its image under ε in U(L). Then U(L) is generated as an R - module by $1 \in R$ and the monomials of degree ≥ 1 in the elements of L. That is the elements of the form $u_1 \ldots u_n$ with $n \geq 1$, $u_i \in L$, $i = 1, \ldots n$ and multiplication in U(L) denoted by juxtaposition.

If M is an L - module then M has a natural U(L)-module structure defined by

$$m(u_1 \ldots u_n) = (\ldots(mu_1)\ldots u_n) \qquad m \in M,\ u_i \in L$$

And conversely any U(L) - module can be interpreted as an L - module. This correspondence preserves submodules (Serre [35] LA 3.2).

We will often write U = U(L) from now on.

LEMMA 2.1.1 (Amayo and Stewart [2] p700)

If I is a Lie ideal of L, then $IU = UIU$. ∎

The following lemma appears as an exercise attributed to Bergman in Serre [35] LA 3.2.

LEMMA 2.1.2

$U = R$ iff $L = 0$. ∎

LEMMA 2.1.3

Let L be a PBW - algebra (over R) and $B \triangleleft L$, $B \leq A \leq L$ and $U = U(L)$ then $BU = AU$ implies $A = B$.

PROOF

By Jacobson [19] p159 - 62 and lemma 2.1.1

$$U(L / B) \cong U / UBU = U / BU$$
$$= U / AU \quad \text{by hypothesis.}$$

Now consider A / B as a subalgebra of L / B. Then $U(A / B)$ is the subalgebra of $U(L / B)$ generated by $(A + BU) / BU$ and 1 (Jacobson [19] p153).

Now $(A + BU) / BU = (A + AU) / AU = 0$. Thus $U(A / B) = R$ and by lemma 2.1.2, $A / B = 0$. ∎

Let <u>Max - u</u> denote the class of Lie algebras L such that U(L) is right Noetherian (i.e. it satisfies the maximal condition on right ideals).

LEMMA 2.1.4

If L is a PBW - algebra then $L \in \text{Max} - u$ implies $L \in \text{Max} - \triangleleft$.

PROOF

Consider a sequence of ideals in L

$$0 \leqslant H_1 \leqslant H_2 \leqslant \ldots\ldots$$

Then $0 \leqslant H_1U \leqslant H_2U \leqslant \ldots\ldots$ is a sequence of (right) ideals in $U = U(L)$ so there exists $r > 0$ such that

$$H_rU = H_{r+1}U = \ldots\ldots$$

Thus by lemma 2.1.3 $H_r = H_{r+1} = \ldots\ldots$ and the result is proved. ■

The following result is well known and the proof is similar to the corresponding result for commutative rings.

LEMMA 2.1.5

(i) If R is a right Noetherian ring and M is a f.g. right R - module then M satisfies Max - R (i.e. M is a Noetherian module).

(ii) If L is a Lie algebra over a commutative ring (R is an associative ring) and

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a short exact sequence of L - modules (R - modules) such that A and C satisfy Max - L (Max - R) then B satisfies Max - L (Max - R). ■

For the rest of this section $L = \langle x_1, \ldots, x_n \rangle$ will denote a f.g. PBW - algebra over R with $U = U(L)$.

Let Z be the ideal of U spanned by all monomials of degree $\geqslant 1$ (i.e. spanned by L).

A Lie algebra over a ring R is <u>finitely presented</u> if it can be generated by finitely many elements $x_1, \dots, x_m$ subject to a finite number of defining relations $f_1(x_1, \dots, x_m) = 0, \dots, f_n(x_1, \dots, x_m) = 0$. Thus it is the quotient of a free Lie algebra on the set $\{x_1, \dots, x_m\}$ by the ideal generated by the elements $f_1(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m)$. (For a discussion on free Lie algebras including the question of existence see 'Free Rings' by P. M. Cohn).

LEMMA 2.1.6

A finite presentation for a Lie algebra L is independent of the finite set of generators chosen.

PROOF

Carry over the notation from the definition above and suppose that $y_1, \dots, y_s$ is any other finite set of generators for L.

Then for certain words φ_i and ψ_j

$$x_i = \varphi_i(y_1, \dots, y_s) \qquad i = 1, \dots, m$$
$$y_j = \psi_j(x_1, \dots, x_m) \qquad j = 1, \dots, s.$$

Then the relations

$$y_j = \psi_j(\varphi_1(y_1, \dots, y_s), \dots, \varphi_m(y_1, \dots, y_s))$$
$$f_k(\varphi_1(y_1, \dots, y_s), \dots, \varphi_m(y_1, \dots, y_s)) = 0$$

for $j = 1, \dots, s$, $k = 1, \dots, n$ certainly hold in L.

Let $\overline{L}$ be the Lie algebra generated by $\overline{y}_1, \dots, \overline{y}_s$

say subject to the relations above (in the $\bar{y}_i$'s). Then in fact $\bar{L} \cong L$. Indeed the defining relations of $\bar{L}$ hold in L so the map $\bar{y}_j \longmapsto y_j$ extends to a homomorphism Θ of $\bar{L}$ onto L. Now let

$$\bar{x}_i = \varphi_i(\bar{y}_1, \dots, \bar{y}_s)$$

Then from above we have

$$\bar{y}_j = \Psi_j(\bar{x}_1, \dots, \bar{x}_m)$$

and so $\bar{L} = \langle \bar{x}_1, \dots, \bar{x}_m \rangle$. Then since

$$f_k(\bar{x}_1, \dots, \bar{x}_m) = 0$$

the map $x_i \longmapsto \bar{x}_i$ extends to a homomorphism η of L onto $\bar{L}$. Finally $\eta\Theta$ and $\Theta\eta$ are the identity maps of L and $\bar{L}$ respectively, so Θ and η are isomorphisms. ∎

<u>LEMMA 2.1.7</u>

If $L \in \mathcal{G}$ and $I \lhd L$ such that L/I is finitely presented then I is f.g. as an ideal of L.

PROOF

Let $a_1, \dots, a_m$ generate L. Then L/I is generated by $a_1 + I, \dots, a_m + I$ (in fact finitely presented by Lemma 2.1.6).

Let F be the free Lie algebra on the set $\{x_1, \dots, x_m\}$. Define a homomorphism $\Theta : F \longrightarrow L$ by $x_i \longmapsto a_i$. Let $K = \Theta^{-1}(I)$.

The map $x_i \longmapsto a_i + I$ extends to a homomorphism of F onto L/I with kernel K. Now as L/I is finitely presented K is f.g. as an ideal of F by

$y_1, \ldots, y_n$ say with $y_i \in K$. Hence I is generated as an ideal of L by $\Theta(y_1), \ldots, \Theta(y_n)$. $\square$

<u>THEOREM 2.1.10</u>

Let L be a PBW - algebra. If $L \in \mathcal{G}$ with an abelian ideal A such that $L/A \in \text{Max} - u$ and is finitely presented then $L \in \text{Max} - \triangleleft$.

PROOF

Since A is abelian we can consider it as an L/A - and hence $U(L/A)$ - module. By Lemma 2.1.7 A is a f.g. ideal of L and hence is f.g. as a $U(L/A)$ - module. But $U(L/A)$ is Noetherian by hypothesis so by Lemma 2.1.5 (i) $A \in \text{Max} - U(L/A)$ and hence $A \in \text{Max} - L/A$. Thus $A \in \text{Max} - L$ since the L - submodules are just the L/A - submodules (since $A^2 = 0$).

By Lemma 2.1.4 $L/A \in \text{Max} - u \leq \text{Max} - \triangleleft$ and so $L/A \in \text{Max} - L$. Hence $L \in \text{Max} - L$ by Lemma 2.1.5. That is $L \in \text{Max} - \triangleleft$. $\blacksquare$

It is shown in [2] that if L is a f.g. Lie algebra over a field with an abelian ideal A and Z is the ideal of $U = U(L)$ generated by L then $A \cap ZA = 0$ but this is not true in general and the proof in [2] cannot be used. Indeed let L be the Lie ring given by $L = \mathbb{Z} \oplus \mathbb{Z} \oplus C_{p^2} = \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \langle x_3 \rangle$ say, with $[x_1, x_2] = x_3$, $[x_1, x_3] = 0$ and $[x_2, x_3] = 0$. Then $L \in \mathcal{G} \cap \mathfrak{N}$ and $A = pL \in \mathfrak{U}$ but

$$0 \neq p(x_1 x_2 - x_2 x_1) = x_1(px_2) - x_2(px_1) = p[x_1, x_2]$$
$$= px_3 \in A \cap ZA.$$

COROLLARY 2.1.11

Theorem 2.1.10 holds when L is a Lie ring ■

We will now show a way of finding algebras which satisfy Max - u.

Let Max - s denote the class of Lie algebras L such that the symmetric algebra S(L) of L is Noetherian.

LEMMA 2.1.12

A PBW - algebra which satisfies Max - s satisfies Max - u .

PROOF

If L is a PBW - algebra $S(L) \cong \text{gr } U(L)$. But by Jacobson [19] p164 Theorem 4 if gr U(L) is right Noetherian then so is U(L) as required. ■

Suppose that L is f.g. as an R - module, say by $x_1, \ldots, x_n$, then S(L) is a quotient of $R[x_1, \ldots, x_n]$ and consequently by the Hilbert basis theorem (Lang [24] p144 Theorem 1) is Noetherian if R is Noetherian and clearly L is finitely presented.

Since $\mathbb{Z}$ is Noetherian this enables us to state immediately ;

THEOREM 2.1.13

Let L be a Lie ring. If $L \in \mathfrak{G} \cap \mathfrak{U}\mathfrak{G}^*$ then $L \in \text{Max} - \triangleleft$. ■

COROLLARY 2.1.14

If L is a Lie ring and $L \in \mathfrak{G} \cap \mathfrak{U}\mathfrak{F}$ then $L \in \text{Max} - \triangleleft$. ■

COROLLARY 2.1.15

If L is a Lie ring and $L \in \mathfrak{G} \cap \mathfrak{A}\mathfrak{N}$ then $L \in \mathrm{Max}\text{-}\lhd$.

PROOF

It is sufficient to prove that $\mathfrak{G} \cap \mathfrak{N} \leq \mathfrak{G}^*$, but this is true by the same argument as Hartley [16] p261 lemma 1(i). ∎

COROLLARY 2.1.16

If L is a Lie ring and $L \in \mathfrak{G} \cap \mathfrak{A}(E\mathfrak{C})$ then $L \in \mathrm{Max}\text{-}\lhd$.

PROOF

Since $E\mathfrak{C} = E\mathfrak{A} \cap \mathfrak{G}^*$. ∎

§ 2.2 RESIDUAL FINITENESS - THE FIRST STEP

We now return to the consideration of Lie rings. Following Hall [13] we define classes $\mathfrak{A}(\pi)$ of $\mathbb{Z}$ - modules (i.e. abelian groups). Recall that if A is a $\mathbb{Z}$ - module and A_0 is the free submodule generated by a maximal family of $\mathbb{Z}$ - linearly independent elements of A then A/A_0 is a torsion module.

We say a $\mathbb{Z}$ - module A is contained in $\mathfrak{A}(\pi)$ where π is a set of primes iff the free submodule A_0 of A defined above, can be chosen so that A/A_0 is a π - torsion module (i.e. the order of every element of A/A_0 is a π - number). Note that if π is the complete set of primes then $\mathfrak{A}(\pi)$ is the class of all $\mathbb{Z}$ - modules (i.e. by abuse of language $\mathfrak{A}$). Also if $\pi = \emptyset$ then $\mathfrak{A}(\pi)$ is the class of free $\mathbb{Z}$ - modules.

<u>LEMMA 2.2.1</u> (Hall [13] lemmas 4.1, 4.2, 4.3)

(i) If $A \cong \mathbb{Q}$ (as $\mathbb{Z}$ - modules) then $A \in \mathfrak{M}(\pi)$ iff π is the complete set of primes.

(ii) If $A \in \mathfrak{M}(\pi)$ then every $\mathbb{Z}$ - submodule of A also belongs to $\mathfrak{M}(\pi)$.

(iii) Let A be expressed as the union of a well ordered ascending series of submodules $\{A_\alpha\}_{\alpha \leq \rho}$ where

$$A = A_\rho \,, \quad A_0 = 0 \,, \quad A_\alpha \leq A_\beta \,, \quad \alpha < \beta \leq \rho$$

and $A_\lambda = \bigcup_{\alpha<\lambda} A_\alpha$ for limit crdinals λ.

Then if $A_{\alpha+1}/A_\alpha \in \mathfrak{M}(\pi)$ for all $\alpha < \rho$ we have $A \in \mathfrak{M}(\pi)$. ∎

When L is a Lie ring we can consider L as a $\mathbb{Z}$ - module and could reinterpret the above lemma in terms of L. For example (ii) would imply that if $L \in \mathfrak{M}(\pi)$ (as a $\mathbb{Z}$ -module) then every Lie subring also belongs to $\mathfrak{M}(\pi)$.

<u>PROPOSITION 2.2.2</u>

Let $L \in E\mathfrak{C}$ and let P be any right ideal of U(L) = U. Then $U/P \in \mathfrak{M}(\pi)$ for some finite set of primes π. (Considering U/P as a $\mathbb{Z}$ - module).

PROOF

The proof is by induction on the number n of cyclic factors. If n = 0, we have L = 0 and $U(L) = \mathbb{Z} \in \mathfrak{M}(\emptyset)$.

Now suppose n > 0 and let

$$0 = L_0 \lhd L_1 \lhd \ldots \lhd L_n = L \,, \quad L_i/L_{i-1} \in \mathfrak{C}$$

be a cyclic series for L. Put $K = L_{n-1}$. Then $L = \langle K, x \rangle$. By the induction hypothesis we may suppose that if Q is any right ideal of U(K) then $U(K)/Q \in \mathfrak{M}(\pi_0)$ for some finite set of primes $\pi_0 = \pi_0(Q)$. We will think of $\overline{U(K)}$ as the subring of $U(L)$ generated by 1 and K. It is a homomorphic image of $U(K)$.

Now consider multiplication inside $U(L)$. If $y \in K$ then

$$xy = yx + [x, y] \qquad \ldots. (1)$$

and $[x, y] \in K \lhd L$.

Thus we can always express a monomial of $U(L)$ as a sum of monomials in which all powers of x (if any) always occur on the right hand side. Hence since $U(L) = \langle \overline{U(K)}, x \rangle$

$$\begin{aligned} U(L) &= \overline{U(K)} + \overline{U(K)}x + \ldots. \\ &= \sum_{i=0}^{\infty} \overline{U(K)}x^i \end{aligned}$$

Now define $U_0 = \overline{U(K)}$, $U_i = \sum_{s=0}^{i} \overline{U(K)}x^s$

i.e. $U_k = U_{k-1} + \overline{U(K)}x^k$

Put $P_k = P + U_{k-1}$ for $k = 1, 2, \ldots.$ and $P_0 = P$. Then $P = P_0 \leq P_1 \leq \ldots..$ and $U(L) = \bigcup_{k=0}^{\infty} P_k$. Then the $\mathbb{Z}$ - modules P_k/P form an ascending chain of submodules of $U(L)/P$ with union $U(L)/P$. The Zassenhaus Butterfly lemma (Lang [24] p102) now gives

$$\begin{aligned} P_{k+1}/P_k &= (P + U_k)/(P + U_{k-1}) \\ &\cong_{\mathbb{Z}} U_k/(U_{k-1} + (P \cap U_k)) \end{aligned}$$

Now if $w \in U_m$ then $w = \sum_{i=0}^{m} c_i x^i$, $c_i \in \overline{U(K)}$. Let Q_m be the set of elements $c_m \in \overline{U(K)}$ which occur as coefficients of x^m in the elements $w \in P \cap U_m$. Then Q_m is an additive abelian group. We will now show that Q_m is a right ideal of $\overline{U(K)}$. Let $y \in K$ and $w \in P \cap U_m$ then

$$\begin{aligned} wy &= \left(\sum_{i=0}^{m} c_i x^i \right) y \\ &= \sum_{i=0}^{m} c_i^* x^i \end{aligned}$$

where $c_i^* \in \overline{U(K)}$, $i = 0, \ldots., m$ (using the same argument as for (1)). Thus $wy \in P \cap U_m$. The coefficient of x^m

is $c_m^* = c_m y$ and y can be any element of K so Q_m is a right ideal of $U(K)$ by induction.

Now consider the map

$$\varphi : U_k/(U_{k-1} + Q_{k-1}x^k) \longrightarrow \overline{U(K)}/Q_{k-1}$$
$$\left(\sum_{i=0}^{k} c_i x^i + (U_{k-1} + Q_{k-1}x^k)\right) \longmapsto c_k + Q_{k-1}$$

We must first check that φ is well defined. Suppose

$$\left(\sum_{i=0}^{k} c_i x^i + (U_{k-1} + Q_{k-1}x^k)\right) = \left(\sum_{i=0}^{k} d_i x^i + (U_{k-1} + Q_{k-1}x^k)\right)$$

then

$$\sum_{i=0}^{k} (c_i - d_i)x^i \in U_{k-1} + Q_{k-1}x^k$$
$$\text{i.e.} \quad c_k - d_k \in Q_{k-1}$$

Hence $c_k + Q_{k-1} = d_k + Q_{k-1}$ as required.

The map is clearly a $\mathbb{Z}$ - module homomorphism and clearly onto. Also φ is a monomorphism since if

$$\sum_{i=0}^{k} c_i x^i + (U_{k-1} + Q_{k-1}x^k) \in \ker\varphi$$

then $c_k \in Q_{k-1}$ and so $\sum_{i=0}^{k} c_i x^i \in U_{k-1} + Q_{k-1}x^k$. Thus φ is a $\mathbb{Z}$ - module isomorphism and so

$$U_k/(U_{k-1} + P \cap U_k) \cong_{\mathbb{Z}} \overline{U(K)}/Q_{k-1}$$

for all k.

Now $(P \cap U_m)x \leq P \cap U_{m+1}$ and hence

$$Q_0 \leq Q_1 \leq \ldots\ldots$$

Also $E\mathfrak{C} \leq \mathfrak{G}^*$ and so by lemma 2.1.12 $\overline{U(K)}$ is right Noetherian and so $Q_r = Q_{r+1} = \ldots\ldots$ for some r. Hence there exists an integer m suchthat each P_k/P_{k-1} is $\mathbb{Z}$ - isomorphic to at least one of the additive groups $\overline{U(K)}/Q_i$, $i = 0, 1, \ldots\ldots, m$.

By induction there exists for each i a finite set of primes π_i such that $\overline{U(K)}/Q_i \in \mathfrak{M}(\pi_i)$. Let π be the union of all the sets π_i , $i = 0, 1, \ldots\ldots, m$.

Then $P_k / P_{k-1} \in \mathfrak{N}(\pi)$ for all $k = 1, 2, \ldots$. Lemma 2.2.1(iii) now gives $U(L)/P \in \mathfrak{N}(\pi)$. ■

We now define a class $\mathcal{B}$ by saying an abelian Lie ring $B \in \mathcal{B}$ iff B can be extended to a f.g. Lie ring L such that $L/B \in E\mathcal{C}$.

PROPOSITION 2.2.3

If $B \in \mathcal{B}$ then $B \in \mathfrak{N}(\pi)$ for some finite set of primes $\pi = \pi(B)$.

PROOF

Let $B \lhd L \in \mathcal{G}$ and $H = L/B \in E\mathcal{C}$ (since $B \in \mathcal{B}$). Then by corollary 2.1.16 $L \in \mathrm{Max}-\lhd$. Since $B \in \mathfrak{N}$ we can regard B as a f.g. H - module and hence as a f.g. $U(H)$ - module, with generators $b_o, \ldots, b_r$ say.

Now define $B_o = 0$ and inductively

$$\begin{aligned} B_{i+1} &= \langle B_i, b_{i+1} \rangle, \qquad 0 \leqslant i \leqslant r-1 \\ &= B_i + b_{i+1} U(H) \end{aligned}$$

Then $B = B_r$.

Each B_{i+1}/B_i is $U(H)$ - isomorphic with $U(H)/R_i$ where R_i is some right ideal of $U(H)$. Now by Proposition 2.2.2 $U(H)/R_i \in \mathfrak{N}(\pi)$ for $i = 0, \ldots, r$ where π is a finite set of primes.

