Robust control of networks under discrete disturbances and controls

(Invited Paper)

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***Abstract*—We consider dynamic networks where the disturbances and control actions take discrete values. We briefly survey some of our recent results establishing necessary and sufficient conditions for the existence of robustly globally invariant (hyper box) sets, as well as sufficient conditions for global attractivity of such sets. We then establish connections between these results and existing results in the literature for the setup where all the inputs are analog. Finally, we derive tight upper and lower bounds on the smallest such set in the special case of a degenerate network.**

I. Introduction

Both production networks and distribution networks can be modeled as network flow systems, using graphs or hypergraphs whose nodes are associated with available raw materials, products, or resources, and whose arcs are associated with flows. The accumulated discrepancy over time of the input and output flows at n interconnected nodes is captured by the n-dimensional state of the system: Practically, this represents resources and or products available (stored) at the n warehouses. The control input denotes the controlled flows (i.e. the production or distribution) and the disturbance input denotes the uncontrolled flows (i.e. the raw material or demand). The dynamics of the system are thus linear, with the matrices defining the model describing which and in what quantity resources and products are involved in a unit flow. In this context, the existence of robustly control invariant sets is a question of interest, as are their properties when they do exist.

These and other similar questions have been previously considered in the literature, for instance in [5] [6] where polytopic invariant sets are considered and in [4] where ellipsoidal sets are considered. The existing results typically assume analog control and disturbance actions. In contrast, the novelty of our treatment is in the discrete nature of the inputs, justifiable from practical as well as theoretical perspectives. From a practical standpoint, materials and goods are usually processed in batches. From a theoretical standpoint, the study of systems under discrete controls and disturbances has sparked much interest in recent years as evidenced by the literature on alphabet control [**?**] [**?**], [7], mixed integer model predictive control [1], discrete team theory [**?**] and boolean control [3]. In this case, the problems of interest may often be formulated as min-max games [2].

In this paper, we consider a general discrete-time model of dynamic networks where the control actions and the disturbances are assumed to take their values in prescribed finite alphabet sets. We focus on three questions, namely the existence of robustly control invariant hyperboxes, global attractively of these sets, and the size of the smallest such invariant sets. We begin by surveying our recent results, presented in [**?**], in which we derived a necessary and sufficient condition for the existence of robustly control invariant hyperboxes and we showed that a stricter version of this same condition is sufficient (though not necessary in general) to guarantee robust global convergence of all state trajectories to the invariant set. Next, we show that the combinatorial conditions derived in [?] imply set inclusion conditions that are strikingly similar to existing results in the literature where the inputs are assumed to be analog [5] [6]. Finally, we derive upper and lower bounds on the size of the smallest robustly control invariant set in the simplified scalar setting, and we illustrate our theoretical derivation with a simple example.

The paper is organized as follows: We introduce the system model, state the questions of interest, and explain their significance in Section II. We briefly survey our relevant recent results in Section III. We elaborate on the connections between our results and existing results in the literature in Section IV. We derive tight upper and lower bounds on the size of the smallest invariant set for the special case of a degenerate network in Section V and we present a simple illustrative example in Section VI.

A word on notation: $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{R}_+$ and $\mathbb{Z}_+$ denote the reals, integers, non-negative reals and non-negative integers, respectively. $[x]_i$ denotes the i^{th} component of $x \in \mathbb{R}^n$. $hull\{S\}$ and $int(S)$ denote the convex hull and interior, respectively, of $S \subset \mathbb{R}^n$. $\mathbb{B}^n$ denotes the set of vertices of the unit hypercube, that is $\mathbb{B}^n = \{0,1\}^n$. For $\mathcal{A} \subset \mathbb{Z}$, $M \in \mathbb{Z}^{n\times m}$, $M\mathcal{A}^m$ denotes the image of $\mathcal{A}^m$ by M, that is $M\mathcal{A}^m = \{a \in \mathbb{Z}^n | a = Mb \text{ for some } b \in \mathcal{A}^m\}$. Given a set $X = [0, x_1^+] \times \ldots \times [0, x_n^+] \subset \mathbb{R}^n$, V_X denotes its set of vertices, that is $V_X = \{x \in \mathbb{R}^n | [x]_i \in \{0, x_i^+\}\}$.

