Properties of geometrical realizations of substitutions associated to a family of Pisot numbers

Víctor F. Sirvent

Thesis submitted for the degree of Doctor of Philosophy at the University of Warwick.

Mathematics Institute,
University of Warwick,
Coventry.

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To the memory of my father

To my mother.

Contents

List of Figures

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Declaration

The work in this thesis is, to the best of my knowledge, original, except where attributed to others.

Summary

In this thesis we study some properties of the geometrical realizations of the dynamical systems that arise from the family of Pisot substitutions:

$$\Pi_n : \quad \begin{array}{ccc} 1 & \longrightarrow & 12 \\ 2 & \longrightarrow & 13 \\ \vdots & & \\ (n-1) & \longrightarrow & 1n \\ n & \longrightarrow & 1 \end{array}$$

for n a positive integer greater than 2.

In chapter 1 we compute the Hölder exponent of the Arnoux map, which is the semiconjugacy between the geometrical realization of (Ω, σ), the dynamical system of this substitution, in the circle $(\mathbf{S}^1, f)$ and the $n-1$ dimensional torus $(\mathbf{T}^{n-1}, T)$. Also in this chapter we introduce the notion of the standard partition in the symbolic space Ω and in its geometrical realizations. The cylinders of this partition are classified according to their structure.

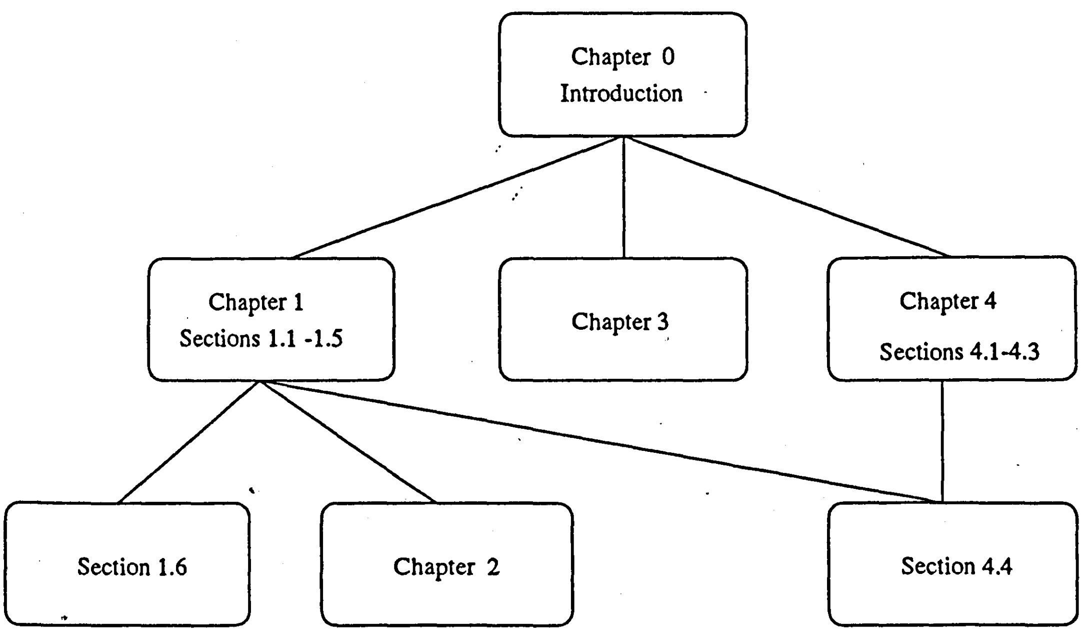
In chapter 2 we construct a geodesic lamination on the hyperbolic disk associated to this standard partition and a transverse measure on the lamination. The interval exchange map f and the contraction h induce maps F and H on the lamination, respectively. The map F preserves the transverse measure and H contracts it.

In chapter 3 we compute the Hausdorff dimension of the boundary of ω, the fundamental domain of the torus $\mathbf{T}^2$ obtained by the realization of the symbolic space Ω that arises from the substitution Π_3. As a corollary we compute the Hausdorff dimension of the pre-image of the boundary of ω under the Arnoux map. We also describe the identifications on the boundary of ω that make it a fundamental domain of the two dimensional torus.

In chapter 4 we study some relationships between the dynamical systems of this family of substitutions. We describe how the dynamics of the systems of this family, corresponding to lower dimensions – i.e. the parameter n in the definition of Π_n – are present in systems of higher dimensions. Also we study the realization of this property in the interval.

Index of Symbols

Interdependence Table



Chapter 0

Introduction

Space filling curves were first introduced by Peano, in 1890 ([34], [35]). Later other examples were introduced by Hilbert ([27]), Lebesgue ([32]), Schoenberg ([44]) and others. These constructions rely on the representation in a integer base of the numbers in the interval and a representation of the points of the unit square using this base, eg. in [34] the base used was 3, in [27] was 4 and in the construction of Schoenberg a different integer base representation of the points in the interval and in the unit square was used. However these constructions have very few dynamical properties.

Substitutions are a source of dynamical systems with very different properties. Using the dynamical systems that arise from a particular family of substitutions a space filling curve can be constructed with interesting dynamical and geometrical properties. A substitution in a finite alphabet $\mathcal{A}$, is a map from the alphabet to a set of words in this alphabet:

$$\begin{array}{rcccl} \Pi & : & \mathcal{A} & \longrightarrow & \cup_{n\geq 1}\mathcal{A}^n \\ & & a & \longrightarrow & v_a. \end{array}$$

This map is extended to a map the set of words in the alphabet $\mathcal{A}$ into itself by juxtaposition, i.e. $\Pi(UV) = \Pi(U)\Pi(V)$ where U and V are words in the alphabet and $\Pi(\emptyset) = \emptyset$. In this way the substitution is extended to a set of infinite sequences in the alphabet $\mathcal{A}$. We are interested in the fixed points of Π or Π^n for some $n \geq 1$. When such points exist, we consider the closure, in the product topology on $\mathcal{A}^{\mathbf{N}^*}$, here $\mathbf{N}^*$ is the set of positive integers of the orbit under the shift map — $\sigma(u_0u_1u_2\ldots) = u_1u_2\ldots$ — of the fixed point, this space is denoted by Ω. One of the main interests of these dynamical systems is that they provide the coding information for certain geometrical dynamical systems.

One of the first substitutions to be studied was

$$\begin{array}{rcccl} \Pi & : & \{0,1\} & \longrightarrow & \{0,1\}^2 \\ & & 0 & \longrightarrow & 01 \\ & & 1 & \longrightarrow & 10 \end{array}$$

It was studied in 1906 by Thue ([45], [46]), who was interested in sequences with non-repetition properties, and rediscovered in 1921 by Morse ([33]) in the study of geodesics flows on surfaces of negative curvature.

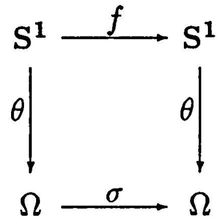
As far as the geometrical realization of these dynamical systems is concerned the substitutions are classified into two groups: substitutions of constant length, i.e. all the words $\Pi(a)$, for a an element of the alphabet $\mathcal{A}$, have the same length and Pisot substitutions, i.e. the Perron-Frobenius eigenvalue of the matrix that represent the substitution is a Pisot number. In the case of substitutions of constant length, Ω is realized as an algebraic extension of the ring of p-adic integers, where p is the length of the substitution. The shift map is realized as a rotation in this non-archimedian space ([42], [40]). In the case of Pisot substitutions, Ω is realized as a compact region in $\mathbf{R}^n$, for some n, which is a fundamental domain of torus $\mathbf{T}^n$. The shift map is realized as a piece exchange map in this compact region of $\mathbf{R}^n$, in fact, the dynamical system (Ω, σ) is semiconjugate to a translation in $\mathbf{T}^n$ ([38]). In the general case the realization of the substitution dynamical system is a product of the spaces that arise in the case of constant length and in the Pisot case([42], [40]).

A comprehensive study of the dynamical systems that arise from substitutions of constant length can be found in [36].

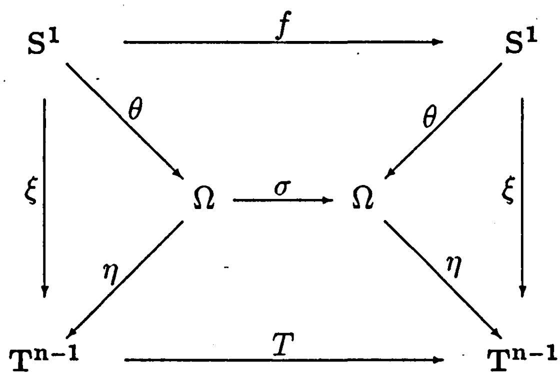
On the other hand, some substitutions can be realized as an interval exchange map. In [3], it was showed that for the family of Pisot substitutions:

$$\Pi_n : \begin{array}{ccc} 1 & \longrightarrow & 12 \\ 2 & \longrightarrow & 13 \\ \vdots & & \\ (n-1) & \longrightarrow & 1n \\ n & \longrightarrow & 1 \end{array}$$

with $n \geq 3$, there exists an interval exchange map $f : \mathbf{S}^1 \to \mathbf{S}^1$ such that the diagram



is commutative, where θ is continuous from the right and therefore there exists a map $\xi : \mathbf{S}^1 \to \mathbf{T}^{n-1}$ such that the following diagram commutes:



where η is the semiconjugacy given in the case of Pisot substitutions. In [3] it was proved that the map ξ is continuous. Hence it is a space filling curve.

On the other hand, a symbolic dynamical system in the alphabet $\mathcal{A}$ can be obtained from an arbitrary sequence $\underline{v} = v_1 v_2 \ldots$ by considering the space:

$$\Sigma = \overline{\{\sigma^n(\underline{v}) \mid n \in \mathbf{N}^*\}}.$$

The dynamical system (Σ, σ) is said to be of complexity $p(n)$ if the cardinality of the set of subwords, of length n, of the sequence $\underline{v}$ is $p(n)$. If $\underline{v}$ is periodic $p(n)$ is constant for large n. The next degree of complexity is when $p(n) = n + 1$. In this case the sequence is called a sturmian sequence([5], [13], [16], [24], [25],[26],[39], [41]). Among these sequences is the one obtained by the Fibonacci substitution, Π_2:

$$\begin{array}{ccc} 1 & \longrightarrow & 12 \\ 2 & \longrightarrow & 1. \end{array}$$

The dynamical systems that arise from the substitutions Π_k are of complexity $(k-1)n+1$ (see proposition 1.2.4).

In [5] it was proved that all the sequences of complexity $(k-1)n+1$ that satisfy a hypothesis, that we will specify next, are realized as an interval exchange map of $2k$ intervals. The additional hypothesis that we have just mentioned is: for every n and any subword of $\underline{v}$, $W = w_1 \ldots w_n$, of length n, it can be extended uniquely to a subword of some length m ($m \geq n$) and this extended subword admits k exactly extensions to a word of length $m+1$. And also, there exists a subword $\hat{W} = \hat{w}_1 \ldots \hat{w}_{m'}$ of length $m' \geq n$ of $\underline{v}$ such that $\hat{w}_{m'-i} = w_{n-i}$ for $0 \leq i \leq n-1$ and $a\hat{W}$ is also a subword of $\underline{v}$ for all $a \in \mathcal{A}$.

The dynamical system that arises from the substitution Π_k satisfies these properties (propositions 1.2.1 and 1.2.2). These dynamical systems are the simplest substitution dynamical systems that can be realized as an interval exchange map and a translation on the torus of some dimension and these two realizations are semiconjugate.

In this thesis we study the dynamical systems, that arise from the family of substitutions Π_k, and their geometrical realizations. Before describing the contents of each chapter, we shall give a summary of the main properties of the dynamical systems that arise from this family of substitutions and their geometrical realizations ([3], [6], [38]). We shall use these results in the thesis.

The substitution Π_k has a unique fixed point since it is a contraction as a map from $\mathcal{A}^{\mathbb{N}^*}$ into itself. Let denote this point by $\underline{u} = (u_0 u_1 \ldots)$. According to previous lines

$$\Omega = \overline{\{\sigma^n(\underline{u}) \mid n \in \mathbb{N}^*\}}.$$

The dynamical system (Ω, σ) is minimal, i.e. every orbit is dense. This space admits a natural self-similar partition $\Omega = \cup_{i=1}^{k} \Omega_i$ where $\Omega_i = \{\underline{v} \in \Omega \,|\, v_0 = i\}$. The self-similarity among the elements of this partition comes from the commutativity of the diagram:

$$\begin{array}{ccc} \Omega & \xrightarrow{\ \sigma\ } & \Omega \\ \Pi \downarrow & & \downarrow \Pi \\ \Omega_1 & \xrightarrow{\ \tilde{\sigma}\ } & \Omega_1 \end{array} \tag{0.1}$$

where $\tilde{\sigma}$ denotes the induced map of σ on Ω_1, i.e.

$$\tilde{\sigma}(\underline{v}) = \sigma^{\min\{l | \sigma^l(\underline{v}) \in \Omega_1\}}(\underline{v}).$$

The construction of the geometrical realization of (Ω, σ) in $\mathbf{R}^{k-1}$ is as follows: Let us define the map $\hat{\eta} : \Omega \to \mathbf{R}^{k-1}$ on the orbit of the fixed point of the substitution, $\underline{\mathbf{u}}$:

$$\hat{\eta}(\sigma^n(\underline{\mathbf{u}})) = n \begin{pmatrix} \alpha \\ \vdots \\ \alpha^{k-1} \end{pmatrix} - \begin{pmatrix} r_1(U_n) \\ \vdots \\ r_{k-1}(U_n) \end{pmatrix} \qquad (0.2)$$

where $U_n = u_0 u_1 \dots u_{n-1}$ and $r_i(U_n)$ is the number of symbols equal to i in U_n and α is the inverse of the real root, greater than 1 of $x^k - x^{k-1} - \cdots - x - 1$. The map $\hat{\eta}$ is extended by continuity to Ω. The image of Ω under $\hat{\eta}$ is denoted by ω, this set is a fundamental domain of $\mathbf{T}^{k-1}$, therefore the map $\hat{\eta}$ could be re-defined as a map from Ω to $\mathbf{T}^{k-1}$, this map is denoted by η. This map is a semiconjugacy between (Ω, σ) and $(\mathbf{T}^{k-1}, T)$, where T is the translation defined by the vector $(\alpha, \dots, \alpha^k)$. The set ω admits a self-similar partition $\{\omega_1, \dots, \omega_k\}$, where each ω_i is the image of Ω_i. When the map T is considered as a map of ω into itself, it exchange the sets ω_i's. In figure 0.1 can be seen ω and its partition, and figure 0.2 shows how ω teselates the plane. for the case $k = 3$.

A space homeomorphic to Ω is obtained using the representation of the non-negative integers, given by the recurrence relation associated to the substitution:

$$g_{n+k} = \sum_{i=0}^{k-1} g_{n+i} \text{ for } n \geq 0 \text{ and } g_i = 2^i \text{ for } 0 \leq i \leq k.$$

See [38] for the details of the construction. We give a summary of it in section 1.3. The space obtained is denoted by $\overline{\mathcal{N}}^n$.

In [6], was introduce the interval exchange map f on $\mathbf{I} = [0, 1)$ defined as: $f = L_{\mathbf{I}} \circ L_{I_1} \circ \cdots \circ L_{I_k}$ where $I_1 = [0, \alpha)$ and, for $j \geq 2$, $I_j = [\sum_{i=1}^{j-1} \alpha^i, \sum_{i=1}^{j} \alpha^i)$. Note that $\sum_{i=1}^{k} \alpha^i = 1$, so that the intervals $I_1, \dots, I_k$ form a partition of I. The map L_J denotes the rotation of order 2 on the interval $J = [a, b)$, i.e.

$$L_J(x) = \begin{cases} x + \frac{b-a}{2} & \text{if } a \leq x < \frac{a+b}{2} \\ x - \frac{b-a}{2} & \text{if } \frac{a+b}{2} \leq x < b \\ x & \text{otherwise} \end{cases}$$

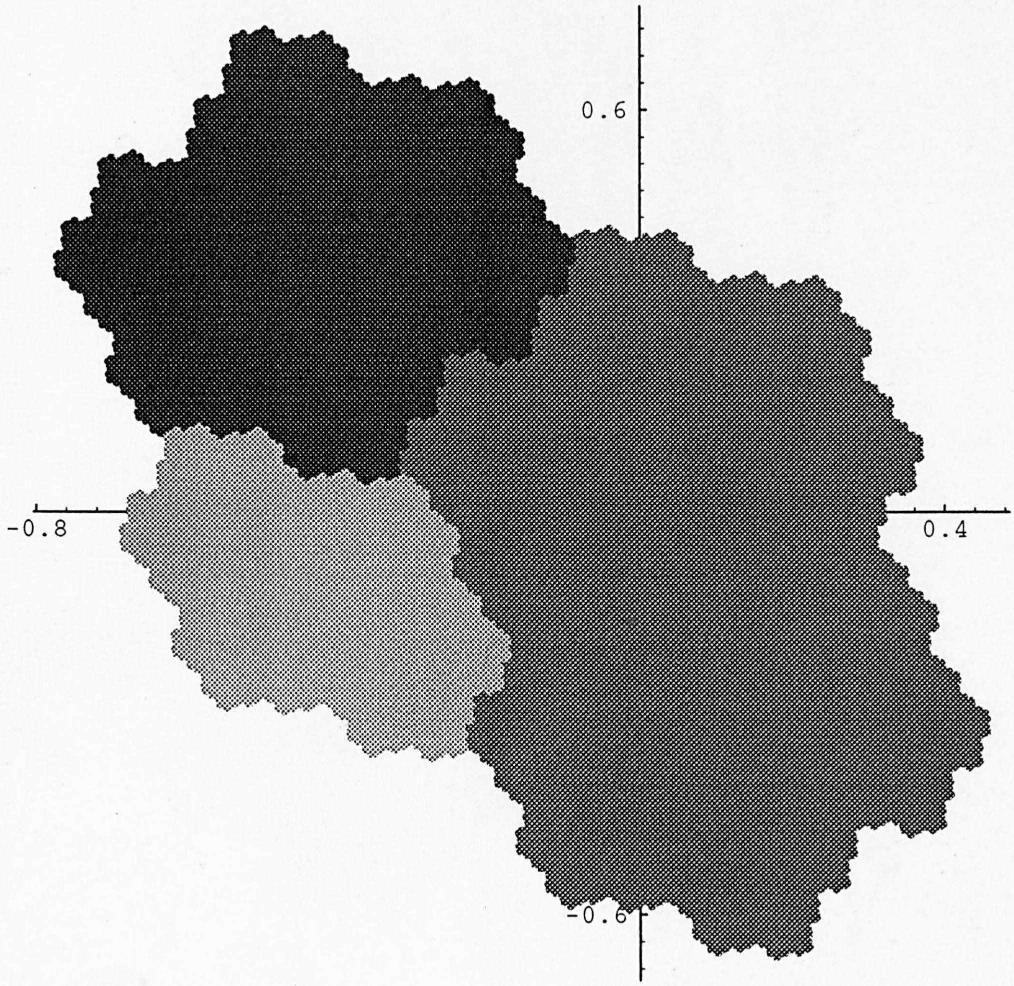


Figure 0.1: The set ω and its partition, in the case of $k = 3$.

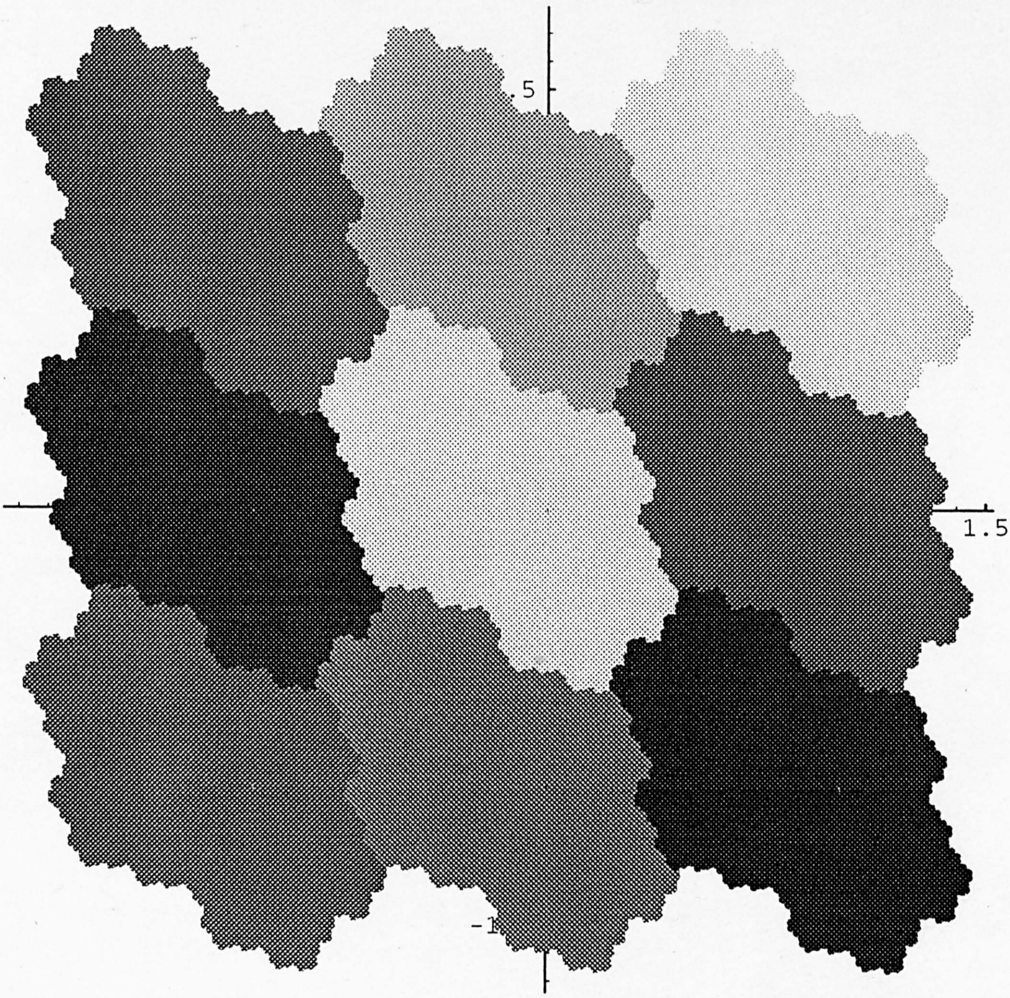


Figure 0.2: The set ω (in the case of $k = 3$) and its translations under the lattice $\mathbf{Z}^2$.

When $k \geq 3$ the dynamical system $(\mathbf{I}, f)$ is minimal ([2]) and is self-similar, i.e. the diagram

$$\begin{array}{ccc} \mathbf{I} & \xrightarrow{\ f\ } & \mathbf{I} \\ h \downarrow & & \downarrow h \\ \mathbf{I}_1 & \xrightarrow{\ \tilde{f}\ } & \mathbf{I}_1 \end{array}$$

is commutative, where the map h is

$$h(x) = \begin{cases} \alpha x + \frac{\alpha+\alpha^{k+1}}{2} & \text{if } 0 \leq x < \frac{1-\alpha^k}{2} \\ \alpha x - \frac{\alpha-\alpha^{k+1}}{2} & \text{if } \frac{1-\alpha^k}{2} \leq x < 1 \end{cases}$$

and $\tilde{f}$ is the induced map of f on I_1.

Let θ be the coding map of the orbits of f, under the partition given by the I_i's, i.e. $\theta(x) = \{Z(f^n(x))\}_{n \in \mathbf{N}^*}$ where the map $Z : \mathbf{I} \rightarrow \mathcal{A}$ is defined as $Z(x) = j$ if x is in I_j. The map θ is not continuous, however it is right-continuous since f is right-continuous.

The map $\xi : \mathbf{S}^1 \rightarrow \mathbf{T}^{k-1}$, defined as the composition of η and θ is a semi-conjugacy between $(\mathbf{S}^1, f)$ and $(\mathbf{T}^{k-1}, T)$, here the circle; $\mathbf{S}^1$, is identified with $\mathbf{I} = [0, 1)$. The continuity of ξ is not obvious, since θ is discontinuous ([3]). In the following chapters we will denote by $\hat{\xi}$ the version map of the ξ when it is considered as a map from $\mathbf{I}$ to the fundamental domain of $\mathbf{T}^{k-1}$ in $\mathbf{R}^k$, obtained before, i.e.

$$\hat{\xi} : \mathbf{I} \rightarrow \omega \quad \hat{\xi} = \hat{\eta} \circ \theta.$$

In chapter 1 we shall compute the Hölder exponent of the Arnoux map $\xi : \mathbf{S}^1 \rightarrow \mathbf{T}^{k-1}$. In order to do that we introduce the notion of standard partition of the symbolic space Ω, that comes from the self-similarity of the Ω_i's. This partition is induced in the geometrical realizations of Π_k in $\mathbf{I}$ and in ω (or in $\mathbf{S}^1$and $\mathbf{T}^{k-1}$). The structure of the cylinders of this partition on $\mathbf{T}^{k-1}$ is trivial; however the structure of the cylinders in the interval is more complex, the cylinders could have many connected components, due to the discontinuity of the map h. In section 1.5 we classify the cylinders of this partition. In order to describe the cylinders of this partition we introduce, in section 1.4, a binary operation on the natural numbers, which reflects the sub-division of a cylinder into sub-cylinders. In this section we introduce a subset of the natural numbers that is a semi-group under this binary operation.

In chapter 2 we construct a geodesic lamination on the hyperbolic disk associated to this standard partition and a transverse measure on the lamination. The interval exchange map f and the contraction h induce maps F and H on the lamination, respectively. The map F preserves the transverse measure and H contracts it and the following commutative diagram arises

$$\begin{array}{ccc} \Lambda & \xrightarrow{F} & \Lambda \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle H} \\ \Lambda_1 & \xrightarrow{\tilde{F}} & \Lambda_1 \end{array}$$

Because of this we can think this lamination as a geometrical realization of (Ω, σ).

