Opinion dynamics in coalitional games with transferable utilities

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Abstract— **This paper studies opinion dynamics in a large number of homogeneous coalitional games with transferable utilities (TU), where the characteristic function is a continuous-time stochastic process. For each game, which we can see as a "small world", the players share opinions on how to allocate revenues based on the mean-field interactions with the other small worlds. As a result of such mean-field interactions among small worlds, in each game, a central planner allocates revenues based on the extra reward that a coalition has received up to the current time and the extra reward that the same coalition has received in the other games. The paper also studies the convergence and stability of opinions on allocations via stochastic stability theory.**

I. Introduction

This paper considers a large number of the same copy of a coalitional game with transferable utilities (TU game). Each single game has a game designer who allocates rewards or revenues based on the excesses of each coalition. In a continuous-time repeated game, the excess of a coalition is the cumulative deviation of the total amount given to the coalition from the value of the coalition up to the current time. The ultimate goal of the game designer is to stabilize the grand coalition. This occurs when the total revenue assigned to all members of any sub-coalition is greater than the value of the sub-coalition itself (see the notion of "core" in [18]).

The coalition's values are time varying and thus the excesses evolve according to controlled uncertain stochastic differential equations. The objective of the game designer is to align the excesses with the average value computed over the infinite copies of the same game. Such a phenomenon is known as crowd-seeking behavior in mean-field games and mirrors a typical attitude in macroeconomics known as *inequity aversion*.

Main result. For the problem at hand, we provide a mean-field game formulation and conduct a heuristic robust control design based on augmentation and regularization of the state space [9]. The mean-field game involves a macroscopic description based on a classical forward Kolmogorov partial differential equation which generates the distribution of the excesses over the horizon. Furthermore, we perform a stability analysis on the microscopic dynamics of the excesses as well as on the average excess. According to this analysis the stochastic processes describing the excesses are mean square bounded.

Related literature. Coalitional games with transferable utilities (TU), introduced first by Von Neuman and Morgenstern [18], have recently sparked much interest in the control and communication engineering communities [16]. In essence, coalitional TU games are comprised of a set of players who can form coalitions and a characteristic function associating a real number with every coalition. This real number represents the value of the coalition and can be thought of as a monetary value that can be distributed among the members of the coalition according to some appropriate fairness allocation rule. The value of a coalition also reflects the monetary benefit demanded by that coalition to be a part of the grand coalition. In the context of coalitional TU games, *robustness* and *dynamics* naturally arise in all the situations where the coalition values are uncertain and time-varying, see e.g., [6], [7], [15].

We also link the approach to the set invariance theory [3] and stochastic stability theory [1], [10], [14], which provides us some useful tools for stability analysis.

The theory on mean-field games originated in the work of M.Y. Huang, P. E. Caines and R. Malhamé [11], [12] and independently in that of J. M. Lasry and P.L. Lions [13], where the now standard terminology of Mean Field Game (MFG) was introduced (see also [17]). The problem we analyze in this paper follows in spirit the study on robust dynamical TU coalitional games in [8] with the additional mean-field interactions between infinite copies of the same game, which was not present in [8]. Explicit solutions in terms of mean-field equilibria are not common unless the problem has a linear-quadratic structure, see [2]. This justifies our solution approach which approximates the original problem by an augmented linear quadratic one.

The rest of the paper is organized as follows. In Section II, we introduce the problem and the model. In Section III, we present the mean-field game. In Section IV, we describe the solution approach. In Section V, we perform numerical analysis. Finally, in Section VI, we draw some conclusions and discuss possible future directions.

Notation. Given a set $N = \{1, \ldots, n\}$ of players and a function $\eta : S \mapsto \mathbb{R}$ defined for each nonempty coalition $S \subseteq N$, we write $< N, \eta >$ to denote the transferable utility (TU) game with players' set N and

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characteristic function η. We let η_S be the value $\eta(S)$ of the characteristic function η associated with a nonempty coalition $S \subseteq N$. Given a TU game $< N, \eta >$, we use $C(\eta)$ to denote the core of the game:

$$C(\eta) = \Big\{ x \in \mathbb{R}^n \,\Big|\, \sum_{i\in N} x_i = \eta_N,\ \sum_{i\in S} x_i \geq \eta_S \text{ for all nonempty } S \subset N \Big\}.$$