Lemma 2.2.1(iii) now gives $B \in \mathfrak{N}(\pi)$. ■

COROLLARY 2.2.4

If $B \in \mathcal{B}$ then B contains no subgroups isomorphic with $\mathbb{Q}$.

PROOF

Proposition 2.2.3 and lemma 2.2.1(i). ■

LEMMA 2.2.5

Suppose $H \lhd L$ with $H \in \mathfrak{E}$ and $L/H \in R\mathfrak{E} \cap \mathfrak{H}$. Then $L \in R\mathfrak{E}$.

PROOF

Clearly $L \in R\mathfrak{E}$ iff $\bigcap_{n=1}^{\infty} nL = 0$, but if L is torsion free then $\bigcap_{n=1}^{\infty} nL$ is divisible and it is easy to deduce that in this case $L \in R\mathfrak{E}$ is equivalent to L being reduced.

Now let L be as in the hypotheses of the lemma and suppose $nH = 0$. Consider the map $\varphi : L \longrightarrow L$, $x \longmapsto nx$, $x \in L$. φ is a $*$ - homomorphism and $\operatorname{im}\varphi = nL$. Clearly nL is torsion free and is a $*$ - quotient of L/H. Hence by the discussion above nL is reduced and so $nL \in R\mathfrak{E}$. Hence $L \in R\mathfrak{E}$. ■

THEOREM 2.2.6

If L is a Lie ring and $L \in \mathfrak{G} \cap \mathfrak{N}(E\mathfrak{C})$ then $L \in R\mathfrak{E}$.

PROOF

By Corollary 2.1.16, $\mathfrak{G} \cap \mathfrak{N}(E\mathfrak{C}) \leq \text{Max} - \lhd$ and hence $\tau(L)$ satisfies the maximal condition for characteristic ideals (i.e. Max - c). If n is an integer > 0, the elements $x \in \tau(L)$ such that $nx = 0$ form a characteristic ideal L_n of $\tau(L)$ (cf L_p with p a prime). We must have $\tau(L) = L_n$ for some n since otherwise there would exist an infinite sequence of integers $n_1, n_2, \ldots$ such that

$L_{n_1} < L_{n_2} < \ldots\ldots$ and $\tau(L)$ would not satisfy Max - c . Thus $\tau(L)$ is of finite exponent $n > 0$, and $\tau(L) \in \mathfrak{E}$. By Corollary 2.2.4 $L / \tau(L)$ is reduced and hence $R\mathfrak{E}$ since it is torsion free. Now apply lemma 2.2.5 and $L \in R\mathfrak{E}$. ∎

The importance of Theorem 2.2.6 is that it reduces the problem of residual finiteness to considering p - rings. To see this, note that the theorem allows us to locate any nonzero element in a periodic top factor which involves only finitely many primes (because of finite exponent). This can be further reduced to a p - ring because a torsion ring is a direct sum of its primary components. Thus we have isolated our nonzero element in a top factor which is a p - ring of finite exponent.

§ 2.3 RESIDUAL FINITENESS FOR LIE RINGS

A Lie ring L is said to be monolithic if the intersection of its nonzero ideals is nonzero. The intersection is called the monolith of L and is denoted by $\mu(L)$.

Following Hall [13] p597 lemma 1 , we have ;

LEMMA 2.3.1

Let $\mathfrak{X}$ be a Q - closed class of Lie rings. Then $\mathfrak{X} \leq R\mathfrak{F}$ iff every monolithic $\mathfrak{X}$ - ring is finite.

PROOF

Exactly as for groups. ∎

Let R be an associative ring and I an ideal of R.

Then I has a <u>centralising set of generators</u> if I is generated (as an ideal) by a finite set of elements $r_1, \ldots, r_n$ such that

(i) $r_1 \in Z(R)$, the centre of R.

(ii) $r_i \in Z(R) \bmod \langle r_1, \ldots, r_{i-1} \rangle$ for $i = 2, \ldots, n$.

(where $\langle r_1, \ldots, r_{i-1} \rangle$ is the ideal of R generated by $r_1, \ldots, r_{i-1}$).

Recall that a submodule N of a right R - module M is <u>essential</u> if it has a nontrivial intersection with every nontrivial submodule of M .

We can now state some results which will be needed for the proof of the main theorem.

<u>LEMMA 2.3.2</u>

Let I be an ideal of an associative ring R . Suppose I has a centralising set of generators. Let M be a right R - module with the maximal condition on submodules (M is Noetherian). Let E be an essential submodule of M .

Then if E is annihilated by some power of I , then M is annihilated by some power of I .

PROOF

Hajarnavis [15] p146-147 in the proof of Theorem 6.46 and attributed to McConnell. ∎

<u>LEMMA 2.3.3</u>

Let $L \in \mathfrak{G} \cap \mathfrak{N}$ and $U = U(L)$. Then every ideal of U has a centralising set of generators.

PROOF

McConnell [25] Theorem 2.3 and Theorem 3.2 . ∎

And finally a result which is crucial to the argument. This result is not true for fields of characteristic 0 , and this is the point that causes the divergence of results for Lie algebras that we observe in § 2.4 .

PROPOSITION 2.3.4

Let L be a Lie algebra over a field k of characteristic $p > 0$. If L is finite dimensional then every irreducible representation of L is finite dimensional.

PROOF

Curtis [7] p952 Theorem 5.1 . ∎

PROPOSITION 2.3.5

Every torsion, monolithic $\mathfrak{G} \cap \mathfrak{U}\mathfrak{N}$ Lie ring is finite.

PROOF

Let L be torsion, monolithic and $L \in \mathfrak{G} \cap \mathfrak{U}\mathfrak{N}$. Say $A \lhd L$, $A \in \mathfrak{U}$ and $\tilde{L} = L/A \in \mathfrak{N}$. Put $M = \mu(L)$. Now $M \leq A$ and is a characteristically simple abelian group, and hence is an elementary p-ring (and hence L must be a p-ring).

Now $\tilde{L}/p\tilde{L}$ is a finite dimensional Lie algebra over $\mathbb{Z}_p$ and M is an irreducible $\tilde{L}/p\tilde{L}$ - module (since $p\tilde{L}$ annihilates M and M is an $\tilde{L}$ - module). By Proposition 2.3.4 M is finite dimensional over $\mathbb{Z}_p$ and hence is finite.

Now let $U = U(L/A)$. Then A is a Noetherian U - module and U - submodules of A are just the ideals of L contained in A (cf § 2.1). So M is an essential submodule of A since it is the monolith of L .

Consider the associated representation $\varphi : U \longrightarrow \mathrm{End}(M)$.

End(M) is finite and so putting $P = \ker \varphi$ we have that U/P is finite. P is the annihlator of M and is a (two - sided) ideal and so by lemma 2.3.3 P has a centralising set of generators. Applying lemma 2.3.2 we get $AP^n = 0$ for some n.

Thus we have a series

$$0 = AP^n < \ldots < AP < A$$

where each factor $F_i = AP^i / AP^{i+1}$ is a f.g. U - module (since A is Noetherian).

But P annihilates F_i so we can consider F_i as a f.g. U/P - module. Thus for each i, F_i is a f.g. module over a finite associative ring and hence F_i is finite. Thus A is finite.

L/A is torsion and $L/A \in \mathcal{G} \cap \mathcal{N}$ so $L/A \in \mathcal{F}$.
Hence $L \in \mathcal{F}$. ■

THEOREM 2.3.6

Every $\mathcal{G} \cap \mathfrak{U}\mathcal{N}$ Lie ring is residually finite.

PROOF

By Theorem 2.2.6 we need only consider the torsion case. The result then follows by Proposition 2.3.5 and lemma 2.3.1 . ■

The question of residual finiteness for $\mathcal{G} \cap \mathfrak{U}(E\mathcal{C})$ Lie rings is still open but we can say the following ;

THEOREM 2.3.7

Let $L \in E\mathcal{C}$. If M is an irreducible L - module then M is finite.

PROOF

M is an irreducible $U = U(L)$ - module and is generated over U by a single element (any $0 \neq m \in M$ will do). Form the split extension $E = L + M$. Then E is a monolithic $\mathfrak{G} \cap \mathfrak{U}(E\mathfrak{C})$ Lie ring with monolith M and the result follows by Theorem 2.1.6 and lemma 2.3.1 . ∎

A <u>chief factor</u> of L is a pair (H , K) of ideals of L such that no ideal of L lies strictly between H and K, and such that $H \lneq K$. A <u>chief series</u> for L is an invariant series for L all of whose factors are chief factors.

<u>COROLLARY 2.3.8</u>

If $L \in \mathfrak{G} \cap \mathfrak{U}(E\mathfrak{C})$ then every chief factor of L is finite.

PROOF

Suppose $H , K \lhd L$ and H/K is a chief factor of L. Then we can consider K trivial, so that H is a minimal abelian ideal of L.

Let A be maximal such that $A \lhd L$, $H \leq A$ and $A \in \mathfrak{U}$. Then $L/A \in E\mathfrak{C}$ and H is an irreducible L/A - module. The result now follows from Theorem 2.3.7 . ∎

§ 2.4 <u>THE LIE ALGEBRA PROBLEM</u>

Throughout this section we consider Lie algebras over fields with notation carrying over in an obvious manner. Note that we are using $\mathfrak{F}$ to mean the class of finite dimensional Lie algebras.

Let S be a noncommutative ring with 1 and R a subring of the centre of S containing 1. We say S is an extension of R and we call it an integral extension if S is a Noetherian R - module.

A representation of a noncommutative ring S is a (ring) homomorphism of S onto a subring T of the ring of endo - morphisms of some abelian group. A representation is irreducible if the group (which clearly can be regarded as a T - module) has no proper T - submodules. An ideal of S is called primitive if it is the kernel of some irreducible representation of S.

In the usual manner representations and modules are associated. We say that an S - module is irreducible if its associated representation is irreducible.

We now have the following important lemma of Curtis.

LEMMA 2.4.1 (Curtis [7] p947 lemma 3.1)

If S is an integral extension of R and P is a primitive ideal of S then $P \cap R$ is a maximal ideal of R. ∎

We will say S is a Curtis ring over a field k if S is an integral extension of a Noetherian ring R where :

(1) R is an extension of the field

and (2) If I is a maximal ideal of R then the dimension of R/I over k is finite.

LEMMA 2.4.2

Let S be an integral extension of the Noetherian ring R. Suppose M is a Noetherian S - module which has an essential irreducible submodule M_o . If $P = \mathrm{Ann}(M_o)$ then $M(R \cap P)^n = 0$ for some n .

PROOF

P is a primitive ideal of S and so by lemma 2.4.1 $P \cap R$ is a maximal ideal of R. Now R is Noetherian and so $P \cap R$ is generated (as an ideal of R) by a finite set of elements of R , say $x_1 ,\ldots, x_n$. Let P_o be the ideal of S generated by $x_1 ,\ldots, x_n$. Then $P_o \leqslant P$ and so P_o annihilates the essential submodule M_o and has a centralising (in fact central) set of generators $x_1 ,\ldots, x_n$, and so by lemma 2.3.2 $MP_o^m = 0$ for some m . Clearly $R \cap P \leqslant P_o$ and so $M(R \cap P)^m = 0$. ■

PROPOSITION 2.4.3

Let S be a Curtis ring over a field k . Suppose M is a Noetherian S - module with an essential irreducible submodule, then M is finite dimensional over k .

PROOF

Suppose S is an integral extension of the Noetherian ring R where R is an extension of the field k such that if I is a maximal ideal of R then the dimension of R/I over k is finite. M is a Noetherian S - module and S is a Noetherian R - module so it follows easily that M is a Noetherian R - module. Let $P = \mathrm{Ann}_S(M_o)$, then by lemma 2.4.2 $M(R \cap P)^n = 0$ for some n . So there exists a (finite) sequence

$$M > M(R \cap P) > \ldots\ldots > M(R \cap P)^n = 0$$

of R - submodules of M . Each factor

$$M(R \cap P)^i/M(R \cap P)^{i+1} \quad , \quad i = 1 ,\ldots, n-1$$

is a f.g. $R/(R \cap P)$ - module. $R \cap P$ is a maximal ideal in R by lemma 2.4.1 and so by hypothesis $R/(R \cap P)$ is finite dimensional over k. Thus each factor is a f.g. module for a finite dimensional ring and hence is finite dimensional. Hence M is finite dimensional. ∎

COROLLARY 2.4.4 (Curtis [7] p949 Theorem 4.2)

Let S be a Curtis ring over a field k. Then every irreducible S - module is finite dimensional over k.

PROOF

An irreducible module is clearly Noetherian. ∎

Let L be a Lie algebra of dimension n over a field k. If char $k = p > 0$ then Curtis [7] §5 p952 shows that $U(L)$ is an integral extension of a subring R of its centre where R is isomorphic to $k[x_1, \ldots, x_n]$ (the polynomial ring in n indeterminates over k). By the Hilbert Nullstellensatz if I is a maximal ideal of R then R/I is finite dimensional over k, and so $U(L)$ is a Curtis ring over k.

If L is abelian and char $k = 0$ then in fact $U(L) \cong k[x_1, \ldots, x_n]$ and so once again $U(L)$ is a Curtis ring over k.

A Lie algebra L is said to be <u>monolithic</u> if the intersection of its nonzero ideals is also nonzero. The intersection is called the <u>monolith</u> of L and is denoted by $\mu(L)$.

Once again (cf lemma 2.3.1) we obtain :

LEMMA 2.4.5

Let $\mathfrak{X}$ be a Q - closed class of Lie algebras. Then $\mathfrak{X} \leq R\mathfrak{F}$ iff every monolithic $\mathfrak{X}$ - algebra is finite dimensional. ∎

PROPOSITION 2.4.6

Suppose L is a Lie algebra over a field k , L monolithic with monolith $\mu(L)$. Then if either

(i) char k = $p > 0$ and $L \in \mathfrak{G} \cap \mathfrak{A}\mathfrak{F}$

or (ii) char k = 0 and $L \in \mathfrak{G} \cap \mathfrak{A}^2$

then L is finite dimensional.

PROOF

Let $A \lhd L$, $A \in \mathfrak{A}$ such that $L/A \in \mathfrak{F}$ if char k = p , and $L/A \in \mathfrak{F} \cap \mathfrak{A}$ if char k = 0 . Then in the usual manner A is an L/A - and hence $U(L/A)$ - module and is Noetherian by Theorem 2.1.10. By the discussion above $U(L/A)$ is a Curtis ring over k . $\mu(L)$ is an essential irreducible submodule of A by definition, and so by Proposition 2.4.3 A is finite dimensional. Hence L is finite dimensional. ∎

THEOREM 2.4.7

Suppose L is a Lie algebra over a field k . If either

(i) char k = $p > 0$ and $L \in \mathfrak{G} \cap \mathfrak{A}\mathfrak{F}$

or (ii) char k = 0 and $L \in \mathfrak{G} \cap \mathfrak{A}^2$

then $L \in R\mathfrak{F}$.

PROOF

Lemma 2.4.5 and Proposition 2.4.6 . ∎

Part (ii) of Theorem 2.4.7 was first proved by Amayo [1] p111 Theorem 8.23 using different methods, and in characteristic 0 he gives an example due to Hartley which shows that not even $\mathfrak{G} \cap \mathfrak{M}$ - algebras need be residually finite.

It is worth noting that we have not used Proposition 2.3.4 and in fact going through Corollary 2.4.3 gives an alternative proof to Curtis's. It is also not hard to see that we have essentially proved that if S is a Curtis ring over k and M is a Noetherian S - module then M is residually k - finite dimensional in the sense that to each $0 \neq x \in M$ there is a submodule N such that $x \notin N$ and M/N is finite dimensional over k.

ERRATUM

The proof of Theorem 3.2.1 is incorrect. On p50 line 9 the statement 'M/K is an irreducible U - module' is wrong. However in the case $L = \mathfrak{V}(L)$ we have that U and U_0 coincide and the difficulty above is trivially avoided. Consequently the results of § 3.4 in which only this version of Theorem 3.2.1 is used are still true. However Lemma 3.3.1 and Theorem 3.3.2 are not proven.

CHAPTER 3 THE FRATTINI STRUCTURE OF FINITELY GENERATED SOLUBLE LIE RINGS

This chapter is a continuation of the investigation begun in chapter 2. Hall in [14] showed that f.g. metanilpotent groups have 'good Frattini structure' in a sense which is explained below. Once again some Lie algebra analogues of the group theoretic results exist. Towers [39] p71 showed that f.g. nilpotent by abelian Lie algebras have good Frattini structure and Stewart in an unpublished paper [38] extended this result to f.g. metanilpotent Lie algebras in prime characteristics. Using different techniques we show that in fact in characteristic $p > 0$, soluble f.g. abelian by finite algebras have good Frattini structure, obtaining at the same time Tower's result for characteristic 0.

Using methods more akin to Hall's for groups we then show that f.g. metanilpotent Lie rings have good Frattini structure. As in chapter 2 we have thus obtained a good analogy with groups in the case of Lie rings and even stronger results for characteristic p Lie algebras.

The first section below owes a great deal to Stewart's paper [38] mentioned above and that in turn to Towers [39]. Since both are unpublished full proofs are given, the results here being, in general, slight generalisations with essentially the same proofs.

§ 3.1 DEFINITIONS AND INITIAL REDUCTION

Throughout this section L will denote either a Lie ring or a Lie algebra and the results and proofs are identical in each case.

We now define the various radicals we will be considering.

The Frattini subring (subalgebra) of L, denoted by $F(L)$, is the intersection of the maximal proper subrings (subalgebras) of L, or is L itself if none such exist. If L is f.g. then (easily) $F(L) < L$.

Now $F(L)$ need not be an ideal of L and so we define the Frattini ideal $\Phi(L)$ of L to be the largest ideal contained in $F(L)$.

Now define $\underline{\varphi(L)} = \bigcap C_L(H/K)$ where the intersection is taken over all chief factors H/K of L.

The Hirsch - Plotkin radical $\rho(L)$ is the unique maximal locally nilpotent ideal of L (cf Hartley [16] p265).

The Fitting radical $\upsilon(L)$ is the sum of the nilpotent ideals of L. Clearly $\upsilon(L) \leq \rho(L)$.

Finally we define $\underline{\tilde{\upsilon}(L)}$ by

$$\tilde{\upsilon}(L) / \Phi(L) = \upsilon(L / \Phi(L))$$

and this corresponds to Hall's υ mod Φ.

Now we say that L has good Frattini structure if

$$\upsilon(L) = \rho(L) = \varphi(L) = \tilde{\upsilon}(L)$$

and if all four are nilpotent.

If L has good Frattini structure then $\Phi(L)$ is nilpotent being contained in $\tilde{\upsilon}(L)$.

LEMMA 3.1.1 (Stewart [37] p317 Theorem 3.2.3)

Let $I \lhd L$, then if both I and L/I^2 are nilpotent then L is nilpotent. ∎

LEMMA 3.1.2

If $L \in \mathfrak{N}.\mathrm{Max}\text{-}\lhd$ then $\nu(L)$ is nilpotent.

PROOF

There exists $N \lhd L$ such that $N \in \mathfrak{N}$ while $L/N \in \mathrm{Max}\text{-}\lhd$, so we can find M maximal with respect to $M \geq N$, $M \lhd L$, $M \in \mathfrak{N}$. Then clearly $M = \nu(L)$. ∎

LEMMA 3.1.3

If $L \in \mathfrak{N}\,\mathrm{Max}\text{-}\lhd$ and $N = \nu(L)$ then $N^2 \leqslant \Phi(L)$.

PROOF

Let M be a maximal subring (subalgebra) of L and suppose $N^2 \nleqslant M$. Since N is nilpotent by lemma 3.1.2 there exists $k \geqslant 2$ maximal with respect to $N^k \nleqslant M$ (since $L/N \in \mathrm{Max}\text{-}\lhd$). Thus $L = N^k + M$. But then

$$N^2 \leqslant L^2 = (N^k + M)^2 \leqslant N^{k+1} + M^2 \leqslant M$$

A contradiction.