In chapter 3 we compute the Hausdorff dimension of the boundary of ω, the fundamental domain of $\mathbf{T}^2$ obtained by the realization of the symbolic space Ω that arises from the substitution Π_3. This result was proved independently by Ito and Kimura ([28]). As a corollary we compute the Hausdorff dimension of the pre-image of the boundary of ω under the Arnoux map. We also describe the identifications on the boundary of ω that make it a fundamental domain of the two dimensional torus.

In chapter 4 we study some relationships between the dynamical systems of this family of substitutions. We describe how the dynamics of the systems of this family, corresponding to lower dimensions – i.e. the parameter k in the definition of Π_k – are present in systems of higher dimensions. In particular we show that there is a subset of $\overline{\mathcal{N}}^k$, whose dynamics resembles the dynamics of $\overline{\mathcal{N}}^{k-1}$, from the topological and metric point of view. We compute the Hausdorff and Billingsley dimensions, with respect to a natural metric and measure on $\overline{\mathcal{N}}^k$, of this set. Also we study the realization of this set in the interval.

Chapter 1

Hölder exponent of Arnoux's semiconjugacy and the standard partition of the geometrical realization of the substitution Π_k.

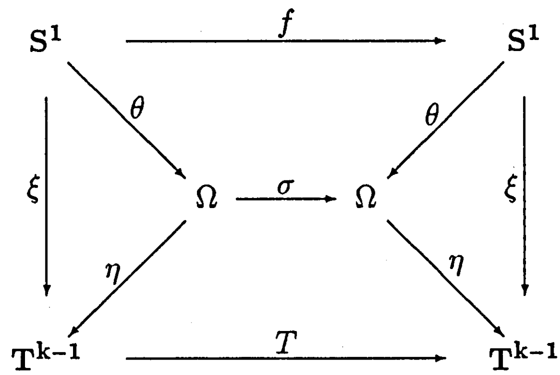
1.1 Introduction

In this chapter we shall compute the Hölder exponent of the semiconjugacy, constructed by P. Arnoux in [3], between an interval exchange map and an irrational translation on $\mathbf{T}^{k-1}$, which are the geometrical realizations of the dynamical system associated with the substitution Π_k (which will be denoted by Π, whenever the parameter k is understood):

$$\Pi : \{1,2,\ldots,k\}^{\mathbf{N}} \to \{1,2,\ldots,k\}^{\mathbf{N}}$$

$$1 \xrightarrow{\Pi} 12,\ 2 \xrightarrow{\Pi} 13, \ldots,\ (k-1) \xrightarrow{\Pi} 1k,\ k \xrightarrow{\Pi} 1.$$

The construction of Arnoux's map gives us a fundamental domain for $\mathbf{T}^{k-1}$ with very irregular – fractal – boundary. This fundamental domain admits a partition into k rectangles, each one associated with one of the symbols in the alphabet $\mathcal{A} = \{1,\ldots,k\}$ on which the substitution is defined. This partition is constructed in such a way that the dynamical system associated with the substitution i.e. $\sigma : \Omega \to \Omega$ gives symbolic dynamics for the irrational translation, T on $\mathbf{T}^{\mathbf{k}-\mathbf{1}}$. If we denote this semiconjugacy by $\xi : \mathbf{S}^{\mathbf{1}} \to \mathbf{T}^{\mathbf{k}-\mathbf{1}}$ the following commutative diagram arises:



(1.1)

where f is an interval exchange map.

Since these k-regions are self-similar, we have a refinement of the partition for all different levels. We call this partition the standard partition, which is study in section 1.5

Given a level n cylinder of these partition, which corresponds to a word of length l, $(l \geq n)$; it turns out that this word can be extended uniquely to a word of length m $(m \geq l)$ such that there are k different possible extensions to a word of length $m+1$ (that correspond to rectangles of level $n+1$)(section 1.2). In the proof of the main theorem of this chapter and for further purposes, we work with words of maximal length for a cylinder of a given level, because the maximal words give more dynamical information. In order to describe and manage this phenomenon properly we define in section 1.4, a binary operation $* : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$. The definition of $*$ uses the representation of the natural numbers into "base Π". We shall show that there are some natural numbers , which are very useful for representing the cylinders of the standard partition; these numbers are called integers compatible with the partition (I.C.P.), and form a semigroup under the binary operation $*$ (Sections 1.4 and 1.5). Also in section 1.4 we use this binary operation for describing other dynamical properties of $\sigma : \Omega \to \Omega$ and its geometrical realizations.

For computing the Hölder exponent of Arnoux's map, we need a good understanding of the standard partition in the symbolic space Ω and in its geometrical realizations. In particular in the interval, the structure of rectangles is not very clear since the interval exchange map f is not continuous. We deal with this topic in section 1.5 and especially in lemma 1.5.3.

1.2 Extension of allowed words in Ω

In this section we are going to show that any allowed word in Ω of length n can be extended uniquely to a word of length m ($m \geq n$) such that this new word admits k different extensions to a word of length $m+1$. Later we shall show that the complexity of (Ω, σ) is $(k-1)n+1$.

Proposition 1.2.1 *Let V be any allowed word of length n. It can be uniquely extended to a word $\tilde{V}$ of some length m ($n \leq m$), such that $\tilde{V}$ admits k possible extensions.*

Proof: For simplicity we are going to prove this proposition when $\mathcal{A}=\{1,2,3\}$.

Consider $\Pi^4(1)$ which is equal to 1213121121312. Here we can see that the symbol 1 admits three possible extensions. It can be followed by 1, 2 and 3. Moreover, the symbol 3 can only be followed by 1. Because 33 is not an allowed word since is not in $\Pi(\mathcal{A})$ and cannot be formed by juxtapositions of elements of $\mathcal{A}$. Similarly 32 is not an allowed word. On the other hand 31 and 131 are subwords of $\Pi^4(1)$. Therefore the extensions of 1 are:

$$\begin{array}{ccc} & & 1 \\ & \nearrow & \\ 1 & \rightarrow & 2 \\ & \searrow & \\ & & 31 \end{array}$$

Hence $\Pi^q(1)$ admits the extensions:

$$\begin{array}{cccl} & & \Pi^q(1) = \Pi^{q-1}(1)\Pi^{q-1}(2) & \\ & \nearrow & & \\ \Pi^q(1) & \rightarrow & \Pi^q(2) = \Pi^{q-1}(1)\Pi^{q-1}(3) & \text{for all } q \geq 1 \\ & \searrow & & \\ & & \Pi^q(31) = \Pi^{q-1}(1)\Pi^{q-1}(1)\Pi^{q-1}(2) & \end{array}$$

On the other hand, any allowed word V of Ω is a subword of $\Pi^q(1)$ for some q. Therefore V can be extended in the same way as $\Pi^q(1)$, moreover this extension is unique when we consider the minimun q, such that V is a subword of $\Pi^q(1)$.

Q.E.D.

In a similar manner we can prove the following proposition:

Proposition 1.2.2 *Let $V = v_1 \dots v_n$ be an allowed word. Then there exists a unique word $\hat{V} = \hat{v}_1 \dots \hat{v}_m$ with $m \geq n$ such that $\hat{v}_{m-i} = v_{n-i}$ for $0 \geq i \geq n-1$ and $a\hat{V}$ is an allowed word for any a in the alphabet $\mathcal{A}$.*

Let $W(n)$ denote the set of allowed words of length n in Ω. In order to prove that the cardinality of $W(n)$ is $(k-1)n+1$, We introduce the following proposition.

Proposition 1.2.3 *Let W be an allowed word of Ω.*

1. *If the word aW is allowed for some a in the alphabet $\mathcal{A}$ and if there exists a unique b in $\mathcal{A}$ such that the word Wb is allowed then aWb is an allowed word.*
2. *If there exists a unique a in $\mathcal{A}$ such that aW is an allowed word and if WB is an allowed word for some b in $\mathcal{A}$ then aWb is an allowed word.*
3. *If aW and Wb are allowed words for all a, b in $\mathcal{A}$ then there exist unique a' and b' in such that $a'Wb'$ is an allowed word.*

Proof:

1. Since the word aW is allowed there exists an allowed word U such that aW is a subword of $\Pi(U)$.

   If Wb is not a subword of $\Pi(U)$ there exists c an element of the $\mathcal{A}$ such that Uc is an allowed word and Wb is a subword of $\Pi(Uc)$. The existence of such symbol c is given by the unicity of b. Therefore aWb is a subword of $\Pi(Uc)$.

2. This proof is similar to 1.

3. Without of lost of generality, we can assume that $\mathcal{A} = \{1, 2, 3\}$. Suppose as an inductive hypothesis that the statement is true for all allowed words of length m, where $1 \leq m < n$.

Consider $W = w_1 \dots w_n$ an allowed word. Since Wb is allowed for all b in $\mathcal{A}$, and since the words 22, 23, 33, 32 are not allowed in Ω, we can conclude that $w_n = 1$. There exists an allowed word U such that Ud ia an allowed word for all d in $\mathcal{A}$ and $U31$ is also allowed such that

- $W1$ is a subword of $\Pi(U31)$
- $W2$ is a subword of $\Pi(U1)$
- $W3$ is a subword of $\Pi(U2)$.

The word U is choosen with minimal length, i.e. if U' is a subword of U then $w_1 \dots w_{n-1}$ is not a subword of $\Pi(U')$. Observe that the length of such word U is smaller than n.

Since U is choosen with minimal length and the fact aW is an allowed word for all a in $\mathcal{A}$, we have that cU is an allowed word for all c in $\mathcal{A}$ and for any a there exists a C such that aW is a subword of $\Pi(cU)$. Therefore U is an allowed word such that cU and Ud are allowed for all c and d in $\mathcal{A}$. Since the length of U is smaller than n, by the inductive hypothesis, there exist unique c' d' such that $c'Ud'$ ia an allowed word, therefore this determines uniquely a' and b' such that $a'Wb'$ is an allowed subword.

Q.E.D.

Proposition 1.2.4 *The cardinality of $W(n)$ is $(k-1)n+1$.*

Moreover (k-1)n elements of $W(n)$ can be extended uniquely to a word of length $n+1$ and only one element of $W(n)$ admits k possible extensions.

Proof: Again for simplicity, we are going to give the proof when $\mathcal{A}=\{1,2,3\}$. The proof is given by induction on the length of the word.

In the case $n = 1$ the elements of the alphabet admit the extensions shown in Proposition 1.2.1

- 1 can be followed by $1, 2, 3$
- 2 can be followed only by 1
- 3 can be followed only by 1.

Let $W(n) = \{V^1, \ldots, V^{2n+1}\}$ where

$$\begin{aligned} V^1 &= v_1^1 \ldots v_n^1 \\ &\vdots \\ V^{2n+1} &= v_1^{2n+1} \ldots v_n^{2n+1}. \end{aligned}$$

By the inductive hypothesis there exists j such that V^j is the only element of $W(n)$ that does not admit a unique extension.

$$\begin{aligned} \tilde{V}^1 &= v_1^1 \ldots v_n^1 v_{n+1}^1 \\ &\vdots \\ \tilde{V}^{j_1} &= v_1^j \ldots v_n^j 1 \\ \tilde{V}^{j_2} &= v_1^j \ldots v_n^j 2 \\ \tilde{V}^{j_3} &= v_1^j \ldots v_n^j 3 \\ &\vdots \\ \tilde{V}^{2n+1} &= v_1^{2n+1} \ldots v_n^{2n+1} v_{n+1}^{2n+1}. \end{aligned}$$

In order to prove that only one of these words cannot be extended uniquely to a word of length $n+2$, consider the following words of length n:

$$W^i = v_2^i \ldots v_n^i v_{n+1}^i$$

Observe that by the inductive hypothesis we have $2n+1$ different words of length n, so there is a unique i' such that $1W^{i'}$, $2W^{i'}$ and $3W^{i'}$ are allowed words, for $i \neq i'$ there exists a unique v_1^i such that $v_1^i W^i$ is allowed.

On the other hand there exists $\hat{i}$ — possibly equal to i'— such that $W^{\hat{i}}1$, $W^{\hat{i}}2$, $W^{\hat{i}}3$ are allowed words, for $i \neq \hat{i}$ there is a unique extension of W^i to a word of length $n+1$, denoted by $W^i w_{n+2}^i$.

If $i' \neq \hat{i}$, by the proposition 1.2.3 parts 1 and 2, the words $v_1^i W^i w_{n+2}^i$ are allowed, having in total $2n+3$ words of length $n+2$.

If $i' = \hat{i}$,by the proposition 1.2.3 part3, for each a in $\mathcal{A}$ there is a unique b_a in $\mathcal{A}$ such that $aW^{i'}b_a$ is allowed. For $i \neq i'$ the word $v_1^i W^i w_{n+2}^i$ is allowed. Obtaining in this way $2n+3$ words of length $n+2$.

Q.E.D.

1.3 A Numeration system associated with the substitution

In this section we are going to mention some results and techniques presented by Rauzy ([38]). The substitution associated with a Pisot number allows us to represent the natural numbers in an 'exotic basis'. This representation is useful for constructing a dynamical system isomorphic to $\sigma : \Omega \to \Omega$ in which some computations and geometrical constructions are easier to do. In this section for simplicity we are going to restrict to the case when the substitution is

$$\Pi : \begin{array}{ccc} 1 & \longrightarrow & 12 \\ 2 & \longrightarrow & 13 \\ 3 & \longrightarrow & 1 \end{array}$$

however the construction and results are valid for all the substitutions of this family, that is, also for $k > 3$.

Let $\underline{u} = u_0 u_1 \ldots$ the fixed point of Π. Observe that $u_0 = 1$ which implies that $\underline{u}$ must start with $\Pi(1)$ and also $\Pi^2(1)$ and so on. Therefore, for all l, the first symbols of $\underline{u}$ agree with $\Pi^l(1)$. Let $U_n = u_0 \ldots u_{n-1}$ be the first n symbols of $\underline{u}$. U_n will be expressed as a juxtaposition of words of $\Pi^{i_j}(1)$.

Proposition 1.3.1 (Rauzy [38]) *Given a positive integer N then*

1. *There exists a unique q and $(i_0, \ldots, i_q)$ such that $0 \le i_0 < i_1 < \ldots < i_q$ with $i_{j+2} > i_j + 2$ for $0 \le j \le k$.*
2. *$U_N = \Pi^{i_q}(1)\Pi^{i_{q-1}}(1) \cdots \Pi^{i_0}(1)$*

This result can be expressed in terms of the recurrence relation associated to the substitution. Let $g_j = |\Pi^j(1)|$ where $|V|$ is the length of the word V.

Since the substitution satisfies

$$\Pi^{n+3}(1) = \Pi^{n+2}(1)\Pi^{n+1}(1)\Pi^n(1) \quad \forall n \ge 0$$

we have the recurrence relation

$$g_{n+3} = g_{n+2} + g_{n+1} + g_n \tag{1.2}$$

with initial conditions $g_0 = 1$, $g_1 = 2$, $g_2 = 4$.

Proposition 1.3.1 permits us to represent each natural number in a unique way as a sum of cetain of the g_i's with no three consecutive g_i's in the present sum. This is a generalization of the Zeckendorf representation of the non-negative integers ([48]) using this recurrence relation instead of the Fibonacci relation.

Let

$$\mathcal{N} = \{\underline{x} \in \{0,1\}^{\mathbf{N}^*} \mid x_j + x_{j+1} + x_{j+2} < 3 \; \forall j \text{ and } \exists K > 0 \text{ s.t. } \forall n \geq K \; x_n = 0\}$$

$$\begin{array}{rcll} \epsilon & : & \mathbf{N}^* & \rightarrow & \mathcal{N} \\ & & N & \rightarrow & \epsilon(N) \end{array}$$

where $\epsilon(N)$ is such that

$$N = g_{i_0} + \cdots + g_{i_k} = \sum_{j \geq 0} \epsilon_j(N) g_j$$

which makes ϵ a bijective map.

Consider $\overline{\mathcal{N}}$ (where it has the topology induced from the product topology on $\{0,1\}^{\mathbf{N}}$) and the dynamical system $(+1) : \overline{\mathcal{N}} \rightarrow \overline{\mathcal{N}}$ where the map $(+1)$ is the induced operation in $\overline{\mathcal{N}}$ of adding one in $\mathcal{N}$., i.e. $(+1)\epsilon(N) = \epsilon(N+1)$

Proposition 1.3.2 (Rauzy [38]) *There exists a homeomorphism $\phi : \overline{\mathcal{N}} \rightarrow \Omega$ such that the diagram*

$$\begin{array}{ccc} \overline{\mathcal{N}} & \xrightarrow{(+1)} & \overline{\mathcal{N}} \\ \phi \downarrow & & \downarrow \phi \\ \Omega & \xrightarrow{\sigma} & \Omega \end{array}$$

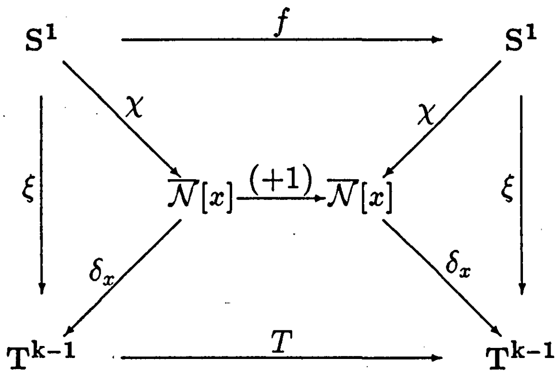
is commutative.

To the set $\overline{\mathcal{N}}$ it is associated the set of formal power series with coefficients zeros and ones, where series with three consecutive coefficients one are not allowed. This set is denoted by $\overline{\mathcal{N}}[x]$. The bijection between then is:

$$X : \begin{cases} \overline{\mathcal{N}} & \rightarrow \quad \overline{\mathcal{N}}[x] \\ \underline{a} = (a_0, a_1, \ldots) & \rightarrow \quad \sum_{i \geq 0} a_i x^i \end{cases}$$

Since X is a bijection, we have the map $X(+1)X^{-1} : \overline{\mathcal{N}}[x] \rightarrow \overline{\mathcal{N}}[x]$, which we will denote by $(+1)$, whenever the context is clear.

According to proposition 1.3.2 the diagram (1.1) can be expressed as:



where the map χ is given by: $\chi = X\phi^{-1}\theta$, which is right continuous, and δ_x is defined as: $\delta_x(\underline{a}) = \sum a_i B^i z /_{\sim}$. The relation $\sim$ is the equivalence relation defined by the lattice $\mathbf{Z}^2$ in $\mathbf{R}^2$. Here B is the matrix

$$\begin{pmatrix} -\alpha & -\alpha \\ 1-\alpha^2 & -\alpha^2 \end{pmatrix}$$

and z is the vector, whose transpose is given by $(\alpha - 1, \alpha^2) = \hat{\eta}(\sigma(\underline{\mathrm{u}}))$. In general the matrix B is the restriction of the matrix that represent the substitution to its contracting eigenspace, which is of codimension 1, since the Perron-Frobenius eigenvalue of this matrix is a Pisot number.

1.4 A binary operation in N compatible with the dynamical systems associated with this family of substitutions

Using the representation of natural numbers described in the last section, we can define a binary operation similar to the Fibonacci multiplication

([31], [4]).

Let n and m be given in the form

$$n = \sum_{i=0}^{N} a_i g_i, \quad m = \sum_{j=0}^{M} b_j g_j$$

where

$$\begin{array}{llll} \epsilon_i(n) = a_i & 0 \le i \le N & \epsilon_i(n) = 0 & \forall i > N \\ \epsilon_j(m) = b_j & 0 \le j \le M & \epsilon_j(m) = 0 & \forall j > M. \end{array}$$

Define $n \diamond m$ by

$$n \diamond m = \sum_{i=0}^{N} \sum_{j=0}^{M} a_i b_j g_{i+j}. \tag{1.3}$$

Like the Fibonacci multiplication this operation is associative.

Now we define a new binary operation in $\mathbf{N}$:

If $n = g_{i_0} + \cdots + g_{i_l}$ with $g_{i_j} < g_{i_q}$ when $j < q$.

Observe that we can write n in the following way:

$$\begin{aligned} n &= g_{i_0} \diamond (1 + g_{i_1 - i_0} + \cdots + g_{i_l - i_0}) \\ &= g_{i_0} \diamond (1 + g_{i_1 - i_0} \diamond (1 + \cdots + g_{i_l - i_1}) \\ &\vdots \\ &= g_{i_0} \diamond (1 + g_{i_1 - i_0} \diamond (1 + \cdots + g_{i_{l-1} - i_{l-2}} \diamond (1 + g_{i_l - i_{l-1}}) \cdots)) \end{aligned}$$

Definition 1.4.1 *Define the binary operaration $*$ by*

$$\begin{array}{lllll} \mathbf{N} & \times & \mathbf{N} & \rightarrow & \mathbf{N} \\ n & * & m & = & g_{i_0} \diamond (1 + g_{i_1 - i_0} \diamond (1 + \cdots + g_{i_{l-1} - i_{l-2}} \diamond (1 + g_{i_l - i_{l-1}} \diamond m) \cdots)) \end{array}$$

Properties:

- $1 * m = m * 1 = m$
- If $n = g_q$ then $n * m = g_q \diamond m$
- $*$ is not commutative: e.g.

$$\begin{aligned}
9 &= 2+7 = g_1 + g_3 = g_1 \diamond (g_0 + g_2) \\
3 &= 1+2 = g_0 + g_1 \\
9 * 3 &= g_1 \diamond (g_0 + g_2 \diamond (1 + g_1)) = g_1 \diamond (g_0 + g_2 + g_3) = \\
&= g_1 + g_3 + g_4 = 2 + 7 + 13 = 22 \\
3 * 9 &= g_0 + g_1 \diamond g_1 \diamond (g_0 + g_2) = g_0 + g_2 \diamond (g_0 + g_2) = \\
&= g_0 + g_2 + g_4 = 1 + 2 + 13 = 16
\end{aligned}$$

- In general it is not associative: e.g.

$$\begin{aligned}
3 * (3 * 2) &= (g_0 + g_1) * ((g_0 + g_1 \diamond g_1)) = (g_0 + g_1) * (g_0 + g_2) = \\
&= g_0 + g_1 \diamond (g_0 + g_2) = g_0 + g_1 + g_3 = \\
&= 1 + 2 + 7 = 10 \\
3 * 3 &= (g_0 + g_1) * (g_0 + g_1) = g_0 + g_1 \diamond (g_0 + g_1) = \\
&= g_0 + g_1 + g_2 = g_3 = 7 \\
(3 * 3) * 2 &= g_3 * g_1 = g_4 = 13
\end{aligned}$$

For this reason, we keep the following convention:

$$m_1 * m_2 * \cdots * m_l \stackrel{\text{def}}{=} m_1 * (m_2 * (\cdots * (m_{l-2} * (m_{l-1} * m_l)) \cdots))$$

However this operation is associative in a subset of the natural numbers. Let $n_1 = g_1 = 2$, $n_2 = g_0 + g_2 = 1 + 4 = 5$, $n_3 = g_0 + g_1 + g_3 = 1 + 2 + 7 = 10$, $n_0 = g_0 = 1$ and $\mathcal{P}$ the set generated by these four numbers under the operation $*$, i.e.

$$\mathcal{P}_l = \{n_{i_1} * \cdots * n_{i_l} | i_j = 0, 1, 2 \text{ or } 3 \text{ for all } j\} \qquad \mathcal{P} = \cup_{l \geq 1} \mathcal{P}_l$$

In section 1.5 we are going to show a geometrical interpretation of this set.

Given any three natural numbers n, m and m' then the associativity in $n * m * m'$ fails when we do the operation $n * m$ and we get an expression with three consecutive g_i's and therefore we have to use the relation 1.2 for expressing the number as in proposition 1.3.1.

Easy calculations show that when we do $n_i * n_j$ for $i, j = 0, 1, 2, 3$ we never get three consecutive g_i's. So the operation $* : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ is associative, which gives:

Proposition 1.4.1 *$(\mathcal{P}, *)$ is a semigroup.*

One of the first applications of this binary operation in N is as follows: given $\tilde{\sigma}^n(\underline{u})$ (which is equal to $\Pi(\sigma^n(\underline{u}))$) — $\tilde{\sigma}$ is the induced map of σ in $\Omega_1 = \{\underline{v} \in \Omega | v_0 = 1\}$— from the definition of $\tilde{\sigma}$ it is clear that $\tilde{\sigma}^n(\underline{u})$ belongs to the orbit of $\underline{u}$ under σ (i.e. $\tilde{\sigma}^n(\underline{u}) = \sigma^m(\underline{u})$ for some m). However what is the relationship between n and m?.

We are going to show how to express $\sigma^n(\underline{u})$ as a composition of powers of Π, applied to $\sigma(\underline{u})$, and σ (without using its powers). In particular we shall associated to each natural number n an operator $O_{\sigma,\Pi}(n)$ such that $\sigma^n(\underline{u}) = O_{\sigma,\Pi}(n)(\sigma(\underline{u}))$. Moreover we shall the property

$$O_{\sigma,\Pi}(m) \circ O_{\sigma,\Pi}(n) = O_{\sigma,\Pi}(n * m)$$

Also we shall show how this property is preserved in the geometrical realizations of $\sigma : \Omega \to \Omega$.

Definition 1.4.2 *If $n = g_{i_0} + \cdots + g_{i_l}$ as in section 1.3 (i.e. no three consecutive g_i's are present) then*

$$n = g_{i_0} \diamond (1 + g_{i_1 - i_0} \diamond (1 + \cdots + g_{i_{l-1} - i_{l-2}} \diamond (1 + g_{i_l - i_{l-1}}) \cdots)).$$

We define:

$$\begin{aligned} O_{\sigma,\Pi}(n) &: \Omega \to \Omega \\ O_{\sigma,\Pi}(n) &= \Pi^{i_0} \sigma \Pi^{i_1 - i_0} \sigma \cdots \Pi^{i_{l-1} - i_{l-2}} \sigma \Pi^{i_l - i_{l-1}}. \end{aligned}$$

Lemma 1.4.1 *The map $O_{\sigma,\Pi}(n)$ satisfies the properties:*

1.

$$O_{\sigma,\Pi}(n)(\sigma(\underline{u})) = \sigma^n(\underline{u}) \quad \textit{for any} \quad n \in \mathbf{N}.$$

2.

$$O_{\sigma,\Pi}(m) \circ O_{\sigma,\Pi}(n) = O_{\sigma,\Pi}(m * n) \quad \textit{for} \quad m, n \in \mathcal{P}.$$

3.

$$O_{\sigma,\Pi}(m)(O_{\sigma,\Pi}(n)(\sigma(\underline{u}))) = \sigma^{m*n}(\underline{u}) \quad \textit{for} \quad m, n \in \mathcal{P}.$$

We are going to prove first the following proposition:

Proposition 1.4.2 *1.* $\sigma^{g_q}(\underline{\mathbf{u}}) = \Pi^q(\sigma(\underline{\mathbf{u}}))$.