Also, $\mathbb{R}_+$ denotes the set of nonnegative real numbers. Given a random vector ξ, the notation $\mathbb{E}[\xi]$ denotes its expected value. Given a Brownian motion (with drift) $\mathcal{B}(t)$, we denote by $d\mathcal{B}(t)$ its infinitesimal increment, i.e., $\mathcal{B}(t) = \int_0^t d\mathcal{B}(\tau)$, the latter being the Itô integral. We use $\bar{\mathcal{B}}(t) = \frac{\mathcal{B}(t)}{t}$ to indicate the average infinitesimal up to time t. If $a(t)$ is the derivative of an almost everywhere differentiable function, the symbol $\tilde{a}(t)$ denotes the function itself, i.e., $\tilde{a}(t) = \int_0^t a(\tau)d\tau$. We also use $\bar{a}(t) = \frac{\tilde{a}(t)}{t}$ to indicate the average up to time t.

II. TU GAMES AS NETWORKS

We consider a large number of replicas of a single coalitional game. The game is an n-player robust dynamical TU game $< N, \eta(t) >$, where $\eta(t)$ is the characteristic function representing the values of different coalitions. The characteristic function is modeled as a diffusion process with drift, and its evolution is described by the stochastic differential equation:

$$\left\{ \begin{array}{l} d\eta(t) = w(t)dt - \sigma d\mathcal{B}(t), \quad \text{in } \mathbb{R}^q, \\ \eta(0) = \eta_0, \end{array} \right. \tag{1}$$

where $q = 2^{n-1}$ is the number of coalitions. The model admits a network representation. For each game, let $\mathcal{H}$ be a corresponding hypergraph involving the vertex set V and edge set E, namely:

$$\mathcal{H} := \{V, E\}, \quad V = \{\mathbf{v}_1, \ldots, \mathbf{v}_q\}, \quad E := \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}.$$

In other words, the vertex set V has one vertex per coalition whereas the edge set E has one edge per player. The incidence relations establish that a generic edge i is

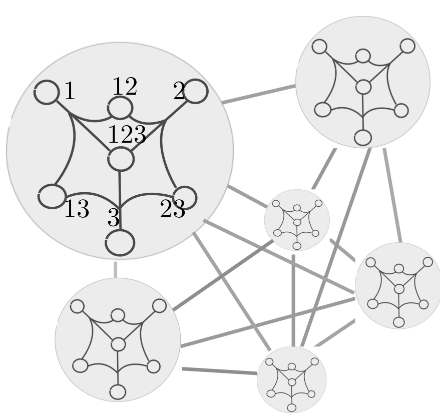

Fig. 1. Infinite copies of hypergraph $\mathcal{H} := \{V, E\}$ for a 3-player coalitional game.

incident to a vertex $\mathbf{v}_j$ if the player i is a member of the the coalition linked to $\mathbf{v}_j$. The hypergraph can then be described by an incidence matrix B whose rows are the characteristic vectors $c^S \in \mathbb{R}^n$. The characteristic vectors are in turn binary vectors where $c_i^S = 1$ if $i \in S$ and $c_i^S = 0$ if $i \notin S$. Figure 1 depicts an example of a hypergraph for a 3-player coalitional game on every single grey node. Following the same approach as in [8], the allocation $u_i(t)$ is represented by the flow on edge $\mathbf{e}_i$ and the coalition value $w_S(t)$ of a generic coalition S is the demand in the corresponding vertex $\mathbf{v}_j$. It is apparent then that any allocation in the *core* of the game $C(\eta(t))$ translates into over-satisfying the demand at the vertices. In particular, we have

$$\tilde{u}(t) \in C(\eta(t)) \quad \Leftrightarrow \quad B_{\mathcal{H}}\tilde{u}(t) \geq \eta(t), \tag{2}$$

where the last inequality is satisfied with equality due to the efficiency condition of the core, i.e, $\sum_{i=1}^n \tilde{u}_i(t) = \eta_q(t)$, where $\eta_q(t)$ denotes the qth component of $\eta(t)$ and is equal to the grand coalition value $\eta_N(t)$. Denote by $x(t) \in \mathbb{R}^q$ the coalition excess; then we can describe the time evolution of $x(t)$ through the following stochastic differential equation:

$$\left\{ \begin{array}{l} dx(t) = (Bu(t) - w(t))dt + \sigma d\mathcal{B}(t), \\ x(0) = x_0. \end{array} \right. \tag{3}$$

