Analytical solutions for tunnels of elliptical cross-section in rheological rock accounting for sequential excavation

H. N. Wang*[1], S. Utili[2], M. J. Jiang[3], P. He[3],

[1] School of Aerospace Engineering and Applied Mechanics, Tongji University, Shanghai, 200092, P.R. China

[2] School of Engineering, University of Warwick, Coventry, CV4 7AL, UK

[3] Department of Geotechnical Engineering, College of Civil Engineering, Tongji University, Shanghai, 200092, P.R. China

* Corresponding author

H. N. Wang

E-mail address: wanghn@tongji.edu.cn (H.N. Wang)

Abstract:

Time dependency in tunnel excavation is mainly due to the rheological properties of rock and sequential excavation. In this paper, analytical solutions for deeply buried tunnels with elliptical cross-section excavated in linear viscoelastic media are derived accounting for the process of sequential excavation. For this purpose, an extension of the principle of correspondence to solid media with time varying boundaries is formulated for the first time. An initial anisotropic stress field is assumed. To simulate realistically the process of tunnel excavation, solutions are developed for a time dependent excavation process with the major and minor axes of the elliptical tunnel growing from zero until a final value according to time dependent functions to be specified by the designers.

Explicit analytical expressions in integral form are obtained assuming the generalized Kelvin viscoelastic model for the rheology of the rock mass, with Maxwell and Kelvin models solved as particular cases.

An extensive parametric analysis is then performed to investigate the effects of various excavation methods and excavation rates. Also the distribution of displacement and stress in space at different times is illustrated. Several dimensionless charts for ease of use of practitioners are provided.

Key words: rheological rock; non-circular tunnel; analytical solution; sequential excavation.

List of symbols

A, A_0 and A_i $(i=1,2,\cdots,\infty)$	Coefficients in inverse conformal mapping
A_i^{jB} and A_{ij}^{kB}	Coefficients correlated to coordinates and material parameters of generalized Kelvin model in Appendix
A_i^{jM} and A_{ij}^{kM}	Coefficients correlated to coordinates and material parameters of Maxwell model in Appendix
a	Function of half major axis with respect to time
a_0	Initial value of half major axis (at time t=0)
a_1	Final value of half major axis
B_i $(i=1,2,\cdots,9)$	Coefficients in displacement solutions
$B_i^j\,(i=1,2;\ j=1,2)$	Terms defined in Eqs. (A.14) and (A.16)
b	Function of half minor axis with respect to time
b_0	Initial value of half minor axis (at time t=0)
b_1	Final value of half minor axis
C_i^B $(i=1,2)$	Coefficients correlated to material parameters of generalized Kelvin model in Appendix
C_1^M	Coefficients correlated to material parameters of Maxwell model in Appendix
$c(t)$	Parameter in conformal mapping (defined in Eq. (26))
D_i $(i=1,2)$	Coefficients in stress solutions (in Eq. (37))
$F_i^j\,(i=1,2;\ j=1,2)$	Terms defined in Eqs. (A.3), (A.4), (A.5) and (A.6)
f_0	Inverse conformal mapping with respect to variable z
f_1	Inverse conformal mapping with respect to variable z_1
G	Time-dependent relaxation shear modulus for viscoelastic model
G_e	Shear modulus of elastic problem
G_H	Shear elastic modulus of the Hookean element in the Generalized Kelvin model
G_K	Shear elastic modulus of the Kelvin element in Generalized Kelvin model
G_S	Permanent shear modulus of the generalized viscoelastic model: $G_S = G_{\mathrm{H}}G_{\mathrm{K}}/(G_{\mathrm{H}}+G_{\mathrm{K}})$
H	Function defined in Eq. (48)
I	Function defined in (A.7)
K	Time-dependent relaxation bulk modulus in the rock viscoelastic model
K_e	Bulk modulus of elastic problem
l	Number of items in inverse conformal mapping
$m(t)$	Parameter in conformal mapping (defined in Eq. (26))
n_j	Vector indicating the direction normal to the boundary
(n_χ, n_τ)	Local coordinates
n_r^K	Normalized excavation rate for generalized Kelvin model

n_r^M	Normalized excavation rate for Maxwell model
P_i ($i = 1,2,3$)	Prescribed time-dependent stresses at stress boundary
P_x (P_y)	Traction (surface force) along the x (y) direction on stress boundary
p_0	Vertical compressive stress at infinity
p_x (p_y)	Boundary tractions (surface forces) applied on the tunnel face to calculate the excavation induced displacements and stresses
q	Number of adopted points in determination of coefficients of inverse conformal mapping
R^*	Radius of axisymmetric problem used in normalization of displacements
S_σ (S_u)	Time-dependent stress (displacement) boundaries
s	Variable in the Laplace transform
s_{ij}^e, e_{ij}^e	Tensors of the stress and strain deviators of elastic case
s_{ij}^v, e_{ij}^v	Tensors of the stress and strain deviators of viscoelastic case
T_K	Retardation time of Kelvin component of generalized Kelvin viscoelastic model
T_M	Relaxation time of Maxwell viscoelastic model
t	Time variable (t=0 is the beginning of excavation)
t_1	End time of excavation
t'	Time variable (t' =0 is the time the initial pressure applied)
t_0'	Beginning time of excavation
u_i	Prescribed displacements at displacement boundary
$u_x^{(A)v}$ ($u_y^{(A)v}$)	Displacement corresponding to viscoelastic problem of case A (in Cartesian coordinates)
$u_x^{(C)v}$ ($u_y^{(C)v}$)	Excavation induced displacement for viscoelastic problem (in Cartesian coordinates)
$u_\chi^{(C)v}$ ($u_\tau^{(C)v}$)	Excavation induced displacement for viscoelastic problem (in local coordinates)
u_x^e (u_y^e)	Displacement along x (y) direction for elastic problems
u_x^s (u_y^s)	Prescribed displacement along x and y direction on displacement boundary
u_x^v (u_y^v)	Displacement along x (y) direction for viscoelastic problems
u_i^v (σ_{ij}^v)	Displacements (stresses) tensor for the viscoelastic problem
u_i^* (σ_{ij}^*)	Displacements (stresses) tensor obtained by replacing G_e with $s\mathscr{L}\left[G(t)\right]$ and K_e with $s\mathscr{L}\left[K(t)\right]$ in the general solution for the associated elastic problem
v_r	Cross-section excavation rate
$\boldsymbol{X}$	Position vector of a point on the plane
$\boldsymbol{X_0}$	Position vector of a point on the boundary
(x, y)	Cartesian coordinates
z	Complex variable: $z=x+iy$
z_A	Generic point on the boundary

z_0	Generic point on the time-dependent boundary at time t'
z_{01}	Point on time-dependent displacement boundary
z_{02}	Point on time-dependent stress boundary
z_1	Complex variable defined in Eq. (32)
z_{1j}	Boundary points in z_1 plane determined by Eq. (33) corresponding to ζ_j

Greek symbols

α	Angle in local coordinates between n_τ and x direction
δ	Dirac delta function
δ_{ij}	Unit tensor
γ	Function with respect to s obtained by replacing elastic moduli G_e with $s\mathscr{L}\left[G(t)\right]$ and K_e with $s\mathscr{L}\left[K(t)\right]$ in κ
ΔP_x (ΔP_y)	Prescribed stresses along the boundaries in calculation of excavation induced displacement and stresses
Δs_{ij}^v (Δe_{ij}^v)	Incremental stresses (strains) induced by the tunnel excavation
$\Delta u_x^v, \Delta u_y^v$	Excavation induced displacements of viscoelastic case (x, y direction)
$\Delta u_\chi^v, \Delta u_\tau^v$	Excavation induced displacements of viscoelastic case (n_χ, n_τ direction)
Δu_s^e	Radial displacement at the inner boundary of axisymmetric elastic problem with radius R^* and shear modulus G_S
Δu_{s0}^e	Radial displacement at the inner boundary of axisymmetric elastic problem with radius R^* used for normalization in Maxwell model and shear modulus G_H
$\Delta\sigma_x^v, \Delta\sigma_y^v, \Delta\sigma_{xy}^v$	Excavation induced stresses of viscoelastic case (x, y direction)
$\Delta\sigma_\chi^v, \Delta\sigma_\tau^v, \Delta\sigma_{\chi\tau}^v$	Excavation induced stresses of viscoelastic case (n_χ, n_τ direction)
ζ	Complex variable: $\zeta = \xi + \eta i$
ζ_j	Points in ζ plane determined by Eq. (34) corresponding to z_1
η	Imaginary part of ζ
η_K	Viscosity coefficient of the dashpot element in the generalized Kelvin model
κ	Material coefficient defined by Eq. (14)
λ	Ratio of horizontal and vertical stresses
ξ	Real part of ζ
(ρ, θ)	Polar coordinates
σ_{ij}^v (ε_{ij}^v)	Stress (strain) tensor for viscoelastic case
σ_{kk}^e (ε_{kk}^e)	Mean stress (strain) for elastic case
σ_{kk}^v (ε_{kk}^v)	Mean stress (strain) for viscoelastic case

σ_x^v, σ_y^v	Normal stress along x and y direction for viscoelastic case
σ_x^e, σ_y^e	Normal stress along x and y direction for elastic case
$\sigma_{xy}^v (\sigma_{xy}^e)$	Shear stress for viscoelastic (elastic) case
$\sigma_x^0, \sigma_y^0, \sigma_{xy}^0$	Initial normal and shear stresses at infinity
$\sigma_x^{(A)}, \sigma_y^{(A)}, \sigma_{xy}^{(A)}$	Stresses corresponding to viscoelastic problem of case A (in Cartesian coordinates)
$\sigma_x^{(C)}, \sigma_y^{(C)}, \sigma_{xy}^{(C)}$	Excavation induced stresses (in Cartesian coordinates)
$\sigma_\chi^{(A)}, \sigma_\tau^{(A)}, \sigma_{\chi\tau}^{(A)}$	Stresses corresponding to viscoelastic problem of case A(in local coordinates)
$\sigma_\chi^{(C)}, \sigma_\tau^{(C)}, \sigma_{\chi\tau}^{(C)}$	Excavation induced stresses (in local coordinates)
φ_1 and ψ_1	Two complex potentials in analysis of elasticity
φ_2 and ψ_2	Two potentials obtained by replacing elastic moduli G_e with $s\mathscr{L}\left[G(t)\right]$ and K_e with $s\mathscr{L}\left[K(t)\right]$ in φ_1 and ψ_1
$\varphi_1^{(A)}$ and $\psi_1^{(A)}$	Two complex potentials for the elastic problem A
$\varphi_1^{(B)}$ and $\psi_1^{(B)}$	Two complex potentials for the elastic problem B
$\varphi_1^{(C)}$ and $\psi_1^{(C)}$	Two complex potentials for calculating the excavation induced displacements and stresses in elastic case
ω	Conformal mapping determined in Eq. (25)

1. Introduction

Analytical solutions are invaluable to gather understanding of the physical generation of deformations and stresses taking place during the excavation of tunnels. Closed form solutions allow highlighting the fundamental relationships existing between the variables and parameters of the problem at hand, for instance between applied stresses and ground displacements. Moreover, although numerical methods such as finite element, finite difference and to a lesser extent boundary element are increasingly used in tunnel design, full 3D analyses for extended longitudinal portions of a tunnel still require long runtimes, so that the conceptual phase of the design process relies on 2D analytical models. In fact, analytical solutions allow performing parametric sensitivity analyses for a wide range of values of the design parameters of the problem so that preliminary estimates of the design parameters to be used in the successive phases of the design process can be obtained. In addition, they provide a benchmark against which the overall correctness of sophisticated numerical analyses performed in the final design stage can be assessed.