Now since $N^2 \lhd L$ we have $N^2 \leqslant \Phi(L)$. ∎

LEMMA 3.1.4

If $L \in \mathfrak{N}\,\mathrm{Max}\text{-}\lhd$ and $I = \nu(L)$ then

$$\nu(L/I^2) = \nu(L)/I^2$$

PROOF

$\nu(L/I^2) = B/I^2$ is nilpotent by lemma 3.1.2. Further $B \geqslant I$. By lemma 3.1.1 B is nilpotent and so $B \leqslant I$. Hence $B = I = \nu(L)$. ∎

We now introduce a temporary notation by defining

$$\mathcal{P} = E\mathfrak{A} \cap \mathfrak{G}^*$$

Thus in the case of Lie rings $\mathcal{P}$ is the class $E\mathfrak{C}$ of polycyclic Lie rings, and in the case of Lie algebras $\mathcal{P}$ is the class of finite dimensional soluble Lie algebras. Clearly $\mathcal{P} \leq \text{Max} - \triangleleft$.

LEMMA 3.1.5

If $L \in \mathfrak{N}\mathcal{P}$ then $\rho(L) \leq \varphi(L)$.

PROOF

Let A/B be a chief factor. We wish to show that $\rho(L)$ centralises A/B. We may work modulo B, so assume $B = 0$ and A is a minimal ideal of L. Let $R = \rho(L)$, $K = \nu(L) \leq R$. Then $L/K \in \mathfrak{G}^*$. Thus for finite r $R = \langle K, t_1, \ldots, t_r \rangle$ with $t_1, \ldots, t_r \in R$. Since A is abelian $A \leq K \leq R$. Now K is nilpotent so by Schenkman [32] lemma 4, $A \cap Z_1(K) \neq 0$. Hence by minimality $A \leq Z_1(K)$.

Let $0 \neq a \in A$. Then $N = \langle a, t_1, \ldots, t_r \rangle \leq R$ and so is nilpotent (since $\rho(L) \in L\mathfrak{N}$), therefore $A \cap Z_1(N) \neq 0$. Thus there exists $c \in A$ such that $[N, c] = 0$. But $[K, c] = 0$, so $[R, c] = 0$ whence $A \cap Z_1(R) \neq 0$ and we have $A \leq Z_1(R)$. So $[A, R] = 0$ as required. ∎

PROPOSITION 3.1.6

Let $L \in \mathfrak{N}\mathcal{P}$, $I = \nu(L)$. If L/I^2 has good Frattini structure, then L has good Frattini structure.

PROOF

We have $I \in \mathfrak{N}$ by lemma 3.1.2 and by hypothesis

$$\nu(L/I^2) = \rho(L/I^2) = \varphi(L/I^2) = \tilde{\nu}(L/I^2) \in \mathfrak{N}$$

Now $\nu(L)/I^2 \in \mathfrak{N}$ by lemma 3.1.4 , so that $\nu(L) \in \mathfrak{N}$ (lemma 3.1.1). Further $\rho(L) \leq \varphi(L)$ by lemma 3.1.5 and clearly $(\varphi(L) + I^2)/I^2 \leq \varphi(L/I^2)$ so that $\varphi(L) \leq \nu(L)$ and we have $\nu(L) = \varphi(L) = \rho(L)$.

Finally $I^2 \leq \Phi(L)$ by lemma 3.1.3 which implies that $\Phi(L/I^2) = \Phi(L)/I^2$ whence

$$\tilde{\nu}(L/I^2) = \tilde{\nu}(L)/I^2$$

and so $\nu(L) = \tilde{\nu}(L)$ and all are nilpotent since $\nu(L)$ is. ∎

This result enables us to concentrate on L/I^2 which lies in the class $\mathfrak{U}\mathfrak{P}$. Furthermore its Fitting radical is I/I^2 which is abelian.

Following Towers [39] we make the following definition. We define for any $b \in L$ the <u>Engel subring</u> (<u>subalgebra</u>)

$$E_L(b) = \{ x \in L \mid [x, {}_r b] = 0 \text{ some } r \}$$

(often called the null component of b). By Liebniz's rule for derivations it follows that $E_L(b)$ is a subring (subalgebra).

<u>LEMMA 3.1.7</u>

Let $b \in L$ be such that $[L, {}_n b] = [L, {}_{n+1} b]$ for some integer n , then

$$L = E_L(b) + [L, {}_n b]$$

PROOF

Let $x \in L$. Then $[L, {}_n b] = [L, {}_{2n} b]$ so there exists $y \in L$ such that

$$[x, {}_n b] = [y, {}_{2n} b]$$

Therefore $x = (x - [y,_n b]) + [y,_n b]$

$$\in E_L(b) + [L,_n b]$$ □

THEOREM 3.1.8

Let $L \in \mathfrak{N}^2$ be monolithic with monolith A, then either

(i) $\Phi(L) = 0$

or (ii) $\nu(L/A) = \nu(L)/A$

PROOF

Let $\nu(L/A) = N/A$. If $N \leq C_L(A)$ then (ii) holds. Assume $N \not\leq C_L(A)$. If $\nu(L) = 0$ then $L = 0$ since $L \in E\mathfrak{U}$ and the result is trivial.

Hence we may assume $A \leq \nu(L)$, A being the monolith. Thus $A \leq Z_1(\nu(L))$ and $\nu(L) \leq C_L(A)$. Choose a nonzero element of $(N + C_L(A))/C_L(A)$ central in $L/C_L(A)$, say $b + C_L(A)$ (possible since $L/C_L(A) \in \mathfrak{N}$). Then $b \in N \setminus C_L(A)$, and $[L,b] \leq C_L(A)$. Let $D = \langle b \rangle + C_L(A)$ which is an ideal of L. Since $[D,A] \neq 0$ we have $[D,A] = A$, so $[b,A] = A$. Then there exists n such that $[L,_n b] = A$, so by lemma 3.1.7 $L = E_L(b) + [L,_n b] = E_L(b) + A$. Now in fact $L = E_L(b) \oplus_R A$ is an R - module direct sum (where $R = \mathbb{Z}$ or a field), so $E_L(b)$ is a maximal subring (subalgebra) (any larger subring intersects A nontrivially and this inter - section is an ideal of L contradicting the minimality of A). Hence $\Phi(L) \leq E_L(b)$. But if $\Phi(L) \neq 0$ we have $A \leq \Phi(L)$, a contradiction. So $\Phi(L) = 0$. ■

We say that L has the property (Δ) if $\varphi(L/K) = \nu(L/K)$ for all ideals K of L.

THEOREM 3.1.9

If $L \in \mathfrak{G} \cap \mathfrak{U}\mathfrak{N}$ and L has property (Δ) then

$$\tilde{\nu}(L) = \nu(L)$$

PROOF

Assume the contrary. By Theorem 2.1.10, $L \in \text{Max} - \lhd$ and so there exists an ideal I of L maximal with respect to $\tilde{\nu}(L/I) \neq \nu(L/I)$. Replacing L by L/I we may assume $\tilde{\nu}(L/J) = \nu(L/J)$ for all $0 \neq J \lhd L$. By hypothesis there exists some chief factor A/B of L not centralised by $\tilde{\nu}(L)$. If $B \neq 0$ then A/B is a chief factor of L/B and so is centralised by $\nu(L/B) = \tilde{\nu}(L/B)$, and hence is centralised by $\tilde{\nu}(L)$. Therefore $B = 0$ and A is a minimal ideal of L. If there is an ideal C such that $A \cap C = 0$ then $A \cong (A + C)/C$ and a similar argument applies. Hence L is monolithic with monolith A. If $\Phi(L) = 0$ we are finished. Otherwise by Theorem 3.1.8 $\nu(L/A) = \nu(L)/A$.

Since $A \leq \Phi(L)$ it is clear that $\Phi(L/A) = \Phi(L)/A$. But then, since $\tilde{\nu}(L/A) = \nu(L/A)$, we have $\tilde{\nu}(L) = \nu(L)$ a contradiction. Hence the result. ∎

COROLLARY 3.1.10

If $L \in \mathfrak{G} \cap \mathfrak{U}\mathfrak{N}$ has property (Δ) then L has good Frattini structure.

PROOF

We have $\nu(L) \leq \rho(L) \leq \varphi(L)$. By (Δ) we have $\varphi(L) \leq \nu(L)$. Thus $\nu(L) = \rho(L) = \varphi(L)$ and clearly these are nilpotent. Finally by Theorem 3.1.9 we have $\tilde{\nu}(L) = \nu(L)$. ∎

There is one case in which (Δ) holds trivially.

PROPOSITION 3.1.11

If L has a finite chief series then L has (Δ).

PROOF

If $K \lhd L$ then L/K has a finite chief series. Since $\varphi(L/K)$ centralises the factors we have $\varphi(L/K) \in \mathfrak{N}$. Thus $\nu(L/K) \leq \varphi(L/K) \leq \nu(L/K)$. ∎

§ 3.2 THE CHIEF ANNIHILATOR PROPERTY

Throughout this section we continue to allow L to be either a Lie ring or a Lie algebra over a field k.

Let S be an associative ring and M a right S - module. A chief factor of M is a module of the form H/K where H and K are submodules of M and H/K is irreducible.

Let $\varphi_S(M)$ be the intersection of the annihilators in S of all chief factors of M.

Now consider a Lie ring (or algebra) L and let $N = \nu(L)$, $U = U(L)$. Then we may consider $U_o = U(N)$ to be the subring (subalgebra) of U generated by 1 and N. In this way any U - module is also a U_o - module.

We say L has the chief annihilator property if whenever M is a Noetherian U - module and $z \in Z(U_o) \cap \varphi_U(M)$ then there exists an integer n such that $Mz^n = 0$ (where $Z(U_o)$ is the centre of U_o).

THEOREM 3.2.1

If any of the following hold :

(1) L is a Lie ring and $L \in E\mathfrak{C}$

(2) L is a Lie algebra over a field k , and either

(a) char $k = p > 0$ and $L \in \mathfrak{F} \cap E\mathfrak{U}$

or (b) char $k = 0$ and $L \in \mathfrak{F} \cap \mathfrak{U}$

then L has the chief annihilator property.

PROOF

Let M be a Noetherian U - module. We must show that if $z \in Z(U_o) \cap \varphi_U(M)$ then $Mz^n = 0$ for some n . If $Mz^n \neq 0$ for all n we can choose a submodule I of M maximal with respect to $Mz^n \nleq I$ for all n . By replacing M by M/I we can assume that $Mz^n \leq J$ for every nonzero submodule J of M .

Let $K = \{ m \in M \mid mz = 0 \}$. If $K \neq 0$ then $Mz^n \leq K$ for some n and then $Mz^{n+1} = 0$. Thus $K = 0$.

Now consider M as a U_o - module (note that $N \neq 0$ for otherwise $L = 0$ since L is soluble). The map

$$\beta : M \longrightarrow M \ , \ m \longmapsto mz$$

is a U_o - module monomorphism (since z is central in U_o). We can therefore define a sequence

$$M = M_o \xrightarrow{\beta_1} M_1 \xrightarrow{\beta_2} M_2 \xrightarrow{\beta_3} \cdots\cdots$$

of U_o - submodules all isomorphic to M , as follows :

Let $m \longmapsto m_k$ be a U_o - isomorphism $M \longrightarrow M_k$. Define

$$\beta_k(m_{k-1}) = (mz)_k$$

Let $\overline{M}$ be the direct limit of this sequence. We may assume that

$$M = M_o \leq M_1 \leq \cdots\cdots \leq \overline{M}$$

and that each M_i is a U_o - submodule of $\overline{M}$. Let Y be an indeterminate and make $\overline{M}$ into a $U_o[Y]$ - module by letting Y act as β^{-1} where $\beta : \overline{M} \longrightarrow \overline{M}$, $m \longmapsto mz$

is now an automorphism.

We will show that there exist maximal $U_o[Y]$ - submodules in $\overline{M}$. Let B be a maximal U - submodule of M (M is a Noetherian U - module). Then M/B is finite by Theorem 2.3.7 when L is a Lie ring and finite dimensional by Proposition 2.3.4 in the cases when L is a Lie algebra. B is also a U_o - submodule of M and so there exists a maximal U_o - submodule containing B, say A. We can now define a sequence of U_o - submodules

$$A = A_o \xrightarrow{\alpha_1} A_1 \xrightarrow{\alpha_2} A_2 \xrightarrow{\alpha_3} \cdots\cdots$$

where $\alpha_k = \beta_k \big|_{A_{k-1}}$, and $A_k \leqslant M_k$ for all k. Further for each i, A_i is isomorphic to A and M_i/A_i is isomorphic to M/A.

For each i

$$0 \longrightarrow A_i \longrightarrow M_i \longrightarrow M_i/A_i \longrightarrow 0$$

is a short exact sequence. So the direct limit $\overline{A}$ is such that

$$0 \longrightarrow \overline{A} \longrightarrow \overline{M} \longrightarrow \overline{M}/\overline{A} \longrightarrow 0$$

is exact. The elememts of $\overline{M}$ are equivalence classes of elements in $\bigcup_{i=0}^{\infty} M_i$ (disjoint union) indexed by the elements of M. Thus $\overline{M}/\overline{A}$ is finite when L is a Lie ring and finite dimensional when L is a Lie algebra. $\overline{A}$ is clearly a $U_o[Y]$ - module and so there exists a maximal $U_o[Y]$ - submodule containing $\overline{A}$.

Hence there exists a maximal $U_o[Y]$ submodule K say, and $M \not\leqslant K$ (since if $M \leqslant K$ then $K = MU_o[Y] = \overline{M}$ contradicting maximality).

Now $U_o[Y]$ is the universal enveloping ring (algebra) of a direct sum $L \oplus T$ where $T \cong \mathbb{Z}$ (or $T \cong k$). Then $\overline{M}/K$ is an irreducible $U_o[Y]$ - module and hence is finite by Theorem 2.3.7 when L is a Lie ring and finite dimensional by Proposition 2.3.4 when L is a Lie algebra.

Let $K_o = M \cap K$, then K_o is a U_o - module and $M/K_o \cong (M+K)/K$ which is finite (respectively finite dimensional). Now $K_o \neq M$ and we can take a U_o - submodule $K_1 \leq M$ maximal with respect to $K_o \leq K_1$. Since $z \in \varphi_U(M)$ we have $Mz \leq K_1$ (since if M/H is an irreducible U_o - module there exists a U - module K' maximal with respect to $K' \leq H$ and then M/K' is an irreducible U - module). Further $K_o z \leq K$. Hence the endomorphism induced by β on M/K_o is not an automorphism. There exists $m \in M$ such that $mz \in K_o$ but $m \notin K_o$ (by finiteness in the Lie ring case and finite dimensionality in the Lie algebra case). However $\beta^{-1}(mz) = m$, so $m \in K$ since K is a $U_o[Y]$ - module. This is a contradiction. Hence the result. ∎

§ 3.3 LIE ALGEBRAS

We now restrict our attention throughout this section to Lie algebras L over a field k.

LEMMA 3.3.1

Let L be a Lie algebra over a field k. Suppose that either

(i) char $k = p > 0$ and $L \in \mathfrak{G} \cap \mathfrak{N}\mathfrak{P}$

or (ii) char $k = 0$ and $L \in \mathfrak{G} \cap \mathfrak{N}\mathfrak{U}$

then $\upsilon(L) = \varphi(L)$.

PROOF

Let $A = \mathcal{V}(L)$. By lemma 3.1.4 we may assume that A is abelian. We know $\mathcal{V}(L) \leqslant \varphi(L)$. We now proceed by induction on the derived length d of L. When $d = 1$ the result is trivial. Suppose the lemma is proved for $d - 1$. Then L/A has derived length $d - 1$ and further by Theorem 3.2.1 $U = U(L/A)$ has the chief annihilator property.

By the remarks before lemma 2.4.5 U is an integral extension of a Noetherian subring R of its centre. Then A is a Noetherian R - module. Since R is central in U we can consider U as acting on A as a subring of $\mathrm{End}_R(A)$.

We now make use of the following result of Small's (Fischer [8] p77 Theorem 2.1) :- If R is an associative ring and M is a Noetherian right R - module then each nil subring of $\mathrm{End}_R(M)$ is nilpotent.

By the remarks above and the chief annihilator property $Z(U_o) \cap \varphi_U(A)$ is a nil subring of $\mathrm{End}_R(A)$ and so by Small's result acts nilpotently on A (where $U_o = U(\mathcal{V}(L/A))$).

If $\mathcal{V}(L) \neq \varphi(L)$ then $\varphi(L)/A$ is nontrivial. Clearly $\varphi(L)/A \leqslant \varphi(L/A)$. Since by the induction hypothesis $\varphi(L/A) = \mathcal{V}(L/A)$ and both are nilpotent we can find an element $a + A \in Z_1(\mathcal{V}(L/A))$ where $a \in \varphi(L) \setminus A$. Hence if $W/A = Z_1(\mathcal{V}(L/A))$ then $T = W \cap \varphi(L) > A$. If $t \in T$ and $z = t + A \in U$ then $z \in Z(U_o) \cap \varphi_U(A)$ and so from above T/A acts nilpotently on A. T/A is nilpotent by the induction hypothesis and so T is a nilpotent ideal strictly containing A which is a contradiction. Hence the result.

THEOREM 3.3.2

Let L be a Lie algebra over a field k . Suppose that either

(i) char k = p > 0 and L $\in \mathfrak{G} \cap \mathfrak{N}\mathfrak{P}$

or (ii) char k = 0 and L $\in \mathfrak{G} \cap \mathfrak{N}\mathfrak{U}$

then L has good Frattini structure.

PROOF

By lemma 3.1.6 we can assume $\nu(L) \in \mathfrak{U}$. Then L has property (Δ) by lemma 3.3.1 . The result now follows by Corollary 3.1.10 . ∎

Part (ii) of Theorem 3.3.2 was originally obtained by Towers [39].

It is worth noting that the situation may be different here from Chapter 2 where there were counterexamples to show that the characteristic 0 case was essentially more restricted than the characteristic p case. The standard counterexamples do not work as such and we know of no $\mathfrak{G} \cap \mathfrak{N}^2$ Lie algebra (of any characteristic) which does not have good Frattini structure. A possibility for characteristic 0 is that soluble f.g. nilpotent by trigonalisable (in the sense of McConnell [26]) Lie algebras have good Frattini structure. In the enveloping algebras of trigonalisable algebras (and also in Curtis rings) the Jacobson radical and the nilradical coincide and so the chief annihilator property (in a stronger form) would follow if a primitive ideal containing the annihilator of a (Noetherian) module always annihilated some chief factor of the module. This is true in the abelian case by Nakayama's lemma.

§ 3.4 LIE RINGS

From now on L will be a Lie ring.

Suppose that $L = \upsilon(L)$ (for example if L is nilpotent). In this case the chief annihilator property means that if M is a Noetherian $U = U(L)$ - module and $z \in Z(U) \cap \varphi_U(M)$ then there exists an integer n such that $Mz^n = 0$.

It is in this form that Hall [14] makes use of the chief annihilator property in the group algebra over an absolutely algebraic field of prime characteristic of a f.g. nilpotent group.

LEMMA 3.4.1

Let $L \in \mathfrak{G} \cap \mathfrak{N}^2$ and suppose that $A = \upsilon(L)$. If $U(L/A)$ has the chief annihilator property then

$$\upsilon(L) = \varphi(L)$$

PROOF

By lemma 3.1.4 we can assume that $A \in \mathfrak{A}$. Now $L/A \in \mathfrak{N}$ so the chief annihilator property takes the form mentioned above.

We know that $\upsilon(L) \leqslant \varphi(L)$. If these are not equal then we can find an element $a + A \in Z_1(L/A)$ where $a \in \varphi(L) \setminus A$. Now let $z = a + A \in U(L/A)$. Clearly $z \in Z(U)$ and since $a \in \varphi(L)$ it follows that $z \in \varphi_U(A)$. Now A is a Noetherian U - module by lemma 2.1.9 . By the chief annihilator property $Az^n = 0$ for some n . Hence $\langle A , a \rangle$ is a nilpotent ideal of L contrary to the definition of A .

COROLLARY 3.4.2

Let $L \in \mathcal{G} \cap \mathcal{N}^2$. Suppose that for every ideal K of L the universal enveloping ring $U((L/K) / \mathfrak{v}(L/K))$ has the chief annihilator property. Then L has good Frattini structure.

PROOF

By lemma 3.4.1 L has property (Δ) and the result follows from lemma 3.1.6 and Corollary 3.1.10 . ∎

THEOREM 3.4.3

Let $L \in \mathcal{G} \cap \mathcal{N}^2$. Then L has good Frattini structure.

