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Mathematical principles for the design of isostatic mount systems for dynamic structures

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Isostatic mounts are used in applications like telescopes and robotics to move and hold part of a structure in a desired pose relative to the rest, by driving some controls rather than driving the subsystem directly. To achieve this successfully requires an understanding of the structure of the coupled space of configurations and controls, and of the singularities of the mapping from the coupled space to the space of controls. It is crucial to avoid such singularities because generically they lead to large constraint forces and internal stresses which can cause distortion. In this paper we outline design principles for isostatic mount systems for dynamic structures, with particular emphasis on robots.

Keywords: Singularity, Configuration Space, Design of Mechanisms, Force Control, Dynamics.

2000 Math Subject Classification: 70E60, 14J17

1. Outline

Our aim is to characterise how to hold and move a linkage consisting of rigid components (e.g. rods) connected by joints in a unique and smoothly controllable configuration without high constraint forces or internal stresses, via coupling to a set of control variables. This problem came to our attention through a TSB-supported [1] collaboration with Metris UK (now Nikon Metrology) who asked us to assist in improving the design of a coordinate measurement robot they were developing, consisting of a seven-axis arm within a six-axis exoskeleton as described by Crampton (2008) and illustrated in Figure 5. Nikon Metrology call such mechanisms "isostatic mounts". The term seems to be used in the literature mainly for vibration isolation (e.g. to hold mirrors on spacecraft or telescopes for astronomy), but we consider the constraints to be stiff, leaving to the end some questions about the effects of compliance. Our paper is a mix of pedagogy aimed at non-specialists in mathematical engineering and original results obtained for the real world problem.

Systems in the general class under consideration can be described by:

- a "configuration space" X for the subsystem to be moved and held; if considered to be discon-

[1] TSB is the Technology Strategy Board, the UK's innovation agency which was recently renamed as "Innovate UK".

nected from the controlling constraints we suppose X to be a smooth [2] manifold (in simple terms, this means that for every configuration the set of all nearby configurations can be described by some number of local Cartesian coordinates, called the dimension of X, denoted $\dim X$); the case of manifold with boundary is also valid, but to avoid technicalities it is easier to ignore the boundaries.

- a "control space" Y for the variables under immediate control; we take Y to be a manifold too.
- a system of constraints that couple the subsystem to the controls; these limit the full system to a "coupled space" Z, which is a subspace of the producy $X \times Y$.

There are natural maps $\pi_X : Z \to X$ and $\pi_Y : Z \to Y$, which take a configuration of the coupled system to the configuration of the subsystem and the state of the controls, respectively.

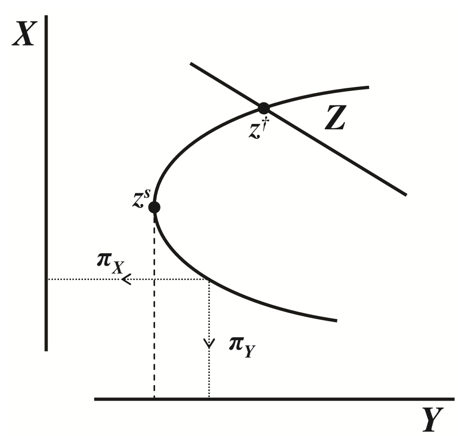

FIG. 1. Sketch of the relations between the coupled space Z, control space Y and subsystem configuration space X, also indicating a singular point $z^\dagger$ of the coupled space (where the two branches of Z cross) and a singularity z^s of the map π_Y (where infinitesimal changes to $z \in Z$ do not explore as many dimensions when projected to Y).

Here are some examples, cf. Craig (2004), Choset (2005) and Angeles (2007):

1. End effector on a six-axis arm, Pieper (1969). The configuration space X is $\mathbb{R}^3 \times SO(3)$, representing the position in 3-space $\mathbb{R}^3$ of a marked point on the end effector and the rotation [3] about the marked point required to bring the end effector into its orientation from a reference orientation. The control space Y is $\mathbb{T}^6$, a 6-dimensional torus representing the joint angles of the six-axis arm, or a subset of $\mathbb{T}^6$ to take into account limits on some of the joint angles or combinations of them. The coupled space Z is the subset of $X \times Y$ corresponding to the forward kinematics from Y to X given by assuming the end of the first axis is fixed to a reference frame. Z is a 6-torus, because each $y \in Y$ determines a unique $x \in X$ which varies smoothly with y.
2. Stewart platform, in which the pose of a hexagonal platform is controlled by the lengths of 6 legs to its corners from the corners of a hexagonal base plate, with universal joints at both ends of

[2] C^1-smoothness suffices for many aspects of this paper, but analysis of singularities generally requires more derivatives.
[3] $SO(3)$ denotes the set of rotations in 3D.

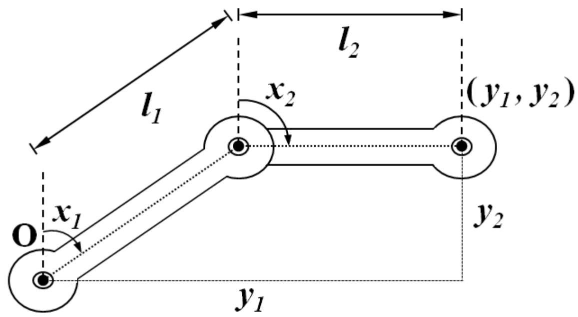

FIG. 2. A two-rod linkage in a vertical plane controlled by its free end, pivoted about a fixed point O (Example 3).

each leg. It has $X = \mathbb{R}^3 \times SO(3)$ again, $Y = \prod_{i=1\ldots6}(\ell_i, L_i)$ corresponding to the allowed range of lengths of the legs, and Z is the subset of $X \times Y$ corresponding to the leg-length constraints of Y on X. For further details, the reader is referred to the original work by Stewart (1965) and a more recent review, Dasgupta & Mruthyunjaya (2000).

3. Two-rod linkage in a vertical plane with one end pivoted about a fixed point, the other end controlled to move in the vertical plane; see Figure 2. The configuration space X is the set of angles (x_1,x_2) (forming a 2-torus) and the control space Y is the set of positions (y_1,y_2) in the vertical plane.

4. Two-axis arm with a rod attached to the second axis whose intersection with a sphere $v_1^2 + v_2^2 + v_3^2 = R^2$ centred near the top of the first axis can be moved over the sphere minus a neighbourhood of the downward axis 1. Then $X = \mathbb{T}^1 \times (x_2^-, x_2^+)$ where $\mathbb{T}^1$ is a circle representing the angle x_1 of joint 1 and $x_2^\pm$ denote the minimum and maximum angles for joint 2, measured from the upward vertical. Y is the sphere S^2 of radius R, minus a neighbourhood of its lowest point; it can be coordinatised by stereographic projection from the lowest point onto a plane tangent to the highest point, or perhaps preferably by (y_1,y_2) in the unit disk via $v_j = 2Ry_j\sqrt{1-|y|^2}$ for $j = 1,2$ and $v_3 = (1-2|y|^2)R$. Z is the subset of $X \times Y$ satisfying the constraint that the rod passes through the chosen point of the sphere; see Fig 3.

