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Optimal Impulse Control Problems and Linear Programming

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Abstract— **Optimal impulse control problems are, in general, difficult to solve. A current research goal is to isolate those problems that lead to tractable solutions. In this paper, we identify a special class of optimal impulse control problems which are easy to solve. Easy to solve means that solution algorithms are polynomial in time and therefore suitable to the on-line implementation in real-time problems. We do this by using a paradigm borrowed from the Operations Research field. As main result, we present a solution algorithm that converges to the exact solution in polynomial time. Our approach consists in approximating the optimal impulse control problem via a binary linear programming problem with a totally unimodular constraint matrix. Hence, solving the binary linear programming problem is equivalent to solving its linear relaxation. It turns out that any solution of the linear relaxation is a feasible solution for the optimal impulse control problem. Then, given the feasible solution, obtained solving the linear relaxation, we find the optimal solution via local search.**

I. INTRODUCTION

This paper is one of the several recent attempts [2], [3], [4], [5], [6], [10], [20] to apply the tools of combinatorial optimization to hybrid optimal control problems. Such problems are, in general, difficult to solve (see, e.g., [8], [10], [21] and references therein). Furthermore, due to the generality and complexity of the models addressed, no theoretical approach is available to study the difficulty of the problems and the computational complexity of the available solution algorithms.

For this reason, a current research goal is to *isolate those problems that lead to tractable solutions* [8]. According to this, the aim of this paper is to identify among the larger set of hybrid optimal control problems dealt in [10], a special class of optimal impulse control problems which are easy to solve. Easy to solve means that, not only discrete optimization techniques can be applied, but also that solution algorithms are polynomial in time and therefore suitable to the on-line implementation in real-time problems. We do this by using a paradigm borrowed from the Operations Research field. For the level of abstractness chosen in our approach, impulsively controlled systems and operations research models are linked together in their simplest form. Any extensions of the presented results to more complex classes of systems is beyond the scope of this work.

As main result, we present a solution algorithm that converges to the exact solution in polynomial time. The system considered is a continuous-time system subject to *controlled impulses* [8], [10], i.e., the state jumps in response to a control command with an associated cost. In particular, the system is an integrator subject to impulsive resets and can describe any storage system in the economic and financial world [13], [15], [16], [17] (see, e.g., [7] for an exhaustive list of applications).

The decision problem (henceforth also optimal impulse control problem) consists in finding the optimal schedule of the impulses to drive and keep the system in a safe operating interval, while minimizing a function related to the cost of the resets. The decision variables are thus binary (whether to reset the state at a given time instant or not). We link the approach to the Input to State Stabilizability (ISS) of impulsively controlled systems, according to the definition provided in [14]. In particular, we focus on ISS systems with dwell time and reverse dwell time.

The decision problem is solved in two steps. First, a related problem is considered, which can be formulated as a binary linear program [9], [19] whose constraints are described by an *interval matrix* [19]. The cost function is linear and the problem can be solved by linear programming (LP), even if binary variables are involved (see, e.g., a previous efficient solution approach based on linear programming in [11]). Then, a local search algorithm [1] is applied to obtain the solution of the original problem by exploiting the solution of the related one. The LP is solved in polynomial time and the local search is shown to have linear complexity w.r.t. the length of the problem horizon. Thus, the total complexity is polynomial, while a "brute-force" approach has a combinatorial complexity because of the binary variables. Numerical illustrations of a queuing system [12], [18] are provided.

This paper is organized as follows. In Section II, we introduce the problem. In Section III, we discuss total unimodularity and connections with (reverse) dwell time conditions. In Section IV, we derive the local search algorithm. Finally, in Section V, we draw some conclusions and discuss future works.