PROOF

Corollary 3.4.2 and Theorem 3.2.1 . ∎

Note that in obtaining Theorem 3.4.3 we have not used the full power of Theorem 3.2.1 and it seems possible that the result can be extended to the class $\mathcal{G} \cap \mathcal{N}(E\mathcal{C})$ although we have not been able to do this.

CHAPTER 4 RESIDUAL PROPERTIES OF CERTAIN CLASSES OF LIE RINGS

We examine analogues of results of Gruenberg [11], Higman [18], and Wehrfritz [40] on residual properties of nilpotent groups. As might be expected nilpotent Lie rings are especially well behaved. The methods of [40] are followed although the linear structure of Lie rings enables us to strengthen and simplify many results.

We also prove a Lie ring analogue of Mal'cev's result [27] that f.g. linear groups are residually finite. The proof is based on a module theoretic argument of Wehrfritz [41].

§ 4.1 RESIDUAL PROPERTIES OF NILPOTENT LIE RINGS

LEMMA 4.1.1

Let B be a Lie ring, $A \leqslant B$ and $B^2 \leqslant A$. Further suppose that A has exponent $m > 0$ and that B/A contains no m - torsion (i.e. for all $x \in B$, $mx \in A$ implies that $x \in A$).

Then $A \cap mB = 0$ and $mB \in \mathfrak{A}$.

PROOF

If $a \in A \cap mB$ then $a = mb$ for some $b \in B$. But B/A contains no m - torsion and so $b \in A$ and $a = 0$ (since A has exponent m).

Further

$$mB \cong mB/(A \cap mB) \cong (A + mB)/A \leqslant B/A$$

and so is abelian. ∎

THEOREM 4.1.2

Let L be a Lie ring. Suppose

$$0 = L_0 \leqslant L_1 \leqslant \;\ldots\ldots\; \leqslant L_n = L$$

is an invariant series of L and for each $i = 1, 2, \ldots, n$ let M_i be a set of integers. Suppose that each factor L_j / L_{j-1} is one of the following two types ;

(i) L_j / L_{j-1} is torsion, and each primary component is of finite exponent dividing some member of M_j .

(ii) $L_j / L_{j-1} \in \mathfrak{U}$ and contains no $\bigcup_{i=1}^{j-1} M_i$ - torsion and there exists $j_o < j$ such that L_i / L_{i-1} is of type (i) whenever $j_o < i < j$ and

$$\bigcap_{m \in M_j} (L_{j_o} + mL_j) = L_{j_o}$$

Then for each $0 \neq x \in L$ there exist $m_1, \ldots\ldots, m_n$ with $m_i \in M_i$ such that if $d = m_1 \ldots\ldots m_n$ then $x \notin dL$.

PROOF

Let $x \in L \setminus \{0\}$. The proof will use induction on the number of factors of type (ii) . Suppose that there are none of these. Then L is torsion and is a direct sum of its primary components. Let p be a prime dividing the order of x . Then for each i there exists an element $m_i \in M_i$ such that $m_i(L_i / L_{i-1})$ has no p - torsion and so $(m_1 \ldots m_n)L$ has no p - torsion and hence $x \notin (m_1 \ldots m_n)L$.

Suppose now that L_{j+1} / L_j is the first factor of type (ii) . If $x \in L_j$, $j > 0$ and x is of finite order then let p be a prime dividing the order of x . Since $x \notin (L_j)_{p'}$ (where $L_{p'}$ denotes the sum of all the primary components of L for all primes not equal to p) we may factor out

by $(L_j)_p$, and so assume L_j is a p - ring. Then for each $i = 1, \ldots, j$ there exists $m_i \in M_i$ such that $m_i L_i \leqslant L_{i-1}$.
By lemma 4.1.1

$$(L_{j-1} + m_j L_{j+1}) \cap L_j = L_{j-1} .$$

Replace the original series by

$$0 = L_0 \leqslant \ldots \leqslant L_{j-1} \leqslant (L_{j-1} + m_j L_{j+1}) \leqslant L_{j+1} \leqslant \ldots \leqslant L .$$

Repeat the process a further $j - 1$ times, so that there exists an ideal Y_0 of L such that $Y_0 \cap L_j = L_0 = 0$ (and so $x \notin Y_0$) and an invariant series

$$Y_0 \leqslant \ldots \leqslant Y_j = L_{j+1} \leqslant L_{j+2} \leqslant \ldots \leqslant L_n = L$$

where $m_i Y_i \leqslant Y_{i-1}$.

This is a series of the given type for L/Y_0 , where for $i = 1, \ldots, j$, Y_i/Y_{i-1} is of type (i) with associated integer set $\{ m_i \}$, and for $i = j+2, \ldots, n$, L_i/L_{i-1} has the same type and integer set as in the original series. By induction there exists $m_{j+2}, \ldots, m_n$ with $m_i \in M_i$ such that $x \notin d_1 L$ where $d_1 = m_1 \ldots m_j m_{j+2} \ldots m_n$.

Suppose now that $x \notin L_j$. There exists $m_{j+1} \in M_{j+1}$ such that $x \notin (L_{(j+1)_0} + m_{j+1} L_{j+1}) = L^{\#}$ say (where $(j+1)_0$ is as in the statement of the theorem). Apply the induction hypothesis to the series

$$L^{\#} \leqslant L_{j+1} \leqslant \ldots \leqslant L_n = L$$

where $L_{j+1}/L^{\#}$ has type (i) and integer set $\{ m_{j+1} \}$ and for $i > j$, L_{j+1}/L_j has the same type and integer set as in the original series. Apply lemma 4.1.1 with $B = L_{j+1}/L_{j-1}$, $A = L_j/L_{j-1}$. Then $L_{j+1}/L_j \in \mathfrak{U}$ so $B^2 \leqslant A$, $m_i L_i \leqslant L_{i-1}$ so $m_j A = 0$, and B/A contains no m_j - torsion by hypothesis. Hence

$$A \cap m_j B = L_j \cap m_j L_{j+1} = L_{j-1} .$$

LEMMA 4.1.3

If L is torsion free then $L/Z_i(L)$ is torsion free for all $i < \omega$.

PROOF

It is clearly sufficient to consider the case $i = 1$. Let $x \in L$ and $nx \in Z_1(L)$, $n \neq 0$. If $y \in L$ then

$$0 = [nx, y] = n[x, y]$$

But L is torsion free so $[x, y] = 0$ and $x \in Z_1(L)$. ∎

Let L be a Lie ring. Then the <u>torsion spectrum</u> of L, denoted by $\pi(L)$, is the set of primes p such that L contains an element of order p.

Suppose that π is a set of primes. We write $L \in R\mathfrak{F}_\pi$ to mean that L is residually a finite π-ring. We write $R\mathfrak{F}_p$ for $R\mathfrak{F}_{\{p\}}$.

THEOREM 4.1.4

Let $L \in \mathfrak{G} \cap \mathfrak{N}$

(a) If $\pi(L) \neq \emptyset$ then $L \in R\mathfrak{F}_{\pi(L)}$ and if $\pi(L) = \emptyset$ then $L \in R\mathfrak{F}_p$ for all primes p.

(b) If π is any infinite set of primes then

$$\bigcap_{p \in \pi} pL \in \mathfrak{F}$$

(c) Suppose that the exponent of $\tau(L)$ divides $m > 0$. Then for any infinite set of primes π

$$mL \cap \Big(\bigcap_{p \in \pi} pL \Big) = 0 .$$

PROOF

Consider the series

$$0 \leq L_0 \leq L_1 \leq \ldots \leq L_n = L$$

where $L_0 = \tau(L)$ and $L_i/L_0 = Z_i(L/L_0)$.

Now $L \in \mathrm{Max}$ so in particular $\tau(L) \in \mathcal{F}$ and there is a $\pi(L)$ - number m such that $mL_o = 0$. By lemma 4.1.3 the factors L_i / L_{i-1} are all torsion free and thus are free abelian of finite rank.

(a) If p is any prime each L_i / L_{i-1} is residually a finite p - ring. By Theorem 4.1.2 if $m \neq 1$ then

$$\bigcap_{s=1}^{\infty} m^s L = 0$$

and if $m = 1$ then

$$\bigcap_{s=1}^{\infty} p^s L = 0$$

for each prime p.

(b) and (c). Let π be any infinite set of primes. There exist infinite disjoint subsets $\pi_1, \ldots, \pi_n$ of π such that every prime in $\bigcup_{i=1}^{n} \pi_i$ does not lie in $\pi(L)$. By Theorem 4.1.2

$$0 = \bigcap (mp_1 \ldots p_n)L$$

where p_i ranges over π_i.

Now m is a $\pi(L)$ - number and none of the p_i is contained in $\pi(L)$ so

$$(mp_1 \ldots p_n)L = mL \cap p_1 L \cap \ldots \cap p_n L$$

($\tau(L)$ is a direct sum of its primary components).
Hence (c) follows.

Now apply Theorem 4.1.2 to L/L_o to get

$$\bigcap_{p_i \in \pi_i} (p_1 \ldots p_n)L \leq L_o$$

and (b) follows. ∎

Let π be a set of primes. A Lie ring L is said to be <u>π - divisible</u> if for each $p \in \pi$ every element of L is a multiple of p ; or equivalently $L = mL$ for all $m > 0$ with m a π - number. If π is the set of all primes then L is divisible. The join of π - divisible subrings is always π - divisible. Hence every ring L has a unique maximal π - divisible ideal which we denote $\partial_\pi(L)$. If this is 0 then L is said to be <u>π - reduced</u> . Thus a π - reduced ring contains no nontrivial π - divisible subrings.

Let $\mathcal{E}_\pi$ be the class of all π - Lie rings of finite exponent (i.e. of exponent a finite π - number).

Now if $L \in R\mathcal{E}_\pi$ then so is every subring and clearly L must be π - reduced. However reduced Lie rings need not be residually of finite exponent in general. There exist reduced abelian p - groups which contain elements of infinite height (Fuchs [9] p118) and considering such a group as an abelian Lie ring provides the necessary counterexample. We will now investigate how far the converse of this result is true.

<u>LEMMA 4.1.5</u>

Let L be a Lie ring then if

(i) $Z_1(L)_p$ has finite exponent dividing p^n , then so does $(Z_{i+1}(L)/Z_i(L))_p$ for all $i < \omega$.

(ii) If $Z_1(L)$ is π - reduced for some set of primes then so is $Z_{i+1}(L)/Z_i(L)$ for all $i < \omega$.

PROOF

(i) Let $x \in Z_2(L)$ and $y \in L$. If $p^m x \in Z_1(L)$ then

$$0 = [y , p^m x] = p^m [y , x]$$

Hence $[y , x] \in Z_1(L)_p$ so for all $y \in L$

$$0 = p^n [y , x] = [y , p^n x]$$

So $p^n x \in Z_1(L)$.

(ii) Let $R/Z_1(L)$ be a π - divisible subring of $Z_2(L)/Z_1(L)$. If $y \in L$ then the map

$$x + Z_1(L) \longmapsto [x, y], \quad x \in R$$

is a Lie homomorphism of $R/Z_1(L)$ onto $[R, y] \leq Z_1(L)$ (since both are abelian). $[R, y]$ is hence π - divisible and so is trivial. Thus $R = Z_1(L)$ and the result follows. ∎

Our first theorem is a simple rewrite of the corresponding result for abelian groups (Wehrfritz [40] p4 lemma 3). It is worth noting that the Lie ring result is considerably stronger than any corresponding result for groups and in fact is true (with the same proof) with the obvious reinterpretation, for generalised rings in the sense of Fuchs [9] Chapter XII .

THEOREM 4.1.6.

Let L be a Lie ring and π a set of primes such that for each $p \in \pi$ the p - component of L has finite exponent. Then $L \in R\mathfrak{E}_\pi$ iff L is π - reduced.

PROOF

Every member of $\mathfrak{E}_\pi$ is π - reduced and so if $L \in R\mathfrak{E}_\pi$ then L is π - reduced.

Suppose that L is π - reduced and let $R = \bigcap_m mL$ where m ranges over all positive π - numbers. Clearly it suffices to prove that R is π divisible . Let $a \in R$ and $p \in \pi$. L_p has finite exponent p^n say. Then $L_p \cap p^nL = 0$. Since $a \in R$ there exists for each i an element $a_i \in p^nL$ such that $p^ia_i = a$. Now $p^i(pa_{i+1} - a_i) = 0$ and yet $(pa_{i+1} - a_i) \in p^nL$,hence $a_i = pa_{i+1}$ for $i = 1, 2, \ldots$ and in particular

$a_1 \in p^i L$ for each i. Let m be any π - number and write $m = p^r s$ for some $r \geq 0$ with s prime to p. Now $a_1 \in p^r L$, $pa_1 = a \in R \leq mL$. However $p^r L / mL$ has exponent dividing s and therefore no elements of order p. Thus $a_1 \in mL$ for every positive π - number m and so $a_1 \in R$. Thus R is π - divisible and the result is proved. ∎

COROLLARY 4.1.7

Let L be a Lie ring and suppose that π is a set of primes such that for every $p \in \pi$ the p - component of L has finite exponent. Then $L / \partial_\pi(L) \in R\mathcal{E}_\pi$.

PROOF

By Theorem 4.1.6 we have only to prove that for $p \in \pi$ if L_p has exponent dividing p^n say then so does $(L / \partial_\pi(L))_p$.

Suppose $x \in L$ and $p^m x \in \partial_\pi(L)$ for some $m > n$. Then there exists $y \in \partial_\pi(L)$ such that $p^m x = p^m y$. Then

$$p^m(x-y) = 0 \Rightarrow p^n(x-y) = 0 \Rightarrow p^n x = p^n y \quad (p^n L_p = 0)$$

Thus $(L / \partial_\pi(L))_p$ has exponent dividing p^n. ∎

COROLLARY 4.1.8

Let L be a Lie ring and suppose that each primary component of L has finite exponent. Then

$$\partial(L) = \bigcap_{m=1}^{\infty} mL .$$ ∎

THEOREM 4.1.9

Let $L \in \mathfrak{N}$ and π a set of primes such that for all $p \in \pi$ the p - component of L has finite exponent. Then the following are equivalent

(i) $L \in R\mathcal{E}_\pi$

(ii) L is π - reduced.

(iii) $Z_1(L)$ is π - reduced.

PROOF

(i) iff (ii). Theorem 4.1.6 .

(ii) implies (iii). Clear .

(iii) implies (ii). Let $Z_i = Z_i(L)$ and let

$$T_i / Z_i = \tau (Z_{i+1} / Z_i) .$$

Now suppose Z_1 is π - reduced. By lemma 4.1.5 (ii) we have Z_{i+1} / Z_i is π - reduced for all i . By lemma 4.1.5 (i) for all $p \in \pi$ the p - component of T_i / Z_i has finite exponent. Hence by Theorem 4.1.6 we have $Z_{i+1} / Z_i \in R \mathcal{E}_\pi$.

Now apply Theorem 4.1.2 to the series

$$Z_0 \leq T_0 \leq Z_1 \leq T_1 \leq \ldots\ldots \leq Z_c = L$$

where the associated integer sets are all taken to be the set of all π - numbers. This gives $L \in R \mathcal{E}_\pi$ as required. ■

§ 4.2 LIE RINGS OF MATRICES

Suppose L is a Lie ring. Let $K \triangleleft H \leq L$ and $\Gamma \leq \mathrm{Der}(L)$. Define

$$C_\Gamma (H / K) = \{ \varphi \in \Gamma \mid \varphi(x) \in K , \text{ for all } x \in H \}$$
$$= C \text{ say.}$$

Note that K and H are C - invariant, and if $x \in H$ then $\varphi \in C$ induces the trivial derivation on H / K .

C is a Lie subring of Γ , and in fact is the largest such that $C(H) \leq K$ (where $C(H)$ denotes the collection of all elements of L of the form $\varphi(x)$ for all $\varphi \in C$ and all $x \in L$).

Let $0 = L_n \leq \ldots\ldots \leq L_0 = L$ be a finite series for L. The stabilizer of this series is defined to be

$$\Sigma = \bigcap_{i=1}^{n} C_{Der(L)}(L_{i-1}/L_i)$$

and consists of those derivations of L under which every term of the series is invariant and which induce trivial derivations on each factor. If $\Phi \leq Der(L)$ we say that Φ stabilizes the series if $\Phi \leq \Sigma$.

We define a class $\mathfrak{S}_n$ by saying that $L \in \mathfrak{S}_n$ iff L is isomorphic to a subring of the stabilizer of an invariant series of length n for some Lie ring (i.e. by abuse of language iff L stabilizes some invariant series of length n).

The following lemma is the reason for introducing stabilizers (the version for groups is due to Kaloujnine [20]).

LEMMA 4.2.1

$$\mathfrak{S}_n \leq \mathfrak{N}_{n-1}$$

PROOF

Suppose $\Gamma \in \mathfrak{S}_n$. Then we can find a Lie ring L on which Γ acts as a Lie ring of derivations, and L has a series $0 = L_n \leq \ldots\ldots \leq L_0 = L$, $L_i \lhd L$ which is stabilized by Γ.

Recall that $\Gamma^n = [\Gamma^{n-1}, \Gamma]$. We will prove by induction on j that $\Gamma^j(L_i) \leq L_{i+j}$ for all i.

If $j = 1$ there is nothing to prove since Γ stabilizes the series. Suppose $j > 1$ and the result is established for j. Then

$$\Gamma^j(\Gamma(L_i)) \leq \Gamma^j(L_{i+1}) \leq L_{i+j+1}$$

and

$$\Gamma(\Gamma^j(L_i)) \leq \Gamma(L_{i+j}) \leq L_{i+j+1}$$

And if $\varphi \in \Gamma^j$, $\psi \in \Gamma$ then

$$[\varphi, \psi](L_i) = \varphi \circ \psi(L_i) - \psi \circ \varphi(L_i) \leqslant L_{i+j+1}$$

So $\Gamma^{j+1}(L_i) \leqslant L_{i+j+1}$.

Now putting $i = 0$ and $j = n$ we find that $\Gamma^n(L) = 0$ and so $\Gamma^n = 0$ and $\Gamma \in \mathcal{N}_{n-1}$. ∎

If R is a commutative ring with identity, and A is an R - module then $\mathrm{End}_R(A)$ is an associative ring and can be regarded as a Lie ring in the usual way.

<u>LEMMA 4.2.2</u>

Let R be a commutative ring with identity 1_R. Further suppose P is an ideal of R, A is an R - module such that $\bigcap_{i=1}^{\infty} AP^i = 0$ and m is a positive integer such that $m1_R \in P$.

If Γ is a Lie ring of R - endomorphisms of A satisfying $\Gamma(A) \subseteq AP$, and $K_i = C_\Gamma(A/AP^{i+1})$, then

(i) $m^i\Gamma \leqslant K_i \lhd \Gamma$

(ii) Γ/K_i is a nilpotent Lie ring of class at most i.

(iii) $\bigcap_{i=1}^{\infty} K_i = 0$.

PROOF

First we will prove by induction that for all i

$$m^i \Gamma(A) \leqslant AP^{i+1}$$

If $i = 0$ then the result is true by hypothesis. Suppose $i > 0$ and $m^{i-1}\Gamma(A) \leqslant AP^i$. Then since $m1_R \in P$

$$m^i \Gamma(A) = m(m^{i-1}\Gamma(A)) \leqslant mAP^i = A(mR)P^i \leqslant AP^{i+1}$$

We can consider A as an abelian Lie ring and then Γ is a Lie ring of derivations of A. Now for each j

$$\Gamma(AP^j) = \Gamma(A)P^j \leq AP^{j+1}$$

So Γ/K_i stabilizes the series

$$AP^{i+1} \leq AP^i \leq \ldots\ldots \leq AP \leq A$$

and so by lemma 4.2.1 $\Gamma/K_i \in \mathfrak{N}_i$. Hence (ii) . Also from above $m^i\Gamma \leq K_i$ giving (i) . Finally if $\gamma \in \bigcap_{i=1}^{\infty} K_i$ then $\gamma(A) \leq \bigcap_{i=1}^{\infty} AP^i = 0$. So $\gamma = 0$. ∎

THEOREM 4.2.3

Let R be a f.g. integral domain and $M = M_n(R)_L$ the full Lie ring of $n \times n$ matrices over R .