5. Two-axis arm contained inside a hollow two-axis arm, coupled by a ring fixed in a tube from the second outer axis through which a tight-fitting rod from the inner second axis is constrained to pass. Then $X = \mathbb{T}^1 \times (x_2^-, x_2^+)$ representing the joint angles of the inner arm, $Y = \mathbb{T}^1 \times (y_2^-, y_2^+)$ representing the joint angles of the outer arm, and Z is the subset of $X \times Y$ satisfying the constraint. See Figure 4. This is a simplified example of the lower third of a six-axis arm with exoskeleton, similar to that under development by Nikon Metrology (formerly Metris), a summary of which is provided by Thornby et al. (2009). The motivation for such a construction is that Nikon are designing a robot-mounted laser measurement arm and it is desirable to decouple the measuring device from the system which drives it, explained in Crampton (2008). The calculations and principles that follow are more easily applied to this simplified system, but may be generalised to the complete system – illustrated in Example 6.

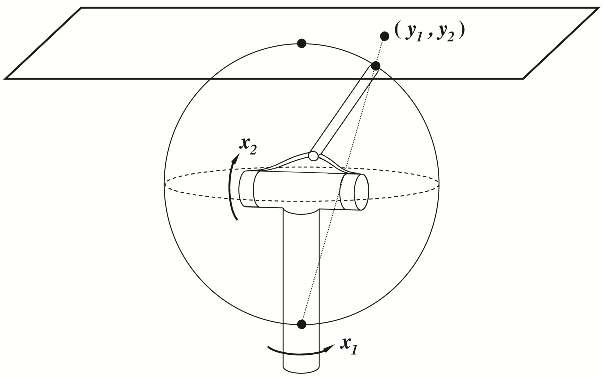

FIG. 3. A two-axis arm controlled by the intersection of a rod from axis 2 with a sphere, showing the use of stereographic projection to coordinatise the sphere minus the lowest point (Example 4).

6. A seven-axis arm within a six-axis exoskeleton. This is an extension of the previous example, repeating the basic plan twice more. The seventh joint of the inner arm simply controls articulation of the "wrist" (and hence the orientation of the end effector) and so the structure can, for all intents and purposes, be considered as a six-axis arm for fixed J_7 (the seventh joint angle).

We consider only holonomic constraints, meaning conditions on configurations, not just on velocities. Thus we exclude examples like controlling a track ball by rolling a plane over it.

The design problems to be solved are:

- to achieve a given configuration of the subsystem repeatably by moving the controls,
- to make the configuration depend smoothly on the controls, and
- to ensure that the stresses resulting from the constraints and external fields like gravity are not excessive.

The projection operator π_Y is, in general, many-to-one, but, mathematically, we would like π_Y to be a local diffeomorphism, so that π_Y^{-1} is a locally defined smooth map, and we would like π_X to be smooth, so that the composition $\pi_X \circ \pi_Y^{-1} : Y \to X$ from controls to subsystem configurations (locally) is smooth. Two obstructions to the desired behaviour are:

- "singular points" of the coupled space Z: these are the points of Z which do not have a neighbourhood diffeomorphic to a ball (e.g. $z^\dagger$ in Figure 1). Let Z^* be the set of non-singular points of Z; then each connected component of Z^* is a manifold, typically each of the same dimension. A recent paper on singular points for linkages is Blanc & Shvalb (2012).
- "singularities" of the map π_Y: these are the points of Z^* at which the rank of the derivative $D\pi_Y$ is less than full, meaning $\min(\dim Z^*, \dim Y)$ (e.g. z^s in Figure 1). We denote the set of singularities of π_Y by Σ, and its image by $\pi_Y(\Sigma)$.

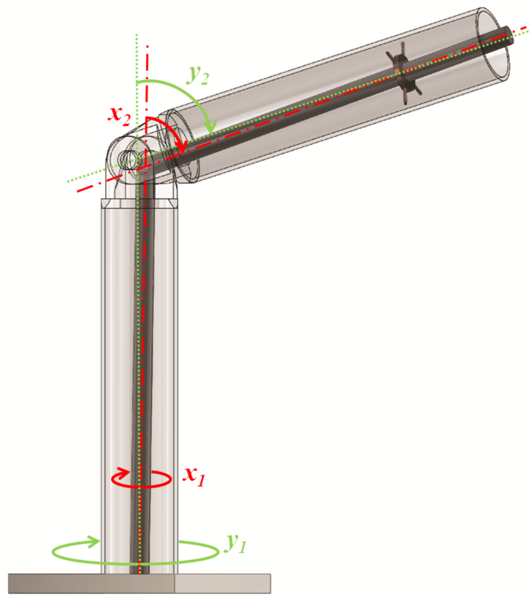

FIG. 4. A two-axis arm inside a two-axis exoskeleton, coupled by a rod from the inner arm passing through a ring in a tube from the outer arm (Example 5). Inner arm axes are shown in red (dot-dash), with joint angles (x_1, x_2); outer arm axes are shown in green (dotted), with joint angles (y_1, y_2).

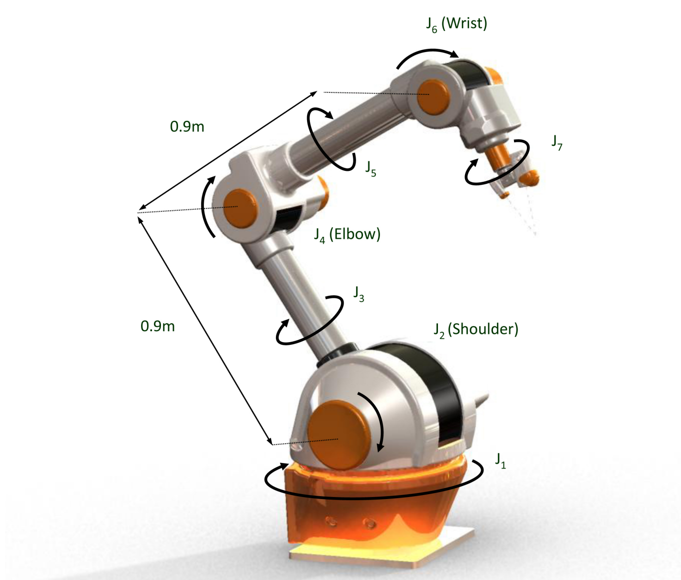

FIG. 5. The Metris (Nikon Metrology) RCA: a seven-axis arm inside a six-axis exoskeleton for coordinate-measuring purposes, using a laser line scanner. Joint angles of the exoskeleton are conventionally labelled J_n, counting upwards from the base.