Most types of rocks including hard rocks exhibit time-dependent behaviors [Malan 2002], which induce gradual deformation over time even after completion of the tunnel excavation process. Elastic and elastoplastic models ignore the effect of time dependency which may contribute in some cases up to 70% of the total deformation [Sulem et al., 1987]. In case of sequential excavation, the observed time-dependent convergence is also a function of the interaction between the prescribed excavation steps and the natural rock rheology. Therefore, proper simulation of the whole sequence of excavation is of great importance for the determination of the optimal values of the tunnelling parameters to achieve optimal design [Tonon, 2010; Sharifzadeh et al., 2012]. Sequential excavation is a technique becoming increasingly popular for the excavation of tunnels with large cross-section in several countries (Tonon, 2010; Miura et al., 2003). For instance, 200 km of tunnels along the new Tomei and Meishin expressways in Japan, have been built via the so-called center drift advanced method.

This sequential excavation technique has been adopted by the Japanese authorities “as the standard excavation method of mountain tunnel” (Miura, 2003).

In this paper, the rock rheology is accounted for by linear viscoelasticity. The so called generalized Kelvin, Maxwell and Kelvin rheological models according to the classical terminology used in rock mechanics (Jaeger et al., 2013) will be considered. Unlike the case of linear elastic materials with constitutive equations in the form of algebraic equations, linear viscoelastic materials have their constitutive relations expressed by a set of operator equations. In general, it is very difficult to obtain analytical solutions for most of the viscoelastic problems, especially in case of time-dependent boundaries although some closed-form solutions have been developed [Brady et al., 1985; Gnirk et al., 1964; Ladanyi et al., 1984]. However, in all these works, only tunnels with circular cross-section are considered, with the excavation being assumed to take place instantaneously. In the literature, the process of sequential excavation is usually ignored since it prevents the use of the principle of correspondence which has been traditionally restricted to solid bodies with time invariant geometrical boundaries [Lee, 1955; Christensen, 1982; Gurtin et al., 1962]. However, recently, analytical methods have been introduced to obtain analytical solutions for circulars tunnels excavated in viscoelastic rock accounting for sequential excavation [Wang and Nie 2010; Wang and Nie 2011; Wang et al. 2013, Wang et al. 2014]. But for tunnels of complex cross-sectional geometries, (e.g. elliptic, rectangular, semi-circular, inverted U-shaped, circular with a notch, etc.), analytical solutions are available only in case of elastic medium [Lei et al., 2001; Exadaktylos et al., 2002; Exadaktylos et al., 2003], hence disregarding the influence of the time-dependent rheological behavior of the rock and sequential excavation. In this paper instead, an analytical solution is derived for sequentially excavated tunnels of non-circular (elliptical) cross-section in linearly viscoelastic rock subject to a non-uniform initial stress state. The stress field considered is anisotropic so that complex geological conditions can be accounted for. The solution is achieved employing complex variable theory and the Laplace transform.

Elliptical and horse-shoe sections with the longer axis in the vertical direction are rather common for railway tunnels (Steiner, 1996; Amberg, 2003; Anagnostou and Ehrbar, 2013) and caverns in rock, e.g. the East Side Access Project in New York (Wone et al., 2003). Sequential excavation is employed for these types of sections much more often than for circular sections since Tunnel Boring Machining is not available for non-circular sections. Also subway tunnels are often featured by elliptical or horse-shoe cross-sections (Hochmuth et al., 1987). Moreover, several road tunnels require an elliptical or nearly elliptical cross-section with the longer axis in the horizontal direction to minimize the excavation volume whilst meeting the geometrical constraints required for the construction of the road and related walk-ways (Miura et al., 2003). In Japan, elliptical sections are specifically prescribed for mountainous regions (Miura, 2003). Finally, elliptical sections can also be the result of ovalisation of circular sections in anisotropic rheological rock (Vu et al., 2013a; Vu et al., 2013b).

A limitation of the analytical solutions here proposed is due to the absence of lining in the cross-section considered. The presence of lining makes the problem mathematically intractable due to the consequent structure – ground interaction. Also in case of non-circular cross-sections the confinement convergence method cannot be applied due to the anisotropy of the displacement field. However, the analytical solutions here introduced can be employed to predict tunnel convergence to assess whether the presence of a lining would be necessary in the preliminary design phase. Also they allow obtaining a first estimate of the magnitude of the excavation induced displacement field. Moreover, for deeply buried tunnels, lining is often not necessary.

In the paper, analytical solutions are provided for a generic time dependent excavation process with the major and minor axes of the cross-section increasing monotonically over time according to a function to be specified by the designers. The analytical solutions have been derived in integral form for the case of a generalized Kelvin viscoelastic rock. The case of Maxwell and Kelvin models can be obtained as particular cases of the solution obtained for the generalized Kelvin model. To calculate the displacement and stress fields, numerical integration of the analytical expressions in integral form has been carried out. Then, a parametric study investigating the influence of various excavation methods, as well as excavation rates, on the excavation induced displacements and stresses are illustrated. Several dimensionless charts of results are plotted for the ease of use of practitioners.

2. Formulation of the problem

The present study focuses on the excavation of an elliptical tunnel in a rheological rock mass. In the analysis, the following assumptions were made:

(1) The rock mass is considered to consist of homogeneous, isotropic, and linearly viscoelastic material under isothermal conditions.

(2) The initial stress field in the rock mass is idealized as a vertical stress p_0 and horizontal stress λp_0, where λ is the ratio of horizontal and vertical stresses, as shown in Figure 1.

(3) The tunnel is deeply buried, hence no linear variation of the stresses with depth is considered.

(4) The excavation speed is low enough that no dynamic stresses are ever induced so that stress changes occur in a quasi-static fashion at all times.

(5) The cross-section of the tunnel is sequentially excavated, that is, the half major and minor axes of the elliptical tunnel section, a and b respectively, are time-dependent. The tunneling process may be divided into two stages: the first (i.e. excavation) stage, spans from time $t=0$ to $t=t_1$, with t_1 being the end time of the cross-section excavation whilst the second stage runs from $t=t_1$ onwards. In the first stage, the size of the major and minor axes varies according to the time dependent functions, $a(t)$ and $b(t)$ respectively, that are likely to be discontinuous over time due to technological requirements since sequential excavations tend to occur step-like. So, an important feature of the analytical solutions provided in this paper is that they are applicable to any type of sequential excavations increasing either stepwise or continuously over time. The second stage spans from $t=t_1$ onwards, with the values of the major and minor elliptical axes being equal to $a(t=t_1)=a_1$ and $b(t=t_1)=b_1$, respectively. Note

that in case the ratio of the ellipse axes remains constant, the section grows homothetically, whereas if the ratio changes over time the shape of the section evolves too (for instance from an initial circular pilot tunnel to a final elliptical section). Since in most of the cases the shape of the cross-section changes over time, the general case of $m(t) = a(t)/b(t)$ will be considered.

In the analysis, the effect of the advancement of the tunnel along the longitudinal direction is not accounted for. The effect of tunnel advancement can easily be considered employing a fictitious pressure as shown in (Wang et al. 2014; Pan and Dong 1991), but it is here omitted for sake of simplicity in the derivation of the solution. So the cross-section considered in the analysis is located at a sufficient distance from the longitudinal tunnel face that stresses and strains are unaffected by three-dimensional effects. According to the aforementioned assumptions, the problem can be formulated as plane strain in the plane of the tunnel cross-section. This plane will be assumed to be of infinite size with an elliptical hole growing over time, subject to a uniform anisotropic stress field, and made of a viscoelastic medium. Since the hole is not circular, polar coordinates are no longer advantageous for the derivation of the analytical solution. Hence, in this paper Cartesian coordinates (x, y) are employed for the derivation of the solution (see Figure 1) which are then transformed into polar coordinates (ρ, θ) to show that the (already known) solution for a circular cross-section can be obtained as a particular case. A system of local coordinates (n_χ, n_τ) is also employed in the paper, with n_χ and n_τ being the normal and tangential directions respectively along the elliptical boundary (see Figure 1). In the following analysis, sign convention is defined as positive for tension and negative for compression.

3. Derivation of the analytical solution

In order to find analytical solutions for boundary value problems of linear viscoelasticity, the most widely used methods are based on the Laplace transform of the differential equations and boundary condition equations governing the problem, which in this case are time-dependent since sequential excavation is accounted for. In Lee [1955] the classical form of the correspondence principle between linear elastic and linear viscoelastic solutions for boundary value problems is described. The principle establishes a correspondence between a viscoelastic solid and an associated fictitious elastic solid of the same geometry. But until now, this method has been applied only to solid bodies with time invariant boundaries because when boundaries are functions of time, the boundary conditions cannot be Laplace transformed. In this section, we describe an extension of the principle to time varying stress boundaries that will be employed to achieve the sought analytical solution for the sequential excavation of tunnels of elliptical cross-sections in viscoelastic rock. In the following the term "general solution" is used to indicate the mathematical solution to the set of differential equations ruling the problem without any boundary conditions imposed whereas "particular solution" indicates a solution which satisfies both the set of differential equations ruling the problem and the boundary conditions.