2. $\tilde{\sigma}^{g_q}(\underline{\mathbf{u}}) = \sigma^{g_{q+1}}(\underline{\mathbf{u}})$.

3. $\sigma^{g_q \diamond n}(\underline{\mathbf{u}}) = \sigma^{n \diamond g_q}(\underline{\mathbf{u}}) = \Pi^q \sigma^n(\underline{\mathbf{u}})$ *for all* $n \in \mathbf{N}^*$

Proof of proposition 1.4.2:

1. This fact is proved by induction on q.

   In the case $q = 1$

   $\underline{\mathbf{u}} = u_0\sigma(\underline{\mathbf{u}}) = 1\sigma(\underline{\mathbf{u}})$ so $\underline{\mathbf{u}} = \Pi(\underline{\mathbf{u}}) = \Pi(1)\Pi(\sigma(\underline{\mathbf{u}})) = 12\Pi(\sigma(\underline{\mathbf{u}}))$.

   Therefore $\sigma^2(\underline{\mathbf{u}}) = \Pi\sigma(\underline{\mathbf{u}})$ but $2 = g_1$ hence $\sigma^{g_1}(\underline{\mathbf{u}}) = \Pi(\sigma(\underline{\mathbf{u}}))$.

   Let the expression of $\underline{\mathbf{u}}$ as

   $\underline{\mathbf{u}} = U(g_q)\sigma^{g_q}(\underline{\mathbf{u}}) = \Pi^q(1)\sigma^{g_q}(\underline{\mathbf{u}})$ since $\underline{\mathbf{u}}$ is the fixed point of the substitution we have $\underline{\mathbf{u}} = \Pi^{q+1}(1)\Pi(\sigma^{g_q}(\underline{\mathbf{u}}))$ therefore we have $\sigma^{g_{q+1}}(\underline{\mathbf{u}}) = \Pi(\sigma^{g_q}(\underline{\mathbf{u}}) = \Pi(\Pi^q(\sigma(\underline{\mathbf{u}})) = \Pi^{q+1}(\sigma(\underline{\mathbf{u}}))$.

2. As we showed in part 1 of this lemma $\sigma^{g_{q+1}}(\underline{\mathbf{u}}) = \Pi(\sigma^{g_q}(\underline{\mathbf{u}}))$ and since $\Pi \circ \sigma = \tilde{\sigma} \circ \Pi$ we have $\Pi(\sigma^{g_q}(\underline{\mathbf{u}})) = \tilde{\sigma}^{g_q}(\Pi(\underline{\mathbf{u}}))$ and since $\underline{\mathbf{u}}$ is the fixed point of the substitution, we have:

$$\sigma^{g_{q+1}}(\underline{\mathbf{u}}) = \tilde{\sigma}^{g_q}(\underline{\mathbf{u}}).$$

3. Let $n = g_{i_0} + \cdots + g_{i_l}$. By Proposition 1.3.1

$$\underline{\mathbf{u}} = U(n)\sigma^n(\underline{\mathbf{u}}) = \Pi^{i_l}(1)\cdots\Pi^{i_0}(1)(\sigma^n(\underline{\mathbf{u}})).$$

   Since $\underline{\mathbf{u}}$ is a fixed point of the substitution Π

$$\underline{\mathbf{u}} = \Pi^q(\underline{\mathbf{u}}) = \Pi^{i_l+q}(1)\cdots\Pi^{i_0+q}(1)\Pi^q\sigma^n(\underline{\mathbf{u}}).$$

   Therefore

$$\Pi^q\sigma^n(\underline{\mathbf{u}}) = \sigma^{g_{i_l+q}+\cdots+g_{i_0+q}}(\underline{\mathbf{u}}) = \sigma^{g_q \diamond n}(\underline{\mathbf{u}}).$$

End of the proof of Proposition 1.4.2

Proof of Lemma 1.4.1:

1. Let

$$\begin{aligned} n &= g_{i_0} + \cdots + g_{i_l} \\ &= g_{i_0} \diamond (1 + g_{i_1 - i_0} \diamond (1 + \cdots + g_{i_{l-1} - i_{l-2}} \diamond (1 + g_{i_l - i_{l-1}}) \cdots)). \end{aligned}$$

By proposition 1.4.2

$$\begin{aligned} \Pi^{i_l - i_{l-1}}(\sigma(\underline{u})) &= \sigma^{g_{i_l - i_{l-1}}}(\underline{u}) \\ \sigma\Pi^{i_l - i_{l-1}}(\sigma(\underline{u})) &= \sigma^{1+g_{i_l - i_{l-1}}}(\underline{u}) \\ \Pi^{i_{l-1} - i_{l-2}}\sigma\Pi^{i_l - i_{l-1}}(\sigma(\underline{u})) &= \sigma^{g_{i_{l-1} - i_{l-2}} \diamond (1+g_{i_l - i_{l-1}})}(\underline{u}) \\ &\vdots \\ \Pi^{i_0}\sigma\Pi^{i_1 - i_0}\sigma \cdots \Pi^{i_l - i_{l-1}}(\sigma(\underline{u})) &= \sigma^{g_{i_0} \diamond (1+g_{i_1 - i_0} \diamond (1+\cdots+g_{i_{l-1} - i_{l-2}} \diamond (1+g_{i_l - i_{l-1}})\cdots))}(\underline{u}) \end{aligned}$$

But the last term is $\sigma^n(\underline{u})$ by using the expression for n given at the beginning of the proof. But, by definition 1.4.2, $O_{\sigma,\Pi}(n) = \Pi^{i_0}\sigma\Pi^{i_1 - i_0}\sigma \cdots \Pi^{i_l - i_{l-1}}$ Therefore

$$O_{\sigma,\Pi}(n)(\sigma(\underline{u})) = \sigma^n(\underline{u})$$

2. Let

$$\begin{aligned} m &= g_{j_0} + \cdots + g_{j_q} \quad \text{and} \qquad m \in \mathcal{P} \\ &= g_{j_0} \diamond (1 + g_{j_1 - j_0} \diamond (1 + \cdots + g_{j_{q-1} - j_{q-2}} \diamond (1 + g_{j_q - j_{q-1}}) \cdots)) \end{aligned}$$

So

$$O_{\sigma,\Pi}(m) = \Pi^{i_0}\sigma\Pi^{j_1 - j_0}\sigma \cdots \Pi^{j_q - j_{q-1}}$$

and

$$O_{\sigma,\Pi}(m) \circ O_{\sigma,\Pi}(n) = \underbrace{\Pi^{j_0}\sigma\Pi^{j_1 - j_0}\sigma \cdots \Pi^{j_q - j_{q-1}}}_{O_{\sigma,\Pi}(m)} \underbrace{\Pi^{i_0}\sigma\Pi^{l_1 - l_0}\sigma \cdots \Pi^{i_l - i_{l-1}}}_{O_{\sigma,\Pi}(n)}$$

Since m and $n \in \mathcal{P}$, $m * n =$

$$g_{j_0} \diamond (1 + \cdots \diamond (1 + g_{j_q - j_{q-1}} \diamond g_{i_0} \diamond (1 + g_{i_1 - i_0} \diamond (1 + \cdots \diamond (1 + g_{i_l - i_{l-1}}) \cdots)))).$$

Therefore

$$O_{\sigma,\Pi}(m * n) = O_{\sigma,\Pi}(m) \circ O_{\sigma,\Pi}(n).$$

3. Follows immediately from 1 and 2.

End of the proof of Lemma 1.4.1

Now we are going to show what is equivalent to lemma 1.4.1 in the geometrical realizations of the dynamical system induced by the substitution i.e. $\sigma : \Omega \to \Omega$.

Definition 1.4.3 *Let n be as in definition 1.4.2 Define:*

$$\begin{array}{rcl} O_{T,B}(n) & : & \omega \to \omega \quad by \\ O_{T,B}(n) & = & B^{i_0}TB^{i_1-i_0}T\cdots B^{i_{l-1}-i_{l-2}}TB^{i_l-i_{l-1}} \end{array}$$

Corollary 1.4.1 *Let $z = T(0,0)$ then*

1. $O_{T,B}(n)z = T^n(0,0)$ *for* $n \in \mathbf{N}$
2. $O_{T,B}(m) \circ O_{T,B}(n) = O_{T,B}(m * n)$ *for all* m *and* $n \in \mathcal{P}$
3. $O_{T,B}(m)(O_{T,B}(n)(z) = T^{m*n}(0,0)$ *for* m *and* $n \in \mathcal{P}$

Proof: From the commutativity of the diagran

$$\begin{array}{ccc} \Omega & \xrightarrow{\sigma} & \Omega \\ \eta \downarrow & & \downarrow \eta \\ \mathbf{T}^2 & \xrightarrow{T} & \mathbf{T}^2 \end{array}$$

we obtain that

$$\hat{\eta}(\sigma(\mathbf{v})) = T(\hat{\eta}(\mathbf{v}))\ ; \quad \text{for all } \mathbf{v} \in \Omega.$$

Since $\hat{\eta}(\Pi(\mathbf{v})) = B(\hat{\eta}(\mathbf{v}))$ for $\mathbf{v}$ in Ω, we obtain $\hat{\eta}(O_{\sigma,\Pi}(m)\sigma(\underline{\mathbf{u}}) = O_{T,B}(m)\hat{\eta}(\sigma(\underline{\mathbf{u}})) = O_{T,B}(m)z$. Therefore the corollary follows from Lemma 1.4.1

Q.E.D.

When we consider the geometrical realization of $\sigma : \Omega \to \Omega$ in $\mathbf{I} = [0,1)$, there is a slight difference that comes from the fact that the preimage of $\underline{\mathbf{u}}$ in the interval — under the map θ (see page 8) — consists of three different points with the property $h(x_i) = x_{i+1 \pmod 3}$, $i = 1,2,3$.

Definition 1.4.4 *Let n be as in definition 1.4.2 Define:*

$$\begin{aligned} O_{f,h}(n) &: \mathbf{I} \to \mathbf{I} \quad by \\ O_{f,h}(n) &= h^{i_0} f h^{i_1 - i_0} f \cdots h^{i_{l-1} - i_{l-2}} f h^{i_l - i_{l-1}} \end{aligned}$$

and define the degree of $O_{f,h}(n)$ as i_l

Corollary 1.4.2 *Let x_j, $j = 1, 2, 3$ be the preimages of $\underline{\mathbf{u}}$ under θ:*

1. $$O_{f,h}(n) f(x_j) = f^n(x_{grad_n(j)})$$
   where $grad_n(j) = j + i_l \pmod 3$ and i_l is the degree of $O_{f,h}(n)$
2. $O_{f,h}(m) \circ O_{f,h}(n) = O_{f,h}(m * n)$ *where m, $n \in \mathcal{P}$*
3. $O_{f,h}(m)(O_{f,h}(n)(f(x_i)) = f^{m*n}(x_{grad_{m*n}(i)})$ *for m and $n \in \mathcal{P}$*

We shall prove this corollary after the following proposition.

Proposition 1.4.3 *1.* $h^q(x_i) = f^{g_q}(x_{grad_q(i)})$
2. $h^{g_q} f^n(x_i) = f^{n \diamond g_q}(x_{grad_q(i)})$

Proof of 1.4.3: We shall prove the two statements of this proposition by induction:

1. If $q = 1$ then $hf(x_i) = \tilde{f}h(x_i) = \tilde{f}(x_{i+1})$. But $\tilde{f}(x_{i+1}) = f^2(x_{i+1})$, in fact $\theta\tilde{f}(x_{i+1}) = \tilde{\sigma}(\underline{\mathbf{u}}) = \sigma^2(\underline{\mathbf{u}})$ (by proposition 1.4.2).

   Consider
   $$h^{q+1} f(x_i) = hh^q f(x_i) = hf^{g_q}(x_{grad_q(i)}) = \tilde{f}^{g_q}(x_{grad_{q+1}(i)})$$
   But $\tilde{f}^{g_q}(x_{grad_{q+1}(i)}) = f^{g_{q+1}}(x_{grad_{q+1}(i)})$ since, by proposition 1.4.2, we have: $\theta(\tilde{f}^{g_q}(x_{grad_{q+1}(i)})) = \tilde{\sigma}^{g_q}(\underline{\mathbf{u}}) = \sigma^{g_{q+1}}(\underline{\mathbf{u}})$.

   Therefore:
   $$h^{q+1}(f(x_i)) = f^{g_{q+1}}(x_{grad_{q+1}(i)})$$

2. If $q = 1$ then $hf^n(x_i) = \tilde{f}^n(x_{i+1})$ and $\tilde{f}^n(x_{i+1}) = f^{n\diamond g_1}(x_{i+1})$ (since $\theta(\tilde{f}^n(x_{i+1})) = \tilde{\sigma}(\underline{\mathrm{u}}) = \Pi(\sigma^n(\underline{\mathrm{u}})) = \sigma^{n\diamond 2}(\underline{\mathrm{u}})$).

   Therefore: $hf^n(x_i) = f^{n\diamond g_1}(x_{i+1})$. Suppose that the statement is true for q, then:

$$\begin{aligned} h^{q+1}f^n(x_i) &= hf^{n\diamond g_q}(x_{grad_q(i)}) \\ &= f^{n\diamond g_q \diamond g_1}(x_{grad_{q+1}(i)}) \\ &= f^{n\diamond g_{q+1}}(x_{grad_{q+1}(i)}) \end{aligned}$$

End of the proof of proposition 1.4.3

Proof of corollary 1.4.2:

1. Let:

$$\begin{aligned} n &= g_{i_0} + \cdots + g_{i_l} \\ &= g_{i_0} \diamond (1 + g_{i_1 - i_0} \diamond (1 + \cdots + g_{i_{l-1}-i_{l-2}} \diamond (1 + g_{i_l - i_{l-1}}) \cdots)) \end{aligned}$$

   On the other hand:

$$\begin{aligned} h^{i_l - i_{l-1}}(f(x_j)) &= f^{g_{i_l - i_{l-1}}}(x_{j+i_l-i_{l-1} \pmod 3}) \\ fh^{i_l - i_{l-1}}(f(x_j)) &= f^{1+g_{i_l - i_{l-1}}}(x_{j+i_l-i_{l-1} \pmod 3}) \\ h^{i_{l-1}-i_{l-2}}fh^{i_l - i_{l-1}}(f(x_j)) &= f^{g_{i_{l-1}-i_{l-2}} \diamond (1+g_{i_l - i_{l-1}})}(x_{j+i_l-i_{l-2} \pmod 3}) \\ &\vdots \end{aligned}$$

$$h^{i_0}fh^{i_1-i_0}f \cdots h^{i_l - i_{l-1}}(f(x_j)) = f^{g_{i_0}\diamond(1+g_{i_1-i_0}\diamond(1+\cdots+g_{i_{l-1}-i_{l-2}}\diamond(1+g_{i_l-i_{l-1}})\cdots))}(x_{j+i_l \pmod 3})$$

   Therefore:

$$O_{f,h}(n)f(x_i) = f^n(x_{grad_n(j)})$$

2. Follows from lemma 1.4.1.

3. Is a straight-forward consequence of part 1 and 2 of this corollary.

End of the proof of corollary 1.4.2

1.5 The Standard Partition

In the symbolic space Ω we have a natural partition into k rectangles, where k is the number of symbols in the alphabet in which the substitution is defined. In the rest of this chapter we are going to work in the case $k = 3$, only for simplicity, the results can be generalized to $k > 3$.

The space Ω admits the partition $\Omega = \cup_{i=1}^{3}\Omega_i$ where

$$\Omega_i = \{\underline{v} \in \Omega | v_0 = i\} \quad i = 1, 2, 3.$$

and each of these sets is self-similar to Ω:

$$\begin{aligned}
\Omega_1 &= \Pi(\Omega) \\
\Omega_2 &= \sigma(\Pi^2(\Omega)) = \sigma(\Pi(\Omega_1)) \\
\Omega_3 &= \sigma(\Pi(\sigma(\Pi^2(\Omega)))) = \sigma(\Pi(\sigma(\Pi(\Omega_1)))) = \sigma(\Pi(\Omega_2))
\end{aligned}$$

This self-similarity induces a partition in each of the Ω_i's and each of these cylinders can be subdivided in three subcylinders according to the maps Π, $\sigma\Pi^2$, $\sigma\Pi\sigma\Pi^2$.

Definition 1.5.1 *The partition of Ω generated by the the system of iterated maps* (Π, $\sigma\Pi^2$, $\sigma\Pi\sigma\Pi^2$) *is called* the standard partition of Ω. *The elements of this partition are called* cylinders.

We are interested in the standard partition because it plays an important rôle in the proof of the the Hölder continuity of the Arnoux map, and in the next chapter.

Let $O_{\sigma,\Pi}(n)$ be as in definition 1.4.2. In the following lines we are going to show that $O_{\sigma,\Pi}(n)\Omega$ gives a l-cylinder of the standard partition in Ω, when $n \in \mathcal{P}$

As we said before the partition is generated by the iterated system of maps describ.ed bellow, so $O_{\sigma,\Pi}(n)$ has to be a composition of members of this family of maps.

In section 1.3, we introduced

$$n_1 = g_1 \qquad\qquad n_2 = g_0 + g_2$$
$$n_3 = g_0 + g_1 + g_3 = g_0 + g_1 \diamond (g_0 + g_2)$$

So

$$O_{\sigma,\Pi}(n_1) = \Pi \quad O_{\sigma,\Pi}(n_2) = \sigma\Pi^2 \quad O_{\sigma,\Pi}(n_3) = \sigma\Pi\sigma\Pi^2$$

So any composition of $O_{\sigma,\Pi}(n_i)$ $i = 1,2,3$ using lemma 1.4.1 can be associated a natural number m such that $O_{\sigma,\Pi}(m)$ is equal to this composition, i.e $O_{\sigma,\Pi}(n_{i_0})\ O_{\sigma,\Pi}(n_{i_1}) \cdots O_{\sigma,\Pi}(n_{i_k})$ is equal to $O_{\sigma,\Pi}(m)$ where $m = n_{i_0} * n_{i_1} * \cdots * n_{i_k}$; since the n_i's are the generators of $\mathcal{P}$, m belongs to this set. Due to this fact, we introduce the following definition:

Definition 1.5.2 *The elements of $\mathcal{P}$ are called* integers compatible with the partition *(ICP)*

Therefore we have:

Lemma 1.5.1 *R is a cylinder of the standard partition if and only if there exists an ICP n such that $R = O_{\sigma,\Pi}(n)\Omega$.*

In section 1.2 we saw that any allowed word in Ω can be extended uniquely to a word that admits three possible extensions. We are going to show that any such maximal word represents the symbols of the standard partition cylinder.

Lemma 1.5.2 *Let $V = v_0 \dots v_{m-1}$ be a maximal allowed word as in section 1.2. Then exists a cylinder R of the standard partition such that $R = \cap_{i=0}^{m-1}\sigma^{-i}(\Omega_{v_i})$*

Proof of lemma 1.5.2:

The set $R = \cap_{i=0}^{m-1}\sigma^{-i}(\Omega_{v_i})$ is not empty since v is an allowed word, we need to find an ICP n such that $R = O_{\sigma,\Pi}(n)(\Omega)$.

Now v_0 can be extended uniquely in Ω to $v_0 \dots v_{i_0}$ and evidently this is a subword of V.

Clearly the cylinder corresponding to this subword $\cap_{l=0}^{i_0}\sigma^{-l}(\Omega_{v_l})$ can be expressed as $O_{\sigma,\Pi}(n_{j_0})$ (since this subword is the extension of a word of length 1). Among the three possible next symbols after v_{i_0}, let v_{i_0+1} be the one in V, and we stop at the next symbol in which the word $v_0 \dots v_{i_0}v_{i_0+1}$ cannot be extended uniquely i.e. $v_0 \dots v_{i_0}v_{i_0+1} \dots v_{i_{i_0+1}}$. Since each symbol of the word $v_{i_0+1} \dots v_{i_{i_0+1}}$ expresses a rectangle Ω_1 Ω_2 or Ω_3 we have

$$\cap_{l=i_0+1}^{i_{i_0+1}}\sigma^{-l}(\Omega_{v_l}) = O_{\sigma,\Pi}(n_{j_1})(\Omega)$$

Therefore
$$\cap_{l=0}^{i_{i_0}+1}\sigma^{-l}(\Omega_{v_l}) = O_{\sigma,\Pi}(n_{j_0} * n_{j_1})\Omega$$
Carrying on this process we find n_{j_0}, $n_{j_1}, \ldots n_{j_q}$ such that
$$R = O_{\sigma,\Pi}(n_{j_0} * n_{j_1} * \cdots * n_{j_q})(\Omega)$$

Q.E.D.

Evidently the partition structure of Ω is translated to its geometrical realizations. The cylinders of the standard partition in $\mathbf{T}^2$ are "easy to understand", all of them are closed, connected and simply connected, since $O_{T,B}(n)$ is continuous. However $O_{f,h}(n)\mathbf{I}$ for n an ICP are "more complicated" since $O_{f,h}(n)$ might not be continuous as a map of the interval into itself so $O_{f,h}(n)\mathbf{I}$ might not be connected. Lemma 1.5.3 deals with the structure of $O_{f,h}(n)\mathbf{I}$, but first we introduce the notion of equivalence of cylinders and some examples of the cylinders of this partition, having different structures.

Definition 1.5.3 *Let n and m be ICP's. We say that the cylinders $O_{f,h}(n)\mathbf{I}$ and $O_{f,h}(m)\mathbf{I}$ are* equivalent *if there exists a homeomorphism that maps one cylinder into the other, in such a way that each subcylinder, belonging to the standard partition, of $O_{f,h}(n)\mathbf{I}$ is map into a subcylinder of the standard partition in $O_{f,h}(m)\mathbf{I}$.*

Remark 1.5.1 *In the following lines we show the structure of $O_{f,h}(n)\mathbf{I}$ for some particular ICP's. In Lemma 1.5.3 we prove that these are all the possible structures of the cylinders of the standard partition in the interval.*

- *Consider the case $n_1 = g_1 = 2$,*
$$O_{f,h}(n_1)\mathbf{I} = h(\mathbf{I}) = I_1 = [0, \alpha]$$
The map h induce a partition in I_1, which is the image of
$$\begin{aligned}
\mathbf{I} &= I_1 \cup I_2 \cup I_3 : \\
I_1 &= h(I_1) \cup h(I_2) \cup h(I_3) \\
h(I_1) &= [0, h(\alpha)) \cup [h(0), \alpha) = [0, \frac{\alpha^2 - \alpha^3}{2}) \cup [\frac{\alpha + \alpha^4}{2}, \alpha) \\
h(I_2) &= [h(\alpha), h(\alpha + \alpha^2)) = [\frac{\alpha^2 - \alpha^3}{2}, \frac{\alpha^2 + \alpha^3}{2}) \\
h(I_3) &= [h(\alpha + \alpha^2), h(0)) = [\frac{\alpha^2 + \alpha^3}{2}, \frac{\alpha + \alpha^4}{2})
\end{aligned}$$

As can be seen here this cylinder consists of one interval and its next level partition consists of two cylinder, which are connected and one cylinder which has two connected components.

