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Analytic Methods In Combinatorics

by

Jan Volec

Thesis

Submitted to the University of Warwick

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Contents

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Declarations

This thesis consists of four chapters.

a) Chapter 1 contains an introduction to flag algebras, which is a framework developed by Razborov [59] for problems in extremal combinatorics. Most of the content of this chapter follows the exposition in [59].

b) The results from Chapter 2 were obtained together with Roman Glebov and Dan Kráľ. The corresponding paper [34] is submitted for publication. An extended abstract was accepted to the proceedings of Erdős Centennial conference [35] and to the proceedings of Eurocomb 2013 conference [36].

c) The results from Chapter 3 were obtained together with Jozef Balogh, Ping Hu, Bernard Lidický, Oleg Pikhurko, and Balász Udvari. The paper [8] contains the results from Section 3.2 and it has been accepted for publication in *Combinatorics, Probability and Computing.* The results from Section 3.2 are also a part of the PhD thesis of Ping Hu. The paper [9] with the results from Section 3.5 and Section 3.6 will be submitted for publication.

d) The results from Chapter 4 were obtained together with Roman Glebov and Dan Kráľ. The corresponding paper [33] is submitted for publication.

With the exception described under the point (c), none of the results appeared in any other thesis.

All the collaborators have agreed with the inclusion of our joint work into this thesis.

Abstract

In the thesis, we apply the methods from the recently emerged theory of limits of discrete structures to problems in extremal combinatorics. The main tool we use is the framework of flag algebras developed by Razborov.

We determine the minimum threshold d that guarantees a 3-uniform hypergraph to contain four vertices which span at least three edges, if every linear-size subhypergraph of the hypergraph has density more than d. We prove that the threshold value d is equal to $1/4$. The extremal configuration corresponds to the set of cyclically oriented triangles in a random orientation of a complete graph. This answers a question raised by Erdős.

We also use the flag algebra framework to answer two questions from the extremal theory of permutations. We show that the minimum density of monotone subsequences of length five in any permutation is asymptotically equal to $1/256$, and that the minimum density of monotone subsequences of length six is asymptotically equal to $1/3125$. Furthermore, we characterize the set of (sufficiently large) extremal configurations for these two problems. Both the values and the characterizations of extremal configurations were conjectured by Myers.

Flag algebras are also closely related to the theory of dense graph limits, where the main objects of study are convergent sequences of graphs. Such a sequence can be assigned an analytic object called a graphon. In this thesis, we focus on finitely forcible graphons. Those are graphons determined by finitely many subgraph densities. We construct a finitely forcible graphon such that the topological space of its typical vertices is not compact. In our construction, the space even fails to be locally compact. This disproves a conjecture of Lovász and Szegedy.

Résumé

Dans cette thèse, nous appliquons à des problèmes de combinatoire extrémale les méthodes de la théorie des limites de structures discrètes, qui a été récemment développée. L'outil principal utilisé est celui des algèbres de drapeaux, développé par Razborov.

Nous déterminons le seuil minimum, d, garantissant que tout hypergraphe 3-uniforme contient quatre sommets induisant au moins trois arêtes, si tout sous-hypergraphe d'ordre linéaire en l'ordre de l'hypergraphe a une densité strictement plus grande que d. Nous prouvons que cette valeur seuil d est égale à $1/4$. La configuration extrémale correspond à un ensemble de triangles orientés de façon cyclique dans une orientation aléatoire d'un graphe complet. Ceci répond à une question posée par Erdős.

Nous utilisons également la théorie des algèbres de drapeaux pour répondre à deux questions de théorie extrémale des permutations. Nous montrons que la densité minimum d'une sous-suite monotone de longueur cinq dans toute permutation est asymptotiquement égale à $1/256$, et que la densité minimum d'une sous-suite monotone de longueur six est asymptotiquement égale à $1/3125$. Par ailleurs, nous caractérisons l'ensemble des configurations extrémales (suffisamment grandes) pour ces deux problèmes. Les deux valeurs ainsi que les caractérisations de configurations extrémales avaient été conjecturées par Myers.

Les algèbres de drapeaux sont également étroitement liées à la théorie des limites de graphes denses, dont les objets d'étude principaux sont les suites de graphes convergentes. On peut associer à une telle suite un objet analytique appelé graphon. Nous nous intéressons aux graphons forçables de façon finie. Il s'agit des graphons déterminés par un ensemble fini de densités de sous-graphes. Nous construisons un graphon forçable de façon finie tel que l'espace topologique de ses sommets typiques n'est pas compact. Dans notre construction, cet espace n'est même pas localement compact. Ceci réfute une conjecture de Lovász et Szegedy.

Notation and preliminaries

We follow the basic graph theory notation from the book of Bondy and Murty [10]. For a graph G, we denote the set of vertices of G by $V(G)$ and the set of edges of G by $E(G) \subseteq \binom{V(G)}{2}$. Analogously, for every integer r and every r-uniform hypergraph H, or an *r-graph* for short, we again let $V(H)$ to be the set of vertices of H and $E(H) \subseteq \binom{V(H)}{r}$ to be the set of (hyper)edges of H. Note that the definition of a 2-graph coincide with the one for a graph. We call the cardinality of $V(H)$ the *order* of H and denote it by $v(H)$, and the cardinality of $E(H)$ the *size* of H and denote it by $e(H)$.

For a vertex v in an graph H, we define $N_H(v) := \{u \in V(H) : uv \in E(H)$ to be the *neighborhood* of v, and $N_H[v] := N_H(v) \cup \{v\}$ to be the *closed neighborhood* of v. If the graph H is clear from the context, we omit the H from the subscript and write only $N(v)$ or $N[v]$. We refer to the size of $N(v)$ as to the *degree of* v. If H is a 3-graph and u and v are two of its vertices, we define the *co-degree of* u *and* v to be the size of the set $N_H(u, v) := \{w \in V(H) : uvw \in E(H)\}$.

For an r-graph H, we denote by $\overline{H}$ the complement of H, i.e., the r-graph with the vertex-set $V(H)$ and the edge-set $\binom{V(H)}{r} \setminus E(H)$. For a subset of vertices $S \subseteq V(H)$, we denote by $H[S]$ the induced subhypergraph (or simply subgraph if $r = 2$), i.e., the r-graph with the vertex-set S and the edge-set $\{e \in E(H) : e \subseteq S\}$.

An *independent set* of an r-graph H is a subset of $I \subseteq V(H)$ such that $e(H[I]) = 0$. The *chromatic number* of H, which we denote by $\chi(H)$, is the smallest integer k such that $V(H)$ can be partitioned into k independent sets.

We say that an r-graph H is *linear* if every two edges intersect in at most one vertex. Note that every 2-graph, i.e., every graph, is linear.

One of the basic questions in extremal combinatorics is to determine the maximum possible number of edges in an n-vertex r-graph that does not contain a copy of some fixed r-graph F. For an r-graph F, we define the *extremal number* of F as

$$\mathrm{ex}(n, F) := \max\{e(H) : \ H \text{ is an } r\text{-graph on } n \text{ vertices with no copy of } F\},$$

and the *Turán density* of F, which is denoted by $\pi(F)$, as $\lim_{n\to\infty} \mathrm{ex}(n,F)/\binom{n}{r}$. Note that for a fixed r-graph F, the function $\mathrm{ex}(n,F)/\binom{n}{r}$ is non-increasing, the hence limit always exists.

For two graphs G and H, we define the *composition* $G \circ H$ (which is sometimes called the *lexicographical product* of G and H) to be the graph on the vertex set $V(G) \times V(H)$ in which a vertex (u,v) is adjacent to a vertex (u',v') if and only if either $uu' \in E(G)$, or $u = u'$ and $vv' \in E(H)$. In other words, we replace each vertex of G by a copy of H, and linking these copies by complete bipartite graphs according to the edges of G. This notion has a naturally generalizes to r-graphs. The composition-closure of a family $\mathcal{F}$ of r-graphs is the smallest family of r-graphs $\mathcal{F}'$ that contains $\mathcal{F}$ as a subfamily and which satisfies $G \circ H \in \mathcal{F}'$ for every $G, H \in \mathcal{F}'$.

For an r-graph H and an integer ℓ, the *ℓ-th blow-up of* H is the r-graph on $\ell \cdot v(H)$ vertices which is constructed from H by replacing each vertex v of H with an independent set of ℓ vertices I_v, and each edge of H with a complete r-partite r-graph, where each part has size ℓ. Analogously, for an r-graph H and an integer k, the *k-th iterated blow-up of* H is the r-graph on the vertex-set $V(H)^k$ isomorphic to

$$\underbrace{H \circ H \circ \cdots \circ H}_{k\text{-times}}.$$

In other words, we first take an ℓ-th blow-up of H for ℓ being $v(H)^{k-1}$, and then we place a copy of the $(k-1)$-th iterated blow-up of H inside I_v for every vertex $v \in V(H)$.

Chapter 1

Flag Algebras

The main tool used in the thesis is the framework of flag algebras. It was introduced by Razborov [59] as a general tool to approach problems from extremal combinatorics. The work of Razborov was inspired by the theory of dense graph limits, which is discussed in Section 4.1.

The flag algebra method have been successfully applied to various problems in extremal combinatorics. To name some of the applications, they were used for attacking the Caccetta-Häggkvist conjecture [43, 63], Turán-type problems in graphs [60, 54, 57, 64, 19, 38, 40, 58, 70, 42] 3-graphs [61, 56, 5, 29, 28, 34] and hypercubes [4, 7], extremal problems in a colored setting [41, 44, 6, 18], or in geometry [45]. More details on these applications can be found in a recent survey of Razborov [62].

In this chapter, we follow the approach of Razborov [59] and introduce the framework of flag algebras for the graphs. Exactly the same scheme can also be used to setup flag algebras for the oriented graphs, the ℓ-uniform hypergraphs (ℓ-graphs), the permutations, and many others. In fact, Razborov introduced in [59] the framework for an arbitrary universal first-order logic theory without constants or function symbols. We decided to present in this section the flag algebra setup for the particular instance of graphs rather than in the general setting, since it might be easier to understand the ideas of the framework in this way.

1.1 Flag algebra setting for graphs

The central notions we are going to introduce are an algebra $\mathcal{A}$ and algebras $\mathcal{A}^\sigma$, where σ is a fixed graph with a fixed labelling of its vertex set. In order to precisely describe the algebras $\mathcal{A}$ and $\mathcal{A}^\sigma$ on formal linear combinations of graphs, we first need to introduce some additional notation. Let $\mathcal{F}$ be the set of all finite non-

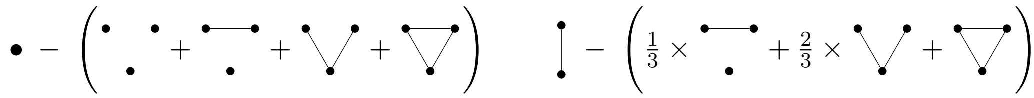


Figure 1.1: Two examples of linear combinations used in generating $\mathcal{K}$.

isomorphic graphs. Next, for every $\ell \in \mathbb{N}$, let $\mathcal{F}_\ell \subset \mathcal{F}$ be the set of all graphs of order ℓ. For convenience, we fix an arbitrary ordering on the elements the set $\mathcal{F}_\ell$ for every $\ell \in \mathbb{N}$, i.e., we always assume that $\mathcal{F}_\ell = \{F_1, F_2, \ldots, F_{|\mathcal{F}_\ell|}\}$.

For $H \in \mathcal{F}_\ell$ and $H' \in \mathcal{F}_{\ell'}$, we define $p(H, H')$ to be the probability that a randomly chosen subset of ℓ vertices in H' induces a subgraph isomorphic to H. Note that $p(H, H') = 0$ if $\ell' < \ell$. Let $\mathbb{R}\mathcal{F}$ be the set of all formal linear combinations of elements of $\mathcal{F}$ with real coefficients. Furthermore, let $\mathcal{K}$ be the linear subspace of $\mathbb{R}\mathcal{F}$ generated by all the linear combinations of the form

$$H - \sum_{H' \in \mathcal{F}_{v(H)+1}} p(H, H') \cdot H'.$$

Two examples of such linear combinations are depicted in Figure 1.1. Finally, we set $\mathcal{A}$ to be the space $\mathbb{R}\mathcal{F}$ factored by $\mathcal{K}$, and the element corresponding to $\mathcal{K}$ in $\mathcal{A}$ to be the zero element of $\mathcal{A}$.

The space $\mathcal{A}$ comes with a natural definition of an addition and a multiplication by a real number. We now introduce the notion of a product of two elements from $\mathcal{A}$. We start with the definition for the elements of $\mathcal{F}$. For $H_1, H_2 \in \mathcal{F}$, and $H \in \mathcal{F}_{v(H_1)+v(H_2)}$, we define $p(H_1, H_2; H)$ to be the probability that a randomly chosen subset of $V(H)$ of size $v(H_1)$ and its complement induce in H subgraphs isomorphic to H_1 and H_2, respectively. We set

$$H_1 \times H_2 := \sum_{H \in \mathcal{F}_{v(H_1)+v(H_2)}} p(H_1, H_2; H) \cdot H.$$

See Figure 1.2 for an example of a product with $H_1 =$ and $H_2 =$. The multiplication on $\mathcal{F}$ has a unique linear extension to $\mathbb{R}\mathcal{F}$, which yields a well-defined multiplication also in the factor algebra $\mathcal{A}$. A formal proof of this is given in [59, Lemma 2.4]. Observe that the one-vertex graph $\bullet \in \mathcal{F}$ is, modulo $\mathcal{K}$, the neutral element of the product in $\mathcal{A}$.

Let us now move to the definition of the algebra $\mathcal{A}^\sigma$, where σ is a fixed finite graph with a fixed labelling of its vertices. The labelled graph σ is usually called

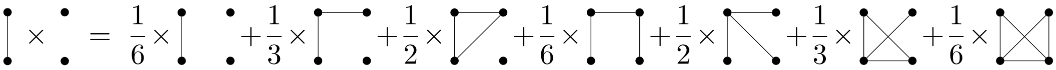


Figure 1.2: An example of a product in the algebra $\mathcal{A}$.

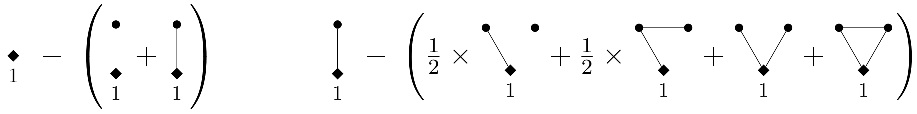


Figure 1.3: Two examples of linear combinations used in generating $\mathcal{K}^\sigma$, where σ is the one-vertex type.

a *type*. We follow the same lines as in the definition of $\mathcal{A}$. Let $\mathcal{F}^\sigma$ be the set of all finite graphs H with a fixed *embedding* of σ, i.e., an injective mapping θ from $V(\sigma)$ to $V(H)$ such that θ is an isomorphism between σ and $H[\mathrm{Im}(\theta)]$. The elements of $\mathcal{F}^\sigma$ are usually called *σ-flags*, the subgraph induced by $\mathrm{Im}(\theta)$ is called the *root* of a σ-flag, and the vertices $\mathrm{Im}(\theta)$ are called the *rooted* or the *labelled* vertices. The vertices that are not rooted are called the *non-rooted* or the *non-labelled* vertices. For every $\ell \in \mathbb{N}$, we define $\mathcal{F}^\sigma_\ell \subset \mathcal{F}^\sigma$ to be the set of all ℓ-vertex σ-flags from $\mathcal{F}^\sigma$. Also, for each type σ and each integer ℓ, we fix an arbitrary ordering on the elements of the set $\mathcal{F}^\sigma_\ell$.

In the analogy to the case of $\mathcal{A}$, for two σ-flags $H \in \mathcal{F}^\sigma$ and $H' \in \mathcal{F}^\sigma$ with the embeddings of σ given by θ and θ', respectively, we set $p(H, H')$ to be the probability that a randomly chosen subset of $v(H) - v(\sigma)$ vertices in $V(H') \setminus \theta'(V(\sigma))$ together with $\theta'(V(\sigma))$ induces a subgraph that is isomorphic to H through an isomorphism f that preserves the embedding of σ. In other words, the isomorphism f has to satisfy $f(\theta') = \theta$. Let $\mathbb{R}\mathcal{F}^\sigma$ be the set of all formal linear combinations of elements of $\mathcal{F}^\sigma$ with real coefficients, and let $\mathcal{K}^\sigma$ be the linear subspace of $\mathbb{R}\mathcal{F}^\sigma$ generated by all the linear combinations of the form

$$H - \sum_{H' \in \mathcal{F}^\sigma_{v(H)+1}} p(H, H') \cdot H'.$$

See Figure 1.3 for two examples of such linear combinations in the case σ is the one-vertex type. We define $\mathcal{A}^\sigma$ to be $\mathbb{R}\mathcal{F}^\sigma$ factored by $\mathcal{K}^\sigma$ and, analogously to the case for the algebra $\mathcal{A}$, we let the element corresponding to $\mathcal{K}^\sigma$ to be the zero element of $\mathcal{A}^\sigma$.

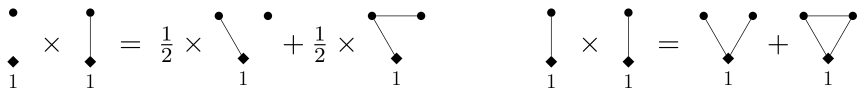


Figure 1.4: Two examples of a product in the algebra $\mathcal{A}^\sigma$, where σ is the one-vertex type.

We now define the product of two elements from $\mathcal{F}^\sigma$. Let $H_1, H_2 \in \mathcal{F}^\sigma$ and $H \in \mathcal{F}^\sigma_{v(H_1)+v(H_2)-v(\sigma)}$ be σ-flags, and θ be the fixed embedding of σ in H. Similarly to the definition of the multiplication for $\mathcal{A}$, we define $p(H_1, H_2; H)$ to be the probability that a randomly chosen subset of $V(H) \setminus \theta(V(\sigma))$ of size $v(H_1) - v(\sigma)$ and its complement in $V(H) \setminus \theta(V(\sigma))$ of size $v(H_2) - v(\sigma)$, extend $\theta(V(\sigma))$ in H to subgraphs isomorphic to H_1 and H_2, respectively. Again, by isomorphic we mean that there is an isomorphism that preserves the fixed embedding of σ. We set

$$H_1 \times H_2 := \sum_{H \in \mathcal{F}^\sigma_{v(H_1)+v(H_2)-v(\sigma)}} p(H_1, H_2; H) \cdot H.$$

Two examples of a product in $\mathcal{A}^\sigma$ for σ being the one-vertex type are depicted in Figure 1.4. The definition of the product for the elements of $\mathcal{F}^\sigma$ naturally extends to $\mathcal{A}^\sigma$. It follows that the unique σ-flag of size $v(\sigma)$ represents, modulo $\mathcal{K}^\sigma$, the neutral element of the product in $\mathcal{A}^\sigma$.

Now consider an infinite sequence $(G_n)_{n\in\mathbb{N}}$ of graphs with increasing orders. We say that the sequence $(G_n)_{n\in\mathbb{N}}$ is *convergent* if the probabilities $p(H, G_n)$ converge for every $H \in \mathcal{F}$. A standard compactness argument (e.g., using Tychonoff's theorem [73]) yields that every such infinite sequence has a convergent subsequence.

Fix a convergent sequence $(G_n)_{n\in\mathbb{N}}$ of graphs with increasing orders. For every $H \in \mathcal{F}$, we set $\phi(H) = \lim_{n\to\infty} p(H, G_n)$, and we then linearly extend ϕ to $\mathcal{A}$. We usually refer to the mapping ϕ as to the *limit* of the sequence. The obtained mapping ϕ is a homomorphism from $\mathcal{A}$ to $\mathbb{R}$, see [59, Theorem 3.3a]. Moreover, for every $H \in \mathcal{F}$, it holds $\phi(H) \geq 0$. Let $\mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$ be the set of all such homomorphisms, i.e., the set of all homomorphisms ψ from the algebra $\mathcal{A}$ to $\mathbb{R}$ such that $\psi(H) \geq 0$ for every $H \in \mathcal{F}$. It is interesting to see that this set is exactly the set of all the limits of convergent sequences of graphs [59, Theorem 3.3b].

Let $(G_n)_{n\in\mathbb{N}}$ be a convergent sequence of graphs and $\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$ its limit. For a type σ and an embedding θ of σ in G_n, we define G_n^θ to be the graph rooted on the copy of σ that corresponds to θ. For every $n \in \mathbb{N}$ and $H^\sigma \in \mathcal{F}^\sigma$, we define $p_n^\theta(H^\sigma) = p(H^\sigma, G_n^\sigma)$. Picking θ at random gives rise to a probability distribution $\mathbf{P}_\mathbf{n}^\sigma$ on mappings from $\mathcal{A}^\sigma$ to $\mathbb{R}$, for every $n \in \mathbb{N}$. Since $p(H, G_n)$ con-

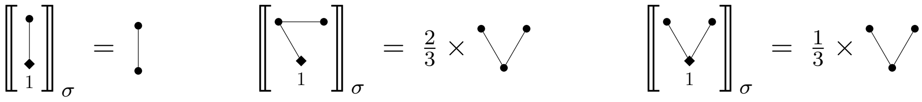


Figure 1.5: Three examples of applying the averaging operator $[\![\cdot]\!]_\sigma$, where σ denotes the one-vertex type.

verges (as n tends to infinity) for every $H \in \mathcal{F}$, the sequence of these probability distributions on mappings from $\mathcal{A}^\sigma$ to $\mathbb{R}$ weakly converges to a Borel probability measure on $\mathrm{Hom}^+(\mathcal{A}^\sigma, \mathbb{R})$, see [59, Theorems 3.12 and 3.13]. We denote the limit probability distribution by $\mathbf{P}^\sigma$. In fact, for any σ such that $\phi(\sigma) > 0$, the homomorphism ϕ itself fully determines the probability distribution $\mathbf{P}^\sigma$ [59, Theorem 3.5]. Furthermore, any mapping ϕ^σ from the support of the distribution $\mathbf{P}^\sigma$ is in fact a homomorphism from $\mathcal{A}^\sigma$ to $\mathbb{R}$ such that $\phi^\sigma(H^\sigma) \geq 0$ for all $H^\sigma \in \mathcal{F}^\sigma$ [59, Proof of Theorem 3.5].

The last notion we introduce is the *averaging* (or downward) operator $[\![\cdot]\!]_\sigma : \mathcal{A}^\sigma \to \mathcal{A}$. It is the linear operator defined on the elements of $H \in \mathcal{F}^\sigma$ by

$$[\![H]\!]_\sigma := p_H^\sigma \cdot H^\emptyset,$$

where $H^\emptyset$ is the (unlabelled) graph from $\mathcal{F}$ corresponding to H after unlabelling all its vertices, and p_H^σ is the probability that a random injective mapping from $V(\sigma)$ to $V(H^\emptyset)$ is an embedding of σ in $H^\emptyset$ yielding a σ-flag isomorphic to H. See Figure 1.5 for three examples of applying the averaging operator $[\![\cdot]\!]_\sigma$, where σ is the one-vertex type.

The key relation between ϕ and ϕ^σ is the following

$$\forall H^\sigma \in \mathcal{A}^\sigma, \quad \phi\left([\![H^\sigma]\!]_\sigma\right) = \phi\left([\![\sigma]\!]_\sigma\right) \cdot \int \phi^\sigma(H^\sigma), \tag{1.1}$$

where the integration is with respect to the probability measure given by the random distribution $\mathbf{P}^\sigma$ on ϕ^σ. Note that

$$\phi\left([\![\sigma]\!]_\sigma\right) = \frac{\left|\mathrm{Aut}\left(\sigma^\emptyset\right)\right|}{v\left(\sigma^\emptyset\right)!} \cdot \phi\left(\sigma^\emptyset\right).$$

Vaguely speaking, the relation 1.1 corresponds to the conditional probability formula $\mathbb{P}[A \cap B] = \mathbb{P}[B] \cdot \mathbb{P}[A \mid B]$, where B is the event that a random injective mapping θ is an embedding of σ, and A is the event that a random subset of $v(H) - v(\sigma)$ vertices extends θ to the σ-flag H^σ. A formal proof is given in [59, Lemma 3.11].

The relation (1.1) implies that if $\phi^\sigma(A^\sigma) \geq 0$ almost surely for some $A^\sigma \in \mathcal{A}^\sigma$, then $\phi\left(\llbracket A^\sigma \rrbracket_\sigma\right) \geq 0$. In particular, for every homomorphism $\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$ and every linear combination $A^\sigma \in \mathcal{A}^\sigma$ it holds

$$\phi\left(\left\llbracket (A^\sigma)^2 \right\rrbracket_\sigma\right) \geq 0. \tag{1.2}$$

We note that a stronger variant of (1.2) can be proven using Cauchy-Schwarz's inequality. Specifically, [59, Theorem 3.14] states that

$$\forall \phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R}), \forall A^\sigma, B^\sigma \in \mathcal{A}^\sigma,\ \phi\left(\left\llbracket (A^\sigma)^2 \right\rrbracket_\sigma \times \left\llbracket (B^\sigma)^2 \right\rrbracket_\sigma\right) \geq \phi\left(\llbracket A^\sigma \times B^\sigma \rrbracket_\sigma\right)^2.$$

Let σ be a type, $A^\sigma \in \mathcal{A}^\sigma$, and m the minimum integer such that

$$A^\sigma = \sum_{F^\sigma \in \mathcal{F}^\sigma_m} \alpha_{F^\sigma} \cdot F^\sigma_i.$$

We say that $\ell := 2m - v(\sigma)$ is the *order* of the expression $\left\llbracket (A^\sigma)^2 \right\rrbracket_\sigma$. It follows that

$$\left\llbracket (A^\sigma)^2 \right\rrbracket_\sigma = \sum_{F \in \mathcal{F}_\ell} \alpha_F \cdot F_i.$$

Since the operator $\llbracket \cdot \rrbracket_\sigma$ is linear, it immediately follows that for every $\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$, $A^\sigma_1, A^\sigma_2, \dots A^\sigma_K \in \mathcal{A}^\sigma$, and non-negative reals $\alpha_1, \alpha_2, \dots, \alpha_K$ we have

$$\phi\left(\left\llbracket \alpha_1 \cdot (A^\sigma_1)^2 + \alpha_2 \cdot (A^\sigma_2)^2 + \cdots + \alpha_K \cdot (A^\sigma_K)^2 \right\rrbracket_\sigma\right) \geq 0.$$

Hence for every finite set $S \subseteq \mathcal{F}^\sigma$ and every real symmetric positive semidefinite matrix M of size $|S| \times |S|$ it holds that

$$\phi\left(\llbracket x_S^T M x_S \rrbracket_\sigma\right) \geq 0,$$

where x_S is the $|S|$-dimensional vector from whose i-th coordinate is equal to the i-th element of S. Note that we used the fact that every real symmetric positive semidefinite matrix M of size $s \times s$ can be written as a sum of squares. In other words, there exists an integer $d \leq s$ such that

$$M = \sum_{j \in [d]} \lambda_j \cdot v_j \times (v_j)^T,$$

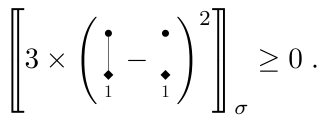
where vectors v_i, for $i \in [d]$, form an orthonormal eigen-basis of M and λ_i are the corresponding (always non-negative) eigenvalues. On the other hand, for every set of d' non-negative reals λ'_j and vectors $v'_j \in \mathbb{R}^s$, the matrix

$$M' := \sum_{j \in [d']} \lambda'_j \cdot v'_j \times \left(v'_j\right)^T$$

is a symmetric positive semidefinite matrix of size $s \times s$.

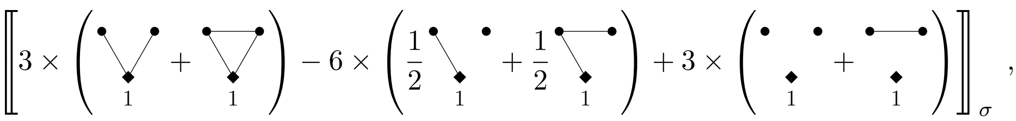
1.2 Semidefinite method

The heart of the semidefinite method are inequalities of the form (1.2). These inequalities turned out to be very useful while dealing with problems from extremal combinatorics. Before moving to the description of the method itself, let us illustrate this on a small example. Consider the following inequality, where σ is the one-vertex type:

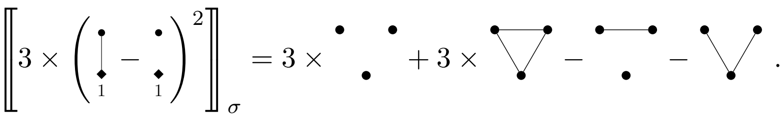


(1.3)

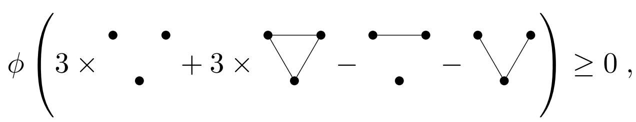
The definition of the product on A^σ implies that the left-hand side of (1.3) is equal to



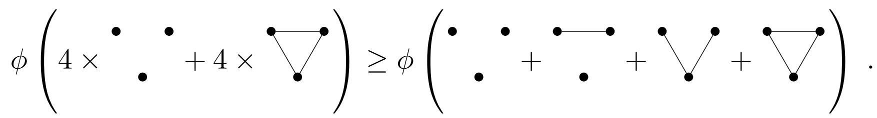
and an application of the averaging operator yields that



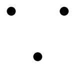
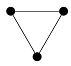
Therefore, every $\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$ satisfies



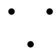
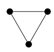
and hence also



Since the right-hand side of the last inequality is equal to one, we conclude that

$\phi\Bigg($  $+$  $\Bigg) \geq \frac{1}{4}$ (1.4)

for every $\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$. This is a well-known inequality due to Goodman [37]. Note that the inequality (1.4) is best possible. This can be seen, for example, by considering the limit of the sequence of Erdős-Renyi random graphs $G_{n,1/2}$ with increasing orders (the sequence is convergent with probability 1). Another example where the inequality (1.4) is tight is the sequence of complete balanced bipartite graphs with increasing orders (it is straightforward to check that this sequence is convergent).

Now consider a general linear combination $A \in \mathcal{A}$. One of the fundamental problems in extremal combinatorics is to determine the smallest value of $\phi(A)$ over all $\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$. The semidefinite method is a tool from the flag algebra framework that systematically searches for inequalities of the form (1.2), like the inequality (1.3) in the case when $A =$  $+$ , in order to find a lower bound on

$$\min_{\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})} \phi(A). \tag{1.5}$$

Note that since $\mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$ is compact, such a minimum exists for every $A \in \mathcal{A}$.

The semidefinite method works as follows. First, fix an upper bound ℓ on the order of flags in all the terms of linear inequalities we are going to consider, including also the terms of the objective function A. Without loss of generality, $A = \sum_{F \in \mathcal{F}_\ell} \alpha_F \cdot F$. Next, fix an arbitrary list of types $\sigma_1, \ldots, \sigma_K$ of order at most ℓ. Recall our aim is to find a lower bound on (1.5). The semidefinite method finds a way how to express A in the algebra $\mathcal{A}$ as follows:

$$A = \underbrace{\left(\sum_{k \in [K]} \sum_{j \in [J_k]} b_j^k \cdot \left[\!\!\left[\left(A_j^{\sigma_k} \right)^2 \right]\!\!\right]_{\sigma_k} \right)}_{R} + \underbrace{\left(\sum_{F \in \mathcal{F}_\ell} \beta_F \cdot F \right)}_{S} + \underbrace{\left(c \cdot \sum_{F \in \mathcal{F}_\ell} F \right)}_{T}, \tag{1.6}$$

where

- $J_1, \ldots J_K$ are non-negative integers,
- $A_j^{\sigma_1} \in \mathcal{A}^{\sigma_1}$ so that the order of $\left[\!\!\left[\left(A_j^{\sigma_1} \right)^2 \right]\!\!\right]_{\sigma_1}$ is at most ℓ for every $j \in [J_1]$,

$$\vdots$$

- $A_j^{\sigma_K} \in \mathcal{A}^{\sigma_K}$ so that the order of $\left[\!\left[\left(A_j^{\sigma_K} \right)^2 \right]\!\right]_{\sigma_K}$ is at most ℓ for every $j \in [J_K]$,
- $b_j^k \geq 0$ for every $k \in [K]$ and $j \in [J_k]$,
- $\beta_F \geq 0$ for every $F \in \mathcal{F}_\ell$, and
- $c \in \mathbb{R}$.

Since $\phi(R) \geq 0$, $\phi(S) \geq 0$, and $\phi(T) = c$ for all $\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$, we conclude that $\phi(A) \geq c$. Note that R is a positive linear combination of inequalities (1.2) of order at most ℓ, hence $R = \sum_{F \in \mathcal{F}_\ell} r_F \cdot F$ for some choice of reals r_F.

For a fixed choice of the parameters ℓ and $\sigma_1, \ldots, \sigma_K$, finding such an expression of A can be formulated as a semidefinite program. Note that all the expressions A, R, S and T can be written as linear combinations of the elements from $\mathcal{F}_\ell$, i.e., they can be viewed as vectors in $\mathbb{R}^{|\mathcal{F}_\ell|}$. Furthermore, the bound obtained by the semidefinite method is "best possible" in the following sense. Let c_0 be the obtained bound. For every expression of A as a linear combination of the form (1.6), the coefficient c in this combination is at most c_0. Note that there might be (and often there are) different combinations that yield the bound c_0.

Let us now describe the corresponding semidefinite program in more detail. Fix one of the types $\sigma_k \in \{\sigma_1, \ldots, \sigma_K\}$. Since the semidefinite method uses inequalities $\left[\!\left[(A^{\sigma_k})^2 \right]\!\right]_{\sigma_k}$ of order at most ℓ, it follows that

$$A^{\sigma_k} = \sum_{F_i \in \mathcal{F}^{\sigma_k}_{m(k)}} \alpha_i \cdot F_i$$

for some integer $m(k)$ such that $m(k) \leq \frac{\ell + v(\sigma_k)}{2}$. Without loss of generality, $m(k) = \left\lfloor \frac{\ell + v(\sigma_k)}{2} \right\rfloor$. Therefore,

$$R = \sum_{k \in [K]} \sum_{j \in [J_k]} b_j^k \cdot \left[\!\left[\left(\sum_{F_i \in \mathcal{F}^{\sigma_k}_{m(k)}} \alpha_{k,j,i} \cdot F_i \right)^2 \right]\!\right]_{\sigma_k} = \sum_{k \in [K]} \left[\!\left[x_{\sigma_k}^T M_{\sigma_k} x_{\sigma_k} \right]\!\right]_{\sigma_k}, \tag{1.7}$$

where

- each vector x_{σ_k} is the $|\mathcal{F}^{\sigma_k}_{m(k)}|$-dimensional vector whose i-th coordinate is equal to the i-th element of $\mathcal{F}^{\sigma_k}_{m(k)}$, and

- each matrix M_{σ_k} is equal to

$$\sum_{j\in[J_k]} b_j^k \cdot \left(\alpha_{k,j,1}, \alpha_{k,j,2}, \ldots, \alpha_{k,j,\left|\mathcal{F}^{\sigma_k}_{m(k)}\right|}\right)^T \times \left(\alpha_{k,j,1}, \alpha_{k,j,2}, \ldots, \alpha_{k,j,\left|\mathcal{F}^{\sigma_k}_{m(k)}\right|}\right).$$

Note that the matrices $M_{\sigma_1}, \ldots, M_{\sigma_K}$ are symmetric positive semidefinite matrices with real entries.

Now recall that $R = \sum_{F\in\mathcal{F}_\ell} r_F \cdot F$. The equation (1.7) implies that all the coefficients r_F depends only on the entries of the matrices $M_{\sigma_1}, \ldots, M_{\sigma_K}$. For a given set of matrices $M_{\sigma_1}, \ldots, M_{\sigma_K}$, we write $r_F(M_{\sigma_1}, \ldots, M_{\sigma_K})$ to denote the coefficient in front of F in R. Using this notation, the semidefinite program for the objective value $A = \sum\limits_{F\in\mathcal{F}_\ell} \alpha_F \cdot F$ can be written as

$$\begin{aligned} \underset{c\in\mathbb{R}}{\text{maximize}} \quad & c \\ \text{subject to} \quad & \alpha_F \geq c + r_F(M_{\sigma_1}, \ldots, M_{\sigma_K}) \quad \forall F \in \mathcal{F}_\ell, \\ & M_{\sigma_1} \succeq 0, \\ & \vdots \\ & M_{\sigma_K} \succeq 0, \end{aligned} \tag{1.8}$$

where the constraints $M_{\sigma_k} \succeq 0$, for $k \in [K]$, denote that the matrices M_{σ_k} are positive semidefinite.

Let us now focus on the dual program of the semidefinite program (1.8). We start with introducing some additional notation. For a homomorphism $\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$ and an integer ℓ, the *local density ℓ-profile* of ϕ is the vector

$$\phi_{\restriction\ell} := \left(\phi(F_1), \phi(F_2), \ldots, \phi(F_{|\mathcal{F}_\ell|})\right).$$

We denote the i-th coordinate of $\phi_{\restriction\ell}$ by $\phi_{\restriction\ell}(F_i)$. Furthermore, for $A = \sum\limits_{F\in\mathcal{F}_\ell} \alpha_F \cdot F$, where α_F are arbitrary fixed reals, we define

$$\phi_{\restriction\ell}(A) := \sum_{F\in\mathcal{F}_\ell} \alpha_F \cdot \phi_{\restriction\ell}(F).$$

With a slight abuse of notation, we use this notion also for an arbitrary vector $z \in \mathbb{R}^{|\mathcal{F}_\ell|}$, i.e., we write $z(F_i)$ for the i-th coordinate of z, and we use $z(A)$ to denote the value of $\sum\limits_{F\in\mathcal{F}_\ell} \alpha_F \cdot z(F)$.

Let $\mathcal{P}_{\mathcal{F}_\ell} := \left\{\phi_{\restriction \ell} : \phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})\right\}$ be the set of all local density ℓ-profiles. Note that $\mathcal{P}_{\mathcal{F}_\ell} \subseteq [0,1]^{|\mathcal{F}_\ell|}$. For a combination $A = \sum\limits_{F \in \mathcal{F}_\ell} \alpha_F \cdot F$, it follows from the definitions that the value of (1.5) is equal to the minimum value of $\phi_{\restriction \ell}(A)$, where the minimum is taken over all $\phi_{\restriction \ell} \in \mathcal{P}_{\mathcal{F}_\ell}$.

Now fix K types $\sigma_1, \ldots, \sigma_K$ of order at most ℓ. Let $\mathcal{S}_{\mathcal{F}_\ell}$ be the set of vectors $z \in \mathbb{R}^{|\mathcal{F}_\ell|}$ that satisfy

- all the linear inequalities of the form $z\left(\left[\!\left[(A^{\sigma_k})^2\right]\!\right]_{\sigma_k}\right) \geq 0$, where $k \in [K]$, $A^{\sigma_k} \in \mathcal{A}^{\sigma_k}$, and the order of the expression $\left[\!\left[(A^{\sigma_k})^2\right]\!\right]_{\sigma_k}$ is at most ℓ,
- the non-negative inequalities $z(F) \geq 0$ for every $F \in \mathcal{F}_\ell$, and
- the equation $z\left(\sum_{F \in \mathcal{F}_\ell} F\right) = 1$.

It immediately follows that $\mathcal{P}_{\mathcal{F}_\ell} \subseteq \mathcal{S}_{\mathcal{F}_\ell}$. It also follows that the set $\mathcal{S}_{\mathcal{F}_\ell}$ is a convex set.

Recall that the aim of the semidefinite method is to find the minimum of $\phi(A)$, where $\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$, which is the same as the minimum of $\phi_{\restriction \ell}(A)$ for $\phi_{\restriction \ell} \in \mathcal{P}_{\mathcal{F}_\ell}$. The duality of semidefinite programing (see, e.g., [39, Theorem 4.1.1]) implies that the dual of (1.8) is the following semidefinite program:

$$\underset{z \in \mathcal{S}_{\mathcal{F}_\ell}}{\text{minimize}} \quad z(A). \tag{1.9}$$

Therefore, if the semidefinite method finds a proof that for every $\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$ it holds that $\phi(A) \geq c$, where $c \in \mathbb{R}$, it also holds that $z(A) \geq c$ for any $z \in \mathcal{S}_{\mathcal{F}_\ell}$. In particular, if $\phi, \psi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$ and $\lambda \in (0,1)$, then

$$\lambda \cdot \phi_{\restriction \ell}(A) + (1-\lambda) \cdot \psi_{\restriction \ell}(A) \geq c. \tag{1.10}$$

Note that the $|\mathcal{F}_\ell|$-dimensional vector $\lambda \cdot \phi_{\restriction \ell} + (1-\lambda) \cdot \psi_{\restriction \ell}$ is usually not a local density ℓ-profile of any convergent sequence of graphs.

Chapter 2

Hypergraphs with positive lower density

One of the most well-known results in extremal graph theory is Turán's theorem [72], which determines the largest possible number of edges in an n-vertex graph without a complete subgraph of a given size. Erdős, Simonovits, and Stone [23, 24] generalized Turán's theorem by showing that the extremal number of a fixed graph F is asymptotically determined by its chromatic number. Specifically, for every graph F with at least one edge,

$$\mathrm{ex}(n, F) = \left(\frac{\chi(F) - 2}{\chi(F) - 1} + o(1)\right)\binom{n}{2}.$$

These extremal questions are dual to determining the minimum number of edges $m(F)$ that guarantees that an n-vertex graph G with at least $m(n, F)$ edges contains a copy of F, since $m(n, F) = \mathrm{ex}(n, F) + 1$.