3.1 Solving procedure

Assuming the Einstein's convention (i.e. repeated indices indicate summation), the constitutive equations of a general linear viscoelastic solid can be expressed in the form of convolution integrals, as shown below:

$$\begin{aligned} s_{ij}^{v}(\boldsymbol{X},t) &= 2G(t) * \mathrm{d}e_{ij}^{v}(\boldsymbol{X},t), \\ \sigma_{kk}^{v}(\boldsymbol{X},t) &= 3K(t) * \mathrm{d}\varepsilon_{kk}^{v}(\boldsymbol{X},t). \end{aligned} \tag{1}$$

where $\boldsymbol{X}$ is the position vector and s_{ij}^{v} and e_{ij}^{v} are the tensors of the stress and strain deviators, respectively for the viscoelastic case (here the superscript 'v' stands for viscoelastic), defined as:

$$\begin{aligned} s_{ij}^{v} &= \sigma_{ij}^{v} - \frac{1}{3}\delta_{ij}\sigma_{kk}^{v}, \\ e_{ij}^{v} &= \varepsilon_{ij}^{v} - \frac{1}{3}\delta_{ij}\varepsilon_{kk}^{v}. \end{aligned} \tag{2}$$

with σ_{ij} and ε_{ij} being the tensors of stresses and strains respectively. $G(t)$ and $K(t)$ in Eq. (2), represent the shear and bulk relaxation modulus, respectively. The asterisk ($*$) in Eq. (1) indicates the convolution integral, defined as:

$$f_1(t) * df_2(t) = f_1(t) \cdot f_2(0) + \int_0^t f_1(t-\tau)\frac{df_2(\tau)}{d\tau}d\tau. \tag{3}$$

The Laplace transform of Eq. (1) yields the following:

$$\mathscr{L}\left[s_{ij}^{v}(\boldsymbol{X},t)\right]=2s\mathscr{L}\left[G(t)\right]\cdot\mathscr{L}\left[e_{ij}^{v}(\boldsymbol{X},t)\right],$$
$$\mathscr{L}\left[\sigma_{kk}^{v}(\boldsymbol{X},t)\right]=3s\mathscr{L}\left[K(t)\right]\cdot\mathscr{L}\left[\varepsilon_{kk}^{v}(\boldsymbol{X},t)\right]. \qquad (4)$$

where $\mathscr{L}\left[f(t)\right]$ is a function of the variable s defined in the Laplace transform of the time function $f(t)$, defined as:

$$\mathscr{L}\left[f(t)\right]=\int_{0}^{\infty}\exp^{-st}f(t)dt\,, \qquad (5)$$

The Laplace transform of the linear elastic constitutive equations is as follows (here the superscript 'e' stands for elastic):

$$\mathscr{L}\left[s_{ij}^{e}(\boldsymbol{X},t)\right]=2G_{e}\mathscr{L}\left[e_{ij}^{e}(\boldsymbol{X},t)\right],$$
$$\mathscr{L}\left[\sigma_{kk}^{e}(\boldsymbol{X},t)\right]=3K_{e}\mathscr{L}\left[\varepsilon_{kk}^{e}(\boldsymbol{X},t)\right]. \qquad (6)$$

with G_e and K_e being the elastic shear and bulk modulus, respectively. Note that Eq. (4) is obtained from Eq. (6) by replacing G_e with $s\mathscr{L}\left[G(t)\right]$ and K_e with $s\mathscr{L}\left[K(t)\right]$. Therefore, the general solution for a viscoelastic isothermal problem, satisfying the set of differential equations governing static equilibrium, kinematic compatibility and the constitutive relationship of the rock in the time-dependent domain, may be obtained by replacing G_e with $s\mathscr{L}\left[G(t)\right]$ and K_e with $s\mathscr{L}\left[K(t)\right]$ in the general solution for the associated elastic problem. Then, performing the Laplace inverse transform, we obtain:

$$u_{i}^{v}(\boldsymbol{X},t)=\mathscr{L}^{-1}\left[\mathscr{L}\left(u_{i}^{*}(\boldsymbol{X},t,s)\right)\right] \qquad (7)\ (7a)$$

$$\sigma_{ij}^{v}(\boldsymbol{X},t)=\mathscr{L}^{-1}\left[\mathscr{L}\left(\sigma_{ij}^{*}(\boldsymbol{X},t,s)\right)\right], \qquad (7b)$$

where $u_{i}^{*}(\boldsymbol{X},t,s)$ and $\sigma_{ij}^{*}(\boldsymbol{X},t,s)$ are the displacements and stresses respectively obtained by replacing G_e with $s\mathscr{L}\left[G(t)\right]$ and K_e with $s\mathscr{L}\left[K(t)\right]$ in the general solution for the associated elastic problem and $\mathscr{L}^{-1}[g(s)]$ indicates the inverse Laplace transform, defined as:

$$\mathscr{L}^{-1}[g(s)]=\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}g(s)\exp^{st}dt\,. \qquad (8)$$

In Eq.(7), the general viscoelastic solution contains yet unknown functions of time t, which have to be determined by imposition of the boundary conditions. Displacement boundary conditions may be expressed as follows:

$$u_i^v(\boldsymbol{X}_0,t)=u_i\left(t\right), \quad \text{with} \quad \boldsymbol{X}_0\in S_u(t), \tag{9)(9b}$$

and stress boundary conditions as:

$$\sigma_{ij}^v(\boldsymbol{X}_0,t)n_j=P_i\left(t\right), \quad \text{with} \quad \boldsymbol{X}_0\in S_\sigma(t), \tag{9a}$$

where n_j is a vector indicating the direction normal to the boundary, $\boldsymbol{X}_0$ is the position of a point on the boundary, $S_\sigma(t)$ and $S_u(t)$ are the boundary surfaces where stress and displacement conditions respectively are applied, and $P_i\,(t)$ and $u_i\,(t)$ are two prescribed functions of time. Unlike problems with time invariant geometrical boundaries, $\boldsymbol{X}_0$ and n_j in Eq. (9), are functions of time, hence they are not constant with respect to the Laplace transform, so that they cannot be taken out of the transform operator. Therefore, the relationship between the particular solution of the viscoelastic problem here examined and the solution of the associated elastic one is unknown. Replacing u_i^v and σ_{ij}^v with the expressions in Eq. (7), Eq. (9) can be rewritten as:

$$u_i^v(\boldsymbol{X},t)\Big|_{X=X_0}=\mathscr{L}^{-1}\left[\mathscr{L}\left(u_i^*(\boldsymbol{X},t,s)\right)\right]\Big|_{X=X_0}=u_i\,, \quad \boldsymbol{X}_0\in S_u(t), \tag{10) (10a}$$

$$\sigma_{ij}^v(\boldsymbol{X},t)n_j\Big|_{X=X_0}=\mathscr{L}^{-1}\left[\mathscr{L}\left(\sigma_{ij}^*(\boldsymbol{X},t,s)\right)\right]n_j\Big|_{X=X_0}=P_i\,, \quad \boldsymbol{X}_0\in S_\sigma(t). \tag{10b}$$

The system of equations (10) together with Eq. (7) define the set of equations to be satisfied by the particular solution that we seek. To find the solution, complex potential theory will be employed (see the next section).

3.2 Problem formulation

Complex potential theory has been widely used to analyze mathematical problems associated with underground constructions, especially in the analysis of non-circular openings. For a two dimensional (2D) elastic problem, displacements and stresses can be expressed in terms of two analytical functions of complex variable, *i.e.* $\varphi_1(z)$ and $\psi_1(z)$ with $z=x+iy$ and $i=\sqrt{-1}$, which are called potential functions. So stresses and displacements can be written as

(Muskhelishvili 1963):

$$2G_e(u_x^e + iu_y^e) = \kappa\varphi_1(z,t) - z\overline{\frac{\partial\varphi_1(z,t)}{\partial z}} - \overline{\psi_1(z,t)}, \tag{11}$$

$$\sigma_x^e + \sigma_y^e = 4\,\mathrm{Re}[\frac{\partial\varphi_1(z,t)}{\partial z}] \ , \tag{12}$$

$$\sigma_y^e - \sigma_x^e + 2i\sigma_{xy}^e = 2\left[\bar{z}\frac{\partial^2\varphi_1(z,t)}{\partial z^2} + \frac{\partial\psi_1(z,t)}{\partial z}\right] \ . \tag{13}$$

with x,y being Cartesian coordinates in the tunnel cross-section plane (see Figure 2) and

$$\kappa = \begin{cases} 1 + \dfrac{6G_e}{3K_e + G_e} & \text{in case of plane strains} \\ \dfrac{15K_e + 8G_e}{9K_e} & \text{in case of plane stresses} \end{cases} , \tag{14}$$

and $\overline{g(z,t)}$ is the conjugate of the complex function $g = g(z,t)$. The potentials $\varphi_1(z)$ and $\psi_1(z)$ in Eqs. (11-13) are time dependent since the geometric boundaries of our problem are time-dependent. According to the formulation of the problem illustrated in the previous section, the Laplace transforms of the equations ruling the viscoelastic problem are performed as follows:

$$\mathscr{L}\left(u_x^v\right) + i\mathscr{L}\left(u_y^v\right) = \frac{1}{2s\mathscr{L}\left[G(t)\right]}\mathscr{L}\left[\gamma(s)\varphi_2(z,s,t) - z\overline{\frac{\partial\varphi_2(z,s,t)}{\partial z}} - \overline{\psi_2(z,s,t)}\right] \tag{15}$$

$$\mathscr{L}\left(\sigma_x^v\right) + \mathscr{L}\left(\sigma_y^v\right) = 4\mathscr{L}\left\{\mathrm{Re}[\frac{\partial\varphi_2(z,s,t)}{\partial z}]\right\} \tag{16}$$

$$\mathscr{L}\left(\sigma_y^v\right) - \mathscr{L}\left(\sigma_x^v\right) + 2i\mathscr{L}\left(\sigma_{xy}^v\right) = 2\mathscr{L}\left[\bar{z}\frac{\partial^2\varphi_2(z,s,t)}{\partial z^2} + \frac{\partial\psi_2(z,s,t)}{\partial z}\right] \tag{17}$$

where the function $\gamma(s)$ appearing in Eq. (15) is obtained by replacing G_e with $s\mathscr{L}\left[G(t)\right]$ and K_e with $s\mathscr{L}\left[K(t)\right]$. Analogously, the analytical expressions for $\varphi_2(z,s,t)$ and $\psi_2(z,s,t)$ are obtained by replacing the elastic moduli with $s\mathscr{L}\left[G(t)\right]$ and $s\mathscr{L}\left[K(t)\right]$ in $\varphi_1(z,t)$ and $\psi_1(z,t)$ respectively. Then, performing the inverse Laplace transform of Eqs. (15)-(17) and imposing the boundary conditions, the equations for the unknown functions will be established, as shown in the following.

Since in our problem only boundary conditions on the stresses are present, from here onwards we consider only the stress boundary, $S_\sigma(t)$. The equation imposing the boundary condition on the stresses is as follows:

$$\mathscr{L}^{-1}\left\{\mathscr{L}\left[\varphi_2(z,s,t)+z\overline{\frac{\partial\varphi_2(z,s,t)}{\partial z}}+\overline{\psi_2(z,s,t)}\right]\right\}\Bigg|_{z=z_\sigma(t)}=i\int_{z_A}^{z_\sigma(t)}(T_x+iT_y)ds\,, \tag{18}$$

where T_x and T_y denote the tractions acting on the (stress) boundary along the x and y directions respectively; $z_\sigma(t)$ is a generic point on the (stress) boundary, *i.e.* $z_\sigma(t)\in S_\sigma(t)$; and z_A is an arbitrary point on the boundary.

According to the theory of complex variable representation (Muskhelishvili 1963), in case of a simply connected domain subject to a constant body force (in our case no body force is present), the two analytical functions φ_1 and ψ_1 are material parameter independent so that $\varphi_1=\varphi_2$ and $\psi_1=\psi_2$. Moreover, also the analytical expressions for the stresses are independent of the material parameters (see Eqs. (16) and (17)). Hence we can simplify Eq. (18) into:

$$\varphi_2(z,t)+z\overline{\frac{\partial\varphi_2(z,t)}{\partial z}}+\overline{\psi_2(z,t)}\Bigg|_{z=z_\sigma(t)}=i\int_{z_A}^{z_\sigma(t)}(P_x+iP_y)ds\,, \tag{19}$$

Therefore, the boundary conditions applied on the viscoelastic medium are the same as the boundary conditions applied on the associated elastic medium. Hence, also the analytical solution for the stress field is the same for both the viscoelastic medium and the associated elastic one. Concerning displacements instead, they can be obtained by replacing G_e with $s\mathscr{L}\left[G(t)\right]$ and K_e with $s\mathscr{L}\left[K(t)\right]$ in the Laplace transformed expressions obtained for the elastic case.