- *Consider the case when* $n = n_1 * n_1$,

$$\begin{aligned} O_{f,h}(n_1 * n_1)\mathbf{I} &= h^2(\mathbf{I}) = h(I_1) = [0, h(\alpha)) \cup [h(0), \alpha) \\ &= [0, \frac{\alpha^2 - \alpha^3}{2}) \cup [\frac{\alpha + \alpha^4}{2}, \alpha) \end{aligned}$$

And this cylinder admits the following partition:

$$\begin{aligned} O_{f,h}(n_1 * n_1)\mathbf{I} &= O_{f,h}(n_1 * n_1)I_1 \cup O_{f,h}(n_1 * n_1)I_2 \cup O_{f,h}(n_1 * n_1)I_3 \\ &= h^2(I_1) \cup h^2(I_2) \cup h^2(I_3) \\ h^2(I_1) &= [0, h(\alpha)) \cup [h(0), h^2(\alpha)) \cup [h^2(\alpha + \alpha^2), \alpha) \\ h^2(I_2) &= [h^2(\alpha), h^2(\alpha + \alpha^2)) \\ h^2(I_3) &= [h^2(\alpha + \alpha^2), \lim_{t \to 1^-} h^2(t)) \end{aligned}$$

So $O_{f,h}(n_1 * n_1)\mathbf{I}$ *consists of two intervals, and its next level partition consists of two cylinders which are connected and one cylinder which has three connected components.*

- *Now, we consider* $n = n_1 * n_1 * n_1$ *and* $O_{f,h}(n_1 * n_1 * n_1)\mathbf{I}$

$$\begin{aligned} O_{f,h}(n_1 * n_1 * n_1)\mathbf{I} &= h^3(\mathbf{I}) = h^2(I_1) \\ &= [0, h(\alpha)) \cup [h(0), h^2(\alpha)) \cup [h^2(0), \alpha) \end{aligned}$$

And the next level partition is

$$\begin{aligned} O_{f,h}(n_1 * n_1 * n_1)\mathbf{I} &= h^3(I_1) \cup h^3(I_2) \cup h^3(I_3) \\ h^3(I_1) &= [h^3(0), h^4(\alpha)) \cup [h^4(0), h^2(\alpha)) \cup [h^2(0), h^3(\alpha)) \\ h^3(I_2) &= [h^4(\alpha), h(\alpha)) \cup [h(0), h^4(\alpha + \alpha^2)) \\ h^3(I_3) &= [h^4(\alpha + \alpha^2), h^4(0)) \end{aligned}$$

- *When we consider the cases* $O_{f,h}(n_1 * n_1 * n_1 * n_1)\mathbf{I}$ *and* $O_{f,h}(n_1 * n_1 * n_1 * n_1 * n_1)\mathbf{I}$ *we also have cylinders consisting of three*

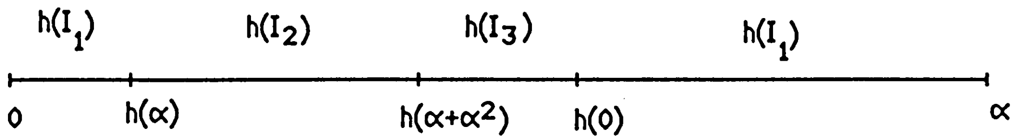


Figure 1.1: Partition of $O_{f,h}(n_1)\mathbf{I}$

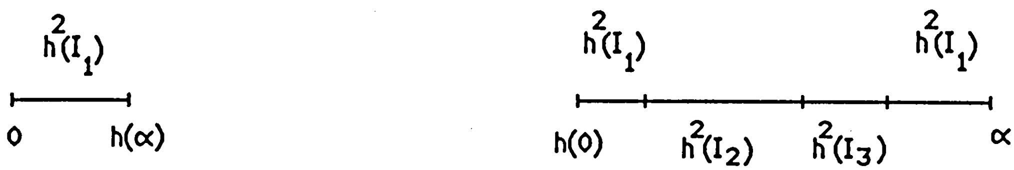


Figure 1.2: Partition of $O_{f,h}(n_1 * n_1)\mathbf{I}$

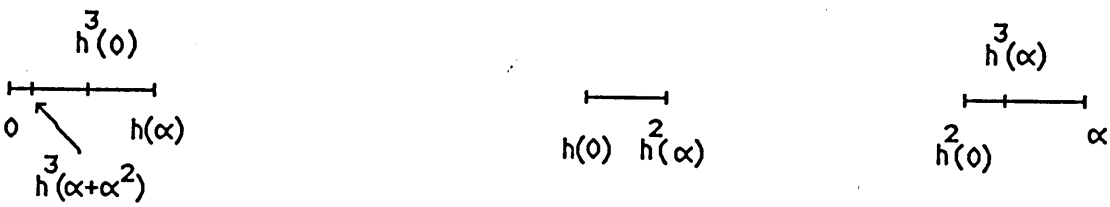


Figure 1.3: Partition of $O_{f,h}(n_1 * n_1 * n_1)\mathbf{I}$

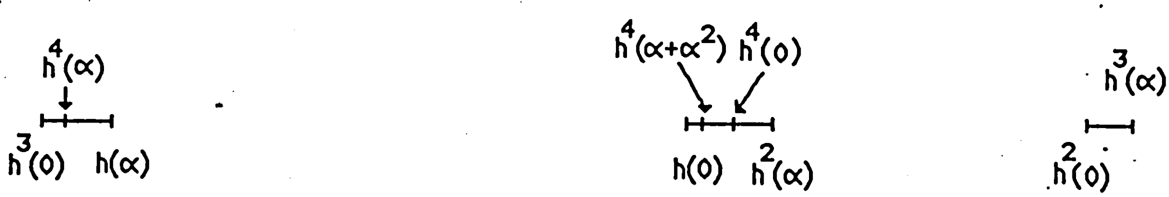


Figure 1.4: Partition of $O_{f,h}(n_1 * n_1 * n_1 * n_1)\mathbf{I}$

Figure 1.5: Partition of $O_{f,h}(n_1 * n_1 * n_1 * n_1 * n_1)\mathbf{I}$

intervals, but the subpartition structure is slightly different between them and the case $O_{f,h}(n_1 * n_1 * n_1)\mathbf{I}$*; as can be seen in the figures 1.4 and 1.5.*

- *However if we consider the case* $O_{f,h}(n_1 * n_1 * n_1 * n_1 * n_1 * n_1)\mathbf{I}$ *we get cylinders of similar structure to* $O_{f,h}(n_1 * n_1 * n_1)\mathbf{I}$*, and all the cylinders coming from* $O_{f,h}(n_1 * n_1 * \cdots * n_1)\mathbf{I}$ *have three intervals and they are of the previous types, as can be seen in the proof of lemma 1.5.3.*

Proposition 1.5.1 *Let be* n, n' *ICP's, such that* $n = n_{i_1} * \cdots * n_{i_k}$ *and* $n' = n_{i_1} * \cdots * n_{i_l}$ *with* $l \leq k$*. If there exists a continuous and bijective function* $\varphi : O_{f,h}(n')\mathbf{I} \rightarrow O_{f,h}(n)\mathbf{I}$ *of the form* $O_{f,h}(m)$ *for some integer* m *(*m *might not be an ICP) such that this map preserves the number of disjoint intervals in each of these cylinders and if*

$$\varphi|_{O_{f,h}(n')I_j} : O_{f,h}(n')I_j \longrightarrow O_{f,h}(n * n_j)\mathbf{I} \tag{1.4}$$

is also bijective for $j = 1, 2, 3$*; then* $O_{f,h}(n)\mathbf{I}$ *and* $O_{f,h}(n')\mathbf{I}$ *are equivalent*

Proof of Proposition 1.5.1: Since φ is a continuous and bijective map from $O_{f,h}(n')\mathbf{I}$ to $O_{f,h}(n)\mathbf{I}$, which preserves the number of disjoint intervals of these cylinders. It can be extended to a homeomorphism of $\mathbf{I}$ into itself.

In order to prove that φ preserves the cylinders of the standard partition, consider a subcylinder of $O_{f,h}(n')\mathbf{I}$, say $O_{f,h}(n' * n_{j_1} * \cdots * n_{j_s})\mathbf{I}$ for some $s \geq 1$. Using corollary 1.4.2 we obtain:

$$\begin{aligned} \varphi O_{f,h}(n' * n_{j_1} * \cdots * n_{j_s})\mathbf{I} &= \varphi O_{f,h}(n' * n_{j_1}) O_{f,h}(n_{j_2} * \cdots * n_{j_s})\mathbf{I} \\ &= O_{f,h}(n * n_{j_1}) O_{f,h}(n_{j_2} * \cdots * n_{j_s})\mathbf{I} \\ &= O_{f,h}(n * n_{j_1} * \cdots * n_{j_s})\mathbf{I} \end{aligned}$$

which is a subcylinder of $O_{f,h}(n)\mathrm{I}$ belonging to the standard partition.

Therefore $O_{f,h}(n')\mathrm{I}$ and $O_{f,h}(n)\mathrm{I}$ are equivalent.

Q.E.D.

Lemma 1.5.3 *Let n be an ICP. The cylinder $O_{f,h}(n)\mathrm{I}$ is either:*

1. *an interval and it is equivalent to $I_1 = O_{f,h}(n_1)\mathrm{I}$. In this case, we say that $O_{f,h}(n)\mathrm{I}$ is of type 1.*
2. *two connected components and it is equivalent to $O_{f,h}(n_1 * n_1)\mathrm{I}$. In this case, we say that $O_{f,h}(n)\mathrm{I}$ is of type 2.*
3. *the union of three connected intervals and one of the following equivalences happens:*
   (a) *$O_{f,h}(n)\mathrm{I}$ is equivalent to $O_{f,h}(n_1 * n_1 * n_1)\mathrm{I}$. In this case, we say that $O_{f,h}(n)\mathrm{I}$ is of type 3-a.*
   (b) *$O_{f,h}(n)\mathrm{I}$ is equivalent to $O_{f,h}(n_1 * n_1 * n_1 * n_1)\mathrm{I}$. In this case, we say that $O_{f,h}(n)\mathrm{I}$ is of type 3-b.*
   (c) *$O_{f,h}(n)\mathrm{I}$ is equivalent to $O_{f,h}(n_1 * n_1 * n_1 * n_1 * n_1)\mathrm{I}$. In this case, we say that $O_{f,h}(n)\mathrm{I}$ is of type 3-c.*

Proof Of Lemma 1.5.3: We use induction on the number of factors in the ICP For $l = 1$:

- In the case $n = n_1$, there is nothing to prove.
- In order to study the case $n = n_2 = 1 + g_2$, we consider the cylinder $O_{f,h}(n_1 * n_1)\mathrm{I} = h^2(\mathrm{I})$ – discussed in remark 1.5.1–, which is of type 2. Since the map f is continuous in $[0, h(\alpha)] \subset [0, \frac{\alpha}{2})$ and $[h(0), \alpha] \subset [\frac{\alpha}{2}, \alpha]$ and also $\lim_{t \to \alpha^-} = f(0)$. Therefore $fh^2(\mathrm{I})$ consists of only one connected component. If we define $\varphi = O_{f,h}(m)$ for $m = 1 + g_1$ then

  $$\varphi : \; O_{f,h}(n_1)\mathrm{I} \longrightarrow O_{f,h}(n_2)\mathrm{I}$$

  by construction is bijective and continuous on $O_{f,h}(n_1)\mathrm{I}$. Furthermore:

  $$\begin{aligned} \varphi O_{f,h}(n_1 * n_j)\mathrm{I} &= O_{f,h}(m) O_{f,h}(n_1 * n_j)\mathrm{I} \\ &= O_{f,h}(m * n_1) O_{f,h}(n_j)\mathrm{I} \\ &= O_{f,h}(n_2) O_{f,h}(n_j)\mathrm{I} \\ &= O_{f,h}(n_2 * n_j)\mathrm{I}. \end{aligned}$$

  Hence $O_{f,h}(n_2)\mathrm{I}$ is of type 1. See figure 1.6

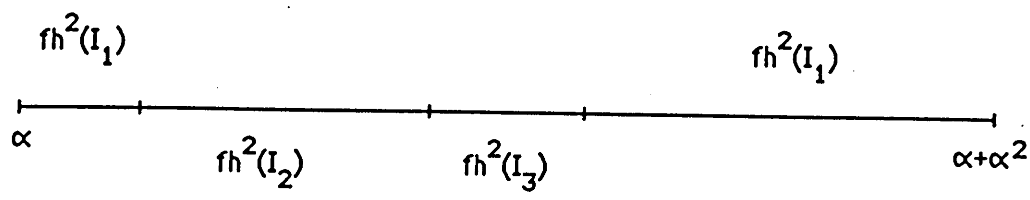


Figure 1.6: Partition of $O_{f,h}(n_2)\mathbf{I}$

- In the case $n = n_3 = 1 + g_1 + g_3$, consider the cylinder $h(fh^2)(\mathbf{I}) = h(I_2)$ and the map f is continuous in it, since $h(I_2) = [h(\alpha), h(\alpha+\alpha^2)) \subset [0, \frac{\alpha}{2}]$ and f is continuous in this interval. Therefore $f(h(I_2))$ is an interval. Moreover $f(h(I_2)) = I_3$, because $fh(\alpha) = \alpha + \alpha^2$ and $fh(\alpha + \alpha^2) = 1$. Let be $\varphi = O_{f,h}(m)$ where $m = 1 + g_1 \diamond (1 + g_1)$
$$\varphi : O_{f,h}(n_1)\mathbf{I} \longrightarrow O_{f,h}(n_3)\mathbf{I}$$
which is a bijection because
$$\varphi(I_1) = fhfh(I_1) = fhfh^2(\mathbf{I}) = O_{f,h}(n_3)\mathbf{I}$$
and continuous. Therefore $O_{f,h}(n_3)\mathbf{I}$ is of type 1.

We are going to show that the lemma holds for $O_{f,h}(n * n_j)\mathbf{I}$ where $n = n_{i_1} * \cdots * n_{i_l}$ and $j = 1, 2, 3$.

- Suppose $O_{f,h}(n)\mathbf{I}$ is of type 1. By the inductive hypothesis, we have a continuous and bijective map
$$\varphi : I_1 \longrightarrow O_{f,h}(n)\mathbf{I}$$
such that
$$\varphi : O_{f,h}(n_1 * n_j)\mathbf{I} \longrightarrow O_{f,h}(n * n_j)\mathbf{I}$$
is bijective for $j = 1, 2, 3$. Also the map φ preserves the number of disjoint intervals. Furthermore φ is of the form $O_{f,h}(m)$ for some positive integer m. By the existence of the map φ we obtain the equality
$$O_{f,h}(m)O_{f,h}(n_1) = O_{f,h}(m * n_1)$$
(which is not always true for all positive integers m).

If $j = 1$ then $O_{f,h}(n * n_1)\mathrm{I}$ is of type 2, because φ maps the partition structure of $O_{f,h}(n * n_1)\mathrm{I}$ into the partion structure of $O_{f,h}(n_1 * n_1)\mathrm{I}$, i.e. satisfies the property (1.4) in proposition 1.5.1:

$$\begin{aligned}
\varphi O_{f,h}(n_1 * n_1 * n_i)\mathrm{I} &= O_{f,h}(m)O_{f,h}(n_1 * n_1 * n_i)\mathrm{I} \\
&= O_{f,h}(m * n_1)O_{f,h}(n_1 * n_i)\mathrm{I} \\
&= O_{f,h}(n)O_{f,h}(n_1 * n_i)\mathrm{I} \\
&= O_{f,h}(n * n_1 * n_i)\mathrm{I} \quad \text{for } i = 0,1,2,3
\end{aligned}$$

From this, we can conclude that the cylinder $O_{f,h}(n * n_1)\mathrm{I}$ consists of two connected components since $O_{f,h}(n_1 * n_1)\mathrm{I}$ is the union of two disjoint intervals.

If $j = 2$ then $O_{f,h}(n * n_2)\mathrm{I}$ is of type 1. Since $O_{f,h}(n_1 * n_2)\mathrm{I}$ is of type 1, in fact

$$O_{f,h}(n_1 * n_2)\mathrm{I} = hfh^2(\mathrm{I}) = h(I_2) = [h(\alpha), h(\alpha + \alpha^2)).$$

Taking $m' = g_1 \diamond (1 + g_1)$, we have $O_{f,h}(m) = hfh$ and $O_{f,h}(m')O_{f,h}(n_1)\mathrm{I}=O_{f,h}(n_1 * n_2)\mathrm{I}$.

We define the map

$$\varphi' : O_{f,h}(n_1)\mathrm{I} \longrightarrow O_{f,h}(n * n_2)\mathrm{I}$$

by $\varphi' = \varphi O_{f,h}(m')$, which is continuous since $O_{f,h}(m')O_{f,h}(n_1)\mathrm{I}$ is the domain of φ, which is continuous. Moreover it is bijective and satisfies the property 1.4, because:

$$\begin{aligned}
\varphi' O_{f,h}(n_1 * n_i)\mathrm{I} &= \varphi O_{f,h}(m')O_{f,h}(n_1 * n_i)\mathrm{I} \\
&= \varphi O_{f,h}(m')O_{f,h}(n_1)O_{f,h}(n_i)\mathrm{I} \\
&= \varphi O_{f,h}(m' * n_1)O_{f,h}(n_i)\mathrm{I} \\
&= \varphi O_{f,h}(n_1 * n_2)O_{f,h}(n_i)\mathrm{I} \\
&= O_{f,h}(n * n_1 * n_2)O_{f,h}(n_i)\mathrm{I} \\
&= O_{f,h}(n * n_1 * n_2 * n_i)\mathrm{I} \\
&= O_{f,h}(n * n_1)O_{f,h}(n_2 * n_i)\mathrm{I} \quad \text{for } i = 0,1,2,3.
\end{aligned}$$

Therefore $O_{f,h}(n * n_2)\mathrm{I}$ is an interval. These properties show that $O_{f,h}(n * n_2)\mathrm{I}$ is of type 1.

In a similar way, we can prove that $O_{f,h}(n * n_3)$ is of type 1.

- When $O_{f,h}(n)\mathrm{I}$ is of type 2, similarly we can prove that $O_{f,h}(n * n_1)\mathrm{I}$ is of type 3-a and $O_{f,h}(n * n_2)\mathrm{I}$, $O_{f,h}(n * n_3)\mathrm{I}$ are of type 1.

- In the same way we can discuss the case when $O_{f,h}(n)\mathrm{I}$ is of type 3. Here, if $O_{f,h}(n)\mathrm{I}$ is of type 3-a then $O_{f,h}(n * n_1)\mathrm{I}$ is of type 3-b, $O_{f,h}(n * n_2)\mathrm{I}$ is of type 2 and $O_{f,h}(n * n_3)\mathrm{I}$ is of type 1. When $O_{f,h}(n)\mathrm{I}$ is of type 3-b then $O_{f,h}(n * n_1)\mathrm{I}$ is of type 3-c, $O_{f,h}(n * n_2)\mathrm{I}$ is of type 2 and $O_{f,h}(n * n_3)\mathrm{I}$ is of type 1. And finally, when $O_{f,h}(n)\mathrm{I}$ is of type 3-c then $O_{f,h}(n * n_1)\mathrm{I}$ is of type 3-a, $O_{f,h}(n * n_2)\mathrm{I}$ is of type 2 and $O_{f,h}(n * n_3)\mathrm{I}$ is of type 1.

Q.E.D.

1.6 Hölder exponent

In this section we compute the Hölder exponent of $\xi : \mathbf{S^1} \to \mathbf{T^{n-1}}$, the semi-conjugacy between the interval exchange map $f : \mathbf{S^1} \to \mathbf{S^1}$ and $T : \mathbf{T^{n-1}} \to \mathbf{T^{n-1}}$, an irrational translation in $\mathbf{T^{n-1}}$.

Theorem 1.6.1 *The map $\xi : \mathbf{S^1} \to \mathbf{T^{n-1}}$ is Hölder continuous with exponent $\rho = -\frac{\log|\beta_1|}{\log\lambda}$, where β_1 is the greatest, in modulus, among those eigenvalues of the matrix associated with the substitution, with norm smaller than one. And λ is the Perron-Frobenius eigenvalue of this matrix.*

Proof:

First we are going to prove this in the case $n = 3$. In this situation $|\beta_1| = \lambda^{-1/2} = \alpha^{1/2}$ so $\rho = \frac{1}{2}$. (At the end of this section we shall show how to prove the theorem in the case when $n > 3$.)

It is enough to prove that

$$\exists C_0 > 0 \; \forall t, t' \in \mathrm{I} \quad \text{such that} \quad |\xi(t) - \xi(t')| \leq C_0|t - t'|^{\frac{1}{2}} \qquad (1.5)$$

Because from the dimensions of the domain and the image, one and two respectively, of ξ we get that the Hölder exponent $\rho \leq 1/2$.

Also we can reduce the proof of the inequality (1.5) to the case when t' is in the interior of a cylinder of the standard partition and t is an extreme point of this cylinder — In this proof we assume that the cylinders of the standard partition are closed intervals, instead of semi-open as we showed in the previous section. This assumption is made in order to facilitate the finding of a extreme point of the cylinder, on the other hand it does not alter the proof. In fact, let $\hat{t}$ be one point that is in closure of a cylinder but not in the cylinder, so $\hat{t} = \lim_{\tau \to \tau_0^-} O_{f,h}(m)\tau$ where τ_0 is either the discontinuity point of the map h or an extreme point of I_1, I_2, I_3. Therefore $O_{f,h}(m)\tau_0$ is a extreme point of the cylinder. Moreover $\chi(t) = \lim_{\tau \to \hat{t}^-} \chi(\tau)$ hence $\xi(t) = \xi(\hat{t})$ — In fact suppose that $\bar{t}, t'$ are in the interior of a cylinder of the standard partition, i.e. $\bar{t}, t' \in O_{f,h}(m)\mathrm{I}$ for some ICP m, such that they are in different subcylinders at the next level: $t' \in O_{f,h}(m * n_i)\mathrm{I}$ and $\bar{t} \in O_{f,h}(m * n_j)\mathrm{I}$ with $i \neq j$, $1 \leq i, j \leq 3$. Consider t the extreme point of $O_{f,h}(m * n_i)\mathrm{I}$ closest to $\bar{t}$:

$$\begin{aligned} |\xi(t') - \xi(\bar{t})| &\leq |\xi(t') - \xi(t)| + |\xi(t) - \xi(\bar{t})| \\ &\leq C_0|t - t'|^{\frac{1}{2}} + C_0|t - \bar{t}|^{\frac{1}{2}} \\ &\leq 2C_0 \max\{|t - t'|^{\frac{1}{2}}, |t - \bar{t}|^{\frac{1}{2}}\} \\ &\leq 2C_0|t' - \bar{t}|^{\frac{1}{2}}. \end{aligned}$$

If t is not a extreme point of $O_{f,h}(m * n_j)\mathrm{I}$, we denote by t'' the extreme point of $O_{f,h}(m * n_j)\mathrm{I}$ closest to t'. We have a similar computation, since the image of t and t' under ξ are the same because $t = O_{f,h}(m * n_j)s$ and $t'' = O_{f,h}(m * n_j)s'$ where s and s' are the extreme points of I_1, I_2, I_3 or the image of the discontinuity point of h under $O_{f,h}(m * n_j)$.

We prove the inequality (1.5) in the case of t an extreme point of a cylinder of the standard partition and t' an interior point of such cylinder. Always the cylinder can be subdivided into sub-cylinders such that t still remains an extreme point for one sub-cylinder and t' is in another sub-cylinder. Thererfore we can reduce the proof, to show that the inequality(1.5) holds for t and extreme point of $O_{f,h}(m * n_l)\mathrm{I}$ and t' is in $O_{f,h}(m * n_j)\mathrm{I}$ for $l \neq j$ and m an ICP, i.e. $m = g_{i_0} + \cdots g_{i_k}$; for all five cases of $O_{f,h}(m)\mathrm{I}$ – according to lemma 1.5.3 –. In each case the idea of the proof is to find an upper bound for $|\xi(t) - \xi(t')|$ using the contraction in $O_{T,B}(m)$ in T^2 and a lower bound for $|t - t'|$ based upon the length of the cylinder $O_{f,h}(m)\mathrm{I}$ and its sub-cylinders. The three cases are:

1. The cylinder $O_{f,h}(m)\mathrm{I}$ is of type 1. Assume that t' belongs to $O_{f,h}(m * n_2)\mathrm{I}$. (If t' is in $O_{f,h}(m * n_3)\mathrm{I}$ the proof follows the same lines). Since t is an extreme point of $O_{f,h}(m)\mathrm{I}$ it is image under $O_{f,h}(m)$, of the discontinuity point of h, denoted by p and $p = \frac{\alpha + \alpha^2}{2}$.

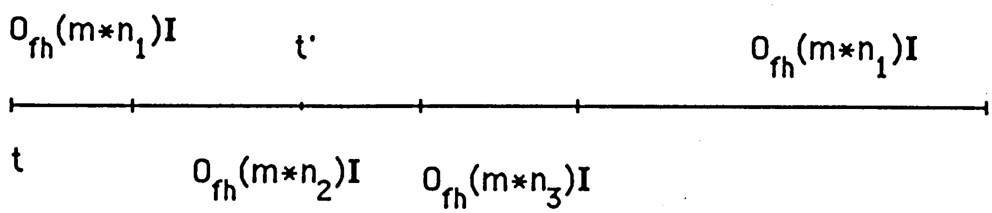


Figure 1.7: The cylinder $O_{f,h}(m)\mathbf{I}$ is of type 1

Let be $\chi : \mathbf{I} \to \overline{\mathcal{N}}[x]$ the map defined in section 1.3, that gives the symbolic expression in $\overline{\mathcal{N}}[x]$ of any point of the interval. The map $\delta_x : \overline{\mathcal{N}}[x] \to \mathbf{T}^2$ was defined in the same section as $\delta_x(\underline{a}) = (\sum a_i B^i z)/_{\sim}$, where $\sim$ is the equivalence relation defined by the lattice $\mathbf{Z}^2$ in $\mathbf{R}^2$.

Since m is an ICP– is of the form $m = g_{i_0} + \cdots g_{i_k}$ – and t' is in $O_{f,h}(m * n_2)\mathbf{I}$, we have $\chi(t') = x^{i_0} + \cdots + x^{i_k} + x^{i_k+2} r(x)$ for some $r(x)$ in $\overline{\mathcal{N}}[x]$.

On the other hand, the point t is the image under $O_{f,h}(m)$ of the discontinuity point of h, denoted by p, since we assume that t is an extreme point of the cylinder $O_{f,h}(m)\mathbf{I}$. A direct computation shows that p is a fixed point of the map $h^2 fh$, therefore $\chi(p) = \sum_{l \geq 0} x^{3l+2}$, so:

$$\chi(t) = x^{i_0} + \cdots + x^{i_{k-1}} + x^{i_k+2} + x^{i_k+5} + \cdots + x^{i_k+3l+2} + \cdots.$$

Now we can find the upper bound for:

$$\begin{aligned}
|\xi(t') - \xi(t)| &= |\delta_x(\chi(t')) - \delta_x(\chi(t))| \\
&= |(B^{i_0} z + \cdots + B^{i_k} z + B^{i_k+2} r(B) z) - \\
&\quad (B^{i_0} z + \cdots + B^{i_{k-1}} z + B^{i_k+2} q(B) z)| \\
&= |B^{i_k} z \ + \ B^{i_k+2} r(B) z \ - \ B^{i_k+2} q(B) z| \\
&\qquad \text{where } q(x) = \textstyle\sum_{l \geq 0} x^{3l} \\
&\leq |B^{i_k}| |(I + B^2(r(B) - q(B))) z| \\
&\leq \alpha^{\frac{i_k}{2}} C \quad \text{where } C = \text{diameter of } \omega \\
&= \alpha^{\frac{i_k+6}{2}} C \lambda^3 \\
&= \left(\frac{\sqrt{2} C \lambda^3}{\sqrt{\alpha + \alpha^2}}\right) \left(\frac{\alpha + \alpha^2}{2}\right)^{1/2} \alpha^{(i_k+6)/2}
\end{aligned}$$

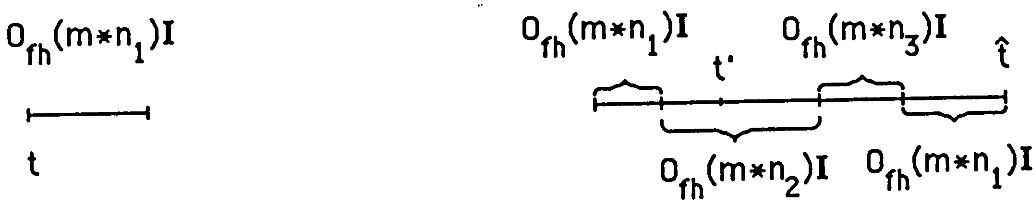


Figure 1.8: The cylinder $O_{f,h}(m)\mathbf{I}$ is of type 2

On the other hand :

$$\begin{aligned} |t-t'| &\geq |t - O_{f,h}(m)\alpha| \\ &= |O_{f,h}(m)p - O_{f,h}(m)\alpha| \\ &= \alpha^{i_k}\,|p-\alpha| \\ &= \alpha^{i_k}(\tfrac{\alpha-\alpha^2}{2}) \\ &= \alpha^{i_k}(\tfrac{\alpha^3+\alpha^4}{2}) \\ &= \alpha^{i_k+2}(\tfrac{\alpha+\alpha^2}{2}). \end{aligned}$$

Therefore

$$\begin{aligned} |\xi(t')-\xi(t)| &\leq C_1(\tfrac{\alpha+\alpha^2}{2})^{1/2}\,\alpha^{(i_k+6)/2} \\ &\leq C_1(\tfrac{\alpha+\alpha^2}{2})^{1/2}\,\alpha^{(i_k+2)/2} \\ &\leq C_1|t-t'|^{\frac{1}{2}} \end{aligned}$$

where

$$C_1 = C\lambda^3(\frac{\alpha+\alpha^2}{2})^{-(1/2)}.$$

2. When the cylinder $O_{f,h}(m)\mathbf{I}$ is of type 2, we assume t' is in $O_{f,h}(m*n_2)\mathbf{I}$. The case t' in $O_{f,h}(m*n_3\mathbf{I})$can be studied in a similar way.