In essence, every component of vector $Bu(t)$ is the total amount given to the members of a coalition at time t, and from this amount the value of the coalition itself, $w(t)$, is subtracted. Then, a positive $x(t)$ means positive cumulative excess. In a first scenario we assume that controls and disturbances are unbounded. In a second scenario we will hypothesize that the control and the disturbances are bounded within polytopes, i.e.,

- $u(t)$ is in the control set $U \subseteq \mathbb{R}^n$, $n > 0$,
$$U := \{u \in \mathbb{R}^n |\, u_i^- \leq u \leq u_i^+\}, \tag{4}$$
where $u_i^- \leq u_i^+$, $u_i^-, u_i^+ \in \mathbb{R}^n$;
- $w(t)$ is in the disturbance set $W \subseteq \mathbb{R}^q$, $q > 0$,
$$W := \{w \in \mathbb{R}^q |\, w_i^- \leq w \leq w_i^+\}, \tag{5}$$
where $w_i^- \leq w_i^+$, $w_i^-, w_i^+ \in \mathbb{R}^q$.

With the above preamble in mind, and given the infinite copies of the same game, we can derive a probability density function $m : \mathbb{R}^q \times [0, +\infty[\to [0, +\infty[$, $(x, t) \mapsto m(x, t)$, for which $\int_{\mathbb{R}^q} m(x, t)dx = 1$ for every time t. We also denote the mean distribution at time t by $\bar{m}(t) := \int_{\mathbb{R}^q} x m(x, t)dx$.

In the spirit of *inequity aversion*, the designer of each game follows a so-called crowd-seeking law in that it readjusts the allocations by targeting the average distribution of the other games.

This is captured by considering a running cost $g : \mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^n \times \mathbb{R}^q \to [0, +\infty[$, $(x, \bar{m}, u, w) \mapsto g(x, \bar{m}, u, w)$ of the quadratic form:

$$g(x, \bar{m}, u, w) = \frac{1}{2}\Big[(\bar{m} - x)^T Q (\bar{m} - x) + u^T(t)Ru(t) - w^T(t)\Gamma w(t) \Big], \tag{6}$$

where $Q, R, \Gamma > 0$, that is positive definite.

We also take as terminal cost the function $\Psi : \mathbb{R}^q \times \mathbb{R}^q \to [0, +\infty[$, $(x, \bar{m}) \mapsto \Psi(x, \bar{m})$ of the form

$$\Psi(x, \bar{m}) = \frac{1}{2}(\bar{m} - x)^T S(\bar{m} - x), \tag{7}$$

where $S > 0$. We are then ready to formalize the problem at hand as follows.

Problem 1: Find the closed-loop optimal control and worst-case disturbance for the problem:

$$\begin{cases} \inf_{u(\cdot)\in\mathcal{U}} \sup_{w(\cdot)\in\mathcal{W}} \Big\{ J(x_0, u(\cdot), w(\cdot), m(\cdot)) \\ = \mathbb{E}\Big[\int_0^T g(x, \bar{m}, u, w)dt + \Psi(x(T), \bar{m}(T)) \Big] \Big\}, \\ dx(t) = (Bu(t) - w(t))dt - \sigma d\mathcal{B}(t), \end{cases} \tag{8}$$

where $\mathcal{U}$ and $\mathcal{W}$ are the sets of all measurable functions $u(\cdot)$ and $w(\cdot)$ from $[0, +\infty[$ to U and W, respectively, and $m(\cdot)$ as a time-dependent function is the evolution of the distribution under the optimal control and the worst-case disturbance.

Note that the problem formulation includes the cases of both bounded and unbounded controls and disturbances.