It is important to distinguish these two types of singular behaviour; to aid in this, we use the term "singular point" for the first and "singularity" for the second. The robotics literature contains a variety of terminology, some of which is reviewed by Ider (2005) for example, but it is not used consistently and the case of singular points is often overlooked. They should also be distinguished from "coordinate singularities", points where a coordinate system is not locally Cartesian, like latitude-longitude coordinates at the poles. There is a large literature on singularities of mechanisms; see for example the database Donelan & Azzato (2014) and the survey by Donelan (2010) for the case of manipulators.

The main content of our paper is five design principles for isostatic mount systems for dynamic structures. In the process of demonstrating them, we notably solve the configuration space of Example 6 in § 2.3. This is followed by sections addressing the questions of how to live with singularities if they can not be avoided, how vibration frequencies behave near singularities, and some concluding remarks.

2. Five design principles

2.1 *Equality constraints*

The first design principle is that the constraints should be equality constraints, not one-sided inequality constraints. Else π_Y is typically locally many-to-one and the motion of the subsystem is typically non-smooth and non-repeatable. Thus constraints with backlash, for example, are not a good idea. One-sided constraints might be used in addition to equality constraints, as safety measures to prevent undesired outcomes which in principle ought not to happen, but they should not be expected to achieve reproducible let alone smooth control if they are ever invoked.

2.2 *Matching the numbers of constraints and degrees of freedom*

The second design principle is that the number N of constraints should equal the number of degrees of freedom of the subsystem ($\dim X$). If there are n fewer constraints than degrees of freedom (i.e. $N = \dim X - n$) then in general the set of compatible configurations for fixed control state is a manifold of dimension n, so the configuration is not locally uniquely determined by the controls. If there are n more constraints than degrees of freedom (i.e. $N = \dim X + n$) then in general there are no compatible configurations, except on a submanifold of the control space of codimension n, which means that the span of n directions of control can not be used; in reality compatible configurations may also be attained outside this subset but at the expense of imposing strains, deforming components which are in principle rigid; the space of deformation modes generated has dimension n and to such strains will correspond large stresses and large constraint forces.

The way to count constraints may require some elaboration. When counted correctly, the number N of constraints governs $\dim Z^*$ by $\dim Z^* = \max(\dim X + \dim Y - N, 0)$ (recall Z^* is the non-singular part of Z). Some constraints are two-dimensional, e.g. that an axis pass through a given point in 3-space, or three-dimensional, e.g. that a point on an axis be a given point in 3-space. It might also be that some constraints are not independent of others, so they should not be counted. For example, a ring that makes an axis pass through a given point adds nothing to the count if the axis is clamped to a fixed base and the given point is on the axis; but if the given point fails to be exactly on the axis then the effect of the ring has to be added to the count. Clamping an axis to a fixed base can itself be regarded as a constraint (of dimension 5 since a point on the axis is fixed in 3-space and the direction of the axis is fixed on a 2-sphere) and this view would allow one to compute the forces on the clamp, but for simplicity we will treat it as fixed.

One might also wish to match the number of constraints to the number of controls. This is not crucial, however. If there are more controls than constraints then one can typically realise a given configuration of the subsystem by a manifold of control states; there is some redundancy, e.g. if a 7-axis arm is used to control the (6-dimensional) pose of an end effector. If there are n more constraints than controls then the set of configurations of the subsystem that can be realised is typically a submanifold of X of codimension n, which would be bad if one wanted to explore all directions in X, but such a restriction might have a valid purpose, so we do not rule it out.

2.3 *Coupled space a manifold*

The third design principle is that the coupled space Z for the whole system should be a manifold. Equivalently, it should have no singular points. Examples of manifolds include spheres and tori. Examples of topological spaces that are not manifolds include figure of eight curves, the union of two intersecting planes, and cones. For theory of manifolds, see Spivak (1965). The problems with a coupled space that is not a manifold are that:

- from a singular point there may be more than one direction the subsystem can move for a given direction of controls, and
- the coupled space is likely to undergo qualitative changes for arbitrarily small changes in design parameters, e.g. a figure of eight curve can deform into a closed loop or two closed loops, if thought of as a level curve of a height function above two dimensions; it is not robust to design a coupled space with singular points.

In contrast, if the coupled space is defined by a level set of N smooth functions from $X \times Y$ to $\mathbb{R}$ whose derivatives are linearly independent on the whole level set, then it is a manifold and any C^1-small [4] change in the functions makes no qualitative change to it.

One might think that singular points would arise in only pathological examples of coupled spaces, but they occur for Example 5 if no offsets are introduced. Denoting the joint angles of the inner arm by (x_1, x_2) and of the outer arm by (y_1, y_2), the coupled space Z is defined by $\{x_1 = y_1, x_2 = y_2\} \cup \{x_1 = y_1 + \pi, x_2 = -y_2\} \cup \{x_2 = y_2 = 0\}$ (we assume $-\pi < x_2^- < 0 < x_2^+ < \pi$ and $-\pi < y_2^- < 0 < y_2^+ < \pi$ and so leave out the unphysical possibility $\{x_2 = y_2 = \pi\}$ because it represents the arms folding back along themselves; we also assume x_2 is not far from y_2, to exclude the unphysical possibility that $x_2 \approx y_2 + \pi$, in which it is the backward extension of the rod that passes through the ring). Each of these pieces is a manifold but the third intersects the first along a circle, and the second along another circle. These two circles form the set of singular points of Z. Adding typical offsets, however, makes the coupled space into a manifold. Examples for some choices of offsets are shown in Figure 6, where $\delta_1 = x_1 - y_1$.

It can be seen that as the controls (y_1, y_2) are varied, x_1 tracks y_1 quite well (i.e. δ_1 is close to 0) except near $y_2 = 0$, which is where the ideal case (free from offsets) has a curve of singular points. Near $y_2 = 0$, large excursions of δ_1 from 0 occur, except for paths near two special choices of y_1 in the first figure.

Figure 6 was computed by noting that the two components of constraint can be written in the form

$$\begin{aligned} A + B\sin x_2 + C\cos x_2 &= 0 \\ a + b\sin x_2 + c\cos x_2 &= 0, \end{aligned} \tag{2.1}$$

[4] A C^1-small function is one whose values and first derivatives are small.