   In this case we compare t' with t and $\hat{t}$, the extreme points of $O_{f,h}(m)\mathbf{I}$ that realize its diameter, see figure 1.8. As we remarked before $O_{f,h}(m)p = t$ and $\lim_{\tau\to p^-} O_{f,h}(m)\tau = \hat{t}$. Moreover $\chi(t) = \chi(\hat{t})$, so $\xi(t) = \xi(\hat{t})$.

   The finding of the upper bound for $|\xi(t')-\xi(t)|$ is the same as in case 1, so:

$$|\xi(t')-\xi(t)| \leq (\frac{\sqrt{2}c\lambda^3}{\sqrt{\alpha+\alpha^2}})\,(\frac{\alpha+\alpha^2}{2})^{1/2}\,\alpha^{(i_k+6)/2}$$

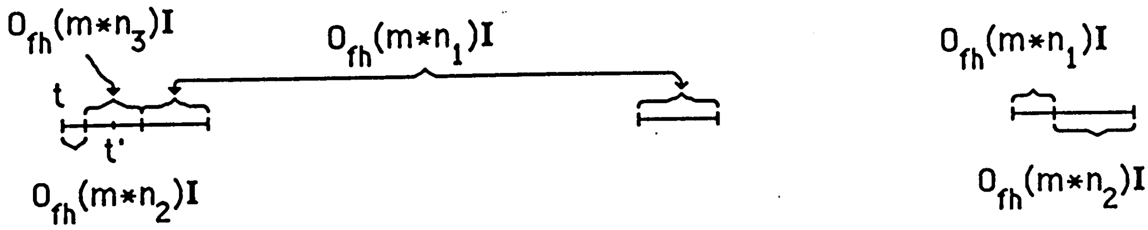


Figure 1.9: The cylinder $O_{f,h}(m)\mathbf{I}$ is of type 3-a

and also $|\xi(t') - \xi(t)| = |\xi(t') - \xi(\hat{t})|$.

On the other hand

$$
\begin{aligned}
|\hat{t} - t'| &\geq |O_{f,h}(m)(\alpha + \alpha^2) - \hat{t}| \\
&= |O_{f,h}(m)(\alpha + \alpha^2) - \lim_{\tau \to p^-} O_{f,h}(m)\tau| \\
&= \alpha^{i_k}|(\alpha + \alpha^2) - p| \\
&= \alpha^{i_k}(\tfrac{\alpha+\alpha^2}{2}).
\end{aligned}
$$

Also

$$
\begin{aligned}
|t - t'| &\geq |O_{f,h}(m)(\alpha) - t| \\
&= |O_{f,h}(m)(\alpha) - O_{f,h}(m)p| \\
&= \alpha^{i_k}|\alpha - p| \\
&= \alpha^{i_k}(\tfrac{\alpha-\alpha^2}{2}) \\
&= \alpha^{i_k+2}(\tfrac{\alpha+\alpha^2}{2}).
\end{aligned}
$$

Therefore

$$|\xi(t') - \xi(t)| \leq C_1|t - t'|^{1/2} \text{ and } |\xi(t') - \xi(\hat{t})| \leq C_1|\hat{t} - t'|^{1/2}.$$

3. Suppose that the cylinder is of type 3. According to lemma 1.5.3 there are three possible cases: 3-a, 3-b, 3-c. We shall prove the inequality 1.5 in case 3-a. In the other two cases the computations are similar.

   In the case 3-a the point t' can be in $O_{f,h}(m * n_3)\mathbf{I}$ or in $O_{f,h}(m * n_1)\mathbf{I}$ and t, $\hat{t}$ are the extreme points of $O_{f,h}(m)\mathbf{I}$ that they are also extreme points of $O_{f,h}(m * n_2)\mathbf{I}$, see figure 1.9.

Suppose that t' is in $O_{f,h}(m * n_3)\mathbf{I}$ then $\chi(t') = x^{i_0} + \cdots + x^{i_k} + x^{i_k+1} + x^{i_k+3}r(x)$ for some $r(x)$ in $\overline{\mathcal{N}}[x]$. Therefore:

$$\begin{aligned}
|\xi(t') - \xi(t)| &= |\delta_x(\chi(t')) - \delta_x(\chi(t))| \\
&= |(B^{i_0}z + \cdots + B^{i_k}z + B^{i_k+1} + B^{i_k+3}r(B)z) - \\
&\quad (B^{i_0}z + \cdots + B^{i_{k-1}}z + B^{i_k+2}q(B)z)| \\
&= |B^{i_k}z \; + |B^{i_k+1} \; B^{i_k+3}r(B)z \; - \; B^{i_k+2}q(B)z| \\
&\qquad \text{where } q(x) = \textstyle\sum_{l\geq 0} x^{3l} \\
&\leq |B^{i_k}||(I + B + B^2(r(B) - q(B))z)| \\
&\leq \alpha^{\frac{i_k}{2}}C \\
&= \alpha^{\frac{i_k+6}{2}}C\lambda^3 \\
&= \left(\tfrac{\sqrt{2}C\lambda^3}{\sqrt{\alpha+\alpha^2}}\right)\left(\tfrac{\alpha+\alpha^2}{2}\right)^{1/2} \alpha^{(i_k+6)/2}
\end{aligned}$$

On the other hand

$$\begin{aligned}
|t - t'| &\geq |t - O_{f,h}(m)(\alpha + \alpha^2)| \\
&= |O_{f,h}(m)p - O_{f,h}(m)(\alpha + \alpha^2)| \\
&= \alpha^{i_k}\,|p - \alpha| \\
&= \alpha^{i_k}(\tfrac{\alpha+\alpha^2}{2}).
\end{aligned}$$

Therefore $|\xi(t') - \xi(t)| \leq C_1|t - t'|$.

If t' is in $O_{f,h}(m * n_1)\mathbf{I}$ we compare it with the two extreme points of the cylinder that realize its diameter, see figure 1.9. As we remarked before these two extreme points have the same image under the map χ and therefore under ξ.

Since t' is in $O_{f,h}(n * n_1)\mathbf{I}$ its image in $\overline{\mathcal{N}}[x]$ is $\chi(t') = x^{i_0} + \cdots + x^{i_k} + x^{i_k+1}r(x)$ for some $r(x)$ in $\overline{\mathcal{N}}[x]$. So:

$$\begin{aligned}
|\xi(t') - \xi(t)| &= |\delta_x(\chi(t')) - \delta_x(\chi(t))| \\
&= |(B^{i_0}z + \cdots + B^{i_k}z + B^{i_k+1}r(B)z) - \\
&\quad (B^{i_0}z + \cdots + B^{i_{k-1}}z + B^{i_k+2}q(B)z)| \\
&= |B^{i_k+1}r(B)z \; - \; B^{i_k+2}q(B)z| \quad \text{where } q(x) = \textstyle\sum_{l\geq 0} x^{3l} \\
&\leq |B^{i_k}||r(B) - B^{i_k}q(B)z| \\
&\leq \alpha^{\frac{i_k}{2}}C \\
&= \alpha^{\frac{i_k+6}{2}}C\lambda^3 \\
&= \left(\tfrac{\sqrt{2}C\lambda^3}{\sqrt{\alpha+\alpha^2}}\right)\left(\tfrac{\alpha+\alpha^2}{2}\right)^{1/2} \alpha^{(i_k+6)/2}
\end{aligned}$$

On the other hand:

$$\begin{aligned}
|t-t'| &\geq |t-O_{f,h}(m)(\alpha+\alpha^2)| \\
&= |O_{f,h}(m)p - O_{f,h}(m)(\alpha+\alpha^2)| \\
&= \alpha^{i_k}\,|p-\alpha| \\
&= \alpha^{i_k}(\tfrac{\alpha+\alpha^2}{2})
\end{aligned}$$

$$\begin{aligned}
|\hat{t}-t'| &\geq |O_{f,h}(m)(\alpha)-\hat{t}| \\
&= |O_{f,h}(m)(\alpha) - \lim_{\tau\to p^-} O_{f,h}(m)\tau| \\
&= \alpha^{i_k}|\alpha-p| \\
&= \alpha^{i_k}(\tfrac{\alpha-\alpha^2}{2}) \\
&= \alpha^{i_k+2}(\tfrac{\alpha+\alpha^2}{2}).
\end{aligned}$$

This ends the proof of theorem 1.6.1 in the case of $n=3$.

As was showed, the proof of this theorem in case $n = 3$, depends on the structure of standard partition cylinders in the interval — studied in lemma 1.5.3 — which allowed us to compare the distance between a point in the interior of the cylinder and its extreme points. As we have seen this distance is of the order of α^{i_k} where i_k is the degree of m, $m = g_{i_0}+\cdots+g_{i_k}$. On the other hand the distance between the images of these points , under the map ξ is of the order $|B^{i_k}| = \alpha^{\frac{i_k}{2}}$.

In the case of n greater or equal to 4, the structure of the cylinders of the standard partition is similar, but could however consists of j connected components for $1 \leq j \leq n$. The distance between a point in the interior and the extremes is still of the order α^{i_k} for α the real root of $x^n + x^{n-1} + \cdots + x - 1$. On the other hand the distance between the images of these points is of the order

$$|B^{i_k}| = |\beta_1|^{i_k} = \alpha^{\frac{i_k \log|\beta_1|}{\log\alpha}} = \lambda^{\frac{i_k \log|\beta_1|}{\log\lambda}}$$

where β_1 is the gratest eigenvalue of B, in modulus.

End of the proof of Theorem 1.6.1

Chapter 2

A Geodesic Lamination on D^2 as a geometrical realization of the substitution Π_q.

2.1 Introduction

In this chapter we construct a geodesic lamination on the disk $\mathbf{D^2}$ with the Poincaré metric, associated to the standard partition on Ω (the symbolic space defined by the substitution Π_q) and therefore on its geometrical realizations on $\mathbf{T^{n-1}}$ and $\mathbf{S^1}$. Some of this lamination can be seen in the figures 2.1 and 2.2. The construction given here is done in the case $q = 3$, i.e. the substitution is defined in three symbols, but can be easily generalized to an arbitrary q.

We consider the circle at infinity of $\mathbf{D^2}$ as the domain of the interval exchange map f, studied in the previous chapter. We shall join by geodesics in $\mathbf{D^2}$, the points of $\mathbf{S^1}$ that are mapped, under Arnoux's map ξ to "the triple point" of $O_{BT}(n)\omega$ (which is the image under $O_{BT}(n)$ of the point where ω_1, ω_2 and ω_3 intersect in the interior of ω) for all each integer n compatible with the partition. Later we define the geodesic lamination Λ as the closure of the set of geodesics defined above. In section 2.2, it is proved that Λ really is a geodesic lamination.

The dynamics on $\mathbf{I}$, given by the commutative diagram

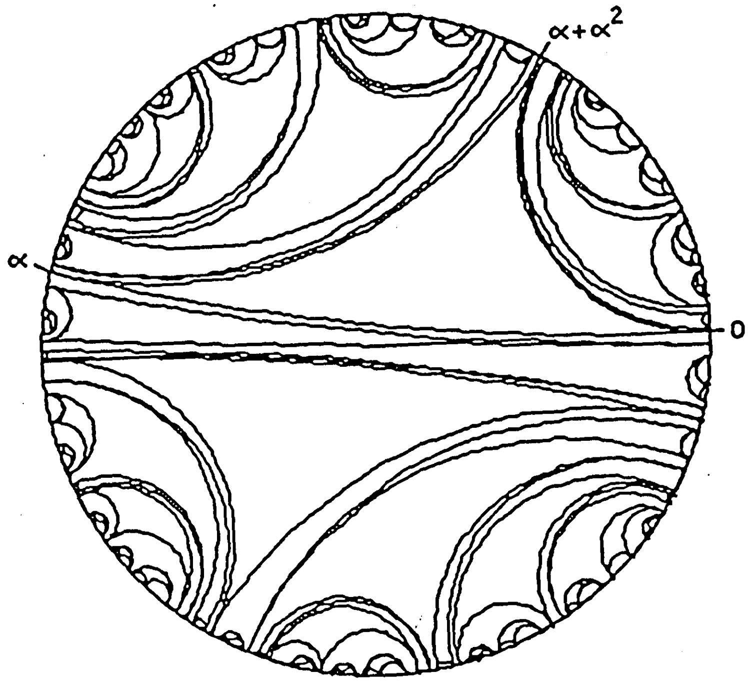


Figure 2.1: The geodesic lamination Λ for $q = 3$

$$
\begin{array}{ccc}
\mathbf{I} & \xrightarrow{\ f\ } & \mathbf{I} \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\
\mathbf{I}_1 & \xrightarrow{\ \tilde{f}\ } & \mathbf{I}_1
\end{array}
$$

is extended to Λ — in section 2.4—, and gives rise to the following commutative diagram

$$
\begin{array}{ccc}
\Lambda & \xrightarrow{\ F\ } & \Lambda \\
{\scriptstyle H}\downarrow & & \downarrow{\scriptstyle H} \\
\Lambda_1 & \xrightarrow{\ \tilde{F}\ } & \Lambda_1
\end{array}
$$

where

$$\Lambda_1 = \{\lambda \in \Lambda \mid \text{ the end points of } \lambda \text{ are in } [0, \alpha]\}.$$

and F is semiconjugate to $\sigma : \Omega \to \Omega$ (the dynamical system defined by the substitution II)

$$
\begin{array}{ccc}
\Lambda & \xrightarrow{\ F\ } & \Lambda \\
{\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \psi} \\
\Omega & \xrightarrow{\ \sigma\ } & \Omega
\end{array}
$$

where ψ is continuous and surjective.

Finally we shall show that Λ admits naturally a transverse measure μ that is invariant under F and $H_*\mu = \lambda^{s_0}\mu$ where λ is the Pisot number associated to the substitution and $s_0 \in (0, 1)$ is computed in section 2.3.

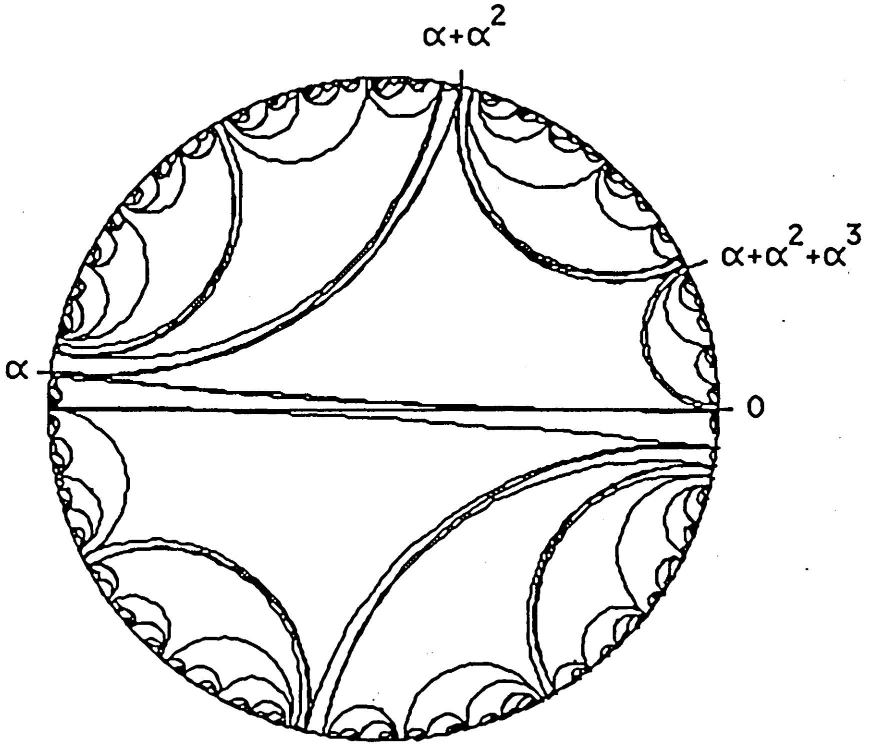


Figure 2.2: The geodesic lamination Λ for $q = 4$

2.2 Construction of the geodesic lamination Λ.

We can think of $\mathbf{S}^1$ as the circle at infinity of $\mathbf{D}^2$ and of the interval exchange map f – defined in page 5 – as acting on it.

On $\mathbf{S}^1$ we are going to distinguish three points $y_1 = 0$, $y_2 = \alpha$, $y_3 = \alpha + \alpha^2$; the boundary points of the standard-partition-rectangles, in $\mathbf{S}^1$.

Proposition 2.2.1 *The images of y_i $i = 1,2,3$ under $\xi : \mathbf{S}^1 \to \mathbf{T}^2$ are the same. Furthermore $\xi(O_{f,h}(n)y_i) = \xi(O_{f,h}(n)y_j)$ for all n ICP and i,j=1,2,3.*

Proof: The points y_i, $i = 1,2,3$ satisfy the relation:

$$fh(y_1) = y_2,\ fh(y_2) = y_3,\ fh(y_3) = y_1 \tag{2.1}$$

Since the maps $(fh)^3$: $I_i \to I_i$ are contractions, the points y_i, $i = 1,2,3$ are the fixed points of these maps. On the other hand $(TB)^3 : \omega_i \to \omega_i$ is a contraction with fixed point $\hat{\xi}(y_i)$ $i = 1,2,3$, since

$$\hat{\xi}(f(x)) = T(\hat{\xi}(x)) \quad \text{and} \quad \hat{\xi}(h(x)) = B((\hat{\xi}(x)) \quad \text{for all } x \in \mathbf{I} \tag{2.2}$$

Since y_i is a boundary point of two rectangles: I_i and I_j for some $1 \leq j \leq 3$, $\hat{\xi}(y_i)$ is also a boundary point of the ω_j. i.e. $\hat{\xi}(y_i) \in \omega_i \cap \omega_j$, so $\hat{\xi}(y_i)$ also satisfies the equation

$$(TB)^3(x) = x,\ x \in \omega_j \tag{2.3}$$

therefore $\hat{\xi}(y_i) = \hat{\xi}(y_j)$. But similarly $\hat{\xi}(y_j) \in \omega_j \cap \omega_k$ for $k \neq i$ (since y_i is the boundary point of I_j and I_k) and therefore $\hat{\xi}(y_j) = \hat{\xi}(y_k) = \hat{\xi}(y_i)$.

On the other hand $\hat{\xi}(O_{f,h}(n)y_i)$ is equal to $O_{T,B}(n)\hat{\xi}(y_i)$ by the property 2.2. Therefore $\hat{\xi}(O_{f,h}(n)y_i) = \hat{\xi}(O_{f,h}(n)y_j)$. Hence $\xi(O_{f,h}(n)y_i) = \xi(O_{f,h}(n)y_j)$.

Q.E.D.

Let $\mathcal{L}$ be the set of geodesics in $\mathbf{D}^2$. The topology on $\mathcal{L} \cup \mathbf{S}^1$ — where $\mathbf{S}^1$ is the circle at infinity of $\mathbf{D}^2$— is given by the following basis of neighbourhoods:

- If γ is an element of $\mathcal{L}$ with end points a and b in $\mathbf{S}^1$, consider the collection of neighbourhoods $(a-\epsilon, a+\epsilon)$ and $(b-\epsilon, b+\epsilon)$ for $\epsilon > 0$. Then the basis elements containing γ are given by the set of geodesics with one end point in $(a-\epsilon, a+\epsilon)$ and the other in $(b-\epsilon, b+\epsilon)$.

- If t is in $\mathbf{S}^1$, consider the collection of neighbourhoods in $\mathbf{S}^1$given by $(t-\epsilon, t+\epsilon)$ for $\epsilon > 0$, then the basis elements containing t are given by the point t and the set of geodesics with one end point in $(t-\epsilon, t)$ and the other in $(t, t+\epsilon)$.

The construction of Λ is as follows: the pair of points $O_{f,h}(n)y_i$, $O_{f,h}(n)y_j$ for n an ICP is joined by a geodesic in $\mathbf{D}^2$, say $\gamma^n_{i,j}$, and then the closure of the union of all these geodesics is taken, i.e.

$$\Lambda = \overline{\cup\{\gamma^n_{ij} | n \in \mathcal{P}\ i,j = 1,2,3\}}. \tag{2.4}$$

The elements of Λ are either geodesics of $\mathbf{D}^2$ or points in $\mathbf{S}^1$. In the later case, those points are called *degenerate geodesics.*

Definition 2.2.1 *A* geodesic lamination *on* $\mathbf{D}^2$ *is a non-empty closed subset of* $\mathcal{L} \cup \mathbf{S}^1$ *whose elements are disjoint.*

Proposition 2.2.2 *If* $t, \bar{t} \in \mathbf{S}^1$ *are joined by a geodesic of* Λ *then* $\xi(t) = \xi(\bar{t})$.

Proof: Let γ denote the geodesic in Λ that joins t and $\bar{t}$. There exists a sequence $\{m_k\} \in \mathcal{P}$ such that $\gamma^k_{ij} \to \gamma$ therefore $O_{f,h}(m_{k_l})y_i \to t$ and $O_{f,h}(m_{k_l})y_j \to \bar{t}$ where $\{m_{k_l}\}$ is a subsequence of $\{m_k\}$. Using proposition 2.2.1 and the continuity of ξ we get $\xi(t) = \xi(\bar{t})$.

Q.E.D.

The converse of this proposition is not true.

Before proving that Λ is a geodesic lamination, we need to introduce more notation and some technical lemmas.

Given a cylinder of the standard partition in $\mathbf{S}^1$, according to Lemma 1.5.3, this cylinder is either:

1. one interval, $[a_1, a_2)$

2. two intervals $[b_1, b_2)$, $[b_3, b_4)$ or

3. three intervals $[c_1, c_2)$, $[c_3, c_4)$, $[c_5, c_6)$.

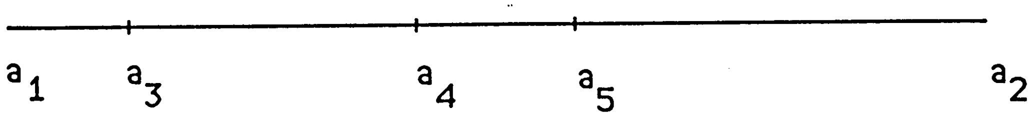


Figure 2.3: $O_{f,h}(n)\mathrm{S}^1 = [a_1, a_2)$

Definition 2.2.2

- *In case 1 we say a_1, a_2 are extreme points of the same type.*
- *In case 2, b_1 and b_4 are extreme points of the same type, and so are b_2 and b_3.*
- *In case 3 c_1, c_6 are extreme points of the same type, and similarly for c_2, c_3 and c_4, c_5.*

This definition is justified by the following lemma:

Lemma 2.2.1 *Let n be an ICP and $O_{f,h}(n)\mathrm{S}^1$ a cylinder of the standard partition. Then the geodesics that join the same type of extreme points of $O_{f,h}(n)\mathrm{S}^1$, belong to Λ.*

Furthermore if $n = n_{i_1} * \cdots * n_{i_k}$ is the factorization of n in $(\mathcal{P}, *)$ the extreme points of $O_{f,h}(n)\mathrm{S}^1$ are of the form $O_{f,h}(m)y_j$ for $m = n_{i_1} * \cdots * n_{i_r}$, for some $r < k$.

Proof: We use induction on the number of factors of n in $(\mathcal{P}, *)$.

When $k = 1$, n is either n_1, n_2 or n_3. If $n = n_1$, $O_{f,h}(n_1)\mathrm{S}^1 = [0, \alpha)$. Here the extreme points 0 and α are joined by a geodesic in Λ, according to its definition, and also $0 = y_1 = O_{f,h}(n_0)y_1$, $\alpha = y_2 = O_{f,h}(n_0)y_2$. Similarly for $n = n_2$ and $n = n_3$.

When $k > 1$, $n = n_{i_1} * \cdots * n_{i_k}$, we have to consider the different cases of $O_{f,h}(n)\mathrm{S}^1$ given by Lemma 1.5.3.

If $O_{f,h}(n)\mathrm{S}^1$ is one interval (See figure 2.3) which is subdivided $O_{f,h}(n)\mathrm{S}^1 = O_{f,h}(n * n_2)\mathrm{S}^1 \cup O_{f,h}(n * n_3)\mathrm{S}^1 \cup O_{f,h}(n * n_1)\mathrm{S}^1$ since $O_{f,h}(n * n_i)\mathrm{S}^1 = O_{f,h}(n)I_i$ for $i = 1, 2, 3$ we have $a_3 = O_{f,h}(n)\alpha = O_{f,h}(n)y_2$ and $a_4 = O_{f,h}(n)\alpha + \alpha^2 = O_{f,h}(n)y_3$ and $a_5 = O_{f,h}(n)0 = O_{f,h}(n)y_1$, the boundary points of the subpartition of the cylinder $O_{f,h}(n)\mathrm{S}^1$.