III. The mean–field game

Let us denote by $v(x, t)$ the (upper) value of the robust optimization problem under worst-case disturbance starting from time t at state x (which in this case also turns out to be the lower value, and hence the *value*, since Isaacs condition [3] holds–see below). Problem 1 results in the following mean-field game system for the unknown functions $v(x, t)$, and $m(x, t)$:

$$\begin{cases} \partial_t v(x, t) + \inf_{u\in U} \sup_{w\in W} \Big\{ (Bu - w)^T \partial_x v(x, t) \\ +g(x, \bar{m}, u, w) \Big\} + \frac{\sigma^2}{2} Tr\Big(\partial^2_{xx} v(x, t) \Big) = 0 \\ \qquad\qquad\qquad \text{in } \mathbb{R}^q \times [0, T[, \\ v(x, T) = \Psi(x, \bar{m}) \ \forall\, x \in \mathbb{R}^q, \\ \\ \partial_t m(x, t) + div(m(x, t) \cdot (Bu - w)) \\ \quad -\frac{\sigma^2}{2} Tr(\partial^2_{xx} m(x, t)) = 0, \text{ in } \mathbb{R}^q \times [0, T[, \\ m(0) = m_0, \\ \frac{d}{dt}\bar{m}_t = B\bar{u}^*_t - \bar{w}^*_t, \text{ in } [0, T[, \end{cases} \tag{9}$$

where $u^*(t, x)$ and $w^*(t, x)$ are the optimal time-varying state-feedback controls and disturbances, respectively, obtained as

$$\begin{cases} u^*(t, x) \in \arg\min_{u\in U} \{ (Bu - w^*)\partial_x v(x, t) \\ \qquad\qquad +g(x, \bar{m}, u, w^*) \}, \\ \\ w^*(t, x) \in \arg\max_{w\in W} \{ (Bu - w)\partial_x v(x, t) \\ \qquad\qquad +g(x, \bar{m}, u^*, w) \}. \end{cases} \tag{10}$$

Note that the minimization and maximization problems above are completely decoupled, and hence in (9) the $\inf\sup$ is the same as $\sup\inf$ (that is, Isaacs condition holds [3]). Further, we have replaced $\inf$ and $\sup$ in (10) with $\min$ and $\max$, respectively, since g is quadratic in u and w.

The first equation in (9) is the Hamilton-Jacobi-Isaacs (HJI) equation with variable $v(x, t)$. Given the boundary condition on final state (second equation in (9)), and assuming a given population behavior captured by $m(\cdot)$, the HJI equation is solved backwards and returns the value function and best-response behavior of the individuals (first equation in (10)) as well as the worst adversarial response (second equation in (10)). The HJI equation is coupled with a second PDE, known as the *Fokker-Planck-Kolmogorov (FPK)* equation (third equation in (9)), defined on variable $m(\cdot)$. Given the boundary condition on initial distribution $m(0) = m_0$ (fourth equation in (9)), and assuming a given individual behavior described by u^*, the FPK equation is solved forward and returns the population behavior time evolution $m(t)$. The last equation in (9) is obtained by averaging the left and right hand sides of the dynamics (3). Any solution of the above system of equations along with (10) is referred to as *worst-disturbance feedback mean-field equilibrium.*

Remark 1: (On the existence of solutions) Analyzing the existence of solutions for the mean-field system (9) is a challenging task. However, under some restrictive sufficient conditions existence of classical solutions can be established using a fixed-point theorem argument as in [13]. Indeed, let us assume that the initial measure m_0 is absolutely continuous with a continuous density function with finite second moment. In this case, we note that the running cost is convex in u. With these conditions, the existence of solution is established in Theorem 2.6 in [13].

Remark 2: (On connections with the finite case) As the cost is Lipschitz continuous on m (given that the control is bounded), the solution to the asymptotic case with infinite number of players relates to the case with a finite number of players as established in [12], [13]. In particular the classical bound of $\frac{1}{\sqrt{N}}$ holds true where N is the number of games.

IV. Augmentation

This section describes a simple heuristic approach toward solving the set of equations (9)), based on state space augmentation [9]. The augmented state space includes the mean distribution, and thus the augmented state variables evolve according to the equations

$$\begin{aligned} \begin{bmatrix} dx(t) \\ d\bar{m}(t) \end{bmatrix} = \Big(B \begin{bmatrix} u^*(x, t) \\ \bar{u}^*(t) \end{bmatrix} \\ - \begin{bmatrix} w^*(x, t) \\ \bar{w}^*(t) \end{bmatrix} \Big) dt + \begin{bmatrix} \sigma d\mathcal{B}_t \\ 0 \end{bmatrix}. \end{aligned} \tag{11}$$

For this system we introduce an assumption on the rate of convergence of the state $\bar{m}(t)$.