3.3 Calculation of stresses and displacements induced by the excavation

Let us consider a rock mass initially subject to the following geostatic anisotropic stress state: $\sigma_x^0=-\lambda p_0$, $\sigma_y^0=-p_0$, $\sigma_{xy}^0=0$, since a reference initial time $t'=0$. The rock mass is subject to growing displacements over time due to its viscosity. In Figure 2 (b), the inner dashed line indicates the boundary $S_\sigma(t')$ of the tunnel at a time $t'\geq t'_0$, with t'_0 being the start time of excavation. Prior to the beginning of the excavation (at time $t'=t_0'^-$), the tractions $p_x\left(z_0(t')\right)$

and $p_y\left(z_0(t^{'})\right)$ (with $z_0(t^{'})$ denoting a generic point on the time-dependent boundary at time $t^{'}$) exchanged between the two bodies along $S_\sigma(t^{'})$ may be easily calculated imposing equilibrium. At the beginning of the excavation, at time $t>t^{'}$, $p_x\left(z_0(t^{'})\right)$ and $p_y\left(z_0(t^{'})\right)$ go to zero along the boundary of the excavated zone inducing displacements in the rock. The excavation induced stress, strain and displacement increments will be calculated since the beginning of the excavation. To this end, the constitutive equation (see Eq. (1)) for the deviatoric stress tensor may be rewritten as follows (the derivation for the isotropic part of the stress tensor is analogous):

$$s_{ij}^v(t^{'}) = 2e_{ij}^v(0^+)G(t^{'}) + 2\int_0^{t^{'}_0{}^-} G(t^{'}-\tau)\frac{de_{ij}^v}{d\tau}d\tau + 2\int_{t^{'}_0{}^+}^{t^{'}} G(t^{'}-\tau)\frac{de_{ij}^v}{d\tau}d\tau + 2[e_{ij}^v(t^{'}_0{}^+) - e_{ij}^v(t^{'}_0{}^-)]G(t^{'}-t^{'}_0) \quad (20)$$

with $t^{'} \geq t^{'}_0$ whilst for $t^{'} = t^{'}_0{}^-$:

$$s_{ij}^v(t^{'}_0) = 2e_{ij}^v(0^+)G(t^{'}_0) + 2\int_0^{t^{'}_0{}^-} G(t^{'}_0-\tau)\frac{de_{ij}^v}{d\tau}d\tau \quad (21)$$

The analytical expressions of the shear relaxation modulus G for the considered viscoelastic models (see Figure 1) are listed in Table 1. The reference time $t^{'}$ can be chosen sufficiently large so that $t' \to \infty$. In case of models with limited viscosity, *e.g.* generalized Kelvin and Kelvin models, here called Type A models, the first two terms in Eq. (20) turn out to be equal to the first two terms in Eq. (21) (for the demonstration of this equality see Appendix A.2), so that:

$$s_{ij}^v(t^{'}) = s_{ij}^v(t^{'}_0) + 2\int_{t^{'}_0{}^+}^{t^{'}} G(t^{'}-\tau)\frac{de_{ij}^v}{d\tau}d\tau + 2[e_{ij}^v(t^{'}_0{}^+) - e_{ij}^v(t^{'}_0{}^-)]G(t^{'}-t^{'}_0) \quad (22)$$

Instead, in case of models with unlimited viscosity, *e.g.* Burgers and Maxwell models, here called Type B models, this is not the case so that Eq. (23) no longer holds true (see Appendix A.3). Now, for Type B models, we define $\Delta s_{ij}^v(t^{'}) \equiv s_{ij}^v(t^{'}) - s_{ij}^v(t^{'}_0{}^-)$ and $\Delta e_{ij}^v(t^{'}) \equiv e_{ij}^v(t^{'}) - e_{ij}^v(t^{'}_0{}^-)$, as incremental stresses and strains respectively induced by the tunnel excavation. Introducing a new reference time *t*, with $t = t^{'} - t^{'}_0$, then Eq. (22) may be rewritten as follows:

$$\Delta s_{ij}^v(t) = 2\int_0^t G(t-\tau)\frac{d\Delta e_{ij}^v}{d\tau}d\tau + 2\Delta e_{ij}^v(0^+)G(t) = 2G(t) * d\Delta e_{ij}^v(t) \quad (23)$$

Note that the relationship between Δs_{ij}^v and Δe_{ij}^v is the same as that in Eq. (1). For the field of induced stresses, strains and displacements, also the same equations of equilibrium and compatibility must be satisfied. However, the corresponding boundary conditions differ from the boundary conditions shown in Figure 2, and the stresses prescribed along the boundaries (see Eq. (18)) may be written as follows:

$$\Delta P_x + i\Delta P_y = -p_x\left(z_0(t)\right) - ip_y\left(z_0(t)\right), \text{ along the inner time-dependent boundary; and}$$

$$\Delta P_x + i\Delta P_y = 0, \text{ along the outer (infinite) boundary} \qquad (24)$$

The boundary conditions for calculating the induced stresses and strains are shown in Figure 2(b). Note that the tractions p_x and p_y applied on the inner boundary in Figure 3 (b) and (c) are of equal absolute value, but of opposite direction.

The solution procedure employed for type A models cannot be used since Eq. (22) no longer holds true. In case of Type B models, the rock before excavation undergoes continuous displacements (see Eq. (A.9) in Appendix A.1). So in order to calculate the excavation induced displacements the rock will be assumed elastic before the excavation takes place.

As outlined in Section 3.2, the solution for the displacements can be obtained from the solution of the associated elastic problem. The elastic solution for our problem will be obtained as the combination of two fictitious cases here called case A and B according to the principle of superposition. In case of no tractions on the inner boundary (see Figure 2(a)), we obtain Solution A-ela (elastic solutions of case A); while the case of a plane without hole subject to the displayed boundary stresses in Figure 2(b) is referred to as Solution B-ela (elastic solutions of case B). Therefore, the elastic induced solutions, i.e. Solution C-ela, may be obtained by subtracting Solution B-ela from Solution A-ela. In the following section, the solutions will be derived by means of complex potential theory.

3.4 Derivation of the analytical solution

The method of conformal mapping provides a very powerful tool to solve problems involving complex geometries. Let us consider the complex plane $z=x+iy$ with x and y representing the horizontal and vertical directions respectively in the plane of the tunnel cross-section (see Fig. 2). Also let us define a function to map the (infinite) domain (in the z plane) of the rock surrounding the elliptical cross-section into a fictitious domain (in the ζ-plane with $\zeta = \eta + i\xi$) with a unit circular hole. Since the elliptical cross-section varies over time, the mapping function is time-dependent too:

$$z = \omega(\zeta, t) = c(t)\left[\zeta + \frac{m(t)}{\zeta}\right] \tag{25}$$

where:

$$c(t) = \frac{a(t)+b(t)}{2} \quad \text{and} \quad m(t) = \frac{a(t)-b(t)}{a(t)+b(t)}. \tag{26}$$

If $\frac{a(t)}{b(t)}$ is constant during the excavation stage, the excavation expands homothetically and *m* remains constant over time. According to the boundary conditions shown in Figure 2a, two complex potentials for the elastic problem A with time-dependent boundaries may be derived as follows (Muskhelishvili, 1963):

$$\varphi_1^{(A)}(\zeta, t) = \frac{-(1+\lambda)p_0 c(t)}{4}\left[\zeta + \frac{m(t)}{\zeta}\right] + \frac{\left[1-\lambda+(1+\lambda)m(t)\right]p_0 c(t)}{2\zeta} \tag{27}$$

$$\begin{aligned}\psi_1^{(A)}(\zeta, t) = {} & \frac{(\lambda-1)p_0 c(t)}{2}\left[\zeta + \frac{m(t)}{\zeta}\right] + \frac{p_0 c(t)}{2\zeta}\left[(1+\lambda)\left(1+m^2(t)\right) + 2(1-\lambda)m(t)\right] \\ & + \frac{\left[1-\lambda+(1+\lambda)m(t)\right]\left[1+m^2(t)\right]p_0 c(t)}{2\zeta\left[\zeta^2 - m(t)\right]}\end{aligned} \tag{28}$$

According to elasticity theory, the two potentials used to calculate the elastic displacements of the infinite plane subject to the anisotropic initial stress state prior to excavation (Solution B-ela) are as follows [Einstein and Schwartz, 1979]:

$$\varphi_1^{(B)}(z) = -\frac{(1+\lambda)p_0 c(t)}{4}\left[\zeta + \frac{m(t)}{\zeta}\right], \quad \psi_1^{(B)}(z) = -\frac{(1-\lambda)p_0 c(t)}{2}\left[\zeta + \frac{m(t)}{\zeta}\right] \tag{29}$$

According to the superposition principle of elasticity, the potentials for calculating the excavation induced displacements are as follows (Solution C-ela):

$$\varphi_1^{(C)}(\zeta,t)=\varphi_1^{(A)}(\zeta,t)-\varphi_1^{(B)}(\zeta,t)=\frac{\left[1-\lambda+(1+\lambda)m(t)\right]p_0c(t)}{2\zeta} \tag{30}$$

$$\begin{aligned}\psi_1^{(C)}(\zeta,t)&=\psi_1^{(A)}(\zeta,t)-\psi_1^{(B)}(\zeta,t)=\\&=\frac{p_0c(t)}{2\zeta}\left[(1+\lambda)\left(1+m^2(t)\right)+2(1-\lambda)m(t)\right]+\frac{\left[1-\lambda+(1+\lambda)m(t)\right]\left[1+m^2(t)\right]p_0c(t)}{2\zeta\left[\zeta^2-m(t)\right]}\end{aligned} \tag{31}$$

After substituting Eqs. (30) and (31) into Eqs. (11), (12) and (13) respectively, the elastic displacements and stresses (Solution C-ela) on the plane ζ may be calculated.