II. PROBLEM SETUP AND SIGNIFICANCE

A. System model

Consider the system described by

$$x(t+1) = x(t) + Bu(t) - Dw(t), \qquad (1)$$

where time index $t \in \mathbb{Z}_+$, state $x(t) \in \mathbb{R}^n$, control input $u(t) \in \mathcal{U}^m$ and disturbance input $w(t) \in \mathcal{W}^p$. The control alphabet set $\mathcal{U} = \{a_1, \ldots, a_r\} \subset \mathbb{Z}$ and the disturbance alphabet set $\mathcal{W} = \{b_1, \ldots, b_q\} \subset \mathbb{Z}$ are discrete, ordered sets with $a_1 < \ldots < a_r$ and $b_1 < \ldots < b_q$. Matrices $B \in \mathbb{Z}^{n \times m}$ and $D \in \mathbb{Z}^{n \times p}$ are given.

B. Questions of Interest

Definition 1: A hyperbox $X = [0, x_1^+] \times \ldots \times [0, x_n^+] \subset \mathbb{R}^n$ is robustly control invariant if there exists a control law $\varphi : X \to \mathcal{U}^m$ such that for every $x(t) \in X$, $x(t+1) = x(t) + B\varphi(x(t)) - Dw(t) \in X$ for any disturbance $w(t) \in \mathcal{W}^p$.

Remark 1: When $X = [0, x_1^+] \times \ldots \times [0, x_n^+]$ is robustly control invariant, then so is any other hyperbox $X' = [x_1^-, x_1^- + x_1^+] \times \ldots \times [x_n^-, x_n^- + x_n^+]$. Indeed, control law $\varphi' : X' \to \mathcal{U}^m$ defined by $\varphi'(x) = \varphi(x - x^-)$, where $[x^-]_i = x_i^-$, verifies this assertion.

Definition 2: A hyperbox $X = [0, x_1^+] \times \ldots \times [0, x_n^+] \subset \mathbb{R}^n$ is robustly globally attractive if there exists a control law $\psi : \mathbb{R}^n \setminus X \to \mathcal{U}^m$ such that for every initial condition $x(0) \in \mathbb{R}^n \setminus X$ and disturbance $w : \mathbb{Z}_+ \to \mathcal{W}^p$, the corresponding state trajectory satisfies $x(\tau) \in X$ for some $\tau \in \mathbb{Z}_+$.

We are interested in answering two questions for the dynamic networks described in (1):

Question 1: Under what conditions does a robustly control invariant hyperbox exist?

Question 2: Under what conditions does a robustly globally attractive and control invariant hyperbox exist?

Question 3: When a robustly control invariant hyperbox exists, what is the size of the smallest such set?

C. Significance of the Problem

The dynamics in (1) describe a unified but fairly general abstract model for three types of logistic networks: Production networks, distribution networks and transportation networks. In the first scenario, the nodes of the network represent "products", be they raw materials, intermediate products or finished products. The i^{th} component of the state vector thus represents the amount of product i. The hyperarcs of the network represent production processes or activities, some of which may be fully or partially controlled by the operator of the network. Additionally, the network may interact with its external environment through both controlled and uncontrolled flows representing (generally uncertain) supply of raw material and demand of various products. The $Bu - Dw$ term thus encodes the various production processes, supplies and demands, with matrices B and D representing the network topology and inputs u and w representing the controlled and uncontrolled flows, respectively. Likewise in the second and third scenarios, the nodes of the network represent warehouses and transportation hubs, respectively, with the i^{th} component of the state vector thus representing the quantity of commodities present in the i^{th} warehouse/hub. The $Bu - Dw$ term encodes the various transportation routes, distribution protocols, supplies and demands, with matrices B, D representing the network topology and inputs u and w again respectively representing the controlled flows and uncertainty in the system.