Assumption 1: There exists a scalar $\theta > 0$ such that

$$\frac{d}{dt}\bar{m}(t) = B\bar{u}^*(t) - \bar{w}^*(t) \geq -\theta \bar{m}_t, \text{ for all } t \in [0, T],$$

where the inequality is to be interpreted component-wise.

The above assumption implies that there exists a variable $\tilde{m}(t)$ which approximates the average mean value from below and evolves according to

$$\begin{cases} \frac{d}{dt}\tilde{m}(t) = -\theta\tilde{m}(t), & \text{for all } t \in [0,T], \\ \tilde{m}_0 = \bar{m}_0. \end{cases} \tag{12}$$

As a result, each single game is described by the dynamical closed-loop system depicted in Fig. 2. The block at the top represents the state dynamics (3), it receives as input the control $u(t)$, and the disturbances $w(t)$ and $\mathcal{B}(t)$. The block on the right includes the internal model for $\tilde{m}$ as in (12) from which we obtain the error $e(t) := \tilde{m}(t) - x(t)$. The block at the bottom describes the closed-loop state feedback control obtained solving the linear quadratic tracking problem. The resulting linear quadratic problem is detailed next.

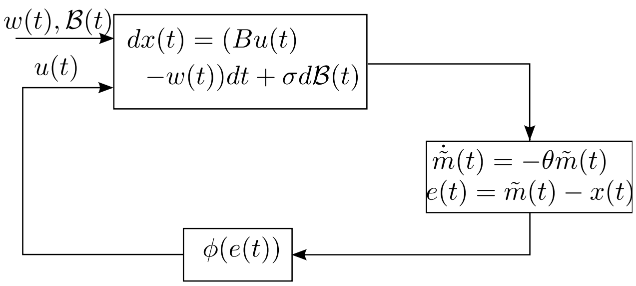

Fig. 2. The dynamical closed-loop system of each single game

By substituting the current mean value $\bar{m}(t)$ by its estimate $\tilde{m}(t)$ the augmented problem becomes

$$\inf_{u(\cdot)\in\mathcal{U}} \sup_{w(\cdot)\in\mathcal{W}} \int_0^T \frac{1}{2}\Big[(\tilde{m}(t) - x(t))^T Q\, (\tilde{m}(t) - x(t)) + u^T(t)Ru(t) - w^T(t)\Gamma w(t)\Big] dt$$

$$\begin{bmatrix} dx(t) \\ d\tilde{m}(t) \end{bmatrix} = \Big(\begin{bmatrix} 0 & 0 \\ 0 & -\theta I \end{bmatrix} \begin{bmatrix} x(t) \\ \tilde{m}(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) - \begin{bmatrix} I \\ 0 \end{bmatrix} w(t) \Big) dt + \begin{bmatrix} \sigma d\mathcal{B}(t) \\ 0 \end{bmatrix}.$$

Reformulating the problem in terms of the augmented state

$$X(t) = \begin{bmatrix} x(t) \\ \tilde{m}(t) \end{bmatrix},$$

we have the linear quadratic problem:

$$\inf_{u(\cdot)\in\mathcal{U}} \sup_{w(\cdot)\in\mathcal{W}} \int_0^T \Big[\frac{1}{2}(X(t)^T \tilde{Q} X(t) + u^T(t)Ru(t) - w^T(t)\Gamma w(t))\Big] dt + \tilde{\Psi}(X(T))$$

$$dX(t) = \Big(FX(t) + Gu(t) + Hw(t)\Big)dt + Ld\mathcal{B}_t,$$

where

$$\tilde{Q} = \begin{bmatrix} Q & -Q \\ -Q & Q \end{bmatrix}, \quad L = \begin{bmatrix} \sigma I \\ 0 \end{bmatrix},$$

$$F = \begin{bmatrix} 0 & 0 \\ 0 & -\theta I \end{bmatrix}, \quad G = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad H = \begin{bmatrix} -I \\ 0 \end{bmatrix},$$