(i) If char $R = 0$ then for all but a finite number of primes p , M contains an ideal of finite index which is residually a finite p - ring .

(ii) If char $R = p > 0$, then M contains an ideal of finite index which is residually a finite p - ring.

PROOF

If P is a maximal ideal of R then R/P is a finite field of characteristic p say, by Bourbaki [3] V 3.4 Cor 1 . R is Noetherian so $\bigcap_{i=1}^{\infty} P^i = 0$ by Zariski and Samuel [42] Chapter 4 , Theorem 12 .

$M_n(R)$ is the endomorphism ring of a free R - module A of finite rank n . Trivially $\bigcap_{i=1}^{\infty} AP^i = 0$. Let $H = C_M(A/AP)$. Now R/P is a finite field so H is of finite index in M . Let $K_i = C_H(A/AP^{i+1})$. $K_i \lhd H$ and by lemma 4.2.2 (i) H/K_i is a p - ring.

We now claim that R/P^i is finite. The proof is by induction on i . R/P is finite. Suppose that R/P^{i-1} is finite. Now R is Noetherian so P^{i-1} is f.g. as an

R - module. Hence P^{i-1}/P^{i} is f.g. as an R/P - module and thus is finite. Thus R/P^{i} is finite.

A/AP^{i+1} is a free R/P^{i+1} - module of rank n and so, since R/P^{i+1} is a finite ring, we have that H/K_i is finite. This shows that H is residually a finite p - ring.

If $\operatorname{char} R = p$ then $\operatorname{char}(R/P) = p$ and this proves (ii).

Suppose $\operatorname{char} R = 0$. If p is a prime of $\mathbb{Z}$ and $p1_R$ is not a unit of R then $p1_R$ is contained in a maximal ideal of R. Now suppose that for $i = 1, 2, \ldots,$ p_i is a prime of $\mathbb{Z}$ such that $p_i 1_R$ is a unit of R. Then $\langle p_i 1_R \mid i = 1, 2, \ldots \rangle$ is a (multiplicative) free abelian group of infinite rank. But the group of units of a f.g. integral domain is f.g. (Samuel [31] Theorem 1). Thus for all but a finite number of primes p, $p1_R$ is not a unit of R and this proves the result. ∎

COROLLARY 4.2.4

Suppose k is a field and let L be a f.g. Lie subring of $M_n(k)_L$. Then

(i) If $\operatorname{char} k = 0$, then for all but a finite number of primes p, L contains an ideal of finite index which is residually a finite p - ring.

(ii) If $\operatorname{char} k = p > 0$, then L contains an ideal of finite index which is residually a finite p - ring.

PROOF

Suppose that L is generated by the matrices $(x_{ij}^{(1)}), \ldots, (x_{ij}^{(s)})$ and R is the subring of k generated by all the $x_{ij}^{(k)}$. Lie multiplication in $M_n(k)_L$ is defined only in terms of the field operations in k and hence L is a Lie subring of

$M_n(R)_L$. The result now follows from Theorem 4.2.3 . ∎

This corollary is essentially a Lie ring analogue of Mal'cev's result [27] that f.g. linear groups are residually finite and we could state the result in this form.

COROLLARY 4.2.5

Suppose k is a field. Then every f.g. k - linear Lie ring is residually finite. ∎

CHAPTER 5 SOLUBLE LIE RINGS OF FINITE RANK

In this chapter we study the residual properties of Lie rings whose underlying abelian groups satisfy certain rank restrictions (in the sense of Fuchs [10] p85) and which are generalisations of polycyclic Lie rings in the soluble case. Our main inspiration is Robinson's work for soluble groups [30] §6. Because of the linear structure of Lie rings and the great influence of the underlying abelian group we are able to obtain results applicable to Lie rings which are not necessarily soluble. For soluble Lie rings the results we obtain are stronger than the corresponding results for groups, for the same reasons.

§ 5.1 THE CLASSES $\mathfrak{M}_0$ AND $E\mathfrak{M}_0$

Let L be a Lie ring. The torsion free rank of L, which is denoted by $r_0(L)$, is the cardinal of a maximal $\mathbb{Z}$ - linearly independent set of elements of L of infinite order (in other words the dimension of $\mathbb{Q} \otimes_{\mathbb{Z}} L$ regarded as a vector space over $\mathbb{Q}$).

If p is a prime, then the p - rank of L, denoted by $r_p(L)$, is the cardinal of a maximal linearly independent set of elements of order p in L.

If we consider L^* as an abelian group it is clear that $r_0(L)$ and $r_p(L)$ are equal to $r_0(L^*)$ and $r_p(L^*)$ respectively (as defined in Fuchs [10] p85) and hence are invariants for L. Using this equality we can state immediately ;

LEMMA 5.1.1

Let L be a Lie ring, $B \leq L$ then

(i) $r_o(L) = r_o(B) + r_o(L^*/B^*)$

(ii) $0 \leq - r_p(B) + r_p(L) \leq r_p(L^*/B^*) \leq r_p(L) + r_o(L)$

for all primes p .

PROOF

As for abelian groups cf Robinson [30] p147. ■

It is clear that $r_o(L)$ is finite iff $(L / \tau(L))^*$ is isomorphic with an additive subgroup of a finite dimensional vector space over $\mathbb{Q}$, and $r_p(L)$ is finite iff $\tau(L)_p^* \in$ Min .

Let $\mathfrak{A}_0$ denote the class of all abelian Lie rings which have finite torsion free rank and finite p - rank for all p . By lemma 5.1.1 $\mathfrak{A}_0$ is S - closed and Q - closed.

$\mathfrak{A}_0^*$ is then the class of all Lie rings with finite torsion free rank and finite p - rank for all p . Once again lemma 5.1.1 gives S - closure, Q - closure and also E - closure. $E\mathfrak{A}_0$ is the class of poly - $\mathfrak{A}_0$ rings. $E\mathfrak{A}_0$ is an S -, Q - and E - closed class containing both polycyclic Lie rings and soluble Lie rings satisfying Min . Easily $\mathfrak{A} \cap E\mathfrak{A}_0 = \mathfrak{A}_0$ and $\mathfrak{A}_0^* \cap E\mathfrak{A} = E\mathfrak{A}_0$.

If $L \in \mathfrak{A}_0$ is of finite exponent then it is finite (cf Robinson [30] p148 considering L as an abelian group). This gives us the very important fact that $\mathfrak{A}_0^*$ Lie rings of finite exponent are finite.

If L is a residually finite π - ring, so is every subring and hence L must be π - reduced. We will investigate how far the converse of this result is true for the classes we have defined above. In this context compare the results of § 4.1 . The example prior to lemma 4.1.5 shows that reduced Lie rings need not in general be residually finite (since elements of infinite height lie in every subring of finite index).

THEOREM 5.1.2

Let $L \in \mathfrak{M}_0^*$ and π be a set of primes. Then $L \in R\mathfrak{F}_\pi$ iff L is π - reduced.

PROOF

Let $L \in \mathfrak{M}_0^*$ and suppose it is π - reduced. $\tau(L)$ is a reduced π - ring and so each of its primary components is finite (since it is of finite rank). L now satisfies the hypotheses of Theorem 4.1.6 and so is $R\mathfrak{E}_\pi$. But $\mathfrak{M}_0^*$ rings of finite exponent are finite and so $L \in R\mathfrak{F}_\pi$. ■

Theorem 5.1.2 is also true for generalised rings in the sense of Fuchs (cf Theorem 4.1.6).

COROLLARY 5.1.3

Suppose $\mathfrak{X}$ is an E - closed subclass of $\mathfrak{M}_0^*$ then $\mathfrak{X} \cap R\mathfrak{F}_\pi$ is E - closed. In particular $\mathfrak{M}_0^* \cap R\mathfrak{F}_\pi$ and $E\mathfrak{M}_0 \cap R\mathfrak{F}_\pi$ are E - closed.

PROOF

Suppose N and $L/N \in \mathfrak{X} \cap R\mathfrak{F}_\pi$ then $L \in \mathfrak{X}$ and L is reduced. Hence by Theorem 5.1.2 $L \in R\mathfrak{F}_\pi$. ■

We will now examine the soluble case where it turns out that a considerably stronger result is possible.

LEMMA 5.1.4

Suppose $B \leq L$ and $|L : B| < \infty$. Then there exists $H \lhd L$ such that $H \leq B$ and $L/H \in \mathfrak{E}$.

PROOF

If $|L : B| < \infty$ then there exists m such that $mL \leq B$ and $mL \lhd L$, so put $H = mL$. ■

LEMMA 5.1.5

$L \in R\mathfrak{F}_\pi \cap \mathfrak{M}_o^*$ iff for all $0 \neq x \in L$ there exists a subring H of L such that $x \notin H$ and H has index in L a finite π - number .

PROOF

Use lemma 5.1.4 and the fact that $\mathfrak{M}_o^*$ rings of finite exponent are finite. ■

LEMMA 5.1.6

Let L be a Lie ring such that $nL = 0$. Then for any derivation φ of L we have $n\varphi = 0$.

Thus any Lie ring of derivations of L has exponent dividing n .

PROOF

Follows from the linearity of φ . ■

LEMMA 5.1.7

Let $N \lhd L$ and assume that every subideal factor of L which has finite exponent is finite. Suppose L/N has an $\mathfrak{M}_o$ series of finite length in which each factor is π - reduced for some set of primes π , Then if x is an element of N which is not contained in every subring of index in N a finite π - number, then there is a subring of L of index a finite π - number to which x does not belong.

PROOF

We can refine the given series for L/N by inserting the torsion ideals in each factor. This gives a series

$$N = N_0 \triangleleft N \triangleleft \dots \triangleleft N_n = L$$

in which each factor N_{i+1}/N_i is either torsion free abelian of finite rank or a direct sum of finite abelian p - rings for different primes $p \in \pi$ (since π - reduced implies $R \mathfrak{F}_\pi$ by Theorem 5.1.2).

Assume that $n > 0$ and that the lemma is true for $M = N_{n-1}$. Thus by hypothesis there is a subring S such that $|M : S|$ is afinite π - number m say, $x \notin S$. Then $mM \leqslant S$ and so by hypothesis M/mM has order a finite π - number.

Let $C = C_L(M/mM)$. Then $C \triangleleft L$ and $L/C \in \mathfrak{F}$ (Corollary 1.3.3). Further $|L/C|$ is a π - number by lemma 5.1.6 . Thus we can assume $x \in C$.

<u>Case (i)</u> L/M is a direct sum of finite abelian p - rings with $p \in \pi$.

Write $\overline{C} = C/mM$, $\overline{M} = M/mM$ and $\overline{x} = x + mM$. Then $\overline{C}$ is a torsion π - ring and so is a direct sum of its primary components, each of which is finite.

Let the order of $\overline{x}$ be a π_0 - number where π_0 is a finite subset of π . Then if $\overline{C}_1$ is the π_0 component of $\overline{C}$, we have $\overline{x} \in \overline{C}_1$. Furthermore $\overline{C}_1$ is a finite π_0 (and hence π) -ring and $\overline{C} = \overline{C}_1 \oplus \overline{C}_2$ for some $\overline{C}_2$ (if $\overline{C}_2 = 0$ then the index of mM in L is a finite π - number). Hence $\overline{x} \notin \overline{C}_2$ and $\overline{C}/\overline{C}_2 \cong \overline{C}_1$ has order a finite π - number. So x is not contained in a subring of index a finite π - number in L.

Case (ii) L/M is torsion free.

Define $D = mC + mM$, $D \lhd L$. Then

$$|L/D| = |L/C|.|C/D| \leq |L/C|.|C/mC| < \infty$$

(since $|C/mC|$ is finite by hypothesis). Further $|L/D|$ divides $|L/C|.|C/mC|$, and $|L/C|$ and $|C/mC|$ are both π - numbers. Hence $|L/D|$ is a π - number.

Suppose $x \in D$ and $x \equiv y \bmod(mM)$ where $y \in mC$. Then $x \equiv mz \bmod(mM)$ where $z \in C$. But $x \in N \leq M$ so $mz \in M$. Since $m > 0$ and L/M is torsion free, we have $z \in M$. Hence $x \in mM$ and this is a contradiction. Hence $x \notin D$. ∎

LEMMA 5.1.8

Let L be a Lie ring with Fitting ideal $N = \nu(L)$, and let π be a set of primes.

If $N \in E\mathfrak{N}_0$ and $Z_1(N)$ is π - reduced, then every abelian ideal of L is π - reduced.

PROOF

By lemma 4.1.5 (ii), if $i < \omega$ then $Z_{i+1}(N)/Z_i(N)$ is π - reduced and so is $R\mathfrak{F}_\pi$ (Theorem 5.1.2).

Suppose $Z_i(N) \in R\mathfrak{F}_\pi$, but $Z_{i+1}(N) \notin R\mathfrak{F}_\pi$. Then there exists $x \in Z_{i+1}(N)$ which belongs to every ideal of $Z_{i+1}(N)$ of index a finite π - number. Then $x \in Z_i(N)$ (since the factors are π - reduced) so $x \notin mZ_i(N)$ for some positive π - number m.

Let $C = C_{Z_{i+1}(N)}(Z_i(N)/mZ_i(N))$. Since $N \in E\mathfrak{N}_0$ subideal factors of N of finite exponent are finite, and so, by corollary 1.3.3 $Z_{i+1}(N)/C$ is finite, and by lemma 5.1.6 it has exponent dividing m. Now m is a π - number and hence so is $|Z_{i+1}(N)/C|$. $x \notin C$ as in 5.1.8 and so does not belong to a subring of finite π - index in $Z_{i+1}(N)$. Hence $Z_j(N) \in R\mathfrak{F}_\pi$ for all $j < \omega$. (Or just use 5.1.3)

Now suppose L contains an abelian ideal that is not π - reduced. Then its maximal π - divisible ideal R is nontrivial. Also $R \leq N$. Let H be any f.g. subring of N. H is contained in some nilpotent ideal of L and so $[L,_n H] = 0$ for some $n = n(H)$.

Let p be any prime and let P be the p - component of R. If P_0 is the maximal divisible ideal of P then P/P_0 is finite (since $N \in \mathfrak{E}\mathfrak{A}_0$). Also $[P,_{n_p} H] = 0$ where $n_p = \log_2(|P:P_0|) + r_0(P_0)$. Further since n_p is independent of H we have $[P,_{n_p} N] = 0$ and $P \leq Z_{n_p}(N)$ which is π - reduced. However P is π - divisible since R is (whether or not $p \in \pi$). So $P = 0$ and R is torsion free.

Now let $\overline{R} = \mathbb{Q} \otimes_{\mathbb{Z}} R$, then $\overline{R}$ is a vector space of dimension $r_0 = r_0(R)$ over $\mathbb{Q}$. H acts as a Lie ring of derivations on R and we can extend the action of H to $\overline{R}$ by

$$(r \otimes \alpha)x = r^x \otimes \alpha \quad , \; r \in R, x \in H, \alpha \in \mathbb{Q}$$

Thus H acts linearly on $\overline{R}$. Now since $[R,_n H] = 0$ then

$$\overline{R} \geqslant [\overline{R}, H] \geqslant \ \dots \ \geqslant [\overline{R},_n H] = 0$$

is a descending chain of subspaces of $\overline{R}$. Since $\dim(\overline{R}) = r_0$ we have $[\overline{R},_{r_0} H] = 0$, and so $[R,_{r_0} H] = 0$. Since r_0 is independent of H, $[R,_{r_0} N] = 0$ and so

$R \leq Z_{r_0}(N)$ which is π - reduced, so $R = 0$. ■

THEOREM 5.1.9

Let $L \in E\mathfrak{U}_0$ and π be a set of primes. Then $L \in R\mathfrak{F}_\pi$ iff the centre of the Fitting ideal of L is π - reduced.

PROOF

Suppose $L \in E\mathfrak{U}_0$ and the centre of the Fitting ideal of L is π - reduced. By lemma 5.1.8 every abelian ideal of L is π - reduced.

Now we define ideals A_i of L as follows : $A_0 = 0$, and if A_i is already defined A_{i+1}/A_i is the maximal abelian ideal of L/A_i containing the last nontrivial term of the derived series of L/A_i . Then

$$0 = A_0 < A_1 < \ldots\ldots < A_d = L$$

is an $\mathfrak{U}_0$ series of ideals of L .

Let R/A_1 be any π - divisible abelian ideal of L/A_1 . If m is any positive π - number A_1/mA_1 is finite and elements of R/A_1 induce derivations on A_1/mA_1 . Now R/A_1 has no proper subrings of index a finite π - number and so R centralises A_1/mA_1 . Hence

$$[A_1, R] \leq \bigcap_m mA_1 = 0$$

since A_1 is π - reduced and so by Theorem 5.1.2 is $R\mathfrak{F}_\pi$ (where m ranges over all positive π - numbers).

If $y \in R$ the map

$$x + A_1 \longmapsto [x, y] , \quad x \in R$$

is a Lie homomorphism of R/A_1 onto $[R, y] \leq A_1$ by the construction of A_1 . Hence $[R, y] = 0$ and $R \in \mathfrak{U}$. By the maximality of A_1 , $R = A_1$. So L/A_1 inherits the properties of L and A_2/A_1 is π - reduced. Similarly every A_{i+1}/A_i is π - reduced.

Let $0 \neq x \in A_1$ then x does not belong to some subring of index a finite π - number in A_1 and so by lemma 5.1.7 x fails to belong to a subring of index a finite π - number in L. Hence $L \in R\mathfrak{F}_\pi$. ∎

Theorem 5.1.9 is not true in general for $\mathfrak{U}_0^*$ - rings. Let L be any finite dimensional simple Lie algebra over $\mathbb{Q}$. Let $\tilde{L}$ be L considered as a Lie ring (by restricting scalar multiplication to $\mathbb{Z}$). The rational completion (§ 1.6) of $\tilde{L}$ is L. L is simple and so Corollary 1.6.2 ensures that the Fitting ideal of $\tilde{L}$ is trivial (and hence reduced). However clearly $\tilde{L} \in \mathfrak{U}_0^*$ and $\tilde{L}$ is divisible and hence by Theorem 5.1.2 $\tilde{L} \notin R\mathfrak{F}$.

<u>COROLLARY 5.1.10</u>

Let $L \in E\mathfrak{U}_0$. Then L is π - reduced iff the centre of the Fitting ideal of L is π - reduced.

PROOF

Theorems 5.1.2 and 5.1.9. ∎

<u>COROLLARY 5.1.11</u>

Let $L \in \mathfrak{N} \cap E\mathfrak{U}_0$ and π any set of primes. Then $L \in R\mathfrak{F}_\pi$ iff the centre of L is π - reduced. ∎

Having an important bearing on the problem of residual finiteness is the structure of minimal ideals and in the case of $E\mathfrak{U}_0$ rings these can be described explicitly.

THEOREM 5.1.12

Let $L \in E\mathfrak{U}_0$. If N is a minimal ideal of L, either

(i) $N \in \mathfrak{U} \cap \mathfrak{F}$ and is of prime exponent

or (ii) $N \in \mathfrak{U} \cap \mathfrak{H}$ and is finite rank and divisible.

PROOF

Let N be a minimal ideal. Since $L \in E\mathfrak{U}_0$, N is clearly abelian. Then N is either torsion or torsion free (otherwise the torsion ideal is strictly contained in N)
If N is torsion it is a p-ring for some p (since primary components are direct). Thus N is finite since $r_p(N) < \infty$.
If N is torsion free it is divisible for otherwise

(a) If it were mixed the divisible ideal of N is strictly contained in N.

(b) If it were reduced then there exists p such that $0 < pN < N$ and $pN \lhd L$.

So N is a direct sum of finitely many copies of $\mathbb{Q}$. ∎

COROLLARY 5.1.13

If $L \in E\mathfrak{U}_0$ then all chief factors are either

(i) $\mathfrak{U} \cap \mathfrak{F}$ of prime exponent

or (ii) $\mathfrak{U} \cap \mathfrak{H}$ and finite rank divisible. ∎

Robinson [30] p183 Theorem 6.45 shows that in the group theory case (ii) does not arise. However it is necessary here as the following example shows. Consider

$$L^* = \mathbb{Q} \oplus \mathbb{Q}$$

with the only nonabelian structure

$$[(x, 0), (0, y)] = (0, xy)$$

Then $L \in E\mathfrak{U}_0$ and $\mathbb{Q}$ is a minimal ideal of L .