In this setting, it is intuitively desirable to contain each component of the state vector within two bounds, a zero lower bound and a positive upper bound. In the case of production networks, the lower bound guards against shortages and interruptions in the production process. In the case of distribution and transportation networks, the lower bound guards against the underuse of distribution and transportation resources. In all scenarios, the upper bound ensures that the storage capabilities of the system are not exceeded. The question of existence of robustly control invariant sets, specifically hyperboxes (Question 1 in Section II-B), thus naturally arises.

Moreover in this setting, the model of uncertainty (specifically the choice of set $\mathcal{W}$) encodes the typical uncertainty encountered in day to day operations. Since it is impossible to rule out rare occurences of large unmodeled uncertainty (be they emergencies or catastrophic events) that would drive the system away from its typical operation, it is reasonable to question whether the system can recover from such events: The question of robust global attractivity of the robustly control invariant hyperboxes (Question 2 in Section II-B) thus naturally arises.

Finally in this setting, it is desirable to make the robustly control invariant sets as small as possible, to cut down on the storage requirements and costs. The question of computing the smallest robustly control invariant set (Question 3 in Section II-B) thus naturally arises. Of course, this question is generally not well defined unless a specific norm is chosen. The particular choice of norm is dependent on practical considerations, with weighted l_1 as well as l_∞ norms presenting reasonable choices.

III. SURVEY OF RECENT RESULTS

In our recent work [?], we derived necessary and sufficient conditions for the existence of robustly control invariant sets, as well as sufficient conditions for ensuring global attractivity. We summarize our main results in this section.

We begin by introducing some relevant notation. Consider the following sets for $i \in \{1, \ldots, n\}$:

$$\begin{aligned}
\mathcal{U}_+^i &= \{u \in \mathcal{U}^m | [Bu - Dw]_i \geq 0, \forall w \in \mathcal{W}^p\}, \\
\mathcal{U}_-^i &= \{u \in \mathcal{U}^m | [Bu - Dw]_i \leq 0, \forall w \in \mathcal{W}^p\}, \\
\mathcal{U}_{+*}^i &= \{u \in \mathcal{U}^m | [Bu - Dw]_i > 0, \forall w \in \mathcal{W}^p\}, \\
\mathcal{U}_{-*}^i &= \{u \in \mathcal{U}^m | [Bu - Dw]_i < 0, \forall w \in \mathcal{W}^p\}.
\end{aligned}$$

Associate with every $x \in \mathbb{R}_+^n$ a *signature*, namely an n-tuple $(s_1, \ldots, s_n)$ with $s_i = +$ if $[x]_i = 0$ and $s_i = -$ if $[x]_i > 0$, and two subsets of $\mathcal{U}^m$ defined by

$$\mathcal{U}_x = \mathcal{U}^1_{s_1} \cap \ldots \cap \mathcal{U}^n_{s_n},$$
$$\mathcal{U}^*_x = \mathcal{U}^1_{s_1*} \cap \ldots \cap \mathcal{U}^n_{s_n*}.$$

We can now state the main results derived in [?]. The first provides a complete answer to Question 1:

Theorem 1: The following two statements are equivalent:

(a) There exists a set $X = [0, x_1^+] \times \ldots \times [0, x_n^+]$ that is robustly control invariant.

(b) The following condition holds

$$\mathcal{U}_z \neq \emptyset, \ \forall z \in \mathbb{B}^n. \tag{2}$$

The second proposes a sufficient condition for the existence of a robustly control invariant set that is globally attractive, thus giving a partial answer to Question 2. In general, this condition need not be necessary!

Theorem 2: If the following condition holds

$$\mathcal{U}^*_z \neq \emptyset, \ z \in \mathbb{B}^n \tag{3}$$

there exists a robustly control invariant and globally attractive set $X = [0, x_1^+] \times \ldots \times [0, x_n^+]$.

IV. Connections to Existing Results

The necessary and sufficient condition (2) for existence of a robustly control invariant set, as well as the sufficient condition (3) for global attractivity are both formulated in terms of combinatorial conditions. In contrast, conditions derived in the literature for the existence of robustly control invariant sets for discrete-time dynamic network flow models with continuous inputs are typically presented as set inclusion conditions. More specifically, in [6] the authors prove that

$$B\mathcal{U}^m \supseteq D\mathcal{W}^p$$

is necessary and sufficient for a robustly control set to exist. In light of this, in this section we attempt to connect our combinatorial conditions with appropriate set inclusion conditions. In particular, we show that (2) implies another condition, formulated in terms of the convex hull of $B\mathcal{U}^m$ and $D\mathcal{W}^p$. Likewise, we show that (3) implies another condition, formulated in terms of the interior of the convex hull of $B\mathcal{U}^m$ and the convex hull of $D\mathcal{W}^p$.