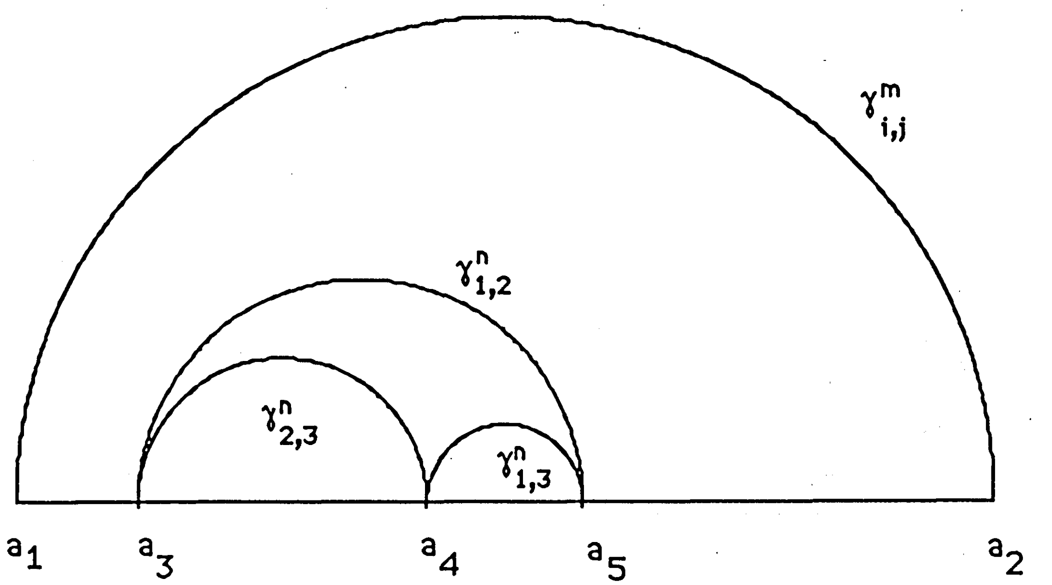


Figure 2.4:

So the extreme points of $O_{f,h}(n * n_2)S^1$ i.e. a_3, a_4 are joined by geodesics of Λ, γ_{23}^n.

Similarly for a_4, a_5 the extreme points of $O_{f,h}(n * n_3)S^1$, also a_3, a_5 are joined by geodesics in Λ, $\gamma_{1,2}^n$ and on the other hand are extreme points of the same type of $O_{f,h}(n * n_1)S^1$.

By the inductive hypothesis a_1 and a_2 —extreme points of $O_{f,h}(n)S^1$ and also for $O_{f,h}(n * n_1)S^1$— are joined by a geodesic of Λ : γ_{ij}^m where $m = n_{i_1} * \cdots * n_{i_r}$ with $r < k$. See figure 2.4

- If $O_{f,h}(n)S^1$ has two connected components:

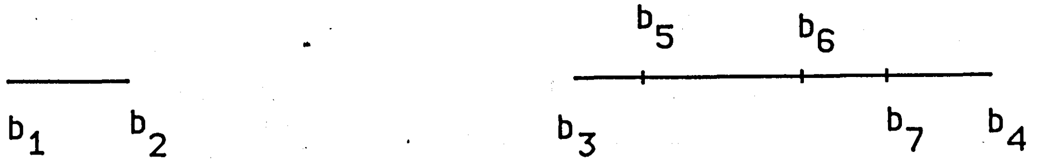


Figure 2.5: $O_{f,h}(n)S^1 = [b_1, b_2) \cup [b_3, b_4)$,

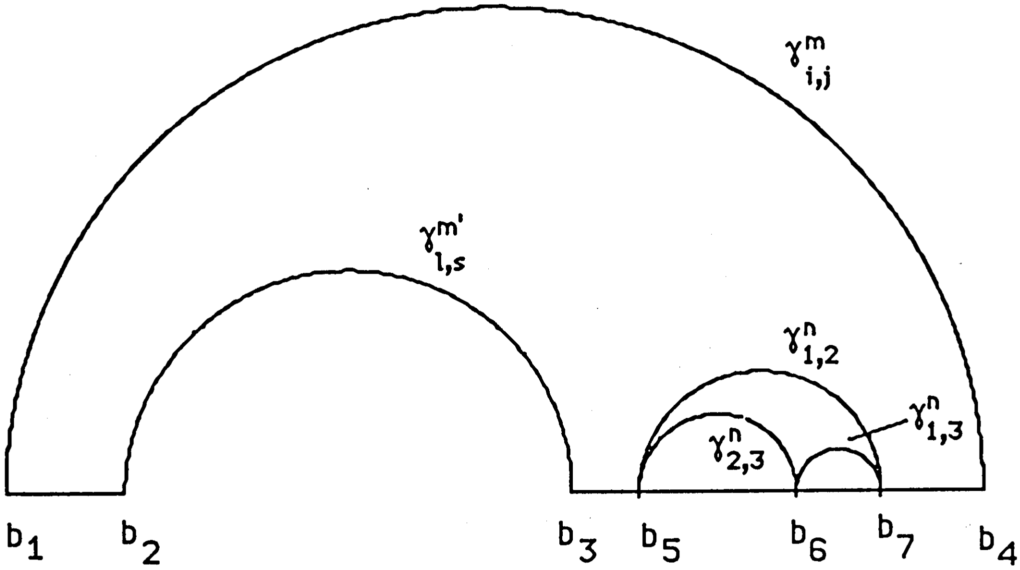


Figure 2.6:

Where $O_{f,h}(n)\mathbf{S}^1 = [b_1, b_2) \cup [b_3, b_4)$,
$O_{f,h}(n)\mathbf{S}^1 = \cup_{i=1}^{3} O_{f,h}(n * n_i)\mathbf{S}^1$
$O_{f,h}(n * n_2)\mathbf{S}^1 = [b_5, b_6)$ $O_{f,h}(n * n_3)\mathbf{S}^1 = [b_6, b_7)$
$O_{f,h}(n * n_1)\mathbf{S}^1 = [b_1, b_2) \cup [b_3, b_5) \cup [b_7, b_4)$
and $b_5 = O_{f,h}(n)\alpha$, $b_6 = O_{f,h}(n)(\alpha + \alpha^2)$, $b_7 = O_{f,h}(n)0$. See figure 2.5

So the extreme points of $O_{f,h}(n * n_2)\mathbf{S}^1$ are joined by geodesics of Λ, $\gamma^n_{2,3}$. Similarly for b_6 and b_7 the extreme points of $O_{f,h}(n * n_3)\mathbf{S}^1$, which are joined by $\gamma^n_{3,1}$. Also b_5 and b_7 are same type extreme points of $O_{f,h}(n * n_1)\mathbf{S}^1$ and are joined by geodesic $\gamma^n_{1,2}$.

The pairs of points b_2, b_3 and b_1, b_4 are extreme points of the same type for $O_{f,h}(n * n_1)\mathbf{S}^1$ and also for $O_{f,h}(n)\mathbf{S}^1$. Therefore they are joined by geodesics $\gamma^m_{i,j}$, $\gamma^{m'}_{ls}$ where $m = n_{i_1} * \cdots * n_{i_r}$, $m' = n_{i_1} * \cdots * n_{i_{r'}}$, $r, r' < k$. See figure 2.6

In the case when $O_{f,h}(n)\mathbf{S}^1$ has three connected components, suppose that $O_{f,h}(n * n_3)\mathbf{S}^1$ is contained in the first component $O_{f,h}(n)\mathbf{S}^1$ i.e. case 3.1 of the Lemma 1.5.3. See figure 2.7. where $O_{f,h}(n)\mathbf{S}^1 = [c_1, c_2) \cup [c_3, c_4) \cup [c_5, c_6)$
$O_{f,h}(n * n_3)\mathbf{S}^1 = [c_8, c_9)$

Figure 2.7: $O_{f,h}(n)\mathbf{S}^1 = [c_1, c_2) \cup [c_3, c_4) \cup [c_5, c_6)$

$O_{f,h}(n * n_2)\mathbf{S}^1 = [c_1, c_8) \cup [c_7, c_6)$
$O_{f,h}(n * n_1)\mathbf{S}^1 = [c_9, c_2) \cup [c_3, c_4) \cup [c_5, c_7)$
and $c_7 = O_{f,h}(n)\alpha$, $c_8 = O_{f,h}(n)(\alpha + \alpha^2)$, $c_9 = O_{f,h}(n)0$. Therefore the points c_8 and c_3 are joined by γ_{ij}^m c_5 and c_4 by $\gamma_{i'j'}^{m'}$ and c_6, c_1 by $\gamma_{ij}^{\tilde{m}}$ where $m = n_{i_1} * \cdots * n_{i_r}$, $m' = n_{i_1} * \cdots * n_{i_{r'}}$, $\tilde{m} = n_{i_1} * \cdots * n_{i_{\tilde{r}}}$ and $r, r', \tilde{r} < k$.

Q.E.D.

Corollary 2.2.1 *The points* $O_{f,h}(n)y_i$ $i = 1, 2, 3$ *are the extreme points of* $O_{f,h}(n * n_j)\mathbf{S}^1$ $j = 1, 2, 3$.

Theorem 2.2.1 Λ *is a geodesic lamination*

Proof: It is sufficient to prove that there are no intersections among the geodesics of the type γ_{ij}^n for $n \in \mathcal{P}$. However, according to Corollary 2.2.1 we can reduce the proof to showing that there is no intersection between geodesics that join extreme points of the same type of standard partition cylinders.

Suppose that such an intersection happens i.e. a geodesic that joins the same type extreme points of $O_{f,h}(n)\mathbf{S}^1$ intersects another geodesic that joins the same type extreme point of $O_{f,h}(m)\mathbf{S}^1$, with n and $m \in \mathcal{P}$.

There are two possible cases:

1. $\text{int}(O_{f,h}(n)\mathbf{S}^1) \cap \text{int}(O_{f,h}(m)\mathbf{S}^1) \neq \emptyset$
2. $\text{int}(O_{f,h}(n)\mathbf{S}^1) \cap \text{int}(O_{f,h}(m)\mathbf{S}^1) = \emptyset$

In 1, the cylinders intersects in a set of positive Lebesgue measure and since they are cylinders of the standard partition one must be contained in the other. But in this case there is no intersection between the geodesics γ_{ij}^n and $\gamma_{i'j'}^m$

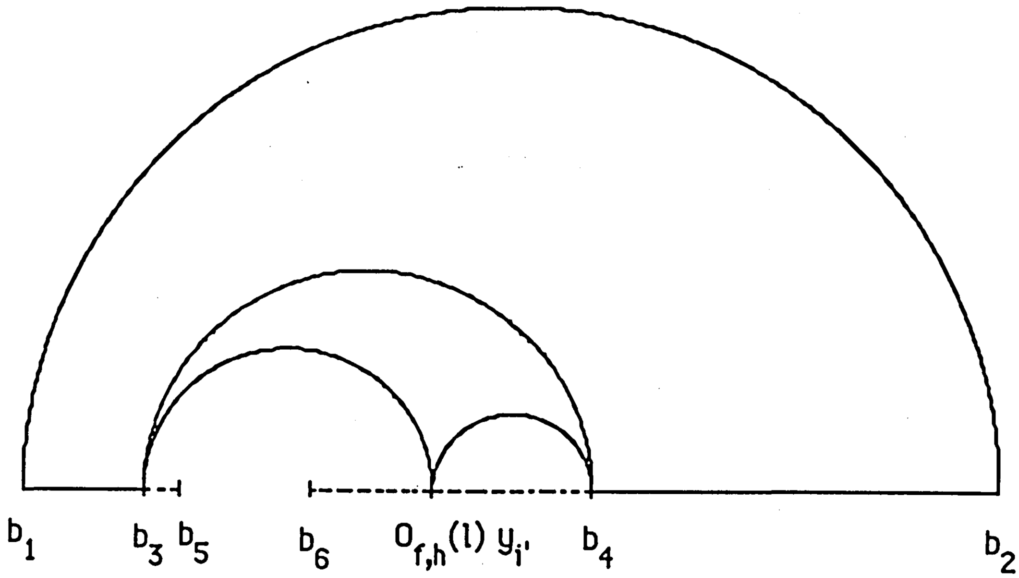


Figure 2.8:

In 2 one connected component of $O_{f,h}(m)\mathbf{S}^1$ lies in one of the gaps of $O_{f,h}(n)\mathbf{S}^1$. In fact, we shall show that we can fill the gaps between two components of any cylinder of the standard partition by other cylinders of this partition.

Suppose that $O_{f,h}(n)\mathbf{S}^1$ has two connected components $[b_1, b_3)$ and $[b_4, b_5)$ (in the case of three connected components the argument is the same) and $n = n_{i_1} * \cdots * n_{i_k}$.

According to lemma 2.2.1 the geodesic that joins b_3 with b_4 is of the type $\gamma^l_{i,j}$ where $l = n_{i_1} * \cdots * n_{i_s}$ $s < k$. Suppose that $O_{f,h}(l)y_{i'}$, where $i' \in \{1,2,3\} \setminus \{i,j\}$, is greater than b_2 or smaller than b_1, so $O_{f,h}(l)\mathbf{S}^1$ and $O_{f,h}(n)\mathbf{S}^1$ have empty intersection and neither is contained in the other. This is a contradiction to the fact that $O_{f,h}(l)\mathbf{S}^1$ and $O_{f,h}(n)\mathbf{S}^1$ are cylinders of the standard partition.

Therefore $O_{f,h}(l)y_{i'} \in [b_1, b_3) \cup [b_4, b_2)$ or $O_{f,h}(l)y_{i'} \in [b_3, b_4)$.

Suppose that $O_{f,h}(l)y_{i'} \in (b_3, b_4)$. According to lemma 2.2.1 there exits b_5 and b_6 (possibly $b_5 = b_6$) such that $b_3 < b_5 \leq b_6 < O_{f,h}(n)y'_i$, and $[b_3, b_5)$, $[b_6, b_4)$ are contained in the connected components of $O_{f,h}(l)\mathbf{S}^1$. See figure 2.8

If $b_5 \neq b_6$ we consider the geodesic that joins these two points and get b_7 and b_8 using the same arguments. And so on.

Either there exists t such that $b_{2t-1} = b_{2t}$ or the sequence $\{b_t\}_{t\in\mathbf{N}}$ converges to a point. In each case we have filled the gap with cylinders of the

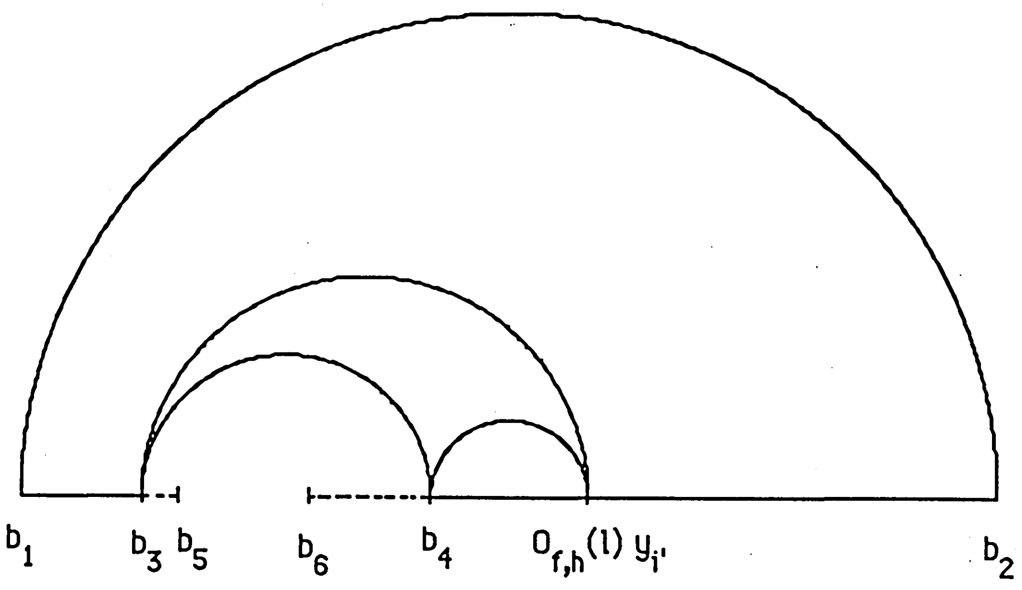


Figure 2.9:

standard partition.

If $O_{f,h}(l)y_i \in [b_1, b_3) \cup [b_4, b_2)$ (See figure 2.9) we apply the same argument to the geodesic that joins b_3 and b_4 and we get b_5 and b_6, and we carry on in the same way.

Q.E.D.

2.3 The transverse measure to Λ

Let δ be any arc in $\mathbf{D}^2$ joining two distinct geodesics of Λ. It can be slid along the geodesic towards the boundary of $\mathbf{D}^2$ according the two possible directions in which the geodesics can be oriented. This procedure gives rise to a Cantor set in the boundary of $\mathbf{D}^2$, say C_δ

Let δ be a transverse arc to Λ. We define

$$\mu(\delta) = \mathcal{M}_{s_0}(C_\delta)$$

where $\mathcal{M}_{s_0}$ is the s_0-Hausdorff measure and s_0 is the Hausdorff dimension of C_δ

Lemma 2.3.1 *For every transverse curve δ to Λ, the Hausdorff dimension of C_δ is $s_0 = \frac{\log\theta}{\log\lambda}$ where θ is the greatest root, in modulus, of the polynomial $x^4 - 2x - 1$*

Proof: Since any geodesic of this lamination is a limit of geodesics of the form γ_{ij}^n with $n \in \mathcal{P}$, we can suppose that the extreme points of δ are in geodesics of this type. Also we can assume that this geodesic joins the same type extreme points of $O_{f,h}(n)\mathbf{S}^1$ for some $n \in \mathcal{P}$. (If this is not the case we can write δ as a union of δ_i's which have the cited property).

In the following lines we are going to show how the Cantor set C_δ is obtained.

Let $K_0(n) = O_{f,h}(n)\mathbf{S}^1$ which admits the partition $O_{f,h}(n)\mathbf{S}^1 = \cup_{i=1}^3 O_{f,h}(n * n_i)\mathbf{S}^1$

When δ is slid along the geodesics towards the boundary, some of the cylinders of this partition do not contribute to C_δ, i.e. the intersection between C_δ and a non-contributing cylinder is empty.

Let $K_1(n) = \cup_{i=1}^{N_1} K_1^i$ where $K_1(n)$ is a contributing cylinder to C_δ of this partition. We carry on this subdivision in $O_{f,h}(m * n_1)\mathbf{S}^1$, $O_{f,h}(m * n_2)\mathbf{S}^1$, $O_{f,h}(m * n_3)\mathbf{S}^1$ for any contributing rectangle $K_j^i(n) = O_{f,h}(m)\mathbf{S}^1$.

It is clear that $C_\delta = \cap_{j\geq 0} K_j(n)$.

Next we study the formation rule of the $K_j(n)$'s. We distinguish two cases:

1. $O_{f,h}(n)\mathbf{S}^1$ has two connected components.

2. $O_{f,h}(n)\mathbf{S}^1$ has three connected components.

In case 1 $O_{f,h}(n)\mathbf{S}^1 = [b_1, b_3) \cup [b_2, b_4)$. and according to Lemma 1.5.3 $O_{f,h}(n)\mathbf{S}^1 = \cup_{i=1}^3 O_{f,h}(n * n_i)\mathbf{S}^1$

where $O_{f,h}(n * n_2)\mathbf{S}^1 = [b_5, b_6)$ $O_{f,h}(n * n_3)\mathbf{S}^1 = [b_6, b_7)$

and $O_{f,h}(n * n_2)\mathbf{S}^1 \cup O_{f,h}(n * n_3)\mathbf{S}^1 \subset [b_2, b_4)$ See figure 2.10

When δ is slid towards (b_2, b_4) the interval (b_5, b_7) is removed. Therefore: $K_0 = O_{f,h}(n)\mathbf{S}^1$, and $K_1 = O_{f,h}(n * n_1)\mathbf{S}^1$

Observe that $|K_1| = \alpha|K_0|$, where $|\ |$ denotes the Lebesgue measure.

And now K_1 has three connected components, so its subdivision is studied in case 2.

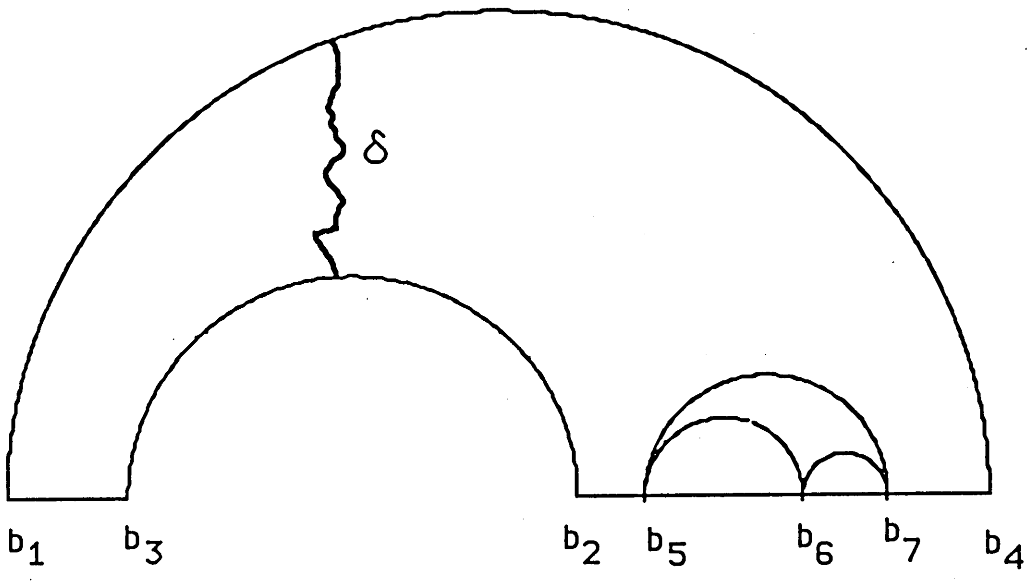


Figure 2.10:

In case 2 we are going to suppose that the cylinder $O_{f,h}(n)\mathbf{S}^1$ is of type 3.1 in the classification given in Lemma 1.5.3. i.e.

$$O_{f,h}(n)\mathbf{S}^1 = \bigcup_{i=1}^{3} O_{f,h}(n * n_i)\mathbf{S}^1$$

$$\begin{aligned} O_{f,h}(n * n_1)\mathbf{S}^1 &= [c_8, c_2) \cup [c_3, c_4) \cup [c_5, c_9) \\ O_{f,h}(n * n_2)\mathbf{S}^1 &= [c_1, c_7) \cup [c_9, c_6) \\ O_{f,h}(n * n_3)\mathbf{S}^1 &= [c_7, c_8) \quad \text{(See figure 2.11)} \end{aligned}$$

therefore: $K_0(n) = O_{f,h}(n)\mathbf{S}^1$, $K_1(n) = K_1^1(n) \cup K_1^2(n)$,
$K_1^1(n) = O_{f,h}(n * n_2)\mathbf{S}^1$ $K_1^2(n) = O_{f,h}(n * n_1)\mathbf{S}^1$

$K_1^1(n)$ has two connected components and therefore its subdivision is according to the description given in case 1.

Lemma 1.5.3 gives three different types of cylinders having three connected components. Since $K_1^2(n)$ is of a different type from $K_0(n)$, we need to subdivide it, until we reach cylinders of the same type as $K_0(n)$.

$K_1^2(n) = O_{f,h}(n * n_1)\mathbf{S}^1 = \bigcup_{i=1}^{3} O_{f,h}(n * n_1 * n_i)\mathbf{S}^1$ where

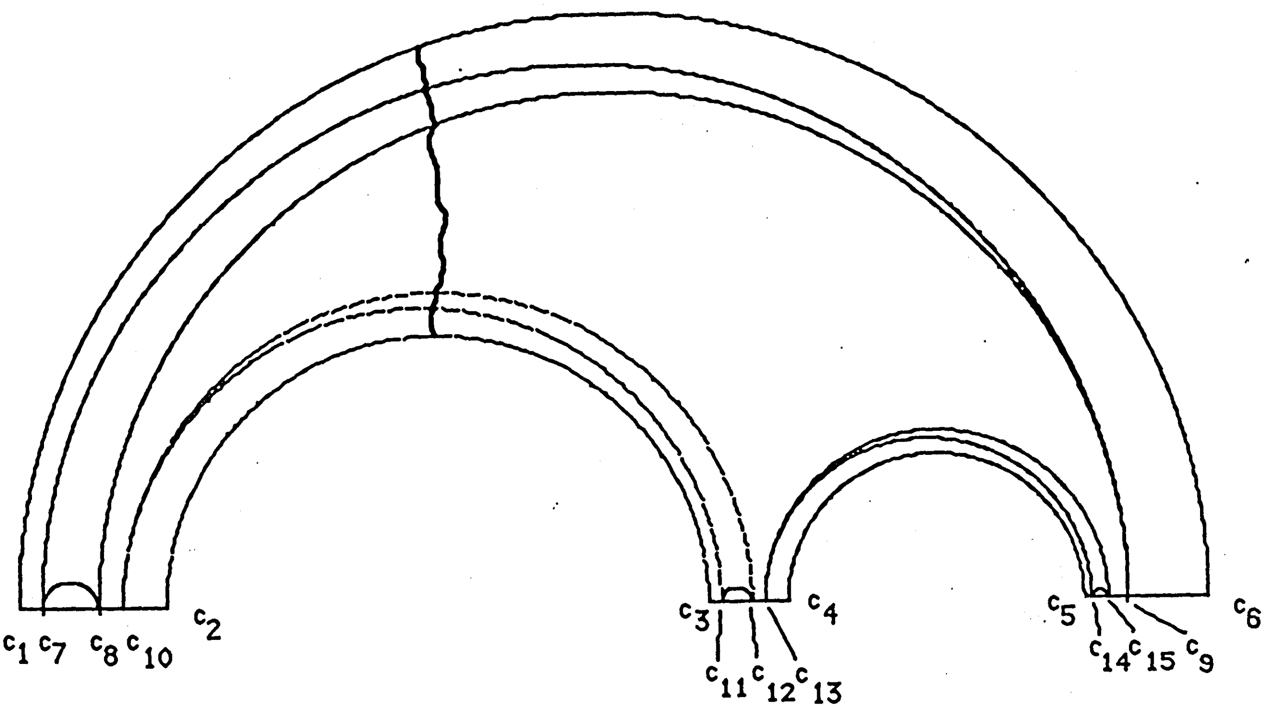


Figure 2.11:

$O_{f,h}(n * n_1 * n_1)\mathbf{S}^1 = [c_8, c_{10}) \cup [c_{12}, c_4) \cup [c_5, c_9)$
$O_{f,h}(n * n_1 * n_2)\mathbf{S}^1 = [c_{10}, c_2) \cup [c_3, c_{11})$
$O_{f,h}(n * n_1 * n_3)\mathbf{S}^1 = [c_{11}, c_{12})$ but this one is a non-contributing cylinder to C_δ.