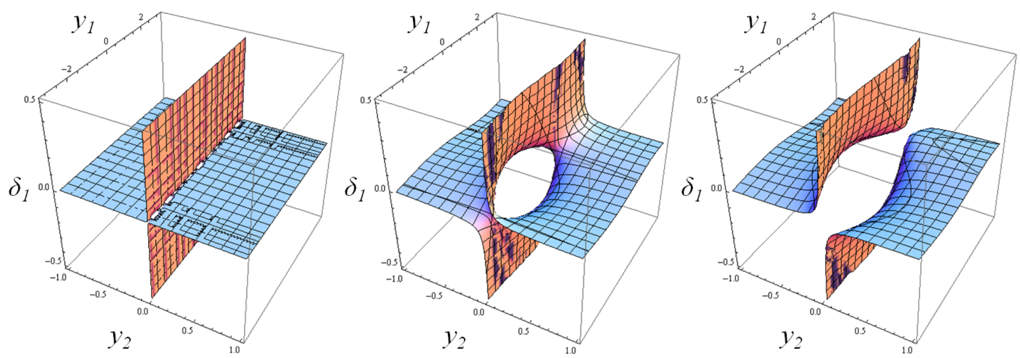

FIG. 6. Projections onto (δ_1, y_1, y_2) of the coupled space for a two-axis arm inside a two-axis arm (Example 5) for the ideal case and two choices of offsets, where $\delta_1 = x_1 - y_1$ and angles are measured in radians. In the "ideal" case with no offsets the coupled space consists of two intersecting planes; as offsets are introduced these planes deform and eventually separate.

with the coefficients A, B, C, a, b, c being functions of x_1, y_1, y_2 and various length and offset parameters – in the manner of Denavit & Hartenberg (1964), illustrated in Figure 7. Putting $t = \tan(x_2/2)$, they become quadratic equations in t:

$$\begin{aligned} A(1+t^2)+2Bt+C(1-t^2) &= 0 \\ a(1+t^2)+2bt+c(1-t^2) &= 0. \end{aligned} \tag{2.2}$$

One can eliminate t between the two equations, yielding the single equation

$$(bC - Bc)^2 = (bA - Ba)^2 + (Ac - aC)^2. \tag{2.3}$$

But this includes unphysical configurations with x_2 near $y_2 + \pi$ as well as the desired ones with x_2 near y_2. To select only the desired ones, we instead solved the first of Equations 2.1 for the solution t near $\tan(y_2/2)$:

$$t = \frac{-B + \sqrt{B^2 + C^2 - A^2}}{A - C}, \tag{2.4}$$

and substituted this into the second equation, obtaining

$$(bA - Ba + Bc - bC)(-B + \sqrt{B^2 + C^2 - A^2}) = (A - C)(Ca - cA), \tag{2.5}$$

whose solution surface in the 3D space of (δ_1, y_1, y_2) was plotted using *Mathematica*'s ContourPlot3D command.

The same analysis can be used to study Example 6. Indeed, the configuration space for (x_1, x_2, y_1, y_2) is independent of the angles (x_k, y_k), $k \geqslant 3$. So the configuration space for the first stage of the device is just that for Example 5. The possibilities for the second stage (x_3, x_4, y_3, y_4), however, depends on the configuration, because the state of the first stage affects the offsets for the second stage. But for a given position (x_1, x_2, y_1, y_2) of the first stage, the configuration space for the second stage is of the same form

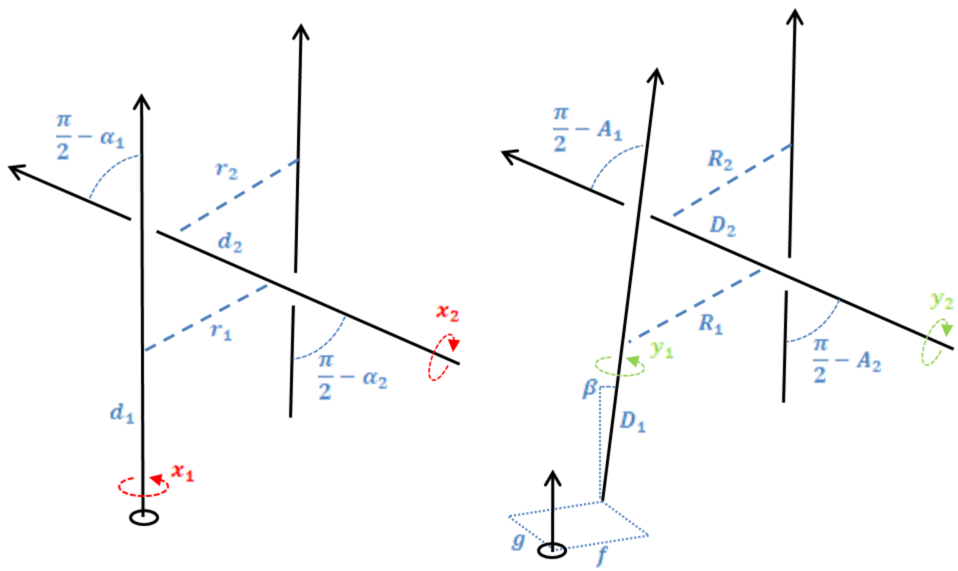

FIG. 7. Paramerisation of offsets considered for the inner arm (left) and outer arm (right). Axes are shown in black, offsets in blue and joint angles in red and green for the inner and outer arms respectively (in correspondence with the labelling convention used in Figure 4). Offsets may be described by a combination of linear translations (such that the axes do not pass through a common point) and rotations (such that successive axes cease to be orthogonal). Additionally, the outer arm may not originate from the same point as the inner arm (described by the linear offsets f and g) and may be skewed (described by the angle β).

again. Similarly, the configuration space for the third stage (x_5, x_6, y_5, y_6) is of the same form again, with offsets determined by the second stage configuration.

Coupled spaces with singular points often fall into a class of topological spaces called "stratified manifolds". These are topological spaces with a decomposition into manifolds of various dimensions, called *strata*, such that the closure of each is its union with some strata of lower dimension. Thus for Example 5 with no offsets, the coupled space is a stratified manifold, decomposing into 6 annuli and 2 circles, the closure of each annulus including one or both of the circles. If one succeeds in following design principle 2, however, there is no need to pursue stratified manifolds further. On the other hand, designs like Example 6 require a deep understanding of the unfolding of stratified manifolds under generic perturbation.

For this section, we henceforth take Z to be a manifold.

2.4 *Avoid singularities of π_Y*

The fourth design principle is that singularities of the map π_Y from the coupled space to the control space should be avoided. This is well known, e.g. Long & Paul (1992); Craig (2004); Choset (2005) and Angeles (2007), but it is important to spell it out.

It is not essential to design systems such that there are no singularities of π_Y, and in many situations their presence may be inevitable (e.g. Gottlieb (1986, 1988)), but positioning and even slow motion control paths should be chosen to avoid any singularities. This may not always be possible, however, as the singularities typically separate Z into pieces between which one might want to move; how to live with singularities will be discussed in section 3.