However, it should be emphasized that this exercise is mainly academic for two reasons: First, the set inclusion conditions are known to be NP-hard to verify in general, as hence do not offer much promise of a substantial reduction in computational burden. Second, the directions of the implications are such that the set inclusion conditions can only be used to draw negative conclusions about the network in the cases where they are violated.

Theorem 3: Condition (2) implies

$$hull\{B\mathcal{U}^m\} \supseteq hull\{D\mathcal{W}^p\}. \tag{4}$$

The converse is not true.

Proof: If $\mathcal{U}_x \neq \emptyset$ for all $x \in V_X$, we have a control $u^j \in \mathcal{U}^m$ such that, for all $w \in \mathcal{W}^p$, the state difference lies in the jth orthant, $Bu^j - Dw \in O_j$ for all orthants O_j, $j = 1, \ldots, 2^n$. We can then consider the subset of 2^n controls (one per each orthant in the n-dimensional state space) $\mathbf{U} = \{u^1, \ldots, u^{2^n}\}$ and, for all $w \in \mathcal{W}^p$, the associated set $B\mathbf{U} - Dw := \{Bu - Dw | u \in \mathbf{U}\}$. Now, as by construction the set $B\mathbf{U} - Dw$ has one point in each orthant then its convex hull will include zero, that is, $0 \in hull\{B\mathbf{U} - Dw\}$ for all $w \in \mathcal{W}^p$. Now, consider the set $B\mathbf{U} - D\mathcal{W}^p := \{Bu - Dw| \, u \in \mathbf{U}, w \in \mathcal{W}^p\}$ and observe that $hull\{B\mathbf{U} - D\mathcal{W}^p\} \supseteq hull\{B\mathbf{U} - Dw\}$, for all $w \in \mathcal{W}^p$. This last condition yields $0 \in hull\{B\mathbf{U} - D\mathcal{W}^p\}$. From $hull\{B\mathbf{U}\} \subseteq hull\{B\mathcal{U}^m\}$ and taking $B\mathcal{U}^m - D\mathcal{W}^p := \{Bu - Dw| \, u \in \mathcal{U}^m, w \in \mathcal{W}^p\}$ then we obtain $0 \in hull\{B\mathcal{U}^m - D\mathcal{W}^p\}$ and therefore also (4).

To see why (4) does not imply $\mathcal{U}_x \neq \emptyset$ for all $x \in V_X$ consider a set parametrized in w $\mathbf{P} = \{p^1(w), \ldots, p^{2^n}(w)\}$ where $p^i(w) \in O_i$ for all $w \in \mathcal{W}$. Observe that $0 \in hull\{\mathbf{P}\}$. By Caratheodory theorem, as $0 \in hull\{\mathbf{P}\}$ then we can always extract from $\mathbf{P}$ $n+1$ points (assume without loss of generality the first $n+1$ points of the set) $p^1(w), \ldots, p^{n+1}(w)$ such that $0 \in hull\{p^1(w), \ldots, p^{n+1}(w)\}$. Now, suppose that the set of controls $\mathcal{U}^m = \{u^1, \ldots, u^{n+1}\}$ so that $Bu^i - Dw = p^i(w)$ for all $w \in \mathcal{W}$ and so we have $0 \in hull\{B\mathcal{U}^m - D\mathcal{W}^p\}$ which implies (4). At the same time, we also have $B\mathcal{U}^m - D\mathcal{W}^p \cap O_h = \emptyset$ for some $h \in \{1, \ldots, 2^n\}$ which means that there does not exist a $u^h \in \mathcal{U}$ such that $Bu^h - Dw \in O_h$ which in turn means that there exists $x \in V_X$ such that $\mathcal{U}_x = \emptyset$ and this concludes our proof. ■

Theorem 4: Condition (3) implies

$$int(hull\{B\mathcal{U}^m\}) \supset hull\{D\mathcal{W}^p\}. \tag{5}$$

The converse is not true.