According to the analysis in Section 3.2, the solution for the viscoelastic case can be obtained by applying the principle of correspondence, and the Laplace inverse transform of the variables (stresses, strains, etc.) calculated for the elastic case with the variable z treated as a constant in the Laplace transform. However, in Eqs. (30) and (31) the variable ζ appears rather than z, hence according to Eq. (25), Eqs. (30) and (31) are time dependent and cannot be Laplace transformed. To replace ζ with z and t, the inverse function of the conformal mapping $\zeta=f_0(z,t)$ needs to be found. If ζ in Eqs. (30) and (31) is replaced with $f_0(z,t)$, then all the time-dependent functions in Eqs. (30) and (31) may be Laplace transformed, and the viscoelastic solution may be derived from Eqs. (15), (16) and (17). Then, defining:

$$z'=\frac{z}{c(t)} \tag{32}$$

and substituting in Eq. (25) the following is obtained:

$$z'=\zeta+\frac{m(t)}{\zeta} \tag{33}$$

If the excavation process is homothetic, *i.e.* m is a constant, then there is no variable t in Eq. (33), and the inverse conformal mapping may be expressed as [Zhang 2001]:

$$\zeta=f_1(z',t)=Az'+\sum_{k=0}^{\infty}A_k(z')^{-k} \tag{34}$$

with the yet undetermined coefficients $A, A_k\left(k=0,1,...,\infty\right)$. For numerical reasons, the series will be truncated to a finite number, l, of terms to calculate the function approximately. Due to the fact that the inverse conformal mapping is derived from the corresponding direct conformal mapping, there is a one-to-one correspondence between all the values of one function, with the values of the other function. Let us choose a number of points ζ_j $(j=1,2\ldots,q)$, with q=160, lying on the inner boundary of the unit circle in the ζ plane to calculate the corresponding points z_j' lying on the inner boundary in the z' plane using Eq. (33). Then, q linear equations for A and A_k can be obtained by substituting ζ_j and z_j' into Eq. (34):

$$\begin{cases} \zeta_1 = Az_1' + \sum_{k=0}^{l} A_k z_1'^{-k} \\ \zeta_2 = Az_2' + \sum_{k=0}^{l} A_k z_2'^{-k} \\ \vdots \\ \zeta_j = Az_j' + \sum_{k=0}^{l} A_k z_j'^{-k} \\ \vdots \\ \zeta_q = Az_q' + \sum_{k=0}^{l} A_k z_q'^{-k} \end{cases} \tag{35}$$

Since the number of independent equations is larger than the number of unknown coefficients (A, A_0, A_1 ,……, A_l), the system is indeterminate. To solve the system, i.e. to determine the unknown coefficients, we employed the method of minimum least squares. The non-zero coefficients obtained for the elliptical shapes here considered, are listed in Table 2 for l=15. In Figure 4, the curves on plane z' and ζ determined by direct and inverse conformal mapping respectively are plotted for various shapes of the elliptical tunnel boundary. The ellipses on the z' plane (plotted in Figures 4 (a-1), (b-1) and (c-1)), map into the circles plotted as dashed lines on the ζ plane (Figures 4a-2, b-2 and c-2), which are determined via Eq. (25). The curves with continuous line on the ζ plane have been obtained by inverse conformal mapping (see Eq. (34)), applied to the ellipses on the z' plane. It can be observed that curves determined by inverse conformal mapping, are very close to circular. However, we can observe that the inverse conformal mapping is less accurate for the inner boundary when m is larger than 0.4. According to the direct and inverse conformal mappings, a one-to-one correspondence for points on the z and ζ plane is established. For a general non-homothetic excavation process, the parameter m is a function of time, so that an analytical expression for the inverse conformal mapping cannot be obtained. However, discrete values of the inverse conformal mapping over time may be calculated according to the prescribed $m(t)$ and $c(t)$.

Substituting Eqs. (30), (31), (34) into Eqs. (15) and (16), the excavation induced displacements and stresses in linearly viscoelastic rock (Solution C-vis) can be derived as follows:

$$\mathscr{L}\left(u_x^{(C)v}\right)+i\mathscr{L}\left(u_y^{(C)v}\right)=p_0\cdot\left[B_1(z,s)+B_2(z,s)+B_3(z,s)+B_4(z,s)\right] \tag{36}$$

$$\begin{matrix}\sigma_x^{(C)}\\ \sigma_y^{(C)}\end{matrix}=p_0\cdot\mathrm{Re}\{D_1(z,t)\}\mp p_0\cdot\mathrm{Re}\{D_2(z,t)\}, \tag{37}$$

$$\sigma_{xy}^{(C)}=p_0\cdot\mathrm{Im}\{D_2(z,t)\}. \tag{38}$$

with $B_1(z,s)=\dfrac{\gamma(s)}{s\mathscr{L}\left[G(t)\right]}\mathscr{L}\left[\dfrac{[1-\lambda+(1+\lambda)m(t)]c(t)}{f_1(z')}\right]$,

$$B_2(z,s)=\frac{z}{s\mathscr{L}\left[G(t)\right]}\mathscr{L}\left[\frac{1-\lambda+(1+\lambda)m(t)}{f_1^2(\overline{z}')-m(t)}\right],$$

$$B_3(z,s)=-\frac{1}{s\mathscr{L}\left[G(t)\right]}\mathscr{L}\left\{\frac{\left[\left(1+m^2(t)\right)(1+\lambda)+2m(t)(1-\lambda)\right]c(t)}{f_1\left(\overline{z}'\right)}\right\},$$

$$B_4(z,s)=\frac{1}{s\mathscr{L}\left[G(t)\right]}\mathscr{L}\left[\frac{[\lambda-1-(1+\lambda)m(t)]\left[1+m^2(t)\right]c(t)}{f_1(\overline{z}')\left[f_1^2(\overline{z}')-m(t)\right]}\right],\quad D_1(z,t)=\frac{\lambda-1-(1+\lambda)m(t)}{2[f_1^2(z')-m(t)]},$$

$$\begin{aligned}D_2(z,t)=&-\frac{\overline{z}[\lambda-1-(1+\lambda)m(t)]f_1^3(z')}{c(t)[f_1^2(z')-m(t)]^3}-\frac{(1+\lambda)\left[1+m^2(t)\right]+2(1-\lambda)m(t)}{2[f_1^2(z')-m(t)]}\\&+\frac{\left[1+m^2(t)\right][\lambda-1-(1+\lambda)m(t)][3f_1^2(z')-m(t)]}{2\left[f_1^2(z')-m(t)\right]^3}.\end{aligned}$$

Because the stresses of the viscoelastic and elastic cases are the same, the stresses of case A (solution A) are the total stresses in the rock, and can be calculated by the two potentials of Solution A, as:

$$\begin{matrix}\sigma_x^{(A)}\\ \sigma_y^{(A)}\end{matrix}=p_0\cdot\mathrm{Re}\left\{\frac{\mp(\lambda-1)-1-\lambda}{2}+p_0\cdot D_1(z,t)\right\}\mp p_0\cdot\mathrm{Re}\{D_2(z,t)\}. \tag{39}$$

$$\sigma_{xy}^{(A)}=p_0\cdot\mathrm{Im}\{D_2(z,t)\} \tag{40}$$

If α is the angle between the horizontal axis x and the normal direction (see Figure 2), the tangential and normal displacements and stresses around the boundary of the excavation may be calculated as follows:

$$\mathscr{L}\left(u_{\chi}^{(C)v}\right)+i\mathscr{L}\left(u_{\tau}^{(C)v}\right)=e^{-i\alpha}\left[\mathscr{L}\left(u_{x}^{(C)v}\right)+i\mathscr{L}\left(u_{y}^{(C)v}\right)\right] \quad (41)$$

$$\begin{matrix}\sigma_{\chi}^{(C)}\\ \sigma_{\tau}^{(C)}\end{matrix}=p_0\cdot\mathrm{Re}\left\{D_1(z,t)\right\}\mp p_0\cdot\mathrm{Re}\left\{e^{2i\alpha}D_2(z,t)\right\}, \quad (42)$$

$$\sigma_{\chi\tau}^{(C)}=p_0\cdot\mathrm{Im}\left\{e^{2i\alpha}D_2(z,t)\right\}. \quad (43)$$

$$\begin{matrix}\sigma_{\chi}^{(A)}\\ \sigma_{\tau}^{(A)}\end{matrix}=p_0\cdot\mathrm{Re}\left\{\frac{\mp(\lambda-1)-1-\lambda}{2}+p_0\cdot D_1(z,t)\right\}\mp p_0\cdot\mathrm{Re}\left\{e^{2i\alpha}D_2(z,t)\right\}. \quad (44)$$

$$\sigma_{\chi\tau}^{(A)}=p_0\cdot\mathrm{Im}\left\{e^{2i\alpha}D_2(z,t)\right\} \quad (45)$$

The expressions for stresses here provided are suitable for all linear viscoelastic models, since the stress state depends only on the shape and size of the opening; conversely displacements depend on the viscoelastic model considered. The analytical solution for the displacements is provided in the next section.

3.5 Solution for the displacements

Rock masses which have strong mechanical properties or are subject to low stresses exhibit limited viscosity. For this type of behavior, the generalized Kelvin viscoelastic model (see Figure 1a) is commonly employed [Dai 2004]. On the other hand, weak, soft or highly jointed rock masses and/or rock masses subject to high stresses are prone to excavation induced continuous viscous flows. In this case, the Maxwell model (see Figure 1b) is suitable to simulate their rheology, due to the fact that this model is able to account for secondary creep. In this section, the analytical solution for the generalized Kelvin model is developed. The constitutive parameters of this model are as follows: i) the elastic shear moduli G_H, due to the Hookean element in the model; ii) G_K, due to the spring element of the Kelvin component; iii) the viscosity coefficient η_K, due to the dashpot element of the Kelvin component (see Figure 1c). The solution for the Maxwell model may be obtained as a particular case of the generalized Kelvin model, for G_K=0. Note that the solution for the Kelvin model (see Figure 1c) may also be obtained as another particular case of the Generalized Kelvin model for $G_H \to \infty$.

Assuming that the rock is incompressible, *i.e.* $K(t) \to \infty$, the two relaxation moduli appearing in the constitutive equations (see Eq. (1)) are as follows:

$$G(t) = \frac{G_{\mathrm{H}}{}^2}{G_{\mathrm{H}} + G_{\mathrm{K}}} e^{-\frac{G_{\mathrm{H}}+G_{\mathrm{K}}}{\eta_{\mathrm{K}}} t} + \frac{G_{\mathrm{H}} G_{\mathrm{K}}}{G_{\mathrm{H}} + G_{\mathrm{K}}} \quad , \quad K(t) = \infty \tag{46}$$

The induced displacements, Solution *C*-vis, may be derived by substituting Eq. (46) into Eq. (41):

$$u_{\chi}^{(C)v} + i u_{\tau}^{(C)v} = \frac{e^{-i\alpha} p_0}{4} \left[B_5(z,t) + B_6(z,t) + B_7(z,t) + B_8(z,t) \right] \tag{47}$$

with $B_5(z,t) = \int_0^t \frac{H(t,\tau) c(\tau) \left[1 - \lambda + (1+\lambda) m(\tau) \right]}{f_1[z'(\tau)]} d\tau$, $B_6(z,t) = z \int_0^t H(t,\tau) \left[\frac{1 - \lambda + (1+\lambda) m(\tau)}{f_1^2[\overline{z'(\tau)}] - m(\tau)} \right] d\tau$,

$$B_7(z,t) = \int_0^t \frac{H(t,\tau) c(\tau) \left[\left(1 + m^2(\tau) \right)(1+\lambda) + 2m(\tau)(1-\lambda) \right]}{f_1[\overline{z'(\tau)}]} d\tau,$$

$$B_8(z,t) = \int_0^t \frac{H(t,\tau) c(\tau) \left[\lambda - 1 - (1+\lambda) m(\tau) \right] \left[1 + m^2(\tau) \right]}{f_1[\overline{z'(\tau)}] \left\{ f_1^2[\overline{z'(\tau)}] - m(\tau) \right\}} d\tau, \text{ and}$$

$$H(t,\tau) = \frac{1}{G_{\mathrm{H}}} \delta(t-\tau) + \frac{1}{\eta_{\mathrm{K}}} e^{-\frac{G_{\mathrm{K}}}{\eta_{\mathrm{K}}}(t-\tau)}. \tag{48}$$

When m=0 and λ=1, the problem reduces to a circular tunnel subject to a hydrostatic state of stress, and the degenerate solution in Eq. (47) coincides with the solution provided in (Wang and Nie 2010), hence the problem becomes axisymmetric.