We can however say the following ;

THEOREM 5.1.14

Let $L \in \mathfrak{G} \cap E\mathfrak{U}_0$ then any minimal ideal (and hence any chief factor) of L is contained in $\mathfrak{U} \cap \mathfrak{F}$.

PROOF

Let L be generated by $x_1 , \ldots\ldots , x_m$ say , and suppose I is a minimal ideal of L which is finite rank divisible. Let $y_1 , \ldots\ldots , y_n$ be a basis for I (considered as a vector space over $\mathbb{Q}$). We have equations of the form

$$[x_i , y_j] = \sum_{k=1}^{n} q_{ijk} y_k$$

where i runs from 1 to m, and j runs from 1 to n . The mn^2 coefficients q_{ijk} are rational numbers. They can be placed over a common denominator d . If $r_1 y_1 + \ldots . + r_n y_n$ is any vector belonging to the ideal I_1 of L generated by y_1 , then none of the rational numbers r_i when reduced to their lowest terms can involve in its denominator any prime not dividing d . Hence I_1 is strictly contained in I , contradicting minimality. Hence the result. ∎

§ 5.2 THE CLASSES $\mathfrak{U}_1$ AND $E\mathfrak{U}_1$

Let L be a Lie ring. Then the total rank of L , denoted $r(L)$, is defined by

$$r(L) = r_0(L) + \sum_p r_p(L)$$

Let $\mathfrak{U}_1$ be the class of all abelian Lie rings of finite total rank. Thus $L \in \mathfrak{U}_1$ iff $\tau(L) \in$ Min and $L / \tau(L)$ has finite torsion free rank.

$\mathfrak{U}_1^*$ is the class of all Lie rings with finite total rank. This class is S - closed and E - closed and $E\mathfrak{U}_1$ is the class of poly $\mathfrak{U}_1$ - rings and it is a proper subclass of $E\mathfrak{U}_0$ containing the polycyclic Lie rings and soluble Lie rings with Min . $E\mathfrak{U}_1$ is S - and E - closed.

None of $\mathfrak{U}_1$, $\mathfrak{U}_1^*$ and $E\mathfrak{U}_1$ is Q - closed (consider the abelian Lie ring with additive subgroup the rationals mod 1). However the torsion subclasses of these classes are Q - closed. Finally $\mathfrak{U} \cap E\mathfrak{U}_1 = \mathfrak{U}_1$.

Once again (cf Theorem 4.1.6) the theorem for abelian groups (Robinson [30] p160) provides us with a result which is also true for generalised rings.

<u>THEOREM 5.2.1</u>

Let $L \in \mathfrak{U}_1^*$ and suppose L is π -reduced for some set of primes π . Then $L \in R\mathfrak{F}_{\pi_0}$ where π_0 is a finite subset of π with cardinality at most $r(L)$.

PROOF

L is π - reduced and consequently $\tau(L)$ is finite. By Kaplansky [21] p18 $\tau(L)^*$ is an abelian group direct factor of L^* . By Theorem 5.1.2 $L \in R\mathfrak{F}_{\pi}$ so there exists a $\bar{\pi}$ - number $\bar{m}$ where $\bar{\pi}$ is a finite subset of π , such that $\bar{L} = \bar{m}L$ is torsion free. It suffices now to find a finite subset π_0 of π such that $R = \bigcap_m m\bar{L}$, where m runs over all positive π_0 - numbers, is trivial (since $m\bar{L}$ ch $\bar{L}$).

Suppose therefore that L is torsion free and nontrivial. L is not π - divisible so there exists $p \in \pi$ such that $L > pL$. Put $P = \bigcap_{i=0}^{\infty} p^i L$. Since L is torsion free it follows easily that L/P is also. By lemma 5.1.1

$$r = r_0(L) = r_0(P) + r_0(L/P)$$

Hence $r(P) < r$ and by induction on r, $P \in R\mathfrak{F}_{\pi_1}$ where $\pi_1 \subseteq \pi$ and $|\pi_1| \leq r - 1$. Let $\pi_0 = \pi_1 \cup \{p\}$ so that $|\pi_0| \leq r$ and define $R = \bigcap_m mL$ where m runs over all positive π_0 - numbers. It is now sufficient to prove that R is π_0 - divisible for then it will be trivial, since $R \leq P$, $P \in R\mathfrak{F}_{\pi_1}$ and $\pi_1 \subseteq \pi_0$.

Let $q \in \pi_0$ and $a \in R$, then $a = qa_1 = q^2a_2 = \ldots.$ where $a_i \in L$. L is torsion free so $a_1 = qa_2 = q^3a_3 = \ldots.$ and $a_1 \in q^iL$ for all i. Let m be any positive π_0 - number and write $m = q^in$ where n is prime to q and $i \geq 0$. $a_1 \in q^iL$ and $qa_1 = a \in R \leq mL$. Now q^iL/mL has exponent dividing n and so $a_1 \in mL$. Hence $a_1 \in R$ and R is π_0 - divisible. Hence $R = 0$ and the result follows. ∎

We can now combine this result with those of § 5.1 to find out what happens in the soluble case.

<u>THEOREM 5.2.2</u>

Let $L \in E\mathfrak{U}_1$. Then L is residually a finite π - ring for some finite set of primes π, iff the centre of the Fitting ideal of L is reduced.

PROOF

($\Rightarrow$) If $L \in R\mathfrak{F}_\pi$ then the centre of its Fitting ideal is π - reduced by Theorem 5.1.9 and hence reduced.

($\Leftarrow$) Suppose the centre of the Fitting ideal of L is reduced then since $L \in E\mathfrak{U}_1 < E\mathfrak{U}_0$, Corollary 5.2.1 gives

$L \in R\mathfrak{F}_{\pi}$ for some finite set of primes π (which can in fact be chosen with cardinality at most $r(L)$). ■

COROLLARY 5.2.3

A polycyclic Lie ring is residually a finite π - ring for some finite set of primes π . ■

Now a converse to Theorem 5.2.2

THEOREM 5.2.4 .

Let $L \in E\mathfrak{N}_0$ be a residually finite π - ring for some finite set of primes π , then $L \in E\mathfrak{N}_1$.

PROOF

Let $L \in E\mathfrak{N}_0 \cap R\mathfrak{F}_{\pi}$ where $|\pi| < \infty$. Every element of finite order in L must have its order equal to a π - number. Thus $\tau(L)^* \in$ Min and consequently $L \in E\mathfrak{N}_1$. ■

We will now collect together some facts (not all of which we will use, but which are worth noting in their own right) about divisible Lie rings.

LEMMA 5.2.5

Suppose L is a torsion Lie ring. Then $\partial(L) \leq Z_1(L)$.

PROOF

Let $y \in L$ and suppose that $ny = 0$. Now $\partial(L)$ is divisible so for any $x \in \partial(L)$ we can find $z \in \partial(L)$ such that $x = nz$. Then

$$[x, y] = [nz, y] = [z, ny] = 0$$

and $x \in Z_1(L)$. ∎

PROPOSITION 5.2.6

Let L be a divisible Lie ring then

(i) $\tau(L)$ is divisible and contained in the centre of L .

(ii) Every term of the upper central series of L is divisible.

PROOF

(i) Since L is divisible and $L/\tau(L)$ is torsion free we have that $\tau(L)$ is divisible.

If $y \in L$, $x \in \tau(L)$ say $nx = 0$. Then

$$0 = [y, nx] = [ny, x] \quad , \text{for all } y \in L$$

Thus $x \in Z_1(L)$.

(ii) L/Z_1 is torsion free, for if not then Z_2/Z_1 is not torsion free and there exists $x \in Z_2 \setminus Z_1$ such that $nx \in Z_1$ for some $n > 0$ (where $Z_i = Z_i(L)$). Then $[ny, x] = 0$ for all $y \in L$, so $x \in Z_1$ which is a contradiction. Since L is divisible and L/Z_1 is torsion free Z_1 is divisible.

For any $\alpha > 0$, $Z_\alpha(L/Z_1) = Z_{\alpha+1}/Z_1$ and $L/Z_{\alpha+1}$ is torsion free by lemma 4.1.3 . Hence $Z_{\alpha+1}$ is divisible. ∎

Let $\mathfrak{A}_{-1}$ be the class of abelian Lie rings with $r_0(L)$ finite .

THEOREM 5.2.7

If $L \in E\mathfrak{A}_{-1}$ then $\partial(L) \in \mathfrak{N}\mathfrak{A}$

PROOF

Let $R = \partial(L)$ the divisible ideal. Now $R/\tau(R)$

is torsion free of finite rank. Consider its $\mathbb{Q}$-completion which is a soluble Lie algebra of finite dimension over $\mathbb{Q}$. Lie's theorem and Corollary 1.6.2 give $R/\tau(R) \in \mathfrak{N}\mathfrak{U}$ (cf Proposition 1.8.1). But $\tau(R) \leq Z_1(R)$ by Proposition 5.2.6 (i), and hence $R \in \mathfrak{N}\mathfrak{U}$. ∎

LEMMA 5.2.8

Let L be a divisible Lie ring and $\Gamma \leq \mathrm{Der}(L)$. Then if either

(i) L is torsion free and $r_0(L) < \infty$

or (ii) L is a p - ring for some prime p and $r_p(L) < \infty$

then $\Gamma \in \mathcal{R}_0$.

PROOF

(i) L^* is a direct sum of finitely many, say n, copies of $\mathbb{Q}$ and so $\mathrm{End}(L^*)$ is the associative ring of $n \times n$ matrices over $\mathbb{Q}$ (Fuchs [9] 55 p210 ff). Thus since any Lie ring of derivations of L is a Lie subring of $\mathrm{End}(L^*)_L$ (by lemma 1.3.1) we have $\Gamma \in \mathcal{R}_0$.

(ii) L^* is a direct sum of finitely many, say n, copies of C_{p^∞} and so $\mathrm{End}(L^*)$ is the associative ring of $n \times n$ matrices over the field of p - adic integers (Fuchs [9] 55 p210 ff). Thus $\Gamma \in \mathcal{R}_0$. ∎

We will now examine in closer detail the structure of $E\mathfrak{U}_1$ rings.

LEMMA 5.2.9

If L is torsion free and $\Gamma \leq \mathrm{Der}(L)$ then Γ is torsion free.

PROOF

If L^* is torsion free then $\mathrm{End}(L^*)$ is torsion free (Fuchs [10] p182). Then use lemma 1.3.1 . ■

LEMMA 5.2.10

Let $L \in \mathfrak{H} \cap E\mathfrak{U}$. Then L has a finite characteristic series with torsion free and abelian factors.

PROOF

Let n be the derived length of L and use induction on n .

If $n = 1$ the result is clear. Suppose $n > 1$. Then $L/L^{(n-1)} \in E\mathfrak{U}$ and has derived length $n - 1$. Thus if $T/L^{(n-1)} = \tau(L/L^{(n-1)})$ then $T \text{ ch } L$ and L/T has a finite characteristic series with $\mathfrak{H} \cap \mathfrak{U}$ factors by induction.

Let $C = C_T(L^{(n-1)})$. Then $C \text{ ch } L$ and T/C is a subring of $\mathrm{Der}(L^{(n-1)})$. Now $L^{(n-1)}$ is torsion free and so by lemma 5.2.9 T/C is torsion free. However T/C is a quotient of $T/L^{(n-1)}$ and hence is torsion. Thus $C = T$. Hence $[T, L^{(n-1)}] = 0$ and $L^{(n-1)} \leq Z_1(T)$.

Suppose $T/L^{(n-1)} \neq 0$. We also know that T is torsion free and $T/L^{(n-1)}$ is torsion. Now $T/Z_1(T)$ is a quotient of $T/L^{(n-1)}$ and hence is torsion, but by lemma 4.1.3 we know it is torsion free. Hence $T/Z_1(T) = 0$ and $T \in \mathfrak{U}$. The result now follows by the case $n = 1$. ■

LEMMA 5.2.11

Let $L \in E\mathfrak{U}_1$. Then L has a characteristic series of finite length with $\mathfrak{U}_1$ factors.

PROOF

Let $T = \tau(L)$. Then $L/T \in \mathcal{H} \cap E\mathfrak{M}_{-1}$ and so by lemma 5.2.10 has a characteristic series of finite length with $\mathcal{H} \cap \mathfrak{M}_{-1}$ factors.

Let $D = \partial(T)$. Then D ch T ch L so D ch L. Now $D \leqslant Z_1(T)$ by lemma 5.2.5 and so $D \in \mathfrak{M}_1$. Further since $L \in E\mathfrak{M}_1$, $T/D \in \mathfrak{F}$ so the derived series for T/D will have finite factors and since its terms are characteristic in T they will be characteristic in L. This then gives a series of the required form.

Note that we could further refine this series by inserting a series for D whose factors are direct sums of finitely many copies of C_{p^∞} with a different prime p for each factor. This is possible since D is a direct sum of its primary components and their are only finitely many of these since $D \in \mathfrak{M}_1$. ■

We are now in a position to prove an analogue of a result of Mal'cev's [28] for groups.

THEOREM 5.2.12

$$E\mathfrak{M}_1 \leqslant \mathfrak{N}\mathfrak{M}\mathfrak{F}$$

PROOF

By lemma 5.2.11 $L \in E\mathfrak{M}_1$ has a characteristic series of finite length whose factors are either finite, direct sums of finitely many C_{p^∞} groups or torsion free abelian of finite rank.

Let F be any factor of this series and write $\bar{L} = L/C_L(F)$. If F is finite so is $\bar{L}$. If F is a direct sum of finitely many C_{p^∞}'s then $\bar{L} \in \mathcal{R}_0$ by lemma 5.2.8 (ii) and so by the argument of Proposition 1.8.1 (that is essentially Lie's Theorem) $\bar{L}^2$ acts nilpotently on F i.e.

$$[F, {}_m \bar{L}^2] = 0$$

for some $m > 0$. If F is torsion-free abelian of finite rank then $\bar{L}$ is torsion free by lemma 5.2.9 , soluble and of finite rank. Consider the $\mathbb{Q}$ - completion of L , then by the same argument as above (Lie's Theorem and Corollary 1.6.2) we have $[F, {}_m \bar{L}^2] = 0$ for some $m > 0$.

Thus for each factor F , there is $N_F \triangleleft L$ such that $L/N_F \in \mathfrak{F}$ and $(N_F/C_L(F))^2$ acts nilpotently on F. Take $N = \bigcap N_F$ with the intersection taken over all factors F. Then $L/N \in \mathfrak{F}$ and $[F, {}_k N^2] = 0$ for all factors F and some $k > 0$. Then $N^2 \in \mathfrak{N}$ and so $L \in \mathfrak{N}\mathfrak{A}\mathfrak{F}$.

$\blacksquare$

CHAPTER 6 $\mathfrak{A}$ - CLOSURE RELATIVE TO $E\mathfrak{A}$

In the first section we investigate the structure of soluble Lie rings satisfying the minimal condition for subideals and find that these are somewhat better behaved than their group theoretic counterparts.

The rest of the chapter is devoted to finding what properties are inherited by soluble Lie rings from their abelian subrings. Firstly various finiteness conditions are examined. The group theoretic version of this work is due to Mal'cev [28] and Schmidt [33] . We then look at the various rank conditions defined in Chapter 5 and prove analogues of theorems of Čarin [5] , [6] and Kargapolov [22] .

§ 6.1 SOLUBLE LIE RINGS WITH Min - si

LEMMA 6.1.1

Let $L \in E\mathfrak{A} \cap$ Min - si . Let N be the unique minimal ideal of finite index in L (which exists by Min - si). Then N is divisible.

PROOF

We will first show that N has no proper subrings of finite index. Suppose T is a proper subring of N of finite index. Then for some m , $mL \leq T$ and $N/mL \in \mathcal{E}$. There is a characteristic abelian series

$$mL = L_0 < L_1 < \ldots\ldots < L_n = N$$

(e.g. the derived series of N/mL).

Then $L_n/L_{n-1} \in \mathfrak{A} \cap \text{Min} \cap \mathcal{E}$ and hence is finite. But L_{n-1} ch N and so $L_{n-1} \lhd L$, contradicting the definition of N . Hence N has no proper subrings of finite index.

If $m > 0$ and $mN < N$ then as above there exists a characteristic abelian series from mN to N . Looking at the top factor again gives a contradiction and so $mN = N$ for all m . Hence N is divisible. $\blacksquare$

COROLLARY 6.1.2

Let $L \in E\mathfrak{A} \cap \text{Min - si}$. If $L \in \mathfrak{E}$ then $L \in \mathfrak{F}$. $\blacksquare$

THEOREM 6.1.3

Let $L \in E\mathfrak{A} \cap \text{Min - si}$. Then L is a finite extension of a divisible abelian ring satisfying Min . Consequently L is countable and satisfies Min .

PROOF

Let $L \in E\mathfrak{A} \cap \text{Min - si}$. Then L has an invariant abelian series each of whose factors satisfies Min (since $\mathfrak{A} \cap \text{Min - si} \leq \text{Min}$). Hence L is torsion. Let N be the unique minimal ideal of finite index in L . By lemma 6.1.1 N is divisible and hence central by lemma 5.2.5 . This completes the result. $\blacksquare$

COROLLARY 6.1.4

Let $L \in E\mathfrak{A} \cap \text{Min - si}$. Then L is centre by finite.

PROOF

As for Theorem 6.1.3 . $\blacksquare$

COROLLARY 6.1.5

Let $L \in \mathfrak{N} \cap \text{Min - } \triangleleft$. Then

(i) $L \in$ Min

(ii) L is centre by finite .

PROOF

(i) If $Z_i(L) \leq H \leq Z_{i+1}(L)$ then $H \lhd L$ since $L \in \mathfrak{N}$. Hence every upper central factor of L and so L itself satisfies Min .

(ii) Min implies Min - si and so the result follows by Corollary 6.1.4 . ■

LEMMA 6.1.6

If L is a divisible Lie ring then any Lie ring of derivations of L is torsion free.

PROOF

If L^* is divisible then $\mathrm{End}(L^*)$ is torsion free (Fuchs [9] p207). The result follows since any Lie ring of derivations of L is a Lie subring of $\mathrm{End}(L^*)_L$. ■

PROPOSITION 6.1.7

Let $L \in E\mathfrak{U} \cap$ Min - si . Suppose Γ is a torsion Lie ring of derivations of L . Then $\Gamma \in \mathfrak{F}$.

PROOF

By Theorem 6.1.3 $L^* \in$ Min and hence $L^* = \bigoplus_j A_j$ where $A_j \cong C_{p_j^\infty}$ or $C_{q_j^k}$ for some primes p_j, q_j and integer $k > 0$. Hence (Fuchs [9] p212 Theorem 55.1) $\mathrm{End}(L^*)$ is isomorphic to the associative ring of all $n \times n$ matrices (a_{ij}) such that each $a_{ij} \in \mathrm{Hom}(A_i, A_j)$. But in this case $\mathrm{Hom}(A_i, A_j)$ is always either torsion free or finite and hence $\tau(\mathrm{End}(L^*)) \in \mathfrak{F}$. The result now follows by Lemma 1.3.1. ■

We now briefly consider soluble Lie rings satisfying the minimal condition for ideals. All we can say is that such rings are torsion.

LEMMA 6.1.8

Let Γ be a torsion Lie ring of derivations of a Lie ring L. If L contains no proper nontrivial Γ - invariant subrings then L is torsion.

PROOF

Suppose L is not torsion. For any $m > 0$, mL is Γ - invariant and so is $\tau(L)$ and hence L is torsion free and divisible. Thus by lemma 6.1.6 any Lie ring of derivations of L is torsion free. A contradiction. ∎

PROPOSITION 6.1.9

Let $L \in E\mathfrak{A} \cap \text{Min} - \triangleleft$. Then L is torsion.