Proof: If $\mathcal{U}^*_x \neq \emptyset$ for all $x \in V_X$, we have a control $u^j \in \mathcal{U}^m$ such that, for all $w \in \mathcal{W}^p$, the state difference lies in the interior of the jth orthant, $Bu^j - Dw \in int(O_j)$ forall orthants O_j, $j = 1, \ldots, 2^n$. We can then consider the subset of 2^n controls (one per each orthant in the n-dimensional state space) $\mathbf{U} = \{u^1, \ldots, u^{2^n}\}$ and, for all $w \in \mathcal{W}^p$, the associated set $B\mathbf{U} - Dw := \{Bu - Dw | u \in \mathbf{U}\}$. Now, as by construction the set $B\mathbf{U} - Dw$ has one point in the interior of each orthant then its convex hull will include zero in its interior, that is, $0 \in int(hull\{B\mathbf{U} - Dw\})$ for all $w \in \mathcal{W}^p$. Now, consider the set $B\mathbf{U} - D\mathcal{W}^p := \{Bu - Dw| \, u \in \mathbf{U}, w \in \mathcal{W}^p\}$ and observe that $hull\{B\mathbf{U} - D\mathcal{W}^p\} \supseteq hull\{B\mathbf{U} - Dw\}$, for all $w \in \mathcal{W}^p$. This last condition yields $0 \in int(hull\{B\mathbf{U} - D\mathcal{W}^p\})$. From $hull\{B\mathbf{U}\} \subseteq hull\{B\mathcal{U}^m\}$ and taking $B\mathcal{U}^m - D\mathcal{W}^p := \{Bu - Dw| \, u \in \mathcal{U}^m, w \in \mathcal{W}^p\}$ then we obtain $0 \in int(hull\{B\mathcal{U}^m - D\mathcal{W}^p\})$ and therefore also (5).

To see why (5) does not imply $\mathcal{U}^*_x \neq \emptyset$ for all $x \in V_X$ consider a set parametrized in w $\mathbf{P} = \{p^1(w), \ldots, p^{2^n}(w)\}$ where $p^i(w) \in int(O_i)$ for all $w \in \mathcal{W}^p$. Observe that $0 \in int(hull\{\mathbf{P}\})$. By Caratheodory theorem, as $0 \in int(hull\{\mathbf{P}\})$ then we can always extract from $\mathbf{P}$ $n+1$ points (assume without loss of generality the first $n+$

1 points of the set) $p^1(w), \ldots, p^{n+1}(w)$ such that $0 \in int(hull\{p^1(w), \ldots, p^{n+1}(w)\})$. Now, suppose that the set of controls $\mathcal{U} = \{u^1, \ldots, u^{n+1}\}$ so that $Bu^i - Dw = p^i(w)$ for all $w \in \mathcal{W}^p$ and so we have $0 \in int(hull\{B\mathcal{U}^m - D\mathcal{W}^p\})$ which implies (5). At the same time, we also have $B\mathcal{U}^m - D\mathcal{W}^p \cap O_h = \emptyset$ for some $h \in \{1, \ldots, 2^n\}$ which means that there does not exists a $u^h \in \mathcal{U}^m$ such that $Bu^h - Dw \in int(O_h)$ which in turn means that there exists $x \in V_X$ such that $\mathcal{U}_x^* = \emptyset$ and this concludes our proof. ∎

V. Smallest Invariant Set

In this section, we consider the special case of scalar dynamics where the smallest invariant set is independent of the choice of norm. In this setting, we explicitly derive lower and upper bounds on the size of the smallest invariant set.

Consider the dynamics described by

$$x(t+1) = x(t) + \alpha u(t) - \eta w(t) \tag{6}$$

where α and η are given non-zero scalars, $u(t) \in \mathcal{U}$ and $w(t) \in \mathcal{W}$.