For problems studied in this chapter, we impose a stronger density assumption on a (hyper)graph, where we wish to find a copy of some fixed (hyper)graph F. Instead of only assuming that the whole graph has a sufficiently large density, we assume that also every sufficiently large subset of vertices, say of size at least δn, has large relative density. Formally, we define the *δ-linear density* of a graph G to be the smallest density induced by a δ-fraction of the set of vertices, i.e.,

$$d(G, \delta) := \min\left\{\frac{e(G[A])}{\binom{|A|}{2}} : \; A \subseteq V(G), |A| \geq \delta \cdot v(G)\right\}.$$

Note that for any graph G, the function $d(G, \delta)$ is a non-decreasing function of δ taking values in $[0, 1]$.

However, requiring a positive δ-linear density immediately forces large graphs to contain every given graph as a subgraph.

Observation 2.1. *For every $\varepsilon > 0$ and a fixed graph F, there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that every graph G on at least n_0 vertices with $d(G, \delta) \geq \varepsilon$ contains F as a subgraph.*

A proof of this observation follows, e.g., from [65, Theorem 1].

The notion of the δ-linear density has a natural generalization to r-graphs. For an r-graph H, we define

$$d(H, \delta) := \min \left\{ \frac{e(H_n[A])}{\binom{|A|}{r}} \; : \; A \subseteq V(H), |A| \geq \delta \cdot v(H) \right\}.$$

Since in this chapter we mostly deal with sequences of 3-graphs, it is natural to define the *lower density* of an increasing sequence of 3-graphs $(H_n)_{n \in \mathbb{N}}$ to be the smallest δ-linear density and denote it by $\lambda\left((H_n)_{n \in \mathbb{N}}\right)$. Formally,

$$\lambda\left((H_n)_{n \in \mathbb{N}}\right) := \lim_{\delta \to 0+} \left\{ \liminf_{n \to \infty} d(H_n, \delta) \right\}.$$

It is natural to ask for what r-graphs F we can generalize Observation 2.1, i.e., what is the set of r-graphs F such that that every sufficiently large r-graph H with a positive δ-linear density contains a copy of F. Frankl and Rödl [30] showed that for every r-graph F from the composition-closure of the set of all r-partite r-graphs, there exists a positive constant δ such that every large r-graph H with a positive δ-linear density contains a copy of F. Until now, these are the only r-graphs F with this property we know.

2.1 Random tournaments construction

In order to simplify the notation in this chapter, we denote by $K_4^{(3)}$ the complete 3-graph on 4 vertices, and by K_4^- the 3-graph on 4 vertices with 3 edges; see Figure 2.1. Erdős and Sós [26, Problem 5] asked whether a sufficiently large 3-graph H with a positive δ-linear density contains a copy of $K_4^{(3)}$, or at least a copy of K_4^-. However, Füredi observed that the following construction of Erdős and Hajnal [22] gives a negative answer to the above question in a very strong sense.

Construction 2.2. *Consider a random tournament T_n on n vertices. Let H_n be the 3-graph on the same vertex set consisting of exactly those triples that span a cyclically oriented triangle in T_n.*

For the exact reference to the observation, we refer the reader to [30], where Frankl and Rödl cite a personal communication with Füredi in 1983. Additional

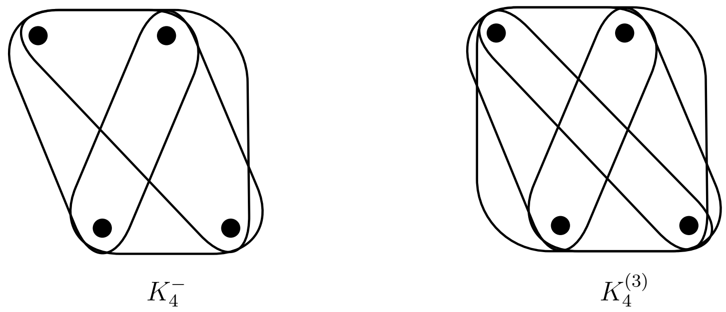


Figure 2.1: The 3-graphs K_4^- and $K_4^{(3)}$.

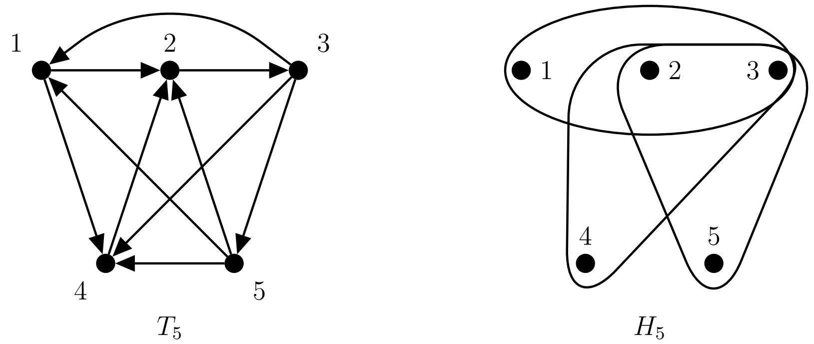


Figure 2.2: An example of the Construction 2.2 for $n = 5$.

information about this problem and its history can be found in [69, Section 5]. One can check that in every 3-graph obtained in this way, any four vertices span at most two edges, i.e., for every $n \in \mathbb{N}$ the 3-graph H_n does not contain K_4^-. It remains to show that for every $\delta > 0$, the δ-linear density of a typical H_n tends to $1/4$ as n goes to infinity. In fact, we prove that even $(1/\log n)$-linear density almost surely tends to $1/4$.

We start with the following standard concentration lemma.

Lemma 2.3. *Let $p \in [0,1]$, $r \in \mathbb{N}$ such that $r \geq 2$, and $n \in \mathbb{N}$. In an n-vertex r-graph H, we associate with every edge $e \in E(H)$ a random event $\mathcal{E}(e)$ such that*

- $\mathbb{P}\big[\mathcal{E}(e)\big] = p$ *for every $e \in E(H)$, and*
- *The events $\mathcal{E}(e)$ and $\mathcal{E}(e')$ are independent whenever $|e \cap e'| \leq 1$.*

Then the probability that the number of occurred events on some large vertex set is far from its expectation tends to zero super-exponentially fast in n. More precisely, *there exists a positive constant c such that*

$$\mathbb{P}\left[\exists A \subseteq V(H), |A| \geq \frac{n}{\log n} : \left| |\{e \in E(H[A]) :\ \mathcal{E}(e)\}| - p|E(H[A])| \right| > \frac{\binom{|A|}{r}}{\log n}\right] = e^{\frac{-c \cdot n^2}{\log^4(n)}}.$$

Proof. Fix an arbitrary set $A \subseteq V(H)$ with $|A| \geq n/\log n$. We claim that

$$\mathbb{P}\left[\left| |\{e \in E(H[A]) :\ \mathcal{E}(e)\}| - p|E(H[A])| \right| > \frac{\binom{|A|}{r}}{\log n}\right] = e^{\frac{-c' \cdot n^2}{\log^4(n)}} \tag{2.4}$$

for some $c' > 0$. The statement of the lemma then follows easily by a union bound over all subsets of $V(H)$ of size at least $n/\log n$, so it remains to prove the claim.

Consider an edge-coloring of $H[A]$ with $\binom{r}{2}\binom{|A|-2}{r-2}$ colors, such that every color class is a linear r-graph, i.e., any two edges of the same color intersect in at most one vertex. Since every fixed edge intersects in at least two vertices less than $\binom{r}{2}\binom{|A|-2}{r-2}$ other edges, such a coloring can be found in a greedy way. Since

$$\binom{r}{2}\binom{|A|-2}{r-2} \cdot \frac{|A|(|A|-1)}{r^4 \cdot \log n} < \frac{\binom{|A|}{r}}{\log n},$$

we infer that the event inside the probability formula in (2.4) implies that in at least one of the color classes, say C, the size of $\{e \in C : \mathcal{E}(e)\}$ deviates from its expectation by at least $|A|(|A|-1)/(r^4 \cdot \log n)$.

Fix a color class C. First, observe that $|C| \leq \binom{|A|}{2}/\binom{r}{2} = \frac{|A|(|A|-1)}{r(r-1)}$. The events $\mathcal{E}(e)$ and $\mathcal{E}(e')$ are independent for every $e, e' \in C$, hence by Chernoff's inequality (see, e.g., [2, Corollary A.1.7]) we obtain

$$\mathbb{P}\left[\left| |\{e \in C :\ \mathcal{E}(e)\}| - p\,|C| \right| > \frac{|A|(|A|-1)}{r^4 \cdot \log n}\right] < 2 \cdot \exp\left(\frac{-2|A|^2(|A|-1)^2}{|C| \cdot r^8 \cdot \log^2 n}\right),$$

which is equal to $e^{\frac{-c' \cdot n^2}{\log^4 n}}$ for c' sufficiently small (recall that $|A| \geq n/\log n$ and $|C| < |A|(|A|-1)$). The claim now follows by a union bound over all color classes. □

We are ready to show that the lower density in Construction 2.2 is equal to 1/4.

Observation 2.5. *Consider a random sequence $(H_n)_{n \in \mathbb{N}}$ from Construction 2.2. With probability 1, the lower density of $(H_n)_{n \in \mathbb{N}}$ is equal to 1/4.*

Proof. Recall that $|V(H_n)| = n$. Let H be the complete 3-graph on n vertices, and $\mathcal{E}(e)$, for $e = uvw$, be the event that the three arcs on the set $\{u, v, w\}$ in the underlying tournament of H_n form an oriented cycle. Lemma 2.3 yields that with probability $1 - e^{\frac{-c \cdot n^2}{\log^4 n}}$, for all subsets $A \subseteq V(H_n)$ of size at least $n/\log n$, the number of edges in $H_n[A]$ is between $(1/4 - 1/\log n)\binom{|A|}{3}$ and $(1/4 + 1/\log n)\binom{|A|}{3}$. Therefore, by Borel-Cantelli Lemma (see, e.g., [2, Lemma 8.6.1]), with probability one there exists $n_0 \in \mathbb{N}$ such that the property above is true for every $n \geq n_0$. But this immediately implies that the lower density of $(H_n)_{n \in \mathbb{N}}$ is $1/4$. □

The main result of this chapter is that the Construction 2.2 is essentially the best possible. This answers a question of Erdős from [21], which is based on the original problem of Erdős and Sós, positively.

2.2 Flag algebra setting

The proof of the main theorem is based on the semidefinite method described in Section 1.2. Through the whole chapter, we will use $\mathcal{F}$ to denote the set of all non-isomorphic finite K_4^--free 3-graphs, and $\mathcal{F}_k$ to denote the subset of $\mathcal{F}$ containing all k-vertex K_4^--free 3-graphs. Next, for a fixed K_4^--free 3-graph σ with a given labelling of its vertices, we define $\mathcal{F}^\sigma$ to be the set all finite K_4^--free 3-graphs with a fixed embedding of σ, and $\mathcal{F}_k^\sigma$ to be the appropriate subsets. Analogously to the graph setting presented in Section 1.1, we construct an algebra $\mathcal{A}$, algebras $\mathcal{A}^\sigma$, where σ is a fixed labelled K_4^--free 3-graph, and averaging operators $[\![\cdot]\!]_\sigma : \mathcal{A}^\sigma \to \mathcal{A}$. Finally, we denote by $\mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$ the set of all algebra homomorphisms ψ from $\mathcal{A}$ to $\mathbb{R}$ such that $\psi(F) \geq 0$ for every $F \in \mathcal{F}$.

A sequence of K_4^--free 3-graphs $(H_n)_{n \in \mathbb{N}}$ of increasing order is convergent if the limit $\lim_{n \to \infty} p(F, H_n)$ exists for every $F \in \mathcal{F}$. As in the graph case, there is a one-to-one correspondence between the set $\mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$ and the set of all vectors in $[0,1]^{\mathcal{F}}$ that represents the limit probabilities of convergent sequences.

We derive now some additional inequalities that are valid for any convergent sequence of 3-graphs $(H_n)_{n \in \mathbb{N}}$ with lower density at least $1/4$.

Let us first informally explain the main idea behind the inequalities. Consider an arbitrary type σ that has a positive density in the limit, and let F be a σ-flag with exactly one non-rooted vertex. Furthermore, assume that n is sufficiently large. Now fix a copy S of σ in H_n and consider the set $U(S)$ of all vertices $u \in V(H_n)$ that extends S to a copy of F. The following two outcomes can happen:

- The size of $U(S)$ is small, i.e., $o(n)$. But then, for this choice of S, both the probability that three random points belongs to $U(S)$, and the probability that three random points belongs to $U(S)$ and span an edge, are $o(1)$.

- The size of $U(S)$ is large, i.e., $\Omega(n)$. But then, by the assumption on the lower density, for this choice of S, the probability that three random points belong to $U(S)$ and span an edge is asymptotically at least a quarter of the probability that three random points belongs to $U(S)$.

Analysis of these two outcomes is given in the following lemma.

Lemma 2.6. *Let $(H_n)_{n\in\mathbb{N}}$ be a convergent sequence of 3-graphs with lower density at least 1/4. Let ϕ be its limit, σ a type and $F \in \mathcal{F}^{\sigma}_{|\sigma|+1}$ a σ-flag. In addition, let*

$$\kappa := F^3 = \sum_{G\in\mathcal{F}^{\sigma}_{v(\sigma)+3}} \alpha_G \cdot G \qquad \text{and} \qquad \kappa^+ := \sum_{G\in\mathcal{F}^{\sigma}_{v(\sigma)+3}} \alpha^+_G \cdot G,$$

where

$$\alpha^+_G := \begin{cases} \alpha_G & \text{if the three non-labelled vertices of } G \text{ span an edge, and} \\ 0 & \text{otherwise.} \end{cases}$$

It holds that

$$\phi\left(\llbracket 4\kappa^+ - \kappa \rrbracket_\sigma\right) \geq 0. \tag{2.7}$$

Note that the values of the coefficients α_G are uniquely determined by the choice of F. Also note that $\alpha_G \in \{0,1\}$ for every $G \in \mathcal{F}^{\sigma}_{v(\sigma)+3}$.

Proof. First observe that if $\phi\left(\sigma^{\emptyset}\right) = 0$, then all the terms in the left-hand side of (2.7) are equal to zero. For the rest of the proof, we assume $\phi\left(\sigma^{\emptyset}\right) > 0$.

Suppose for the contrary that (2.7) is not true, i.e., $\phi\left(\llbracket 4\kappa^+ - \kappa \rrbracket_\sigma\right) = -\varepsilon_r$ for some $\varepsilon_r > 0$. Let q_n be the values of the expression $\llbracket 4\kappa^+ - \kappa \rrbracket_\sigma$ evaluated on the densities of H_n. Since the sequence $(H_n)_{n\in\mathbb{N}}$ is convergent, there exists $n_0 \in \mathbb{N}$ such that $q_{n_0} \leq -\varepsilon_r/2$. Moreover, since (H_n) has lower density at least 1/4, we may assume that $d\left(H_{n_0}, \beta_e\right) \geq 1/4$ for $\beta_e := \sqrt[3]{\varepsilon_r/13}$. It follows that one can get $\beta_e\varepsilon_r$ for free.

Fix an embedding θ of σ in H_{n_0}. Let $q^{\theta}_{n_0}$ be the value of the expression $(4\kappa^+ - \kappa)$ evaluated on the rooted densities of $H^{\theta}_{n_0}$. If there are less than $\beta_e n$ vertices that extend θ to F, then $q^{\theta}_{n_0}$ is at least $-6\left(\beta_e\right)^3 - o(1)$. On the other hand, if at least $\beta_e n$ vertices extend θ to F, then the density of the subhypergraph induced by those vertices is at least 1/4 and hence $4\kappa^+ \geq \kappa - o(1)$. In other words, $q^{\theta}_{n_0} \geq -o(1)$.

Recall that $p^{\sigma}_{\sigma^{\emptyset}}$ denotes the probability that a random labelling of $V\left(\sigma^{\emptyset}\right)$ with labels $\left\{1, \ldots, v\left(\sigma^{\emptyset}\right)\right\}$ produces the type σ, and note that $p^{\sigma}_{\sigma^{\emptyset}} \cdot p\left(\sigma^{\emptyset}, H_{n_0}\right) \leq 1$.

The definition of the conditional probability yields that

$$q_{n_0} = p^{\sigma}_{\sigma^\emptyset} \cdot p\left(\sigma^\emptyset, H_{n_0}\right) \cdot \mathbb{E}\left[q^{\theta}_{n_0}\right] \geq -6\left(\beta_e\right)^3 - o(1),$$

where the average is taken over all possible embeddings θ of σ in H_{n_0}. However, the choice of β_e implies that $q_{n_0} \geq -6\varepsilon_r/13 - o(1)$ which contradicts the fact that $q_{n_0} \leq -\varepsilon_r/2$.

□

We are now ready to present the two inequalities we are going to use in the next step. Let σ be the unique type on 2 vertices; from now on, we also write "**2**" to denote this type. We denote the 3-vertex σ-flag with no edges by F_0, and the 3-vertex σ-flag with one edge by F_1. The first inequality is an instance of (2.7) for $\sigma = \mathbf{2}$ and $F = F_1$. Let κ_1 be the corresponding κ and let κ_1^+ be the corresponding κ^+. The first line in Figure 2.3 shows the flag F_1 and the corresponding linear combinations κ_1 and κ_1^+. Since all σ-flags in $\mathcal{F}^\sigma$ are K_4^--free, it follows that $\kappa_1 = L_1 + L_1^+$ and $\kappa_1^+ = L_1^+$. The σ-flags L_1 and L_1^+ are depicted in Figure 2.4.

Analogously, let κ_2 be κ and let κ_2^+ be κ^+ from (2.7) for $F = F_0$ (again, σ is equal to **2**). The second line in Figure 2.3 shows the flag F_0, and the combinations κ_2 and κ_2^+. It holds that

$$\kappa_2 = \sum_{i=2}^{14} L_i + \sum_{i=2}^{6} L_i^+ \qquad \text{and} \qquad \kappa_2^+ = \sum_{i=2}^{6} L_i^+,$$

where the σ-flags $L_2, \ldots, L_{14}$ and $L_2^+, \ldots, L_6^+$ are again depicted in Figure 2.4. Lemma 2.6 yields that

$$\phi\left(\llbracket 4\kappa_1^+ - \kappa_1 \rrbracket_{\mathbf{2}}\right) \geq 0, \tag{2.8}$$

and

$$\phi\left(\llbracket 4\kappa_2^+ - \kappa_2 \rrbracket_{\mathbf{2}}\right) \geq 0. \tag{2.9}$$

2.3 3-graphs with 4 vertices spanning at most 2 edges

In this section, we present the main result of this chapter. Specifically, we prove the following theorem.

Theorem 2.10. *For every $\varepsilon_{\mathrm{THM}} > 0$ there exist $\delta_{\mathrm{THM}} > 0$ and $n_{\mathrm{THM}} \in \mathbb{N}$ such that every 3-graph H on at least n_{THM} vertices with $d(H, \delta_{\mathrm{THM}}) \geq 1/4 + \varepsilon_{\mathrm{THM}}$ contains a copy of K_4^-.*

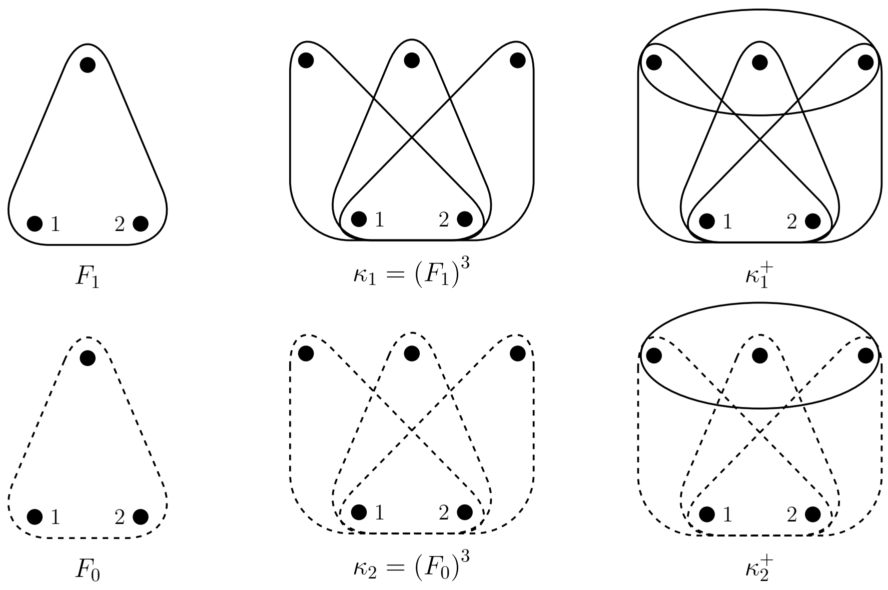


Figure 2.3: The expressions used in the inequalities (2.8) and (2.9). A solid line denotes an edge, a dashed line a non-edge, and finally we sum over all the possible choices edge/non-edge for the triples where there is neither a dashed nor a solid line.

Recall that $\mathcal{F}_7$ is the set of all K_4^--free 3-graphs of size 7. It holds that $|\mathcal{F}_7| = 8157$. Let EXT_7 be the set of all 3-graphs from $\mathcal{F}_7$ that can possibly appear as induced subhypergraphs of the 3-graphs from Construction 2.2. It holds that $|\text{EXT}_7| = 247$. We start the proof of Theorem 2.10 with the following lemma.

Lemma 2.11. *There exist rational numbers $(\alpha_G)_{G\in\mathcal{F}_7}$, where $\alpha_G = 0$ for $G \in \text{EXT}_7$, and $\alpha_G < 0$ otherwise, such that the following holds. If ϕ is an element of $Hom^+(\mathcal{A}, \mathbb{R})$ that satisfies* (2.8) *and* (2.9)*, then*

$$\phi\left(\sum_{G\in\mathcal{F}_7} \alpha_G \cdot G\right) \geq 0.$$

Proof. Using the method from Section 1.2, an instance of the SDP was used to find positive rationals γ_1 and γ_2 and 8 symmetric positive semidefinite matrices $M_1, M_2, \dots M_8$ with rational entries such that the following holds:

$$\phi\left(\gamma_1 \cdot [\![4\kappa_1^+ - \kappa_1]\!]_{\mathbf{2}} + \gamma_2 \cdot [\![4\kappa_2^+ - \kappa_2]\!]_{\mathbf{2}} + \sum_{i\in[8]} [\![x_i^T M_i x_i]\!]_{\sigma_i}\right) = \phi\left(\sum_{G\in\mathcal{F}_7} \alpha_G \cdot G\right),$$

where

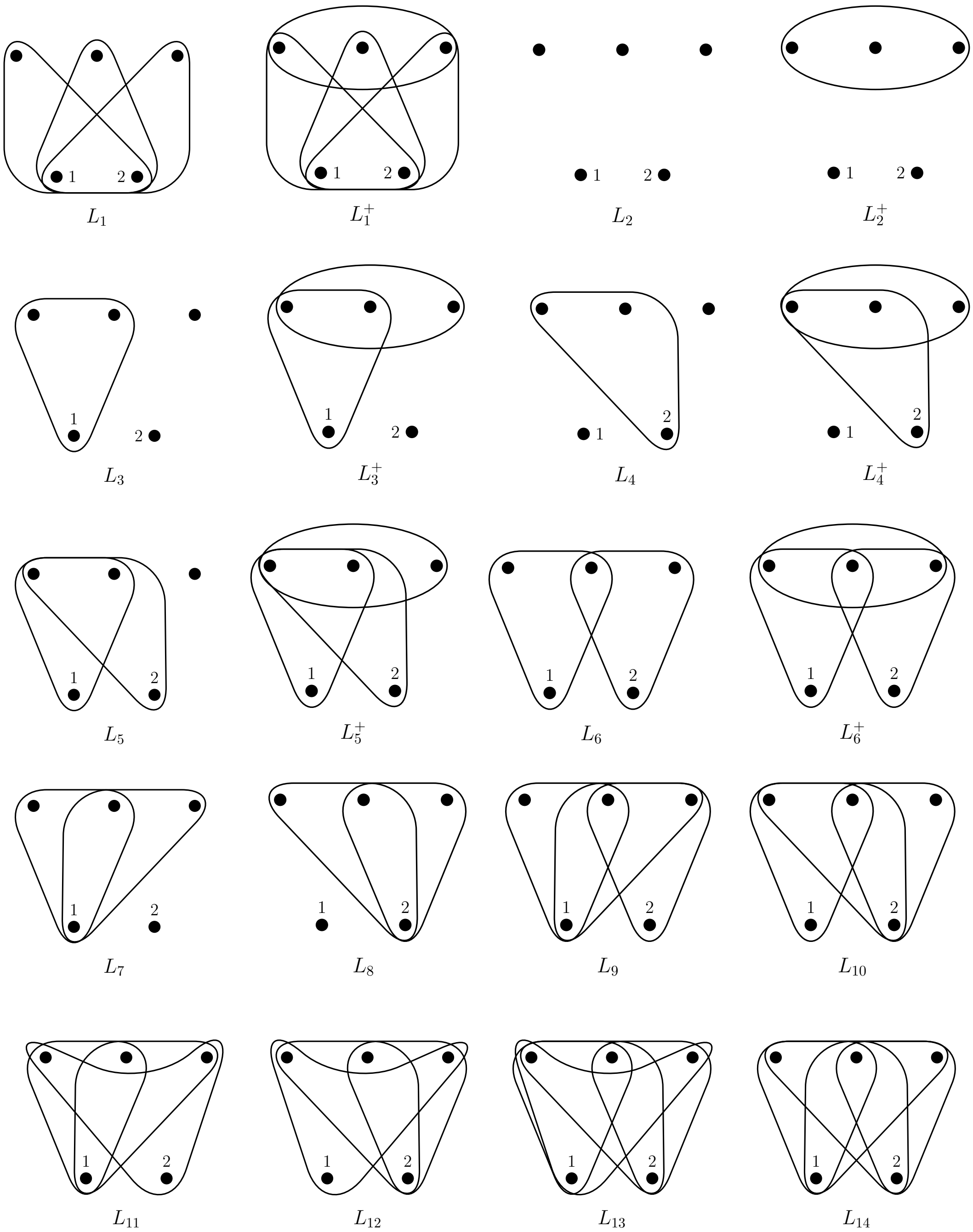


Figure 2.4: The σ-flags from the inequalities (2.8) and (2.9).

- the type σ_1 is the type with one vertex,
- the type σ_2 is the type with three vertices and no edge,
- the type σ_3 is the type with three vertices and one edge,
- the types $\sigma_4, \ldots, \sigma_8$ are the five specific types on five vertices given in Figure 2.5,
- the vector $x_1 \in (\mathbb{R}\mathcal{F}_4^{\sigma_1})^{|\mathcal{F}_4^{\sigma_1}|}$ is the vector whose j-th coordinate is equal to the j-th element of the canonical base of $\mathbb{R}\mathcal{F}_4^{\sigma_1}$,
- for $i = 2, 3$, the vector $x_i \in (\mathbb{R}\mathcal{F}_5^{\sigma_i})^{|\mathcal{F}_5^{\sigma_i}|}$ is the vector whose j-th coordinate is equal to the j-th element of the canonical base of $\mathbb{R}\mathcal{F}_5^{\sigma_i}$,
- for $i = 4, 5, \ldots, 8$, each vector $x_i \in (\mathbb{R}\mathcal{F}_6^{\sigma_i})^{|\mathcal{F}_6^{\sigma_i}|}$ is the vector whose j-th coordinate is equal to the j-th element of the canonical base of $\mathbb{R}\mathcal{F}_6^{\sigma_i}$, and
- $\alpha_G = 0$ if $G \in \mathrm{EXT}_7$, and $\alpha_G < 0$ otherwise.

The left-hand side of the inequality above is non-negative by (1.2), (2.8), and (2.9). □

The proof of the previous lemma was found with a computer assistance. We used semidefinite programming libraries CSDP [11] and SDPA [74] to find an approximate solution of the corresponding semidefinite program. The approximate solution was then turned into an exact one by a careful rounding. For the rounding part, we used a mathematical software Sage [71]. Also note that the sizes a_i of the sets $\mathcal{F}_4^{\sigma_1}, \mathcal{F}_5^{\sigma_2}, \mathcal{F}_5^{\sigma_3}, \mathcal{F}_6^{\sigma_4}, \mathcal{F}_6^{\sigma_5}, \mathcal{F}_6^{\sigma_6}, \mathcal{F}_6^{\sigma_7}$, and $\mathcal{F}_6^{\sigma_8}$, which coincide with the order of the matrices M_i, are $5, 95, 47, 191, 135, 95, 101$, and 148, respectively.

The numerical values of γ_1, γ_2, and the entries of the matrices $M_1, \ldots, M_8$ can be downloaded from `http://honza.ucw.cz/phd/`. Each matrix M_i is not stored directly but as an appropriate number of vectors $v_i^j \in \mathbb{Q}^{a_i}$ and positive rationals w_i^j such that

$$M_i = \sum_{j=1}^{r_i} w_i^j \cdot v_i^j \times \left(v_i^j\right)^T,$$

where $r_1 = 3, r_2 = 45, r_3 = 21, r_4 = 63, r_5 = 60, r_6 = 43, r_7 = 31$, and $r_8 = 25$.

In order to make an independent verification of our computations easier, we created a sage script called "lemma_2_13-verify.sage", which is also available on the web page mentioned above.

Lemma 2.11 immediately yields the following.

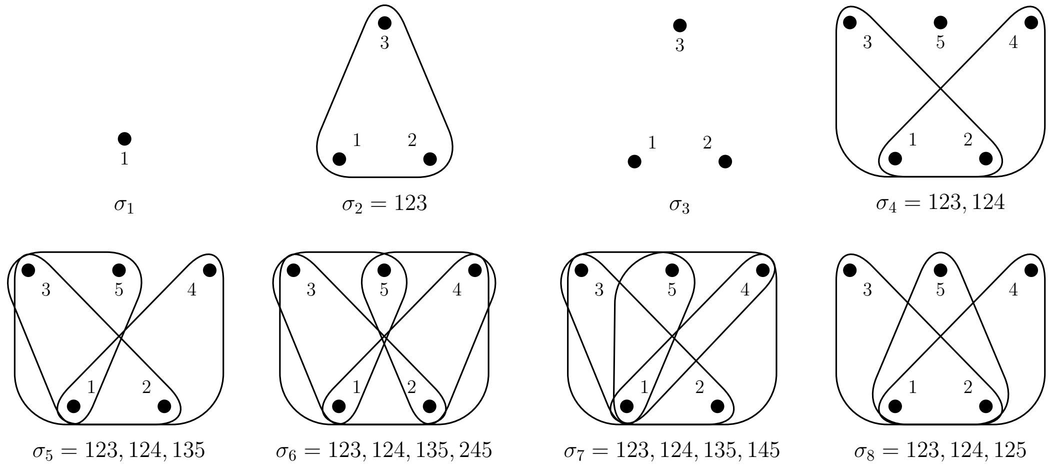


Figure 2.5: The types σ_1 through σ_8. A solid line denotes an edge, a triple without a solid line spans a non-edge.

Corollary 2.12. *Let F be a 3-graph of size at most 7 such that its induced density in the 3-graphs from Construction 2.2 is zero. If $\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$ satisfies* (2.8) *and* (2.9)*, then $\phi(F) = 0$.*

We are now ready to present the proof of Theorem 2.10.

Proof of Theorem 2.10. Suppose to the contrary that there exists $d > 1/4$ and a sequence $(H_n)_{n\in\mathbb{N}}$ of K_4^--free 3-graphs of increasing orders with lower density d. In other words

$$\lambda\left((H_n)_{n\in\mathbb{N}}\right) = d > 1/4.$$

Let us assume, without loss of generality, that $(H_n)_{n\in\mathbb{N}}$ converges and ϕ_C is the limit; We aim to show that the edge density of ϕ_C is zero, contradicting the fact that it must be at least the lower density of the sequence.

Let B be the 3-graph depicted in Figure 2.6, which we call the butterfly 3-graph. Since the 3-graphs from Construction 2.2 are B-free, Corollary 2.12 implies that $\phi(B) = 0$ whenever a homomorphism $\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$ satisfies (2.8) and (2.9). In particular, for the limit ϕ of any convergent sequence of K_4^--free 3-graphs with lower density at least $1/4$, we have $\phi(B) = 0$.

Instead of applying the claim directly to ϕ_C, we first construct from $(H_n)_{n\in\mathbb{N}}$ a new sequence $(H'_n)_{n\in\mathbb{N}}$ such that $\lambda\left((H'_n)_{n\in\mathbb{N}}\right) = 1/4$. The new sequence is obtained by a *random sparsification* of $(H_n)_{n\in\mathbb{N}}$, i.e., removing each edge of each H_n at random with the appropriately chosen probability. This is formulated in the

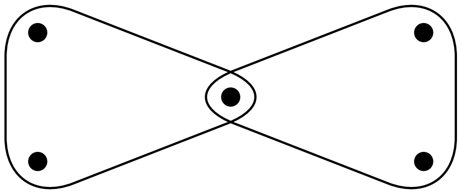


Figure 2.6: The butterfly 3-graph B.

following observation, which is an immediate corollary of Lemma 2.3.

Observation 2.13. *Let $(H_n)_{n\in\mathbb{N}}$ be a sequence of 3-uniform K_4^--free 3-graphs with $d := \lambda\left((H_n)_{n\in\mathbb{N}}\right) > 1/4$. Furthermore, for every $n \in \mathbb{N}$, let H'_n be a random subhypergraph of H_n obtained by removing every hyperedge of H_n independently with probability $1-\frac{1}{4d}$. Then*

$$\mathbb{P}\left[\lambda\left(\left(H'_n\right)_{n\in\mathbb{N}}\right) = 1/4\right] = 1.$$

Let $(H'_n)_{n\in\mathbb{N}}$ be a sequence of 3-graphs obtained from $(H_n)_{n\in\mathbb{N}}$ by a random sparsification with probability $1-\frac{1}{4d}$. All the following holds with probability one. Let ϕ'_C be its limit (the sequence $(H'_n)_{n\in\mathbb{N}}$ must be convergent). Lemma 2.6 implies that ϕ'_C satisfies both (2.8) and (2.9). Hence, by Corollary 2.12, the induced density of B in ϕ'_C is equal to zero. But this implies that $\phi_C(F) = 0$ for any $F \in \mathcal{F}$ that contains B as a non-induced subhypergraph. This holds because otherwise there would be some $F \in \mathcal{F}$ that contains a non-induced copy of B and $\phi_C(F) > 0$. However, a positive proportion of the induced copies of F from ϕ_C will be then turned, by a random sparsification, to induced copies of B in ϕ'_C. We conclude that the *non-induced* density of the butterfly 3-graph B in ϕ_C is equal to zero. In particular, any non-negative combination of the elements of $\mathcal{F}_5$ that contain the butterfly as a non-induced subhypergraph must be zero in ϕ_C.

We are now ready to conclude that the edge density of ϕ_C is zero. Let $\rho \in \mathcal{F}_3$ be the 3-graph on three (non-rooted) vertices that span an edge. Let σ be the one-vertex type and $\rho_1 \in \mathcal{F}_3^\sigma$ the flag corresponding to ρ with exactly one rooted vertex. Observe that all the elements of $\mathcal{F}_5^\sigma$ with positive coefficients in the expression $(\rho_1)^2$ contain B as a subhypergraph. So, $\phi_C\left(\llbracket(\rho_1)^2\rrbracket_\sigma\right) = 0$, which implies that

$$\phi_C(\rho)^2 = \phi_C\left(\llbracket\rho_1\rrbracket_\sigma\right)^2 \leq \phi_C\left(\llbracket(\rho_1)^2\rrbracket_\sigma\right) = 0.$$

□

Theorem 2.10 together with the 3-uniform version of the Hypergraph Removal Lemma (see, e.g., [67, Theorem 3] for the general r-uniform version of the lemma) immediately implies that if the δ-linear density is more than 1/4, we do not have only one but actually many copies of K_4^-. Let us first precisely state the version of the removal lemma for 3-graphs, which actually follows already from the results in [31].

Lemma 2.14 (Hypergraph Removal Lemma, 3-uniform case)**.** *For every* $\varepsilon_{\mathrm{RL}} > 0$ *and fixed* k*-vertex* 3*-graph* F*, there exists* $\gamma_{\mathrm{RL}} > 0$ *such that every* 3*-graph* H *on* n *vertices with less than* $\gamma_{\mathrm{RL}} \cdot \binom{n}{k}$ *copies of* F *can be made* F*-free by removing less than* $\varepsilon_{\mathrm{RL}} \cdot \binom{n}{3}$ *edges.*

We are now ready to derive the counting version of Theorem 2.10.

Corollary 2.15. *For every* $\varepsilon_{\mathrm{CNT}} > 0$ *there exist* $\gamma_{\mathrm{CNT}} > 0$, $\delta_{\mathrm{CNT}} > 0$ *and* $n_{\mathrm{CNT}} \in \mathbb{N}$ *such that every* 3*-graph* H *on at least* n_{CNT} *vertices with* $d(H, \delta_{\mathrm{CNT}}) \geq 1/4 + \varepsilon_{\mathrm{CNT}}$ *contains* $\gamma_{\mathrm{CNT}} \cdot \binom{v(H)}{4}$ *copies of* K_4^-*.*

Proof. Suppose for a contradiction there is $\varepsilon_0 > 0$ such that for every $\gamma > 0, \delta > 0$ and $n \in \mathbb{N}$ we can find a 3-graph $H(n, \delta, \gamma)$ on at least n vertices, with δ-linear density at least $1/4 + \varepsilon_0$, and with less than $\gamma \cdot \binom{n}{4}$ copies of K_4^-. Fix such ε_0 and for every γ, δ and n, fix one such 3-graph $H(n, \delta, \gamma)$.

Let n_{THM} and δ_{THM} be the constants from Theorem 2.10 applied for $\varepsilon_{\mathrm{THM}} := \varepsilon_0/2$. Fix an integer k so that $k \log k > n_{\mathrm{THM}}$ and $1/k < \delta_{\mathrm{THM}}$. Next, let $\gamma_{\mathrm{RL}} := \gamma_{\mathrm{RL}}(k)$ be the constant given by the Hypergraph Removal Lemma for K_4^- applied for $\varepsilon_{\mathrm{RL}} := \varepsilon_0/(3k^3)$. Finally, let

$$G := H(k \cdot \log k, 1/k, \gamma_{\mathrm{RL}}(k)).$$

This means that every $S \subseteq V(G)$ of size at least $v(G)/k \geq \log k$ contains at least $(1/4 + \varepsilon_0)\binom{|S|}{3}$ edges. Moreover, G can be transformed to a K_4^--free 3-graph G' by removing less than $\frac{\varepsilon_0}{3k^3} \cdot \binom{v(G')}{3}$ edges. Consequently, every $S \subseteq V(G')$ of size at least $v(G')/k$ contains at least $(1/4 + 2\varepsilon_0/3)\binom{|S|}{3} - O(|S|^2)$ edges. Since k is large, it follows that

$$\left(\frac{1}{4} + \frac{2\varepsilon_0}{3}\right) \cdot \binom{|S|}{3} - O\left(|S|^2\right) > \left(\frac{1}{4} + \frac{\varepsilon_0}{2}\right) \cdot \binom{|S|}{3}.$$

Therefore, $d(G', 1/k) > 1/4 + \varepsilon_0/2$ and Theorem 2.10 imply that G' contains a copy of K_4^-, a contradiction. □

Also, exploiting the fact that the semidefinite method seeks for a vector minimizing a given linear function over a convex superset of the set $\mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$, we can replace the $d(H, \delta) \geq 1/4 + \varepsilon$ condition by another that controls only the densities inside subsets that are common neighborhood of two vertices in H and inside subsets that are common neighborhoods in the complement of H.

Theorem 2.16. *For every $\varepsilon > 0$ there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that the following is true. Every 3-graph H on $n \geq n_0$ vertices such that for every $\{u, v\} \in \binom{V(H)}{2}$ satisfies the following two conditions:*

- *if $|N_H(u, v)| \geq \delta n$, then $H[N_H(u, v)]$ has density at least $(1/4 + \varepsilon)$, and*
- *if $|N_{\overline{H}}(u, v)| \geq \delta n$, then $H[N_{\overline{H}}(u, v)]$ has density at least $(1/4 + \varepsilon)$,*

contains K_4^- as a subhypergraph.

Proof. Suppose for a contradiction the theorem is false. Hence there exist a positive ε_0 and a convergent sequence of K_4^--free 3-graphs $(H_n)_{n \in \mathbb{N}}$ of increasing orders such that each H_n satisfies the two conditions from the statement of this theorem. We denote the limit of this sequence by ϕ_C.

First, we claim that the edge density of ϕ_C is positive (in fact, it is strictly more than $1/32$). Fix two arbitrary vertices u and v in H_n. Since either $uvw \in E(H_n)$, or $uvw \in E(\overline{H_n})$ for every $w \in V(H_n) \setminus \{u, v\}$, it follows that there is a set $S \subseteq V(H_n)$ with at least $v(H_n)/2 - 1$ vertices such that $H_n[S]$ has edge density at least $(1/4 + \varepsilon_0)$. But this means that H_n has edge density at least $1/32 + \varepsilon_0/8 - o(1)$.