We begin with the elementary Example 3. Then X is a 2-torus, which can be parametrised by the angles x_1, x_2 of the two rods from the vertical. Y is the vertical plane, and can be parametrised by horizontal coordinate y_1 and vertical coordinate y_2, relative to the fixed pivot. Z can be considered to be the same as X because each configuration $x \in X$ determines a unique $y \in Y$. Then the map $\pi_Y : (x_1, x_2) \mapsto (y_1, y_2)$ is given by

$$\begin{aligned} y_1 &= \ell_1 \sin x_1 + \ell_2 \sin x_2 \\ y_2 &= \ell_1 \cos x_1 + \ell_2 \cos x_2, \end{aligned} \tag{2.6}$$

where ℓ_1, ℓ_2 are the lengths of the two rods (between the pivot points). So the derivative $D\pi_Y$ is represented by the matrix

$$\begin{bmatrix} \ell_1 \cos x_1 & \ell_2 \cos x_2 \\ -\ell_1 \sin x_1 & -\ell_2 \sin x_2 \end{bmatrix}, \tag{2.7}$$

the determinant of which is $\ell_1 \ell_2 \sin(x_1 - x_2)$, and is zero if and only if $x_1 - x_2 \in \{0, \pi\}$ modulo 2π. Thus the set Σ of singularities of π_Y consists of two circles $\Sigma_0 = \{x_1 = x_2\}$ and $\Sigma_\pi = \{x_1 = x_2 + \pi\}$ on Z, corresponding respectively to the fully extended configurations and the doubled-back configurations. Σ separates Z into two parts where $x_1 - x_2 \in (0, \pi)$ or $(\pi, 2\pi)$. The image of Σ under π_Y consists of two circles in Y bounding an annulus A of accessible control states. To each interior point of A correspond two compatible configurations in Z, which merge as the controls go to either boundary of A. See Figure 8.

Next we explain what goes wrong near singularities. A consequence of the second design principle is that $\dim Z = \dim Y$, so the derivative $D\pi_Y$ is represented by a square matrix, and we will restrict attention to this case. A square matrix has full rank if and only if it is invertible. Equivalent formulations are that it has non-zero determinant or its kernel is zero. Thus singularities of π_Y are the places where $D\pi_Y$ is not invertible. Away from singularities, $D\pi_Y$ is invertible and by the implicit function theorem this implies that the controls y determine a locally unique configuration $z(y)$ of the coupled system. Furthermore, it depends smoothly on y, and using the chain rule, the velocity $\dot{z}$ of response of the system to a velocity $\dot{y}$ of controls is given by[5]

$$\dot{z} = D\pi_Y^{-1} \dot{y}. \tag{2.8}$$

We deduce also that the controls y determine a locally unique configuration $x = \pi_X z(y)$ of the subsystem, depending smoothly on y and with velocity

$$\dot{x} = D\pi_X D\pi_Y^{-1} \dot{y}. \tag{2.9}$$

In contrast, at a singularity of π_Y the possible velocities of control are limited to a subspace of lower dimension than $\dim Y$, because $\dot{y} = D\pi_Y \dot{z}$ and $D\pi_Y$ does not have full rank. Thus there is certainly not a smooth local map from controls to configuration.

[5]Note that $D\pi_Y^{-1}$ can be interpreted as either $(D\pi_Y(z))^{-1}$ or $D(\pi_Y^{-1})(y)$, since they are equal.

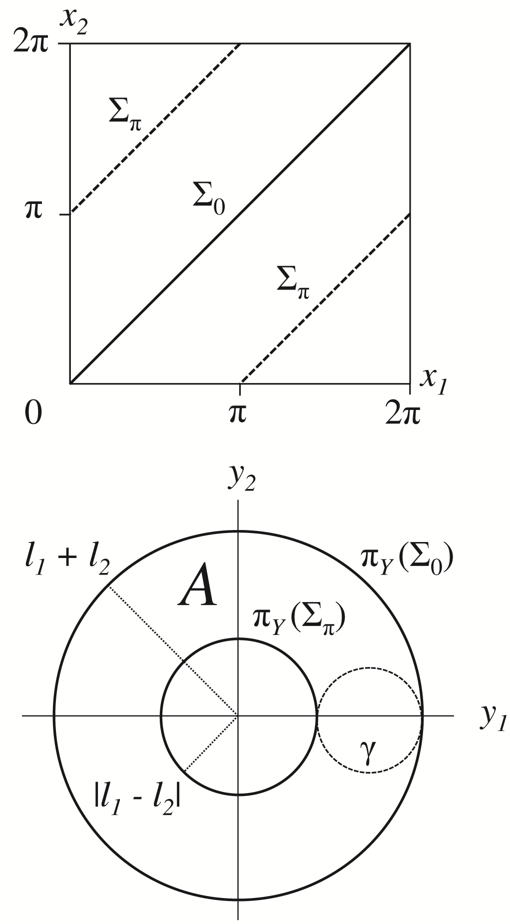

FIG. 8. The singularities of the map π_Y from the coupled space Z to the control space Y for the two-link Example 3: (a) the set of singularities form two circles Σ_0, Σ_π in Z, (b) the image of Z in Y is an annulus A bounded by the images of Σ_0 and Σ_π; the circle γ is the track of (y_1, y_2) as x_2 makes one revolution at fixed $x_1 = \pi/2$ (case $\ell_2 < \ell_1$).

The typical [6] situation, known as a "fold singularity", is that in suitable local coordinate systems for Z and Y centred on the singularity and its image, π_Y takes the form

$$\begin{aligned} y_1 &= z_1^2 \\ y_j &= z_j, j > 1. \end{aligned} \tag{2.10}$$

Thus locally, only a half-space $\{y_1 \geqslant 0\}$ of controls is accessible, and as y_1 approaches 0, two compatible configurations with $z_1 = \pm\sqrt{y_1}$ merge. For constant $\dot{y}_1 = -v < 0$, the velocities of these two configurations go to infinity like $\mp v/(2\sqrt{y_1})$, until y_1 hits zero, when it has to stop or else deform components of the system. Note that the whole set $\{z_1 = 0\}$ in these coordinates consists of singularities of

[6] C^2-stable.

the same form. It is called a “fold curve”, thinking of the case where Y and Z have two dimensions, but the same term is used in higher dimensions too.