II. IMPULSIVELY-CONTROLLED SYSTEM

Equation (1) describes an impulsively-controlled system where function $f : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ is the dynamics of $x(t)$, function $h(x(t), d(t))$ is the reset value at time t^+, i.e., at the time instant after an impulse has occured at time t, variable $u(t)$ is the impulse control law returning impulses whenever $u(t)$ is set to one, variable $d(t)$ is a disturbance:

$$\begin{array}{ll} \dot{x}(t) = f(x(t), d(t)) & \text{if } u(t) = 0 \\ x(t^+) = h(x(t), d(t)) & \text{if } u(t) = 1. \end{array} \tag{1}$$

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Let $c(x(t),t) = K(t) + \Psi(x(t))$ be the cost of control $u(t)$ for all $t \geq 0$, where $\Psi(\cdot)$ is a function of the state $x(t)$ and $K(\cdot)$ is a function of time. Denote by $u(\cdot)$ the values of $u(t)$ for all $t \geq 0$ and call it (time based) *control law*. Then, after denoting $\delta(t)$ a function returning a Dirac impulse at any time t where $u(t) = 1$, the cost associated to a given control law $u(\cdot)$ is

$$J(u(\cdot)) = \int_0^\infty c(x(t),t)\delta(t)dt. \tag{2}$$

Here dependence of $J(u(\cdot))$ on the initial state is omitted. The cost functional (2) sums the costs $c(x(\tau_i),\tau_i)$ at the times $t = \tau_i$, $i = 1,2,\ldots$, where impulses occur. Then, if we assume both costs and number of impulses bounded, convergence of (2) is not an issue.

Assumption 1: Assume that i) function $f(\cdot,\cdot)$ satisfies $\frac{\partial f_i}{\partial x_j} > 0$ for all $i \neq j$, and ii) $h(x(t),d(t)) \leq x(t)$ componentwise where the last inequality holds strictly for at least one component, and iii) function $\Psi(\cdot)$ satisfies $\Psi(\eta) \geq 0$ for all $\eta \in \mathbb{R}^n$ and $\frac{\partial \Psi}{\partial x_j} \leq 0$ for all j.

Assumption i) and ii) mean that dynamics $f(\cdot,\cdot)$ makes the state x in the positive orthant to diverge from zero while impulses drive x near to zero. This is typical of systems with an unstable dynamics subject to stabilizing impulses. Assumption iii) is used only to facilitate the local search procedure discussed later on (see, e.g., the proof of Lemma 1).

Problem 1: Find an impulse control law $u(\cdot)$ that minimizes the cost (2) and such that system (1) is input to state stable (ISS) according to the definition of [14].

Note that cost (2) is non linear and the control law $u(\cdot)$ is discontinuous. The idea is then to reformulate the problem in a receding horizon framework.

A. Receding horizon

Let a finite set of times $\{r_0,\ldots,r_h\}$ be arbitrarily chosen and consider a receding horizon from time r_i to time r_{i+1}, with $i = 0,\ldots,h-1$. Optimization is carried out on each interval $[r_i,r_{i+1}]$ at a time (control and prediction horizons coincide). In particular, take a sample interval $\Delta t = \frac{r_{i+1}-r_i}{N}$ with the number of samples N chosen arbitrarily and extract the associated discrete times $r_i + k\Delta t$ with $k = 0,\ldots,N$. Let the discrete time continuous state be $\xi(k)$, with the initial condition $\xi(0) = x(r_i)$. Also, assume that control impulses can occur only at discrete times and let the discrete time control $\mu(k)$ and disturbance $\gamma(k)$ be obtained by sampling $u(t)$ and $d(t)$ at time $r_i + k\Delta t$, i.e., $\mu(k) = u(r_i + k\Delta t)$ and $\gamma(k) = d(r_i + k\Delta t)$.

Then, for $k = 0,\ldots,N-1$, the sampled counterpart of system (1) is

$$\begin{array}{rcl} \xi(k+1) & = & \xi(k) + w(\xi(k),\gamma(k)) + \\ & + & (h(\xi(k),\gamma(k)) - \xi(k))\mu(k), \quad \xi(0) = x(r_i) \\ \mu(k) & \in & \{0,1\}, \end{array} \tag{3}$$

where we denote by

$$w(\xi(k),\gamma(k)) = \int_{r_i+k\Delta t}^{r_i+(k+1)\Delta t} f(x(t),d(t))dt. \tag{4}$$

Feasible solutions for fixed horizon $[r_i,r_{i+1}]$, are $\mathbf{u}$, $\mathbf{d}$, $\mathbf{x}$, and $\mathbf{w}(\mathbf{x},\mathbf{d})$, such that system (3) is ISS where we define

$$\begin{array}{ll} \mathbf{u} = [\mu(0),\ldots,\mu(N-1)] & \mathbf{d} = [\gamma(0),\ldots,\gamma(N-1)] \\ \mathbf{x} = [\xi(0),\ldots,\xi(N)] & . \end{array}$$

For a compact description, define the feasible solution set

$$\mathscr{F}(x(r_i)) = \{\mathbf{u},\mathbf{d},\mathbf{x} \in \{0,1\}^N \times \mathbb{R}^{N\times m} \times \mathbb{R}^{(N+1)\times n} : \text{system (3) is ISS}\}.$$

Note that the feasible solution set depends on $x(r_i)$ because of the initial conditions on the discrete time state ξ in (3). Also $x(r_i)$ is measured and full known at the beginning of the horizon and therefore it can be dealt with as known parameter.