Let us now analyze some further properties of ϕ_C. Let $\sigma = \mathbf{2}$ be the two-vertex type. Recall that F_0 is the 3-vertex σ-flag with no edges, and F_1 is the edge with two rooted vertices. If $\mathbb{P}\left[\phi_{\mathbf{C}}^{\sigma}(F_1) = 0\right] = 1$, then, by averaging over all pairs of the vertices, we conclude that the edge density of ϕ_C is zero. However, we know the edge density is more than $1/32$. Analogously, if $\mathbb{P}\left[\phi_{\mathbf{C}}^{\sigma}(F_0) = 0\right] = 1$, then the edge density of ϕ_C must be one. However, since $\phi_C\left(K_4^-\right) = \phi_C\left(K_4^{(3)}\right) = 0$, the edge density of ϕ_C is trivally at most $1/2$, a contradiction. Therefore, there exists a positive γ such that

$$\mathbb{P}\left[\phi_C^{\sigma}(F_1) \geq \gamma\right] \geq \gamma,$$

and similarly

$$\mathbb{P}\left[\phi_C^{\sigma}(F_0) \geq \gamma\right] \geq \gamma.$$

Recall the inequalities (2.8) and (2.9). The reasoning from the previous paragraph yields that a positive proportion of pairs $\{u, v\}$ have a large co-degree of $\{u, v\}$ in the underlying sequence of ϕ_C. Analogously, a positive proportion of pairs $\{u, v\}$ have large co-degree in the complement. Since the density on every such subset is at least $1/4 + \varepsilon_0 > 1/4$, the inequalities (2.8) and (2.9) have some slack for $\phi = \phi_C$. More precisely, there exists a positive ζ such that

$$\phi_C\left(\llbracket 4\kappa_1^+ - \kappa_1 \rrbracket_{\mathbf{2}}\right) \geq \zeta, \tag{2.17}$$

and

$$\phi_C\left(\llbracket 4\kappa_2^+ - \kappa_2 \rrbracket_{\mathbf{2}}\right) \geq \zeta. \tag{2.18}$$

Now recall the semidefinite program (1.8) and its dual (1.9). In the proof of Lemma 2.11, we used the semidefinite method to find two positive rationals γ_1 and γ_2 and an expression

$$R = \sum_{i=1}^{8} \sum_{j=1}^{r_i} b_j^i \cdot \left\llbracket \left(A_j^{\sigma_i}\right)^2 \right\rrbracket_{\sigma_i},$$

where $b_j^i \geq 0$, $A_j^{\sigma_i} \in \mathcal{A}^{\sigma_i}$, and the order of the expression $\left\llbracket \left(A_j^{\sigma_i}\right)^2 \right\rrbracket_{\sigma_i}$ is at most 7 for every $i \in [8]$ and $j \in [r_i]$, such that the following is true:

$$\gamma_1 \cdot \llbracket 4\kappa_1^+ - \kappa_1 \rrbracket_{\mathbf{2}} + \gamma_2 \cdot \llbracket 4\kappa_2^+ - \kappa_2 \rrbracket_{\mathbf{2}} + R = \sum_{G \in \mathcal{F}_7} \alpha_G \cdot G,$$

where $\alpha_G = 0$ for every $G \in \mathrm{EXT}_7$, and $\alpha_G < 0$ for every $G \in \mathcal{F}_7 \setminus \mathrm{EXT}_7$. It holds that $\phi(R) \geq 0$ for every $\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$. Furthermore, the reasoning from Section 1.2 yields that the bound $z(R) \geq 0$ holds also for every vector $z \in \mathbb{R}^{|\mathcal{F}_7|}$ that satisfies the following:

- all the linear inequalities of the form $z\left(\left\llbracket (A^{\sigma_1})^2 \right\rrbracket_{\sigma_1}\right) \geq 0$, where $A^{\sigma_1} \in \mathcal{A}^{\sigma_1}$ and the order of the expression $\left\llbracket (A^{\sigma_1})^2 \right\rrbracket_{\sigma_1}$ is at most 7,

$$\vdots$$

- all the linear inequalities of the form $z\left(\left\llbracket (A^{\sigma_8})^2 \right\rrbracket_{\sigma_8}\right) \geq 0$, where $A^{\sigma_8} \in \mathcal{A}^{\sigma_8}$ and the order of the expression $\left\llbracket (A^{\sigma_8})^2 \right\rrbracket_{\sigma_8}$ is at most 7,
- the non-negative inequalities $z(F) \geq 0$ for every $F \in \mathcal{F}_7$, and
- the equation $z\left(\sum_{F \in \mathcal{F}_7} F\right) = 1$.

The types $\sigma_1, \ldots, \sigma_8$ are the same types as in Lemma 2.11, i.e., the types depicted in Figure 2.5. We denote the set of all such vectors $z \in \mathbb{R}^{|\mathcal{F}_7|}$ by $\mathcal{S}_{\mathcal{F}_7}$.

Finally, Recall the 3-graph B depicted in Figure 2.6. We slightly abuse the notation and use B also to denote the linear combination of the elements of $\mathcal{F}_7$ that is equal to B in $\mathcal{A}$. Our aim is to use the homomorphism ϕ_C for constructing a vector $z \in \mathcal{S}_{\mathcal{F}_7}$ that satisfies the inequalities (2.8), (2.9) and $z(B) > 0$. This is an immediate contradiction, because on one hand

$$z\left(\gamma_1 \cdot [\![4\kappa_1^+ - \kappa_1]\!]_{\mathbf{2}} + \gamma_2 \cdot [\![4\kappa_2^+ - \kappa_2]\!]_{\mathbf{2}} + R\right) \geq 0,$$

but on the other hand every 3-graph $G \in \mathrm{EXT}_7$ does not contain B as an induced subhypergraph, and hence

$$z\left(\sum_{G \in \mathcal{F}_7} \alpha_G \cdot G\right) < 0.$$

Let us emphasize that in the argument we do not need that the vector $z \in \mathcal{S}_{\mathcal{F}_7}$ to be a 7-local density profile of some convergent sequence of 3-graphs.

Let $\phi_b \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$ be the limit corresponding to the sequence of 3-graphs $(B_k)_{k \in \mathbb{N}}$, where B_k is the k-th balanced blow-up of the butterfly 3-graph B. Every 3-graph B_k is K_4^--free, and the density of B tends to 24/625, i.e., $\phi_b(B) = 24/625$. For every $\xi \in [0,1]$, let $\phi_\xi := \xi \cdot \phi_C + (1-\xi) \cdot \phi_b$. Since the set $\mathcal{S}_{\mathcal{F}_7}$ is convex, it follows that $(\phi_\xi)_{\restriction \ell} \in \mathcal{S}_{\mathcal{F}_7}$ for every $\xi \in [0,1]$. However, since $\phi_1 = \phi_C$ satisfies even the inequalities (2.17) and (2.18), there exists a point $\xi_0 \in (0,1)$ such that

$$\phi_{\xi_0}\left([\![4\kappa_1^+ - \kappa_1]\!]_{\mathbf{2}}\right) \geq \zeta/2 \geq 0,$$

and

$$\phi_{\xi_0}\left([\![4\kappa_2^+ - \kappa_2]\!]_{\mathbf{2}}\right) \geq \zeta/2 \geq 0.$$

However, $\phi_{\xi_0}(B) = (1-\xi_0) \cdot \phi_b(B) > 0$, a contradiction. □

2.4 Uniqueness of the tournament construction

In the last section of this chapter, we show that the limit of any convergent sequence of K_4^--free 3-graphs with lower density 1/4 has to be equal to the limit of the sequence of 3-graphs from Construction 2.2. In order to do so, we use the argument of Falgas-Rarvy, Pikhurko and Vaughan [27], who proved the following closely related result.

Theorem 2.19 (Falgas-Rarvy, Pikhurko and Vaughan). *For every $\varepsilon > 0$ there exists*

$n_0 \in \mathbb{N}$ *such that every* 3*-graph* H *on* $n \geq n_0$ *vertices with the minimum co-degree at least* $(1/4 + \varepsilon) \cdot n$ *contains a copy of* K_4^-.

The relation between Theorems 2.10 and 2.19 is that the 3-graphs from Construction 2.2 serve as extremal configurations for the corresponding two problems. The entire section is devoted to a proof of the following theorem.

Theorem 2.20. *Let* $(H_n)_{n\in\mathbb{N}}$ *be a convergent sequence of* K_4^-*-free* 3*-graphs so that* $\lambda((H_n)) = 1/4$*, and let* $\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$ *be its limit. It holds that* ϕ *is equal to the homomorphism* ψ*, which is the limit of the sequence of* 3*-graphs from Construction 2.2 with increasing orders.*

Proof. Without loss of generality, $v(H_n) = n$. Recall that the set EXT_7 is the set of all 7-vertex induced subhypergraphs of the 3-graphs from Construction 2.2. Corollary 2.12 and the induced version of the Hypergraph Removal Lemma [66, Theorem 6] implies that there exists a sequence of 3-graphs $(H'_n)_{n\in\mathbb{N}}$ such that each H'_n can be obtained from H_n by adding or removing $o(n^3)$ edges, and every 7-vertex induced subhypergraph of H'_n is isomorphic to one of the 3-graphs from EXT_7. Note that $(H'_n)_{n\in\mathbb{N}}$ is a convergent sequence of 3-graphs. Also note that the density of any fixed 3-graph G in $(H'_n)_{n\in\mathbb{N}}$ tends to $\phi(G)$. Since the application of the Hypergraph Removal Lemma changed only $o(n^3)$ edges, it holds that $\lambda((H'_n)) = 1/4$. Therefore, it is enough to show that the limit of $(H'_n)_{n\in\mathbb{N}}$ is equal to ψ.

The induced subhypergraph property satisfied by the 3-graphs H'_n yields the following important fact: for every induced subhypergraph Z of any H'_n on at most 7 vertices, there exists a tournament T_Z such that the edges of Z are in one-to-one correspondence to the cyclically oriented triangles in T_Z.

We now setup a variant of the flag algebra framework for convergent sequences of 3-graphs that satisfy this property on 7-vertex induced subhypergraphs. Let $\mathcal{G}$ be the set of all 3-graphs such that each of their 7-vertex induced subhypergraphs is isomorphic to one of the 3-graphs from EXT_7. Also, for any type σ with the underlying 3-graph from $\mathcal{G}$, let $\mathcal{G}^\sigma$ be the corresponding set of all the σ-flags with this property. Finally, let $\mathcal{A}'$ be the corresponding flag algebra defined by $\mathcal{G}$. It follows that there exists a homomorphism $\phi' \in \mathrm{Hom}^+(\mathcal{A}', \mathbb{R})$ which represents the limit of the sequence $(H'_n)_{n\in\mathbb{N}}$.

Let $\sigma = \mathbf{2}$ be the 2-vertex type and let F_0 and F_1 be the σ-flags on 3 vertices with zero edges and one edge, respectively. Since H'_n was obtained from H_n by adding or removing $o(n^3)$ edges, the homomorphism ϕ' satisfies the following two inequalities, which are direct analogues of (2.8) and (2.9) in the algebra $\mathcal{A}'$:

$$\phi'\left(\llbracket 4\iota_1^+ - \iota_1 \rrbracket_{\mathbf{2}}\right) \geq 0 \qquad \text{and} \qquad \phi'\left(\llbracket 4\iota_2^+ - \iota_2 \rrbracket_{\mathbf{2}}\right) \geq 0,$$

where

$$\iota_1 := (F_1)^3 = \sum_{G\in\mathcal{G}_5^\sigma} \alpha_G \cdot G, \qquad \iota_1^+ := \sum_{G\in\mathcal{G}_5^\sigma} \alpha_G^+ \cdot G,$$
$$\iota_2 := (F_0)^3 = \sum_{G\in\mathcal{G}_5^\sigma} \beta_G \cdot G, \qquad \text{and} \qquad \iota_2^+ := \sum_{G\in\mathcal{G}_5^\sigma} \beta_G^+ \cdot G.$$

The values of the coefficients α_G and β_G are uniquely determined,

$$\alpha_G^+ := \begin{cases} \alpha_G & \text{if the three non-labelled vertices of } G \text{ span an edge, and} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\beta_G^+ := \begin{cases} \beta_G & \text{if the three non-labelled vertices of } G \text{ span an edge, and} \\ 0 & \text{otherwise.} \end{cases}$$

Note that all the values of $\alpha_G, \alpha_G^+, \beta_G$ and β_G^+ are either zero or one for every $G \in \mathcal{G}_5^\sigma$.

Let $\rho \in \mathcal{G}$ be the 3-graph on 3 vertices with one edge. Using the semidefinite method, we find four positive definite matrices $M_1', \ldots, M_4'$ such that the following inequality in the algebra $\mathcal{A}'$ holds:

$$\rho = \sum_{F\in\mathcal{G}_6} \alpha_G \cdot G \leq \frac{1}{4} \cdot \sum_{F\in\mathcal{G}_6} F - \sum_{i\in\{1,2\}} \gamma_i' \cdot [\![4\iota_i^+ - \iota_i]\!]_{\mathbf{2}} - \sum_{i\in[4]} [\![(x_i I_i)^T \cdot M_i' \cdot (I_i x_i)]\!]_{\sigma_i}, \tag{2.21}$$

where

- $\alpha_G = e(G)/20$ for every $G \in \mathcal{G}_6$,
- $\gamma_1' = 148803/16384$ and $\gamma_2' = 70943/16384$,
- the type $\sigma_1 = \mathbf{2}$ is the two-vertex type,
- the type σ_2 is the type with four vertices a, b, c, d and no edge,
- the type σ_3 is the type with four vertices a, b, c, d and the edge abc,
- the type σ_4 is the type with four vertices a, b, c, d and the edges abc and abd,
- the vector $x_1 \in (\mathbb{R}\mathcal{G}_4^{\sigma_1})^{|\mathcal{G}_4^{\sigma_1}|}$ is the vector whose j-th coordinate is equal to the j-th element of the canonical base of $\mathbb{R}\mathcal{G}_4^{\sigma_1}$,
- for $i = 2, 3, 4$, the vector $x_i \in (\mathbb{R}\mathcal{G}_5^{\sigma_i})^{|\mathcal{G}_5^{\sigma_i}|}$ is the vector whose j-th coordinate is equal to the j-th element of the canonical base of $\mathbb{R}\mathcal{G}_5^{\sigma_i}$, and

- the matrices $I_1, \dots, I_4$ are some specific matrices of sizes 8×7, 38×26, 17×15 and 17×15, respectively. The entries of the matrices I_i can be found in Appendix A.1.

Similarly to the case of Lemma 2.11, the numerical values of the entries of the matrices $M'_1, \dots, M'_4$ are available online and they can be downloaded from `http://honza.ucw.cz/phd/`. A sage script called "theorem_2_22-verify.sage", which is also available on the web page, can be used to verify the computations.

The equation (2.21) implies that the edge density of ϕ' is at most 1/4. On the other hand, since $\lambda((H'_n)) = 1/4$, the edge density of ϕ' must be at least 1/4. We conclude that $\phi'(\rho) = 1/4$. Therefore, all the inequalities that were used in (2.21) must be in fact equations for the homomorphism ϕ'. In particular, for every $i \in [4]$ it holds that

$$\phi'\left(\left[\!\left[(x_i I_i)^T \cdot M'_i \cdot (I_i x_i)\right]\!\right]_{\sigma_i}\right) = 0.$$

Since the matrices M'_i are positive definite, we conclude that for a random homomorphism ϕ^{σ_i} drawn the probability distribution $\mathbf{P}^{\sigma_i}$ given by ϕ', the vector $\phi^{\sigma_i}(x_i I_i)$ has all the coordinates equal to zero with probability one. By $\phi^{\sigma_i}(x_i I_i)$ we mean the random vector $x'_i \in \mathbb{R}^{J_i}$ whose j-th coordinate is equal to ϕ^{σ_i} applied to the j-th coordinate of $x_i I_i$, where the values of $J_1, \dots, J_4$ are equal to 7, 26, 15, and 15, respectively.

Therefore, asymptotically almost surely each coordinate of $\phi_n^{\sigma_i}(x_i I_i)$ is equal to $o(1)$, where $\phi_n^{\sigma_i}$ is drawn from the probability distribution $\mathbf{P}_n^{\sigma_i}$ (recall that the probability distributions $\mathbf{P}_n^{\sigma_i}$ arise from picking a copy of σ_i in H'_n at random, and the sequence $(\mathbf{P}_n^{\sigma_i})$ weakly converges to the distribution $\mathbf{P}^{\sigma_i}$).

The next step of our proof is to deduce some structural properties of the 3-graphs H'_n from the fact that the vectors $\phi_n^{\sigma_i}(x_i I_i)$ have for most of the choices of the quadruple a, b, c and d all its coordinates equal to $o(1)$. Before doing so, we need to introduce some additional notation. We say that a pair of 2-sets of vertices $(\{a,b\},\{c,d\})$ is *tightly connected* in H'_n if H'_n contains (at least) one of the following 12 configurations:

- a vertex e and the edges abc, ace, cde
- a vertex e and the edges abc, bce, cde
- a vertex e and the edges abd, ade, cde
- a vertex e and the edges abd, bde, cde
- a vertex e and the edges abe, ade, cde
- a vertex e and the edges abe, ace, cde
- a vertex e and the edges abe, bde, cde
- a vertex e and the edges abe, bce, cde
- the edges abc, acd
- the edges abc, bcd
- the edges abd, bcd
- the edges abd, acd.

Note that we do allow the subhypergraph induced by the vertices a, b, c, d and e

to contain also some other edges. Also note that the definition is symmetric, i.e., $(\{a,b\},\{c,d\})$ is tightly connected if and only if $(\{c,d\},\{a,b\})$ is. In other words, the pair $(\{a,b\},\{c,d\})$ is tightly connected if there exists a tight path from $\{a,b\}$ to $\{c,d\}$ on at most 5 vertices. Our aim is to show that most of the pairs $(\{a,b\},\{c,d\})$ in H'_n are tightly connected.

Claim 1. *For every H'_n, the number of pairs $(\{a,b\},\{c,d\})$ that are not tightly-connected is $o(n^4)$.*

Fix one pair $(\{a,b\},\{c,d\})$ that is not tightly-connected and let σ be the subhypergraph induced by a,b,c,d in H'_n labelled with a,b,c and d. Since there is no tight path of length four between the pairs, we may assume by the symmetry that σ is equal to σ_2, σ_3 or σ_4.

First, let us consider the case when $\sigma = \sigma_4$. We know that for all but $o(n^4)$ copies of σ_4 it holds that $\phi_n^{\sigma_4}(x_4 I_4)$ is an almost zero-vector. It holds that there are 6 (out of 17) σ_4-flags of size 5 that contain a tight path from a,b to c,d. For brevity, we let the last 6 coordinates of x_4 to be their corresponding densities. Also, we assume that $\mathcal{G}_5^{\sigma_4} = \{G_1^{\sigma_4}, \dots, G_{17}^{\sigma_4}\}$ so that the j-th coordinate of x_i is equal to the density of $G_j^{\sigma_4}$. Taking the sum of the 13th and the 15th column of I_4, we conclude that asymptotically almost surely

$$\phi_n^{\sigma_4}\left(\sum_{G^{\sigma_4}\in\mathcal{G}_5^{\sigma_4}} \alpha_{G^{\sigma_4}} \cdot G^{\sigma_4}\right) = o(1),$$

where $\alpha_{G^{\sigma_4}} < 0$ for $G^{\sigma_4} \in \{G_1^{\sigma_4}, \dots, G_{11}^{\sigma_4}\}$, i.e., for those G^{σ_4}s that do not contain a tight path from $\{a,b\}$ to $\{c,d\}$, and, additionally also for $G^{\sigma_4} \in \{G_{12}^{\sigma_4}, G_{15}^{\sigma_4}, G_{17}^{\sigma_4}\}$. In the other cases, i.e., for $G^{\sigma_4} \in \{G_{13}^{\sigma_4}, G_{14}^{\sigma_4}, G_{16}^{\sigma_4}\}$, the value of $\alpha_{G^{\sigma_4}}$ is positive. See Appendix A.1 for the entries of the matrix I_4 as well as the vector corresponding to the sum of the 13th and the 15th column of I_4. However, the numbers $\phi_n^{\sigma_4}(G_1^{\sigma_4}), \dots, \phi_n^{\sigma_4}(G_{17}^{\sigma_4})$ form a probability distribution so the fact that $(\{a,b\},\{c,d\})$ is not tightly-connected implies that a,b,c,d must induce one of these exceptional $o(n^4)$ copies of σ_4.

If $\sigma = \sigma_3$, an analogous reasoning with a different linear combination of the 13th and 15th column of I_3 yields that

$$\phi_n^{\sigma_3}\left(\sum_{G^{\sigma_3}\in\mathcal{G}_5^{\sigma_3}} \alpha_{G^{\sigma_3}} \cdot G^{\sigma_3}\right) = o(1)$$

asymptotically almost surely. This time, $\alpha_{G^{\sigma_3}}$ is negative for $G^{\sigma_3} \in \{G_1^{\sigma_3}, \dots, G_{11}^{\sigma_3}\} \cup$

$\{G_{12}^{\sigma_3}, G_{13}^{\sigma_3}, G_{15}^{\sigma_3}, G_{17}^{\sigma_3}\}$, and positive otherwise. Note that $\mathcal{F}_5^{\sigma_3}$ also has size 17 and again, there are 6 σ_3-flags of size 5 that contain the desired tight path. As in the case $\sigma = \sigma_4$, we used $G_{12}^{\sigma_3}, \dots, G_{17}^{\sigma_3}$ to denote these σ_3-flags. See Appendix A.1 for the additional details.

Finally, if $\sigma = \sigma_2$, then there are exactly 38 σ_2-flags of size 5. Also in this case, six of them contain a tight path from $\{a, b\}$ to $\{c, d\}$. We denote these σ_2-flags by $G_{33}^{\sigma_2}, \dots, G_{38}^{\sigma_2}$. A particular linear combination of 8 columns of I_2 shows that asymptotically almost surely

$$\phi_n^{\sigma_2}\left(\sum_{G^{\sigma_2} \in \mathcal{G}_5^{\sigma_2}} \alpha_{G^{\sigma_2}} \cdot G^{\sigma_2}\right) = o(1),$$

where $\alpha_{G^{\sigma_2}} > 0$ if and only if $G \in \{G_{33}^{\sigma_2}, \dots, G_{38}^{\sigma_2}\}$. Note that this linear combination was found with an assistance of a software for solving linear programs called QSopt_ex [3]. Analogously to the previous two cases, the exact computation is given in Appendix A.1. This finishes the proof of Claim 1.

Our final goal is to establish the following claim:

Claim 2. *For every H'_n, there exists an oriented graph O_n on the same set of vertices such that its arc density is at least $1 - o(1)$, and if three vertices u, v, w in O_n span a transitive triangle, then uvw is a non-edge in H'_n.*

Before proving this claim, let us look at how it implies the theorem. A simple application of Cauchy-Schwarz's inequality shows that the density of cyclically oriented triangles in any oriented graph is at most $1/4 + o(1)$. Furthermore, the equality holds if and only if the oriented graph is an almost balanced almost tournament, i.e., for all but $o(n)$ vertices v both the out-degree and the in-degree of v are equal to $n/2 \pm o(n)$. Since every three vertices that span an edge in H'_n either correspond to a cyclically oriented triangle in O_n, or contain at least one of the $o(n^2)$ pairs that do not span an arc in O_n, we conclude that the density of cyclically oriented triangles in O_n is at least $1/4 - o(1)$ (hence it must be equal to $1/4 \pm o(1)$). Therefore, O_n is an almost balanced almost tournament. Moreover, since $\lambda((H'_n)) = 1/4$, the same reasoning can be also applied to every subgraph of O_n of size $n/2$.

Now since every $(n/2)$-vertex subgraph of O_n is an almost balanced almost tournament, a classical result of Chung and Graham [16] on quasi-random tournaments yields that the sequence $(O_n)_{n \in \mathbb{N}}$ must be (after adding $o(n^2)$ arcs to each O_n) a sequence of quasi-random tournaments (see the property P6 in [16, Theorem 1]). This immediately concludes the theorem, since the density of any subtournament in a sequence of quasi-random tournaments is the same as the corresponding density in the sequence of the tournaments that are truly random [16].

It only remains to prove Claim 2. For every $n \in \mathbb{N}$, we use the tightly-connected property proved in Claim 1 to define an orientation for almost every pair of vertices $u, v \in V(H'_n)$. For two vertices u and v, we write $u \to v$ to denote the fact that we put to O_n an arc from u to v. Fix two vertices $a, b \in V(H'_n)$ such that for all but $o(n^2)$ subsets $\{c, d\} \subseteq V(H'_n)$ the pair $(\{a, b\}, \{c, d\})$ is tightly-connected. Claim 1 implies that there exists such a choice of a and b. First, we orient $a \to b$. Next for any $\{c, d\}$ that forms a tightly-connected pair with $\{a, b\}$, we use a tight path P on at most 5 vertices to define the orientation of $\{c, d\}$ as follows. Consider the induced subhypergraph $H'_n[V(P)]$. There is always a unique way how to orient the pairs of the vertices of $H'_n[V(P)]$ that are contained in some of the edges of P so that $a \to b$, and every edge of P corresponds to a cyclically oriented triangle in this orientation. In particular, there is a unique orientation of $\{c, d\}$, which is the way how we orient $\{c, d\}$ in O_n.

Let us first show that the orientation of almost all $\{c, d\} \subseteq V(H'_n)$ is well-defined. For any two tight paths P_1, P_2 from $\{a, b\}$ to $\{c, d\}$ on at most 5 vertices, consider the induced subhypergraph $H'_n[V(P_1) \cup V(P_2)]$. It follows that the sub-hypergraph has at most 6 vertices. Recall that for every induced subhypergraph Z of H'_n on at most 7 vertices, there must exist an underlying tournament T_Z on $V(Z)$ such that the edges of Z are in one-to-one correspondence with the cyclically oriented triangles in T_Z. Therefore, both paths P_1 and P_2 must define the same orientation of $\{c, d\}$ in O_n.

The last part of Claim 2 we need to establish is that if three vertices u, v, w span a transitive triangle in O_n, then uvw is a non-edge in H'_n. The argument is very similar to the one used in the last paragraph. Without loss of generality, the arcs on $\{u, v\}$ and $\{u, w\}$ in O_n are $u \to v$ and $u \to w$. Let P_{uv} and P_{uw} be tight paths on at most 5 vertices from $\{a, b\}$ to $\{u, v\}$ and $\{u, w\}$, respectively. Since the induced subhypergraph $Z := H'_n[V(P_{uv} \cup V(P_{uw})]$ has at most 7 vertices, there is an underlying tournament T_Z on $V(Z)$ so that the edges of Z correspond to the cyclically oriented triangles in T_Z. Without loss of generality, $a \to b$ in T_Z. But then also $u \to v$ and $u \to w$ in T_Z and hence uvw cannot be an edge of Z. □

Chapter 3

Monotone subsequences in permutations

The problem we study here is inspired by a well-known result of Erdős and Szekeres [25] which states that every permutation on $[n] = \{1, \ldots, n\}$, where $n \geq (k-1)^2 + 1$, contains a monotone subsequence of length k. When n is much larger than k^2, one expects that the number of monotone subsequences of length k is much more than just one. In fact, a simple double-counting argument implies that the result of Erdős and Szekeres guarantees that there are at least $\binom{n}{k} / \binom{(k-1)^2+1}{k}$ such subsequences. A natural question to ask is what is the minimum number of monotone subsequences of length k inside a permutation of length n.

According to Myers [53], this problem was first posed by Atkinson, Albert and Holton. In this chapter, we use $F_k(\tau)$ to denote the number of monotone subsequences of length k in a permutation τ. Note that $F_k(\tau) = F_k\left(\tau^{-1}\right)$, where τ^{-1} is the reverse permutation of τ. The minimum of $F_k(\tau)$ over all permutations $\tau \in S_n$ is denoted by $F_k(n)$. For brevity, we also define $f_k(\tau)$ to denote the density of monotone subsequences of length k in τ, i.e., $f_k(\tau) := F_k(\tau)/\binom{n}{k}$.

Myers [53] described a permutation $\tau_k(n) \in S_n$ which gives an upper bound on $F_k(n)$ of the form $(k-1)^{1-k} \cdot \binom{n}{k} + O(n^{k-1})$. It consists of $k-1$ increasing sequences $A_1, A_2, \ldots, A_{k-1}$ whose sizes differ by at most one, and, every monotone subsequence of length k is entirely contained in A_i for some $i \in [k-1]$. In other words, with $a_j = \lfloor jn/k \rfloor$, an example of such a permutation is

$$\begin{array}{llllll} \tau_k(n) = (& a_{k-2}+1, & a_{k-2}+2, & \ldots, & n-1, & n, \\ & a_{k-3}+1, & a_{k-3}+2, & \ldots, & a_{k-2}-1, & a_{k-2}, \\ & \cdots & \cdots & \cdots & \cdots & \cdots \\ & 1, & 2, & \ldots, & a_1 - 1, & a_1 \quad). \end{array}$$

See Figure 3 for $\tau_4(15)$.

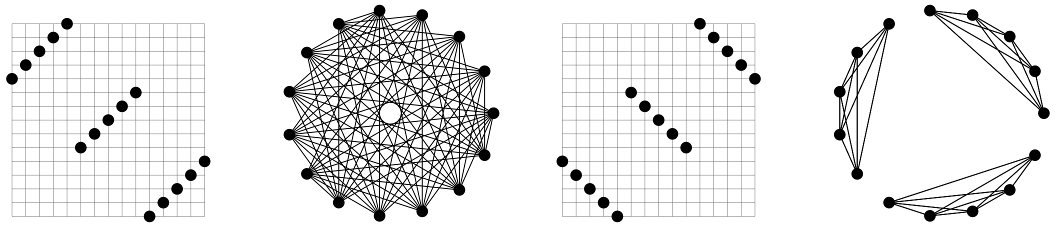


Figure 3.1: Permutation $\tau_4(15)$, its permutation graph $T_3(15)$, the reverse of $\tau_4(15)$, and its permutation graph $\overline{T_3(15)}$.

Let $r \equiv n \pmod{k-1}$, where $0 \leq r < k-1$. It follows that value of $F_k(\tau_k(n))$ is equal to

$$r\binom{\lceil n/(k-1)\rceil}{k} + (k-1-r)\binom{\lfloor n/(k-1)\rfloor}{k} = \frac{\binom{n}{k}}{(k-1)^{k-1}} + O\left(n^{k-1}\right).$$

Myers [53] proved that $F_3(n) = F_3(\tau_3(n))$ and he described all permutations $\tau \in S_n$ where $F_3(\tau) = F_3(n)$. He conjectured that the upper bound given by $F_k(\tau_k(n))$ actually holds for every $k \in \mathbb{N}$.

Conjecture 3.1 (Myers [53])**.** *Let n and k be positive integers. In any permutation $\tau \in S_n$ there are at least $F_k(\tau_k(n))$ monotone subsequences of length k.*

The main result of this chapter is to show that the conjecture is true for $k = 5$ and $k = 6$, when n is sufficiently large. In fact, for these values of k and n, we also determine a full description of the set of extremal permutations $\tau \in S_n$, i.e., $\tau \in S_n$ such that $F_k(\tau) = F_k(n) = F_k(\tau_k(n))$.

As we already mentioned, Myers showed the conjecture is true for $k = 3$, which is actually a consequence of Goodman's formula. In [8], the conjecture was verified for $k = 4$ and n sufficiently large. In Section 3.2, we present a slightly different proof of this result as a "warm-up" for the next two cases. Very recently, Samotij and Sudakov [68] confirmed the conjecture if $n \leq k^2 + ck^{3/2}/\log k$ for some absolute positive constant c, provided k is sufficiently large.

Subject to the additional constraint that all the monotone subsequences of length k are either only increasing or only decreasing (and also $n \geq (k-1)(2k-3)$), Myers proved that every such a permutation contains at least the conjectured number of monotone subsequences of length k. He also gave the list $\mathcal{W}_k(n)$ of all such permutations $\tau \in S_n$ that satisfy the additional constraint and have $F_k(\tau) = F_k(\tau_k(n))$. Every permutation from $\mathcal{W}_k(n)$ can be decomposed into k disjoint monotone subsequences $A_1, \ldots, A_{k-1}$ that are either all increasing or all decreasing, and their sizes differ by at most one. Moreover, every monotone subsequence of length k is a subsequence of A_i for some $i \in [k-1]$. These permutations look very similar to $\tau_k(n)$ or $\tau_k^{-1}(n)$ and the only parts where they can vary are at the first / last $k-1$ elements of each block. It turns out that $|\mathcal{W}_k(n)|$ has size $2 \cdot \binom{k-1}{n \bmod (k-1)} \cdot C(k-1)^{2k-4}$, where $C(k)$ is the k^{th} Catalan number. The precise description of the sets $\mathcal{W}_k(n)$ can be found in [53].

Theorem 3.2 ([53, Theorem 9])**.** *Let $n \geq (k-1)(2k-3)$. If a permutation $\tau \in S_n$ contains no increasing subsequence of length k, then $F_k(\tau) \geq F_k(\tau_k(n))$. Furthermore, there are exactly $\binom{k-1}{n \bmod (k-1)} \cdot C(k-1)^{2k-4}$ permutations τ such that $F_k(\tau) = F_k(\tau_k(n))$. We denote the set of all such extremal permutations by $\mathcal{W}_k^-(n)$.*

Symmetrically, if a permutation $\tau \in S_n$ contains no decreasing subsequence of length k, then $F_k(\tau) \geq F_k(\tau_k(n))$. Furthermore, there are exactly $\binom{k-1}{n \bmod (k-1)} \cdot C(k-1)^{2k-4}$ permutations τ such that $F_k(\tau) = F_k(\tau_k(n))$. We denote the set of all such extremal permutations by $\mathcal{W}_k^+(n)$.

Note that it immediately follows that $\mathcal{W}_k(n) = \mathcal{W}_k^+(n) \,\dot\cup\, \mathcal{W}_k^-(n)$.

3.1 Flag algebra setting

Instead of working directly with permutations, we formulate the problem in the language of permutation graphs and tweak the flag algebra framework for this particular setting. The permutation graph of a permutation $\pi \in S_n$ is a graph G_π with the vertex set $[n]$ and two vertices $i, j \in [n]$ form an edge if and only if the pair $\{i, j\}$ is an inversion in π. For a permutation graph G and an integer k, we define $F_k(G)$ to denote the number of induced subgraphs of size k that are either complete, or empty, and we let $f_k(G) := F_k(G)/\binom{n}{k}$.

For an integer k, we let K_k to be the k-vertex complete graph. Since our problem is symmetric under taking the graph complement, we apply flag algebras in the complement-blind setting, i.e., we do not distinguish between a graph G and its complement. Formally, we say that two graphs G_1 and G_2 are blindly isomorphic, if G_1 is isomorphic either to G_2, or to the complement of G_2. In particular, for a permutation graph G, the value of $F_k(G)$ is exactly the number of k-vertex subgraphs blindly isomorphic to K_k.

Let $\mathcal{F}$ be the set of all finite permutation graphs up to a blind isomorphism, and let $\mathcal{F}_\ell \subseteq \mathcal{F}$ be the set of permutation graphs of order exactly ℓ. Note that $\mathcal{F}_3 = 2, \mathcal{F}_4 = 6, \mathcal{F}_5 = 17, \mathcal{F}_6 = 71, \mathcal{F}_7 = 388$ and $\mathcal{F}_8 = 2852$.

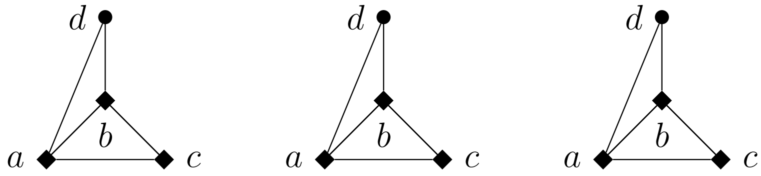


Figure 3.2: The possible extensions of the triangle abc such that $a < b < c$ by a vertex d to a permutation graph that contains the edge bd.

Finally, we want to exploit the fact that the vertex set of a permutation graph comes with a natural linear order, which gives an additional restriction on possibilities how to extend a fixed small subgraph in a permutation graph to a larger one. Therefore, we do a more careful setting of flag algebras compared to the one in Chapter 1. Let us describe the differences in more detail.

For the first step, we follow the standard approach and construct an algebra $\mathcal{A}$, which is the set $\mathbb{R}\mathcal{F}$ factored by the space of all linear combinations of the form $H - \sum_{H' \in \mathcal{F}_{v(H)+1}} p(H, H') \cdot H'$. The space $\mathcal{A}$ has naturally defined linear operations of an addition, and we follow Chapter 1 in order to define also a multiplication on the elements of $\mathcal{A}$.

Our next task is to define algebras $\mathcal{A}^\sigma$, where σ is a permutation graph with a fixed labelling of its vertex set. Again, the labelled graph σ is called a type, any supergraph of σ that preserves the labelling of the vertices of σ is called a σ-flag, and the labelled part of a σ-flag is called the root. At this point, we are going to differ from the Chapter 1 so before proceeding further, let us demonstrate our approach on a simple example.

Suppose abc is a triangle in an n-vertex permutation graph G_π such that $a < b < c$. That means, all three pairs $\{a, b\}$, $\{b, c\}$ and $\{a, c\}$ are inversions in the permutation π. Now observe that for every point $d \in [n] \setminus \{a, b, c\}$, if $\{b, d\}$ is an inversion in π, then $\{a, d\}$ or $\{c, d\}$ has to be an inversion as well. Translated back to the graph language, there is no vertex d in G_π that would extend the triangle abc to a 4-vertex graph with the edge set $\{ab, bc, ac, bd\}$, see Figure 3.1. Note that the other two options of adding a vertex of degree 1 to abc, i.e., the edge sets $\{ab, bc, ac, ad\}$ and $\{ab, bc, ac, cd\}$, can be realized. On the other hand, if we consider a triangle abc with $b < a < c$, there might be a vertex d such that the 4-vertex subgraph induced by $\{a, b, c, d\}$ has the edge set $\{ab, bc, ac, bd\}$. The non-realizable extension for this case is $\{ab, bc, ac, ad\}$. Our plan is to construct the algebra $\mathcal{A}^\sigma$, for σ being a triangle, in such a way that it will be generated by a set of σ-flags $\mathcal{F}^\sigma$ that in particular contains only two σ-flags with 4 vertices and 4 edges.

Let us now move to the general definition of $\mathcal{A}^\sigma$. Fix σ a type and let $\tau_1, \tau_2, \ldots, \tau_t \in S_{v(\sigma)}$ be all the permutations such that their permutation graph is blindly isomorphic to σ. We denote the set $\{\tau_1, \ldots, \tau_t\}$ by $T(\sigma)$. For each $\tau \in T(\sigma)$, we choose a bijection b_τ from the points of τ to the vertices of σ that maps the permutation graph of τ to σ with respect to blind isomorphism. For example, if σ is a triangle abc, we have $\tau_1 = 321$ and $\tau_2 = 123$. If we set $b_{\tau_1} := b_{\tau_2} := abc$, by which we mean the function that maps 1 to a, 2 to b and 3 to c, then we infer that in both cases we cannot realize any σ-flag that contains a vertex adjacent with b but not adjacent with a and c.

In general, let $B(\tau)$ be the set of all blind isomorphisms $b : V(G_\tau) \to V(\sigma)$. Note that $B(\tau) \cong \mathrm{Aut}(\sigma)$. For every $\tau \in T(\sigma)$ and $b \in B(\tau)$, we define $\mathcal{F}^{\tau,b}$ as the set of all finite σ-flags that satisfy the following:

a) they can be realized as permutation graphs with the root σ,

b) the vertices of the root induce τ in at least one of the choices of an underlying permutation, and

c) the bijection between the root and the points of τ is equal to b.

We set $\mathcal{F}^\sigma := \bigcup_{\tau \in T(\sigma)} \mathcal{F}^{\tau,b_\tau}$. Note that if σ has a non-trivial automorphism group, we had some freedom how to choose the bijections $(b_\tau)_{\tau \in T(\sigma)}$. For each σ, the particular choice of (b_τ) was made (with a computer assistance) in order to make the sets $\mathcal{F}^\sigma$ as minimal as possible. Next, for any blind isomorphism $b : V(G_\tau) \to V(\sigma)$, we define an injective mapping $z_{\tau,b}$ from $\mathcal{F}^{\tau,b}$ to $\mathcal{F}^\sigma$ as the relabelling function of the vertices of the root of $H \in \mathcal{F}^{\tau,b}$ using $b^{-1} \circ b_\tau$.

Now, we again follow the scheme described in Chapter 1 and construct the algebras $\mathcal{A}^\sigma$ by factoring the space $\mathbb{R}\mathcal{F}^\sigma$ with all linear combinations of the form $H - \sum_{H' \in \mathcal{F}^\sigma_{v(H)+1}} p(H, H') \cdot H'$, and equip $\mathcal{A}^\sigma$ with a multiplication. Following the same lines, we also construct algebras $\mathcal{A}^{\tau,b}$ for every $\tau \in T(\sigma)$ and $b \in B(\tau)$.