4. Comparison with FEM results

Two types of FEM analyses were run employing the FEM code ANSYS (version 11.0, employing the module of structure mechanics). The first FEM analysis wants to replicate the viscoelastic problem of solution A whereas the second one the problem of solution C. All FEM analyses were carried out with a small displacement formulation to be consistent within the derivation of the analytical solution.

Analytical solution A-vis for generalized Kelvin viscoelastic model can be derived by substituting Eqs. (27), (28), (34) and (46) into Eqs. (15)-(17). The expressions for displacements are as follows:

$$u_x^{(A)v} + iu_y^{(A)v} = \frac{p_0}{4}\left[B_5(z,t) + B_6(z,t) + B_7(z,t) + B_8(z,t) + B_9(z,t)\right] \quad (49)$$

where $B_9(z,t) = (1-\lambda)\bar{z}\int_0^t H(t,\tau)d\tau$. Displacements and stresses of solution C-vis and stresses of solution A-vis can be found in Eqs. (47), (37), (38), (39) and (40), respectively.

First, we shall compare displacements and stresses of solution A-vis obtained by the analytical solution with the FEM analysis along 3 directions (horizontal, vertical, 45° over the horizontal). Second, the excavation induced stresses and displacements from the analytical solution C-vis and FEM along Line 2 (45° over the horizontal) will be compared to validate the correctness of the analytical solution here achieved. In the FEM analysis of case A-vis, initial stresses are applied on a planar domain having an elliptical hole with the major axis being $2a_0$ long and minor axis $2b_0$ long (Part Ⅰ in Figure 5). Then, the rock is sequentially excavated at different times (see Part Ⅱ to Ⅶ in Figure 5), as listed in Table 3. In the second simulation instead, initial stresses are first applied on the finite rectangular domain without hole, then an excavation starting after 50 days is simulated. Part Ⅰ to Ⅶ are excavated at t'=50th day, 51 th day, ……, 56 th day, respectively. In the end, the excavation induced stresses and displacements can be obtained by subtracting the initial values before excavation from the ones calculated in the excavation stage. In FEM analysis, elements are deleted at the time of excavation by setting the stiffness of the deleted elements to zero (by multiplying the stiffness matrix by 10^{-6}).

A vertical stress, $p_0 = 10MPa$, and a horizontal stress, λp_0 with $\lambda = 0.5$, were applied at the boundaries of the domain of analysis. The rock was simulated as a generalized Kelvin medium, with the following constitutive parameters adopted: $G_{\mathrm{H}} = 2000MPa$, $G_{\mathrm{K}} = 1000MPa$ and $\eta_{\mathrm{K}} = 10000MPa \cdot day$. The excavation sequence here considered is specified by the values of the major and minor axes of the elliptical section listed in Table 3 with an initial value of $2a_0$=3.0m for the major axis and $2b_0$=2.0m for the minor axis. Note that the ratio $m(t)$=*const*, *i.e.* the elliptical section evolves homothetically. The FEM mesh nearby the hole is plotted in Figure 5.

The points and lines selected for comparison between the FEM analysis and the analytical solution are plotted in Figure 6: three points on the inner boundary (points 1, 2, 3 in the Figure) and three lines, one horizontal (line 1), one vertical (line 3) and one inclined at 49.8° over the horizontal (line 2), were chosen. In Figure 7, displacements and stresses for points 1, 2 and 3 are plotted versus time. In Figures 8 and 9 displacements and stresses respectively at four different times (t=1st, 3rd, 6th and 20th days) are plotted for lines 1, 2 and 3 versus the distance to the centre of the ellipse. It emerges that the predictions from the analytical solution are in excellent agreement with the results from the FEM analysis. In Figure 7 it can be noted that displacements and stresses undergo a stepwise increase following instantaneous excavation events (1st, 2nd, 3rd,... 6th days).

In Figure 10 the excavation induced displacements and stresses along Line 2 obtained from the analytical solution and FEM analysis, are plotted. A good agreement in terms of both stresses and displacements can be observed. Unlike the case of solution A, almost all the induced displacements are decreasing functions of the distance to the centre of the ellipse.

5. Parametric investigation

In order to study the influence of sequential excavation rate and methods, as well as the time-dependent distribution of displacements and stresses, a parametric investigation is illustrated in this section. With the same notation as in Section 4, a_0 and b_0 are values of the half major and minor axes at time $t=0$, respectively, and a_1, b_1 are the values of the axes when $t \geq t_1$, with t_1 being the end time of excavation. Assuming an axisymmetric elastic problem, *i.e.* circular tunnel in infinite plane, subjecting to hydrostatic initial stress p_0, with tunnel radius $R^* = (a_1 + b_1)/2$ and shear modulus $G_S = G_{\mathrm{H}} G_{\mathrm{K}} / (G_{\mathrm{H}} + G_{\mathrm{K}})$ which is the permanent modulus of generalized Kelvin model (see Figure 1a), the excavation induced radial displacement at the inner boundary of the tunnel can be calculated as follows:

$$u_s^e = -p_0 R^* / (2G_S) \tag{50}$$

In the following analysis, the induced displacements of viscoelastic cases for elliptical tunnel excavation will be normalized by the displacement listed in Eq. (50), and stresses are normalized by $-p_0$. Therefore, positive dimensionless normal stress is compression in the following figures. Now, let us define the dimensional parameter $T_K = \eta_K / G_K$, which expresses the retardation time of the Kelvin component of the generalized Kelvin model. It is convenient to normalize the time as t/T_K for the generalized Kelvin model. For Maxwell model, G_K is equal to zero (see Figure 1), hence T_K can not be used in normalization; instead, the relaxation time $T_M = \eta_K / G_H$ will be employed to normalize the time as t/T_M.

5.1 Influence of the excavation rate

Concerning sequential excavation, the values of half major and minor axes grow from zero to the final values. In this case, a linear increase of the tunnel axis over time is assumed when t is less than t_1, i.e. $a(t) = \begin{cases} a_0 + v_r t & 0 \le t \le t_1 \\ a_1 & t > t_1 \end{cases}$, where v_r is the (constant) speed of cross-section excavation. It is convenient to express the half major axis in dimensionless form as:

$$\frac{a(t)}{a_1} = \begin{cases} \dfrac{a_0}{a_1} + n_r^K \dfrac{t}{T_K} & 0 \le t < t_1 \\ 1 & t \ge t_1 \end{cases} \tag{51}$$

where n_r^K is the dimensionless excavation speed, defined as follows:

$$n_r^K = \frac{v_r T_K}{a_1} \tag{52}$$

In the parametric analysis $a_0/a_1 = 1/4$ was assumed together with the following dimensionless excavation speeds: (1) $n_r^K \to \infty$, corresponding to the case of instantaneous excavation (implying $t_1/T_K = 0$); (2) $n_r^K = 1.5$ (implying $t_1/T_K = 0.5$); (3) $n_r^K = 0.75$ (implying $t_1/T_K = 1.0$); and (4) $n_r^K = 0.5$ (implying $t_1/T_K = 1.5$). Concerning the excavation method, a homothetic excavation with the constant ratio $a(t)/b(t) = 2.0$ ($m = 1/3$) is assumed in the analysis, with the ratio of horizontal and vertical stresses $\lambda = 1/3$.

In order to cover the wide range of responses for rock types of different viscous characteristics, the time-dependent displacements and stresses were analyzed for two types of rocks of different stiffness ratios: $G_K / G_H = 0.5$ and 2.0. In Figures 11 and 12 the time-dependent radial and tangential displacements for the rock at the final tunnel face (*i.e.* the face at the end of excavation of cross-section) with angle $\theta = 0^{\circ}$, 45° and 90° are plotted for the types of rock and excavation rates considered. The symbol '●' represents the end time of excavation, t_1. The figures show that the normal displacement increases with time and reaches a constant value after a certain period of time; however, the tangential displacement first decreases with time and then increases rapidly towards the end of the excavation, then eventually reaches a constant value. Comparing Figure 11 with Figure 12, the final displacements are reached later for rocks with smaller stiffness ratios (Figure 11). It can also be noted that the bigger the stiffness ratio is, the larger the after excavation displacements are. For both types of rock, the results show that a lower excavation rate implies a longer excavation time, which in turn leads to a larger value of normal displacement at the tunnel face with $\theta = 45^{\circ}$ and 90° when $t = t_1$; however, the tangential displacement at $\theta = 45^{\circ}$ and the normal displacement at $\theta = 0^{\circ}$ show no significant difference among the various excavation rates at time $t = t_1$. It can also be observed that higher excavation rates imply larger normal displacement at any time, and the maximum absolute value of the tangential displacement during the excavation stage will be larger.

The Maxwell model is suitable to simulate the rheology of weak, soft or highly jointed rock, with continuous linear viscous response when constant stresses are applied. When G_K=0, the Maxwell model is obtained (Figure 1b). In this case, according to Eq. (50), $G_S = 0$, and $u_s^e \rightarrow \infty$. Hence, in order to normalize the displacements, a different normalization must be employed. To achieve this, we chose to replace G_S with the initial elastic modulus G_H of Maxwell model in

Eq. (50) to calculate the radial displacement at the tunnel face for the axisymmetric elastic problem, i.e. $u_{s0}^{e} = -p_0 R^* / (2G_H)$. In this case, we adopt $n_r^M = \dfrac{v_r T_M}{a_1}$ as the dimensionless excavation rate with $T_M = \eta_K / G_H$, and we consider the following four excavation rates in our analysis: (1) $n_r^M \to \infty$; (2) $n_r^M = 1.5$; (3) $n_r^M = 0.75$; and (4) $n_r^M = 0.5$. In Figure 13, the normalized displacements at the final tunnel face at point $\theta = 45^{\circ}$ are plotted against the normalized time t/T_M. Since the stresses of the rock are constant after excavation (see Eqs. (44) and (45)), in Figure 13, the displacements after excavation grow linearly over time. It also emerges that the influence of the excavation rate for Maxwell model is similar to that for the generalized Kelvin model.

Observing Eqs. (44) and (45), it is shown that the stresses depends only on the size and shape of the opening, hence given a prescribed sequential excavation the stress field is identical for all the viscoelastic models. In Figure 14, the principal stresses of the rock at the tunnel face at points $\theta = 0^{\circ}$, $\theta = 45^{\circ}$ and $\theta = 90^{\circ}$, are presented for various excavation rates. As it can be expected, the variations of stress with time are more gradual for lower excavation rates. In all the cases, the maximum difference between the two principal stresses occurs after excavation.