Now, given the set $H = \{0,1,2,\ldots,N\}$ of possible values of the index k spanning over the horizon window, consider a generic set of subsets $\{C_1,\ldots,C_m\}$ such that $\bigcup_j C_j = H$ and each C_j is made by consecutive elements of H, i.e., given any pair $y,z \in C_j$ with $y < z$ this implies $v \in C_j$ for any integer number $y < v < z$ and for all $j = 1,\ldots,m$. Sets C_j's may overlap one each other.

We claim that in a number of cases there exists a specific set of subsets $\{C_1,\ldots,C_m\}$ with $m \leq N$ such that system (3) is ISS under certain linear conditions on the binary controls and on the initial states $x(r_i)$ of the horizon. Some of these cases are based on the notions of dwell time and reverse dwell time [14] and will be discussed in Section III-A and III-B.

At the initial time r_i of the horizon, the aforementioned conditions take on the form

$$\sum_{k\in C_j} \mu(k) \geq l_j(x(r_i)), \quad \text{for all } j = 1,\ldots,m \tag{5}$$

where function $l_j : \mathbb{R}^n \to \{0,1\}$ models some logical conditions for $x(r_i)$.

Then, we can get rid of $\mathbf{x},\mathbf{w}(\mathbf{x},\mathbf{d})$ and rewrite the feasible solution set in a simplified manner as shown below

$$\mathscr{F}(x(r_i)) = \{\mathbf{u} \in \{0,1\}^N : \text{conditions (5) satisfied}\}.$$

Rewriting the solution set as above requires sets C_j's to be a priori known and has the advantage of converting the original dynamic problem (because of the presence of the state variable) into a static one. This is possible as in a receding horizon setting, variable $x(r_i)$ once measured at time r_i enters as parameter in the right-hand side of (5).

To complete the formulation of the receding horizon problem, let the following vectors of sampled costs (index s means "sampled") and approximated costs ("tilde" means approximate) be given

$$\mathbf{c}_s = [c_s(0),\ldots,c_s(N-1)], \quad c_s(k) = K(r_i + k\Delta t) + \Psi(\xi(k)), \forall k \tag{6}$$

$$\tilde{\mathbf{c}}_s = [\tilde{c}_s(0),\ldots,\tilde{c}_s(N-1)], \quad \tilde{c}_s(k) = K(r_i + k\Delta t), \forall k \tag{7}$$

The receding horizon problem with exact costs is then

$$\min_{\mathbf{u}\in\mathscr{F}(x(r_i))} \mathbf{c}_s\mathbf{u}, \tag{8}$$

which we next approximate by solving the simpler problem with state independent costs

$$\min_{\mathbf{u}\in\mathscr{F}(x(r_i))} \tilde{\mathbf{c}}_s\mathbf{u}. \tag{9}$$

Finally, let $\mu(0),\ldots,\mu(N-1)$ be the optimal sequence of discrete controls, we need to reconstruct the continuous time controls $u(t)$. We can do this through the following function $\theta:\{0,1\}^N \mapsto \{u(t), r_i \le t < r_{i+1}\}$ returning, for each interval $[r_i+k\Delta t, r_i+(k+1)\Delta t)$, the control $u(r_i+k\Delta t)=\mu(k)$ and $u(t)=0$ for all $t\in(r_i+k\Delta t, r_i+(k+1)\Delta t)$.