The algebras $\mathcal{A}^\sigma$ and $\mathcal{A}^{\tau,b}$ are closely related. Recall the injective mapping $z_{\tau,b} : \mathcal{F}^{\tau,b} \to \mathcal{F}^\sigma$. We slightly abuse the notation and will use $z_{\tau,b}$ both for the mapping from $\mathcal{F}^{\tau,b}$ to $\mathcal{F}^\sigma$ and its unique extension to a linear operator from $\mathcal{A}^{\tau,b}$ to $\mathcal{A}^\sigma$. The operator $z_{\tau,b}$ has the following property for every $H_1, H_2 \in \mathcal{F}^{\tau,b}$:

$$H_1^\sigma \times H_2^\sigma = z_{\tau,b}(H_1 \times H_2) + \sum_{\substack{H^\sigma \in \\ \mathcal{F}^\sigma_\ell \setminus z_{\tau,b}\left(\mathcal{F}^{\tau,b}_\ell\right)}} p(H_1^\sigma, H_2^\sigma; H^\sigma) \cdot H^\sigma, \tag{3.3}$$

where $H_1^\sigma = z_{\tau,b}(H_1), H_2^\sigma = z_{\tau,b}(H_2)$ and $\ell = v(H_1) + v(H_2) - v(\sigma)$. We also define the "inverse" operator $\overline{z_{\tau,b}}$ from $\mathcal{A}^\sigma$ to $\mathcal{A}^{\tau,b}$ as the unique linear extension of the following map of the basis of $\mathcal{A}^\sigma$:

- $z_{\tau,b}^{-1}(H^\sigma)$ for $H^\sigma \in z_{\tau,b}\left(\mathcal{F}^{\tau,b}\right)$, and
- 0 for $H^\sigma \in \mathcal{F}^\sigma \setminus z_{\tau,b}\left(\mathcal{F}^{\tau,b}\right)$.

Now consider an infinite sequence $(\pi_n)_{n\in\mathbb{N}}$ of permutations of increasing size. We say that the sequence is *convergent* if for every fixed permutation τ, the probabilities that a random $|\tau|$ points in π_n induce the subpermutation equal to τ has a limit as n tends to infinity. We denote this limit probability by $q(\tau, (\pi_n))$. In particular, for every convergent sequence of permutations $(\pi_n)_{n\in\mathbb{N}}$, the probabilities $p(H, G_{\pi_n})$ have a limit for every $H \in \mathcal{F}$. As in Chapter 1, we know that every infinite sequence of permutations has a convergent subsequence. Fix a convergent increasing sequence $(\pi_n)_{n\in\mathbb{N}}$ of permutations. For every $H \in \mathcal{F}$, we set $\phi(H) := \lim_{n\to\infty} p(H, G_{\pi_n})$ and linearly extend ϕ to $\mathcal{A}$. The obtained mapping ϕ is a homomorphism from $\mathcal{A}$ to $\mathbb{R}$ such that $\phi(H) \geq 0$ for every $H \in \mathcal{F}$. Let $\mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$ be the set of all homomorphisms ψ from the algebra $\mathcal{A}$ to $\mathbb{R}$ such that $\psi(H) \geq 0$ for every $H \in \mathcal{F}$.

Fix a convergent sequence of permutations $(\pi_n)_{n\in\mathbb{N}}$ and let $\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$ be the corresponding homomorphism from $\mathcal{A}$ to $\mathbb{R}$. Fix a type σ. Recall that $T(\sigma)$ is the set of all permutations $\tau \in S_{v(\sigma)}$ with the permutation graph G_τ blindly isomorphic to σ, and $B(\tau)$ is the set of bijections between the vertices of σ and the points of τ that preserve the blind isomorphism between σ and G_τ. For an embedding θ of a permutation $\tau \in T(\sigma)$ in π_n, we define $G_{\pi_n}^\theta$ to be the permutation graph of π_n rooted on the copy of σ that corresponds to θ. Furthermore, let $b \in B(\tau)$ be the bijection between the points of θ and the vertices of the copy of σ. For every $n \in \mathbb{N}$ and $H^{\tau,b} \in \mathcal{F}^{\tau,b}$, we define $p_n^\theta(H^{\tau,b}) = p(H^{\tau,b}, G_{\pi_n}^\theta)$. Picking such θ with a fixed bijection b in π_n at random gives rise to a probability distribution $\mathbf{P}_\mathbf{n}^{\tau,\mathbf{b}}$ on mappings from $\mathcal{A}^{\tau,b}$ to $\mathbb{R}$, and, via the operator $z_{\tau,b}$ also on mappings from $\mathcal{A}^\sigma$ to $\mathbb{R}$.

Since the sequence $(\pi_n)_{n\in\mathbb{N}}$ is convergent, the sequence of probability distributions $\mathbf{P}_\mathbf{n}^{\tau,\mathbf{b}}$ on mappings from $\mathcal{A}^{\tau,b}$ to $\mathbb{R}$ also converge. As in Chapter 1, this follows from [59, Theorems 3.12 and 3.13]. We denote the limit probability distribution by $\mathbf{P}^{\tau,\mathbf{b}}$. Furthermore, the relation (3.3) implies that for the labelled permutation graph σ of the permutation τ, any mapping $\phi^{\tau,b}\left(\overline{z_{\tau,b}}(\cdot)\right)$, where $\phi^{\tau,b}$ is taken from the support of the distribution $\mathbf{P}^{\tau,\mathbf{b}}$, is a homomorphism from $\mathcal{A}^\sigma$ to $\mathbb{R}$ such that $\phi^\sigma(H^\sigma) \geq 0$ for all $H^\sigma \in \mathcal{F}^\sigma$ [59, Proof of Theorem 3.5].

The remaining bit we need to introduce is the averaging operator $[\![\cdot]\!]_\sigma : \mathcal{A}^\sigma \to \mathcal{A}$. Again, it is an alagoue of the averaging operator from Chapter 1. The operator is the linear extension of a mapping defined on the elements of $H^\sigma \in \mathcal{F}^\sigma$ by

$$[\![H^\sigma]\!]_\sigma := p_H^\sigma \cdot H^\emptyset,$$

where $H^\emptyset$ is the (unlabelled) permutation graph from $\mathcal{F}$ corresponding to H^σ, and p_H^σ is probability that a random injective mapping from $V(\sigma)$ to $V(H^\emptyset)$ is an embedding of σ in $H^\emptyset$ yielding a σ-flag blindly isomorphic to H^σ. However, the corresponding relation between the homomorphism ϕ and the homomorphisms $\phi^{\tau,b}$, where $\tau \in T(\sigma)$ and $b \in B(\tau)$, is slightly different from the analogue in Chapter 1. Specifically,

$$\forall H^\sigma \in \mathcal{A}^\sigma, \quad \phi\left([\![H^\sigma]\!]_\sigma\right) = \phi([\![\sigma]\!]_\sigma) \cdot \sum_{\substack{\tau \in T(\sigma) \\ b \in B(\tau)}} \frac{q(\tau,(\pi_n))}{\phi(\sigma^\emptyset)|B(\tau)|} \cdot \int \phi^{\tau,b}\left(\overline{z_{\tau,b}}\left(H^\sigma\right)\right) \mathrm{d}\mathbf{P}^{\tau,\mathbf{b}}.$$

Therefore, if for some fixed $A^\sigma \in \mathcal{A}^\sigma$ and every $\tau \in T(\sigma), b \in B(\tau)$ we have

$$\phi^{\tau,b}\left(\overline{z_{\tau,b}}\left(A^\sigma\right)\right) \geq 0$$

almost surely, then $\phi\left([\![A^\sigma]\!]_\sigma\right) \geq 0$. In particular, the analogue of the inequality (1.2), i.e.,

$$\phi\left(\left[\!\!\left[(A^\sigma)^2\right]\!\!\right]_\sigma\right) \geq 0,$$

holds for every $A^\sigma \in \mathcal{A}^\sigma$.

3.2 Monotone subsequences of length four

In this section, we reprove the main result of [8]. Our proof essentially follows the same lines as in [8]. There are the following two differences between the proofs:

- We use the flag algebra setup presented in the previous section, which is a slightly different than the one used in [8].
- Our flag algebra proof provides a somewhat better control on the substructures appearing in (almost) extremal configurations, which helps to simplify our stability arguments presented in Section 3.3.

In Section 3.5 and Section 3.6, we use the same approach to resolve also the question of minimizing monotone subsequences of length 5 and 6.

Recall we use the flag algebra framework in the complement-blind way. In particular, $\phi(K_4)$ is the sum of the density of K_4 and the density of the complement of K_4. Let EXT_7^4 be the set of all 7-vertex subgraphs up to a blind isomorphism that have a positive density in $T_3(n)$, or in $\overline{T_3(n)}$. It follows that $\mathrm{EXT}_7^4 = \{H_1, H_2, \ldots, H_8\}$. The set EXT_7^4 is depicted in Figure 3.3. Next, let

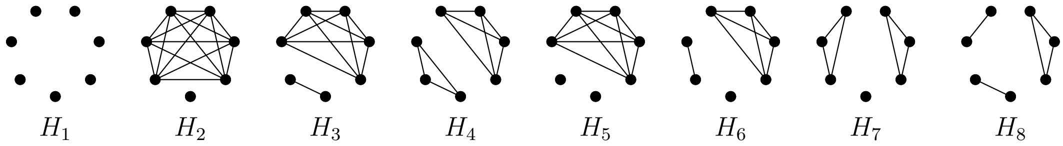


Figure 3.3: The set of graphs EXT_7^4.

$\mathcal{E}_7^4 := \mathcal{F}_7 \setminus \mathrm{EXT}_7^4$. We are now ready to present the main theorem of this section.

Theorem 3.4. *There exists a positive rational α such that the following is true. If $(\pi)_{n\in\mathbb{N}}$ is a convergent sequence of permutations and $\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$ is its limit, then*

$$\phi\left(K_4 - \alpha \cdot \sum_{H \in \mathcal{E}_7^4} H\right) \geq \frac{1}{27}.$$

Theorem 3.4 immediately implies the following two corollaries.

Corollary 3.5. *For every positive $\varepsilon_{\mathrm{ASYM}}$ there exists $n_{\mathrm{ASYM}} \in \mathbb{N}$ such that the following is true. If G is a permutation graph on $n \geq n_{\mathrm{ASYM}}$ vertices, then $f_4(G) > (1/27 - \varepsilon)$.*

Corollary 3.6. *For every positive $\varepsilon_{\mathrm{CONF}}$ there exist a positive δ_{CONF} and $n_{\mathrm{CONF}} \in \mathbb{N}$ such that the following is true. If G is a permutation graph on $n \geq n_{\mathrm{CONF}}$ vertices that satisfies $f_4(G) \leq (1/27 + \delta_{\mathrm{CONF}})$, then G contains at most $\varepsilon_{\mathrm{CONF}} \cdot \binom{n}{7}$ induced copies of F and at most $\varepsilon_{\mathrm{CONF}} \cdot \binom{n}{7}$ induced copies of $\overline{F}$, where $F \in \mathcal{E}_7^4$.*

Proof of Theorem 3.4. We use the flag algebra framework presented in Section 3.1 in the following way. Let σ_1 is the 3-vertex type with no edges and σ_2 is the 3-vertex type with the edge bc. The types σ_1 and σ_2 are depicted in Figure 3.4.

Next, the set $T(\sigma_1)$ contains two permutations $\tau_{1,1} = 123$ and $\tau_{1,2} = 321$. We set $b_{\tau_{1,1}} := b_{\tau_{1,2}} := abc$. It follows that $|\mathcal{F}_5^{\sigma_1}| = 54$ and $|\mathcal{F}_7^{\sigma_1}| = 5388$.

Analogously, the set $T(\sigma_2)$ contains the remaining four permutations from S_3. Specifically, $T(\sigma_2)$ contains the permutations $\tau_{2,1} = 132$, $\tau_{2,2} = 231$, $\tau_{2,3} = 231$ and $\tau_{2,4} = 312$. We set $b_{\tau_{2,1}} := b_{\tau_{2,4}} := abc$ and $b_{\tau_{2,2}} := b_{\tau_{2,3}} := bca$. It follows that $|\mathcal{F}_5^{\sigma_2}| = 71$ and $|\mathcal{F}_7^{\sigma_2}| = 9055$.

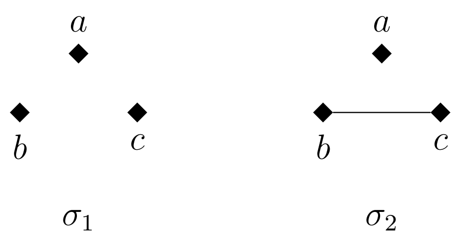


Figure 3.4: The types σ_1 and σ_2 used in the proof of Theorem 3.4.

Using the method from Section 1.2, an instance of the SDP program was used to find two symmetric positive semidefinite matrices M_1 and M_2 with rational entries such that for every $\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$ we have

$$\phi\left(\llbracket x_1^T M_1 x_1 \rrbracket_{\sigma_1} + \llbracket x_2^T M_2 x_2 \rrbracket_{\sigma_2}\right) \leq \phi\left(K_4 - \frac{1}{27} - \alpha \cdot \sum_{H \in E_7^4} H\right),$$

where

- the vector $x_1 \in (\mathbb{R}\mathcal{F}_5^{\sigma_1})^{|\mathcal{F}_5^{\sigma_1}|}$ is the vector whose j-th coordinate is equal to the j-th element of the canonical base of $\mathbb{R}\mathcal{F}_5^{\sigma_1}$,
- the vector $x_2 \in (\mathbb{R}\mathcal{F}_5^{\sigma_2})^{|\mathcal{F}_5^{\sigma_2}|}$ is the vector whose j-th coordinate is equal to the j-th element of the canonical base of $\mathbb{R}\mathcal{F}_5^{\sigma_2}$, and
- $\alpha = 2629661/9106944 \sim 0.28875$.

The left-hand side of the inequality above is non-negative by (1.2). □

As in Chapter 2, the proof was found with an assistance of computer programs CSDP [11], SDPA [74] and SAGE [71]. We also used a software package nauty [52] for graph isomorphism tests performed while the appropriate sets of flags were being generated.

The numerical values of the entries of M_1 and M_2 can be downloaded from the web page `http://honza.ucw.cz/phd/`. In fact, the 54×54 matrix M_1 is not stored directly, but as a pair of matrices (D_1, M_1) such that N_1 is a 52×52 positive definite matrix and $M_1 = D_1 \cdot N_1 \cdot (D_1)^T$. Analogously, the 71×71 matix M_2 is stored as a pair of matrices (D_2, M_2) such that N_2 is a 70×70 positive definite matrix and $M_2 = D_2 \cdot N_2 \cdot (D_2)^T$.

Similarly to the flag algebra computations presented in Chapter 2, we created a sage script called "thm_3_4-verify.sage", which can be used for an independent verification of the computations in Theorem 3.4. The script is also available on the web page mentioned above.

3.3 Stability of almost extremal configurations

In this section, we show that the following two properties that are satisfied by (at this moment only conjectured) extremal graphs – Turán graph and its complement – force every almost extremal graph to be close in the edit distance to one of the conjectured extremal graphs.

We say that a graph G satisfies *Property A(k)*, if for every k-vertex subgraph H of G that is either complete or empty, and every vertex $v \in V(G') \setminus V(H)$, we have either $V(H) \subseteq N(v)$, or $V(H) \cap N(v) = \emptyset$. See Figure 3.5 for an illustration of Property A(5). Next, we say that a graph G satisfies *Property B*, if G does not

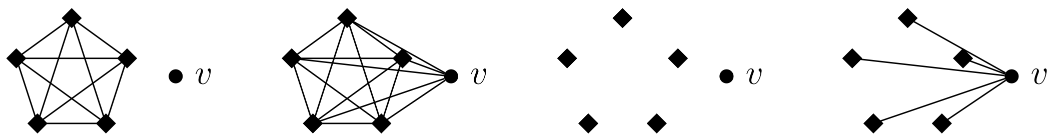


Figure 3.5: The possible extensions of H in Property A(5).

contain a copy of the graph E_4 and $\overline{E}_4$, where E_4 is the paw graph, i.e., the 4-vertex graph containing a triangle and one pendant edge. The graph E_4 is depicted in Figure 3.6.

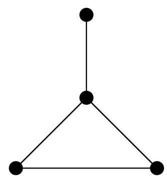


Figure 3.6: The graph E_4 from Property B.

We start with the following lemma that helps us to describe the global structure of almost extremal configurations from the two (local) properties described in the previous paragraph.

Lemma 3.7. *For a fixed integer $k \geq 3$, let G be a graph and H a k-vertex subgraph of G that is a clique or an independent set. If G satisfies Property A(k) and Property B, then G is either a disjoint union of at most $k-1$ cliques, or the complement, i.e, a complete r-partite graph for some $r < k$.*

Proof. Without loss of generality, the graph H is a clique on k vertices. Let $h_1, h_2, \ldots, h_k$ be the vertices of H. Property A(k) implies that every vertex $u \in N(h_1) \setminus V(H)$ extends H to a clique on $k+1$ vertices. Therefore, $N[h_i] = N[h_j]$ for every $1 \leq i, j \leq k$. Furthermore, any two neighbors $u, v \in N(h_1)$ are connected by an edge, otherwise the graph induced by $(V(H) \setminus \{h_k\}) \cup \{u, v\}$ violates Property A(k). Therefore, the vertices $N[h_1]$ form a clique and every vertex $z \in V(G) \setminus N[h_1]$ has its neighborhood $N(z)$ disjoint from $N[h_1]$.

If $N[h_1] = V(G)$, we are done. Otherwise, we claim that the graph $G[V(G) \setminus N[h_1]]$ is a disjoint union of cliques. Indeed, suppose there exist distinct vertices $x, y, z \in V(G) \setminus N[h_1]$ such that $\{x, y\} \subseteq N(z)$ and $xy \notin E(G)$. However, the subgraph $\{h_1, x, y, z\}$ is isomorphic to the graph $\overline{E}_4$, a contradiction.

So the whole graph G is a disjoint collection of cliques and we only need to show that the number of components must be less than k. Suppose for a contradiction the number of components of G is at least k. However, any independent set of size k containing h_1 is extended by the vertex h_2 to a subgraph that violates Property A(k). □

Our next tool is the Infinite Removal Lemma of Alon and Shapira [1].

Lemma 3.8 (Infinite Removal Lemma [1]). *For any (possibly infinite) family of graphs $\mathcal{H}$ and $\varepsilon_{\mathrm{RL}} > 0$, there exists $\delta_{\mathrm{RL}} > 0$ such that if a graph G on n vertices contains at most $\delta_{\mathrm{RL}} \cdot \binom{n}{v(H)}$ induced copies of H for every graph H in $\mathcal{H}$, then it is possible to make G induced $\mathcal{H}$-free by adding and/or deleting at most $\varepsilon_{\mathrm{RL}} \cdot \binom{n}{2}$ edges.*

We are ready to present the stability result for the function $F_4(G)$.

Theorem 3.9. *For every $\varepsilon_{\mathrm{STAB}} > 0$ there exist $\delta_{\mathrm{STAB}} > 0$ and n_{STAB} such that the following is true. If G is a permutation graph on $n_{\mathrm{STAB}} \geq n_0$ vertices with $f_4(G) \leq \frac{1}{27} + \delta_{\mathrm{STAB}}$, then G is isomorphic to either $T_3(n)$ or $\overline{T_3(n)}$ after adding and/or deleting at most $\varepsilon_{\mathrm{STAB}} \cdot \binom{n}{2}$ edges.*

Note that we do not optimize the proof for the best possible values of δ_{STAB} and n_{STAB} but rather try to keep our computations as simple as possible.

Proof. Without loss of generality, $\varepsilon_{\mathrm{STAB}} < 1/2$. Fix C a sufficiently large constant ($C \geq 25 \times 3^3 = 675$ will suffice).

Let $\mathcal{X}$ be the set of all non-permutation graphs, i.e.,

$$\mathcal{X} := \{G : G \text{ is a graph such that } G \notin \mathcal{F} \text{ and } \overline{G} \notin \mathcal{F}\},$$

and δ_{RL} the value from Infinite Removal Lemma applied for $\varepsilon_{\mathrm{RL}} := (\varepsilon_{\mathrm{STAB}})^2/C$ and the family $\mathcal{X} \cup \mathcal{E}^4_7$. Next, let δ_{CONF} and n_{CONF} be the values from Corollary 3.6 applied for $\varepsilon_{\mathrm{CONF}} := \delta_{\mathrm{RL}}$. Finally, let n_{ASYM} be the value from Corollary 3.5 for $\varepsilon_{\mathrm{ASYM}} := (\varepsilon_{\mathrm{STAB}})^2/C$. We set $\delta_{\mathrm{STAB}} := \min\{(\varepsilon_{\mathrm{STAB}})^2/C, \delta_{\mathrm{CONF}}\}$ and $n_{\mathrm{STAB}} := \max\{C/\varepsilon_{\mathrm{STAB}}, n_{\mathrm{CONF}}, 2 \cdot n_{\mathrm{ASYM}}\}$.

Let G be a given graph on n vertices satisfying the assumptions of the theorem. Corollary 3.6 and Infinite Removal Lemma imply that we can change at most $\varepsilon_{\mathrm{RL}} \cdot \binom{n}{2} < \frac{1}{5} \cdot \varepsilon_{\mathrm{STAB}} \cdot \binom{n}{2}$ edges and obtain a permutation graph G' that does not contain neither a copy of a graph E, nor a copy of a graph $\overline{E}$, where $E \in \mathcal{E}^4_7$. In particular, G' satisfies Property A(4) and Property B. Furthermore, $f_4(G') \le 1/27 + \delta_{\mathrm{STAB}} + \varepsilon_{\mathrm{RL}}$. Lemma 3.7 implies we can partition G' into at most 3 parts such that either every part is a clique and there are no edges between the parts, or every part is an independent set and all pairs of vertices from different parts are connected by an edge.

Without loss of generality, G' is a disjoint union of at most 3 cliques. We claim that every clique has size at most $(1/3 + \varepsilon_{\mathrm{STAB}}/5) \cdot n$. This immediately concludes the theorem since the number of cliques must be 3, and in order to balance the sizes of the cliques we need to add/remove edges incident to at most $\frac{2}{5} \cdot \varepsilon_{\mathrm{STAB}} \cdot n$ vertices, which means changing at most $\frac{4}{5} \cdot \varepsilon_{\mathrm{STAB}} \cdot \binom{n}{2}$ edges. Therefore, it is enough to show the claim.

Let $\gamma := \varepsilon_{\mathrm{STAB}}/5$. Suppose for a contradiction G' contains a clique of size more than $(1/3 + \gamma) \cdot n$. Let $H_0 := G'$. For each $i \in [\gamma \cdot n/3]$, let v_i be an arbitrary vertex from a maximum clique inside H_{i-1}, and let $H_i := H_{i-1} - v_i$. Let $Z := \{v_1, v_2, \ldots, v_{\gamma \cdot n/3}\}$. It follows that every vertex $v \in Z$ is contained in at least $\binom{(1/27+2\gamma/3)\cdot n}{3}$ copies of K_4 that are disjoint from $Z \setminus \{v\}$. Since $n_{\mathrm{STAB}} > 12/\gamma = 60/\varepsilon_{\mathrm{STAB}}$, we have

$$\binom{(1/3 + 2 \cdot \gamma/3) \cdot n}{3} > \left(\frac{1}{3} + \frac{\gamma}{2}\right)^3 \cdot \frac{n^3}{6} > \left(\frac{1}{27} + \frac{\gamma}{6}\right) \cdot \frac{n^3}{6}.$$

Furthermore,

$$f_4(H_{i+1}) \le f_4(H_i) \le f_4(G') \le \frac{1}{27} + 2 \cdot \frac{(\varepsilon_{\mathrm{STAB}})^2}{C} = \frac{1}{27} + 2 \cdot \frac{\gamma^2}{27}.$$

Our aim is to show that $f_4(H_i) - f_4(H_{i+1}) > \gamma/(3 \cdot n)$. Indeed, we have

$$
\begin{aligned}
f_4(H_i) - f_4(H_{i+1}) &> \frac{\left(\frac{1}{27} + \frac{\gamma}{6}\right) \cdot \frac{n^3}{6} + f_4(H_{i+1}) \cdot \binom{v(H_i)-1}{4} - f_4(H_{i+1}) \cdot \binom{v(H_i)}{4}}{\binom{v(H_i)}{4}} \\
&> \frac{\left(\frac{1}{27} + \frac{\gamma}{6}\right) \cdot \frac{n^3}{6} - f_4(H_{i+1}) \cdot \binom{v(H_i-1)}{3}}{\binom{v(H_i)}{4}} > \frac{4 \cdot \left(\frac{1}{27} + \frac{\gamma}{6} - f_4(H_{i+1})\right)}{n} \\
&\geq \frac{4}{n} \cdot \left(\frac{\gamma}{6} - \frac{2\gamma^2}{27}\right) > \frac{4}{n} \cdot \left(\frac{\gamma}{6} - \frac{2\gamma}{27}\right) \\
&> \frac{\gamma}{3n}.
\end{aligned}
$$

Since $n - \gamma \cdot n/3 \geq n/2 \geq n_{\text{ASYM}}$, Corollary 3.5 implies that $f_4\left(H_{\gamma \cdot n/3}\right) \geq 1/27 - (\varepsilon_{\text{STAB}})^2/C$. However,

$$
\frac{1}{27} + 2 \cdot \frac{(\varepsilon_{\text{STAB}})^2}{C} \geq f_4(G') = f_4(H_0) > f_4(H_{\gamma \cdot n/3}) + \frac{\gamma^2}{27} \geq \frac{1}{27} - \frac{(\varepsilon_{\text{STAB}})^2}{C} + \frac{(\varepsilon_{\text{STAB}})^2}{25 \times 9},
$$

which contradicts the choice of C. □

3.4 Extremal configurations

For a fixed integer k, we say a permutation $\tau \in S_n$ is *k-extremal* if $F_k(\tau) = F_k(n)$. Analogously, we say an n-vertex permutation graph G is *k-extremal* if $F_k(G) = F_k(n)$. In this section, we present a method for obtaining the exact description of k-extremal permutations for $k \geq 4$. Using the stability result from the previous section and the asymptotic result given by Theorem 3.4, we apply this method to the case $k = 4$. The same analysis will be then used for the cases $k = 5$ and $k = 6$ in Section 3.5 and Section 3.6, respectively.

We start with the following definition. We call $u \in V(G)$ a *clone* of $v \in V(G)$ if $N(u) = N(v)$. In particular, uv is not an edge of G. The next simple proposition shows that if we add a clone of a vertex x in a permutation graph G, then the new graph G' is still a permutation graph.

Proposition 3.10. *Let G be a permutation graph of order n. If we add a clone x' of some $x \in V(G)$ to form a new graph G' of order $n+1$, i.e., $N_{G'}(x') = N_G(x)$, then G' is still a permutation graph.*

Proof. The graph G comes from some permutation $\tau \in S_n$. Let k be the number in $[n]$ that corresponds to x, then we can construct a new permutation $\tau' \in S_{n+1}$ as follows:

$$\tau'(i) = \begin{cases} \tau(i) & \text{if } i \le k \text{ and } \tau(i) \le \tau(k) \\ \tau(i)+1 & \text{if } i \le k \text{ and } \tau(i) > \tau(k) \\ \tau(k)+1 & \text{if } i = k+1 \\ \tau(i-1) & \text{if } i > k \text{ and } \tau(i-1) < \tau(k) \\ \tau(i-1)+1 & \text{if } i > k \text{ and } \tau(i-1) \ge \tau(k). \end{cases}$$

The permutation graph of τ' is G' with $k+1$ corresponding to the new vertex x'. □

For an integer k, a graph G and a vertex $u \in V(G)$, let $I_k^u(G)$ be the number of independent sets of size k that contain u. Analogously, let $J_k^u(G)$ be the number of cliques of size k that contain u. We let $F_k^u(G) := I_k^u(G) + J_k^u(G)$ and $f_k^u(G) := F_k^u(G)/\binom{n-1}{k-1}$. An immediate corollary of the previous proposition is that in a k-extremal graph G, every two vertices contributes to $F_k(G)$ by roughly the same amount.

Corollary 3.11. *Fix an integer k and let G be a k-extremal graph of order n. For every two vertices u and v, we have $|F_k^u(G) - F_k^v(G)| \le \binom{n-2}{k-2}$. Therefore,*

$$f_k(G) - \frac{k}{n} < f_k^u(G) < f_k(G) + \frac{k}{n}$$

for every $u \in V(G)$.

Proof. Without loss of generality, $F_k^u(G) \ge F_k^v(G)$. Let G' be the n-vertex permutation graph obtained from G by removing u and adding a clone of v. It follows that

$$0 \le F(G') - F(G) \le F_k^v(G) - F_k^u(G) + \binom{n-2}{k-2}.$$

□

Let E_7 be the 7-vertex graph obtained by gluing three paths $x - y_i - z_i$, $i = 1, 2, 3$, at the common vertex x, see Figure 3.7. We continue our exposition by observing that there is no permutation $\tau \in S_7$ such that the permutation graph of τ would be isomorphic to E_7.

Observation 3.12. *The graph E_7 is not a permutation graph.*

Proof. Suppose there is a permutation $\tau_D \in S_7$ such that its permutation graph is isomorphic to E_7. Without loss of generality, $y_1 < y_2 < y_3$. Since the only neighbor of z_2 is y_2, it follows that $y_1 < z_2 < y_3$ and $\tau(y_1) < \tau(z_2) < \tau(y_3)$. However, the points $\{y_1, y_2, y_3\}$ form an independent set in E_7 and they are all connected to the point x, hence either $x > y_i$ and $\tau(x) < \tau(y_i)$ for every $i \in [3]$, or $x < y_i$

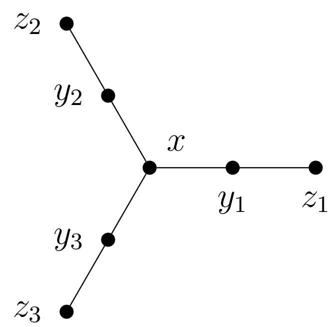


Figure 3.7: The graph E_7.

and $\tau(x) > \tau(y_i)$ for every $i \in [3]$. In the first case, we conclude that $x > z_2$ and $\tau(x) < \tau(z_2)$. In the latter case $x < z_2$ and $\tau(x) > \tau(z_2)$, a contradiction.

Alternatively, it is possible to check all the permutation graphs on 7 vertices, i.e., the set $\mathcal{F}_7$, and conclude that it does not contain neither E_7, nor the complement of E_7. $\square$

Now let us state a simple auxiliary statement.

Observation 3.13. *Fix an integer $k \geq 4$. The minimum value of the polynomial $p(x, y) = x^{k-1}+y^{k-1}+(k-1)! \cdot (1-x) \cdot (1-y)$ on $[0, 1]^2$ is equal to 1, and $p(x_0, y_0) = 1$ if and only if $\{x_0, y_0\} = \{0, 1\}$. Furthermore, if the value of $p(x_1, y_1)$ is close to 1, then $\{x_1, y_1\}$ is close $\{0, 1\}$.*

Proof. If $x \in \{0, 1\}$ is fixed, then $p(x, y)$ is clearly minimized when $y = 1 - x$, and symmetrically for $y \in \{0, 1\}$. Now suppose there are $x_0, y_0 \in (0, 1)$ such that $\frac{\partial p}{\partial x}(x_0, y_0) = 0$ and $\frac{\partial p}{\partial y}(x_0, y_0) = 0$. It follows that

$$(k-1) \cdot \left(x^{k-2} - y^{k-2}\right) = (x - y) \cdot (k-1)!,$$

and hence

$$\sum_{i=0}^{k-3} x^i \cdot y^{k-3-i} = (k-2)!\ .$$

However, since both x and y are less than one and $k \geq 4$, the sum on the left-hand side is equal to $(k - 2)$ summands, each of them less than one; a contradiction.

Since $p(x, y)$ is continuous, the second part of the statement follows from the compactness of $[0, 1]^2$. $\square$

The main result of this section is the following theorem.

Theorem 3.14. *For every integer $k \geq 4$ there exist a positive ε and an integer n_0 such that the following is true. If G is a k-extremal permutation graph on $n \geq n_0$ vertices that can be transformed to $T_{k-1}(n)$ or $\overline{T_{k-1}(n)}$ by adding and/or removing at most $\varepsilon\binom{n}{2}$ edges, then the vertices of G can be partitioned into either $k-1$ independent sets, or $k-1$ cliques.*

Note that combining Theorems 3.9 and 3.14 yields that every extremal permutation $\tau \in S_n$, where n is sufficiently large, does not contain either any increasing subsequence of length 4, or any decreasing subsequence of length 4. Therefore, using Theorem 3.2 we conclude that $\tau \in \mathcal{W}_4(n)$.

Proof of Theorem 3.14. Let $\gamma := 1/(100 \cdot k^k)$ be an auxiliary constant and fix the choice of ε small enough so that any solution (x, y) of the polynomial

$$p(x,y) := x^{k-1} + y^{k-1} + (k-1)! \cdot (1-x) \cdot (1-y)$$

that has value at most $1 + 20k^{2k} \cdot \varepsilon^{1/6}$ satisfy either $x \in [0,\gamma]$ and $y \in [1-\gamma, 1]$, or $x \in [1-\gamma, 1]$ and $y \in [0,\gamma]$. Such a choice of $\varepsilon > 0$ exists by Observation 3.13. Furthermore, we also assume that $\varepsilon < 1/k^{100k}$. Finally, let $n_0 := k^2/\varepsilon$.

Without loss of generality, we can modify $\varepsilon\binom{n}{2}$ pairs in $\binom{V(G)}{2}$ to obtain $\overline{T_{k-1}(n)}$. Our aim is to show that G can be partitioned into $k-1$ cliques. Since G is close to $\overline{T_{k-1}(n)}$, it follows that

$$\frac{1}{(k-1)^{k-1}} - \frac{k^2}{2} \cdot \varepsilon < f_k(G) < \frac{1}{(k-1)^{k-1}} + \frac{k^2}{2} \cdot \varepsilon,$$

and, by Corollary 3.11,

$$\frac{1}{(k-1)^{k-1}} - k^2 \cdot \varepsilon < f_k^v(G) < \frac{1}{(k-1)^{k-1}} + k^2 \cdot \varepsilon \tag{3.15}$$

for every $v \in V(G)$.

Let C be a graph on n vertices that is a union of $k-1$ disjoint cliques $A_1, A_2, \ldots, A_{k-1}$ such that the size of the symmetric difference $E(C)\Delta E(G)$ is as small as possible. In particular, $|E(C)\Delta E(G)| \leq \varepsilon\binom{n}{2}$. Furthermore, every clique A_i, where $i \in [k-1]$, has to have size between $\left(\frac{1}{k-1} - 2\varepsilon\right) \cdot n$ and $\left(\frac{1}{k-1} + 2\varepsilon\right) \cdot n$.

Fix a vertex $v \in V(G)$. We claim that there are at most $\gamma \cdot n$ edges e in $E(C)\Delta E(G)$ such that v is one of the endpoints of e. We call such edges v-wrong.

Consider the partition of the set $V(G)$ into parts $P_1, P_2, \ldots, P_{k-1}$ according to the cliques $A_1, A_2, \ldots, A_{k-1}$ of C, and let $d_i^v := e(v, P_i)/|P_i|$. Furthermore, let $\gamma' := 10 \cdot k^{k+1} \cdot \varepsilon^{1/6}$. By our choice of ε, it follows that $\gamma > \gamma'$. First, suppose for a contradiction there exist 3 distinct numbers $i, i', i'' \in [k-1]$ such that for every $j \in \{i, i', i''\}$, we have $\frac{\gamma'}{2} < d_j < 1 - \frac{\gamma'}{2}$. That means there are at least

$$\left(\frac{\gamma'}{2}\right)^6 \cdot \left(\frac{1}{k-1} - 2\varepsilon\right)^6 \cdot n^6 \geq \left(\frac{\gamma'}{2}\right)^6 \cdot \left(\frac{1}{(k-1)^{k-1}} - 12\varepsilon\right) \cdot n^6 \geq \varepsilon \cdot n^6$$

choices of $y_i, z_i \in P_i$, $y_{i'}, z_{i'} \in P_{i'}$, and $y_{i''}, z_{i''} \in P_{i''}$ such that $vy_j \in E(G)$ and $vz_j \notin E(G)$ for every $j \in \{i, i', i''\}$. However, at most $\varepsilon \cdot \binom{n}{2}\binom{n-2}{4} < \varepsilon \cdot n^6$ such choices of $y_i, y_{i'}, y_{i''}, z_i, z_{i'}, z_{i''}$ can contain an edge from $E(C)\Delta E(G)$. Therefore, G contains a copy of E_7, which contradicts Observation 3.12.

Without loss of generality, let $d_i^v \in [0,1] \setminus \left(\frac{\gamma'}{2}, 1 - \frac{\gamma'}{2}\right)$ for every $i \in [k-3]$. If there is an $i \in [k-3]$ such that $d_i^v \geq 1 - \frac{\gamma'}{2}$, then $d_j^v < \gamma'$ for every $j \in [k-1] \setminus \{i\}$. Indeed, otherwise $f_k^v(G)$ is at least

$$(1 - \frac{\gamma'}{2} + \gamma') \cdot \left(\frac{1}{(k-1)^{k-1}} - 2k \cdot \varepsilon\right) - \varepsilon \geq \frac{1}{(k-1)^{k-1}} + k^2 \cdot \varepsilon,$$

which contradicts (3.15). Therefore, we conclude that $v \in P_i$ (otherwise moving v to A_i would decrease $|E(G)\Delta E(C)|$), and hence the number of v-wrong edges is at most $\gamma' \cdot n < \gamma \cdot n$.

Now suppose $d_i^v \leq \gamma'/2$ for every $i \in [k-3]$. It follows that $f_k^v(G)$ is at least

$$(k-1)! \cdot \left(\frac{1}{(k-1)^{k-1}} - 2k \cdot \varepsilon\right) \cdot \left(1 - \frac{\gamma'}{2}\right)^{k-3} \cdot \left(1 - d_{k-2}^v\right) \cdot \left(1 - d_{k-1}^v\right) +$$
$$\left(\frac{1}{(k-1)^{k-1}} - 2k \cdot \varepsilon\right) \cdot \left(d_{k-2}^v\right)^{k-1} + \left(\frac{1}{(k-1)^{k-1}} - 2k \cdot \varepsilon\right) \cdot \left(d_{k-1}^v\right)^{k-1} - \varepsilon, \tag{3.16}$$

where the first summand corresponds to the independent sets of size k that contain v and one other vertex from each P_i, the next two summands correspond to the cliques of size k with all the other $k-1$ vertices inside the part P_{k-2} or P_{k-1}, and finally the summand $-\varepsilon$ comes from an upper bound for the $(k-1)$-sets that contain at least one of the edges from $E(G)\Delta E(C)$. It follows that (3.16) is at least

$$\frac{p(d_{k-2}^v, d_{k-1}^v)}{(k-1)^{k-1}} - 7 \cdot k! \cdot \varepsilon - \frac{(k-3) \cdot (k-1)! \cdot \gamma'}{2 \cdot (k-1)^{k-1}} > \frac{p(d_{k-2}^v, d_{k-1}^v)}{(k-1)^{k-1}} - \left(7 \cdot k! \cdot \varepsilon + \gamma'\right).$$

However, $f_k^v(G) - \frac{1}{(k-1)^{k-1}} < 2k^2 \cdot \varepsilon$, so our choice of the parameters imply that either $d_{k-2}^v \in [0, \gamma]$ and $d_{k-1}^v \in [1-\gamma, 1]$, or $d_{k-2}^v \in [1-\gamma, 1]$ and $d_{k-1}^v \in [0, \gamma]$. In the first case, v must be in P_{k-1} (otherwise moving v to A_{k-1} would decrease $|E(G)\Delta E(C)|$), symmetrically in the other case $v \in P_{k-2}$. Furthermore, the number of v-wrong edges is at most $\gamma \cdot n$.

Suppose there are two vertices $u \in P_i$ and $v \in P_i$ for some $i \in [k-1]$ so that $uv \notin E(G)$. The number of independent sets of size k in G that contain both u and v is then at least

$$\left((1-2\gamma) \cdot \frac{1-2\varepsilon}{k-1} \cdot n\right)^{k-2} - \varepsilon \cdot \binom{n}{k-2} > \left(\frac{1}{(k-1)^{k-2}} - 5k \cdot \gamma\right) \cdot n^{k-2}. \quad (3.17)$$

On the other hand, the number of cliques of size k in $G + uv$ that contain both u and v is at most

$$\binom{\frac{1+2\varepsilon}{k-1}}{k-2} + \binom{\gamma \cdot n}{k-2} < \left(\frac{1+2^k \cdot \varepsilon}{(k-1)^{k-2} \cdot (k-2)!} + \gamma\right) \cdot n^{k-2}. \quad (3.18)$$

By the choice of γ, the value of (3.17) - (3.18) is positive. But that means $F_k(G + uv) < F_k(G)$, a contradiction. We conclude that for every $i \in [k-1]$, the set of vertices P_i form a clique in G. $\square$

3.5 Monotone subsequences of length five

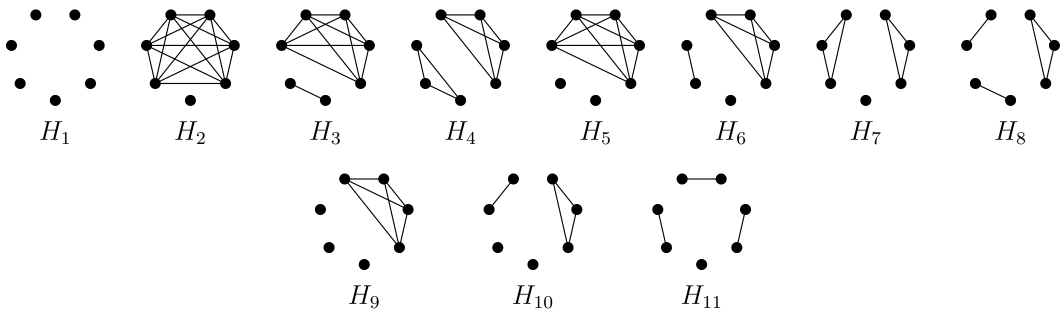


Figure 3.8: The set of graphs EXT_7^5.

In this section, we follow the approach presented in Sections 3.2, 3.3 and 3.4 in order to fully characterize all sufficiently large 5-extremal permutations.