Therefore $K_2(n) = K_2^1(n) \cup K_2^2(n) \cup K_2^3(n)$ where $K_2^1(n) = O_{f,h}(n * n_2 * n_1)\mathbf{S}^1$ is the contributing cylinder that arises in the subdivision of $K_1^1(n)$.

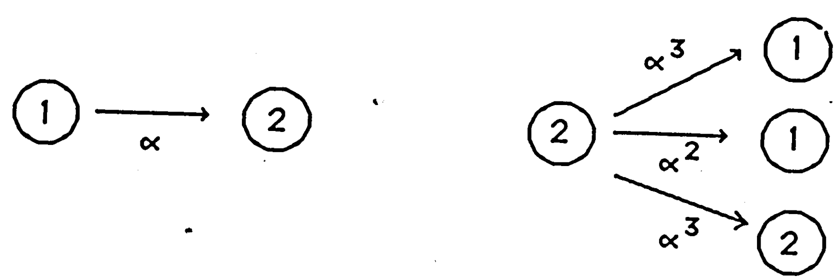
$K_2^2(n) = O_{f,h}(n * n_1 * n_1)\mathbf{S}^1$ and $K_2^3(n) = O_{f,h}(n * n_1 * n_2)\mathbf{S}^1$ this last one has two connected components.

The cylinder $K_2(n)$ is of type 3.3, therefore we need to subdivide it.
$O_{f,h}(n * n_1 * n_1)\mathbf{S}^1 = \bigcup_{i=1}^3 O_{f,h}(n * n_1 * n_1 * n_i)\mathbf{S}^1$
$O_{f,h}(n * n_1 * n_1 * n_1)\mathbf{S}^1 = [c_8, c_{10}) \cup [c_{12}, c_{13}) \cup [c_{15}, c_9)$
$O_{f,h}(n * n_1 * n_1 * n_2)\mathbf{S}^1 = [c_{13}, c_4) \cup [c_5, c_{14})$
$O_{f,h}(n * n_1 * n_3)\mathbf{S}^1 = [c_{14}, c_{15})$

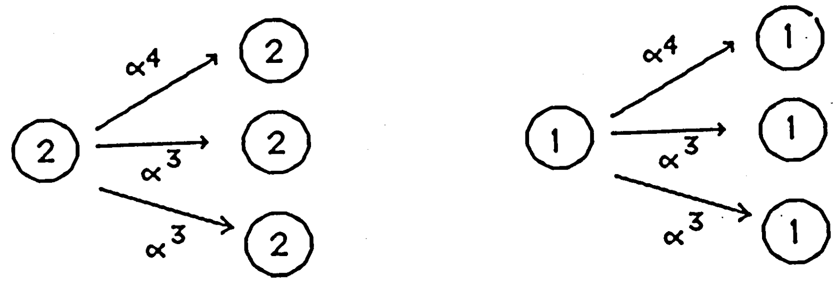
Here the only contributing cylinder is $O_{f,h}(n * n_1 * n_1 * n_1)\mathbf{S}^1$ which is of the same type of K_0.

Observe that $|O_{f,h}(n * n_1 * n_1 * n_1)\mathbf{S}^1| = \alpha^3 |O_{f,h}(n)\mathbf{S}^1|$
$|K_2^2(n)| = \alpha^3 |K_0|$
$|K_1^1(n)| = \alpha^2 |K_0|$

Hence the structure of C_δ can be described by an infinite labelled tree. Vertices correspond to cylinders and are labelled 1 or 2 corresponding to a cylinder of two or three components. Directed edges are labelled α, α^2, etc corresponding to a reduction of Lebesgue measure by this factor when passing to a sub-cylinder. The edges from vertices labelled 1 or 2 are:



In the infinite tree we can either suppress the vertices labelled 1 while joining successive edges and multiplying their labels or suppress the vertices labelled 2. This corresponds to subdividing a cylinder further into sub-cylinders with same number of components. Two new self-similar infinite trees arrise according to these two possible procedures. In these new trees the edges emanating from a vertex are:



that defines the Cantor sets C_δ^+ and C_δ^-, $C_\delta = C_\delta^+ \cup C_\delta^-$.

The sets of function that defines these trees are the same but they are applied to different kind of sets i.e. sets of type (-) and(+) we get that C_δ^- and C_δ^+ are disjoint.

The Hausdorff dimension of the sets C_δ^- and C_δ^+ are given by the theorem.

Theorem 2.3.1 (Falconer [19], pag. 118) *Let $\phi_i : \mathbf{R}^n \to \mathbf{R}^n$, $i = 1, \ldots, k$ a system of iterated functions with ratios c_i and satisfying the open set condition.*

If X is an invariant set for the system of iterated functions (i.e. $X = \bigcup_{i=1}^k \phi_i(X)$) then the Hausdorff dimension of X is the solution of $\sum_{i=1}^k c_i^s = 1$.

Moreover, for this value of s, $0 < \mathcal{M}_s(X) < \infty$

Clearly the open set condition (i.e. there exists a bounded non-empty open set V such that $\bigcup_{i=1}^k \phi_i(V) \subset V$) is satisfied by the system of functions that define C_δ^- and C_δ^+ i.e. either of the two descriptions indicated in the self-similar trees discussed above.

Therefore the Hausdorff dimension of these two sets is given by the solution of $\alpha^{4s} + 2\alpha^{3s} = 1$ which is $s_0 = \frac{\log \nu}{\log \alpha}$ where ν is the real solution smaller than one, in absolute value, of $x^4 + 2x^3 - 1 = 0$.

End of proof of Lemma 2.3.1

2.4 Induced Dynamical Systems on Λ

Theorem 2.4.1 *There exist*

- *a continuous map $F : \Lambda \to \Lambda$ that preserves the transverse measure μ and*
- *a continuous map $H : \Lambda \to \Lambda_1$ with the property $H_*\mu = \lambda^{s_0}\mu$ where*

$$\Lambda_1 = \{\gamma \in \Lambda | \textit{the end points of } \gamma \textit{ are in } [0, \alpha]\}$$

such that the following diagram commutes

$$\begin{array}{ccc} \Lambda & \xrightarrow{F} & \Lambda \\ H \downarrow & & \downarrow H \\ \Lambda_1 & \xrightarrow{\tilde{F}} & \Lambda_1 \end{array}$$

where $\tilde{F}$ is the map induced by F in Λ_1.

Proof: We are going to define $F : \Lambda \to \Lambda$ and $H : \Lambda \to \Lambda_1$ as the extensions of $f : \mathbf{I} \to \mathbf{I}$ and $h : \mathbf{I} \to I_1$, respectively, to the geodesic lamination Λ.

Let $\gamma \in \Lambda$ with end points in I, $a_\gamma < b_\gamma$.

$F(\gamma)$ (and similarly $H(\gamma)$) is defined as the geodesic with end points $f(a_\gamma)$ and $\lim_{t \to b_\gamma^-} f(t)$ ($h(a_\gamma)$, $\lim_{t \to b_\gamma^-} h(t)$).

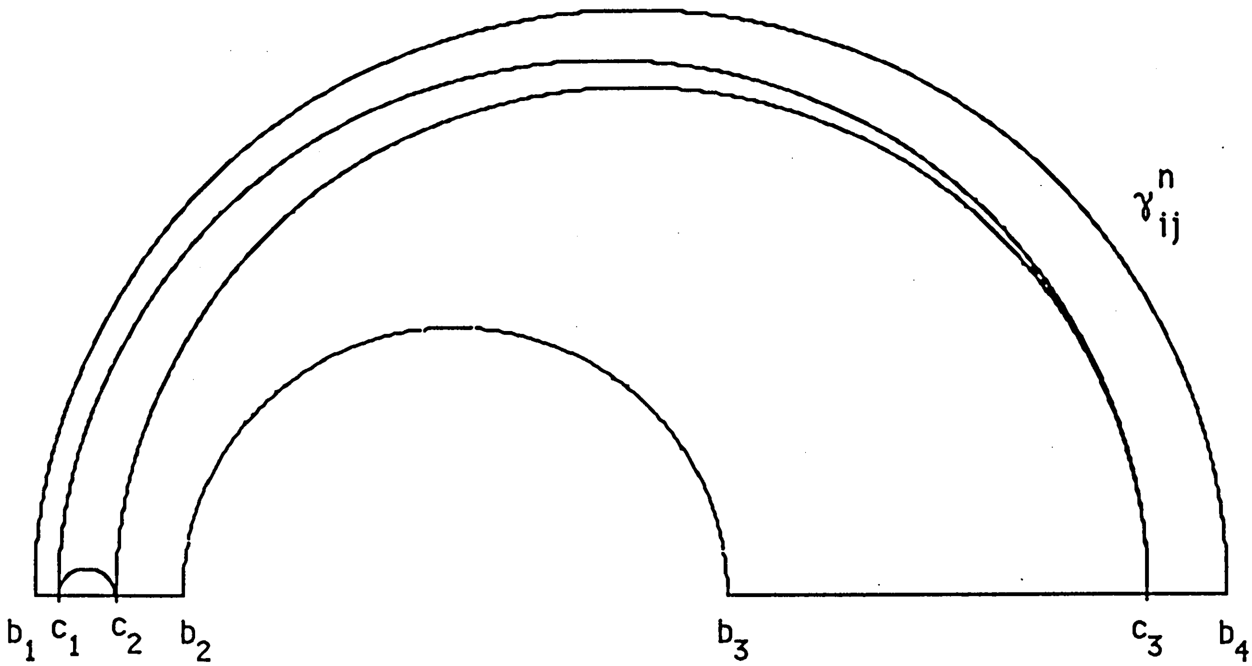
$H(\gamma) \in \Lambda_1$: Suppose that γ is of the form γ_{ij}^n for some $n \in \mathcal{P}$ with end points y_i, y_j. Then $H(\gamma_{ij}^n) = \gamma_{ij}^{n_1 * n}$ and clearly $n_1 * n \in \mathcal{P}$, therefore $H(\gamma_{ij}^n) \in \Lambda$ and since the end points are

$$h(O_{f,h}(n)y_j) \text{ and } \lim_{t \to [O_{f,h}(n)y_j]^-} h(O_{f,h}(n)(y_j))$$

if $O_{f,h}(n)y_i < O_{f,h}(n)y_j$.

Hence $H(\gamma_{ij}^n) \in \Lambda_1$.

Next we prove that $F(\gamma) \in \Lambda$. Suppose that $\gamma = \gamma_{ij}^n$ for some $n \in \mathcal{P}$, so γ is a geodesic that joins two extreme points of the same type of the cylinder



$$O_{f,h}(n)\mathbf{S}^1 \doteq [b_1, b_2) \cup [b_3, b_4)$$
$$O_{f,h}(n')\mathbf{S}^1 = [b_1, c_1) \cup [c_3, b_4)$$

Figure 2.12:

$O_{f,h}(n)\mathbf{S}^1$. By the arguments used in theorem 2.2.1 we can suppose that γ is the "exterior geodesic" of the geodesics that join the same type extreme points of the cylinder $O_{f,h}(n)\mathbf{S}^1$ (i.e. the geodesic that joins the greatest extreme point of $O_{f,h}(n)\mathbf{S}^1$ with the smallest).

If $n+1 \in \mathcal{P}$, which implies that $f(O_{f,h}(n)\mathbf{S}^1)$ is a cylinder of the standard partition, then $F(\gamma_{ij}^n) = \gamma_{ij}^{n+1}$ and therefore $\gamma_{ij}^{n+1} \in \Lambda$.

If $n+1 \notin \mathcal{P}$ we subdivide $O_{f,h}(n)\mathbf{S}^1$ such that γ^n is still an "exterior geodesic" of a new cylinder, say $O_{f,h}(n')\mathbf{S}^1$ with $n' \in \mathcal{P}$ so that $f(O_{f,h}(n')\mathbf{S}^1)$ is a cylinder of the standard partition, i.e. $n'+1 \in \mathcal{P}$. See figure 2.12.

This can be done by using the next lemma:

Lemma 2.4.1 *Given n an ICP (i.e. $n \in \mathcal{P}$) such that $n+1$ is also an ICP then $1 + n * n_1 \in \mathcal{P}$.*

In its proof the following proposition is needed

Proposition 2.4.1 *Let m be a positive integer and $m = g_{i_0} + \cdots + g_{i_l}$ be its expression as a sum of the g_i's.*
If $m \notin \mathcal{P}$ then $i_l = i_{l-1} + 1$

Proof of proposition 2.4.1: We use induction on l.
When $l = 2$, $m = g_{i_1} + g_{i_2} = g_{i_1} * (1 + g_{i_2 - i_1})$ if $i_2 - i_1 = r > 1$ then

$$1 + g_r = (1 + g_2) * \underbrace{g_1 * \cdots * g_1}_{r-2 \text{ times}}, \text{ so } 1 + g_r \in \mathcal{P}.$$

Hence $m \in \mathcal{P}$.
Now suppose:

$$m = g_{i_0} + \cdots + g_{i_{l+1}} = g_{i_0} + \cdots + g_{i_l} * (1 + g_{i_{l+1} - i_l})$$

where $i_{l+1} - i_l = r > 1$. We shall prove that $m \in \mathcal{P}$. If $g_{i_0} + \cdots + g_{i_l} \in \mathcal{P}$ then $m \in \mathcal{P}$ since $1 + g_r \in \mathcal{P}$ for $r \geq 2$. However if $g_{i_0} + \cdots + g_{i_l} \notin \mathcal{P}$ then, by the inductive hypothesis, $i_l = i_{l-1} + 1$ so

$$g_{i_0} + \cdots + g_{i_l} = g_{i_0} + \cdots + g_{i_{l-1}} * (1 + g_1).$$

Observe that $(1 + g_1) * (1 + g_r) \in \mathcal{P}$ for all $r \geq 2$ because

$$\begin{aligned} (1 + g_1) * (1 + g_r) &= 1 + g_1 + g_{r+1} \\ &= (1 + g_1) * (1 + g_2) * g_{r-2} \\ &= (1 + g_1 + g_3) * g_{r-2} \\ &= n_3 * \underbrace{n_1 * \cdots * n_1}_{r-2}. \end{aligned}$$

so if $g_{i_0} + \cdots + g_{i_{l-1}} \in \mathcal{P}$ then $g_{i_0} + \cdots + g_{i_l} * (1 + g_1) * (1 + g_r) \in \mathcal{P}$.. Therefore if $g_{i_0} + \cdots + g_{i_{l-1}} \in \mathcal{P}$ then $m \in \mathcal{P}$.

Next, we are going to prove that $g_{i_0} + \cdots + g_{i_{l-1}} \in \mathcal{P}$. Suppose that it is not an ICP then $i_{l-1} = i_{l-2} + 1$ by the inductive hypothesis.

Therefore

$$\begin{aligned} m &= g_{i_0} + \cdots + g_{i_{l-2}} * (1 + g_1) * (1 + g_1) * (1 + g_r) \\ &= g_{i_0} + \cdots + g_{i_{l-2}} * (1 + g_1 + g_2 + g_{r+2}) \\ &= g_{i_0} + \cdots + g_{i_{l-2}} + g_{i_{l-2}+1} + g_{i_{l-2}+2} + g_{i_{l-2}+r+2} \\ &= g_{i_0} + \cdots + g_{i_{l-3}} + g_{i_{l-2}+3} + g_{i_{l-2}+r+2} \end{aligned}$$

Having a different expression of m, contradicting in this way the uniqness of the expression of m as a sum of the g_i's.

Therefore $m \in \mathcal{P}$.

End of the proof of Proposition 2.4.1

Proof of Lemma 2.4.1:

Let the expressions of n and $n+1$ as sums of the g_i's be

$$n = g_{i_0} + \cdots + g_{i_l} \quad n + 1 = g_{j_0} + \cdots + g_{j_k}$$

When 1 is added to n, there could be cancellations of the g_i's according to the relation:

$$g_r + g_{r+1} + g_{r+2} = g_{r+3}$$

so i_l might be increased by 1. However $n \in \mathcal{P}$ so that $i_l > i_{l-1} + 1$ (by proposition 2.4.1); thus such cancellations cannot affect i_l, therefore $g_{j_k} = g_{i_l}$. Hence $g_{j_0} + \cdots + g_{j_{k-1}} = 1 + g_{i_0} + \cdots + g_{i_{l-1}}$.

On the other hand $n * n_1 = g_{i_0} + \cdots + g_{i_{l-1}} + g_{i_l+1}$, when 1 is added we obtain $1 + n * n_1 = 1 + g_{i_0} + \cdots + g_{i_{l-1}} + g_{i_l+1}$. Since $g_{j_0} + \cdots + g_{j_{k-1}} = 1 + g_{i_0} + \cdots + g_{i_{l-1}}$ and $g_{i_k} = g_{i_l}$ then $1 + n * n_1 = (1 + n) * n_1$; therefore $1 + n * n_1 \in \mathcal{P}$.

End of the proof of Lemma 2.4.1

We return to the proof of theorem 2.4.1. If γ is not of the type γ_{ij}^n, then γ could still be a geodesic that joins two different points of $\mathbf{S}^1$, in this case it is straight-forward to prove that $F(\gamma) \in \Lambda$, since it can be approximated by a geodesic of the type γ_{ij}^n. But it can happen that γ is a *degenerate geodesic* i.e. consists of only one point in the boundary of $\mathbf{D}^2$. In this case two different situations could occur :

1. f is continuous at this point. Then γ is mapped to another degenerate geodesic and clearly this is in Λ, since the approximations γ^n to γ have the property $F(\gamma_n) \in \Lambda$ and approximate $F(\gamma)$.

2. f is discontinuous at this point.

We know that the discontinuities of f are 0, $\frac{\alpha}{2}$, α, $\alpha + \frac{\alpha^2}{2}$, $\alpha + \alpha^2$ and $\alpha + \alpha^2 + \frac{\alpha^3}{2}$.

But at the points 0, α and $\alpha + \alpha^2$ there are no degenerate geodesics. In fact these points are joined by $\gamma^1_{1,2}$, $\gamma^1_{1,3}$, $\gamma^1_{2,3}$. However the points $\frac{\alpha}{2}$, $\alpha + \frac{\alpha^2}{2}$, $\alpha + \alpha^2 + \frac{\alpha^3}{2}$ are degenerate geodesics as is proved in the following proposition:

Proposition 2.4.2 *The elements of Λ with extreme points at $\frac{\alpha}{2}$, $\alpha + \frac{\alpha^2}{2}$ and $\alpha + \alpha^2 + \frac{\alpha^3}{2}$ are degenerate geodesics.*

Proof of Proposition 2.4.2: The proof is done for the point $\frac{\alpha}{2}$ and it is entirely similar for the other points. $\frac{\alpha}{2}$ is the fixed point of the function $g = hfhfh : I_1 \to I_1$. Since this function is increasing, $(g^n(0))$ approaches increasingly to $\frac{\alpha}{2}$ and $(g^n(\alpha))$ decreasingly. So the geodesics that join these points which belong to Λ, namely

$$\left(\gamma_{g^n(0)g^n(\alpha)} = \gamma^m_{0,\alpha} \text{ where } m = n_1 * \underbrace{n_3 * \cdots * n_3}_{n-\text{times}}\right).$$

collapse, in the limit, to the point $\frac{\alpha}{2}$.

End of the proof of proposition 2.4.2

According to the definition of F, $F(x)$ — where x is a point in the boundary of $\mathbf{D}^2$ that represents a degenerate geodesic in Λ — is the geodesic that joins

$$f(x) \quad \text{and} \quad \lim_{t \to x^-} f(t)$$

In the particular case when

$$x \in \{\frac{\alpha}{2}, \alpha + \frac{\alpha^2}{2}, \alpha + \alpha^2 + \frac{\alpha^3}{2}\}$$

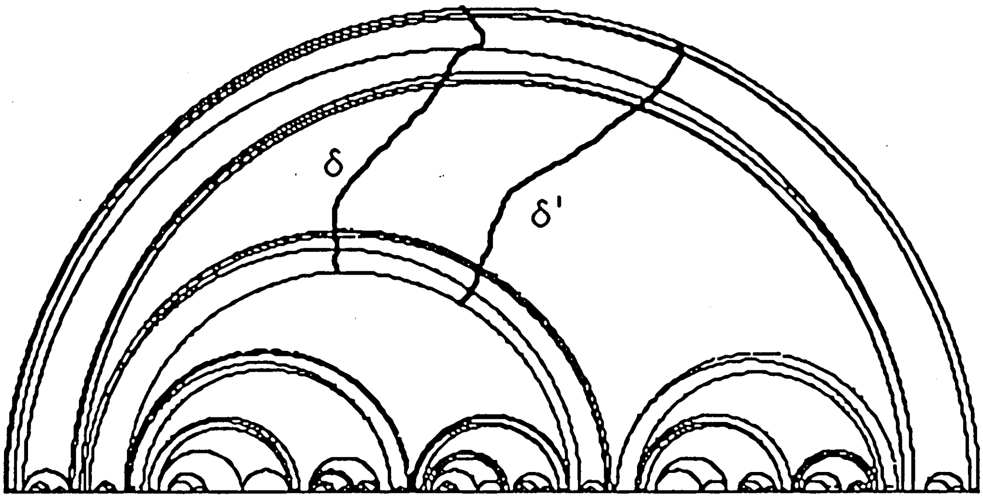


Figure 2.13:

$F(x)$ belongs to Λ since

$$h^{3n}(0) \to f(\frac{\alpha}{2}) \text{ and } h^{3n}(\alpha) \to f(\alpha + \frac{\alpha^3}{2})$$

therefore

$$\gamma_{h^{3n}(0),h^{3n}(\alpha)} = \gamma_{0,\alpha}^{m} \to F(\frac{\alpha}{2}) \quad \text{where} \quad m = n_1 * \cdots * n_1$$

and similarly for other points.

This finishes the proof of the fact that F is well defined.

The domain of F (and similarly H) can be extended to the set of equivalence classes of transverse curves to the geodesic lamination Λ.

Given δ and δ' two transverse curves to Λ we say that $\delta \sim \delta'$ if the end points of each curve lie in the same pair of distinct geodesics and $C_\delta = C_{\delta'}$. See figure 2.13 Since $C_\delta = C_{\delta'}$ if $\delta \sim \delta'$ we get $\mu(\delta) = \mu(\delta')$.

Let

$$\mathcal{T} = \{ \text{ transverse curves to } \Lambda\}$$

Given $\delta \in \mathcal{T}$, $F(\delta)$ is defined as a curve transversal only to all $F(\gamma)$ where γ are the geodesics in Λ that they are transversal to δ. (Similarly we can define $H(\delta)$.)

It is clear that

$$F : \mathcal{T}/_\sim \to \mathcal{T}/_\sim \quad \text{and} \quad H : \mathcal{T}/_\sim \to \mathcal{T}/_\sim$$

are well defined.

Lemma 2.4.2

$$fC_\delta = C_{F(\delta)} \quad and \quad \mu(\delta) = \mu(F(\delta))$$

Proof: Let δ be a transverse curve to the geodesics of Λ with extreme points in $O_{f,h}(n)\mathbf{S}^1$ for some $n \in \mathcal{P}$. Let C_δ be the Cantor set defined in section 2.3. According to lemma 2.3.1

$$C_\delta = \bigcap_{i=0}^{\infty} K_i \qquad K_i = \bigcup_{j=1}^{N_i} K_i^{ij}$$
$$\text{and} \quad C_{F(\delta)} = \bigcap_{i=0}^{\infty} J_i \qquad J_i = \bigcup_{j=1}^{M_i} J_i^{ij}.$$

Observe that the subcylinder of $O_{f,h}(n)\mathbf{S}^1$ contributes to C_δ if and only if its image under f contributes to $C_{F(\delta)}$. Therefore $J_i = f(K_i)\ \forall i$.

Hence $f(C_\delta) = C_{F(\delta)}$.

If the image of the map $O_{f,h}(n)$ does not contain a discontinuity point of f each K_1^j is translated by f:

$$f(K_1^j) = K_1^j + \kappa_j \text{ where } \kappa_j \in (0,1).$$

So $f(C_\delta) = \cap_{i=0}^{\infty} f(K_i)$ and $f(K_i) = \cup_{j=0}^{N_i} K_i^{ij} + \kappa_k$ (if $K_i^{ij} \subset K_1^k$).

Therefore $\mathcal{M}_{s_0}(f(C_\delta)) = \mathcal{M}_{s_0}(C_\delta)$ and $\mu(F(\delta)) = \mu(\delta)$. See figure 2.14. If p is a discontinuity point of f such that is in $O_{f,h}(n)\mathbf{S}^1$, suppose that belongs to one of the contributing rectangles e.g. K_1^j (If not the argument is the same as before, since f translates each rectangle).