PROOF

Let $L \in E\mathfrak{A} \cap \text{Min} - \triangleleft$. Proceed by induction on the derived length d of L. Write $A = L^{(d-1)}$ and assume $d > 1$ and L/A is torsion. By Min - $\triangleleft$ there exists an ascending chief series of L from 0 to A, say $\{A_\alpha \mid \alpha \leq \beta\}$. Now $A_{\alpha+1}/A_\alpha$ has no proper nontrivial L - invariant subrings. Further $L/C_L(A_{\alpha+1}/A_\alpha)$ is torsion (since $C_L(A_{\alpha+1}/A_\alpha) \geqslant A$). The result now follows from lemma 6.1.8. ∎

§ 6.2 $\mathfrak{A}$ - CLOSURE AND FINITENESS CONDITIONS

Suppose $\mathfrak{X}$ and $\mathfrak{Y}$ are classes of Lie rings closed with respect to taking abelian subrings. We say $\mathfrak{X}$ is $\mathfrak{A}$ - closed relative to $\mathfrak{Y}$ if given $L \in \mathfrak{Y}$ and every abelian subring of L is contained in $\mathfrak{X} \cap \mathfrak{Y}$ then $L \in \mathfrak{X} \cap \mathfrak{Y}$.

In the case $\mathfrak{X} \leq \mathfrak{Y}$ we say <u>$\mathfrak{X}$ is $\mathfrak{A}$ - closed in $\mathfrak{Y}$</u> if it is $\mathfrak{A}$ - closed relative to $\mathfrak{Y}$.

Throughout the rest of this chapter we will investigate what classes are $\mathfrak{A}$ - closed relative to $E\mathfrak{A}$.

<u>THEOREM 6.2.1</u>

Min is $\mathfrak{A}$ - closed relative to $E\mathfrak{A}$.

PROOF

Let L be soluble with each of its abelian subrings satisfying Min . Suppose L has derived length $d > 1$, and put $N = L^2$. By induction on d , $N \in$ Min . The hypothesis implies that L is torsion (just look at the cyclic subring generated by each element). Hence if we put $C = C_L(N)$ then L/C can be considered as a torsion Lie ring of derivations of N and hence by Proposition 6.1.7 $L/C \in \mathfrak{F}$ and hence satisfies Min .

Now $C^2 = [C, C] \leq [L, L] = L^2$ and so

$$[C^2, C] \leq [L^2, C] = 0$$

and $C \in \mathfrak{N}_2$. Let M denote one of its maximal abelian ideals. M satisfies Min and $M = C_C(M)$, so a further application of Proposition 6.1.7 gives $C/M \in$ Min . Hence $L \in$ Min . ∎

<u>LEMMA 6.2.2</u>

If $L \in \mathfrak{G} \cap \mathfrak{A}$ and $\Gamma \leq \mathrm{Der}(L)$ then $\Gamma \in \mathfrak{G}^* \leq \mathfrak{G}$.

PROOF

Suppose $L = F_1 \oplus \ldots\ldots \oplus F_m \oplus T_1 \oplus \ldots\ldots \oplus T_n$

where $F_i \cong \mathbb{Z}$ for all $i = 1,\ldots,m$ and $T_i \cong C_{p_i^{k_i}}$ for some p_i and k_i for all $i = 1,\ldots,n$.

By Fuchs [9] p212 Theorem 55.1 if A is an abelian group and $A = \bigoplus_{i=1}^{n} A_i$ then $\mathrm{End}(A)$ is isomorphic to the associative ring of all $n \times n$ matrices (a_{ij}) such that each $a_{ij} \in \mathrm{Hom}(A_i , A_j)$. Now

$$\begin{aligned}
\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) &\cong \mathbb{Z} \\
\mathrm{Hom}(\mathbb{Z}, C_{p^k}) &\cong C_{p^k} \\
\mathrm{Hom}(C_{p^k}, \mathbb{Z}) &\cong 0 \\
\mathrm{Hom}(C_{p^k}, C_{q^j}) &\cong \begin{cases} 0 & p \neq q \\ \text{finite} & p = q \end{cases}
\end{aligned}$$

So in particular all are f.g. abelian groups. So $\mathrm{End}(L^*)$ is a f.g. additive abelian group and hence so is every additive subgroup. Thus $\Gamma \leq \mathrm{Der}(L) = \mathrm{End}(L^*)_L \in \mathfrak{G}^*$. ∎

COROLLARY 6.2.3

If $L \in \mathfrak{G}^*$ and $\Gamma \leq \mathrm{Der}(L)$ then $\Gamma \in \mathfrak{G}^* \leq \mathfrak{G}$.

PROOF

Lemma 1.3.1 and lemma 6.2.2 . ∎

COROLLARY 6.2.4

Let $L \in E\mathfrak{C}$ and $\Gamma \leq \mathrm{Der}(L)$ then $\Gamma \in \mathfrak{G}^* \leq \mathfrak{G}$ and further if $\Gamma \in E\mathfrak{A}$ then $\Gamma \in E\mathfrak{C}$.

PROOF

Since $\mathfrak{G}^* \cap E\mathfrak{A} = E\mathfrak{C}$. ∎

THEOREM 6.2.5

Let $L \in E\mathfrak{A}$ and suppose that each of its abelian subideals is f.g. Then $L \in E\mathfrak{C}$.

PROOF

Assume $L \notin \mathfrak{A}$. Let N be the last nontrivial term of the derived series of L . By hypothesis $N \in \mathfrak{G}$. Let H/N be an abelian subideal of L/N and let $C = C_H(N)$. Since $C \geqslant N$, $H/C \in \mathfrak{A}$ and by lemma 6.2.2 $H/C \in \mathfrak{G}$.

Now $C^2 \leqslant N$ and so $[C^2, C] = 0$ and $C \in \mathfrak{N}_2$. Let M be a maximal abelian ideal of C . M si L and so $M \in \mathfrak{G}$. By the maximality of M , $M = C_C(M)$, and since $C^2 \leqslant Z_1(C) \leqslant M$ we have $C/M \in \mathfrak{A}$. Hence by lemma 6.2.2 C/M is f.g. and so H is also .

Thus $H/N \in \mathfrak{G}$ and L/N satisfies the initial hypotheses of the theorem. By induction on the derived length L/N and hence L is polycyclic. ∎

We can restate this result in a number of forms.

<u>COROLLARY 6.2.6</u>

$E\mathfrak{C}$ is $\mathfrak{A}$ - closed relative to $E\mathfrak{A}$.

PROOF

Since $E\mathfrak{C} \leqslant \mathfrak{G}$. ∎

<u>COROLLARY 6.2.7</u>

If $L \in E\mathfrak{A}$ is such that all its abelian subrings are f.g. then $L \in \mathfrak{G}$.

(Note that we cannot use the terminology of $\mathfrak{A}$ - closure since $\mathfrak{G}$ is not closed with respect to taking abelian subrings).

PROOF

Since $E\mathfrak{C} \leqslant \mathfrak{G}$. ∎

COROLLARY 6.2.8

Max is $\mathfrak{U}$ - closed relative to $E\mathfrak{U}$.

PROOF

Since $E\mathcal{C} \leq \text{Max} \cap \mathfrak{U} \leq \mathcal{G}$. ∎

THEOREM 6.2.9

$\mathcal{F}$ is $\mathfrak{U}$ - closed relative to $E\mathfrak{U}$.

PROOF

Let $L \in E\mathfrak{U}$ with all its abelian subrings finite. Now $\mathcal{F} \leq \mathcal{G} \cap \text{Min}$ and so by Theorem 6.2.1 and Theorem 6.2.5 $L \in E\mathcal{C} \cap \text{Min}$ and thus by Theorem 6.1.3 $L \in \mathcal{F}$. ∎

§ 6.3 $\mathfrak{U}$ - CLOSURE AND FINITE RANK

We say an abelian Lie ring L is in the class $\mathfrak{U}_2$ iff $r_0(L) < \infty$ and $\tau(L) \in \mathcal{F}$.

We have now defined the following classes (cf Chapter 5)

$$\mathfrak{U}_2 < \mathfrak{U}_1 < \mathfrak{U}_0 < \mathfrak{U}_{-1}$$

All the classes are clearly distinct and all are S - closed. $\mathfrak{U}_{-1}$ and $\mathfrak{U}_0$ are Q - closed but the others are not.

LEMMA 6.3.1

Let $A \in \mathcal{H} \cap \mathfrak{U}_{-1}$, say $r_0(A) = n$. Let Γ be an abelian subring of $\text{Der}(A)$. Further suppose that there are no Γ - invariant subrings of A of rank $< n$. Then if $0 \neq \xi \in \Gamma$, $\ker \xi = 0$.

PROOF

Put $V = \mathbb{Q} \otimes_{\mathbb{Z}} A$, a vector space over $\mathbb{Q}$ of dimension n. Γ acts on V by

$$\gamma(q \otimes a) = q \otimes \gamma(a)$$

where $q \in \mathbb{Q}$, $a \in A$, $\gamma \in \Gamma$.

Let W be a Γ - invariant subspace with $\dim W = r < n$ and with basis $1 \otimes x_1, \ldots, 1 \otimes x_r$ say. We can consider A as being embedded in V (as $1 \otimes A$). $W \cap A$ is Γ - invariant of rank r (it contains $1 \otimes x_1, \ldots, 1 \otimes x_r$) and so $r = 0$ and V is Γ - irreducible in the usual sense.

Suppose $\xi \in \Gamma$ and $\det \xi = 0$. Put

$$U = \{ v \in V \mid \xi(v) = 0 \}$$

If $U \neq 0$ then U is a subspace and if $u \in U$, $\alpha \in \Gamma$ then

$$(u\alpha)\xi = (u\xi)\alpha = 0 \cdot$$

(since $\Gamma \in \mathfrak{M}$).

So U is Γ - invariant and $U = V$, so $\xi = 0$. Thus if $\xi \neq 0$, $\det \xi \neq 0$ and $\ker \xi = 0$. ∎

<u>LEMMA 6.3.2</u>

Suppose $L \in \mathfrak{M}_0^*$ and L is torsion. Then every finite set of elements of L lies in a finite characteristic ideal of L.

PROOF

Suppose $L \in \mathfrak{M}_0^*$ is torsion, then $L = \bigoplus_p L_p$ where each L_p^* is a direct sum of finitely many C_{p^k} and C_{p^∞} groups.

Let $x_1, \ldots, x_n \in L$ with each x_i of order m_i. Let $m = m_1 m_2 \ldots m_n$. Clearly m involves only finitely many primes and so

$$L[m] = \{ x \in L \mid mx = 0 \}$$

is finite and characteristic by lemma 1.3.4 .

LEMMA 6.3.3

Let $L \in \mathcal{H} \cap \mathfrak{U}_0^*$. Suppose $H \leq L$ and $H \cong L$ then $|L : H| < \infty$.

PROOF

This follows immediately from a result of Čarin [5] p899 Theorem 2 which states that if A is a torsion free abelian group of finite rank and $B \leq A$ and $B \cong A$ then $|A : B| < \infty$.

LEMMA 6.3.4

Let $H \lhd L$, $H \in E\mathfrak{U}_0$, $L/H \in \mathfrak{U} \setminus \mathfrak{U}_{-1}$. Then L contains a free abelian subring of infinite rank.

PROOF

Case (i) $H = 0$

Then $L \in \mathfrak{U} \setminus \mathfrak{U}_{-1}$. Let $T = \tau(L)$ then $r_0(L/T)$ is infinite. So there exists an infinite $\mathbb{Z}$- linearly independent set $x_1 + T$, $x_2 + T$,..... of elements of L/T . Consequently x_1 , x_2 ,..... are $\mathbb{Z}$- linearly independent in L and so generate a free abelian subring of L of infinite rank.

In view of this case, since L/H will always contain a free abelian subring of infinite rank we may assume without loss of generality that it is such a ring.

Case (ii) $H \in \mathcal{H} \cap \mathfrak{U}_0$ and $H \leq Z_1(L)$.

Let A be a maximal abelian subring of L with $A \geq H$. Let $r_0(H) = n$ say. Suppose $r_0(A) = m$ ($\geq n$).

By the maximality of A, $A = C_L(A)$ and since $L/H \in \mathfrak{A}$ we have $A \lhd L$. Thus L/A may be considered a subring of $\mathrm{Der}(A)$. Let $x_1, x_2, \ldots, x_n$ be a maximal $\mathbb{Z}$- linearly independent set in H and extend it to one in A. Then if $u \in L$

$$[x_i, u] = \begin{cases} 0 & i \leq n \\ 0 \bmod H & i > n \end{cases}$$

(since $H \leq Z_1(L)$ and L is metabelian).

Now consider A as being embedded in $V = \mathbb{Q} \otimes_{\mathbb{Z}} A$ a vector space over $\mathbb{Q}$ of dimension m, with basis $1 \otimes x_1, \ldots, 1 \otimes x_m$. Define an action of L on V by

$$(q \otimes a)^u = q \otimes a^u \quad , \; q \in \mathbb{Q} \; , \; a \in A, \; u \in L$$

This action is represented by a matrix over $\mathbb{Q}$ of the form

$$\begin{array}{cc} \left(\begin{array}{c|c} 0 & 0 \\ \hline ? & 0 \end{array}\right) & \begin{array}{l} \updownarrow n \\ \updownarrow m-n \end{array} \\ \begin{array}{cc} \leftarrow n \rightarrow & \leftarrow m-n \rightarrow \end{array} & \end{array}$$

(Note that an additive group of matrices of this form forms an abelian Lie ring under the usual commutation).

L/A is isomorphic to an abelian Lie ring of matrices of this form. Thus L/A is isomorphic to an abelian subring of $M_n(\mathbb{Q})_L$ (where $M_n(\mathbb{Q})$ is the ring of $n \times n$ matrices over $\mathbb{Q}$). Thus $L/A \in \mathfrak{A}_0$ and so $r_0(L) < \infty$, a contradiction.

<u>Case (iii)</u> $H \in \mathcal{H} \cap \mathfrak{A}_0$ and is rank irreducible in the sense that if $K \leq H$, $K \lhd L$ with $r_0(K) < r_0(H)$ then $K = 0$.

By case (ii) we can assume that $H \nleq Z_1(L)$.

Choose $x \in L$ such that $[H, x] \neq 0$. Now consider the map $\xi : H \longrightarrow H$, $h \longmapsto [h, x]$.

This is a derivation of H. Consider $\Gamma = \langle \xi \rangle \leq \mathrm{Der}(H)$. Then there are no Γ - invariant subrings of A of rank less than n (by the initial hypotheses), so by lemma 6.3.1 $\ker \xi = 0$. Hence $[H, x] \cong H$ (both are abelian). Thus by lemma 6.3.3 $|H : [H, x]| = k < \infty$.

Suppose $L/H = \bigoplus_i \langle x_i + H \rangle$ (since we are assuming L/H is free abelian of infinite rank). For each i $[x_i, x] \in H$ (since $L/H \in \mathfrak{U}$). Thus

$$[kx_i, x] = k[x_i, x] \in [H, x]$$

So there exists $h_i \in H$ such that

$$[kx_i, x] = [h_i, x] \qquad \text{for all } i.$$

i.e. $[(kx_i - h_i), x] = 0$

Put $y_i = kx_i - h_i$ for all i. Take $A = \langle y_1, y_2, \dots \rangle$

Then $A \leq C_L(x)$ and $C_L(x) \cap H = 0$ so $A \cap H = 0$.

Thus

$$A \cong (A + H)/H \cong \bigoplus_i \langle kx_i + H \rangle$$

which is free abelian of infinite rank.

<u>Case (iv)</u> $H \in \mathfrak{T} \cap \mathfrak{U}$

Suppose (as above) that $L/H \cong \bigoplus_i \langle x_i + H \rangle$. We will now construct $y_1, y_2, \dots$ such that $y_i = k_i x_i$ $k_i \neq 0$, and $[y_i, y_j] = 0$ for all i, j.

Now $[x, y] \in H$ for all $x, y \in L$ since $L/H \in \mathfrak{U}$.

Take $y_1 = x_1$ and suppose that $y_1, \dots, y_n$ have been constructed. Now by lemma 6.3.2 $[y_i, x_{n+1}]$ $i = 1, \dots, n$ all lie in a finite characteristic subring $F \leq H$. So $F \lhd L$. Suppose $|F| = m$. Then for all $i = 1, \dots, n$

$$[y_i, mx_{n+1}] = m[y_i, x_{n+1}] = 0$$

Take $y_{n+1} = mx_{n+1}$ and y_{n+1} is as required.

Let $A = \langle y_1 , y_2 , \ldots\ldots \rangle$. The natural homomorphism $\xi : L \longrightarrow L/H$ maps A onto $(A + H)/H$ and $\ker \xi|_A = 0$ since L/H is torsion free (i.e. $nx_i \notin H$ for all n). Hence A is free abelian of infinite rank.

<u>Case (v)</u> $H \in \mathfrak{M}$

Let $T = \tau(H)$ and use induction on $r_o(H/T) = n$. If $n = 0$, then $T = H$ and case (iv) applies. If $n > 0$ choose K with $T \leq K \leq H$, $K \lhd L$ and K of maximal rank subject to $r_o(K/T) < r_o(H/T)$. We may assume K is torsion free (otherwise just factor out the torsion ideal). Case (iii) now applies and L/K has a subring A/K which is free abelian of infinite rank.

The induction hypothesis now shows that A has a free abelian subring of infinite rank.

<u>Case (vi)</u> The general case.

Now use induction on the derived length d of H . If $d = 1$ use case (v) . Suppose $d > 1$. Then by induction $L/H^{(d-1)}$ has a free abelian subring of infinite rank. But $H^{(d-1)} \in \mathfrak{M}$ and so case (v) finishes the argument. ■

Let $\mathfrak{X}$ be any class of torsion abelian Lie rings. Define a class $\overline{\mathfrak{X}}$ by :-

$$A \in \overline{\mathfrak{X}} \quad \text{iff} \quad A \in \mathfrak{M} , \quad \tau(A) \in \mathfrak{X} \quad \text{and} \quad r_o(A/\tau(A)) < \infty$$

THEOREM 6.3.5

Let $\mathfrak{X}$ be a class of torsion abelian Lie rings such that

(i) $\mathfrak{X} \geqslant \mathfrak{F} \cap \mathfrak{A}$

(ii) $\mathfrak{X} = s\mathfrak{X}$

(iii) $\mathfrak{X} \leq \mathfrak{A}_0$

Then if $E\mathfrak{X}$ is $\mathfrak{A}$ - closed in $E\mathfrak{A}$ so is $E\overline{\mathfrak{X}}$.

PROOF

Let $L \in E\mathfrak{A}$ and suppose that all its abelian subrings lie in $\overline{\mathfrak{X}}$. We will use induction on the derived length d of L .

If $d = 1$ then $L \in \overline{\mathfrak{X}}$. If $d > 1$ then by induction we may assume $L^2 \in E\overline{\mathfrak{X}} \leqslant E\mathfrak{A}_0$

If $L/L^2 \notin \mathfrak{A}_{-1}$ then by lemma 6.3.4 L has a free abelian subring of infinite rank, a contradiction. Thus $L/L^2 \in \mathfrak{A}_{-1}$ and so $L \in E\mathfrak{A}_{-1}$. Let $T = \mathfrak{T}(L)$. By lemma 5.2.10 $L/T \in E(\mathfrak{H} \cap \mathfrak{A}_{-1}) \leqslant E\overline{\mathfrak{X}}$ (since clearly $\mathfrak{H} \cap \mathfrak{A}_{-1} \leqslant \overline{\mathfrak{X}}$). Now every abelian subring of T lies in $\overline{\mathfrak{X}} \cap \mathfrak{T} = \mathfrak{X}$ and so $T \in E\mathfrak{X}$.

Hence $L \in E\overline{\mathfrak{X}}$. ■

THEOREM 6.3.6

$E\mathfrak{A}_0$ is $\mathfrak{A}$ - closed in $E\mathfrak{A}$.

PROOF

Take $\mathfrak{X} = \mathfrak{A}_0 \cap \mathfrak{T}$. Let $L \in \mathfrak{T} \cap E\mathfrak{A}$ and suppose that all abelian subrings of L are in $\mathfrak{T} \cap \mathfrak{A}_0$. Now $L \in E\mathfrak{A}_0$ iff $L_p \in \mathrm{Min} \cap E\mathfrak{A}$ for all primes p . (Use induction on the derived length of L_p , together with the fact that if L_p has derived length d then $L_p^{(d-1)} \in \mathfrak{A}_0 \cap \mathfrak{T} \leqslant \mathrm{Min}$).

Since L_p is a direct factor of L, the abelian subrings of L_p are precisely the abelian subrings of L intersected with L_p. Hence Theorem 6.2.1 together with Theorem 6.3.5 gives the result. ∎

<u>THEOREM 6.3.7</u>

$E\mathfrak{A}_1$ is $\mathfrak{A}$ - closed in $E\mathfrak{A}$.