$R > 0$, $\Gamma > 0$, and $\tilde{\Psi}(X) := \Psi(x, \tilde{m})$.
The idea is therefore to consider a new value function $\mathcal{V}_t(x, \tilde{m})$ (in compact form $\mathcal{V}_t(X)$) in the augmented state space, which satisfies

$$\begin{cases} \partial_t \mathcal{V}_t(X) + H(X, \partial_X \mathcal{V}_t(X)) \\ +\frac{1}{2}\sigma^2 \mathrm{Tr}\partial^2_{xx}\mathcal{V}_t(X) = 0, \text{ in } \mathbb{R}^{2q} \times [0, T[, \\ \mathcal{V}_T(X) = \tilde{\Psi}(X) \quad \text{in } \mathbb{R}^{2q}, \end{cases}$$

where $H(X, \partial_X \mathcal{V}_t(X))$ is the robust Hamiltonian [5]:

$$H(X, \partial_X \mathcal{V}_t(X)) = \frac{1}{2}\, X^T \tilde{Q} X + \partial_X \mathcal{V}_t(X) F X - \frac{1}{2}\, \partial_X \mathcal{V}_t(X)[GR^{-1}G^T - H\Gamma^{-1}H^T](\partial_X \mathcal{V}_t(X))^T.$$

This PDE admits the unique solution given by

$$\mathcal{V}_t(X) = \frac{1}{2}\, X(t)^T \underbrace{\begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{12}(t)^T & P_{22}(t) \end{bmatrix}}_{P(t)} X(t) + \frac{1}{2}\, p(t),$$

where the symmetric matrix $P(t)$ satisfies (is the unique nonnegative-definite solution of) the generalized (game) Riccati differential equation

$$\begin{aligned} &\dot{P}(t) + P(t)F + F^T P(t) \\ &-P(t)(GR^{-1}G^T - H\Gamma^{-1}H^T)P(t) + \tilde{Q} = 0\,, \\ &P(T) = \begin{bmatrix} S & -S \\ -S & S \end{bmatrix}, \end{aligned} \tag{13}$$

and $p(\cdot)$ is solved from

$$\dot{p}(t) + \sigma^2\, \mathrm{Tr}P(t)\,, \quad p(T) = 0\,.$$

Then, the corresponding optimal control is given by

$$\begin{aligned} \tilde{u}(t) &= -R^{-1}G^T P(t) X(t) \\ &= -R^{-1}B^T(P_{11}(t)x(t) + P_{12}(t)\bar{m}(t)), \end{aligned} \tag{14}$$

and the worst-case disturbance is given by

$$\begin{aligned} \tilde{w}(t) &= \Gamma^{-1}H^T P X(t) \\ &= -\Gamma^{-1}(P_{11}(t)x(t) + P_{12}(t)\bar{m}(t)). \end{aligned} \tag{15}$$

A. Bounded controls and disturbances

Consider now a second scenario where both controls and disturbances are bounded. Let us introduce the *sat* function as in [4]:

$$sat_{[u_i^-, u_i^+]}\{\xi\} \doteq \begin{cases} u_i^- & \text{if} \quad \xi < u_i^- \\ u_i^+ & \text{if} \quad \xi > u_i^+ \\ \xi & \text{if} \quad u_i^- \le \xi \le u_i^+ \end{cases},$$

where u_i^- and u_i^+ are, respectively, the lower and upper bounds on u_i. Then a sub-optimal control is given by

$$\begin{aligned} \tilde{u}(t) &= sat\Big\{ - R^{-1}G^T P(t) X(t) \Big\} \\ &= sat\Big\{ - R^{-1}B^T(P_{11}(t)x(t) + P_{12}(t)\bar{m}(t)) \Big\}, \end{aligned}$$

and the worst-case disturbance can be approximated by

$$\begin{aligned} \tilde{w}(t) &= sat\Big\{ \Gamma^{-1}H^T P X(t) \Big\} \\ &= sat\Big\{ - \Gamma^{-1}(P_{11}(t)x(t) + P_{12}(t)\bar{m}(t)) \Big\}. \end{aligned}$$

The underlying idea of the approximation above is to consider the solution of the soft-constrained linear quadratic problem when the hard constraints are not active, while saturating every single component as soon as it reaches its upper or lower bounds.