Let EXT_7^5 be the set of all 7-vertex subgraphs up to a blind isomorphism that have a positive density in the conjectured extremal construction. It follows that $\mathrm{EXT}_7^5 = \{H_1, H_2, \ldots, H_{11}\} = \mathrm{EXT}_7^4 \cup \{H_9, H_{10}, H_{11}\}$. The set EXT_7^5 is depicted in Figure 3.8. Let $\mathcal{E}_7^5 := \mathcal{F}_7 \setminus \mathrm{EXT}_7^5$. The main theorem of this section is the following.

Theorem 3.19. *There exists a positive rational α such that the following is true. If $(\pi)_{n\in\mathbb{N}}$ is a convergent sequence of permutations and $\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$ is its limit,* *then*

$$\phi\left(K_5 - \alpha \cdot \sum_{H \in \mathcal{E}_7^5} H\right) \geq \frac{1}{256}.$$

Proof. As in the proof of Theorem 3.4, we let σ_1 to be the 3-vertex type with no edges and σ_2 the 3-vertex type with the edge bc. Additionally, let σ_3 and σ_4 be two specific types of order 5 given in Figure 3.9.

Again, we set $b_{\tau_{1,1}} := b_{\tau_{1,2}} := abc$, $b_{\tau_{2,1}} := b_{\tau_{2,4}} := abc$, and $b_{\tau_{2,2}} := b_{\tau_{2,3}} := bca$. The set $T(\sigma_3)$ contains 16 permutations $\tau_{3,j}$. For each $j \in [16]$, we define $b_{\tau_{3,j}}$ in the following way:

- for $\tau_{3,1} = 13542$, set $b_{\tau_{3,1}} := abcde$,
- for $\tau_{3,2} = 14352$, set $b_{\tau_{3,2}} := acdbe$,
- for $\tau_{3,3} = 15243$, set $b_{\tau_{3,3}} := aebdc$,
- for $\tau_{3,4} = 15324$, set $b_{\tau_{3,4}} := aedcb$,
- for $\tau_{3,5} = 24315$, set $b_{\tau_{3,5}} := bcdea$,
- for $\tau_{3,6} = 24531$, set $b_{\tau_{3,6}} := edcba$,
- for $\tau_{3,7} = 25341$, set $b_{\tau_{3,7}} := ebdca$,
- for $\tau_{3,8} = 32415$, set $b_{\tau_{3,8}} := cdbea$,
- for $\tau_{3,9} = 34251$, set $b_{\tau_{3,9}} = cdbea$,
- for $\tau_{3,10} = 41325$, set $b_{\tau_{3,10}} = ebdca$,
- for $\tau_{3,11} = 42135$, set $b_{\tau_{3,11}} = edcba$,
- for $\tau_{3,12} = 42351$, set $b_{\tau_{3,12}} = bcdea$,
- for $\tau_{3,13} = 51342$, set $b_{\tau_{3,13}} = aedcb$,
- for $\tau_{3,14} = 51423$, set $b_{\tau_{3,14}} = aebdc$,
- for $\tau_{3,15} = 52314$, set $b_{\tau_{3,15}} = acdbe$,
- for $\tau_{3,16} = 53124$, set $b_{\tau_{3,16}} = abcde$.

It holds that $|\mathcal{F}_6^{\sigma_3}| = 26$ and $|\mathcal{F}_7^{\sigma_3}| = 574$. Similarly, the set $T(\sigma_4)$ has size 12. For each $j \in [12]$, we set $b_{\tau_{4,j}}$ as follows:

- for $\tau_{4,1} = 14532$, set $b_{\tau_{4,1}} := abcde$,
- for $\tau_{4,2} = 15342$, set $b_{\tau_{4,2}} := adbce$,
- for $\tau_{4,3} = 15423$, set $b_{\tau_{4,3}} := aedbc$,
- for $\tau_{4,4} = 23541$, set $b_{\tau_{4,4}} := edcba$,
- for $\tau_{4,5} = 24351$, set $b_{\tau_{4,5}} := dcbea$,
- for $\tau_{4,6} = 32451$, set $b_{\tau_{4,6}} := cbdea$,
- for $\tau_{4,7} = 34215$, set $b_{\tau_{4,7}} := cbdea$,
- for $\tau_{4,8} = 42315$, set $b_{\tau_{4,8}} := dcbea$,
- for $\tau_{4,9} = 43125$, set $b_{\tau_{4,9}} := edcba$,
- for $\tau_{4,10} = 51243$, set $b_{\tau_{4,10}} := aedbc$,
- for $\tau_{4,11} = 51324$, set $b_{\tau_{4,11}} := adbce$,
- for $\tau_{4,12} = 52134$, set $b_{\tau_{4,12}} := abcde$.

In this case, $|\mathcal{F}_6^{\sigma_4}| = 28$ and $|\mathcal{F}_7^{\sigma_4}| = 624$.

Based on the semidefinite method presented in Section 1.2, an instance of the SDP was used to find 4 symmetric positive semidefinite matrices M_1, M_2, M_3 and M_4 with rational entries such that for every $\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$ we have

$$\phi\left(\sum_{i=1}^{4} [\![x_i^T M_i x_i]\!]_{\sigma_i}\right) \leq \phi\left(K_5 - \frac{1}{256} - \alpha \cdot \sum_{H \in \mathcal{E}_7^5} H\right),$$

where

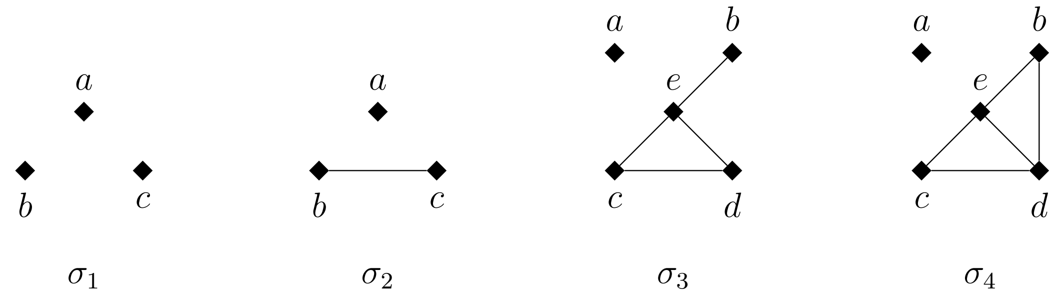


Figure 3.9: The types σ_1 through σ_4 used in the proof of Theorem 3.19.

- the vector $x_1 \in (\mathbb{R}\mathcal{F}_5^{\sigma_1})^{|\mathcal{F}_5^{\sigma_1}|}$ is the vector whose j-th coordinate is equal to the j-th element of the canonical base of $\mathbb{R}\mathcal{F}_5^{\sigma_1}$,
- the vector $x_2 \in (\mathbb{R}\mathcal{F}_5^{\sigma_2})^{|\mathcal{F}_5^{\sigma_2}|}$ is the vector whose j-th coordinate is equal to the j-th element of the canonical base of $\mathbb{R}\mathcal{F}_5^{\sigma_2}$,
- the vector $x_3 \in (\mathbb{R}\mathcal{F}_6^{\sigma_3})^{|\mathcal{F}_6^{\sigma_3}|}$ is the vector whose j-th coordinate is equal to the j-th element of the canonical base of $\mathbb{R}\mathcal{F}_6^{\sigma_3}$,
- the vector $x_4 \in (\mathbb{R}\mathcal{F}_6^{\sigma_4})^{|\mathcal{F}_6^{\sigma_4}|}$ is the vector whose j-th coordinate is equal to the j-th element of the canonical base of $\mathbb{R}\mathcal{F}_6^{\sigma_4}$, and
- $\alpha = 22961176619/6306641510400 \sim 0.00364$.

The left-hand side of the inequality above is non-negative by (1.2). □

As in the case of Theorem 3.4, the numerical values of the entries of the matrices M_1, M_2, M_3 and M_4 can be downloaded from the web page `http://honza.ucw.cz/phd/`. We also created a sage script called "thm_3_19-verify.sage", which can be used to verify the computations.

The methods presented in Section 3.3 can be straightforwardly adopted also to the case of monotone subsequences of length 5. Specifically, they yield the following stability result, which is an analogue of Theorem 3.9.

Theorem 3.20. *For every $\varepsilon_{\mathrm{STAB}} > 0$ there exist $\delta_{\mathrm{STAB}} > 0$ and n_{STAB} such that the following is true. If G is a permutation graph on $n_{\mathrm{STAB}} \geq n_0$ vertices with $f_5(G) \leq \frac{1}{256} + \delta_{\mathrm{STAB}}$, then G is isomorphic to either $T_4(n)$ or $\overline{T_4(n)}$ after adding and/or deleting at most $\varepsilon_{\mathrm{STAB}} \cdot \binom{n}{2}$ edges.*

The reasoning is essentially the same as for Theorem 3.9. For the sake of completeness, we give a proof in Appendix B.1.

The stability result together with Theorem 3.14 and Theorem 3.2 gives a complete characterization of 5-extremal permutations.

Theorem 3.21. *There exists an integer n_0 such that for every permutation $\tau \in S_n$, where $n \geq n_0$, we have $F_5(\tau) \geq F_5(\tau_5(n))$. Furthermore, if $F_5(\tau) = F_5(\tau_5(n))$, then $\tau \in \mathcal{W}_5(n)$.*

3.6 Monotone subsequences of length six

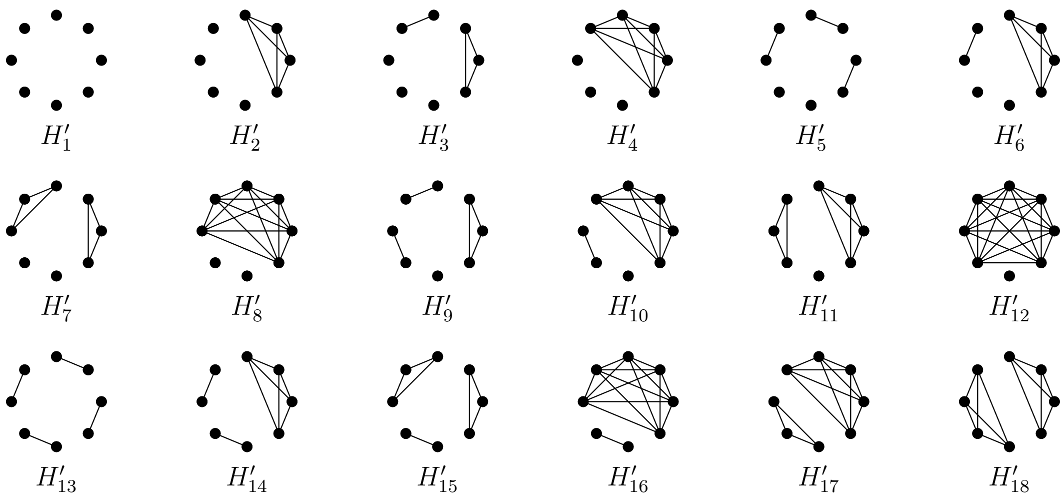


Figure 3.10: The set of graphs EXT_8^6.

In the last section of this chapter, we use our methods to characterize also all large 6-extremal permutations. Analogously to the previous cases, we let EXT_8^6 to be the set of all non-blindly-isomorphic 8-vertex graphs that have a positive density in the conjectured extremal example. It holds that $\left|\mathrm{EXT}_8^6\right| = 18$ and the graphs $H'_1, \ldots, H'_{18}$ are depicted in Figure 3.10. We also define $\mathcal{E}_8^6$ to be the complement of the set EXT_8^6, i.e., $\mathcal{E}_8^6 := \mathcal{F}_8 \setminus \mathrm{EXT}_8^6$.

We start with the main theorem of this section.

Theorem 3.22. *There exists a positive rational α such that the following is true. If $(\pi)_{n\in\mathbb{N}}$ is a convergent sequence of permutations and $\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$ is its limit,* *then*

$$\phi\left(K_6 - \alpha \cdot \sum_{H \in \mathcal{E}_8^6} H\right) \geq \frac{1}{3125}.$$

Proof. Let $\sigma_1, \dots, \sigma_{31}$ be the types depicted in Figure 3.11. Note that the type σ_1 is the only type (up to a blind isomorphism) of order 2, the types $\sigma_2, \dots, \sigma_7$ are all 6 the types of order 4, and the remaining types $\sigma_8, \dots, \sigma_{31}$ are 24 types of order 6. Note we do not use all the possible types of order 6 (there are 71 non-isomorphic types of order 6 in total) and the particular choice was suggested by a computer.

Going through the lists of all the permutations of sizes 2, 4, and 6 and their corresponding permutation graphs yields that the sets $T(\sigma_1), \dots, T(\sigma_{31})$ have the following sizes: 2, 2, 6, 8, 4, 2, 2, 2, 8, 8, 8, 8, 8, 8, 8, 8, 8, 4, 8, 8, 4, 8, 4, 8, 4, 8, 8, 8, 8, 4, and 2, respectively. Note that

$$\sum_{i=2}^{7} |T(\sigma_i)| = |S_4| = 24 \qquad \text{and} \qquad \sum_{i=8}^{31} |T(\sigma_i)| = 160.$$

The second sum is less than $|S_6| = 720$, since we do not use all the types of size six.

Now we need to choose the bijections $b_{\tau_{i,j}}$, where $i \in [31]$ and $\tau_{i,j} \in T(\sigma_i)$. The choice of $b_{\tau_{i,j}}$ has been made with a computer assistance and they bijections are given in Appendix B.3. It follows that

- the set $\mathcal{F}_5^{\sigma_1}$ has size 119, the set $\mathcal{F}_8^{\sigma_1}$ has size 99440,
- the sets $\mathcal{F}_6^{\sigma_i}$, where $i \in \{2, \dots, 7\}$, have sizes $122, 243, 191, 170, 191$, and 220, respectively,
- the sets $\mathcal{F}_8^{\sigma_i}$ have sizes 22361, 66186, 44698, 39286, 44698, and 51540, respectively,
- the sets $\mathcal{F}_7^{\sigma_i}$, where $i \in \{8, \dots, 31\}$, have sizes 22, 29, 34, 34, 32, 31, 31, 32, 34, 33, 33, 34, 33, 35, 34, 34, 32, 35, 35, 35, 35, 34, 34, and 35, respectively,
- the sets $\mathcal{F}_8^{\sigma_i}$ have sizes 436, 719, 904, 904, 822, 791, 791, 837, 904, 871, 871, 904, 871, 938, 904, 904, 837, 938, 938, 938, 938, 904, 904, 938, respectively.

We use the semidefinite method to find 31 symmetric positive semidefinite matrices $M_1, \dots, M_{31}$ such that the following is true. If $\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$, then

$$\phi\left(\sum_{i=1}^{31} [\![x_i^T M_i x_i]\!]_{\sigma_i}\right) \leq \phi\left(K_6 - \frac{1}{3125} - \alpha \cdot \sum_{H \in \mathcal{E}_8^6} H\right),$$

where

- the vector $x_1 \in (\mathbb{R}\mathcal{F}_5^{\sigma_1})^{|\mathcal{F}_5^{\sigma_1}|}$ is the vector whose j-th coordinate is equal to the j-th element of the canonical base of $\mathbb{R}\mathcal{F}_5^{\sigma_1}$,
- for $i \in \{2, \dots, 7\}$, the vector $x_i \in (\mathbb{R}\mathcal{F}_6^{\sigma_i})^{|\mathcal{F}_6^{\sigma_i}|}$ is the vector whose j-th coordinate is equal to the j-th element of the canonical base of $\mathbb{R}\mathcal{F}_6^{\sigma_i}$,
- for $i \in \{8, \dots, 31\}$, the vector $x_i \in (\mathbb{R}\mathcal{F}_7^{\sigma_i})^{|\mathcal{F}_7^{\sigma_i}|}$ is the vector whose j-th coordinate is equal to the j-th element of the canonical base of $\mathbb{R}\mathcal{F}_7^{\sigma_i}$, and
- $\alpha =$ 13271039154489448354112691216559213024258505962804294581188506 45991816217012355524712673099993094496421736848347313984284148255 1112122065075838596697425682443380032858294928639014516332583745304 406174382777990388793236389 / 2068666133260833193020736966223634816 713964175096859278237533746755570020517472993373482674882607444989 153783350542609132600136129827503488843545087803632723102183361388 90262647318509494337222405421715841786157274759168000000 $\sim 6 \cdot 10^{-5}$.

□

The numerical values of the entries of the matrices $M_1, \dots, M_{31}$ can be downloaded from the web page `http://honza.ucw.cz/phd/`. A sage script called “thm_3_22-verify.sage”, which is also available on the web page, can be used to verify our computations.

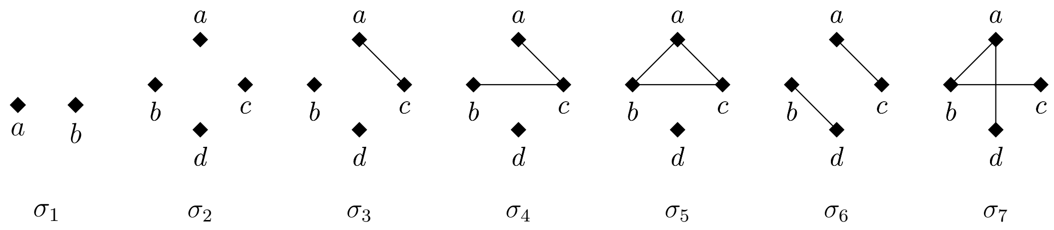
Analogously to the case of monotone subsequences of length 4 and 5, the methods from Section 3.3 yield the following stability result.

Theorem 3.23. *For every $\varepsilon_{\mathrm{STAB}} > 0$ there exist $\delta_{\mathrm{STAB}} > 0$ and n_{STAB} such that the following is true. If G is a permutation graph on $n_{\mathrm{STAB}} \geq n_0$ vertices with $f_6(G) \leq \frac{1}{3125} + \delta_{\mathrm{STAB}}$, then G is isomorphic to either $T_5(n)$ or $\overline{T_5(n)}$ after adding and/or deleting at most $\varepsilon_{\mathrm{STAB}} \cdot \binom{n}{2}$ edges.*

The proof of Theorem 3.23 is given in Appendix B.2.

As in the case of 5-extremal permutations, the stability result together with Theorem 3.14 and Theorem 3.2 immediately gives a complete characterization of sufficiently large 6-extremal permutations.

Theorem 3.24. *There exists an integer n_0 such that for every permutation $\tau \in S_n$, where $n \geq n_0$, we have $F_6(\tau) \geq F_6(\tau_6(n))$. Furthermore, if $F_6(\tau) = F_6(\tau_6(n))$, then $\tau \in \mathcal{W}_6(n)$.*



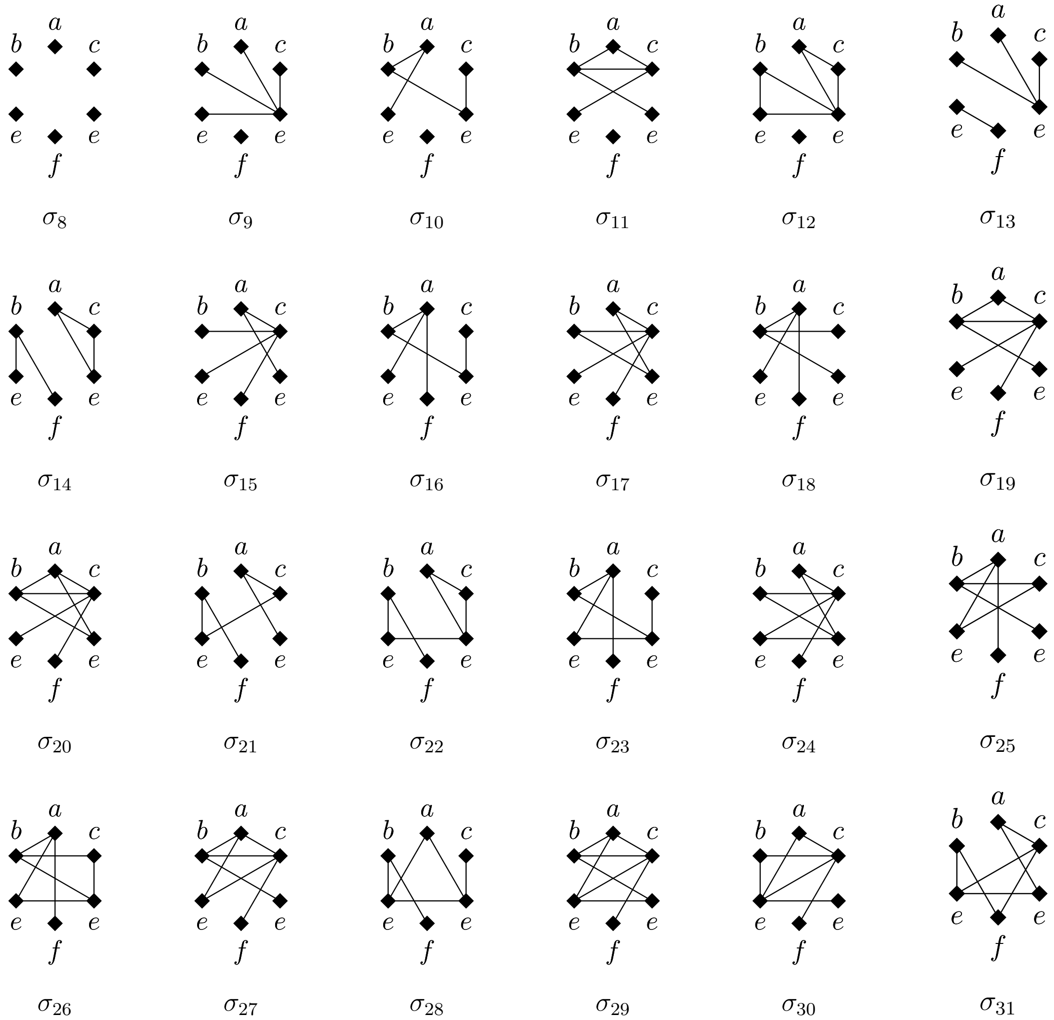


Figure 3.11: The types σ_1 through σ_{31} used in the proof of Theorem 3.22.

Chapter 4

Finitely forcible graphons

Recently, a theory of limits of combinatorial structures emerged and attracted substantial attention. In this chapter, we address limits of dense graphs, which is the most studied case. Its study was initiated in a series of papers by Borgs, Chayes, Lovász, Sós, Szegedy and Vesztergombi [12, 13, 14, 47, 49]. Dense graph limits are also closely related to the framework of flag algebras, which we discussed in Chapter 1.

As in Section 1.1, our object of study are convergent sequences of graphs, i.e., sequences, where the induced densities of every fixed graph as a subgraph in the graphs from the sequence converge. In Section 1.1, such a convergent sequence of graphs was assigned an algebra homomorphism from the set of all graphs to reals. The homomorphism naturally represents the limit subgraph densities. A convergent sequence of graphs can also be associated with an analytic object (graphon), which is a symmetric measurable function from the unit square $[0,1]^2$ to $[0,1]$. Note that a graphon also contains the information about the limit subgraph densities in the sequence.

In this chapter, we are concerned with finitely forcible graphons, i.e., those graphons that are uniquely determined (up to a natural notion of equivalence) by finitely many subgraph densities. Such graphons are related to uniqueness of extremal configurations in extremal graph theory as well as to other problems.

4.1 Dense graph limits

In this section, we introduce basic notions from the theory of dense graph limits. We follow a recent monograph on the topic by Lovász [46].

4.1.1 Graphons

Recall from Section 1.1 that $p(H, G_i)$ denotes the probability that random subset of $V(G_i)$ of size $v(H)$ induces a copy of H. Also recall a sequence of graphs $(G_i)_{i\in\mathbb{N}}$ is convergent if the subgraph density of every graph in G_i converges, i.e., the limit $\lim_{i\to\infty} p(H, G_i)$ exists for every graph $H \in \mathcal{F}$. As we discussed in Chapter 1, such a sequence naturally defines a limit object – a homomorphism $\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$. In the whole chapter, we study a different (and more analytical) representation of the limit object of a convergent sequence of graphs.

A *graphon* W is a symmetric measurable function from $[0,1]^2$ to $[0,1]$. Here, symmetric stands for the property that $W(x,y) = W(y,x)$ for every $x, y \in [0,1]$. Given a graphon W, we define a *W-random graph* of order k in the following way. First, we sample k random points $v_1, v_2, \ldots, v_k \in [0,1]$ uniformly and independently, and then we join the i-th and the j-th vertex by an edge with probability $W(v_i, v_j)$ (again independently of the other edges and on the sampling of the vertices $v_1, v_2, \ldots, v_k$). Since the points of $[0,1]$ play the role of vertices, we refer to them as to the *vertices of W*.

The *density* $p(H, W)$ of a graph H in a graphon W is equal to the probability that a W-random graph of order $v(H)$ is isomorphic to H. Clearly, the following holds:

$$p(H,W) = \frac{v(H)!}{|\operatorname{Aut}(H)|} \int\limits_{[0,1]^{v(H)}} \prod_{ij\in E(H)} W(v_i,v_j) \prod_{ij\notin E(H)} (1 - W(v_i,v_j))\ \mathrm{d}\lambda_{v(H)},$$

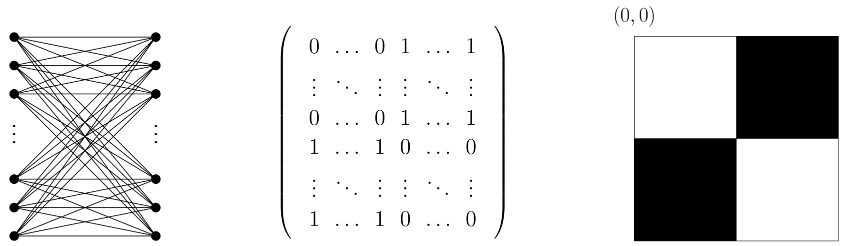
where $\mathrm{Aut}(H)$ is the automorphism group of H and $\lambda_{v(H)}$ is the uniform Borel measure on $[0,1]^{v(H)}$.

One of the key results of the theory of dense graph limits asserts that for every convergent sequence $(G_i)_{i\in\mathbb{N}}$ of graphs with the limit $\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$, there exists a graphon W such that

$$p(H, W) = \phi(H) = \lim_{i\to\infty} p(H, G_i)$$

for every graph $H \in \mathcal{F}$. For a proof, see, e.g., [46, Theorem 11.21]. Conversely, if W is a graphon, then the sequence of W-random graphs with increasing orders converges with probability one and its limit is W [46, Corollary 11.15].

An example of a graphon corresponding to the sequence of complete balanced bipartite graphs $K_{n,n}$ is depicted in Figure 4.1. Through of this chapter, we use the following convention when drawing graphons. The point $(0,0)$ is always in the top-left corner of the square $[0,1]^2$, black points represent the value one, gray points,



$$\begin{pmatrix} 0 & \dots & 0 & 1 & \dots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & \dots & 1 \\ 1 & \dots & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \dots & 1 & 0 & \dots & 0 \end{pmatrix}$$

Figure 4.1: A complete balanced bipartite graph $K_{n,n}$, its adjacency matrix, and a graphon representing the sequence $(K_{n,n})_{n\in\mathbb{N}}$.

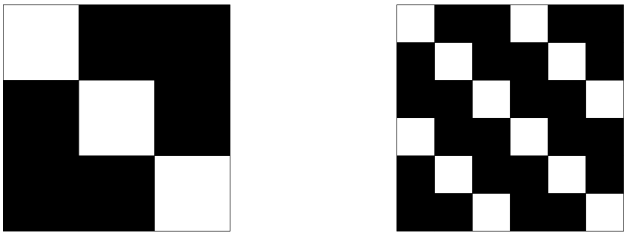


Figure 4.2: Two graphons weakly isopmorphic to each other. They both represent the limit of a sequence of complete balanced tripartite graphs.

depending on their shade, represent values between zero and one, and white points represent the value zero.

Two graphons W_1 and W_2 are *weakly isomorphic* if $p(H, W_1) = p(H, W_2)$ for every graph H. If $\varphi : [0,1] \to [0,1]$ is a measure preserving map, then the graphon $W^\varphi(x, y) := W(\varphi(x), \varphi(y))$ is always weakly isomorphic to W. The opposite is true in the following sense [15]: if two graphons W_1 and W_2 are weakly isomorphic, then there exist measure preserving maps $\varphi_1 : [0,1] \to [0,1]$ and $\varphi_2 : [0,1] \to [0,1]$ such that $W_1^{\varphi_1} = W_2^{\varphi_2}$ almost everywhere. An example of two different graphons, both representing the limit of the sequence of complete balanced tripartite graphs of increasing order, is depicted in Figure 4.2.

4.1.2 Finite forcibility

A graphon W is *finitely forcible* if there exist finitely many graphs $H_1, \dots, H_k$ such that every graphon W' satisfying $p(H_i, W) = p(H_i, W')$ for every $i \in [k]$ is weakly isomorphic to W. Such graphons are related to uniqueness of extremal configurations in extremal graph theory as well as to other problems. For example, the classical result of Chung, Graham and Wilson [17] asserting that a large graph is pseudorandom if and only if the non-induced densities of K_2 and C_4 are the same as in the Erdős-Rényi random graph $G_{n,1/2}$ can be cast in the language of graphons as follows: the graphon identically equal to 1/2 is uniquely determined by the non-induced densities of K_2 and C_4. In other words, it is finitely forcible. Another example that can be cast in the language of finite forcibility is the asymptotic version of the theorem of Turán [72]: there exists a unique graphon with edge density $\frac{r-1}{r}$ and zero density of K_{r+1}, namely the graphon corresponding to the sequence of Turán graphs $T_r(n)$.

A systematic study of finitely forcible graphons, which was started by Lovász and Szegedy in [50], was motivated by a possibility of a better understanding of extremal configurations for problems in extremal graph theory. For every finitely forcible graphon W, there exists an extremal graph theory problem such that W is its (unique) solution. Perhaps the most important open problem in the area of finite forcibility is whether there exists also a certain converse of this statement. Specifically, Lovász and Szegedy asked the following question:

Conjecture 4.1 ([50, Conjecture 7])**.** *Let k be an integer, let $F_1, \ldots F_k$ be k fixed graphs and let $a_1, \ldots, a_k$ be k fixed reals. If a finite set of constraints of the form $p(F_i, W) = a_i$ is satisfied by some graphon, then it is satisfied by a finitely forcible graphon.*

If an extremal problem has a unique solution then clearly the graphon corresponding to the solution is a finitely forcible graphon. However, the conjecture for a general extremal graph theory problem remains open.

Let us now describe some other known examples of finitely forcible graphons. A *stepfunction* is a graphon W such that its vertex-set $[0, 1]$ can be partitioned into finitely many measurable sets (also called parts) $A_1, A_2, \ldots A_k$ in such a way that for every $i, j \in [k]$ if $u, u' \in A_i$ and $v, v' \in A_j$, then $W(u, v) = W(u', v')$. Note that a stepfunction with one part is a graphon of a sequence of Erdős-Renyi graphs $G_{n,p}$ for a fixed $p \in [0, 1]$, and vice versa. An example of a stepfunction with four parts is depicted in Figure 4.3. The result of Chung, Graham, and Wilson was generalized by Lovász and Sós [51] who proved that any graphon that is a stepfunction is finitely forcible.

Next, a result of Diaconis, Homes, and Janson [20] asserts that the half graphon $W_\Delta(x, y)$ defined as $W_\Delta(x, y) = 1$ if $x + y \geq 1$, and $W_\Delta = 0$, otherwise, is finitely forcible; see Figure 4.4. This was probably the the first example of a finitely forcible graphon that is not a stepfunction. Further examples of finitely forcible graphons were found by Lovász and Szegedy in [50].

When dealing with finitely forcible graphons, we usually give a set of equality constraints that uniquely determines W instead of specifying the finitely many

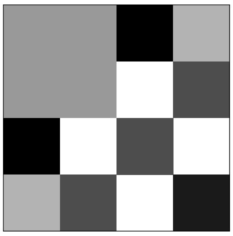


Figure 4.3: A stepfunction with four parts.

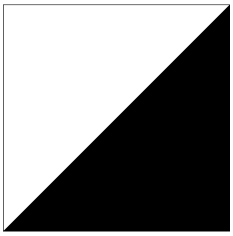


Figure 4.4: The half graphon W_Δ.

subgraphs that uniquely determine W. A *constraint* is an equality between two density expressions, where a *density expression* is recursively defined as follows: a real number or a graph $F \in \mathcal{F}$ are density expressions, and if D_1 and D_2 are two density expression, then the sum $D_1 + D_2$ and the product $D_1 \times D_2$ are also density expressions. The value of a density expression is the value obtained by substituting for every subgraph F its density in the graphon. Observe that if W is a unique (up to weak isomorphism) graphon that satisfies a finite set $\mathcal{C}$ of constraints, then it is finitely forcible. In particular, W is the unique (up to weak isomorphism) graphon with densities of subgraphs appearing in $\mathcal{C}$ equal to their densities in W. This holds since any graphon with these densities satisfies all constraints in $\mathcal{C}$ and thus it must be weakly isomorphic to W.

Following [48], each graphon W can be assigned a topological space of so-called typical vertices of W. To simplify our notation, if $A \subseteq [0,1]$ is measurable, we use $|A|$ to denote its measure. For $x \in [0,1]$, we define the neighborhood function of x as $f_x^W(y) := W(x,y)$. For an open set $A \subseteq L_1[0,1]$, we write A^W for $\left\{x \in [0,1],\, f_x^W \in A\right\}$. Let $T(W)$ be the set formed by the functions $f \in L_1[0,1]$ such that $\left|U^W\right| > 0$ for every neighborhood U of f. The set $T(W)$ inherits topology from $L_1[0,1]$. The vertices $x \in [0,1]$ with $f_x^W \in T(W)$ are called *typical vertices* of a graphon W. Almost every vertex of a graphon is typical [48].

4.1.3 Finite forcibility and non-compactness

If W is a finitely forcible graphon, how complicated can the space of its typical vertices $T(W)$ be? If the structure of the space $T(W)$ would be somewhat "simple", we might hope that together with a positive answer to Conjecture 4.1 we will be able to establish a general machinery for solving problems in extremal graph theory. Unfortunately, it turned out that there are finitely forcible graphons W where the structure of the space $T(W)$ is rather complicated.

The known examples of finitely forcible graphons led Lovász and Szegedy to make the following two conjectures.

Conjecture 4.2 ([50, Conjecture 9]). *If W is a finitely forcible graphon, then $T(W)$ is a compact space.*

They noted that they could not even prove that $T(W)$ had to be locally compact. The main result of this chapter is a construction of a finitely forcible graphon W_R, which we call Rademacher graphon, such that $T(W_R)$ fails to be locally compact. In particular, $T(W)$ is not compact.

Theorem 4.3. *There exists a finitely forcible graphon W_R such that the topological space $T(W_R)$ is not locally compact.*

The other conjecture of Lovász and Szegedy asks whether the dimension of $T(W)$ of a finitely forcible graphon W is always finite.

Conjecture 4.4 ([50, Conjecture 10]). *If W is a finitely forcible graphon, then $T(W)$ is finite dimensional.*

Note that Lovász and Szegedy stated in their paper that they intentionally did not specify the notion of dimension they had in mind. This conjecture has been recently disproved by Glebov, Klimošová and Král' [32]. Specifically, they constructed a finitely forcible graphon W such that the space $T(W)$ contains a subset A which is homeomorphic to $[0, 1]^\infty$.

4.2 Partitioned graphons

In this section, we introduce the notion of partitioned graphons. Some of the methods presented in this section are analogous to those used by Lovász and Sós in [51], and by Norin in [55]. In particular, they used similar types of arguments to specialize their constraints to parts of graphons they were forcing as we do in this section.

We adapt the notion of rooted densities from the flag algebra framework to graphons in order to extend the notion of density expressions to the rooted case. Recall from Chapter 1 that a type σ is a graph with a fixed labelling of its vertex-set and a σ-flag $F^\sigma \in \mathcal{F}^\sigma$ is a graph containing a fixed labelled embedding of σ. The subgraph induced by the labelled vertices is called the root of F^σ and the labelled vertices are also referred to as the rooted vertices of F^σ.

Fix a type σ, a σ-flag $F \in \mathcal{F}^\sigma$, and let $m = v(\sigma)$. Recall $\sigma^\emptyset$ is the unlabelled graph from $\mathcal{F}$ that corresponds to σ. For a graphon W with $p(\sigma^\emptyset, W) > 0$, we let the auxiliary function $c_\sigma : [0,1]^m \to [0,1]$ denote the probability that an m-tuple $(x_1, \ldots, x_m) \in [0,1]^m$ induces a copy of σ in W respecting the labeling of vertices of σ:

$$c_\sigma(x_1, \ldots, x_m) := \left(\prod_{ij \in E(\sigma)} W(x_i, x_j) \right) \cdot \left(\prod_{ij \notin E(\sigma)} (1 - W(x_i, x_j)) \right).$$

We next define a probability measure μ_σ on $[0,1]^m$. If $A \subseteq [0,1]^m$ is a Borel set, then

$$\mu_\sigma(A) := \frac{\int\limits_A c_\sigma(x_1, \ldots, x_m) \,\mathrm{d}\lambda_m}{\int\limits_{[0,1]^m} c_\sigma(x_1, \ldots, x_m) \,\mathrm{d}\lambda_m} = \frac{m!}{\left|\mathrm{Aut}\left(\sigma^\emptyset\right)\right|} \cdot \frac{\int_A c_\sigma(x_1, \ldots, x_m) \,\mathrm{d}\lambda_m}{p(\sigma^\emptyset, W)}.$$

Note that the probability measure μ_σ is an analogue of the probability distribution $\mathbf{P}^\sigma$ on $\mathrm{Hom}^+(\mathcal{A}^\sigma, \mathbb{R})$ from Section 1.1.

When $x_1, \ldots, x_m \in [0,1]$ are fixed, then the density of F with the rooted vertices being $x_1, \ldots, x_m$ is the probability that a random sample of the non-roots yields a copy of F conditioned on the event that the roots induce σ. Noticing that an automorphism of a σ-flag has all the rooted vertices as fixed points, we obtain that this is equal to

$$\frac{(v(F) - m)!}{|\,\mathrm{Aut}(F)|} \int\limits_{[0,1]^{v(F)-m}} \prod_{ij \in E(F) \setminus E(\sigma)} W(x_i, x_j) \prod_{ij \notin E(F) \cup \binom{V(\sigma)}{2}} (1 - W(x_i, x_j)) \,\mathrm{d}\lambda_{v(F)-m}.$$

Two flags F_1 and F_2 are *compatible* if the their types are isomorphic, i.e., both $F_1 \in \mathcal{F}^\sigma$ and $F_2 \in \mathcal{F}^\sigma$ for some type σ. A *rooted density expression* D is a density expression such that all flags that appear in it are mutually compatible rooted graphs. Note we will also speak about compatible rooted density expressions to emphasize that the flags in all of them are mutually compatible. First let $x_1, \ldots, x_m \in [0,1]$ be fixed. Analogously to the non-rooted case we use the notion

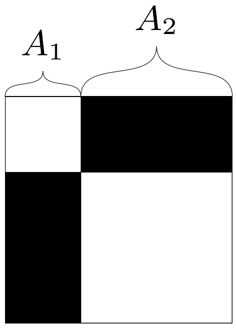


Figure 4.5: A partitioned graphon with two two parts A_1 and A_2. The measure of A_1 is 1/3 and the degree of each of its vertices is 2/3, and the measure of A_2 is 2/3 and the degree of each of its vertices is 1/3.

of density of $F \in \mathcal{F}^\sigma$ from the previous paragraph to determine the value of D with the rooted vertices being $x_1, \ldots, x_m$. For different choices of $x_1, \ldots, x_m$, we obtain different values. Finally, the *rooted density* of D is then a random variable determined by the choice of the rooted vertices according to the probability measure μ_σ.

We now consider a constraint such that both the left and the right hand sides D and D' are compatible rooted density expressions. Let σ be the corresponding type of all the flags in D and D'. Such a constraint should be interpreted to mean that it holds that $D - D' = 0$ with probability one, where the randomness comes from picking the root according to μ_σ. It follows from (1.1) that the expected value of a rooted density expression D with the root σ is equal to $\llbracket D \rrbracket_\sigma / p(\sigma^\emptyset, W)$, where $\llbracket D \rrbracket_\sigma$ is an ordinary density expression (in particular, it does not dependent of W). Observe that if D and D' are compatible rooted density expressions, both with the roots σ, then a graphon satisfies $D = D'$ if and only if it satisfies the (ordinary) constraint $\llbracket (D - D') \times (D - D') \rrbracket_\sigma = 0$. This allows us to express constraints involving rooted density expressions as ordinary constraints, hence we will not distinguish between the two types of constraints in what follows.

A *degree* of a vertex $x \in [0,1]$ of a graphon W is equal to

$$\int_{[0,1]} W(x,y)\,\mathrm{d}y = \int_{[0,1]} f_x^W(y)\,\mathrm{d}y\ .$$

Note that the degree is well-defined for almost every vertex of W. A graphon W is *partitioned* if there exist $k \in \mathbb{N}$, positive reals $a_1, \ldots, a_k$ with $\sum_i a_i = 1$ and distinct reals $d_1, \ldots, d_k \in [0,1]$ such that the set of vertices of W with degree d_i has measure a_i. An example of a partitioned graphon with two parts is depicted in Figure 4.5. We will often speak about partitioned graphons when having in mind fixed values of k, $a_1, \ldots, a_k$, and $d_1, \ldots, d_k$. Being a partitioned graphon can be finitely forced as shown in the next lemma.