PROOF

Take $\mathfrak{X} = \mathfrak{A} \cap \mathrm{Min}$ in Theorem 6.3.5 and use Theorem 6.2.1 . ∎

<u>THEOREM 6.3.8</u>

$E\mathfrak{A}_2$ is $\mathfrak{A}$ - closed in $E\mathfrak{A}$.

PROOF

Take $\mathfrak{X} = \mathfrak{A} \cap \mathfrak{F}$ in Theorem 6.3.5 and use Theorem 6.2.9 . ∎

CHAPTER 7 THE SUBIDEAL INTERSECTION PROPERTY

We now have a short chapter in which we examine a class of Lie rings which satisfy the subideal intersection property (i.e. an arbitrary intersection of subideals is always a subideal). It turns out that we cannot learn as much about the Lie ring situation as is possible in the group theoretic counterpart (cf Robinson [30](7.1)).

This sort of result (together with the lack of coalescence theorems for example) would seem to imply that the concept of a subideal in the study of Lie rings is not as powerful a tool as the subnormal subgroup in group theory.

§ 7.1 THE SUBIDEAL INTERSECTION PROPERTY

Let $H \leq L$, then the ideal closure series of H in L is defined as follows ; $H^{L,0} = L$ and inductively

$$H^{L,i+1} = \langle H^{H^{L,i}} \rangle$$

the smallest ideal of $H^{L,i}$ which contains H . We refer to $H^{L,i}$ as the i^{th} ideal closure of H in L . It is easy to see that H si L iff H equals some term of its ideal closure series in L ($H \lhd^n L$ iff H equals the n^{th} ideal closure).

We say $L \in \mathcal{L}$ iff the intersection of an arbitrary collection of subideals of L is a subideal itself. We also say that L has the subideal intersection property.

LEMMA 7.1.1

$L \in \mathcal{L}$ iff for each $H \leq L$ there exists a nonnegative

integer $n = n(H)$ such that
$$H^{L,n} = H^{L,n+1} = \ldots.$$

PROOF

Let $L \in \mathcal{L}$ and $H \leq L$. For each i, $H^{L,i} \lhd^{i} L$. So $H \leq H^{L,\omega}$ si L. Suppose that $H^{L,\omega} \lhd^{r} L$, then since $H \leq H^{L,\omega}$ we get
$$H^{L,r} \leq (H^{L,\omega})^{L,r} = H^{L,\omega} \leq H^{L,r}$$
Hence $H^{L,r} = H^{L,r+1} = \ldots..$

Now let L satisfy the condition and let $H = \bigcap_{\lambda\in\Lambda} H_{\lambda}$ where H_{λ} si L for all $\lambda \in \Lambda$. For some $n \geq 0$
$$H^{L,n} = H^{L,n+1} = \ldots.$$
Hence if $H_{\lambda} \lhd^{r_{\lambda}} L$ then $H^{L,n} \leq H_{\lambda}^{L,r_{\lambda}} = H_{\lambda}$ for all λ so that $H^{L,n} = H$ and $H \lhd^{n} L$. ∎

LEMMA 7.1.2

Suppose $A \lhd L$, $B \leq L$ and $L = A + B$ and $[A, [L, B]] = 0$. Then
$$B^{L,i} = B + [A,_{i} B]$$

PROOF

We prove this by induction on i. If $i = 0$ then
$$B^{L,0} = B + A = L$$
and the result is true.

Suppose $i > 0$ and the result is proved for $i - 1$. Then $B^{L,i}$ is the smallest ideal of $B^{L,i-1} = B + [A,_{i-1} B]$ containing B.

Clearly $B \leq B + [A,_{i} B] \leq B^{L,i}$. Also
$$[B + [A,_{i-1} B], B + [A,_{i} B]]$$
$$= [B, B] + [A,_{i} B]$$
$$+ [[A,_{i-1} B], [A,_{i} B]] + [B, [A,_{i} B]]$$

$$\leq B + [A,_i B]$$

(Since $[[A,_{i-1} B], [A,_i B]] \leq [A, [L, B]] = 0$).

Thus $(B + [A,_i B]) \lhd B^{L,i-1}$ and the result is proved. ∎

LEMMA 7.1.3

Let L be an abelian Lie ring and $\varphi \in \mathrm{End}(L)$. Suppose there exists $k > 0$ such that

$$\varphi^k(L) = \varphi^{k+1}(L) = \ldots\ldots = I \text{ say}$$

and $\ker^k(\varphi) = \ker^{k+1}(\varphi) = \ldots\ldots = K$ say.

Then $L = I \oplus K$.

PROOF

This is just a special case of Fitting's lemma (Scott [34] p79). ∎

LEMMA 7.1.4

Suppose L is a Lie ring, $L \in \mathrm{Max}-\lhd$ and all chief factors of L are finite. Let $A \lhd L$, $A \in \mathfrak{A}$. If $x \in L$ such that $\langle x, A\rangle \in \mathfrak{L}$ and $x + A \in Z_1(L/A)$ then there exists a positive integer m such that

$$\langle mx, A\rangle \in \mathfrak{N}$$

PROOF

Suppose the result is false. Then $\langle x\rangle \cap A = 0$. Let ε_t be the endomorphism of A given by

$$a \longmapsto [a, tx] \quad , \quad t > 0$$

Then $a \in \ker(\varepsilon_t^i)$ iff $[a,_i tx] = 0$. Now $x + A \in Z_1(L/A)$ and $A \in \mathfrak{A}$ so $[A, [L, x]] = 0$. Thus $\ker(\varepsilon_t^i) \lhd L$ for if $a \in \ker(\varepsilon_t^i)$ and $y \in L$ then

$$[[a, tx], y] = -[[y, a], tx] + [[y, tx], a]$$
$$= [[a, y], tx]$$

Now $\ker(\varepsilon_t) \leq \ker(\varepsilon_t^2) \leq \ldots\ldots \leq A$

and $L \in \text{Max}-\lhd$ so there exists an integer s such that

$$\ker(\mathcal{E}_t^s) = \ker(\mathcal{E}_t^{s+1}) = \ldots\ldots = K_t \quad \text{say} .$$

Now easily $\langle mx , A\rangle \lhd \langle x , A\rangle \in \mathcal{L}$ which implies that $\langle mx , A\rangle \in \mathcal{L}$. Hence by lemma 7.1.1 there exists an integer r such that

$$\langle tx\rangle^{H,r} = \langle tx\rangle^{H,r+1} = \ldots\ldots$$

where $H = \langle tx , A\rangle$.

By lemma 7.1.2

$$\begin{aligned} \langle tx\rangle^{H,i} &= \langle tx\rangle + [A ,_i tx] \\ &= \langle tx\rangle + \mathcal{E}_t^i(A) \end{aligned}$$

So $\mathcal{E}_t^r(A) = \mathcal{E}_t^{r+1}(A) = \ldots\ldots = J_t$ say ($\langle x\rangle \cap A = 0$).

By lemma 7.1.3 , $A = J_t \oplus K_t$.

By $\text{Max}-\lhd$ we can choose $t > 0$ so that K_t is maximal.

If $K_t = A$ then $\langle tx , A\rangle \in \mathfrak{N}$, a contradiction. Hence $K_t < A$. By $\text{Max}-\lhd$ we can choose an $A_1 \lhd L$ maximal with respect to $K_t \leq A_1 < A$. Then A/A_1 is a chief factor of L so $A/A_1 \in \mathfrak{F}$. For some $m > 0$, mtx centralises A/A_1 so

$$J_{tm} \leq [A , tmx] \leq A_1$$

Now $K_t \leq K_{tm}$ and by the maximality of K_t , $K_t = K_{tm}$. Hence $A = J_{tm} \oplus K_{tm} \leq A_1 < A$. This final contradiction completes the proof. ∎

LEMMA 7.1.5

Let L be a Lie ring, $A \lhd L$, A torsion free with $L = \langle x , A\rangle$. If there exists $m > 0$ such that $\langle mx , A\rangle \in \mathfrak{N}$ then $L \in \mathfrak{N}$.

PROOF

If $\langle mx , A\rangle \in \mathfrak{N}$ then there exists $r > 0$ such that
$$[A,_r \langle mx\rangle] = 0$$
i.e.
$$m^r [A,_r \langle x\rangle] = 0$$
But A is torsion free which implies
$$[A,_r \langle x\rangle] = 0$$
Hence $\langle x , A\rangle \in \mathfrak{N}$. ∎

THEOREM 7.1.6

Let L be a Lie ring with $L \in \mathfrak{G} \cap E\mathfrak{A} \cap \mathfrak{H}$. Then if $L \in \mathfrak{L}$ then $L \in \mathfrak{N}$.

PROOF

If $L \in \mathfrak{H} \cap E\mathfrak{A}$ then by lemma 5.2.10 L has a finite characteristic series with torsionfree abelian factors. Let
$$0 < L_1 < \ldots\ldots < L_n = L$$
be such a series. We will use induction on the length n. Suppose the theorem is proved for $n - 1$. Then
$$L/L_1 \in \mathfrak{G} \cap E\mathfrak{A} \cap \mathfrak{H} \cap \mathfrak{L}$$
and so by the induction hypothesis $L/L_1 \in \mathfrak{N}$. Thus $L \in \mathfrak{G} \cap \mathfrak{A}\mathfrak{N}$.

Let N be the Fitting ideal of L. Then $N \geq L_1$, and N is nilpotent by lemma 3.1.2. Let $x + N \in Z_1(L/N)$. Then $\langle x , N\rangle \triangleleft L$ and so $\langle x , N\rangle \in \mathfrak{L}$. Apply Lemma 7.1.4 to L/N^2 and then for some $m > 0$ $\langle mx , N\rangle / N^2$ is nilpotent. Now N is nilpotent and so $\langle mx , N\rangle$ is nilpotent by lemma 3.1.1. N is torsion free and so by lemma 7.1.5 $\langle x , N\rangle \in \mathfrak{N}$ and hence by the definition of N, $\langle x , N\rangle \leq N$ and $x \in N$ and so $Z_1(L/N) = 0$. Thus $L = N$ since $L/N \in \mathfrak{N}$ and the result is proved. ■

PROPOSITION 7.1.7

If $L \in \mathfrak{G} \cap E\mathfrak{A}_0 \cap \mathfrak{L}$ then $L \in \mathfrak{F}\mathfrak{N}$.

PROOF

Let L have derived length $d > 1$ and let $A = L^{(d-1)}$. By induction on d , $L/A \in \mathfrak{F}\mathfrak{N}$ and since L/A is f.g. it is polycyclic.

Thus $L \in \mathfrak{G} \cap \mathfrak{A}(F\mathfrak{C})$ and as in Theorem 2.2.6 $\tau(L) \in \mathfrak{E}$ but $\tau(L) \in E\mathfrak{A}_0$ and so $\tau(L) \in \mathfrak{F}$. Then $L/\tau(L) \in \mathfrak{G} \cap E\mathfrak{A}_0 \cap \mathfrak{H} \cap \mathfrak{L}$ and so by Theorem 7.1.6 $L/\tau(L) \in \mathfrak{N}$. Hence $L \in \mathfrak{F}\mathfrak{N}$.
(In fact what we have shown is that L is polycyclic). ■

Let $L \in \mathfrak{K}_i$ ($i \geqslant 0$) if H si L implies that the subideal index of H in L is less than or equal to i . (The subideal index of H in L is the length of the ideal closure series of H in L).

Define

$$\mathfrak{K} = \bigcup_{i \geq 0} \mathfrak{K}_i$$

This is the class of all Lie rings having an upper bound on their subideal indices.

LEMMA 7.1.8

$L \in \mathfrak{K}$ iff given $H \leq L$ there exists a nonnegative integer n (independent of H) such that

$$H^{L,n} = H^{L,n+1} = \ldots .$$

PROOF

As for lemma 7.1.1 . ∎

LEMMA 7.1.9

Let $N \triangleleft L$ and assume N has a subideal composition series of finite length. If $L/N \in \mathfrak{K}$ then $L \in \mathfrak{K}$.

PROOF

Let m be the subideal composition length of N and let $L/N \in \mathfrak{K}_r$. If H si L then certainly $H + N \triangleleft^r L$. Let s be the subideal index of H in $H + N$ and write

$$H = H_s < \ldots\ldots < H_o = H + N$$

for the ideal closure series of H in $H + N$. If $N_i = H_i \cap N$ then $H_i = H_i \cap (H + N) = H + N_i$. Since $H_{i+1} < H_i$ we have $N_{i+1} < N_i$ and also of course $N_{i+1} \triangleleft N_i$. The series

$$H \cap N = N_s < N_{s-1} < \ldots\ldots < N_o = N$$

may be refined to a subideal composition series of N and the Jordan - Holder Theorem (cf Higgins [17]) implies $s \leq m$. Thus $H \triangleleft^m H + N$. Hence $H \triangleleft^{m+r} L$ and $L \in \mathfrak{K}_{m+r}$. ∎

THEOREM 7.1.10

$$L \in \mathfrak{G} \cap {}_E\mathfrak{A}_o \cap \mathfrak{L} \text{ iff } L \in \mathfrak{G} \cap (\mathfrak{F} \cap {}_E\mathfrak{A})\mathfrak{N}$$

PROOF

($\Rightarrow$) Proposition 7.1.7 .

($\Leftarrow$) By lemma 7.1.8 $\mathfrak{K} \leq \mathfrak{L}$. Now clearly $\mathfrak{N} \leq \mathfrak{K}$, so by lemma 7.1.9 $\mathfrak{F}\mathfrak{N} \leq \mathfrak{K} \leq \mathfrak{L}$. ∎

COROLLARY 7.1.11

$L \in \mathfrak{G} \cap {}_E\mathfrak{A}_o \cap \mathfrak{L}$ iff L can be embedded in the direct sum of a finite soluble Lie ring and a f.g. torsion free nilpotent Lie ring

PROOF

Let $L \in \mathcal{G} \cap (\mathcal{F} \cap E\mathcal{U})\mathcal{N}$. Then $L/\tau(L) \in \mathcal{G} \cap \mathcal{H} \cap \mathcal{N}$.

Now $L \in E\mathcal{C}$ so $\tau(L) \in \mathcal{F}$ and there exists $N \lhd L$ such that $N \in \mathcal{H}$ and $L/N \in \mathcal{F}$. Then $\tau(L) \cap N = 0$ and so the mapping

$$y \longmapsto (y + N, y + \tau(L))$$

is a monomorphism of $L \longrightarrow (L/N) \oplus (L/\tau(L))$ and the result follows. ∎

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REFERENCES

[1] R. K. Amayo — 'Infinite dimensional Lie algebras' Ph.D. Thesis (University of Warwick) 1972 , University of Warwick Duplicated Notes .

[2] R. K. Amayo, I. N. Stewart — 'Finitely generated Lie algebras' J. London Math. Soc. (2), 5 (1972) 697 - 703 .

[3] N. Bourbaki — 'Commutative Algebra' Addison Wesley, 1972 .

[4] N. Bourbaki — 'Groupes et Algèbres de Lie' Hermann, 1960 .

[5] V. S. Čarin — 'On soluble groups of type A_4 ' Mat. Sb. 52 (1960) 895 - 914 .

[6] V. S. Čarin — 'On soluble groups of type A_3 ' Mat. Sb. 54 (1961) 489 - 499 .

[7] C. W. Curtis — 'Noncommutative extensions of Hilbert rings' Proc. Amer. Math. Soc. 4 (1953) 945 - 955 .

[8] J. W. Fisher — 'Nil subrings of endomorphism rings of modules' Proc. Amer. Math. Soc. 34 (1972) 75 - 78 .

[9] L. Fuchs — 'Abelian Groups' Pergamon Press, 1960.

[10] L. Fuchs — 'Infinite Abelian Groups Vol I ' Academic Press, 1970 .

[11] K. W. Gruenberg 'Residual properties of infinite soluble groups'
Proc. London Math. Soc. (3) 7 (1957)
29 - 62 .

[12] P. Hall 'Finiteness conditions for soluble groups'
Proc. London Math. Soc. (3) 4 (1954)
419 - 436 .

[13] P. Hall 'On the finiteness of certain soluble groups'
Proc. London Math. Soc. (3) 9 (1959)
595 - 622 .

[14] P. Hall 'The Frattini subgroups of finitely generated groups'
Proc. London Math. Soc. (3) 11 (1961)
327 - 352 .

[15] C. R. Hajarnavis 'Orders in noncommutative quotient rings'
Ph.D. Thesis (University of Leeds)
1969 , University of Warwick
Duplicated Notes .

[16] B. Hartley 'Locally nilpotent ideals of a Lie algebra'
Proc. Cambridge Philos. Soc. 63 (1967)
257 - 272 .

[17] P. J. Higgins 'Group with multiple operators'
Proc. London Math. Soc. (3) 6 (1956)
366 - 416 .

[18] G. Higman 'A remark on finitely generated nilpotent groups' Proc. Amer. Math. Soc. 6 (1955) 284 - 285 .

[19] N. Jacobson 'Lie algebras' Interscience, 1962 .

[20] L. A. Kaloujnine 'Über gewisse Beziehungen zwischen einer Gruppe und ihren Automorphismen' Berliner Mat. Tagung. (1953) 164 - 172 .

[21] I. Kaplansky 'Infinite Abelian Groups' University of Michigan Press, 1971.

[22] M. I. Kargapolov 'On soluble groups of finite rank' Alg. i. Log. 1 (1962) 37 - 44 .

[23] M. Lazard 'Sur les algèbres enveloppantes universelles de ceraines algèbres de Lie' Publications Scientifiques de l'Université d'Alger, Série A, Vol I fasc 2 (1954) 281 - 294 .

[24] S. Lang 'Algebra' Addison Wesley, 1965 .

[25] J. C. McConnell 'The intersection theorem for a class of noncommutative rings' Proc. London Math. Soc. (3) 17 (1967) 487 - 493 .

[26] J. C. McConnell 'Representations of solvable Lie algebras and the Gelfand-Kirillov conjecture' Preprint (University of Leeds).

[27] A. I. Mal'cev — 'On faithful representations of infinite groups by matrices'
Mat. Sb. 8 (1940) 405 - 422 .
Amer. Math. Soc. Trans. (2) 45 (1965) 1 - 18 .

[28] A. I. Mal'cev — 'On certain classes of soluble groups'
Mat. Sb. 28 (1951) 567 - 588 .
Amer. Math. Soc. Trans. (2) 2 (1956) 1 - 21 .

[29] S. Moran — 'The Lie ring associated with certain groups'
Państwowe Wydawnictwo Naukowe (Warszawa) 1965 .

[30] D. J. S. Robinson — 'Infinite Soluble and Nilpotent Groups'
Queen Mary College Mathematics Notes.

[31] P. Samuel — 'Apropos du théorème des unités'
Bull. Sci. Math. 90 (1966) 89 - 96 .

[32] E. Schenkman — 'Infinite Lie algebras'
Duke Math. J. 19 (1952) 529 - 535 .

[33] O. J. Schmidt — 'Infinite soluble groups'
Mat. Sb. 17 (1945) 145 - 162 .

[34] W. R. Scott — 'Group Theory'
Prentice Hall, 1964.

[35] J. P. Serre — 'Lie Algebras and Lie Groups'
Benjamin, 1965 .

[36] A. I. Širšov — 'On the representations of Lie rings in associative rings'
Uspekhi Mat. Nauk. 5 (1953) 173 - 175 .

[37] I. N. Stewart 'Infinite dimensional Lie algebras in the spirit of infinite group theory' Compositio Math. (22) 3 (1970) 313 - 331 .

[38] I. N. Stewart 'The Frattini structure of soluble Lie algebras' Unpublished .

[39] D. A. Towers 'A Frattini theory for algebras' Ph.D. Thesis (University of Leeds) 1972 .

[40] B. A. F. Wehrfritz 'A note on the residual properties of nilpotent groups' J. London Math. Soc. (2) 5 (1972) 1 - 7 .

[41] B. A. F. Wehrfritz 'Groups of automorphisms of soluble groups' Proc. London Math. Soc. (3) 20 (1972) 101 - 122 .

[42] O. Zariski, P. Samuel 'Commutative Algebra Vol 1 ' Princeton, 1958 .

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