B. Asymptotic stability and mean-field equilibrium

Using the optimal control and worst-case disturbance (14)-(15) in the SDE (3) we obtain

$$\begin{array}{rcl} dx(t) & = & (-R^{-1}B^T - \Gamma^{-1})P_{11}(t)x(t) \\ & + & (-R^{-1}B^T - \Gamma^{-1})P_{12}(t)\bar{m}(t) \\ & & +\sigma d\mathcal{B}(t),\ t \in (0,T],\ x_0 \in \mathbb{R}. \end{array} \tag{16}$$

Assumption 2: There exists $\kappa > 0$ such that

$$\begin{array}{rcl} -\kappa x(t) & \geq & (-R^{-1}B^T - \Gamma^{-1})P_{11}(t)x(t) \\ & + & (-R^{-1}B^T - \Gamma^{-1})P_{12}(t)\bar{m}(t) \end{array} \tag{17}$$

Under the above assumption, the SDE is linear and time-varying and the corresponding stochastic process can be analyzed in the context of stochastic stability theory [14].

Definition 1 (cf. Definition (11.3.1) in [1]): (stability in pth moment) The equilibrium solution of a stochastic process $\xi(t)$ is said to be *stable in pth moment*, $p > 0$, if given $\varepsilon > 0$, there exists a $\delta(\varepsilon, t_0) > 0$ so that $\|x(0)\| \leq \delta$ guarantees that

$$\mathbb{E}\{\sup_{t \geq t_0} \|x(t)\|^p < \varepsilon\}.$$

When $p = 1$ or 2 we speak of *stability in mean* or *in mean square* respectively.

Theorem 1: The stochastic process (16) describing the time evolution of the excesses is mean square stable.

Proof: Let us consider as Lyapunov function the quadratic function $V(x) = x^2$ and observe that $V(x) \geq 0$, $V(0) = 0$.

In addition to this, let the infinitesimal generator be

$$\mathcal{L} = \frac{1}{2}\sigma^2 \frac{d^2}{dx^2} - \kappa x(t)\frac{d}{dx}. \tag{18}$$

We recall that for a Brownian motion we have $\mathbb{E}d\mathcal{B}_t = 0$ and $\mathbb{E}d\mathcal{B}_t^2 = dt$ and dropping the second-order terms (in dt^2) one obtains (18).

Then the stochastic derivative of $V(x)$ can be obtained by applying the infinitesimal generator to $V(x)$, which yields

$$\begin{array}{rcl} \mathcal{L}V(x(t)) & = & \lim\limits_{dt \to 0} \dfrac{\mathbb{E}V(x(t+dt)) - V(x(t))}{dt} \\ & = & \sigma^2 - 2\kappa x^2(t). \end{array}$$

Then we have that $\mathcal{L}V(x) \leq 0$ on $Q_\epsilon := \{x : V(x) \geq \epsilon\}$ for some $\epsilon > 0$. Hence the 2nd moment is bounded and the process is mean square stable. ■

The interpretation of the above result is that the variance of the excesses in each game is bounded.

C. Mean-field equilibrium

We can approximate the mean-field equilibrium, which is captured by the evolution of $\bar{m}_t$ over the horizon $(0, T]$, as

$$\begin{array}{l} \frac{d}{dt}\bar{m}_t = ((-R^{-1}B^T - \Gamma^{-1})P_{11}(t)\int x(t)dm \\ \quad +(-R^{-1}B^T - \Gamma^{-1})P_{12}(t)\bar{m}(t) \\ \quad = (-R^{-1}B^T - \Gamma^{-1})(P_{11}(t) + P_{12})\bar{m}_t \\ \qquad t \in (0,T],\ x_0 \in \mathbb{R}. \end{array}$$

Actually, we can derive an expression based on the matrix exponential $e^{\rho t}$ describing the evolution of the mean distribution which represents a bound, namely

$$\left\{ \begin{array}{l} \bar{m}_t = \bar{m}_0 e^{\rho t} \\ \rho = (-R^{-1}B^T - \Gamma^{-1})(P_{11}(t) + P_{12}). \end{array} \right.$$

The equation above corresponds to saying that the mean distribution converges exponentially to zero in the absence of the stochastic disturbances (the Brownian motion).