III. Total unimodularity

There is an important aspect that needs to be emphasized and represents the main result of this work (see also [2]). The set of feasible solutions $\mathscr{F}(x(r_i))$ is a discrete set in the sense that it contains only integer points. However we can replace the integrality constraints $\mathbf{u}\in\{0,1\}^N$ by the relaxed and more tractable constraints $0\le\mathbf{u}\le 1$ and consider the resulting polytope

$$\mathscr{F}(x(r_i)) = \{\mathbf{u}\in\mathbb{R}^N : \sum_{k\in C_j}\mu(k) \ge l_j(x(r_i)), \forall j=1,\ldots,m, 0\le\mathbf{u}\le 1\}.$$

We clarify this aspect more in details next. Let us rewrite the inequalities (5) in matrix form. We can do this by using a matrix $A\in\{0,1\}^{m\times N}$, with only entries 0 and 1, one row for each inequality of type (5), one column for each time k. Observe that the constraint matrix is an *interval matrix*, i.e., it has 0-1 entries and each row is of the form

$$(0,\ldots,0 \quad \underbrace{1,\ldots\ldots\ldots,1}_{\text{consecutive 1's}} \quad 0,\ldots,0).$$

It is well known from the literature [19] that each interval matrix is totally unimodular where we remind here that a matrix is *totally unimodular* if the determinant of any square sub-matrix is equal to -1, 0 or 1.

This means that the polyhedron obtained from $Proj(\mathscr{F})$ by replacing the integrality constraints $\mathbf{u}\in\{0,1\}^N$ with the linear constraint $0\le\mathbf{u}\le 1$ is an integral polyhedron. As a consequence we have that the linear relaxation of the receding horizon problem (9) has an integral optimal solution as established in the next theorem. Let the vector of logical conditions be defined as $l=[l_1(x(r_i)),\ldots,l_m(x(r_i))]^T$.

Theorem 1: Solving the receding horizon problem (9) is equivalent to solving the linear programming problem

$$\min_{\mathbf{u}} \quad \tilde{\mathbf{c}}_s\mathbf{u} \tag{10}$$

$$s.t. \quad A\mathbf{u}\ge l \tag{11}$$

$$0\le\mathbf{u}\le 1. \tag{12}$$

Proof: Apply a standard technique in linear programming to turn the constraints (11) into equalities of type

$$[A \quad I]\begin{bmatrix}\mathbf{u}\\ \mathbf{s}\end{bmatrix} = l \tag{13}$$

where $\mathbf{s}\in\mathbb{R}^m$ is the surplus vector and $I\in\mathbb{R}^{m\times m}$ is the identity matrix. From the properties of total unimodular matrices one knows that if A is totally unimodular then also $[A \quad I]$ is totally unimodular. Then, take a generic square sub-matrix $R\in\mathbb{R}^{m\times m}$ and observe that $det(R)\in\{0,\pm 1\}$. Any feasible base solution of (13) is of the form $\bar{v}=R^{-1}l=\frac{adj(R)}{det(R)}l$ where $adj(R)$ is the adjoint matrix of R. Hence, because of the integrality of l and $det(R)$ we have that $\bar{v}$ is integer. This means that constraints (11)-(12) define an integral polyhedron, and that the optimal solution of the linear programming problem (10)-(12) is also integer.

■

A. Dwell time

In [14] it has been shown that for a number of systems Problem 1 can be solved by any impulse control law $u(t)$ satisfying some so-called *dwell time* conditions. A typical dwell time condition requires that intervals between consecutive impulses must be no shorter than T time units.

Consider the sampled counterpart (3) starting at time r_i, take for simplicity $\Delta t=1$, and assume that $r_i-\hat{k}$ is the time of the last switch. The following linear programming problem of type (10)-(12) returns a switching control satisfying the above dwell time condition:

$$\min_{\mathbf{u}} \tilde{c}\mathbf{u}, \quad \text{s.t.} \quad 0\le\mathbf{u}\le 1,$$
$$\underbrace{\begin{bmatrix}\overbrace{1\ldots\ldots 1}^{b} & 0\ldots 0\\ 0\ldots\ldots 0 & 1\ldots 1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}\mu(0)\\ \vdots\\ \mu(N-1)\end{bmatrix}}_{\mathbf{u}} \ge \underbrace{\begin{bmatrix}0\\ 1\end{bmatrix}}_{l}, \tag{14}$$

where $b=T-\hat{k}$. The above problem derives from taking $C_1=\{0,\ldots,T-\hat{k}\}$, and $C_2=\{(T-\hat{k})+1,\ldots,N\}$. Note that the above constraint matrix A does not exclude multiple switchings between $(T-\hat{k})+1$ and N which possibly violate the dwell time condition. However such solutions though feasible, are not optimal for problem (10)-(12) as multiple switchings increase the cost.