In this setting, the sets defined in Section III can be computed by simply checking the signs of the entries of the following table, reminiscent of the payoff table in a zero sum game between two non-cooperating players

u/w	b_1	$\ldots$	b_q
a_1	$\alpha a_1 - \eta b_1$	$\ldots$	$\alpha a_1 - \eta b_q$
$\vdots$	$\vdots$		$\vdots$
a_r			

Specifically, we have four subsets of $\mathcal{U}$ as follows:

$$\begin{aligned}
\mathcal{U}_+ &= \{u \in \mathcal{U} | \alpha u - \eta w \geq 0, \forall w \in \mathcal{W}\} \\
\mathcal{U}_- &= \{u \in \mathcal{U} | \alpha u - \eta w \leq 0, \forall w \in \mathcal{W}\} \\
\mathcal{U}_+^* &= \{u \in \mathcal{U} | \alpha u - \eta w > 0, \forall w \in \mathcal{W}\} \\
\mathcal{U}_-^* &= \{u \in \mathcal{U} | \alpha u - \eta w < 0, \forall w \in \mathcal{W}\}
\end{aligned}$$

as well as one additional subset defined as

$$\mathcal{U}_c = \{u \in \mathcal{U} | u \notin \mathcal{U}_- \cup \mathcal{U}_+\}.$$

Note that $\mathcal{U}_c$, $\mathcal{U}_+$ and $\mathcal{U}_-$ are pairwise disjoint and their union is $\mathcal{U}$: Thus they partition $\mathcal{U}$. Also note that $\mathcal{U}_+^* \subseteq \mathcal{U}_+$ and $\mathcal{U}_-^* \subseteq \mathcal{U}_-$.

We are now ready to derive upper and lower bounds on the size of the smallest invariant set. We begin by defining

$$\begin{aligned}
\delta^+(u) &= \max_{w \in \mathcal{W}} \alpha u - \eta w \\
\delta^-(u) &= \min_{w \in \mathcal{W}} \alpha u - \eta w.
\end{aligned}$$

Note that $\delta^+(u) > 0$ when $u \in \mathcal{U}_c \cup \mathcal{U}_+$, and $\delta^+ \leq 0$ when $u \in \mathcal{U}_-$. Likewise, note that $\delta^-(u) < 0$ when $u \in \mathcal{U}_- \cup \mathcal{U}_c$ and $\delta^-(u) \geq 0$ when $u \in \mathcal{U}_+$.

Now let

$$\begin{aligned}
L_1^o &= \min_{u \in \mathcal{U}_+} \delta^+(u) \\
L_2^o &= \min_{u \in \mathcal{U}_-} |\delta^-(u)| \\
L_3^o &= \min_{u \in \mathcal{U}_c} \delta^+(u) \\
L_4^o &= \min_{u \in \mathcal{U}_c} |\delta^-(u)|.
\end{aligned}$$

Theorem 5: (Lower Bound) Let l be the size of the smallest robustly control invariant set. We have

$$l \geq \max\left\{L_1^o + L_4^o, L_2^o + L_3^o\right\} - \beta, \ \forall \beta > 0 \tag{7}$$

.

Proof: Suppose without loss of generality that the interval $[0, l]$ is robustly control invariant. Then in particular when $x(t) = l - \epsilon$, where

$$0 \leq \epsilon < \min_{u \in \mathcal{U}_c \cup \mathcal{U}_+} \delta^+(u)$$

we require $x(t+1) \in [0, l]$. Equivalently, we require

$$\begin{aligned}
& 0 \leq l - \epsilon + \alpha u_{l-\epsilon} - \eta w \leq l, \ \forall w \in \mathcal{W} \\
\Leftrightarrow \quad & -l + \epsilon \leq \alpha u_{l-\epsilon} - \eta w \leq \epsilon, \ \forall w \in \mathcal{W} \\
\Leftrightarrow \quad & \begin{cases} \alpha u_{l-\epsilon} - \eta w \leq \epsilon, & \forall w \in \mathcal{W} \\ \alpha u_{l-\epsilon} - \eta w \geq -l + \epsilon, & \forall w \in \mathcal{W} \end{cases} \\
\Leftrightarrow \quad & \begin{cases} \max_{w \in \mathcal{W}} \alpha u_{l-\epsilon} - \eta w \leq \epsilon \\ \min_{w \in \mathcal{W}} \alpha u_{l-\epsilon} - \eta w \geq -l + \epsilon \end{cases} \\
\Leftrightarrow \quad & \begin{cases} \delta^+(u_{l-\epsilon}) \leq \epsilon \\ \delta^-(u_{l-\epsilon}) \geq -l + \epsilon \end{cases}.
\end{aligned}$$