V. Simulations

In this section we provide simulations of a game with three players. Matrix $B \in \{0,1\}^{7\times 3}$ takes the form

$$B^T = \left[\begin{array}{ccccccc} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array} \right].$$

step size	horizon length	σ	θ
0.1	500	0.01	0.01

TABLE I
Simulations data

Given that (3) is an overdetermined system, where B is a tall matrix, we take the error as

$$e(t) = m(t) - x(t)$$

and calculate the least square approximation as

$$e_{ls}(t) = (B^T B)^{-1} B^T e(t).$$

We simulate (3) using the discrete-time expression

$$x(t + dt) = x(t) + (Bu)dt + \sigma(\mathcal{B}(t + dt) - \mathcal{B}(t)) \tag{19}$$

where the control $u(t) = e_{ls}(t)$, the step size $dt = 0.1$ and $\sigma = 0.01$. The initial state is randomly selected, and in this specific example takes the value

$$x(0) = [1\ 3\ 2\ 3\ 2\ 5\ 4],$$

while for the initial average distribution we take

$$\bar{m}(0) = [10\ 20\ 50\ 30\ 20\ 50\ 40].$$

We also approximate the time evolution of the average (12) by using the discrete-time expression

$$\left\{ \begin{array}{l} \tilde{m}(t + dt) = \tilde{m}(t) - \theta\tilde{m}(t)dt, \quad \text{for all } t \in [0,T], \\ \tilde{m}_0 = \bar{m}_0. \end{array} \right.$$

where $\theta = 0.01$.

The temporal evolution of the state is depicted in Figure 3. As to be expected, the state converges to a neighborhood of the origin. For a second scenario we take $\theta = 0$, which implies that the average $\bar{m}$ is constant and simulate the state evolution in absence of disturbance ($\sigma = 0$). The resulting time-plot is depicted in Fig. 4. We observe that the state converges to the least squares approximation of $\bar{m}(0)$.

Algorithm

Input: Set of parameters as in Table I.
Output: State trajectory $x(t)$

```
Initialize. Generate x_0 and m̄_0
for time t = 0, 1, ..., T − 1 do
    if t > 0, then compute m̄_t
    end if
    compute least-square error e_ls(t),
    compute new state x(t + 1) by executing (19)
end for
STOP
```

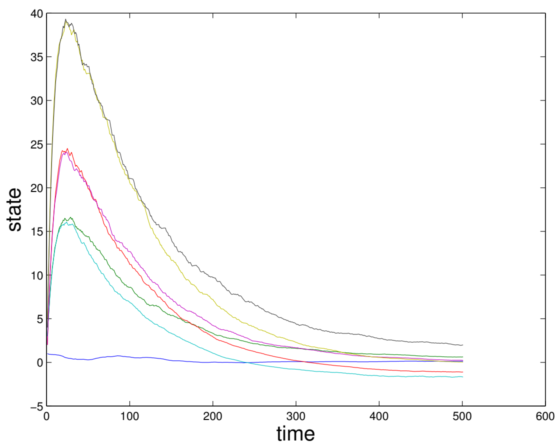

Fig. 3. First simulation scenario: time plot of state $x(t)$.

VI. Conclusions and future directions

We have provided a mean-field game formulation of infinite copies of “small worlds”, each one described as a TU coalitional game. The problem has connections to recent research on robust dynamic coalitional TU games [8] and robust mean-field games [5], [9]. A quantitative analysis of the approximation error of the solution presented is left as future work.

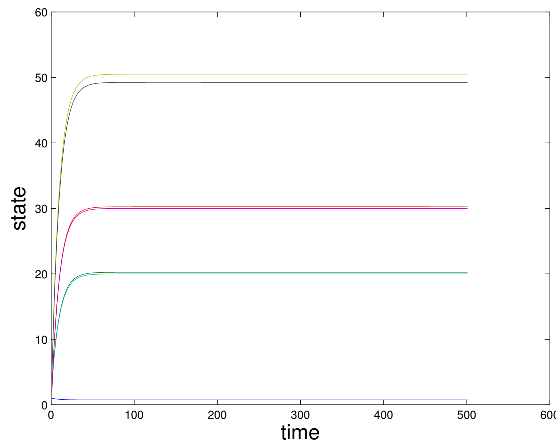

Fig. 4. Second simulation scenario: time plot of state $x(t)$.

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