Lemma 4.5. *Let k be an integer, $a_1, \ldots, a_k$ positive real numbers summing to one, and $d_1,\ldots,d_k$ distinct reals between zero and one. There exists a finite set of constraints $\mathcal{C}$ such that any graphon W satisfying $\mathcal{C}$ also satisfies the following:*

The set of vertices of W with degree d_i has measure a_i.

In other words, such W must be a partitioned graphon with parts of sizes $a_1, \ldots, a_k$ and degrees $d_1, \ldots, d_k$.

Proof. The desired property of being a partitioned graphon with the given choice of parameters is forced by the following set of constraints:

$$\prod_{i=1}^{k}(e^1 - d_i) = 0 \text{ , and}$$

$$\left[\!\!\left[\prod_{i=1,\, i\neq j}^{k} (e^1 - d_i) \right]\!\!\right]_1 = a_j \prod_{i=1,\, i\neq j}^{k} (d_j - d_i) \text{ for every } j,\ 1 \leq j \leq k,$$

where "1" denotes the 1-vertex type and e^1 is an edge with one rooted and one non-rooted vertex. The first constraint says that the degree of almost every vertex is equal to one of the numbers $d_1, \ldots, d_k$. For $j \leq k$, the left hand side of the second constraint before applying the $[\![\cdot]\!]$-operator is non-zero only if the degree of the rooted vertex is d_j, assuming the degree is one of $d_1, \ldots, d_k$. Hence, the left hand side is equal to

$$\prod_{i=1,i\neq j}^{k} (d_j - d_i)$$

in that case. Therefore, the measure of vertices of degree d_j is forced to be a_j. □

Assume that W is a partitioned graphon. We write A_i for the set of vertices of degree d_i for i, $1 \leq i \leq k$ and identify A_i with the interval $[0, a_i)$ (note that the measure of A_i is a_i). This will be convenient when defining partitioned graphons. For example, we can use the following when defining a graphon W: $W(x, y) = 1$ if $x \in A_1$, $y \in A_2$ and $x \geq y$.

A graph H is *decorated* if its vertices are labelled with parts $A_1, \ldots, A_k$. The density of a decorated graph H in a partitioned graphon W is the probability that randomly chosen $v(H)$ vertices induce a subgraph isomorphic to H with its vertices

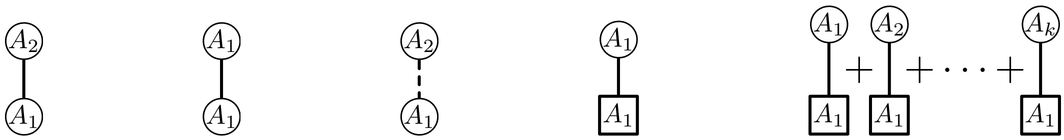


Figure 4.6: Examples of decorated constraints.

contained in the parts corresponding to the labels. For example, if H is an edge with vertices decorated with parts A_1 and A_2, then the density of H is the density of edges between the parts A_1 and A_2, i.e.,

$$p(H,W) = \int_{A_1}\int_{A_2} W(x,y)\ \mathrm{d}x\ \mathrm{d}y\ .$$

If W is the graphon depicted in Figure 4.5, then $p(H,W) = 4/9$.

Similarly as in the case of non-decorated graphs, we define rooted decorated subgraphs. A constraint that uses (rooted or non-rooted) decorated subgraphs is referred to as decorated. In this chapter, we use the following convention for drawing graphs in density expressions: edges of graphs are always drawn solid, non-edges dashed, and if two vertices are not joined, then the picture represents the sum over both possibilities. If a graph contains some roots, the roots are depicted by square vertices, and the non-root vertices by circles. If there are more roots from the same part of a graphon, then the squares are rotated to distinguish the roots. If a graph is decorated, then the decorations of its vertices are always drawn inside their circles or squares. See Figure 4.6 for the following five examples of decorated expressions:

- the first expression from the left denotes the edge density between the parts A_1 and A_2,
- the next one denotes the edge density inside the part A_1 multiplied by $|A_1|^2$,
- the third one is the non-edge density between the parts A_1 and A_2,
- the fourth one corresponds to the degree of a fixed vertex from A_1 inside the part A_1 multiplied by $|A_1|$, and
- the last expression is equal to the degree of a fixed vertex from A_1 inside the entire partitioned graphon (assuming the graphon has k parts).

The next lemma shows that decorated constraints are not more powerful than non-decorated ones, and therefore they can be used to show that a graphon is

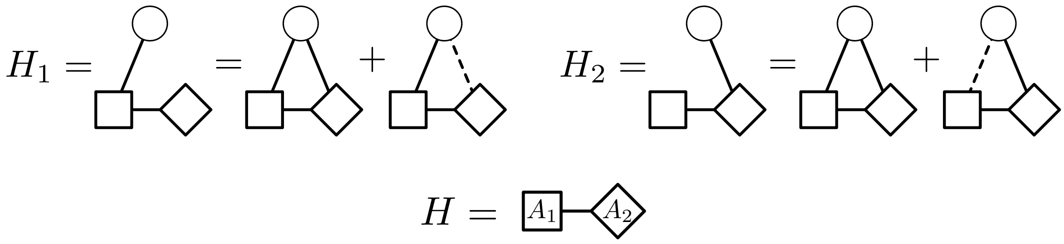


Figure 4.7: The graphs H_1 and H_2 from the proof of Lemma 4.6 if H is an edge with roots decorated with A_1 and A_2.

finitely forcible. We will always apply this lemma after forcing a graphon W to be partitioned using Lemma 4.5.

Lemma 4.6. *Let k be an integer, $a_1, \dots, a_k$ positive real numbers summing to one, and $d_1, \dots, d_k$ distinct reals between zero and one. If W is a partitioned graphon with k parts formed by vertices of degree d_i and measure a_i each, then a decorated (rooted or non-rooted) constraint can be expressed as a non-decorated one. In other words, W satisfies the decorated constraint if and only if it satisfies the non-decorated constraint.*

Proof. By the argument analogous to the non-decorated case, it is enough to show that the density of a non-rooted decorated subgraph can be expressed as a combination of densities of non-decorated subgraphs. Fix a non-rooted decorated subgraph H with vertices $v_1, \dots, v_n$ such that v_i is labelled with a part A_{ℓ_i}. Let H^U be the ordinary subgraph corresponding to H after removing the decorations, σ the type corresponding to H^U labelled with $1, \dots, n$ (the vertex v_i has label i), and H_i the sum of all σ-flags on $n+1$ vertices where the only non-rooted vertex is always adjacent to v_i (an example is given in Figure 4.7). We claim that the proportion of copies of H^U that have their vertices inside the parts according to the decoration of H is equal to

$$\int_{\mu_\sigma} \prod_{i=1}^{n} \prod_{j=1, j\neq \ell_i}^{k} \frac{H_i - d_j}{d_{\ell_i} - d_j} ,$$

hence the density of H is equal to

$$\frac{v(H)!}{|\operatorname{Aut}(H)|} \times \left[\!\!\left[\prod_{i=1}^{n} \prod_{j=1, j\neq \ell_i}^{k} \frac{H_i - d_j}{d_{\ell_i} - d_j} \right]\!\!\right]_\sigma . \tag{4.7}$$

Indeed, if the n rooted vertices are chosen on a copy of H^U such that the i-th rooted vertex is not from A_{ℓ_i}, then the second product of the above expression is zero. Otherwise, the second product is one. Hence the value of (4.7) is exactly the probability that randomly chosen n vertices induce a labelled copy of H^U such that the i-th vertex belong to A_{ℓ_i}. □

Since a decorated constraint can be expressed by a non-decorated one, we will not distinguish between decorated and non-decorated constraints in what follows.

We finish this section with two lemmas that are straightforward corollaries of Lemma 4.6. The first one says that we can finitely force a copy of a finitely forcible graphon inside one of the parts of a partitioned graphon.

Lemma 4.8. *Let W_0 be a finitely forcible graphon. Then for every choice of $k \in \mathbb{N}$, positive reals $a_1, \ldots, a_k$ summing to one, distinct reals $d_1, \ldots, d_k$ between zero and one, and $\ell \leq k$, there exists a finite set of constraints $\mathcal{C}$ such that the graphon induced by the ℓ-th part of every graphon W that is a partitioned graphon with k parts $A_1, \ldots, A_k$ of measures $a_1, \ldots, a_k$ and degrees $d_1, \ldots, d_k$, respectively, and that satisfies $\mathcal{C}$ is weakly isomorphic to W_0. More precisely, there exist measure preserving maps φ and φ' from A_ℓ to itself such that*

$$W_0\left(\frac{\varphi(x)}{|A_\ell|}, \frac{\varphi(y)}{|A_\ell|}\right) = W\left(\varphi'(x), \varphi'(y)\right)$$

for almost every $x, y \in A_\ell$.

Proof. Assume that W_0 is forced by some m constraints of the form $p(H_i, W) = d_i$, where $i \in [m]$ and $H_i \in \mathcal{F}$. The set $\mathcal{C}$ is then formed by constraints of the form

$$p(H_i', W) = a_\ell^{v(H_i)} \cdot d_i \ ,$$

where H_i' is the graph H_i with all vertices decorated with A_ℓ. □

The second lemma asserts finite forcibility of pseudorandom bipartite graphs between different parts of a partitioned graphon.

Lemma 4.9. *For every choice of $k \in \mathbb{N}$, positive reals $a_1, \ldots, a_k$ summing to one, distinct reals $d_1, \ldots, d_k$ between zero and one, $\ell, \ell' \leq k$, $\ell \neq \ell'$, and $p \in [0,1]$, there exists a finite set of constraints $\mathcal{C}$ such that every graphon W that is a partitioned graphon with k parts $A_1, \ldots, A_k$ of measures $a_1, \ldots, a_k$ and degrees $d_1, \ldots, d_k$, respectively, and that satisfies $\mathcal{C}$ also satisfies that $W(x, y) = p$ for almost every $x \in A_\ell$ and $y \in A_{\ell'}$.*

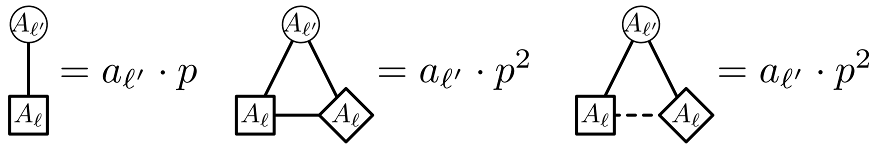


Figure 4.8: The constraints used in the proof of Lemma 4.9.

Proof. Let H be a rooted edge with the root decorated with A_ℓ and the non-root decorated with $A_{\ell'}$, let H_1 be a triangle with two roots such that the roots are decorated with A_ℓ and the non-root with $A_{\ell'}$, and let H_2 be a cherry (a path on three vertices) with two roots on its non-edge such that the roots are decorated with A_ℓ and the non-root with $A_{\ell'}$. The set $\mathcal{C}$ is formed by three constraints: $H = p$, $H_1 = p^2$, and $H_2 = p^2$ (also see Figure 4.8). These constraints imply that

$$\int_{A_{\ell'}} W(x,y)\ \mathrm{d}y = a_{\ell'} \cdot p \qquad \text{and} \qquad \int_{A_{\ell'}} W(x,y) \cdot W(x',y)\ \mathrm{d}y = a_{\ell'} \cdot p^2$$

for almost every $x, x' \in A_\ell$. Following the reasoning given in [50, proof of Lemma 3.3], the second equation implies that

$$\int_{A_{\ell'}} W^2(x,y)\ \mathrm{d}y = a_{\ell'} \cdot p^2$$

for almost every $x \in A_\ell$. Cauchy-Schwarz's inequality yields that $W(x,y) = p$ for almost every $x \in A_\ell$ and $y \in A_{\ell'}$. □

4.3 Rademacher graphon

In this section, we introduce the graphon W_R which we refer to as *Rademacher graphon*. The name comes from the fact that the adjacencies between its parts A and C resemble the Rademacher system of functions on the unit interval. Note that such adjacencies also appear in [46, Example 13.30] as an example of a graphon with non-compact space of typical vertices.

The graphon W_R has eight parts and we use A, A', B, B', B'', C, C' and D to denote the parts. All the parts except for C have the same measure $a := 1/9$; the measure of C is $2a = 2/9$.

For $x \in [0,1)$, let us denote by $\langle x \rangle$ the smallest integer k such that $x+2^{-k} < 1$. The graphon W_R is then defined as follows (also see Figure 4.9). Note that whenever

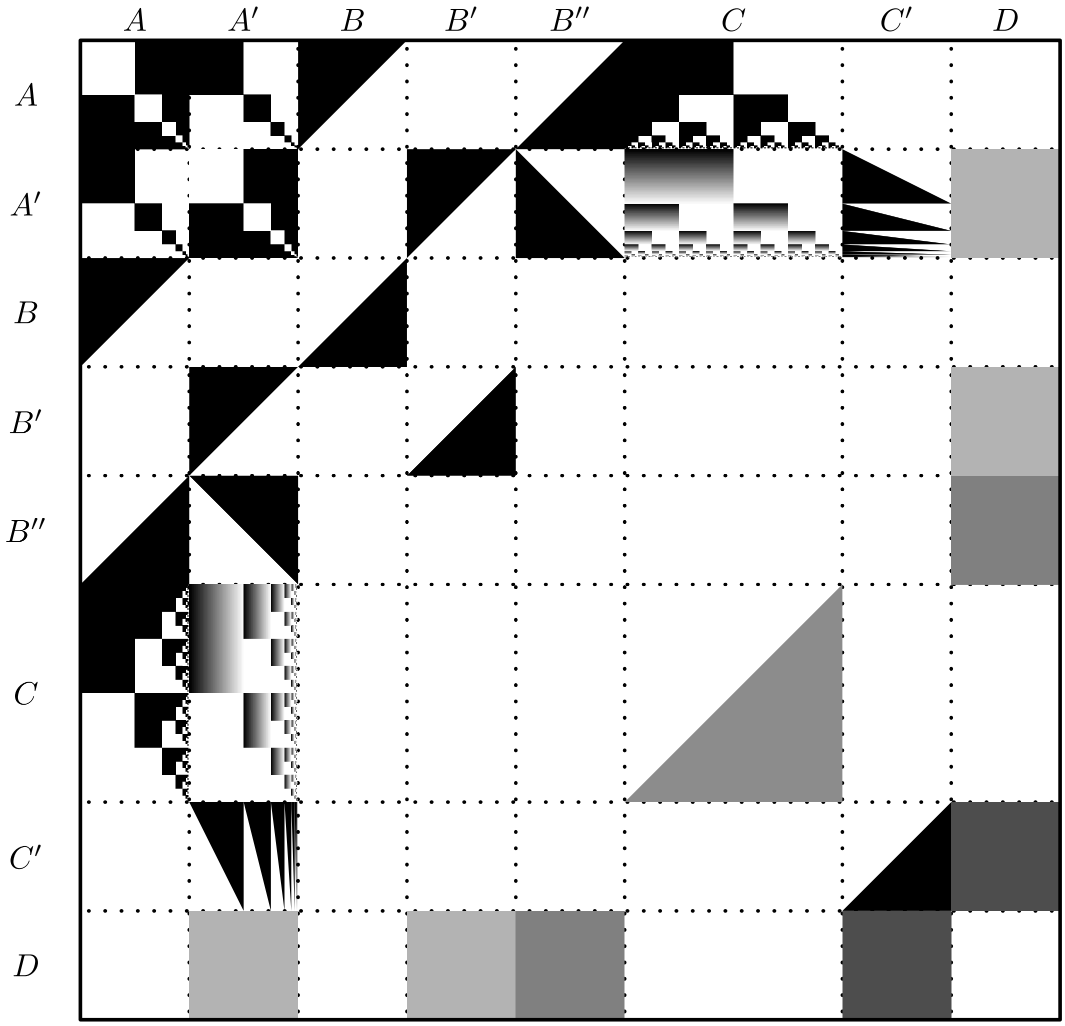


Figure 4.9: Rademacher graphon W_R.

Part	A	A'	B	B'	B''	C	C'	D
Degree	$3a$	$3.2a$	a	$1.2a$	$1.4a$	$1.5a$	$1.8a$	$1.6a$
	$1/3$	$16/45$	$1/9$	$2/15$	$7/45$	$1/6$	$1/5$	$8/45$

Table 4.1: The degrees of vertices in the nine parts of Rademacher graphon W_R.

we prescribe the value of $W_R(x,y)$ for some pair $(x,y) \in [0,1]^2$, we implicitly assume that we also defined the value $W_R(y,x) := W_R(x,y)$. Let x and y be two vertices of W_R. The value $W_R(x,y)$ is equal to one in the following cases:

- $x, y \in A$ and $\langle x/a \rangle \neq \langle y/a \rangle$,
- $x, y \in A'$ and $\langle x/a \rangle \neq \langle y/a \rangle$,
- $x \in A$, $y \in A'$ and $\langle x/a \rangle = \langle y/a \rangle$,
- $x \in A$, $y \in B$ and $x + y \leq a$,
- $x \in A$, $y \in B''$ and $x + y \geq a$,
- $x \in A'$, $y \in B'$ and $x + y \leq a$,
- $x \in A'$, $y \in B''$ and $y \leq x$,
- $x, y \in B$ and $x + y \geq a$,
- $x, y \in B'$ and $x + y \geq a$,
- $x, y \in C'$ and $x + y \geq a$,
- $x \in A$, $y \in C$ and $\left\lfloor \frac{y}{2a} \cdot 2^{\langle x/a \rangle} \right\rfloor$ is even, and
- $x \in A'$, $y \in C'$ and $(1 - 2^{-\langle x/a \rangle} - x/a) \cdot 2^{\langle x/a \rangle} + y/a \leq 1$.

If $x \in A'$, $y \in C$ and $\left\lfloor \frac{y}{2a} \cdot 2^{\langle x/a \rangle} \right\rfloor$ is even, then

$$W_R(x,y) := \left(1 - 2^{-\langle x/a \rangle} - x/a\right) \cdot 2^{\langle x/a \rangle}.$$

For $x, y \in C$ such that $x + y \geq 2a$, the value $W_R(x,y)$ is equal to $3/4$. If $y \in D$, then

$$W_R(x,y) := \begin{cases} 0.2 & \text{if } x \in A' \text{ or } x \in B', \\ 0.4 & \text{if } x \in B'', \text{ and} \\ 0.8 & \text{if } x \in C'. \end{cases}$$

Finally, $W_R(x,y) := 0$ if neither (x,y) nor the symmetric pair fall in any of the described cases.

The degrees of vertices in the eight parts of Rademacher graphon W_R are routine to compute and they are given in Table 4.1.

We finish this section with establishing that Rademacher graphon, assuming its finite forcibility, yields Theorem 4.3.

Proposition 4.10. *The topological space $T(W_R)$ is not locally compact.*

Proof. We understand the interval $[0,1]$ to be partitioned by the intervals A, A', B,

B', B'', C, C' and D. Let $g : [0,1] \to [0,1]$ be the function defined as follows:

$$g(x) := \begin{cases} 1 & \text{if } x \in A' \cup B'' \cup C', \\ 0.2 & \text{if } x \in D, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Further, let $g_{i,\delta} : [0,1] \to [0,1]$ for $i \in \mathbb{N}$ and $\delta \in [0,1]$ be defined as follows:

$$g_{i,\delta}(x) := \begin{cases} 1 & \text{if } x \in A \text{ and } \langle x/a \rangle = i, \\ 1 & \text{if } x \in A' \text{ and } \langle x/a \rangle \neq i, \\ 1 & \text{if } x \in B' \text{ and } x \leq (1+\delta) \cdot 2^{-i}, \\ 1 & \text{if } x \in B'' \text{ and } x \leq 1 - (1+\delta) \cdot 2^{-i}, \\ \delta & \text{if } x \in C \text{ and } \lfloor 2^i \cdot x/2a \rfloor \text{ is even,} \\ 1 & \text{if } x \in C' \text{ and } x/a \leq 1 - \delta, \\ 0.2 & \text{if } x \in D, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $W_R\big(x, 2/9 - (1+\delta) \cdot 2^{-i}/9\big) = g_{i,\delta}(x)$ for every $i \in \mathbb{N}$, $\delta \in (0,1)$, and $x \in [0,1]$. It follows that

$$\left\|g_{i,\delta} - g\right\|_1 = \frac{1}{9} \cdot \big(2 \cdot 2^{-i} + 2 \cdot (1+\delta) \cdot 2^{-i} + 2 \cdot \delta\big) = \frac{(4+2\delta) \cdot 2^{-i} + 2\delta}{9},$$

and

$$\left\|g_{i,\delta} - g_{i',\delta'}\right\|_1 \geq \int_C \left|g_{i,\delta}(x) - g_{i',\delta'}(x)\right| \mathrm{d}x \geq \frac{\delta + \delta'}{18} \qquad \text{for } i \neq i'.$$

Hence, since $g_{i,\delta} \in T(W_R)$ for every $i \in \mathbb{N}$ and $\delta \in (0,1)$, we obtain that $g \in T(W_R)$. However, for every $\varepsilon > 0$, all the functions $g_{i,\varepsilon}$ with $i > \log_2 \varepsilon^{-1}$ are at L_1-distance at most ε from g and the L_1-distance between any pair of them is at least $\varepsilon/9$. We conclude that no neighborhood of g in $T(W)$ is compact. □

4.4 Forcing the graphon

In this section, we prove that Rademacher graphon W_R defined in the previous section is finitely forcible. We first describe the set of constraints $\mathcal{C}_R$ we use to force a graphon to be weakly isomorphic to W_R. We give names to the different kinds of the constraints to refer to them in our exposition.

- The *partition constraints* forcing the existence of eight parts of sizes as in W_R and with vertex degrees as in W_R (the existence of such constraints follows

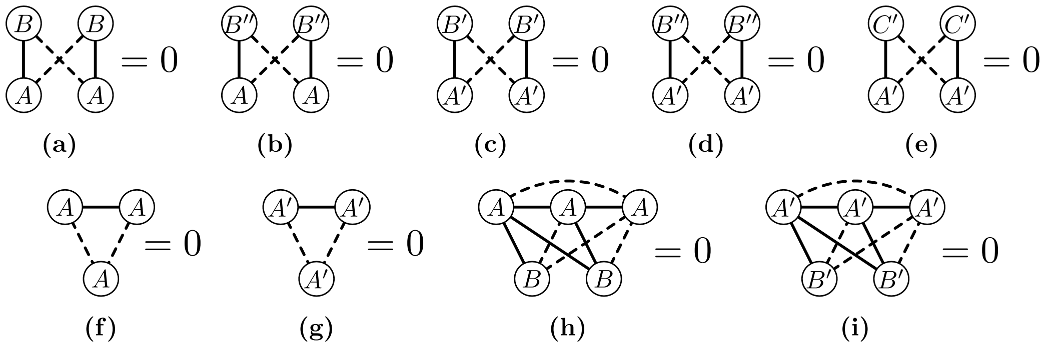


Figure 4.10: The monotonicity constraints.

from Lemma 4.5),

- the *zero constraints* setting the edge density inside B'' and D to zero as well as setting the edge density between the following pairs of parts to zero: A and C', A and D, A' and B, B and B', B and B'', B and C, B and C', B and D, B' and B'', B' and C, B' and C', B'' and C, B'' and C', and C and D,

- the *triangular constraints* forcing the half graphons on B, B', C, and C' with densities 1, 1, 1 and 3/4 (see Lemma 4.8 and [50, Corollaries 3.15 and 5.2] for their existence), respectively,

- the *pseudorandom constraints* forcing the pseudorandom bipartite graph between D and the parts A', B', B'', and C' with densities 0.2, 0.2, 0.4, and 0.8, respectively (see Lemma 4.9 for their existence),

- the *monotonicity constraints* depicted in Figure 4.10,

- the *split constraints* depicted in Figure 4.11,

- the *infinitary constraints* depicted in Figure 4.12, and

- the *orthogonality constraints* depicted in Figure 4.13.

The existence of the corresponding monotonicity, split, infinitary, and orthogonality constraints as ordinary constraints follows from Lemma 4.6. Also note that the first five monotonicity constraints imply that the graphon has values zero and one almost everywhere between the parts A and B, A and B'', A' and B', A' and B'', and A' and C' (see [50, Lemma 3.3] for further details).

Before we proceed to the proof of the main theorem of this section, let us recall a useful proposition for one-variable measurable functions called Monotone Reordering Theorem (see, e.g., [46, Proposition A.19]).

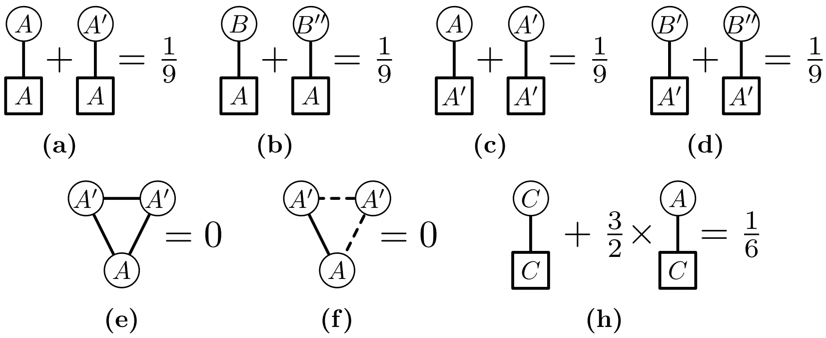


Figure 4.11: The split constraints.

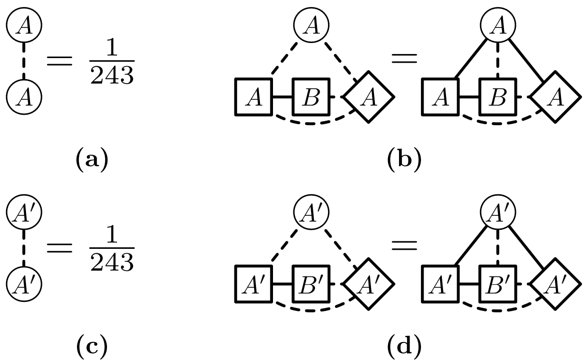


Figure 4.12: The infinitary constraints.

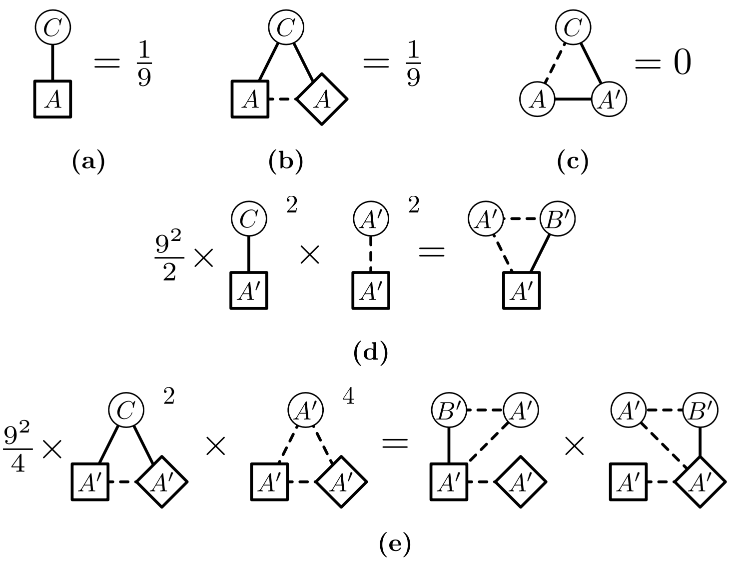


Figure 4.13: The orthogonality constraints.

Proposition 4.11 (Monotone Reordering Theorem)**.** *For every measurable function $f : [0,1] \to [0,\infty)$ there exist a monotone decreasing measurable function $h : [0,1] \to [0,\infty)$ and a measure preserving map $\phi : [0,1] \to [0,1]$ such that $f = \phi \circ h$. The function h is uniquely determined up to a set of measure zero.*

We are now ready to show that Rademacher graphon is finitely forcible.

Theorem 4.12. *If W is a graphon satisfying all constraints in $\mathcal{C}_R$, then there exist measure preserving maps $\varphi, \psi : [0,1] \to [0,1]$ such that W^φ and W_R^ψ are equal almost everywhere.*

Proof. Since W satisfies the partition constraints contained in $\mathcal{C}_R$, Lemma 4.5 yields that the interval $[0,1]$ can be partitioned into eight parts all but one having measure $1/9$ and the remaining one with measure $2/9$ such that almost all vertices in the parts have degrees as those in the corresponding parts of W_R. In particular, there exists a measure preserving map $\varphi : [0,1] \to [0,1]$ such that the subintervals of $[0,1]$ corresponding to the parts of W_R are mapped to the corresponding parts of W. From now on, we use A, A', B, B', B'', C, C', and D for the subintervals of $[0,1]$ corresponding to the parts.

We next construct a measure preserving map ψ consisting of measure preserving maps on the intervals A, A', B, B', B'', C and C'. We choose these maps such that there exist decreasing functions $f_A : A \to [0,1]$ and $f_{A'} : A' \to [0,1]$, and increasing functions $f_B : B \to [0,1]$, $f_{B'} : B' \to [0,1]$, $f_{B''} : B'' \to [0,1]$, $f_C : C \to [0,1]$ and $f_{C'} : C' \to [0,1]$ such that the following holds almost everywhere (the existence of such maps and functions follows from Monotone Reordering Theorem):

$$\begin{array}{llll}
\forall x \in A & f_A(\psi(x)) = \int\limits_B W^\varphi(x,y)\,\mathrm{d}y & \forall x \in A' & f_{A'}(\psi(x)) = \int\limits_{B'} W^\varphi(x,y)\,\mathrm{d}y \\
\forall x \in B & f_B(\psi(x)) = \int\limits_B W^\varphi(x,y)\,\mathrm{d}y & \forall x \in B' & f_{B'}(\psi(x)) = \int\limits_{B'} W^\varphi(x,y)\,\mathrm{d}y \\
\forall x \in B'' & f_{B''}(\psi(x)) = \int\limits_A W^\varphi(x,y)\,\mathrm{d}y & \forall x \in C & f_C(\psi(x)) = \int\limits_C W^\varphi(x,y)\,\mathrm{d}y \\
\forall x \in C' & f_{C'}(\psi(x)) = \int\limits_{C'} W^\varphi(x,y)\,\mathrm{d}y & &
\end{array}$$

In the rest of the proof, we establish that W^φ and W_R^ψ are equal almost everywhere.

The pseudorandom and zero constraints in $\mathcal{C}_R$ imply that W^φ and W_R^ψ agree almost everywhere on $D \times [0,1]$ and $[0,1] \times D$. The zero and triangular constraints and the choice of ψ on B, B', C, and C' yield the same conclusion for $(B \cup B' \cup B'' \cup C \cup C')^2$, $A \times B'$, $B' \times A$, $A \times C$, $C \times A$, $A' \times B$, and $B \times A'$ (see Figure 4.14).

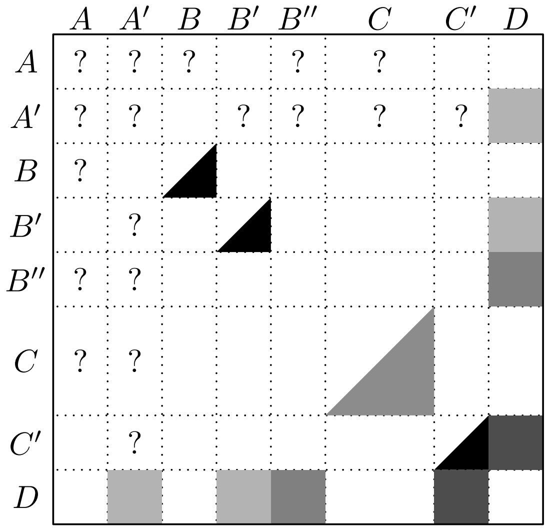


Figure 4.14: The forced structure of W after the first step of the proof of Theorem 4.12. Question marks denote the parts with no forced structure so far.

Let us now introduce some additional notation. If x is a vertex and Y is one of the parts, let $N_Y(x)$ denote the set of $y \in Y$ such that $W^\varphi(x,y) > 0$. If x and y belong to the same part, then we write $x \preceq y$ iff $\psi(x) \leq \psi(y)$. Observe that the monotonicity constraint (a) from Figure 4.10 and the choice of ψ implies the existence of a set Z of measure zero such that $N_B(x') \setminus N_B(x)$ has measure zero for $x, x' \in A \setminus Z$ if and only if $x \preceq x'$. Since the degree of every vertex in B is 1/9, this yields that the graphons W^φ and W_R^ψ agree almost everywhere on $A \times B$. The same reasoning applies to A' and B'. Thus, we conclude that the graphons W^φ and W_R^ψ agree almost everywhere on $(A \cup A') \times (B \cup B')$ and $(B \cup B') \times (A \cup A')$.

We now apply the same reasoning using the monotonicity constraint (b) and the split constraints (b) to deduce the existence of a zero measure set Z such that $N_{B''}(x) \setminus N_{B''}(x')$ has measure zero if and only if $x \preceq x'$ for $x, x' \in A \setminus Z$. The monotonicity constraint also imply that W^φ has only values zero and one almost everywhere on $A \times B''$. Since the measure of $N_B(x) \cup N_{B''}(x)$ is 1/9 for almost all $x \in A$ by the split constraint (b), the choice of ψ on B'' implies that the graphons W^φ and W_R^ψ agree almost everywhere on $A \times B''$. The degree regularity in B'', the split constraint (d), and the monotonicity constraint (d), which yields that W^φ has values zero and one almost everywhere on $A' \times B''$, yield the agreement almost everywhere on $A' \times B''$. Symmetrically, they agree almost everywhere on $B'' \times (A \cup A')$. The forced structure of W forced so far is depicted in Figure 4.15.

We now focus on the graphon W^φ on A^2. Observe first that the measure of

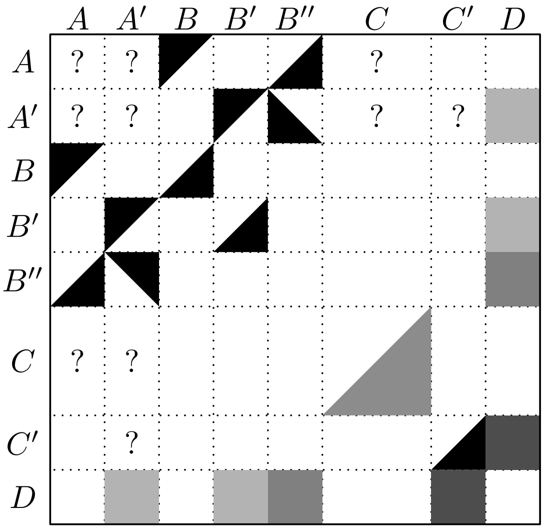


Figure 4.15: The forced structure of W after the second step of the proof of Theorem 4.12. Again, question marks denote the parts with no forced structure so far.

$N_B(x)$ is equal to $\psi(x)$ for almost all $x \in A$. The monotonicity constraints (f) and (h) from Figure 4.10 imply that there exists a set Z of measure zero such that every point $x \in A \setminus Z$ can be associated with a unique open interval $J_x \subseteq A$ such that $W^\varphi(x, x') = 0$ for almost every $x' \in \psi^{-1}(J_x)$, and $W^\varphi(x, x') = 1$ for almost every $x' \in A \setminus \psi^{-1}(J_x)$. The interval J_x can be empty for some choice of x. Recall that $|J_x|$ is the measure of the interval J_x, and let $\mathcal{J}$ be the set of all intervals J_x, $x \in A$, with $|J_x| > 0$. Since the intervals in $\mathcal{J}$ are disjoint, the set $\mathcal{J}$ is equipped with a natural linear order.

Let us now focus on the infinitary constraint (b) from Figure 4.12. Fix three vertices (two from A and one from B) as in the figure and let x be the left vertex from A. Observe that if $x \in A$ is fixed, then the set of choices of the other two vertices has non-zero measure unless $\psi(x) = \sup J_x$. The left hand side of the constraint is equal to the measure of J_x, i.e., $\sup J_x - \inf J_x$. The right hand side is equal to $1/9 - \sup J_x$. We conclude that $\inf J_x = 1/9 - 2|J_x|$. This implies that the set $\mathcal{J}$ is well-ordered and countable.

Let us write J_k for the k-th interval contained in $\mathcal{J}$. Furthermore, for $k \geq 1$, define

$$\beta_k = \frac{2(1 - 9 \inf J_{k+1})}{1 - 9 \inf J_k} = \frac{2|J_{k+1}|}{|J_k|} ,$$

and let β_0 be equal to $1 - 9 \inf J_1$. Note that by the observations made in the last paragraph and since $\inf J_{k+1} \geq \sup J_k$, we obtain $\beta_k \leq 1$ for every $k \geq 0$. In case that $\mathcal{J}$ is finite, we define $\beta_k = 0$ for $k > |\mathcal{J}|$. We can now express the density of non-edges with both end-vertices in A as

$$\sum_{J\in\mathcal{J}} |J|^2 = \sum_{k=1}^{\infty} \left(\frac{1}{9\cdot 2^k} \prod_{k'=0}^{k-1} \beta_{k'} \right)^2 .$$

Since the sum is forced to be $1/243$ by the infinitary constraint (a), we get that $\beta_k = 1$ for every k. This implies that for every k, $J_k = \left(\frac{1-2^{-k+1}}{9}, \frac{1-2^{-k}}{9}\right)$. In particular, the graphons W^φ and W_R^ψ agree almost everywhere on A^2.

The same reasoning as for A^2 yields that the graphons W^φ and W_R^ψ agree almost everywhere on A'^2. Let $\mathcal{J}'$ be the corresponding set of intervals for A' and let $J'_1, J'_2, \dots$ be their ordering. The split constraints (e) and (f) from Figure 4.11 imply that for almost every $x \in A$ with $|N_{A'}(x)| > 0$, there exists $J' \in \mathcal{J}'$ such that $N_{A'}(x) \,\Delta\, \psi^{-1}(J')$ has measure zero and $W^\varphi(x, y) = 1$ for almost every $y \in \psi^{-1}(J')$.

Let $x \in \psi^{-1}(J_k)$. The split constraint (a) from Figure 4.11 yields that $|N_{A'}(x)| = \frac{1}{9\cdot 2^k}$. Consequently, $N_{A'}(x) \,\Delta\, \psi^{-1}(J'_k)$ has measure zero for almost every $x \in \psi^{-1}(J_k)$ and $W(x, x') = 1$ for almost every $x \in \psi^{-1}(J_k)$ and $x' \in \psi^{-1}(J'_k)$. We conclude that the graphons W^φ and W_R^ψ agree almost everywhere on $A \times A'$ and $A' \times A$.

The orthogonality constraints (a) and (b) from Figure 4.13 yield that there exist measurable subsets $I_k \subseteq C$ with $|I_k| = 1/9$ for every $k \geq 1$ such that it holds for almost every $x \in \psi^{-1}(J_k)$ that $N_C(x)$ differs from I_k on a set of measure zero and $W^\varphi(x, y) = 1$ for almost every $y \in I_k$. The construction of ψ and the split constraint (h) from Figure 4.11 imply that $|N_A(x)| = 1/9 - \psi(x)/2$ for almost every $x \in C$. Since $\psi^{-1}(J_1) \setminus N_A(x)$ has measure zero for almost every $x \in I_1$, we get that $|J_1| \leq |N_A(x)|$ for almost every $x \in I_1$. This implies that I_1 and $\psi^{-1}([0, 1/9])$ differ on a set of measure zero (also see Figure 4.16). Since $\psi^{-1}(J_2) \setminus N_A(x)$ has measure zero for almost every $x \in I_2$ and $J_1 \cap J_2$ has measure zero, we get that $|J_1| + |J_2| \leq |N_A(x)|$ for almost every $x \in I_1 \cap I_2$ and that $|J_2| \leq |N_A(x)|$ for almost every $x \in I_2 \setminus I_1$. This implies that I_2 and $\psi^{-1}([0, 1/18] \cup [1/9, 1/6])$ differ on a set of measure zero. Iterating the argument, we obtain that I_k differs from the preimage with respect to ψ of the set

$$\bigcup_{i=1}^{2^{k-1}} \left[\frac{2i-2}{9\cdot 2^{k-1}}, \frac{2i-1}{9\cdot 2^{k-1}} \right]$$

on a set of measure zero for every $k \in \mathbb{N}$. This yields that the graphons W^φ and W_R^ψ agree almost everywhere on $A \times C$.

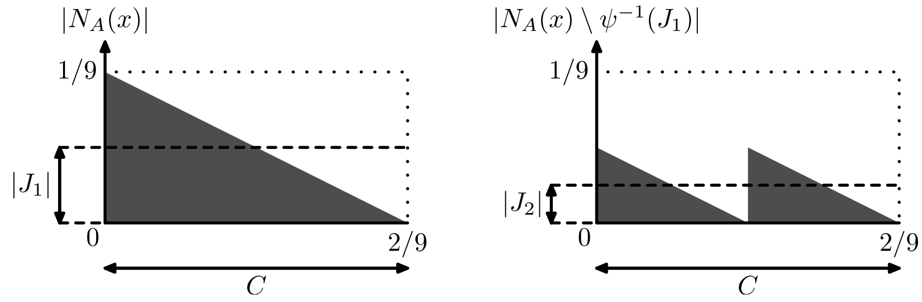


Figure 4.16: Illustration of the argument used in the proof of Theorem 4.12 to establish that the graphons W^φ and W_R^ψ agree almost everywhere on $A \times C$.

The orthogonality constraint (c) from Figure 4.13 implies that $(C \setminus N_C(x)) \cap N_C(x')$ has measure zero for every k, almost every $x \in A \setminus \psi^{-1}(J_k)$, and almost every $x' \in \psi^{-1}(J'_k)$. In particular, almost every $x' \in \psi^{-1}(J'_k)$ satisfies that $N_C(x') \setminus I_k$ has measure zero, i.e., $W^\varphi(x', y) = 0$ for almost every $x' \in \psi^{-1}(J'_k)$ and $y \notin I_k$.