In control spaces of dimension greater than 1 one is likely to come across more complicated types of singularity than just folds. These are typically singularities at which several fold curves meet in a non-trivial way. The simplest example is a “cusp singularity”, around which coordinates can be chosen so that

$$\begin{aligned} y_1 &= z_1^3 - z_2 z_1 \\ y_j &= z_j, j > 1. \end{aligned} \tag{2.11}$$

Then two fold curves $z_2 = 3z_1^2$ for $z_1 \neq 0$ join in such a way that their images in Y form a semi-cubic cusp $y_1 = -2z_1^3, y_2 = 3z_1^2$ (parametrised by z_1) or $27y_1^2 = 4y_2^3$. For controls in the region between the images of the two fold curves there are locally three compatible configurations; two of these merge at the fold curve and annihilate each other, leaving just one on the other side; all three configurations merge as the controls approach the cusp point. For an introduction to singularity theory, see Bruce & Giblin (1992) and for a definitive survey, see Arnol’d et al. (1998).

A second problem with singularities of π_Y is that static forces are typically infinite there. Let us start by considering just the control forces. These are the forces F conjugate to the control variables y, required to maintain the system in equilibrium against all other forces G, like gravity and static friction. The forces F are measured in units such that the work they do by an infinitesimal displacement δy in the controls is the scalar product $F^T \delta y$ (where superscript T denotes transpose). Thus conjugate to a linear displacement is a linear force, conjugate to an angular displacement is a torque, and so on. Similarly for G with respect to changes in configuration δz. Then d’Alembert’s principle of virtual work leads to the force balance equation:

$$G^T = -F^T D\pi_Y. \tag{2.12}$$

It follows that away from singularities of π_Y, bounded forces G can be balanced by bounded control forces F. At singularities of π_Y, however, there are directions of forces G which cannot be balanced by any finite control force F, and as one approaches a singularity, typically F goes to infinity.

The two-link system Example 3 provides a simple illustration. Under gravitational force given by the negative gradient of the potential

$$V = \frac{1}{2} m_1 \ell_1 g \cos x_1 + m_2 g(\ell_1 \cos x_1 + \frac{1}{2}\ell_2 \cos x_2), \tag{2.13}$$

the equilibrium control force is readily computed by the principle of virtual work to have horizontal and vertical components

$$\begin{aligned} F_1 &= \frac{(m_1 + m_2) g \sin x_1 \sin x_2}{2\sin(x_1 - x_2)} \\ F_2 &= \frac{(\frac{1}{2}m_1 + m_2) g \sin x_1 \cos x_2 - \frac{1}{2} m_2 g \sin x_2 \cos x_1}{\sin(x_1 - x_2)}, \end{aligned} \tag{2.14}$$

for which the radial component goes to infinity as $x_1 - x_2$ goes to 0 or π (except at $x_1, x_2 \in \{0, \pi\}$).

The problem with forces going to infinity is not only for control forces but also most internal forces in the system, in particular the forces on the constraints that couple the controls to the subsystem. A simple way to extend the analysis to compute the (equal and opposite) internal forces at some location is to imagine disconnecting the system there, augmenting the control space Y by additional control variables measuring the displacement between the disconnected parts (which could be a linear displacement in 3D to find an internal linear force or an angular displacement to find an internal torque or bending force). Then Z is also augmented by the effect of this displacement, and π_Y is augmented. The effect on $D\pi_Y$ at undisplaced configurations is to augment its matrix by adding blocks in the following form:

$$D\pi_Y = \begin{bmatrix} A & A' \\ 0 & I \end{bmatrix}, \tag{2.15}$$

where A is the original matrix for $D\pi_Y$, A' is a matrix representing how the new displacements affect the configuration, 0 is a matrix of zeroes, and I the identity matrix of dimension corresponding to the new displacements. The static force balance equation (2.12) with F, G augmented to $(F, F'), (G, G')$ gives $G'^T = -F^T A' - F'^T$ and hence

$$F'^T = -F^T A' - G'^T. \tag{2.16}$$

Thus, as F goes to infinity at a singularity, so typically does F', the only exception being if A' happens to give zero in the direction of F.

It follows that most internal forces typically go to infinity at singularities. As an illustration, we compute the linear force at the joint between rods 1 and 2 of the two-link Example 3. In this case, disconnecting the joint by a displacement (u_1, u_2) the matrix A' is the identity, and the potential energy is augmented by $m_2 g u_2$, so $G'^T = (0, -m_2 g)$ and $F'^T = -F^T + (0, m_2 g)$, which goes to infinity at the singularities in the same way as F.

A standard example of computation of singularities is for the control of a 6-axis arm of "321 structure" by motion of its end effector, Bruyninckx & De Schutter (1998), a case of Example 1. The first axis is assumed to be clamped to a fixed base plate. The subsystem configuration space X is a 6-torus representing the joint angles x_j, $j = 1 \dots 6$ of the 6 axes (or that part of the 6-torus that can be achieved without steric hindrance). The control space Y is the set of accessible poses (positions and attitudes) of an end effector; it is 6-dimensional (three coordinates for position of a reference point on the end effector and three coordinates for its attitude) and can be written as $\mathbb{R}^3 \times SO(3)$. The coupled space Z is essentially the same 6-torus as X, because the end effector is attached rigidly to the sixth axis. The joint angles determine the end effector pose, but not necessarily vice versa. For example, there are end effector poses for which some of the axes can spin round freely. The mapping π_Y from the coupled space to $\mathbb{R}^3 \times SO(3)$ has $\det D\pi_Y = -d^h l_2 l_3 s_3 s_5$, where l_j are arm lengths, s_j are the sines of the joint angles x_j and $d^h = s_2 l_2 + \sin(x_2 + x_3) l_3$ is the horizontal distance in the arm plane from axis 1 to the wrist. Thus the singularities of π_Y correspond to three situations:

- $x_3 = 0$ "arm-extended singularity" ($x_3 = \pi$ is excluded by steric hindrance)
- $x_5 = 0$ "wrist-extended singularity" ($x_5 = \pi$ is excluded)
- $d^h = 0$ "wrist-above-shoulder singularity": the wrist centre lies on the first axis

With offsets or other designs, the singularities move and are in general more complicated to compute. Considerable energy and ingenuity has gone into designs with smaller singularity sets. But some singularities are inevitable: configurations of maximum reach for a given point on the end effector are always singularities, so there will always be a corresponding 5-dimensional singularity set. Even if reach is ignored, there may be inevitable singularities for attitude (e.g. Gottlieb (1986, 1988) for Example 1).

The roles of X and Y are usually inverted in most treatments of this example: the controls Y are taken to be the 6 joint angles and the end effector is the subsystem X to be controlled (as in Example 1). Then there are no singularities of π_Y, because Z is just the same 6-torus as Y with the implied end effector poses added on. So what has singularities in that interpretation is π_X. They do not cause any infinite velocities or forces, but they do restrict the range of achievable velocities in X, by equation (2.9). A way they could be construed as giving infinite velocities is if a desired motion of the end effector is specified (e.g. to scan with a laser measurement head) and π_X is singular somewhere along the path then to attain the desired motion will require infinite control velocity there (and typically the motion will be unrealisable thereafter).