The receding horizon process repeats at time $r_{i+1}=r_i+(N-1)\Delta t$ (regular starting times) or at time $r_{i+1}=r_i+(\hat{k}+1)\Delta t$ where $\hat{k}$ is the last switching time returned by the problem solved at time r_i (time-varying starting times).

B. Reverse dwell time

On the contrary, a typical reverse dwell time condition requires that intervals between consecutive impulses must be no longer than T time units. We can generalize the approach by considering m different dwell times of $T_1, T_2,\ldots,T_m$ over the horizon. We expand more on this topic next.

Consider the sampled counterpart (3) starting at time r_i, take for simplicity $\Delta t=1$, and assume that r_i-1 is the time of the last impulse. The following linear programming problem of type (10)-(12) returns a switching control satisfying the above reverse dwell time condition:

$$\min_{\mathbf{u}} \tilde{c}\mathbf{u}, \quad \text{s.t.} \quad 0 \le \mathbf{u} \le 1,$$

$$\underbrace{\begin{bmatrix} \overbrace{1\ldots1}^{T_1} & 0\ldots0\ldots & \overbrace{0\ldots0}^{T_m} \\ 01\ldots\ldots\ldots & 1\ldots & 0\ldots0 \\ \vdots & \ddots & \vdots \\ 0\ldots0 & 0\ldots0\ldots & 1\ldots1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \mu(0) \\ \vdots \\ \mu(N-1) \end{bmatrix}}_{\mathbf{u}} \ge \underbrace{\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}}_{l}. \tag{15}$$

The above problem derives from taking $C_1 = \{0,\ldots,T_1\}$, $C_2 = \{1,\ldots,2+T_2\}$, …, $C_i = \{i-1,\ldots,i+T_i\}$, …, $C_m = \{m-1,\ldots,N\}$.

IV. Local Search

Back to Problem (8), we can use the solution of the linear programming problem (10)-(12), as initial solution and improve it via local search. We will prove that such a solution dominates a number of other solutions (see Lemma 2) and this facilitates (reduces the number of computations in) the search for the optimal solution. The local search algorithm solves a sequence of the following subproblems on a moving horizon.

Let set $C_j = \{\alpha, \alpha+1, \ldots, \beta-1, \beta\}$ be given with $1 \le \alpha \le \beta \le N$. We can redefine Problem (8) over the interval $[\alpha, \beta]$ as follows

$$\min_{\mathbf{u} \in \mathscr{F}_j(x(r_i))} \mathbf{c}_s^j \mathbf{u}, \tag{16}$$
$$\xi(k) := \xi_0,$$

with pre-assigned $\xi_0 \in \mathbb{R}^n$ and where $\mathbf{c}_s^j = [c_s(k)]_{k \in C_j}$ and

$$\mathscr{F}_j(x(r_i)) = \{\mathbf{u} \in \{0,1\}^{|C_j|} : \sum_{k \in C_j} \mu(k) \ge l_j(x(r_i))\}.$$

Now, let $\tilde{\mu}_{[\alpha,\beta]} = (\tilde{\mu}(\alpha), \ldots, \tilde{\mu}(\beta))$ be the solution returned by the linear programming problem (10)-(12) restricted to the interval $[\alpha, \beta]$. The solution may be not unique only in pathological situations, for instance, when $c_s(k) = c_s(k+1)$ for a given pair k and $k+1$ of successive time instants. In this case an impulse at time k or $k+1$ returns the same cost. Nevertheless, as it will be clearer later on, the existence of multiple solutions does no compromise the validity of the approach as we can take any of the multiple solutions to initialize the local search. Note that A is now simply a row vector as we are considering just one set C_j. Also, observe that the above solution must have only one non null component in addition to the first one, that is $\tilde{\mu}_{[\alpha,\beta]} = (1, 0, \ldots\ldots\ldots, 0, 1, 0\ldots0)$. This is evident as any other solution with additional non null components $(1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0\ldots0)$ provides a higher cost for the linear programming problem. When this happens, we say that this last solution *is dominated* by $\tilde{\mu}_{[\alpha,\beta]}$. Let $\tilde{\mu}(i)$ be the non null component, then, we can state the following lemma.