5.2 Influence of the excavation methods

In this section, the final values of the major and minor axes and ratio of horizontal and vertical stresses λ are the same as in the previous section with the end time of excavation being $t_1 / T_{\mathrm{K}} = 1.0$. The time-dependent tunnel inner boundaries, which simulate the real across-section excavation process as center drift advanced method [Katushi and Hiroshi 2003] (e.g. method C shown in Figure 15), drilling and blasting method [Tonon 2010] (e.g methods A, B1 and B2 shown in Figure 15), are shown in Figures 15 (a), (b) and (c). The functions $a(t)$ and $b(t)$ are plotted in Figures 16 with their analytical expressions provided in Table 4. In real project application, $a(t)$ and $b(t)$ may be determined by accounting for the actual excavation process, as prescribed by the designers.

Sequential excavation methods A and C are stepwise excavations, in which parts ① to ⑤ (or ① to ④) are excavated instantaneously in succession. In Figure 15 it is shown that the shape of the opening in method A first changes from ellipse to circle, and then to ellipse, by sequential excavation along the major axis direction. Obviously, the excavation is nonhomothetic with time-dependent ratio $a(t)/b(t)$ during excavation. Figure 6 shows that the adopted excavation rate is faster in the beginning and slower toward the end of the excavation in this method. In method C, the initial shape of the opening is circular, then gradually changes to elliptical with the increase of the ratio $a(t)/b(t)$ with time. The excavation rate is slower in the beginning and becomes faster toward the end of excavation, which is opposite of that in method A. Excavation methods B1 and B2 instead, are continuous homothetic excavations ($a(t)/b(t)=2.0$). method B1 consists of a linear excavation at uniform speed, whereas method B2 consists of an excavation function $a(t)$ in quadratic form, with a faster excavation rate toward the end.

For the two types of rock (i.e. $G_{\mathrm{K}}/G_{\mathrm{H}}=0.5$ and $G_{\mathrm{K}}/G_{\mathrm{H}}=2.0$), the time dependent normal and tangential displacements at the final tunnel face with angles 0°, 45° and 90° are plotted in Figures 17 and 18 for the four excavation methods. It emerges that the induced displacements are sensitive to the excavation method adopted. In particular it can be observed that the methods with faster speeds in the early stages lead to larger normal displacement at generic time (except for the cases of $\theta=0^{\circ}$), as well as at the end of excavation time t_1. For all the excavation methods, the normal displacements at $\theta=45^{\circ}$ and 90° increase over time and reach a constant positive value after a certain period of time; whereas the normal displacements at $\theta=0^{\circ}$ are approximately zero in the early stages of excavation, and increase rapidly toward the end of excavation for methods B1, B2 and C. The tangential displacements at $\theta=45^{\circ}$ in Figure 17(d) are negative and first decrease in the early stages and then increase to positive values.

In order to analyse the maximum difference of displacements among various excavation methods, the normalized displacements and difference radios between methods A and C (the difference between this two methods is the maximum according to Figures 17 and 18) at time $t = t_1$ are listed in Table 5. The difference ratios of normal displacement for the rock with $G_K / G_H = 0.5$ range from 26% to 33% , and reach up to 60% for tangential displacement. The ratios range from 7% to 13% for normal displacement and 20% for tangential displacement for the type of rock with $G_K / G_H = 2.0$, which are less than the ones in cases where $G_K / G_H = 0.5$.

Figure 19 presents the normalized principal stresses calculated at the final tunnel face. It may be observed that the stresses show no difference for all of the excavation methods when $t \geq t_1$, because the final shape and size of the tunnel are the same. However, during the excavation stage the stress field is clearly affected by the excavation method adopted. This stress analysis accounting for sequential excavation is valuable to check for potential failure mechanisms since it provides the stress state at any time for any point in the rock.

5.3 Distribution of displacements and stresses for different excavation methods

In this section, the distributions of displacement and stress for the rock with $G_K / G_H = 0.5$ are analyzed, adopting sequential excavation methods A and C with the same end time of excavation. Four points in time are considered in the following analysis: time $t_{(1)}$: $t_{(1)} / T_K = 0.0$, the beginning of excavation; time $t_{(2)}$: $t_{(2)} / T_K = 0.5$, during the excavation stage; time $t_{(3)}$: $t_{(3)} / T_K = 1.0$, the end of excavation; and time $t_{(4)}$: $t_{(4)} / T_K = 2.5$, the time after excavation at which no further displacements practically occur.

Figure 20 presents the contour plots of the normal displacement at times $t_{(3)}$ and $t_{(4)}$ for methods A and C, respectively; and Figures 21 presents the contour plots of the tangential displacements. The Figure 20 shows that, the distribution regularities of normal displacement at same time after excavation, e.g the distributions in Figures 20 (a) and (c) or in (b) and (d), are very similar, whatever method is adopted. It can also be noted that the values of displacement at same position corresponding to different methods have significant difference around the tunnel crown when $t_{(3)}$, whereas at $t_{(4)}$, the difference is very small. Figure 21 shows that the maximum negative tangential displacement occurs inside of the ground, and the maximum positive one occurs at the tunnel face with θ approximately equal to 10-30 degree. Furthermore, the distributions of tangential displacement with different excavation methods are very similar except for the values, e.g. the ones in Figures 21 (a) and (c) or in (b) and (d). In Figures 22(a) and (b), the contours of the major and minor principal stresses respectively are plotted at time $t_{(3)}$.

Figures 23(a) and (b) present the distribution of normal and tangential displacements at the final tunnel face as a function of the angle θ for excavation method A and C at various times, which due to symmetry of the problem, is illustrated in the range $\theta = 0^{\circ}$ to 90°only. It emerges that the normal displacement is a monotonically increasing function of the angle, and the curve shapes are similar for various excavation methods. However, at times $t_{(2)}$and $t_{(3)}$, the values of normal displacement for the two excavation methods are significantly different. Unlike the normal displacement, the tangential displacement increases with θ for $0 < \theta < \theta_{\max}$, then decreases for $\theta_{\max} < \theta < \pi/2$. Furthermore, the angle corresponding to the maximum displacement, $\theta_{\max}$, decreases over time. At the time $t_{(2)}$ (in the excavation stage), the sign of the tangential displacement is opposite in the two excavation methods, exhibiting approximately the same $\theta_{\max}$. Considering the difference of induced tangential displacements between the two excavation methods, at the end of excavation it is smaller, whereas it becomes larger afterwards with a rapid decrease of displacement in method C. In addition, the angle corresponding to the maximal value, $\theta_{\max}$, is larger in method A than that in method C. The difference between the displacements of the two methods is smallest when $t/T_K = 2.5(t_{(4)})$.

In Figures 24(a) and (b), the principal stresses at the final tunnel face as a function of the angle θ are plotted for method A and C. Because the stresses depends only on the size and shape of the opening, the stresses at time $t_{(4)}$, which are the same as the ones at time $t_{(3)}$, are not included in Figure 24. It can be noted that at the end time of excavation, $t/T_K = 1.0$, the distribution of stresses is the same whatever excavation method is adopted, with largest compressive major principal stress at $\theta = 0^{\circ}$. Conversely, the distribution of stresses during excavation is significantly different for the two excavation methods.

6. Conclusions

Analytical expressions for the rock stress and displacement of deeply buried elliptical tunnels excavated in viscoelastic media were derived accounting for sequential excavation processes. An initial anisotropic stress field was assumed so that complex geological conditions can be accounted for, with the rock mass modeled as linearly viscoelastic. Solutions were derived for a sequential excavation process, with the major and minor axes of the tunnel growing monotonically, according to a time-dependent function to be specified by the designers.

First, an extension of the principle of correspondence to solve viscoelastic problems involving time-dependent stress boundaries was laid out employing the Laplace transform technique and complex potential theory. From the problem formulation it emerges that the stress field depends only on the shape and size of the opening, whereas displacements are a function of the rock rheological properties. The methodology described in this paper may in principle be applied to obtain analytical solutions for any other arbitrary cross-sectional shapes of tunnels excavated in viscoelastic rock.

The solution for sequentially excavated tunnels of elliptical cross-section was derived by introducing an inverse conformal mapping which allows eliminating the variable t from the conformal mapping in the two complex potentials. The analytical integral expressions of the solution obtained for the generalized Kelvin viscoelastic model include the Maxwell and Kelvin models as particular cases. To validate the methodology, FEM analyses were run. A good agreement between analytical solution and FEM analyses was shown.

Finally, a parametric analysis for various excavation rates and excavation methods was performed from which the following conclusions may be drawn:

- Slow excavation rates lead to larger normal displacements at the end of the excavation time, whilst tangential displacements show no significant difference for various excavation rates. The maximum absolute value of the tangential displacement during the sequential excavation stage is larger for slower excavation rates.
- For rocks of small stiffness ratios, G_K/G_H , the final steady state is reached later with the displacements occurring after the end of the excavation process being larger.
- Sequential excavation methods with faster excavation rate in the early stages lead to larger normal displacements and smaller absolute values of negative tangential displacements but larger positive ones after excavation.
- The normal displacement increases with the angle from the horizontal of the direction considered, whereas the tangential displacement shows first an increase then a decrease. The angle of the orientation corresponding to the maximal tangential displacement becomes smaller over time.

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Appendix A.

This appendix presents the analytical derivation of the first two terms in Eq. (20) and Eq. (21) for the purpose of calculating the excavation induced stresses and displacements. The two equations (20) and (21) are listed in the following as Eq. (A.1) and Eq. (A.2):

$$s_{ij}^{v}(t') = 2e_{ij}^{v}(0^{+})G(t') + 2\int_{0}^{t_0'^{-}} G(t'-\tau)\frac{de_{ij}^{v}}{d\tau}d\tau + 2\int_{t_0'^{+}}^{t'} G(t'-\tau)\frac{de_{ij}^{v}}{d\tau}d\tau + 2[e_{ij}^{v}(t_0'^{+}) - e_{ij}^{v}(t_0'^{-})]G(t'-t_0') \quad (A.1)$$

$$s_{ij}^{v}(t'_0) = 2e_{ij}^{v}(0^+)G(t'_0) + 2\int_0^{t'_0{}^-} G(t'_0 - \tau)\frac{de_{ij}^{v}}{d\tau}d\tau \tag{A.2}$$

The first and second terms in Eq. (A.1) are as follows:

$$\text{The first term:} \quad F_1^1 = 2e_{ij}^{v}(0^+)G(t') \tag{A.3}$$

$$\text{The second term:} \quad F_2^1 = 2\int_0^{t'_0{}^-} G(t' - \tau)\frac{de_{ij}^{v}}{d\tau}d\tau \tag{A.4}$$

The corresponding terms in Eq. (A.2) are as follows:

$$\text{The first term:} \quad F_1^2 = 2e_{ij}^{v}(0^+)G(t'_0) \tag{A.5}$$

$$\text{The second term:} \quad F_2^2 = 2\int_0^{t'_0{}^-} G(t'_0 - \tau)\frac{de_{ij}^{v}}{d\tau}d\tau \tag{A.6}$$

The expressions for displacements and strain rates in the rock before the excavation starts needed in the derivation, are obtained in the next section A.1. Note that the exact expressions of the coefficients are not given, since only the form of functions of the coefficients with respect to the given parameters are necessary in the demonstration. .