Example 4 provides an illuminating illustration where the effects of offsets can be analysed in some detail. Recall that it is a two-axis arm with control of the point at which an end effector (which we call axis 3, though no rotation takes place about it) passes through a sphere. If all is "ideal" (rotational symmetry of control sphere about axis 1, axes 1 and 3 perpendicular to axis 2, all three axes intersecting in a common point), then rank of $D\pi_Y$ is lost if and only if $x_2 = 0$ or π; let us ignore the latter as unrealisable. The singularity set Σ is a circle (it can be parametrised by x_1) and its image in the control space is a single point (N pole). For each control point, which we label by longitude $\hat{y}_1$ and colatitude $\hat{y}_2$ (i.e. angle from the N pole), other than the N pole, there are two possible configurations: $x_1 = \hat{y}_1, x_2 = \hat{y}_2$ and $x_1 = \hat{y}_1 + \pi$, $x_2 = -\hat{y}_2$. If we resolve the longitude-colatitude coordinate singularity in the control space by using Cartesian position η_1, η_2 of axis 3 in the plane tangent to the N pole, then the mapping from (x_1, x_2) to (η_1, η_2) has Jacobian determinant $\Delta = r^2 \frac{\sin x_2}{\cos^3 x_2}$ (where r is the radius of the sphere), which passes 0 transversely at $x_2 = 0$ ($\frac{\partial \Delta}{\partial x_2} \neq 0$). So if one introduces offsets, the circle of singularities moves a little (by the implicit function theorem), but its image in control space may change qualitatively. The simplest form of offset, displacing axis 3 by distance ℓ along axis 2 from axis 1 but keeping right angles between axis 2 and the other two, preserves rotational symmetry about axis 1 and turns $\pi_Y(\Sigma)$ into a circle about the N pole; the disc it surrounds has no preimages and points of Y just outside this disc have two preimages. This is the case also for all choices of offset preserving rotational symmetry about axis 1 (i.e. for which axis 1 passes through the centre of the sphere), except those special combinations for which the radius of the circle is zero. Breaking the rotational symmetry a little deforms $\pi_Y(\Sigma)$ from a round circle but makes no qualitative change. We had thought that breaking rotational symmetry from cases where the image of Σ was just the N pole might produce more complicated image sets, like the 4-cusped "astroid" of Goryunov (2001) and Chekanov & Pushkar (2005), but we did not find such a case.

Figure 9 shows the control space (accessible end effector positions) for the "ideal" case with no offsets and some scenarios involving basic offsets (taken in the same sense as Figure 7) in which axes 2 and 3 are not orthogonal ($\alpha_2 \neq 0$) and axes 2 and 3 are not concurrent ($r_2 \neq 0$). In the ideal scenario the end effector explores the surface of a sphere; the image of the singularity set Σ is the N pole ($x_2 = 0$), at which point all values for x_1 yield the same position in control space. The image of Σ in control space deforms in the presence of certain offsets; in particular, when a "skew" is introduced ($\alpha_2 \neq 0$) the singularity at the N pole deforms to a set of points on a bounded surface, resulting in a set of inaccessible physical configurations (a hole at the top of the configuration space). When axes cease to be concurrent, the degeneracy of the control space is broken; the two possible configurations ($x_1 = \hat{y}_1$, $x_2 = \hat{y}_2$ and $x_1 = \hat{y}_1 + \pi$, $x_2 = -\hat{y}_2$) no longer map to the same point and the surface divides in two.

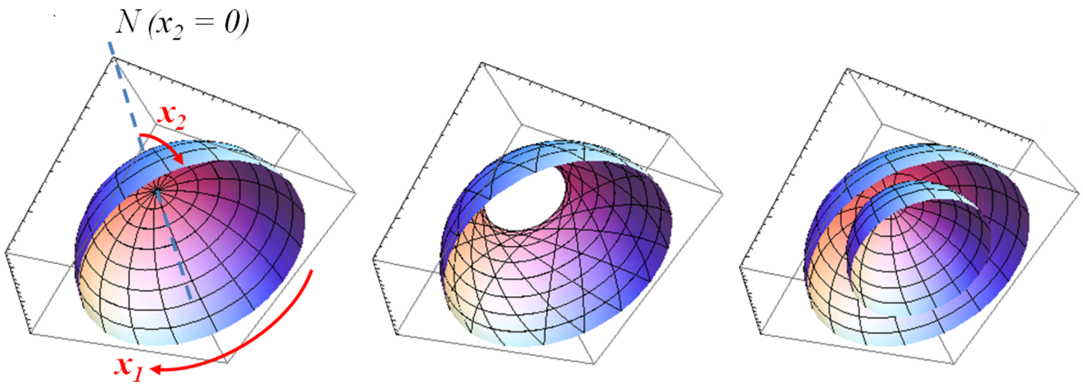

FIG. 9. Control space for a two-axis arm for various choices of offset: the “ideal” case without offsets (left), the case when axes 2 and 3 are not orthogonal (centre) and the case when axes 2 and 3 are not concurrent (right). For the purposes of clarity only half of the control space is shown, i.e. $-\pi/2 \leqslant x_2 \leqslant \pi/2$, with $x_2 = 0$ at the N pole.

2.5 *Keep norm of inverse matrix moderate*

The fifth principle is that the constraints should act in directions where the effects of configuration change are significant. More formally, they should be chosen to make $D\pi_Y^{-1}$ bounded by a not too large constant (with respect to suitable norms on tangent spaces to the control and coupled spaces), or operation should be restricted to a domain where this holds. This is because even if singularities of π_Y are avoided, many of the undesirable things that happen at singularities also happen when $D\pi_Y^{-1}$ is large (large forces and velocities).

3. How to live with singularities

Notwithstanding the above principles, there may be systems for which it is infeasible to avoid singularities.

The set Σ of singularities is typically of codimension one in the coupled space Z, so may separate Z into more than one component. If applications of the device require it to pass from one component of $Z \setminus \Sigma$ (the set of points of Z which are not in Σ) to another, then one has to cross Σ (we suppose Z is connected). First we address the question whether one really needs to pass from one component of $Z \setminus \Sigma$ to another. Take the two-axis Example 4: Z is a two-torus, generated by angles x_1, x_2, and Σ consists of the sets $x_2 = 0$ or π (corresponding to the rod pointing vertically up or down). Thus, if crossing singularities is forbidden, we are stuck in $x_2 \in (0, \pi)$ or $(-\pi, 0)$, and one cannot go from positive to negative values of x_2. An equivalent effect can be achieved, however, by reducing x_2 to a small positive value, rotating x_1 by π and then increasing x_2 again. So apart from having to program a more complicated path, nothing is lost here by requiring singularities to be avoided. Similarly for Example 6, one can achieve the same end effector pose in eight different ways, as illustrated in Figure 10.