We now interpret the orthogonality constraint (d) from Figure 4.13. Fix an integer $k \geq 1$ and a typical vertex $x' \in \psi^{-1}(J'_k)$. The left term in the product on the left hand side of the constraint is equal to the square of

$$\int\limits_C W^\varphi(x', y)\, \mathrm{d}y = \int\limits_{I_k} W^\varphi(x', y)\, \mathrm{d}y\ .$$

The right term in the product is equal to the square of $|J'_k| = 2^{-\langle \psi(x')/a \rangle}/9$. The term on the right hand side is equal to the probability that randomly chosen x'' and y satisfy $x'' \in A'$, $y \in B'$, $x'' \in \psi^{-1}(J'_k)$, and $\psi(x') \leq \psi(y) < \psi(x'')$. This is equal to

$$\frac{\left(1 - 2^{-\langle \psi(x')/a \rangle} - \psi(x')/a\right)^2}{2 \cdot 9^2}\ .$$

We deduce that almost every $x' \in \psi^{-1}(J'_k)$ satisfies

$$\int\limits_{I_k} W^\varphi(x', y)\, \mathrm{d}y = \frac{1 - 2^{-\langle \psi(x')/a \rangle} - \psi(x')/a}{9 \cdot 2^{-\langle \psi(x')/a \rangle}}\ . \tag{4.13}$$

We apply the same reasoning to the orthogonality constraint (e) from Figure 4.13

and deduce that almost every pair of vertices $x', x'' \in \psi^{-1}(J'_k)$ satisfies

$$\frac{9^2}{4} \cdot \left(\int_{I_k} W^\varphi(x', y) W^\varphi(x'', y) \,\mathrm{d}y \right)^2 \cdot \left(2^{-\langle \psi(x')/a \rangle} \right)^4 = \frac{\left(1 - 2^{-\langle \psi(x')/a \rangle} - \psi(x')/a\right)^2}{2} \cdot \frac{\left(1 - 2^{-\langle \psi(x'')/a \rangle} - \psi(x'')/a\right)^2}{2} .$$

This implies (similarly as in the proof of Lemma 4.9) that almost every $x' \in \psi^{-1}(J'_k)$ satisfies:

$$\left(\int_{I_k} W^\varphi(x', y)^2 \,\mathrm{d}y \right)^{1/2} = \frac{1 - 2^{-\langle \psi(x')/a \rangle} - \psi(x')/a}{3 \cdot 2^{-\langle \psi(x')/a \rangle}} . \tag{4.14}$$

Using Cauchy-Schwartz Inequality, we deduce from (4.13) and (4.14) (recall that $|I_k| = 1/9$) that the following holds for almost every $x' \in \psi^{-1}(J'_k)$ and $y \in I_k$,

$$W^\varphi(x', y) = \frac{1 - 2^{-\langle \psi(x')/a \rangle} - \psi(x')/a}{2^{-\langle \psi(x')/a \rangle}} .$$

In other words, $W^\varphi(x', y)$ is constant almost everywhere on I_k for almost every $x' \in \psi^{-1}(J'_k)$ and its value linearly decreases from one to zero almost everywhere inside $\psi^{-1}(J'_k)$. Hence, the graphons W^φ and W^ψ_R agree almost everywhere on $A' \times C$ and $C \times A'$ (recall that $W^\varphi(x', y) = 0$ for almost every pair $x' \in \psi^{-1}(J'_k)$ and $y \notin I_k$).

The monotonicity constraint (e) from Figure 4.10 yields that at least one of the sets $N_{C'}(x) \setminus N_{C'}(x')$ or $N_{C'}(x') \setminus N_{C'}(x)$ has measure zero for every k and almost every pair $x, x' \in A'$, and also that the graphon W^φ has values zero and one almost everywhere on $A' \times C'$. This together with the regularity on A' and C' imply that the graphons W^φ and W^ψ_R agree almost everywhere on $A' \times C'$. We have shown that the graphons W^φ and W^ψ_R agree almost everywhere on $(A \cup A') \times (C \cup C')$ and $(C \cup C') \times (A \cup A')$. Since these were the last subsets of their domains that remained to be analyzed, we proved that the graphon W^φ is equal to W^ψ_R almost everywhere. □

Theorem 4.12 immediately yields the following.

Corollary 4.15. *The graphon W_R is finitely forcible.*

Appendix A

Supplementary computations for Chapter 2

A.1 Matrices I_1, I_2, I_3 and I_4 from proof of Theorem 2.20

In this appendix, we display the matrices I_1, I_2, I_3 and I_4 that appear in the proof of Theorem 2.20. For the matrices I_2, I_3 and I_4, we also present the appropriate linear combinations of their rows that were used in the proof of Claim 1.

We start with the matrix I_1. It has size 8×7 and corresponds to the type $\sigma_1 = \mathbf{2}$.

$$I_1 = \begin{pmatrix} \frac{1}{37} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-6}{37} \\ \frac{-3}{19} & \frac{37}{38} & 0 & 0 & 0 & 0 & 0 & \frac{-1}{38} \\ \frac{-2}{13} & \frac{-1}{39} & \frac{38}{39} & 0 & 0 & 0 & 0 & \frac{-1}{39} \\ \frac{-3}{20} & \frac{-1}{40} & \frac{-1}{40} & \frac{39}{40} & 0 & 0 & 0 & \frac{-1}{40} \\ \frac{-3}{11} & \frac{-1}{22} & \frac{-1}{22} & \frac{-1}{22} & \frac{10}{11} & 0 & 0 & \frac{-1}{22} \\ \frac{-1}{4} & \frac{-1}{24} & \frac{-1}{24} & \frac{-1}{24} & \frac{-1}{12} & \frac{11}{12} & 0 & \frac{-1}{24} \\ \frac{-3}{13} & \frac{-1}{26} & \frac{-1}{26} & \frac{-1}{26} & \frac{-1}{13} & \frac{-1}{13} & \frac{12}{13} & \frac{-1}{26} \end{pmatrix}$$

The matrix I_2 has size 26×38. It corresponds to the type σ_2, which is the 4-vertex type with no edges. Unfortunately, the whole matrix is too large to be fully displayed here. Therefore, we display its transpose I_2^T decomposed into 3 block in the following way:

$$I_2^T = \left(\begin{array}{c|c|c} I_{2A} & I_{2B} & I_{2C} \end{array} \right),$$

where I_{2A}, I_{2B} and I_{2C} are the corresponding blocks of sizes 38×10, 38×7 and 38×9, respectively.

Now let us move to the linear combination of the rows of I_2 used in the proof Claim 1. Let $r_1, \ldots, r_8$ be the first, 7th, 14th, 19th, 20th, 23rd, 24th and the last row of I_2, respectively. In other words, r_1 is the first column and r_2 is the 7th column of the matrix I_{2A}, r_3 is the 4th column of I_{2B}, and $r_4, \ldots, r_8$ are the second, third, 6th, 7th and 9th column of I_{2C}, respectively. Next, let $w_1, \dots, w_8$ be the following 8 rationals:

$$w_1 := -\frac{17848347258759844}{667105311709433}, \qquad w_2 := -\frac{29077073547217221}{1007329020681243833},$$
$$w_3 := -\frac{182634053866072}{667105311709433}, \qquad w_4 := -\frac{250349838727759}{2001315935128299},$$
$$w_5 := -\frac{100121758433266968721}{197509869637811828310}, \qquad w_6 := -\frac{561895905711673}{1334210623418866},$$
$$w_7 := -\frac{216649503358540325}{695123734801229186}, \qquad w_8 := -\frac{21227849574516}{667105311709433}.$$

Recall that the last six coordinates of the vector x_2 are the ones that correspond to six σ_2-flags on five vertices containing a tight path from $\{a,b\}$ to $\{c,d\}$. The vector $q_2 := \sum_{i \in [8]} w_i \cdot r_i$ is a 38-dimensional vector with rational entries such that the last six coordinates of q_2 are positive and all the other ones are negative. Note that there is no linear combination of (at most) 7 rows from I_2 with such a property. The vector q_2 is also generated and then verified in the script "theorem_2_22-verify.sage".

The next matrix is I_3. It has size 15×17 and corresponds to the type σ_3, the 4-vertex type that contains a single edge abc. For brevity, we display the transpose I_3^T of I_3 instead of I_3 itself. Let r_1 be the 13th and r_2 the 15th row of I_3, i.e., the 13th and 15th column of I_3^T. The vector $q_3 := 6 \cdot r_2 - 5 \cdot r_1$ has the desired property that its first 11 entries are negative. We write the last six coordinates of the vector q_3 in a **bold** font.

Finally, the last matrix is I_4. It corresponds to the type σ_4, which is the 4-vertex type with the edge-set $\{abc, abd\}$. Like the matrix I_3, the matrix I_4 also has size 15×17 and again, we display rather its transpose I_4^T. The sum of the 13th and the 15th row of I_4, i.e., the 13th and the 15th column of I_4^T, is equal to the vector q_4. One more time, since the last six coordinates of the vector x_4 correspond to the σ_4-flags on five vertices that contain a tight path from $\{a,b\}$ to $\{c,d\}$, we just need to check that that q_4 has positive values only on the last six coordinates. Note that the last six coordinates of q_4 are again written in a **bold** font.

$$q_3 = \begin{pmatrix} -\frac{66}{1225} \\ -\frac{44}{1225} \\ -\frac{11}{1225} \\ -\frac{11}{1225} \\ -\frac{44}{1225} \\ -\frac{11}{1225} \\ -\frac{11}{1225} \\ -\frac{44}{1225} \\ -\frac{11}{1225} \\ -\frac{11}{1225} \\ -\frac{22}{1225} \\ \mathbf{-\frac{11}{1225}} \\ \mathbf{-\frac{6136}{1225}} \\ \mathbf{+\frac{11}{25}} \\ \mathbf{-\frac{11}{1225}} \\ \mathbf{+\frac{136}{25}} \\ \mathbf{-\frac{11}{1225}} \end{pmatrix} \qquad q_4 = \begin{pmatrix} -\frac{99}{1225} \\ -\frac{271}{7350} \\ -\frac{13}{3675} \\ -\frac{13}{3675} \\ -\frac{99}{1225} \\ -\frac{13}{3675} \\ -\frac{271}{7350} \\ -\frac{271}{7350} \\ -\frac{99}{1225} \\ -\frac{297}{2450} \\ -\frac{99}{2450} \\ \mathbf{-\frac{13}{3675}} \\ \mathbf{+\frac{7079}{7350}} \\ \mathbf{+\frac{11}{150}} \\ \mathbf{-\frac{271}{7350}} \\ \mathbf{+\frac{68}{75}} \\ \mathbf{-\frac{13}{3675}} \end{pmatrix}$$

$$I_4^T = \begin{pmatrix}
\frac{1}{9} & \frac{-1}{7} & \frac{-1}{5} & \frac{-2}{13} & \frac{-10}{23} & \frac{-1}{14} & \frac{-2}{17} & \frac{-1}{10} & \frac{-2}{7} & \frac{-6}{23} & \frac{-1}{12} & \frac{-4}{109} & \frac{-2}{49} & \frac{-5}{136} & \frac{-1}{25} \\
0 & \frac{9}{14} & \frac{1}{5} & \frac{2}{13} & \frac{-3}{23} & \frac{1}{7} & \frac{-3}{17} & \frac{-3}{20} & \frac{-1}{14} & \frac{-3}{46} & \frac{-1}{48} & \frac{11}{109} & \frac{-27}{245} & \frac{-27}{272} & \frac{11}{150} \\
0 & 0 & \frac{7}{10} & \frac{-3}{13} & \frac{-2}{23} & \frac{-5}{28} & \frac{2}{17} & \frac{1}{10} & \frac{-1}{14} & \frac{-3}{46} & \frac{-1}{48} & \frac{-13}{109} & \frac{22}{245} & \frac{11}{136} & \frac{-7}{75} \\
0 & 0 & 0 & \frac{10}{13} & \frac{-2}{23} & \frac{-5}{28} & \frac{2}{17} & \frac{1}{10} & \frac{-1}{14} & \frac{-3}{46} & \frac{-1}{48} & \frac{-13}{109} & \frac{22}{245} & \frac{11}{136} & \frac{-7}{75} \\
0 & 0 & 0 & 0 & \frac{13}{23} & \frac{-1}{14} & \frac{-2}{17} & \frac{-1}{10} & \frac{-2}{7} & \frac{-6}{23} & \frac{-1}{12} & \frac{-4}{109} & \frac{-2}{49} & \frac{-5}{136} & \frac{-1}{25} \\
0 & 0 & 0 & 0 & 0 & \frac{23}{28} & \frac{2}{17} & \frac{1}{10} & \frac{-1}{14} & \frac{-3}{46} & \frac{-1}{48} & \frac{-13}{109} & \frac{22}{245} & \frac{11}{136} & \frac{-7}{75} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{14}{17} & \frac{-3}{20} & \frac{-1}{14} & \frac{-3}{46} & \frac{-1}{48} & \frac{11}{109} & \frac{-27}{245} & \frac{-27}{272} & \frac{11}{150} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{17}{20} & \frac{-1}{14} & \frac{-3}{46} & \frac{-1}{48} & \frac{11}{109} & \frac{-27}{245} & \frac{-27}{272} & \frac{11}{150} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{7} & \frac{-6}{23} & \frac{-1}{12} & \frac{-4}{109} & \frac{-2}{49} & \frac{-5}{136} & \frac{-1}{25} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{14}{23} & \frac{-1}{8} & \frac{-6}{109} & \frac{-3}{49} & \frac{-15}{272} & \frac{-3}{50} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{23}{24} & \frac{-2}{109} & \frac{-1}{49} & \frac{-5}{272} & \frac{-1}{50} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{96}{109} & \frac{22}{245} & \frac{11}{136} & \frac{-7}{75} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{218}{245} & \frac{-27}{272} & \frac{11}{150} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{245}{272} & \frac{11}{150} \\
\frac{-2}{9} & \frac{-5}{14} & \frac{1}{5} & \frac{2}{13} & \frac{-3}{23} & \frac{1}{7} & \frac{-3}{17} & \frac{-3}{20} & \frac{-1}{14} & \frac{-3}{46} & \frac{-1}{48} & \frac{11}{109} & \frac{-27}{245} & \frac{-27}{272} & \frac{11}{150} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{68}{75} \\
\frac{-2}{9} & \frac{2}{7} & \frac{-3}{10} & \frac{-3}{13} & \frac{-2}{23} & \frac{-5}{28} & \frac{2}{17} & \frac{1}{10} & \frac{-1}{14} & \frac{-3}{46} & \frac{-1}{48} & \frac{-13}{109} & \frac{22}{245} & \frac{11}{136} & \frac{-7}{75}
\end{pmatrix}$$

$$I_3^T = \begin{pmatrix}
\frac{1}{19} & \frac{-4}{9} & \frac{-3}{41} & \frac{-3}{55} & \frac{-24}{71} & \frac{-9}{91} & \frac{-3}{37} & \frac{-4}{15} & \frac{-3}{53} & \frac{-3}{46} & \frac{-1}{8} & \frac{-6}{109} & \frac{-3}{49} & \frac{-15}{272} & \frac{-3}{50} \\
0 & \frac{19}{27} & \frac{-2}{41} & \frac{-2}{55} & \frac{-16}{71} & \frac{-6}{91} & \frac{-2}{37} & \frac{-8}{45} & \frac{-2}{53} & \frac{-1}{23} & \frac{-1}{12} & \frac{-4}{109} & \frac{-2}{49} & \frac{-5}{136} & \frac{-1}{25} \\
0 & 0 & \frac{27}{41} & \frac{-14}{55} & \frac{-2}{71} & \frac{17}{91} & \frac{17}{111} & \frac{-2}{45} & \frac{7}{53} & \frac{-25}{184} & \frac{-1}{48} & \frac{11}{109} & \frac{-27}{245} & \frac{11}{136} & \frac{-7}{75} \\
0 & 0 & 0 & \frac{41}{55} & \frac{-2}{71} & \frac{17}{91} & \frac{17}{111} & \frac{-2}{45} & \frac{7}{53} & \frac{-25}{184} & \frac{-1}{48} & \frac{11}{109} & \frac{-27}{245} & \frac{11}{136} & \frac{-7}{75} \\
0 & 0 & 0 & 0 & \frac{55}{71} & \frac{-6}{91} & \frac{-2}{37} & \frac{-8}{45} & \frac{-2}{53} & \frac{-1}{23} & \frac{-1}{12} & \frac{-4}{109} & \frac{-2}{49} & \frac{-5}{136} & \frac{-1}{25} \\
0 & 0 & 0 & 0 & 0 & \frac{71}{91} & \frac{-20}{111} & \frac{-2}{45} & \frac{-8}{53} & \frac{21}{184} & \frac{-1}{48} & \frac{-13}{109} & \frac{22}{245} & \frac{-27}{272} & \frac{11}{150} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{91}{111} & \frac{-2}{45} & \frac{-8}{53} & \frac{21}{184} & \frac{-1}{48} & \frac{-13}{109} & \frac{22}{245} & \frac{-27}{272} & \frac{11}{150} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{37}{45} & \frac{-2}{53} & \frac{-1}{23} & \frac{-1}{12} & \frac{-4}{109} & \frac{-2}{49} & \frac{-5}{136} & \frac{-1}{25} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{45}{53} & \frac{21}{184} & \frac{-1}{48} & \frac{-13}{109} & \frac{22}{245} & \frac{-27}{272} & \frac{11}{150} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{159}{184} & \frac{-1}{48} & \frac{11}{109} & \frac{-27}{245} & \frac{11}{136} & \frac{-7}{75} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{23}{24} & \frac{-2}{109} & \frac{-1}{49} & \frac{-5}{272} & \frac{-1}{50} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{96}{109} & \frac{22}{245} & \frac{-27}{272} & \frac{11}{150} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{218}{245} & \frac{11}{136} & \frac{-7}{75} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{245}{272} & \frac{11}{150} \\
\frac{-3}{19} & \frac{-2}{27} & \frac{13}{41} & \frac{13}{55} & \frac{-6}{71} & \frac{-20}{91} & \frac{-20}{111} & \frac{-2}{45} & \frac{-8}{53} & \frac{21}{184} & \frac{-1}{48} & \frac{-13}{109} & \frac{22}{245} & \frac{-27}{272} & \frac{11}{150} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{68}{75} \\
\frac{-3}{19} & \frac{-2}{27} & \frac{-14}{41} & \frac{-14}{55} & \frac{-2}{71} & \frac{17}{91} & \frac{17}{111} & \frac{-2}{45} & \frac{7}{53} & \frac{-25}{184} & \frac{-1}{48} & \frac{11}{109} & \frac{-27}{245} & \frac{11}{136} & \frac{-7}{75}
\end{pmatrix}$$

$$I_{2A} = \begin{pmatrix}
\frac{1}{151} & \frac{-20}{657} & \frac{-8}{137} & \frac{-5}{21068} & \frac{-2765}{47034} & \frac{215}{16853} & \frac{-3710}{38553} & \frac{-920}{9931} & \frac{-4900}{169689} & \frac{525}{734257} \\
0 & \frac{151}{657} & \frac{156}{685} & \frac{433}{5267} & \frac{4868}{23517} & \frac{944}{16853} & \frac{-2180}{38553} & \frac{3490}{69517} & \frac{13450}{169689} & \frac{1356}{734257} \\
0 & 0 & \frac{292}{685} & \frac{-153}{10534} & \frac{-337}{23517} & \frac{-50558}{589855} & \frac{1724}{38553} & \frac{12449}{69517} & \frac{13574}{169689} & \frac{-104946}{734257} \\
0 & 0 & 0 & \frac{6165}{21068} & \frac{-419}{5226} & \frac{36459}{589855} & \frac{3493}{12851} & \frac{732}{69517} & \frac{-85981}{509067} & \frac{-6719}{734257} \\
0 & 0 & 0 & 0 & \frac{10534}{23517} & \frac{-16852}{589855} & \frac{1724}{38553} & \frac{12449}{69517} & \frac{-16219}{169689} & \frac{31107}{734257} \\
0 & 0 & 0 & 0 & 0 & \frac{188136}{589855} & \frac{3493}{12851} & \frac{732}{69517} & \frac{33191}{509067} & \frac{-5381}{734257} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{16853}{38553} & \frac{-2752}{69517} & \frac{49850}{509067} & \frac{12205}{734257} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{38553}{69517} & \frac{-4255}{56563} & \frac{72693}{734257} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{347585}{509067} & \frac{-41783}{734257} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{274113}{734257} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{5267}{23517} & \frac{-8426}{589855} & \frac{862}{38553} & \frac{-13052}{69517} & \frac{-1727}{169689} & \frac{-20793}{734257} \\
0 & 0 & 0 & 0 & \frac{-5267}{23517} & \frac{-35942}{117971} & \frac{5512}{38553} & \frac{9568}{69517} & \frac{560}{13053} & \frac{38379}{734257} \\
0 & \frac{-151}{657} & \frac{136}{685} & \frac{-1019}{10534} & \frac{62}{23517} & \frac{96112}{589855} & \frac{-536}{12851} & \frac{-87}{9931} & \frac{-7156}{169689} & \frac{129432}{734257} \\
0 & 0 & \frac{146}{685} & \frac{-153}{21068} & \frac{-337}{47034} & \frac{-25279}{589855} & \frac{862}{38553} & \frac{-13052}{69517} & \frac{-134284}{509067} & \frac{-67928}{734257} \\
0 & \frac{-151}{657} & \frac{-2}{137} & \frac{1070}{5267} & \frac{1204}{7839} & \frac{-38712}{589855} & \frac{-536}{12851} & \frac{-87}{9931} & \frac{-11537}{509067} & \frac{-98884}{734257} \\
0 & 0 & \frac{-146}{685} & \frac{-1503}{5267} & \frac{158}{1809} & \frac{-2236}{117971} & \frac{5512}{38553} & \frac{9568}{69517} & \frac{-77470}{509067} & \frac{-145478}{734257} \\
\frac{-5}{151} & \frac{-17}{219} & \frac{44}{685} & \frac{-1707}{21068} & \frac{1363}{15678} & \frac{-2019}{16853} & \frac{3877}{38553} & \frac{-1013}{9931} & \frac{21595}{509067} & \frac{-88879}{734257} \\
\frac{-5}{151} & \frac{-17}{219} & \frac{44}{685} & \frac{-1707}{21068} & \frac{1363}{15678} & \frac{-2019}{16853} & \frac{3877}{38553} & \frac{-1013}{9931} & \frac{21595}{509067} & \frac{185234}{734257} \\
\frac{-5}{151} & \frac{-17}{219} & \frac{44}{685} & \frac{-1707}{21068} & \frac{-6445}{47034} & \frac{125897}{589855} & \frac{-3359}{38553} & \frac{9445}{69517} & \frac{10117}{509067} & \frac{-85672}{734257} \\
\frac{-5}{151} & \frac{100}{657} & \frac{-92}{685} & \frac{331}{21068} & \frac{305}{3618} & \frac{21359}{589855} & \frac{-889}{38553} & \frac{-2998}{69517} & \frac{26404}{509067} & \frac{38216}{734257} \\
\frac{-5}{151} & \frac{-17}{219} & \frac{-102}{685} & \frac{4611}{21068} & \frac{655}{47034} & \frac{-8927}{589855} & \frac{-3359}{38553} & \frac{9445}{69517} & \frac{20048}{509067} & \frac{-39875}{734257} \\
\frac{-5}{151} & \frac{100}{657} & \frac{54}{685} & \frac{89}{10534} & \frac{-1151}{7839} & \frac{4506}{589855} & \frac{-889}{38553} & \frac{-2998}{69517} & \frac{-11411}{56563} & \frac{-8919}{734257}
\end{pmatrix}$$

$$I_{2B} = \begin{pmatrix}
\frac{-204315}{20500507} & \frac{-47490}{2854093} & \frac{1495875}{72542753} & \frac{-3703870}{103518683} & \frac{-1883725}{212445379} & \frac{-10841565}{350591602} & \frac{-149185}{30095913} \\
\frac{-3506702}{20500507} & \frac{-20441}{1861365} & \frac{5023798}{72542753} & \frac{7918837}{103518683} & \frac{3968553}{212445379} & \frac{19831601}{350591602} & \frac{-135233065}{692205999} \\
\frac{-24662}{20500507} & \frac{274210}{2854093} & \frac{-154320}{72542753} & \frac{-8488879}{103518683} & \frac{5828531}{212445379} & \frac{-21807811}{350591602} & \frac{34816967}{230735333} \\
\frac{1314159}{20500507} & \frac{1024592}{14270465} & \frac{8697444}{72542753} & \frac{5713367}{103518683} & \frac{-4422871}{212445379} & \frac{2413587}{350591602} & \frac{-857369}{30095913} \\
\frac{2705057}{20500507} & \frac{-101941}{8562279} & \frac{-3258280}{72542753} & \frac{8367141}{103518683} & \frac{-38869737}{212445379} & \frac{10301552}{175295801} & \frac{5561534}{230735333} \\
\frac{-800939}{20500507} & \frac{1125191}{42811395} & \frac{-7501556}{72542753} & \frac{-16075062}{103518683} & \frac{2575000}{212445379} & \frac{18408871}{175295801} & \frac{539898}{10031971} \\
\frac{-11789}{436181} & \frac{-4341437}{42811395} & \frac{-896713}{72542753} & \frac{9620921}{103518683} & \frac{1471126}{212445379} & \frac{-20699821}{175295801} & \frac{-264054}{10031971} \\
\frac{744581}{20500507} & \frac{-1139414}{14270465} & \frac{-1012848}{72542753} & \frac{-9278647}{103518683} & \frac{28319163}{212445379} & \frac{-11481760}{175295801} & \frac{12725060}{692205999} \\
\frac{-186607}{20500507} & \frac{-621733}{14270465} & \frac{-3662001}{72542753} & \frac{3694536}{103518683} & \frac{-3664944}{212445379} & \frac{17817417}{350591602} & \frac{7855799}{692205999} \\
\frac{2286791}{20500507} & \frac{-174394}{3891945} & \frac{-414766}{72542753} & \frac{19981}{103518683} & \frac{24997917}{212445379} & \frac{58377987}{350591602} & \frac{-42502033}{692205999} \\
\frac{9545341}{20500507} & \frac{895038}{14270465} & \frac{-6466074}{72542753} & \frac{1997188}{103518683} & \frac{-7557276}{212445379} & \frac{-12386267}{175295801} & \frac{5913764}{230735333} \\
0 & \frac{20500507}{42811395} & \frac{17371913}{72542753} & \frac{4439694}{103518683} & \frac{25070034}{212445379} & \frac{46929219}{350591602} & \frac{-37320545}{692205999} \\
0 & 0 & \frac{42811395}{72542753} & \frac{-6458525}{103518683} & \frac{-24312107}{212445379} & \frac{-31525389}{350591602} & \frac{64895831}{692205999} \\
0 & 0 & 0 & \frac{72542753}{103518683} & \frac{-7875065}{212445379} & \frac{-11749613}{350591602} & \frac{-709643}{30095913} \\
0 & 0 & 0 & 0 & \frac{103518683}{212445379} & \frac{20470699}{350591602} & \frac{11997811}{230735333} \\
0 & 0 & 0 & 0 & 0 & \frac{212445379}{350591602} & \frac{192103}{30095913} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{350591602}{692205999} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{980238}{20500507} & \frac{11704522}{42811395} & \frac{-13842457}{72542753} & \frac{-21999373}{103518683} & \frac{-4965847}{212445379} & \frac{-39337267}{350591602} & \frac{-43176707}{692205999} \\
\frac{-733382}{20500507} & \frac{3329357}{42811395} & \frac{-4992182}{72542753} & \frac{-13949178}{103518683} & \frac{-42399504}{212445379} & \frac{30853405}{350591602} & \frac{-5834087}{30095913} \\
\frac{-3043128}{20500507} & \frac{-1116504}{14270465} & \frac{5865372}{72542753} & \frac{-4435745}{103518683} & \frac{-1570824}{12496787} & \frac{-9812995}{350591602} & \frac{-1057397}{10031971} \\
\frac{-291318}{20500507} & \frac{-5551958}{42811395} & \frac{-6425692}{72542753} & \frac{-3672231}{103518683} & \frac{-21947861}{212445379} & \frac{37213613}{175295801} & \frac{20934826}{230735333} \\
\frac{-1943265}{20500507} & \frac{-545324}{8562279} & \frac{-2309540}{72542753} & \frac{-5434133}{103518683} & \frac{36730867}{212445379} & \frac{-30372771}{350591602} & \frac{-115679647}{692205999} \\
\frac{-3677108}{20500507} & \frac{1920278}{42811395} & \frac{908302}{72542753} & \frac{3865268}{103518683} & \frac{6508885}{212445379} & \frac{9004760}{175295801} & \frac{-22240114}{692205999} \\
\frac{4337779}{20500507} & \frac{-1741787}{8562279} & \frac{14845870}{72542753} & \frac{-639980}{103518683} & \frac{1752725}{12496787} & \frac{40755477}{350591602} & \frac{-789823}{10031971} \\
\frac{-2920771}{20500507} & \frac{7188124}{42811395} & \frac{-4542304}{72542753} & \frac{8281032}{103518683} & \frac{8214976}{212445379} & \frac{-15277736}{175295801} & \frac{1351048}{10031971} \\
\frac{3110685}{20500507} & \frac{1851226}{14270465} & \frac{-3355998}{72542753} & \frac{6561629}{103518683} & \frac{-18270496}{212445379} & \frac{1924329}{175295801} & \frac{-4281590}{30095913} \\
\frac{-124499}{20500507} & \frac{-4496243}{42811395} & \frac{8426963}{72542753} & \frac{-23877425}{103518683} & \frac{-13359283}{212445379} & \frac{-20979701}{350591602} & \frac{-8906611}{230735333} \\
\frac{-3048002}{20500507} & \frac{524374}{14270465} & \frac{-5479602}{72542753} & \frac{3586034}{103518683} & \frac{-25799111}{212445379} & \frac{22984352}{175295801} & \frac{113159020}{692205999} \\
\frac{-1396055}{20500507} & \frac{-1252216}{42811395} & \frac{-9595754}{72542753} & \frac{5347936}{103518683} & \frac{19040844}{212445379} & \frac{-41205979}{350591602} & \frac{-27373393}{692205999}
\end{pmatrix}$$

$$I_{2C} = \begin{pmatrix}
\frac{895}{26229} & \frac{-874}{9869} & \frac{-213170}{1587953} & \frac{-1462375}{56082871} & \frac{-58495}{2009423} & \frac{-6940}{88049} & \frac{-154}{1781} & \frac{-460}{8489} & \frac{-20}{351} \\
\frac{9400}{603267} & \frac{-120}{9869} & \frac{151400}{1587953} & \frac{2854375}{56082871} & \frac{114175}{2009423} & \frac{-15204}{88049} & \frac{-150}{1781} & \frac{575}{8489} & \frac{25}{351} \\
\frac{-1761}{201089} & \frac{3473}{49345} & \frac{-201972}{1587953} & \frac{-3257175}{56082871} & \frac{-130287}{2009423} & \frac{7043}{88049} & \frac{596}{8905} & \frac{18}{8489} & \frac{-56}{351} \\
\frac{-18442}{218575} & \frac{-69}{9869} & \frac{87055}{1587953} & \frac{-939158}{56082871} & \frac{-210129}{2009423} & \frac{-646}{88049} & \frac{631}{8905} & \frac{-506}{25467} & \frac{-22}{1053} \\
\frac{6982}{201089} & \frac{3473}{49345} & \frac{-201972}{1587953} & \frac{-3257175}{56082871} & \frac{-130287}{2009423} & \frac{7043}{88049} & \frac{596}{8905} & \frac{-104}{653} & \frac{-2}{351} \\
\frac{67076}{655725} & \frac{-69}{9869} & \frac{87055}{1587953} & \frac{-5703017}{56082871} & \frac{-55558}{2009423} & \frac{-646}{88049} & \frac{631}{8905} & \frac{-506}{25467} & \frac{-22}{1053} \\
\frac{-97}{8743} & \frac{-184}{49345} & \frac{-216744}{1587953} & \frac{6349700}{56082871} & \frac{253988}{2009423} & \frac{-96}{88049} & \frac{-1416}{8905} & \frac{736}{25467} & \frac{32}{1053} \\
\frac{-16829}{603267} & \frac{-114}{695} & \frac{167276}{1587953} & \frac{3075025}{56082871} & \frac{123001}{2009423} & \frac{-1658}{88049} & \frac{-150}{1781} & \frac{575}{8489} & \frac{25}{351} \\
\frac{-40351}{655725} & \frac{175}{9869} & \frac{157520}{1587953} & \frac{211984}{4314067} & \frac{-62331}{2009423} & \frac{7733}{88049} & \frac{614}{8905} & \frac{-6}{653} & \frac{25}{351} \\
\frac{-628401}{5027225} & \frac{9872}{49345} & \frac{536}{93409} & \frac{6628487}{56082871} & \frac{-585083}{6028269} & \frac{-1478}{264147} & \frac{127}{1781} & \frac{-190}{8489} & \frac{32}{1053} \\
\frac{260311}{3016335} & \frac{-283}{49345} & \frac{11941}{93409} & \frac{-398395}{56082871} & \frac{815006}{6028269} & \frac{9176}{264147} & \frac{-38}{8905} & \frac{4718}{25467} & \frac{-22}{1053} \\
\frac{-43348}{655725} & \frac{-5042}{49345} & \frac{-43602}{1587953} & \frac{-3029334}{56082871} & \frac{223952}{2009423} & \frac{27020}{264147} & \frac{-354}{8905} & \frac{-852}{8489} & \frac{116}{1053} \\
\frac{19441}{218575} & \frac{6262}{49345} & \frac{114067}{1587953} & \frac{6724284}{56082871} & \frac{-5858}{154571} & \frac{-1883}{264147} & \frac{337}{8905} & \frac{2828}{25467} & \frac{-19}{1053} \\
\frac{-954302}{15081675} & \frac{175}{9869} & \frac{157520}{1587953} & \frac{-2008067}{56082871} & \frac{92240}{2009423} & \frac{7733}{88049} & \frac{614}{8905} & \frac{607}{8489} & \frac{-2}{351} \\
\frac{-1299422}{15081675} & \frac{9872}{49345} & \frac{536}{93409} & \frac{-6075137}{56082871} & \frac{651485}{6028269} & \frac{-1478}{264147} & \frac{127}{1781} & \frac{800}{25467} & \frac{-22}{1053} \\
\frac{2552654}{15081675} & \frac{-5042}{49345} & \frac{-43602}{1587953} & \frac{6498384}{56082871} & \frac{-85190}{2009423} & \frac{27020}{264147} & \frac{-354}{8905} & \frac{2924}{25467} & \frac{-100}{1053} \\
\frac{-56032}{1005445} & \frac{-283}{49345} & \frac{11941}{93409} & \frac{7541370}{56082871} & \frac{42151}{6028269} & \frac{9176}{264147} & \frac{-38}{8905} & \frac{-254}{8489} & \frac{194}{1053} \\
\frac{10031971}{15081675} & \frac{6262}{49345} & \frac{114067}{1587953} & \frac{-2803434}{56082871} & \frac{232988}{2009423} & \frac{-1883}{264147} & \frac{337}{8905} & \frac{-199}{8489} & \frac{116}{1053} \\
0 & \frac{28727}{49345} & \frac{-60858}{1587953} & \frac{-845825}{56082871} & \frac{-33833}{2009423} & \frac{-1208}{264147} & \frac{-1424}{8905} & \frac{-506}{25467} & \frac{-22}{1053} \\
0 & 0 & \frac{1036245}{1587953} & \frac{-4187050}{56082871} & \frac{-167482}{2009423} & \frac{-3038}{88049} & \frac{-786}{8905} & \frac{-1288}{8489} & \frac{-56}{351} \\
0 & 0 & 0 & \frac{39698825}{56082871} & \frac{449792}{6028269} & \frac{-1471}{88049} & \frac{63}{1781} & \frac{-2300}{25467} & \frac{-100}{1053} \\
0 & 0 & 0 & 0 & \frac{4314067}{6028269} & \frac{-1471}{88049} & \frac{63}{1781} & \frac{-2300}{25467} & \frac{-100}{1053} \\
0 & 0 & 0 & 0 & 0 & \frac{154571}{264147} & \frac{-1424}{8905} & \frac{-506}{25467} & \frac{-22}{1053} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{521}{685} & \frac{-46}{8489} & \frac{-2}{351} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{15755}{25467} & \frac{32}{1053} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{653}{1053} \\
\frac{-99747}{1005445} & \frac{443}{9869} & \frac{197705}{1587953} & \frac{7467820}{56082871} & \frac{33325}{6028269} & \frac{-4370}{264147} & \frac{-38}{8905} & \frac{4718}{25467} & \frac{-22}{1053} \\
\frac{-298577}{15081675} & \frac{4622}{49345} & \frac{5158}{54757} & \frac{-2118392}{56082871} & \frac{87827}{2009423} & \frac{960}{88049} & \frac{614}{8905} & \frac{-6}{653} & \frac{25}{351} \\
\frac{-1115354}{15081675} & \frac{17}{49345} & \frac{127297}{1587953} & \frac{-2619559}{56082871} & \frac{240343}{2009423} & \frac{31982}{264147} & \frac{337}{8905} & \frac{2828}{25467} & \frac{-19}{1053} \\
\frac{-526559}{3016335} & \frac{443}{9869} & \frac{197705}{1587953} & \frac{-471945}{56082871} & \frac{806180}{6028269} & \frac{-4370}{264147} & \frac{-38}{8905} & \frac{-254}{8489} & \frac{194}{1053} \\
\frac{19441}{218575} & \frac{17}{49345} & \frac{127297}{1587953} & \frac{6908159}{56082871} & \frac{-68799}{2009423} & \frac{31982}{264147} & \frac{337}{8905} & \frac{-199}{8489} & \frac{116}{1053} \\
\frac{-40351}{655725} & \frac{4622}{49345} & \frac{5158}{54757} & \frac{2645467}{56082871} & \frac{-66744}{2009423} & \frac{960}{88049} & \frac{614}{8905} & \frac{607}{8489} & \frac{-2}{351} \\
\frac{-4936}{218575} & \frac{-5267}{49345} & \frac{13173}{1587953} & \frac{-2951414}{56082871} & \frac{-181607}{6028269} & \frac{-9208}{88049} & \frac{-358}{8905} & \frac{-437}{25467} & \frac{-19}{1053} \\
\frac{30036}{218575} & \frac{-5267}{49345} & \frac{13173}{1587953} & \frac{-1363461}{56082871} & \frac{-336178}{6028269} & \frac{-9208}{88049} & \frac{-358}{8905} & \frac{-437}{25467} & \frac{-19}{1053} \\
\frac{-725971}{15081675} & \frac{990}{9869} & \frac{-3810}{93409} & \frac{6204184}{56082871} & \frac{-96958}{2009423} & \frac{-27164}{264147} & \frac{-354}{8905} & \frac{-852}{8489} & \frac{116}{1053} \\
\frac{-1955147}{15081675} & \frac{-24}{9869} & \frac{30280}{1587953} & \frac{-5780937}{56082871} & \frac{686789}{6028269} & \frac{52706}{264147} & \frac{127}{1781} & \frac{-190}{8489} & \frac{32}{1053} \\
\frac{-43348}{655725} & \frac{990}{9869} & \frac{-3810}{93409} & \frac{-3323534}{56082871} & \frac{212184}{2009423} & \frac{-27164}{264147} & \frac{-354}{8905} & \frac{2924}{25467} & \frac{-100}{1053} \\
\frac{683049}{5027225} & \frac{-24}{9869} & \frac{30280}{1587953} & \frac{6922687}{56082871} & \frac{-549779}{6028269} & \frac{52706}{264147} & \frac{127}{1781} & \frac{800}{25467} & \frac{-22}{1053}
\end{pmatrix}$$

Appendix B

Supplementary computations for Chapter 3

B.1 Proof of Theorem 3.20

Up to a few changes in the choice of the parameters, we follow the lines of the proof of Theorem 3.9 presented in Section 3.3. As in Section 3.2, Theorem 3.19 has the following two corollaries.

Corollary B.1. *For every positive $\varepsilon_{\mathrm{ASYM}}$ there exists $n_{\mathrm{ASYM}} \in \mathbb{N}$ such that the following is true. If G is a permutation graph on $n \geq n_{\mathrm{ASYM}}$ vertices, then $f_5(G) > (1/256 - \varepsilon)$.*

Corollary B.2. *For every positive $\varepsilon_{\mathrm{CONF}}$ there exist a positive δ_{CONF} and $n_{\mathrm{CONF}} \in \mathbb{N}$ such that the following is true. If G is a permutation graph on $n \geq n_{\mathrm{CONF}}$ vertices that satisfies $f_5(G) \leq (1/256 + \delta_{\mathrm{CONF}})$, then G contains at most $\varepsilon_{\mathrm{CONF}} \cdot \binom{n}{7}$ induced copies of F and at most $\varepsilon_{\mathrm{CONF}} \cdot \binom{n}{7}$ induced copies of $\overline{F}$, where $F \in \mathcal{E}_7^5$.*

Proof of Theorem 3.20. Let $C := 49 \times 140 = 6860$. Recall $\mathcal{X}$ is the set of all non-permutation graphs. We do the analogous set-up of the parameters as in Theorem 3.9. Specifically, let δ_{RL} be the value from Infinite Removal Lemma applied for $\varepsilon_{\mathrm{RL}} := (\varepsilon_{\mathrm{STAB}})^2/C$ and the family $\mathcal{X} \cup \mathcal{E}_7^5$, let δ_{CONF} and n_{CONF} be the values from Corollary B.2 applied for $\varepsilon_{\mathrm{CONF}} := \delta_{\mathrm{RL}}$, and let n_{ASYM} be the value from Corollary B.1 for $\varepsilon_{\mathrm{ASYM}} := (\varepsilon_{\mathrm{STAB}})^2/C$. We set $\delta_{\mathrm{STAB}} := \min\{(\varepsilon_{\mathrm{STAB}})^2/C, \delta_{\mathrm{CONF}}\}$ and $n_{\mathrm{STAB}} := \max\{C/\varepsilon_{\mathrm{STAB}}, n_{\mathrm{CONF}}, 2 \cdot n_{\mathrm{ASYM}}\}$.