Denote by $\mu_{[\alpha,\beta]} = (\tilde{\mu}(\alpha), \mu(\alpha+1), \ldots, \mu(i), \tilde{\mu}(i+1), \ldots, \tilde{\mu}(\beta))$ for all $\mu(\alpha+1), \ldots, \mu(i) \in \{0,1\} \times \ldots \times \{0,1\}$ the sequences of controls that keep unchanged the last $\tilde{\mu}(i+1), \ldots, \tilde{\mu}(\beta)$ control components.

Lemma 1: Solution $\tilde{\mu}_{[\alpha,\beta]}$ dominates any other solution of type $\mu_{[\alpha,\beta]}$.

Proof: This proof is based on Assumption II. Actually, it holds $c_s(k) = K(r_i + k\Delta t) + \Psi(\xi(k)) \ge \tilde{c}_s(k) = K(r_i + k\Delta t)$ for all k. Now, if the index i gives the minimum to the cost with $\tilde{c}_s(k)$, then it gives also the minimum to the cost with $c_s(k)$, because the sequence $\Psi(\xi(k))$ is decreasing for increasing $\xi(k)$. ■

The interpretation of the above lemma is that no benefits derive from resetting before time i.

As a consequence of the above lemma, the optimal solution for the subproblem must be found among solutions of type $(\tilde{\mu}(\alpha), \ldots, \tilde{\mu}(i-1), \mu(i), \ldots, \mu(\beta))$ for all $\mu(i), \ldots, \mu(\beta) \in \{0,1\} \times \ldots \times \{0,1\}$. In other words, the solutions candidate for the optimum have the components α to $i-1$ unchanged and equal to $\tilde{\mu}(\alpha), \ldots, \tilde{\mu}(i-1)$. Then, in the search for the optimum it suffices to let the rest of the components from i to β be varying. In particular, the optimal solution, if different from $\tilde{\mu}_{[\alpha,\beta]}$, can be found by shifting the non null component forward in time. It makes sense then to define a *neighborhood* as follows. Given a solution $(\mu(\alpha), \ldots, 1, 0, \ldots, \mu(\beta))$ with just one non null component, the neighbor solution is $(\mu(\alpha), \ldots, 0, 1, \ldots, \mu(\beta))$ obtained by shifting the non null component at the next time instant. In the space of solutions with one null component, we define the distance $\|x-y\|$ between two solutions as the number of *shifting forward* operations to obtain y from x or viceversa. Searching the optimum has worst-case complexity linear in N as remarked next.

Remark 1: We can solve Problem (16) in polynomial time by first solving the associated linear programming problem to obtain an initial solution $\tilde{\mu}_{[\alpha,\beta]}$ and then by improving the initial solution via shifting forward operations until we obtain the optimal solution $\mu^*_{[\alpha,\beta]}$. Furthermore, shifting forward operations are at most $O(\beta - i)$ as $\|\mu^*_{[\alpha,\beta]} - \tilde{\mu}_{[\alpha,\beta]}\| \le \beta - i$.

Let $\mu^*_{[0,\gamma]} = (\mu^*(0), \mu^*(1), \ldots, \mu^*(\gamma))$ and $\mu^*_{[\gamma+1,\zeta]} = (\mu^*(\gamma+1), \ldots, \mu^*(\zeta))$ be the optimal solutions of Problem 16 restricted to the interval $[0,\gamma]$ and $[\gamma+1,\zeta]$, and associated to the sets $C_1 = \{0,\ldots,\gamma\}$ and $C_2 = \{\gamma+1,\ldots,\zeta\}$. In particular, the two problems above are related according to equation (1) which gives $\xi(\gamma+1)$ as a function of $\xi(\gamma)$ and $\mu^*(\gamma)$.

Also, denote by $\mu_{[0,\zeta]} = (\mu^*_{[0,\gamma]}, \mu^*_{[\gamma,\zeta]})$ the solution obtained merging the two optimal solutions.

Lemma 2: The solution $\mu_{[0,\zeta]}$ is optimal for Problem (16) defined in the interval $[0,\zeta]$.