Nevertheless, in more complicated devices it may indeed be infeasible to avoid singularities, so we address the question of how to traverse them safely.

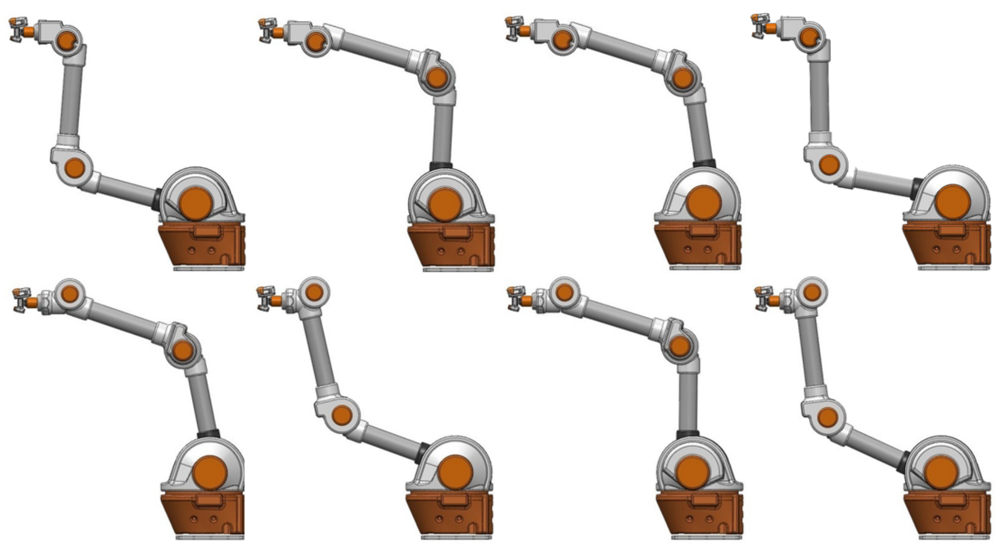

FIG. 10. There are, in general, muliple different “poses” (configurations of joint angles) which yield the same end effector position.

To keep velocities bounded, it suffices to move the controls tangentially to the image of the singularity set whenever it is required to cross a singularity. This strategy requires the controller to have a good knowledge of the singularity set or some automatic detection system that feels where it is when it gets close. Unfortunately, this strategy does not solve the problem of infinite forces at singularities.

To solve the problems of infinite forces and velocities simultaneously, a solution is to use inertia to take the system across singularities. This has the defect that one cannot stop at (or near) a singularity, but at least allows one to explore all components of $Z \setminus \Sigma$. The dynamics of a general system is given by (e.g. Choset (2005))

$$M_{ij}(z)\ddot{z}_j + \Gamma^i_{jk}\dot{z}_j\dot{z}_k = G_i, \tag{3.1}$$

where M_{ij} is the inertia matrix (in general configuration-dependent) which gives the kinetic energy by the expression $\frac{1}{2}\dot{z}^T M\dot{z}$; Γ^i_{jk} are the Christoffel symbols, expressions in the derivatives of M which represent generalisations of centripetal acceleration; and G represents all the tangential forces, including the effects of control forces and frictional forces. The inertia matrix M is assumed to be positive definite everywhere, corresponding to the physically natural condition of positivity of kinetic energy (else an additional type of singularity is encountered: configurations at which M loses rank). At singularities of π_Y, the contribution of some directions of control force F may go to zero (by the formula $G^T = -F^T D\pi_Y$), but if $\dot{z}$ is transverse to Σ and M is non-degenerate then the equation of motion carries the system from one side of Σ to the other with no untoward effects.

There are two requirements to make this work. One is that the transverse speed be large compared with the ratio of forces to mass: then the acceleration term dominates the equation of motion up to the scale of distances from Σ of order the ratio of Γ to M and takes the system across Σ with small change in velocity. The other is that one must apply control forces, not attempt to control the state directly. This may be a major change from standard engineering practice, where for example it is hard to find a motor that produces a desired torque as a function of time but easy to find one that produces a desired angular speed as a function of time. A related approach was presented in Ider (2005).

We propose that the first three design principles should still be respected. In contexts, however, where repeatability is not a requirement one could relax the second principle, to allow some degrees of freedom in the system for fixed controls. This reduces the singularity set to one of higher co-dimension and thereby makes it much easier to avoid. In particular, the singularity set no longer separates the the configuration space.

4. Vibration frequencies

One important further question is how the natural frequencies of vibration of the device vary if its parts are compliant instead of perfectly rigid. In particular, it is important to keep them above some lower limit, else all but the slowest motion of the system may excite large vibrations (see the theory of normally elliptic slow manifolds, e.g. MacKay (2004)). To compute the natural frequencies of vibration about a given configuration does not require the full dynamics. It is enough (in the frictionless case) to know the kinetic energy K for all velocities (but now of all flexible degrees of freedom, including compliance of the controls) and the second variation $\delta^2 V$ of the potential energy with respect to all infinitesimal deformations q. Then the square of the minimum frequency ω_{min} of vibration is given by minimising $\delta^2 V(q)$ subject to $K(q) = 1$. Really friction should be included and then the eigenvalues of the linearised motion have negative real parts as well.

The aspect we address here is how the frequencies might vary near singularities. In particular, any lack of stiffness in the controls has an exaggerated effect on the system near singularities. The control forces F induce an effective tangential force $F^T A$ to Z, where $A = D\pi_Y$. We deduce that a stiffness matrix $-\frac{\partial F_i}{\partial y_j} = k_{ij}$ contributes the terms

$$-F_j \frac{\partial^2 y_j}{\partial z_i \partial z_k} + k_{jm} A_{mk} A_{ji} \tag{4.1}$$

to the stiffness matrix $-\frac{\partial G_i}{\partial z_k}$ for the full tangential forces. So the effect of k, which a priori was large, is softened in the null direction of A, leaving only a residual stiffness from a constrained version of the second derivative of the potential, for example. Thus we see yet another reason to avoid singularities of π_Y.

5. Conclusion

Five design principles have been formulated for isostatic mounts of dynamic structures, and they have been illustrated by a range of examples including a real engineering prototype for a robotic coordinate measurement arm. The distinction between singular points of the coupled space and singularities of the projection to control space has been emphasised. The need to avoid both has been explained. Consequences for design and motion planning have been elaborated.

Two issues we have not addressed here are existence and uniqueness of configuration for given control state. There are several directions in which these questions can be studied but we leave them for future work.

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