Let G be a graph on n vertices satisfying the assumptions of the theorem. Corollary B.2 and Infinite Removal Lemma implies that we can add or remove less than $\frac{1}{7} \cdot \varepsilon_{\mathrm{STAB}} \cdot \binom{n}{2}$ edges in G and obtain a permutation graph G' that satisfies Property A(5) and Property B. Furthermore, $f_5(G') < 1/256 + 2 \cdot \varepsilon_{\mathrm{STAB}}^2/C$. Hence, by Lemma 3.7, we can partition G' into at most 4 parts such that either each part is a clique and there are no edges between the parts, or every edge between the parts is present and the parts themselves form an independent set. Without loss of generality, G' is a disjoint union of at most 4 cliques.

It is enough to show that every clique has size at most $(1/4+\varepsilon_{\text{STAB}}/7)\cdot n$. Let $\gamma := \varepsilon_{\text{STAB}}/7$ and suppose for a contradiction G' contains a clique of size more than $(1/4+\gamma)\cdot n$. Let $H_0 := G'$. For each $i \in [\gamma \cdot n/3]$, let v_i be an arbitrary vertex from a maximum clique inside H_{i-1}, and let $H_i := H_{i-1} - v_i$. Let $Z := \{v_1, v_2, \dots, v_{\gamma\cdot n/3}\}$. It follows that every vertex $v \in Z$ is contained in at least $\binom{(1/256+2\gamma/3)\cdot n}{4}$ copies of K_5 that are disjoint from $Z \setminus \{v\}$. Since $n_{\text{STAB}} > 18/\gamma = 126/\varepsilon_{\text{STAB}}$, we have

$$\binom{(1/4 + 2\cdot\gamma/3)\cdot n}{4} > \left(\frac{1}{4}+\frac{\gamma}{2}\right)^4 \cdot \frac{n^4}{24} > \left(\frac{1}{256}+\frac{\gamma}{32}\right)\cdot\frac{n^4}{24}.$$

Furthermore,

$$f_5(H_{i+1}) \le f_5(H_i) \le f_5(G') \le \frac{1}{256} + 2\cdot\frac{(\varepsilon_{\text{STAB}})^2}{C} = \frac{1}{256} + 2\cdot\frac{\gamma^2}{140}.$$

Our aim is to show that $f_5(H_i) - f_5(H_{i+1}) > \gamma/(12\cdot n)$. Indeed, we have

$$\begin{aligned} f_5(H_i) - f_5(H_{i+1}) &> \frac{\left(\frac{1}{256}+\frac{\gamma}{32}\right)\cdot\frac{n^4}{24} + f_5(H_{i+1})\cdot\binom{v(H_i)-1}{5} - f_5(H_{i+1})\cdot\binom{v(H_i)}{5}}{\binom{v(H_i)}{5}} \\ &> \frac{\left(\frac{1}{256}+\frac{\gamma}{32}\right)\cdot\frac{n^4}{24} - f_5(H_{i+1})\cdot\binom{v(H_i-1)}{4}}{\binom{v(H_i)}{5}} \\ &> \frac{5\cdot\left(\frac{1}{256}+\frac{\gamma}{32} - f_5(H_{i+1})\right)}{n} \ge \frac{5}{n}\cdot\left(\frac{\gamma}{32} - \frac{2\gamma^2}{140}\right) \\ &> \frac{5}{n}\cdot\left(\frac{\gamma}{32}-\frac{\gamma}{70}\right) > \frac{\gamma}{12n}. \end{aligned}$$

However, Corollary B.1 implies that $f_5\left(H_{\gamma\cdot n/3}\right) \ge 1/256 - (\varepsilon_{\text{STAB}})^2/C$. Putting the inequalities together would yield that

$$\frac{1}{256} + \frac{2(\varepsilon_{\text{STAB}})^2}{C} \ge f_5(G') = f_5(H_0) > f_5(H_{\gamma\cdot n/3}) + \frac{\gamma^2}{36} \ge \frac{1}{256} - \frac{(\varepsilon_{\text{STAB}})^2}{C} + \frac{(\varepsilon_{\text{STAB}})^2}{49\times 36},$$

a contradiction. □

B.2 Proof of Theorem 3.23

Again, we adapt the proof of Theorem 3.9. Analogously to Theorems 3.4 and 3.19, Theorem 3.22 has the following two corollaries.

Corollary B.3. *For every positive $\varepsilon_{\text{ASYM}}$ there exists $n_{\text{ASYM}} \in \mathbb{N}$ such that the* *following is true. If G is a permutation graph on $n \geq n_{\text{ASYM}}$ vertices, then $f_6(G) > (1/3125 - \varepsilon)$.*

Corollary B.4. *For every positive $\varepsilon_{\text{CONF}}$ there exist a positive δ_{CONF} and $n_{\text{CONF}} \in \mathbb{N}$ such that the following is true. If G is a permutation graph on $n \geq n_{\text{CONF}}$ vertices that satisfies $f_6(G) \leq (1/3125+\delta_{\text{CONF}})$, then G contains at most $\varepsilon_{\text{CONF}} \cdot \binom{n}{8}$ induced copies of F and at most $\varepsilon_{\text{CONF}} \cdot \binom{n}{8}$ induced copies of $\overline{F}$, where $F \in \mathcal{E}_8^6$.*

Proof of Theorem 3.23. We let $C := 81 \times 900 = 72900$ this time. Again, $\mathcal{X}$ is the set of all non-permutation graphs. We do the following set-up of the parameters. Let δ_{RL} be the value from Infinite Removal Lemma applied for $\varepsilon_{\text{RL}} := (\varepsilon_{\text{STAB}})^2/C$ and the family $\mathcal{X} \cup \mathcal{E}_8^6$, let δ_{CONF} and n_{CONF} be the values from Corollary B.4 applied for $\varepsilon_{\text{CONF}} := \delta_{\text{RL}}$, and let n_{ASYM} be the value from Corollary B.3 for $\varepsilon_{\text{ASYM}} := (\varepsilon_{\text{STAB}})^2/C$. We set $\delta_{\text{STAB}} := \min\{(\varepsilon_{\text{STAB}})^2/C, \delta_{\text{CONF}}\}$ and $n_{\text{STAB}} := \max\{C/\varepsilon_{\text{STAB}}, n_{\text{CONF}}, 2 \cdot n_{\text{ASYM}}\}$.

Let G be a graph on n vertices satisfying the assumptions of the theorem. Corollary B.4 and Infinite Removal Lemma implies that G is in edit distance less than $\frac{1}{9} \cdot \varepsilon_{\text{STAB}} \cdot \binom{n}{2}$ to a permutation graph G' that satisfies Property A(6) and Property B. Furthermore, $f_6(G') < 1/3125 + 2 \cdot \varepsilon_{\text{STAB}}^2/C$. By Lemma 3.7, there is a partition of G' into at most 5 parts such that either each part is a clique and there are no edges between the parts, or every edge between the parts is present and the parts themselves form an independent set. Without loss of generality, G' is a disjoint union of at most 5 cliques.

It is enough to show that every clique has size at most $(1/5 + \varepsilon_{\text{STAB}}/9) \cdot n$. Let $\gamma := \varepsilon_{\text{STAB}}/9$ and suppose for a contradiction G' contains a clique of size more than $(1/5 + \gamma) \cdot n$. Again, let $H_0 := G'$. For each $i \in [\gamma \cdot n/3]$, let v_i be an arbitrary vertex from a maximum clique inside H_{i-1}, and let $H_i := H_{i-1} - v_i$. Let $Z := \{v_1, v_2, \ldots, v_{\gamma \cdot n/3}\}$. It follows that every vertex $v \in Z$ is contained in at least $\binom{(1/3125+2\gamma/3)\cdot n}{5}$ copies of K_6 that are disjoint from $Z \setminus \{v\}$. Since $n_{\text{STAB}} > 24/\gamma = 216/\varepsilon_{\text{STAB}}$, we have

$$\binom{(1/5 + 2 \cdot \gamma/3) \cdot n}{5} > \left(\frac{1}{5} + \frac{\gamma}{2}\right)^5 \cdot \frac{n^5}{120} > \left(\frac{1}{3125} + \frac{\gamma}{250}\right) \cdot \frac{n^5}{120}.$$

Furthermore,

$$f_6(H_{i+1}) \leq f_6(H_i) \leq f_6(G') \leq \frac{1}{3125} + 2 \cdot \frac{(\varepsilon_{\text{STAB}})^2}{C} = \frac{1}{3125} + 2 \cdot \frac{\gamma^2}{900}.$$

Our aim is to show that $f_6(H_i) - f_6(H_{i+1}) > \gamma/(100 \cdot n)$. Indeed, we have

$$\begin{aligned}
f_6(H_i) - f_6(H_{i+1}) &> \frac{\left(\frac{1}{3125} + \frac{\gamma}{250}\right) \cdot \frac{n^5}{120} + f_6(H_{i+1}) \cdot \binom{v(H_i)-1}{6} - f_6(H_{i+1}) \cdot \binom{v(H_i)}{6}}{\binom{v(H_i)}{6}} \\
&> \frac{\left(\frac{1}{3125} + \frac{\gamma}{250}\right) \cdot \frac{n^5}{120} - f_6(H_{i+1}) \cdot \binom{v(H_i-1)}{5}}{\binom{v(H_i)}{6}} \\
&> \frac{6 \cdot \left(\frac{1}{3125} + \frac{\gamma}{250} - f_6(H_{i+1})\right)}{n} \geq \frac{6}{n} \cdot \left(\frac{\gamma}{250} - \frac{2\gamma^2}{900}\right) \\
&> \frac{6}{n} \cdot \left(\frac{\gamma}{250} - \frac{\gamma}{450}\right) > \frac{\gamma}{100n}.
\end{aligned}$$

Corollary B.1 implies that $f_6\left(H_{\gamma \cdot n/3}\right) \geq 1/3125 - (\varepsilon_{\mathrm{STAB}})^2/C$. Therefore,

$$\frac{1}{3125} + 2 \cdot \frac{(\varepsilon_{\mathrm{STAB}})^2}{C} \geq f_6(G') > \frac{1}{3125} - \frac{(\varepsilon_{\mathrm{STAB}})^2}{C} + \frac{(\varepsilon_{\mathrm{STAB}})^2}{81 \times 300},$$

a contradiction. □

B.3 The bijections $\tau_{i,j}$ from the proof of Theorem 3.22

- $\tau_{1,1} = 12$: $b_{\tau_{1,1}} := ab$,
- $\tau_{1,2} = 21$: $b_{\tau_{1,2}} := ab$,
- $\tau_{2,1} = 1234$: $b_{\tau_{2,1}} := abcd$,
- $\tau_{2,2} = 4321$: $b_{\tau_{2,2}} := abcd$,
- $\tau_{3,1} = 1243$: $b_{\tau_{3,1}} := abcd$,
- $\tau_{3,2} = 1324$: $b_{\tau_{3,2}} := acdb$,
- $\tau_{3,3} = 2134$: $b_{\tau_{3,3}} := cdba$,
- $\tau_{3,4} = 3421$: $b_{\tau_{3,4}} := cdba$,
- $\tau_{3,5} = 4231$: $b_{\tau_{3,5}} := acdb$,
- $\tau_{3,6} = 4312$: $b_{\tau_{3,6}} := abcd$,
- $\tau_{4,1} = 1342$: $b_{\tau_{4,1}} := abcd$,
- $\tau_{4,2} = 1423$: $b_{\tau_{4,2}} := adbc$,
- $\tau_{4,3} = 2314$: $b_{\tau_{4,3}} := cbda$,
- $\tau_{4,4} = 2431$: $b_{\tau_{4,4}} := dcba$,
- $\tau_{4,5} = 3124$: $b_{\tau_{4,5}} := dcba$,
- $\tau_{4,6} = 3241$: $b_{\tau_{4,6}} := cbda$,
- $\tau_{4,7} = 4132$: $b_{\tau_{4,7}} := adbc$,
- $\tau_{4,8} = 4213$: $b_{\tau_{4,8}} := abcd$,
- $\tau_{5,1} = 1432$: $b_{\tau_{5,1}} := abcd$,
- $\tau_{5,2} = 2341$: $b_{\tau_{5,2}} := bcda$,
- $\tau_{5,3} = 3214$: $b_{\tau_{5,3}} := bcda$,
- $\tau_{5,4} = 4123$: $b_{\tau_{5,4}} := abcd$,
- $\tau_{6,1} = 2143$: $b_{\tau_{6,1}} := abcd$,
- $\tau_{6,2} = 3412$: $b_{\tau_{6,2}} := abcd$,
- $\tau_{7,1} = 2413$: $b_{\tau_{7,1}} := abcd$,
- $\tau_{7,2} = 3142$: $b_{\tau_{7,2}} := abcd$,
- $\tau_{8,1} = 123456$: $b_{\tau_{8,1}} := abcdef$,
- $\tau_{8,2} = 654321$: $b_{\tau_{8,2}} := abcdef$,
- $\tau_{9,1} = 134562$: $b_{\tau_{9,1}} := abcdef$,
- $\tau_{9,2} = 162345$: $b_{\tau_{9,2}} := afbcde$,
- $\tau_{9,3} = 234516$: $b_{\tau_{9,3}} := edcbfa$,
- $\tau_{9,4} = 265431$: $b_{\tau_{9,4}} := fedcba$,
- $\tau_{9,5} = 512346$: $b_{\tau_{9,5}} := fedcba$,
- $\tau_{9,6} = 543261$: $b_{\tau_{9,6}} := edcbfa$,
- $\tau_{9,7} = 615432$: $b_{\tau_{9,7}} := afbcde$,
- $\tau_{9,8} = 643215$: $b_{\tau_{9,8}} := abcdef$,
- $\tau_{10,1} = 135264$: $b_{\tau_{10,1}} := abcdef$,
- $\tau_{10,2} = 142635$: $b_{\tau_{10,2}} := adbfce$,
- $\tau_{10,3} = 241536$: $b_{\tau_{10,3}} := ecfbda$,
- $\tau_{10,4} = 315246$: $b_{\tau_{10,4}} := fedcba$,
- $\tau_{10,5} = 462531$: $b_{\tau_{10,5}} := fedcba$,
- $\tau_{10,6} = 536241$: $b_{\tau_{10,6}} := ecfbda$,
- $\tau_{10,7} = 635142$: $b_{\tau_{10,7}} := adbfce$,
- $\tau_{10,8} = 642513$: $b_{\tau_{10,8}} := abcdef$,
- $\tau_{11,1} = 136425$: $b_{\tau_{11,1}} := abcdef$,
- $\tau_{11,2} = 152463$: $b_{\tau_{11,2}} := aebdfc$,
- $\tau_{11,3} = 253146$: $b_{\tau_{11,3}} := fedcba$,
- $\tau_{11,4} = 364251$: $b_{\tau_{11,4}} := cfdbea$,
- $\tau_{11,5} = 413526$: $b_{\tau_{11,5}} := cfdbea$,
- $\tau_{11,6} = 524631$: $b_{\tau_{11,6}} := fedcba$,
- $\tau_{11,7} = 625314$: $b_{\tau_{11,7}} := aebdfc$,
- $\tau_{11,8} = 641352$: $b_{\tau_{11,8}} := abcdef$,
- $\tau_{12,1} = 143652$: $b_{\tau_{12,1}} := abcdef$,
- $\tau_{12,2} = 163254$: $b_{\tau_{12,2}} := afcbed$,
- $\tau_{12,3} = 256341$: $b_{\tau_{12,3}} := fedcba$,
- $\tau_{12,4} = 325416$: $b_{\tau_{12,4}} := debcfa$,
- $\tau_{12,5} = 452361$: $b_{\tau_{12,5}} := debcfa$,
- $\tau_{12,6} = 521436$: $b_{\tau_{12,6}} := fedcba$,
- $\tau_{12,7} = 614523$: $b_{\tau_{12,7}} := afcbed$,
- $\tau_{12,8} = 634125$: $b_{\tau_{12,8}} := abcdef$,
- $\tau_{13,1} = 214563$: $b_{\tau_{13,1}} := abcdef$,
- $\tau_{13,2} = 216345$: $b_{\tau_{13,2}} := bafcde$,
- $\tau_{13,3} = 234165$: $b_{\tau_{13,3}} := edcfab$,
- $\tau_{13,4} = 365412$: $b_{\tau_{13,4}} := fedcba$,
- $\tau_{13,5} = 412365$: $b_{\tau_{13,5}} := fedcba$,
- $\tau_{13,6} = 543612$: $b_{\tau_{13,6}} := edcfab$,
- $\tau_{13,7} = 561432$: $b_{\tau_{13,7}} := bafcde$,
- $\tau_{13,8} = 563214$: $b_{\tau_{13,8}} := abcdef$,
- $\tau_{14,1} = 231654$: $b_{\tau_{14,1}} := abcdef$,
- $\tau_{14,2} = 312654$: $b_{\tau_{14,2}} := cabfed$,
- $\tau_{14,3} = 321564$: $b_{\tau_{14,3}} := defbac$,
- $\tau_{14,4} = 321645$: $b_{\tau_{14,4}} := fedcba$,
- $\tau_{14,5} = 456132$: $b_{\tau_{14,5}} := fedcba$,
- $\tau_{14,6} = 456213$: $b_{\tau_{14,6}} := defbac$,
- $\tau_{14,7} = 465123$: $b_{\tau_{14,7}} := cabfed$,
- $\tau_{14,8} = 546123$: $b_{\tau_{14,8}} := abcdef$,
- $\tau_{15,1} = 234615$: $b_{\tau_{15,1}} := abcdef$,
- $\tau_{15,2} = 261345$: $b_{\tau_{15,2}} := fedcba$,
- $\tau_{15,3} = 265413$: $b_{\tau_{15,3}} := eabcfd$,
- $\tau_{15,4} = 314562$: $b_{\tau_{15,4}} := dfcbae$,
- $\tau_{15,5} = 463215$: $b_{\tau_{15,5}} := dfcbae$,
- $\tau_{15,6} = 512364$: $b_{\tau_{15,6}} := eabcfd$,
- $\tau_{15,7} = 516432$: $b_{\tau_{15,7}} := fedcba$,
- $\tau_{15,8} = 543162$: $b_{\tau_{15,8}} := abcdef$,
- $\tau_{16,1} = 235164$: $b_{\tau_{16,1}} := abcdef$,
- $\tau_{16,2} = 241563$: $b_{\tau_{16,2}} := ecfbad$,
- $\tau_{16,3} = 316245$: $b_{\tau_{16,3}} := fedcba$,
- $\tau_{16,4} = 365142$: $b_{\tau_{16,4}} := dabfce$,
- $\tau_{16,5} = 412635$: $b_{\tau_{16,5}} := dabfce$,
- $\tau_{16,6} = 461532$: $b_{\tau_{16,6}} := fedcba$,
- $\tau_{16,7} = 536214$: $b_{\tau_{16,7}} := ecfbad$,
- $\tau_{16,8} = 542613$: $b_{\tau_{16,8}} := abcdef$,
- $\tau_{17,1} = 235614$: $b_{\tau_{17,1}} := abcdef$,
- $\tau_{17,2} = 265143$: $b_{\tau_{17,2}} := eabfcd$,
- $\tau_{17,3} = 341562$: $b_{\tau_{17,3}} := dcfbae$,
- $\tau_{17,4} = 361245$: $b_{\tau_{17,4}} := fedcba$,
- $\tau_{17,5} = 416532$: $b_{\tau_{17,5}} := fedcba$,
- $\tau_{17,6} = 436215$: $b_{\tau_{17,6}} := dcfbae$,
- $\tau_{17,7} = 512634$: $b_{\tau_{17,7}} := eabfcd$,
- $\tau_{17,8} = 542163$: $b_{\tau_{17,8}} := abcdef$,
- $\tau_{18,1} = 236145$: $b_{\tau_{18,1}} := abcdef$,
- $\tau_{18,2} = 365214$: $b_{\tau_{18,2}} := cfebad$,
- $\tau_{18,3} = 412563$: $b_{\tau_{18,3}} := cfebad$,
- $\tau_{18,4} = 541632$: $b_{\tau_{18,4}} := abcdef$,
- $\tau_{19,1} = 236415$: $b_{\tau_{19,1}} := abcdef$,
- $\tau_{19,2} = 263145$: $b_{\tau_{19,2}} := fedcba$,
- $\tau_{19,3} = 265314$: $b_{\tau_{19,3}} := eabdfc$,
- $\tau_{19,4} = 364215$: $b_{\tau_{19,4}} := cfdbae$,

- $\tau_{19,5} = 413562$: $b_{\tau_{19,5}} := cfdbae$,
- $\tau_{19,6} = 512463$: $b_{\tau_{19,6}} := eabdfc$,
- $\tau_{19,7} = 514632$: $b_{\tau_{19,7}} := fedcba$,
- $\tau_{19,8} = 541362$: $b_{\tau_{19,8}} := abcdef$,
- $\tau_{20,1} = 236514$: $b_{\tau_{20,1}} := abcdef$,
- $\tau_{20,2} = 265134$: $b_{\tau_{20,2}} := eabfdc$,
- $\tau_{20,3} = 346215$: $b_{\tau_{20,3}} := cdfbae$,
- $\tau_{20,4} = 362145$: $b_{\tau_{20,4}} := fedcba$,
- $\tau_{20,5} = 415632$: $b_{\tau_{20,5}} := fedcba$,
- $\tau_{20,6} = 431562$: $b_{\tau_{20,6}} := cdfbae$,
- $\tau_{20,7} = 512643$: $b_{\tau_{20,7}} := eabfdc$,
- $\tau_{20,8} = 541263$: $b_{\tau_{20,8}} := abcdef$,
- $\tau_{21,1} = 241635$: $b_{\tau_{21,1}} := abcdef$,
- $\tau_{21,2} = 315264$: $b_{\tau_{21,2}} := caebfd$,
- $\tau_{21,3} = 462513$: $b_{\tau_{21,3}} := caebfd$,
- $\tau_{21,4} = 536142$: $b_{\tau_{21,4}} := abcdef$,
- $\tau_{22,1} = 241653$: $b_{\tau_{22,1}} := abcdef$,
- $\tau_{22,2} = 316254$: $b_{\tau_{22,2}} := cafbed$,
- $\tau_{22,3} = 325164$: $b_{\tau_{22,3}} := debfac$,
- $\tau_{22,4} = 356142$: $b_{\tau_{22,4}} := fedcba$,
- $\tau_{22,5} = 421635$: $b_{\tau_{22,5}} := fedcba$,
- $\tau_{22,6} = 452613$: $b_{\tau_{22,6}} := debfac$,
- $\tau_{22,7} = 461523$: $b_{\tau_{22,7}} := cafbed$,
- $\tau_{22,8} = 536124$: $b_{\tau_{22,8}} := abcdef$,
- $\tau_{23,1} = 245163$: $b_{\tau_{23,1}} := abcdef$,
- $\tau_{23,2} = 361542$: $b_{\tau_{23,2}} := dafbce$,
- $\tau_{23,3} = 416235$: $b_{\tau_{23,3}} := dafbce$,
- $\tau_{23,4} = 532614$: $b_{\tau_{23,4}} := abcdef$,
- $\tau_{24,1} = 245613$: $b_{\tau_{24,1}} := abcdef$,
- $\tau_{24,2} = 261543$: $b_{\tau_{24,2}} := eafbcd$,
- $\tau_{24,3} = 316542$: $b_{\tau_{24,3}} := fedcba$,
- $\tau_{24,4} = 345162$: $b_{\tau_{24,4}} := dcbfae$,
- $\tau_{24,5} = 432615$: $b_{\tau_{24,5}} := dcbfae$,
- $\tau_{24,6} = 461235$: $b_{\tau_{24,6}} := fedcba$,
- $\tau_{24,7} = 516234$: $b_{\tau_{24,7}} := eafbcd$,
- $\tau_{24,8} = 532164$: $b_{\tau_{24,8}} := abcdef$,
- $\tau_{25,1} = 246135$: $b_{\tau_{25,1}} := abcdef$,
- $\tau_{25,2} = 362514$: $b_{\tau_{25,2}} := cfbead$,
- $\tau_{25,3} = 415263$: $b_{\tau_{25,3}} := cfbead$,
- $\tau_{25,4} = 531642$: $b_{\tau_{25,4}} := abcdef$,
- $\tau_{26,1} = 246153$: $b_{\tau_{26,1}} := abcdef$,
- $\tau_{26,2} = 351642$: $b_{\tau_{26,2}} := fedcba$,
- $\tau_{26,3} = 352614$: $b_{\tau_{26,3}} := cebfad$,
- $\tau_{26,4} = 361524$: $b_{\tau_{26,4}} := dafbec$,
- $\tau_{26,5} = 416253$: $b_{\tau_{26,5}} := dafbec$,
- $\tau_{26,6} = 425163$: $b_{\tau_{26,6}} := cebfad$,
- $\tau_{26,7} = 426135$: $b_{\tau_{26,7}} := fedcba$,
- $\tau_{26,8} = 531624$: $b_{\tau_{26,8}} := abcdef$,
- $\tau_{27,1} = 246315$: $b_{\tau_{27,1}} := abcdef$,
- $\tau_{27,2} = 263514$: $b_{\tau_{27,2}} := eadbfc$,
- $\tau_{27,3} = 264135$: $b_{\tau_{27,3}} := fedcba$,
- $\tau_{27,4} = 362415$: $b_{\tau_{27,4}} := cfbdae$,
- $\tau_{27,5} = 415362$: $b_{\tau_{27,5}} := cfbdae$,
- $\tau_{27,6} = 513642$: $b_{\tau_{27,6}} := fedcba$,
- $\tau_{27,7} = 514263$: $b_{\tau_{27,7}} := eadbfc$,
- $\tau_{27,8} = 531462$: $b_{\tau_{27,8}} := abcdef$,
- $\tau_{28,1} = 251463$: $b_{\tau_{28,1}} := abcdef$,
- $\tau_{28,2} = 253164$: $b_{\tau_{28,2}} := ebdfac$,
- $\tau_{28,3} = 316425$: $b_{\tau_{28,3}} := cafdbe$,
- $\tau_{28,4} = 364152$: $b_{\tau_{28,4}} := fedcba$,
- $\tau_{28,5} = 413625$: $b_{\tau_{28,5}} := fedcba$,
- $\tau_{28,6} = 461352$: $b_{\tau_{28,6}} := cafdbe$,
- $\tau_{28,7} = 524613$: $b_{\tau_{28,7}} := ebdfac$,
- $\tau_{28,8} = 526314$: $b_{\tau_{28,8}} := abcdef$,
- $\tau_{29,1} = 256314$: $b_{\tau_{29,1}} := abcdef$,
- $\tau_{29,2} = 263154$: $b_{\tau_{29,2}} := eadfbc$,
- $\tau_{29,3} = 326415$: $b_{\tau_{29,3}} := cbfdae$,
- $\tau_{29,4} = 364125$: $b_{\tau_{29,4}} := fedcba$,
- $\tau_{29,5} = 413652$: $b_{\tau_{29,5}} := fedcba$,
- $\tau_{29,6} = 451362$: $b_{\tau_{29,6}} := cbfdae$,
- $\tau_{29,7} = 514623$: $b_{\tau_{29,7}} := eadfbc$,
- $\tau_{29,8} = 521463$: $b_{\tau_{29,8}} := abcdef$,
- $\tau_{30,1} = 263415$: $b_{\tau_{30,1}} := abcdef$,
- $\tau_{30,2} = 264315$: $b_{\tau_{30,2}} := bfdcae$,
- $\tau_{30,3} = 513462$: $b_{\tau_{30,3}} := bfdcae$,
- $\tau_{30,4} = 514362$: $b_{\tau_{30,4}} := abcdef$,
- $\tau_{31,1} = 351624$: $b_{\tau_{31,1}} := abcdef$,
- $\tau_{31,2} = 426153$: $b_{\tau_{31,2}} := abcdef$.

Bibliography

[1] Noga Alon and Asaf Shapira. A characterization of the (natural) graph properties testable with one-sided error. *SIAM J. Comput.*, 37(6):1703–1727, 2008.

[2] Noga Alon and Joel H. Spencer. *The Probabilistic Method.* Wiley Series in Discrete Mathematics and Optimization. Wiley, 2008.

[3] David L. Applegate, William J. Cook, Sanjeeb Dash, and Daniel G. Espinoza. *QSopt_ex*, 2010. `http://www.dii.uchile.cl/~daespino/ESolver_doc`.

[4] Rahil Baber. Turán densities of hypercubes. Submitted; E-print available at `arXiv:1201.3587`.

[5] Rahil Baber and John Talbot. Hypergraphs do jump. *Combin. Probab. Comput.*, 20(2):161–171, 2011.

[6] Rahil Baber and John Talbot. A Solution to the 2/3 Conjecture. *SIAM J. Discrete Math.*, 28(2):756–766, 2014.

[7] Jozsef Balogh, Ping Hu, Bernard Lidický, and Hong Liu. Upper bounds on the size of 4- and 6-cycle-free subgraphs of the hypercube. *European J. Combin.*, 35:75–85, 2014.

[8] Jozsef Balogh, Ping Hu, Bernard Lidický, Oleg Pikhurko, Balázs Udvari, and Jan Volec. Minimum number of monotone subsequences of length 4 in permutations. *To appear in Combin. Probab. Comput.*

[9] Jozsef Balogh, Ping Hu, Bernard Lidický, Oleg Pikhurko, Balázs Udvari, and Jan Volec. Note on monotone subsequences of length 5 and 6 in permutations. Preprint.

[10] John-Adrian Bondy and U. S. R. Murty. *Graph theory.* Graduate texts in mathematics. Springer, 2007.

[11] Brian Borchers. CSDP, A C library for semidefinite programming. *Optimization Methods and Software*, 11(1-4):613–623, 1999.

[12] Christian Borgs, Jennifer Chayes, László Lovász, Vera T. Sós, Balázs Szegedy, and Katalin Vesztergombi. Graph limits and parameter testing. In *Proceedings of the Thirty-eighth Annual ACM Symposium on Theory of Computing*, STOC '06, pages 261–270, New York, NY, USA, 2006. ACM.

[13] Christian Borgs, Jennifer Chayes, László Lovász, Vera T. Sós, and Katalin Vesztergombi. Convergent sequences of dense graphs I: Subgraph frequencies, metric properties and testing. *Adv. Math*, 219(6):1801–1851, 2008.

[14] Christian Borgs, Jennifer Chayes, László Lovász, Vera T. Sós, and Katalin Vesztergombi. Convergent sequences of dense graphs II. Multiway cuts and statistical physics. *Ann. of Math.*, 176, 2012.

[15] Christian Borgs, Jennifer Chayes, and László Lovász. Moments of two-variable functions and the uniqueness of graph limits. *Geometric and Functional Analysis*, 19(6):1597–1619, 2010.

[16] Fan R.K. Chung and Ronald L. Graham. Quasi-random tournaments. *Journal of Graph Theory*, 15(2):173–198, 1991.

[17] Fan R.K. Chung, Ronald L. Graham, and Richard M. Wilson. Quasi-random graphs. *Combinatorica*, 9(4):345–362, 1989.

[18] James Cummings, Daniel Kráľ, Florian Pfender, Konrad Sperfeld, Andrew Treglown, and Michael Young. Monochromatic triangles in three-coloured graphs. *J. Combin. Theory Ser. B*, 103(4):489–503, 2013.

[19] Shagnik Das, Hao Huang, Jie Ma, Humberto Naves, and Benny Sudakov. A problem of Erdős on the minimum number of k-cliques. *J. Combin. Theory Ser. B*, 103(3):344–373, 2013.

[20] Persi Diaconis, Susan Holmes, and Svante Janson. Threshold graph limits and random threshold graphs. *Internet Math.*, 5(3):267–320, 2008.

[21] Paul Erdős. Problems and results on graphs and hypergraphs: Similarities and differences. In Jaroslav Nešetřil and Vojtěch Rödl, editors, *Mathematics of Ramsey Theory*, volume 5 of *Algorithms and Combinatorics*, pages 12–28. Springer Berlin Heidelberg, 1990.

[22] Paul Erdős and András Hajnal. On Ramsey like theorems, problems and results. In *Combinatorics. Proceedings of the Conference on Combinatorial Mathematics held at the Mathematical Institute, Oxford, 3-7 July, 1972.*, pages 123–140. Southend-on-Sea: The Institute of Mathematics and its Applications, 1972.

[23] Paul Erdős and Miklós Simonovits. An extremal graph problem. *Acta Math. Hungar.*, 22:275–282, 1971.

[24] Paul Erdős and Arthur Harold Stone. On the structure of linear graphs. *Bull. Amer. Math. Soc.*, 52(12):1087–1091, 12 1946.

[25] Paul Erdős and George Szekeres. A combinatorial problem in geometry. *Compositio Math.*, 2:463–470, 1935.

[26] Paul Erdős and Vera T. Sós. On Ramsey-Turán type theorems for hypergraphs. *Combinatorica*, 2(3):289–295, 1982.

[27] Victor Falgas-Ravry, Oleg Pikhurko, and Emil R. Vaughan. The codegree density of K_4^-. In preparation.

[28] Victor Falgas-Ravry and Emil R. Vaughan. Turán H-densities for 3-graphs. *Electron. J. Combin.*, 19(3):P40, 26pp., 2012.

[29] Victor Falgas-Ravry and Emil R. Vaughan. Applications of the semi-definite method to the Turán density problem for 3-graphs. *Combin. Probab. Comput.*, 22(1):21–54, 2013.

[30] Peter Frankl and Vojtěch Rödl. Some Ramsey-Turán type results for hypergraphs. *Combinatorica*, 8(4):323–332, 1988.

[31] Peter Frankl and Vojtěch Rödl. Extremal problems on set systems. *Random Structures Algorithms*, 20(2):131–164, 2002.

[32] Roman Glebov, Tereza Klimošová, and Daniel Kráľ. Infinite dimensional finitely forcible graphon. Submitted; E-print available at `arXiv:1404.2743`.

[33] Roman Glebov, Daniel Kráľ, and Jan Volec. Compactness and finite forcibility of graphons. Submitted; E-print available at `arXiv:1309.6695`.

[34] Roman Glebov, Daniel Kráľ, and Jan Volec. A problem of Erdős and Sós on 3-graphs. Submitted; E-print available at `arXiv:1303.7372`.

[35] Roman Glebov, Daniel Kráľ, and Jan Volec. An application of flag algebras to a problem of Erdős and Sós. *Electronic Notes in Discrete Mathematics*, 43(0):171 – 177, 2013.

[36] Roman Glebov, Daniel Kráľ, and Jan Volec. A problem of Erdős and Sós on 3-graphs. In Jaroslav Nešetřil and Marco Pellegrini, editors, *The Seventh European Conference on Combinatorics, Graph Theory and Applications*, volume 16 of *CRM Series*, pages 3–8. Scuola Normale Superiore, 2013.

[37] Adolph W. Goodman. On sets of acquaintances and strangers at any party. *Amer. Math. Monthly*, 66(9):778–783, 1959.

[38] Andrzej Grzesik. On the maximum number of five-cycles in a triangle-free graph. *J. Combin. Theory Ser. B*, 102(5):1061–1066, 2012.

[39] Bernd Gärtner and Jiří Matoušek. *Approximation Algorithms and Semidefinite Programming*. Springer, 2012.

[40] Hamed Hatami, Jan Hladký, Daniel Kráľ, Sergey Norin, and Alexander A. Razborov. On the number of pentagons in triangle-free graphs. *J. Combin. Theory Ser. A*, 120(3):722–732, 2013.

[41] Hamed Hatami, Jan Hladký, Daniel Kráľ, Sergey Norin, and Alexander A. Razborov. Non-three-colourable common graphs exist. *Combin. Probab. Comput.*, 21(5):734–742, 2012.

[42] James Hirst. The inducibility of graphs on four vertices. *J. Graph Theory*, 75(3):231–243, 2014.

[43] Jan Hladký, Daniel Kráľ, and Sergey Norin. Counting flags in triangle-free digraphs. In *European Conference on Combinatorics, Graph Theory and Applications (EuroComb 2009)*, volume 34 of *Electron. Notes Discrete Math.*, pages 621–625. Elsevier Sci. B. V., Amsterdam, 2009.

[44] Daniel Kráľ, Chun-Hung Liu, Jean-Sébastien Sereni, Peter Whalen, and Zelealem B. Yilma. A new bound for the 2/3 conjecture. *Combin. Probab. Comput.*, 22(3):384–393, 2013.

[45] Daniel Kráľ, Lukáš Mach, and Jean-Sébastien Sereni. A new lower bound based on Gromov's method of selecting heavily covered points. *Discrete Comput. Geom.*, 48(2):487–498, 2012.

[46] László Lovász. *Large Networks and Graph Limits.*, volume 60 of *Colloquium Publications*. American Mathematical Society, 2012.

[47] László Lovász and Balázs Szegedy. Limits of dense graph sequences. *J. Combin. Theory Ser. B*, 96(6):933–957, 2006.

[48] László Lovász and Balázs Szegedy. Regularity partitions and the topology of graphons. In Imre Bárány, József Solymosi, and Gábor Sági, editors, *An Irregular Mind*, volume 21 of *Bolyai Society Mathematical Studies*, pages 415–446. Springer Berlin Heidelberg, 2010.

[49] László Lovász and Balázs Szegedy. Testing properties of graphs and functions. *Israel J. Math.*, 178(1):113–156, 2010.

[50] László Lovász and Balázs Szegedy. Finitely forcible graphons. *J. Comb. Theory Ser. B*, 101(5):269–301, September 2011.

[51] László Lovász and Vera T. Sós. Generalized quasirandom graphs. *J. Comb. Theory Ser. B*, 98(1):146–163, 2008.

[52] Brendan D. McKay and Adolfo Piperno. Practical graph isomorphism, II. *Journal of Symbolic Computation*, 60(0):94 – 112, 2014.

[53] Joseph Samuel Myers. The minimum number of monotone subsequences. *Electron. J. Combin.*, 9(2):RP 4, 17 pp., 2002/03. Permutation patterns (Otago, 2003).

[54] Vladimir Nikiforov. The number of cliques in graphs of given order and size. *Trans. Am. Math. Soc.*, 363(3):1599–1618, 2011.

[55] Sergey Norin. Personal communication.

[56] Oleg Pikhurko. The minimum size of 3-graphs without a 4-set spanning no or exactly three edges. *European J. Combin.*, 32(7):1142–1155, 2011.

[57] Oleg Pikhurko and Alexander A. Razborov. Asymptotic structure of graphs with the minimum number of triangles. E-print available at `arXiv:1204.2846`.

[58] Oleg Pikhurko and Emil R. Vaughan. Minimum number of k-cliques in graphs with bounded independence number. *To appear in Combin. Probab. Comput.*

[59] Alexander A. Razborov. Flag algebras. *J. Symbolic Logic*, 72(4):1239–1282, 2007.

[60] Alexander A. Razborov. On the minimal density of triangles in graphs. *Combin. Probab. Comput.*, 17:603–618, 7 2008.

[61] Alexander A. Razborov. On 3-hypergraphs with forbidden 4-vertex configurations. *SIAM J. Discrete Math.*, 24(3):946–963, 2010.

[62] Alexander A. Razborov. Flag Algebras: An Interim Report. In *The Mathematics of Paul Erdős II*, pages 207–232. Springer, 2013.

[63] Alexander A. Razborov. On the Caccetta-Häggkvist conjecture with forbidden subgraphs. *J. Graph Theory*, 74(2):236–248, 2013.

[64] Christian Reiher. The clique density theorem. Submitted; E-print available at `arXiv:1212.2454`.

[65] Vojtěch Rödl. On universality of graphs with uniformly distributed edges. *Discrete Math.*, 59(1–2):125–134, 1986.

[66] Vojtěch Rödl and Mathias Schacht. Generalizations of the removal lemma. *Combinatorica*, 29(4):467–501, 2009.

[67] Vojtěch Rödl and Jozef Skokan. Applications of the regularity lemma for uniform hypergraphs. *Random Structures Algorithms*, 28(2):180–194, 2006.

[68] Wojciech Samotij and Benny Sudakov. On the number of monotone sequences. Submitted; E-print available at `arXiv:1405.6894`.

[69] Miklós Simonovits and Vera T. Sós. Ramsey-Turán theory. *Discrete Math.*, 229(1–3):293–340, 2001.

[70] Konrad Sperfeld. The inducibility of small oriented graphs. Submitted; E-print available at `arXiv:1111.4813`.

[71] William A. Stein et al. *Sage Mathematics Software (Version 6.1).* The Sage Development Team, 2013. `http://www.sagemath.org`.

[72] Paul Turán. Egy gráfelméleti szélsöértékfeladatról. *Mat. és Fiz. Lapok*, 48:436–451, 1941.

[73] Andrey Tychonoff. Über die topologische erweiterung von räumen. *Math. Ann.*, 102(1):544–561, 1930.

[74] Makoto Yamashita, Katsuki Fujisawa, Kazuhide Nakata, Maho Nakata, Mituhiro Fukuda, Kazuhiro Kobayashi, and Kazushige Goto. A high-performance software package for semidefinite programs: SDPA 7, 2010.