Proof: The optimal solution in the interval $[0,\zeta]$, call it $\mu^*_{[0,\zeta]}$, is obtainable by merging the optimal solution in the two consecutive intervals $[0,\gamma]$ and $[\gamma+1,\zeta]$, where the initial state for the latter interval $\xi(\gamma+1)$ depends on controls $\mu^*_{[0,\gamma]}$. Now, if $\xi(\gamma+1)$ is set according to (1), $\xi(\gamma)$ and $\mu^*(\gamma)$, we match the initial condition defined in Problem (16). Then, the problem of finding the optimal solution in the interval $[\gamma+1,\zeta]$ is exactly Problem (16) from which we can conclude

the thesis. ■

With all previous results in mind, we can go back to Problem (8) and derive the following *local search algorithm*. At each iteration j, Problem (16) is to be solved with respect to the interval $[\alpha(j),\beta(j)]$ and set $C_j = \{\alpha(j),\ldots,\beta(j)\}$ where the two extreme elements of the cover $\alpha(j)$ and $\beta(j)$ are now function of j. The value of $\alpha(j)$ depends on the solution of iteration $j-1$ as it is explained more formally next (we initialize $\alpha(0) := 0$).

(1) `Assign` $j := 1$`; solve Problem (16) for` C_1 `to obtain the optimal solution`

$$\mu^*_{[0,l]} = (\mu^*(0),0,\ldots,0,\underbrace{\mu^*(\alpha(1)-1)}_{=1},0,\ldots,0,\mu^*(\beta(1))).$$

`Let the new non null component be` $\mu^*(\alpha(1)-1) = 1$ `and assign` $j := j+1$.

(2) `Let` $\mu^*(\alpha(j)-1)$ `be the non null component at the` j`th iteration,`

(2.a) `if there exists` $C_j = \{\alpha(j),\ldots,\beta(j)\}$`, then solve Problem (16) for` C_j `obtaining`

$$\mu^*_{[\alpha(j),\beta(j)]} = (\mu^*(\alpha(j)),0,\ldots,0,\underbrace{\mu^*(\alpha(j+1))}_{=1},0,\ldots,0,\mu^*(\beta(j))),$$

`and combine the latter solution with previous solutions to obtain`

$$\mu^*_{[0,\beta(j)]} = \left(\mu^*_{[0,\alpha(j)-1]},\mu^*_{[\alpha(j),\beta(j)]}\right);$$

(2.b) `otherwise set` $\mu(k) := 0$ `for all` $\alpha(j) \leq k \leq N$ `(no other impulses until the end of the horizon) and STOP the algorithm.`

The local search algorithm converges to the optimal solution in linear time as remarked next.

Remark 2: The above local search algorithm finds the optimal solution to Problem (8) in the worst-case in O(N). This is evident as the worst-case is when we have a minimal cover $C = \{1,\ldots,N\}$, for which we must compare all the $N-1$ shifting forward operations.

V. Conclusions

Using a paradigm borrowed from the Operations Research field, we have identified a special class of hybrid optimal control problems which are easy to solve. We have done this, by finding a solution algorithm that converges to the exact solution in polynomial time.

The system described in this paper is an integrator subject to impulsive resets. The decision problem consists in finding the optimal schedule of the impulses to maintain the system in a safe operating interval, while minimizing a function related to the cost of the resets. The decision variables are thus binary (whether to reset the state at a given time instant or not).

The optimal impulse control problem is solved in two steps. First, a related problem is considered, which can be formulated as a binary programming problem whose constraints are described by an *interval matrix*. The cost function is linear and the problem can be solved by linear programming (LP), even if binary variables are involved, because of its particular structure (the interval matrix is *totally unimodular*). Then, a local search algorithm is applied to obtain the solution of the original problem by exploiting the solution of the related one. The LP is solved in polynomial time and the local search is shown to have linear complexity w.r.t. the length of the problem horizon. Thus, the total complexity is polynomial, while a "brute-force" approach has a combinatorial complexity because of the binary variables.

Future research will extend the use of cutting planes algorithms to all those impulse control problems whose binary linear reformulation does not benefit from total unimodularity. In all these cases, we can no longer solve the linear relaxation and obtain binary solutions. So, cutting planes are introduced iteratively with the aim of eliminating fractional solutions. As done in this paper, cutting planes will be derived by exploiting the structure, if any, of the optimal impulse control problems. With "structure" we mean any type of conditions, as, for instance, the (reverse) dwell time conditions, that may lead to a simplified binary linear program.

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