A.1. Expressions for displacements and strain rates before the excavation

Substituting Eq. (29) into Eq. (15), and assuming that:

$$H(t') = \mathscr{L}^{-1}\left[\frac{1}{s\mathscr{L}\left[G(t')\right]}\right], \quad I(t') = \mathscr{L}^{-1}\left[\frac{\kappa_L(s)}{s\mathscr{L}\left[G(t')\right]}\right], \tag{A.7}$$

The displacements occurred prior to the excavation can be calculated as (solution B-vis):

$$u_x^{(B)v} + iu_y^{(B)v} = \frac{(1+\lambda)p_0 z + 2(1-\lambda)p_0 \bar{z}}{8}\int_0^{t'} H(t' - \tau)d\tau - \frac{(1+\lambda)p_0 z}{8}\int_0^{t'} I(t' - \tau)d\tau \tag{A.8}$$

Substituting the functions of the shear and bulk relaxation moduli of adopted viscoelastic model into Eq. (A.8), the explicit expressions can be obtained. If only shear viscoelasticity is considered, *i.e.* $K(t)=K_e$, displacements can be derived as follows:

$$u_i^{(B)v} = A_i^{1M}(x,y) + A_i^{2M}(x,y)t' + A_i^{3M}(x,y)\exp(-\lambda_{M1}t'), \quad i=1,2, \lambda_{M1}>0 \text{ for the Maxwell model} \tag{A.9}$$

$$u_i^{(B)v} = A_i^{1B}(x,y) + A_i^{2B}(x,y)\exp(-\lambda_{B1}t') + A_i^{3B}(x,y)\exp(-\lambda_{B2}t'),$$

$$i=1,2, \lambda_{B1}>0, \lambda_{B2}>0 \text{ for the generalized Kelvin model} \tag{A.10}$$

where the terms with subscript i=1 denote the components of the x direction, and i=2 corresponds to y direction. By Eq. (A.8), the coefficients A_i^{jM}, A_i^{jB} (i=1,2 and j=1-3) are determined, which is the functions of coordinates and included the material parameters.

According to Eqs. (A.9) and (A.10), and the strain-displacement relations, $\varepsilon_{ij}^v = \frac{1}{2}[u_{i,j}^v + u_{j,i}^v]$, as well as the definition of strain deviators given in Eq. (2), the derivative of strain deviators for time t' can be calculated as follows:

$$\frac{de_{ij}^v}{dt'} = A_{ij}^{4M}(x,y) + A_{ij}^{5M}(x,y)\exp(-\lambda_{M1}t'),\quad i=1,2,\quad j=1,2 \quad \text{for the Maxwell model} \tag{A.11}$$

$$\frac{de_{ij}^v}{dt'} = A_{ij}^{4B}(x,y)\exp(-\lambda_{B1}t') + A_{ij}^{5B}(x,y)\exp(-\lambda_{B2}t'),\quad i=1,2,\quad j=1,2$$

$$\text{for the generalized Kelvin model} \tag{A.12}$$

For models with unlimited viscosity (Type B models), e.g. Burgers model, the expressions as a function of time for displacement and strain rate are analogous to Eqs. (A.9) and (A.11), which have different coefficients A for different models; conversely for models of limited viscosity, e.g. Kelvin model, the expressions with time are analogous to Eqs. (A.10) and (A.12).

A.2. Derivation for the generalized Kelvin model

According to the expression of $G(t)$ for the generalized Kelvin model in Eq. (46), when the time tend to infinity or is large enough, $G(t)$ will be a constant, which is a general law for Type A viscoelastic models. Because the excavation started at the time much later than that the initial stresses applied, the time t_0' (beginning of excavation,), and the generic time $t' \geq t_0'$, can be treated as infinity. Therefore, $G(t') = G(t_0')$ and the first terms from Eq. (A.1) and (A.2) are equal, that is $F_1^1 = F_1^2$.

Substituting Eq. (46) into Eq. (A.4), yields

$$\begin{aligned} F_2^1 &= 2\int_0^{t_0'^-} G(t'-\tau)\frac{de_{ij}^v}{d\tau}d\tau = 2\int_0^{t_0'^-}\left[C_1^B\exp\left[-\lambda_B(t'-\tau)\right] + C_2^B\right]\frac{de_{ij}^v}{d\tau}d\tau \\ &= 2\left\{\int_0^{t_0'^-} C_1^B\exp\left[-\lambda_B(t'-\tau)\right]\frac{de_{ij}^v}{d\tau}d\tau + \int_0^{t_0'^-} C_2^B\frac{de_{ij}^v}{d\tau}d\tau\right\} \\ &= 2\left(B_1^1 + B_2^1\right) \end{aligned} \tag{A.13}$$

where C_1^B and C_2^B are coefficients which is independent of time, and:

$$\begin{aligned} B_1^1 &= \int_0^{t_0^{'-}} C_1^B \exp\left[-\lambda_B(t'-\tau)\right] \frac{de_{ij}^v}{d\tau} d\tau \\ B_2^1 &= \int_0^{t_0^{'-}} C_2^B \frac{de_{ij}^v}{d\tau} d\tau \end{aligned} \tag{A.14}$$

Substituting Eq. (46) into Eq. (A.6), yields:

$$\begin{aligned} F_2^2 &= 2\int_0^{t_0^{'-}} G(t_0' - \tau)\frac{de_{ij}^v}{d\tau} d\tau = 2\int_0^{t_0^{'-}} \left[C_1^B \exp\left[-\lambda_B(t_0'-\tau)\right] + C_2^B\right] \frac{de_{ij}^v}{d\tau} d\tau \\ &= 2\left\{\int_0^{t_0^{'-}} C_1^B \exp\left[-\lambda_B(t_0'-\tau)\right]\frac{de_{ij}^v}{d\tau} d\tau + \int_0^{t_0^{'-}} C_2^B \frac{de_{ij}^v}{d\tau} d\tau\right\} \\ &= 2\left(B_1^2 + B_2^2\right) \end{aligned} \tag{A.15}$$

where:

$$\begin{aligned} B_1^2 &= \int_0^{t_0^{'-}} C_1^B \exp\left[-\lambda_B(t_0'-\tau)\right] \frac{de_{ij}^v}{d\tau} d\tau \\ B_2^2 &= \int_0^{t_0^{'-}} C_2^B \frac{de_{ij}^v}{d\tau} d\tau \end{aligned} \tag{A.16}$$

It can be noted from Eqs. (A.14) and (A.16) that $B_2^1 = B_2^2$. Substituting Eq. (A.12) into the expression of B_1^1 (in Eq. (A.14)), yields:

$$B_1^1 = \int_0^{t_0^{'-}} C_1^B \exp\left[-\lambda_B(t'-\tau)\right]\left[A_{ij}^{4B}\exp(-\lambda_{B1}\tau) + A_{ij}^{5B}\exp(-\lambda_{B2}\tau)\right] d\tau \tag{A.17}$$

After integration, then rearranging:

$$B_1^1 = D_{ij}^{1B}\left[\exp\left(-\lambda_3(t'-t_0')\right)\exp\left(-\lambda_{B1}t'\right) - \exp\left(-\lambda_B t'\right)\right] + D_{ij}^{2B}\left[\exp\left(-\lambda_4(t'-t_0')\right)\exp\left(-\lambda_{B2}t'\right) - \exp\left(-\lambda_B t'\right)\right] \tag{A.18}$$

where: $\lambda_3 = \lambda_B - \lambda_{B1} > 0$, $\lambda_4 = \lambda_B - \lambda_{B2} > 0$. When $t' \to \infty$ and $t_0' \to \infty$, Eq. (A.18) becomes:

$$B_1^1\Big|_{\substack{t' \to \infty \\ t_0' \to \infty}} = 0 \tag{A.19}$$

Substituting Eq. (A.12) into the expression of B_1^2 yields:

$$B_1^2 = D_{ij}^{1B}\left[\exp\left(-\lambda_{B1}t_0'\right) - \exp\left(-\lambda_B t_0'\right)\right] + D_{ij}^{2B}\left[\exp\left(-\lambda_{B2}t_0'\right) - \exp\left(-\lambda_B t_0'\right)\right] \tag{A.20}$$

when $t_0' \to \infty$, $B_1^2 = 0$. According to Eqs. (A.13) and (A.15), as well as the conclusion that $B_2^1 = B_2^2$, $B_1^1 = B_1^2 = 0$, the second term of Eq. (A.1) is equal to that of Eq. (A.2), that is, $F_2^1 = F_2^2$. Owing to the fact that the first and second term in Eqs (A.1) are equal to the corresponding terms in Eq. (A.2), the equality of sum of the first and second terms in Eqs. (A.1) and (A.2) has been demonstrated, which is used in Section 3.3. An analogous demonstration can be carried out for the rheological models Type A with limited viscosity, achieving the same conclusion.

A.3. Derivation for the Maxwell model

The expression of *G*(*t*) for the Maxwell model is in form of exponential function as shown in Table 1. It is obvious that $G(t^{'}) = G(t_0^{'})$ because $t^{'}$ ($t^{'} > t_0$) and $t_0^{'}$ can be regarded as infinity. Substituting into Eqs. (A.3) and (A.5), the first terms from two equations are equally as $F_1^1 = F_1^2$. For any values of $t^{'}$ and $t_0^{'}$, the second term of Eq. (A.1) can be written as:

$$\begin{aligned} F_2^1 &= 2\int_0^{t_0^{'-}} \left[C_1^M \exp\left(-\lambda_{M2}(t^{'} - \tau)\right)\right] \frac{de_{ij}^v}{d\tau} d\tau \\ &= \exp\left(-\lambda_{M2}(t^{'} - t_0^{'})\right) \cdot 2\int_0^{t_0^{'-}} G(t_0^{'} - \tau) \frac{de_{ij}^v}{d\tau} d\tau \\ &= \exp\left(-\lambda_{M2}(t^{'} - t_0^{'})\right) F_2^2 \end{aligned} \tag{A.21}$$

The term $\exp\left(-\lambda_{M2}(t^{'} - t_0^{'})\right)$ is not zero. The second term of Eq. (A.2) can be written as:

$$F_2^2 = 2\int_0^{t_0^{'-}} \left[C_1^M \exp\left(-\lambda_{M2}(t_0^{'} - \tau)\right)\right]\left[A_{ij}^{4M} + A_{ij}^{5M} \exp(-\lambda_{M1}\tau)\right] d\tau \tag{A.22}$$

After integration and rearranging, yields:

$$F_2^2 = D_{ij}^{1M}\left[1 - \exp\left(-\lambda_{M2} t_0^{'}\right)\right] + D_{ij}^{2M}\left[\exp\left(-\lambda_{M1} t_0^{'}\right) - \exp\left(-\lambda_{M2} t_0^{'}\right)\right] \tag{A.23}$$

When $t^{'} \to \infty$ and $t_0^{'} \to \infty$, the above equation becomes:

$$F_2^2 = D_{ij}^{1M} \neq 0 \tag{A.24}$$

According to Eqs. (A.21) and (A.24), the second terms in Eqs. (A.1) and (A.2) are not equal, but have the relationships shown in Eq. (A.21).

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