The first inequality implies

$$\delta^+(u_{l-\epsilon}) < \min_{u \in \mathcal{U}_c \cup \mathcal{U}_+} \delta^+(u),$$

which holds iff $u_{l-\epsilon} \in \mathcal{U}_-$. In this case, the second inequality reduces to

$$\epsilon - \delta^-(u_{l-\epsilon}) \leq l \Leftrightarrow \epsilon + |\delta^-(u_{l\epsilon})| \leq l,$$

which implies that

$$\epsilon + \min_{u \in \mathcal{U}_-} |\delta^-(u_{l\epsilon})| \leq l \Leftrightarrow \epsilon + L_2^o \leq l.$$

Finally, noting that this inequality holds for any $\epsilon \in \left[0, \min_{u \in \mathcal{U}_c \cup \mathcal{U}_+} \delta^+(u)\right)$ and noting additionally that

$$\min_{u \in \mathcal{U}_c \cup \mathcal{U}_+} \delta^+(u) = \min_{u \in \mathcal{U}_c} \delta^+(u) = L_3^o,$$

we conclude that

$$l \geq L_2^o + L_3^o - \beta, \ \forall \beta > 0.$$

Likewise, a similar argument can be made for $x(t) = \epsilon$ where

$$0 \leq \epsilon < \min_{u \in \mathcal{U}_c \cup \mathcal{U}_-} |\delta^-(u)|$$

to show that

$$l \geq L_1^o + L_4^o - \beta, \ \forall \beta > 0.$$

We thus conclude that

$$l \geq \max\left\{L_1^o + L_4^o, L_2^o + L_3^o\right\} - \beta, \ \forall \beta > 0.$$

■

Theorem 6: (Upper Bound) Let l be the size of the smallest robustly control invariant set. We have

$$l \leq \min_{u \in \mathcal{U}_c} \left(\max\left\{L_1^o + |\delta^-(u)|, L_2^o + \delta^+(u)\right\}\right). \quad (8)$$

Proof: We begin by noting that any control law φ that renders the set $X = [0, l_\varphi]$ robustly control invariant may be used to establish an upper bound on l. Let

$$u_1^o = argmin_{u \in \mathcal{U}_+} \delta^+(u),$$

$$u_2^o = argmin_{u \in \mathcal{U}_-} |\delta^-(u)|,$$

and for each $u \in \mathcal{U}_c$ consider the control law $\varphi_u : [0, l_u] \to \mathcal{U}$ defined by

$$\varphi_u(x) = \begin{cases} u_1^o & x \in [0, |\delta^-(u)|) \\ u & x \in [|\delta^-(u)|, l(u) - \delta^+(u)] \\ u_2^o & x \in (l(u) - \delta^+(u), l(u)] \end{cases}.$$

It is straightforward to ascertain that φ_u renders $[0, l(u)]$ robustly control invariant provided

$$\begin{cases} |\delta^-(u) + L_1^o \leq l \\ l - \delta^+(u) - L_2^o \geq 0 \end{cases} \Leftrightarrow \begin{cases} l \geq |\delta^-(u) + L_1^o \\ l \geq \delta^+(u) + L_2^o \end{cases}.$$

Next, noting that for any $u \in \mathcal{U}_c$, we have $\delta^-(u) \geq L_4^o$ and $\delta^+(u) \geq L_3^o$, we conclude that picking

$$l(u) = \max\left\{L_1^o + |\delta^-(u)|, L_2^o + \delta^+(u)\right\}$$

is sufficient to ensure robust invariance. Finally, the best upper bound corresponds to minimizing this choice of $l(u)$ over all $u \in \mathcal{U}_c$. ■

VI. Illustrative Example

Consider the scalar dynamics ($n = m = p = 1$) given by

$$x(t+1) = x(t) + Bu(t) - Dw(t)$$

with alphabets $\mathcal{U} = \{-100, -2, 3, 150\}$ and $\mathcal{W} = \{-6, 4\}$. We begin by computing sets $\mathcal{U}_+$ and $\mathcal{U}_-$ (no need for indices 'i' in this case) by inspecting the table below, whose entries are simply the values of '$Bu - Dw$':

u/w	-6	4
-100	$-100B + 6D$	$-100B - 4D$
-2	$-2B + 6D$	$-2B - 4D$
3	$3B + 6D$	$3B - 4D$
150	$150B + 6D$	$150B - 4D$

We have $\mathcal{U}_+ \neq \emptyset$ and $\mathcal{U}_- \neq \emptyset$ iff

$$\begin{cases} -100B + 6D \leq 0 \\ -100B - 4D \leq 0 \end{cases} \text{and} \begin{cases} 150B + 6D \geq 0 \\ 150B - 4D \geq 0 \end{cases}.$$

We thus conclude that a robustly control invariant set indeed exists iff

$$B \geq \max\{0.6D, -0.04D\}$$

and is moreover globally attractive provided strict equality holds. Now consider for example the case where $B = D = 1$, for which a robustly control invariant set is guaranteed to exist. We now proceed to compute

$$L_1^o = \min_{u \in \mathcal{U}_+} \delta^+(u) = 156,$$
$$L_2^o = \min_{u \in \mathcal{U}_-} |\delta^-(u)| = 104,$$
$$L_3^o = \min_{u \in \mathcal{U}_c} \delta^+(u) = 4,$$
$$L_4^o = \min_{u \in \mathcal{U}_c} |\delta^-(u)| = 1$$

from which we conclude that $X = [0, 157]$ is the smallest robustly control invariant set.

VII. Conclusions & Future Work

We considered dynamic networks where the disturbances and control actions take discrete values. We surveyed our recent results establishing necessary and sufficient conditions for the existence of robustly globally invariant hyperboxes and sufficient conditions for global attractivity of these sets. We showed that each of these conditions implies a set inclusion condition reminiscent of existing results in the literature for the setup where all the inputs are analog. We then derived tight upper and lower bounds on the smallest robustly control invariant set in the special case of a degenerate network, and we concluded with a simple illustrative example.

Future work will focus on extending the established lower and upper bounds to the general setup, as well as exploring special structures that may allow us to verify the combinatorial conditions in a more computationally efficient manner.

Acknowledgment

D. C. Tarraf's research was supported by NSF CAREER award ECCS-0954601 and AFOSR YIP award FA9550-11-1-0118.

References

[1] D. Axehill, L. Vandenberghe, and A. Hansson, "Convex relaxations for mixed integer predictive control," *Automatica*, vol. 46, pp. 1540–1545, 2010.

[2] T. Basar and G. Olsder, *Dynamic Noncooperative Game Theory*. SIAM, 1999.

[3] D. Bauso, "Boolean-controlled systems via receding horizon and linear programming," *Journal of Mathematics of Control, Signals and Systems*, vol. 21, no. 1, pp. 69–91, 2009.

[4] D. Bauso, L. Giarrè, and R. Pesenti, “Robust control of uncertain multi-inventory systems via linear matrix inequality,” *International Journal of Control*, vol. 83, no. 8, pp. 1723–1740, 2010.
[5] F. Blanchini, S. Miani, and W. Ukovich, “Control of production-distribution systems with unknown inputs and system failures,” *IEEE Trans. on Automatic Control*, vol. 45, no. 6, pp. 1072–1081, 2000.
[6] F. Blanchini, F. Rinaldi, and W. Ukovich, “Least inventory control of multi-storage systems with non-stochastic unknown input,” *IEEE Trans. on Robotics and Automation*, vol. 13, no. 5, pp. 633–645, 1997.
[7] D. Tarraf, A. Megretski, and M. Dahleh, “A framework for robust stability for systems over finite alphabets,” *IEEE Transactions on Automatic Control*, vol. 53, no. 5, pp. 1133–1146, June 2008.