The interval exchange map f translates a subset of K_1^j by a constant, and the complement of this subset in K_1^j is translated by another constant - i.e. $K_1^j = K_1^j(-) \cup K_1^j(+)$ where

$$K_1^j(-) = \{x \in K_1^j | x < p\}, \quad K_1^j(+) = \{x \in K_1^j | x \geq p\}$$

and

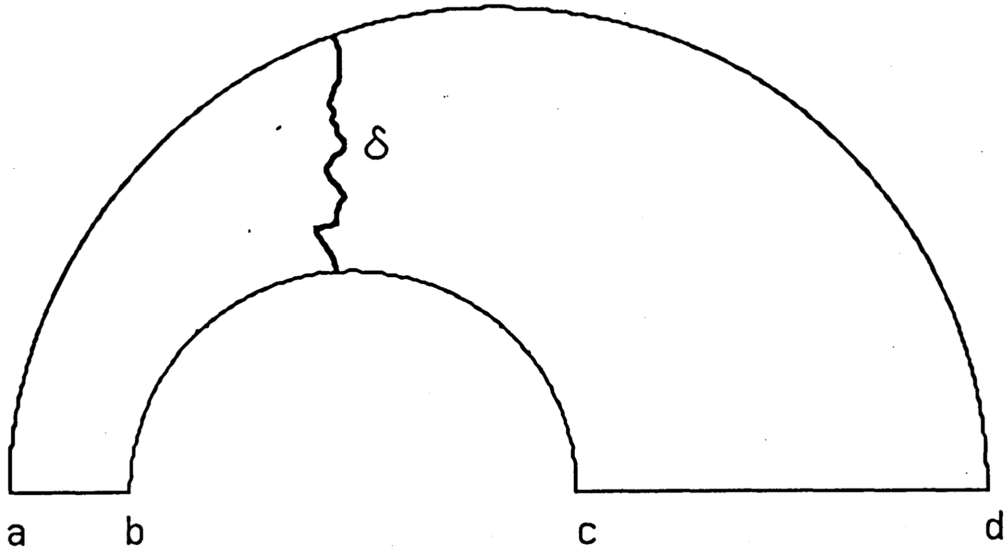
$$f(K_1^j(-)) = K_1^j(-) + \kappa_-^j \text{ and } f(K_1^j(+)) = K_1^j(+) + \kappa_+^j \quad \text{where } \kappa_-^j,\ \kappa_+^j \in (0,1)$$

so

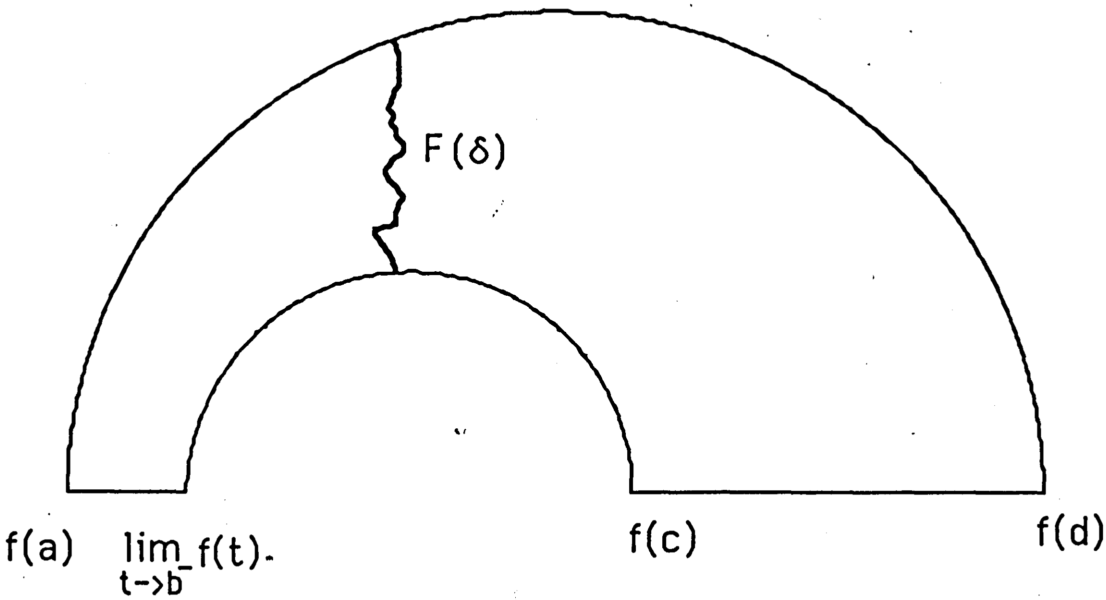
$$f(C_\delta) = \cap_{i=0}^{\infty} f(K_i)$$

and

$$K_i = (\cup_{\{k | K_i^{ik} \subset K_1^l,\, l \neq j\}} K_i^{ik} + \kappa_l) \cup_{r \in \{+,-\}} (\cup_{\{k | K_i^{ik} \cap K_1^j(r) \neq \emptyset\}} K_i^{ik} \cap K_1^j(r) + \kappa_r^j).$$



$$O_{f,h}(n) = [a,b) \cup [c,d)$$



$$f(O_{f,h}(n)\, S^1) = [f(a), \lim_{t \to b^-} f(t)) \cup [f(c), f(d))$$
$$= [a + \kappa_1, b + \kappa_1) \cup [c + \kappa_2, d + \kappa_2)$$

Figure 2.14:

Therefore

$$\mathcal{M}_{s_0}(f(C_\delta)) = \mathcal{M}_{s_0}(C_\delta)$$

.

End of the proof of Lemma 2.4.2

Lemma 2.4.3

$$h(C_\delta) = C_{H(\delta)} \quad \textit{and} \quad H_*\mu = \lambda^{s_0}\mu$$

Proof of Lemma 2.4.3: Let δ be a transverse curve to Λ as in lemma 2.4.2. Similarly:

$$\begin{array}{lcl} C_\delta = \cap_{i=0}^{\infty} K_i & \text{and} & K_i = \cup_{j=0}^{N_i} K_i^{ij} \\ C_{H(\delta)} = \cap_{i=0}^{\infty} L_i & \text{and} & L_i = \cup_{j=0}^{M_i} L_i^{ij} \end{array}$$

In the same way as before a subcylinder of $O_{f,h}(n)\mathbf{S}^1$ contributes to C_δ if and only if its image under h contributes to $C_{H(\delta)}$. Therefore $h(K_i) = L_i$, hence $C_{H(\delta)} = h(C_\delta)$.

Also $|h(K_i^{ij})| = \alpha|K_i^{ij}|\ \forall i, j$. See figure 2.15.

Therefore

$$\mathcal{M}_{s_0}(h(C_\delta)) = \alpha^{s_0}\mathcal{M}_{s_0}(C_\delta)$$

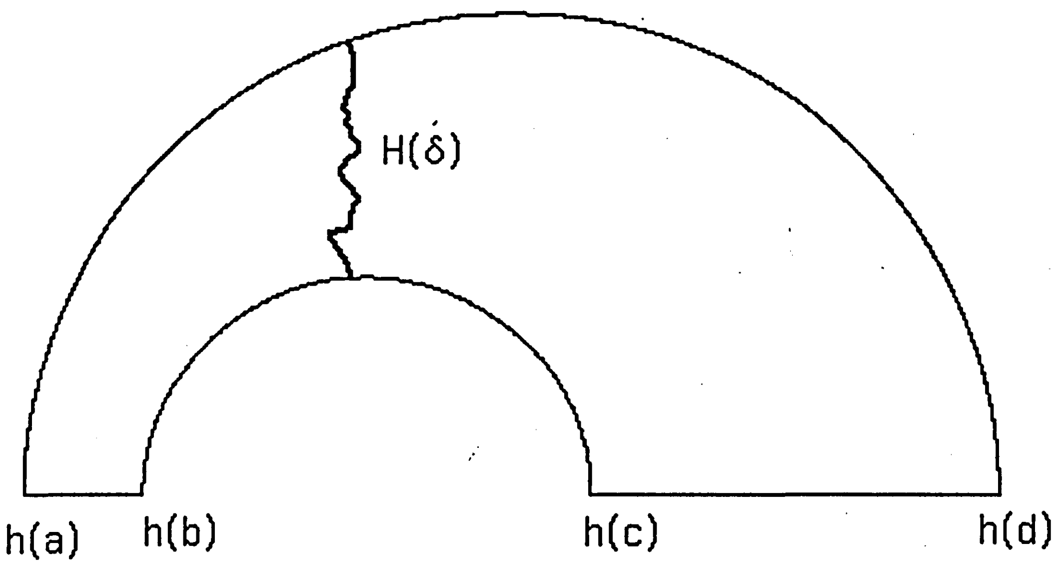
Note that $h(K_i^{ij})$ has one more connected component than K_i^{ij} if and only if the discontinuity point of h is in K_i^{ij}.

End of the proof of proposition 2.4.3

Finally the commutativity of the diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{F} & \Lambda \\ H\downarrow & & \downarrow H \\ \Lambda_1 & \xrightarrow{\tilde{F}} & \Lambda_1 \end{array}$$

is a straight-forward consequence of the commutativity of the diagram:



$$h(O_{f,h}(n)\,S^1) = [h(a),h(b)) \cup [h(c),h(d))$$

Figure 2.15:

$$
\begin{array}{ccc}
\mathbf{I} & \xrightarrow{f} & \mathbf{I} \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\
\mathbf{I}_1 & \xrightarrow{\tilde{f}} & \mathbf{I}_1
\end{array}
$$

End of the proof of Theorem 2.4.1

Chapter 3

Boundary of ω

3.1 Introduction

In this chapter the Hausdorff dimension of the boundary of ω is computed, where ω is the geometrical realization on the plane of the dynamical system associated to the substitution:

$$\Pi : \begin{array}{ccc} 1 & \longrightarrow & 12 \\ 2 & \longrightarrow & 13 \\ 3 & \longrightarrow & 1 \end{array}$$

First we describe the identifications on the boundary of ω that makes ω a fundamental domain of the two dimensional torus, for the action of the lattice $\mathbf{Z}^2$ on the plane. We define a system of maps which is related to the inverses of the iterated system of maps which generates the standard partition, studied in section 1.5. We shall show that the boundary of ω is invariant under this system of maps and compute the transitions under it. Finally we use the spectral information of the transition matrix for computing the Hausdorff dimension of the boundary of ω and also the dimension of the pre-image of this boundary under Arnoux's map.

The methods expounded here can be generalized to other substitutions, associated to Pisot numbers, which are realizable on $\mathbf{T}^2$.

Theorem 3.3.1 has been proved independently in [28].

3.2 Triple points and identifications on the boundary

In the first part of this section we are going to define a equivalence relation $\overline{R}$ in the space $\overline{\mathcal{N}}[x]$, which together with the map $(+1) : \overline{\mathcal{N}}[x] \to \overline{\mathcal{N}}[x]$ gives the symbolic dynamics for the dynamical system $T : \mathbf{T}^2 \to \mathbf{T}^2$ defined in page 5. This equivalence relation identifies the points that have the same image on the torus, under the semi-conjugacy between these two maps . We shall prove that the equivalence class contains at most three points. The image of such points will be called triple points. In order to do this we shall introduce some auxiliary spaces: $\mathbf{N}^*[x]$, $\mathcal{N}^-[x]$, $\mathbf{N}^*_{\mathcal{B}}\{x\}$, $\{-1,0,1\}\{x\}$, and $\mathbf{Z}_{\mathcal{B}}\{x\}$.

Section 1.3 introduced the bijection $\epsilon : \mathbf{N}^* \to \mathcal{N}$ given by the representations of the non-negative integers in the base associated to the recurrence relation established by the substitution Π, i.e. $g_{n+3} = g_{n+2} + g_{n+1} + g_n$, with the conditions $g_0 = 1$, $g_1 = 2$, $g_2 = 4$.

To the set $\mathcal{N}$ we associate $\mathcal{N}[x]$— the set of polynomials with coefficients in $\{0,1\}$ where polynomials with three consecutive coefficients equal to 1 are not allowed — according to the bijection:

$$\begin{array}{ccc} \mathcal{N} & \to & \mathcal{N}[x] \\ (a_0, a_1, \ldots, a_n) & \to & \sum_{i=0}^{n} a_i x^i \end{array}$$

and let ϵ_x be the composition of the bijection ϵ and this new map:

$$\begin{array}{ccc} \mathbf{N}^* & \xrightarrow{\epsilon} & \mathcal{N} \\ & \searrow^{\epsilon_x} & \downarrow \\ & & \mathcal{N}[x] \end{array}$$

We consider the binary operation on $\mathcal{N}[x]$, induced by the standard addition on $\mathbf{N}^*$:

$$\begin{array}{ccccc} \mathcal{N}[x] & + & \mathcal{N}[x] & \longrightarrow & \mathcal{N}[x] \\ a(x) & + & b(x) & = & \epsilon_x(\epsilon_x^{-1}(a(x)) + \epsilon_x^{-1}(b(x))) \end{array}$$

Let $\mathbf{N}^*[x]$ be the set of polynomials with non-negative integer coefficients and R the equivalence relation defined as follows: $a(x)\,R\,b(x)$ if $b(x)$ can be obtained from $a(x)$ using the rules:

- $x^{n+3}\,R\,(x^{n+2}+x^{n+1}+x^n)$ for all $n \geq 0$
- $2\,R\,x$
- $2x\;R\;x^2$
- $2x^2\;R\;(x^3+1)$
- if $a(x)\,R\,b(x)$ then $(a(x)+c(x))\,R\,(b(x)+c(x))$ for all $c(x)$ in $\mathbf{N}^*[x]$.

Proposition 3.2.1

$$\mathcal{N}[x] = \mathbf{N}^*[x]/_R$$

Proof: Let E be the map:

$$\begin{array}{ccc} \mathbf{N}^*[x] & \xrightarrow{E} & \mathcal{N}[x] \\ \sum p_i x^i & \longrightarrow & \sum(\epsilon_x(p_i))x^i \end{array}$$

where the sum is according to that defined in $\mathcal{N}[x]$.

Let $a(x), b(x) \in \mathbf{N}^*[x]$, if $a(x)\,R\,b(x)$ then $E(a(x)) = E(b(x))$, since E does not distinguish the transformation rules that define the relation R. On the other hand, the rules that define the relation R are the same ones under which the addition in $\mathcal{N}[x]$ is done, therefore $a(x)\,R\,b(x)$ if $E(a(x)) = E(b(x))$.

Q.E.D.

Observe that we have a finite representation of the negative integers if we allow the symbol -1 in the alphabet. Let

$$\mathcal{N}^-[x] = \mathcal{N}[x] \cup \{\sum -a_i x^i \mid \sum a_i x^i \in \mathcal{N}[x]\}$$

$$\begin{array}{rcll} \epsilon_x & : & \mathbf{Z} \quad \longrightarrow & \mathcal{N}^-[x] \\ \epsilon_x(n) & = & \sum_{i\geq 0} -\epsilon_i(-n)x^i & \text{if } n < 0 \\ \epsilon_x(n) & = & \sum_{i\geq 0} \epsilon_i(n)x^i & \text{if } n \geq 0 \end{array}$$

Therefore we can consider the equivalence relation R in the set of polynomials with integer coefficients, $\mathbf{Z}[x]$.

Also in section 1.3 was introduced the dynamical system $(+1) : \overline{\mathcal{N}} \to \overline{\mathcal{N}}$ where the space $\overline{\mathcal{N}}$ is the closure of $\mathcal{N}$ in the product topology of $\{0,1\}^{\mathbf{N}}$ and the map $(+1)$ is the induced operation in $\overline{\mathcal{N}}$ of adding 1 on $\mathcal{N}$. This map gives the symbolic dynamics of $T : \mathbf{T^2} \to \mathbf{T^2}$, since proposition 1.3.2 shows that this dynamical system is conjugate to the dynamical system induced by the substitution i.e. $\sigma : \Omega \to \Omega$.

In a similar way the set $\overline{\mathcal{N}}[x]$ is defined as the set of formal power series with coefficients zeros and ones, where series with three consecutive coefficients one are not allowed. The bijection between $\overline{\mathcal{N}}$ and $\overline{\mathcal{N}}[x]$ is:

$$\begin{array}{ccc} \overline{\mathcal{N}} & \to & \overline{\mathcal{N}}[x] \\ \underline{a} = (a_0, a_1, \ldots) & \to & \sum_{i \geq 0} a_i x^i \end{array}$$

We introduce the topology in $\overline{\mathcal{N}}[x]$, that makes this bijection a homeomorphism. We denote $(1, 0, 0 \ldots)$ by $\underline{1}$ and its image under this map by $\underline{1}(x)$, however, in order to simplify the notation we will denote both elements just by 1, whenever the context is clear.

Let $\mathbf{N}^*_B\{x\}$ denote the set of bounded power series with coefficients in $\mathbf{N}^*$, with the norm $\|\underline{a}(x)\| = \sup\{a_i\}$. If we allow to consider the equivalence relation R an infinite number of times, we get a relation $\overline{R}$. Unlike the finite case:

$$\overline{\mathcal{N}}[x] \neq \mathbf{N}^*_B\{x\}/_{\overline{R}}$$

because when the relation R is taken an infinite number of times, new identifications turn out, as can be seen in the following example: consider $a(x) = \sum_{n \geq 1} x^{3n}$ and $b(x) = \sum_{n \geq 0} x^{3n+1}$ which are two different elements of $\overline{\mathcal{N}}[x]$; however $a(x) \, \overline{R} \, b(x)$, since:

$$\begin{aligned} a(x) &= \sum_{n \geq 1} x^{3n} \\ &R \sum_{n \geq 1} (x^{3n-3} + x^{3n-2} + x^{3n-1}) \\ &= \sum_{n \geq 0} (x^{3n} + x^{3n+1} + x^{3n+2}) \\ &= 1 + \sum_{n \geq 0} (x^{3n+1} + x^{3n+2} + x^{3n+3}) \\ &R \; 1 + \sum_{n \geq 0} x^{3n+4} \end{aligned}$$

$$= \sum_{n\geq 0} x^{3n+1}$$
$$= b(x)$$

However this definition of $\overline{R}$ is vague, it will be defined properly in definition 3.2.2; in order to do this we need to introduce some auxiliary spaces and give some additional definitions.

Given any formal power series $q(x) = \sum_{i\geq 0} q_i x^i$ we denote by $q_N(x)$ the polynomial obtained by truncating the power series at the N-th term, i.e. $q_N(x) = \sum_{i=0}^{N} q_i x^i$. Also we denote the series with all its coefficients equal to zero by $0(x)$ or simply by 0.

Let $\{-1,0,1\}\{x\}$ be the set of formal power series with coefficients in $\{-1,0,1\}$.

Definition 3.2.1 *Let $e(x)$ be an element of $\{-1,0,1\}\{x\}$. We say $e(x)$ is* $\overline{R}$-equivalent to 0 *(or simply $e(x)\overline{R}0$) if either*

- *there exists $N > 0$ such that for all $n \geq N$ exists N'_n with the property $\{N'_n | n \geq N\}$ has no upper bound and $e_n R \pm x^{N'_n} p(x)$ where $p(x) \in \mathcal{N}[x]$ and $p_0 = 1$.*
- *or $e(x) = 0$*

We can subtract two formal power series in $\overline{\mathcal{N}}[x]$ term by term, however the result might not lie in $\overline{\mathcal{N}}[x]$ but certainly it is in $\{-1,0,1\}\{x\}$.

Definition 3.2.2
1. *Let $a(x)$ and $b(x) \in \overline{\mathcal{N}}[x]$, $a(x)$* is $\overline{R}$-equivalent to *$b(x)$ (or simply $a(x)\overline{R}b(x)$) if $(a(x) - b(x))\overline{R}0$*
2. *Let $a(x)$ and $b(x)$ be elements of $\overline{\mathcal{N}}[x]$ (or $\{-1,0,1\}\{x\}$), we say $a(x)$* is R-equivalent to *$b(x)$ if for all $n > 0$ there exist N'_n, $N''_n \geq 0$ such that $a_{N'_n}(x) R b_{N''_n}(x)$ and the sets of the N'_n's and N''_n's are not bounded above.*

There is no difficulty in proving that $\overline{R}$ is an equivalence relation.

As we showed in the previous example $\overline{R}$-equivalence does not imply R-equivalence.

Proposition 3.2.2 *Let $\mathcal{Q}$ be the projection:*

$$\mathcal{Q} \; : \; \overline{\mathcal{N}}[x] \; \longrightarrow \; \overline{\mathcal{N}}[x]/_{\overline{R}}$$

then

$$\#\{\mathcal{Q}^{-1}(\underline{a}(x))\} \leq 3 \quad \textit{for any} \quad \underline{a}(x) \in \overline{\mathcal{N}}[x]/_{\overline{R}}$$

Proof: Let

$$C = \{e(x) \in \{-1, 0, 1\}\{x\} | e(x)\overline{R}0\}$$

and $l(x) = 1 + \sum_{n \geq 0} x^{3n+1} + x^{3n+2}$.

Clearly $L = \{\pm x^n l(x) | n \geq 0\} \subset C$.

On the other hand every element of C is R-equivalent to a element of L. In fact, let $e(x) \in C$ and n sufficient large so that exists $N_n > 0$ with the property $e_n(x)Rx^{N_n}p(x)$ for some $p(x)$ in $\mathcal{N}[x]$ and $p_0 = 1$. Since the sets of the N_n's is not bounded above, exists $m > n$ such that $e_m(x)Rx^{N_m}q(x)$ for some $q(x)$ in $\mathcal{N}[x]$ with $q_0 = 1$, $N_m > N_n$ and $N_m > n$. We can express $e_m(x)$ as: $e_m(x) = \sum_{i=0}^{m} e_i x^i = e_n(x) + \sum_{i=n+1}^{m} e_i x^i$ therefore:

$$\begin{array}{rcl} e_m(x) & R & x^{N_n}p(x) + \sum_{i=n+1}^{m} e_i x^i \\ & = & x^K(x^{N_n - K}p(x) + \sum_{i=n+1-K}^{m-K} e_i x^i) \quad K = \min\{N_n, n\} \end{array}$$

Since $e_m(x)Rx^{N_m}q(x)$, there exists $k > 1$ such that the first k-terms of $x^{N_n-K}p(x) + \sum_{i=n+1-K}^{m-K} e_i x^i$ are R-equivalent to x^{N_m-K}. Therefore $e(x)$ is R-equivalent to an element of L.

Since for any $n \geq 0$ $x^{3n+j}l(x)Rx^j l(x)$ with $j = 0, 1$ or 2, we have that every element of L is R-equivalent to either $\pm l(x)$, $\pm xl(x)$ or $\pm\, x^2 l(x)$.

Hence for any par of elements of $\overline{\mathcal{N}}[x]$, say $a(x)$ and $b(x)$ such that $a(x)\overline{R}b(x)$, we have that $a(x) - b(x)$ is R-equivalent to either $\pm l(x)$, $\pm xl(x)$ or $\pm$ $x^2 l(x)$.

So we conclude there can not be more than three elements of $\overline{\mathcal{N}}[x]$ in any $\overline{R}$-class.

Q.E.D.

On the other hand Rauzy's construction of ω is obtained as the image of $\overline{\mathcal{N}}[x]$ under the map

$$\hat{\delta}_x(\textstyle\sum_{i \geq 0} a_i x^i) = \sum_{i \geq 0} a_i B^i z \quad \text{for } \sum_{i \geq 0} a_i x^i \in \overline{\mathcal{N}}[x]$$

where B and z were explained in page 18.

Remark 3.2.1 *Since ϵ_x is compatible with the additive structure of* $\mathbf{N}^*$ *and* $\mathcal{N}[x]$ *we have*

$$\underline{1}(x) + \underline{1}(x) = \epsilon_x(1) + \epsilon_x(1) = \epsilon_x(1+1) = \epsilon_x(2)$$

hence

$$\underline{1}(x) + \underline{1}(x) = x$$

and their images under $\hat{\delta}_x$ satisfy the relation

$$\begin{array}{rcl} \hat{\delta}_x(\underline{1}(x)) + \hat{\delta}_x(\underline{1}(x)) & = & \hat{\delta}_x(x) + (-1,1) \\ z+z & = & Bz + (-1,1) \end{array}$$

similarly

$$\begin{array}{rcl} x+x & = & x^2 \\ \hat{\delta}_x(x) + \hat{\delta}_x(x) & = & \hat{\delta}_x(x^2) + (0,-1) \\ Bz + Bz & = & B^2 z + (0,-1) \end{array}$$

On the other hand, the characteristic polynomial of the matrix B is $x^3 - x^2 - x - 1$, therefore the points in the images under $\hat{\delta}_x$ of two $\overline{R}$-equivalent power series in $\overline{\mathcal{N}}[x]$ are the same point in ω or they differ by a vector of integer coordinates. In the later case, the point belongs to the boundary of ω, since this set is a fundamental domain of $\mathbf{T}^2$.

From proposition 3.2.2 and remark 3.2.1 the preimage, under δ_x, of any point in ω, consists at most of three points, facts that allow us to introduce the definition of triple point:

Definition 3.2.3 *A point p in* $\mathbf{T}^2$ *is a* triple point *if it is in the intersection of three different cylinders of the standard partition.*

For the definition of cylinder of the standard partition, see section 1.5

Definition 3.2.4 *A point p in* $\mathbf{T}^2$ *is a* 0-triple point *if it is in* $\omega_1 \cap \omega_2 \cap \omega_3$ *or* $\omega_{12} \cap \omega_{13} \cap \omega_2$ *or* $\omega_{12} \cap \omega_{13} \cap \omega_3$

Later we are going to find the 0-triple points and use them for describing the identifications in the boundary of ω.

The previous definitions of R and $\overline{R}$ equivalence extend in a straight forward manner to $\mathbf{N}^*_B\{x\}$and, also to $\mathbf{Z}_B\{x\}$ (the set of bounded power series with integer coefficients).

Proposition 3.2.3

$$\mathbf{N}^*_B\{x\}/_{\overline{R}} = \overline{\mathcal{N}}[x]/_{\overline{R}}$$

Proof: Clearly $\overline{\mathcal{N}}[x]/_{\overline{R}} \subset \mathbf{N}^*_B\{x\}/_{\overline{R}}$.

On the other hand, consider $a(x)$ and $b(x)$ in $\mathbf{N}^*_B\{x\}$ such that they are $\overline{R}$-equivalent. Let $a'(x)$ be an element of $\overline{\mathcal{N}}[x]$ in the closure of $\cup_{n>0}\{c_n(x) \in \mathcal{N}[x] | c_n(x) R a_m(x)$ for some $m\}$ so $a'(x)$ is $\overline{R}$-equivalent to $a(x)$, similarly we get $b'(x)$. By transitivity we conclude that $a'(x)\overline{R}b'(x)$.

Q.E.D.

The group structure of $(\mathbf{T}^2,+)$ almost induces a binary operation on ω. However it is not well defined, since the addition of two points might lie on different pieces of the boundary, which are identified under $\mathbf{Z}^2$, this fact is reflected in $\overline{\mathcal{N}}[x]$, where we introduce the operation:

$$\begin{array}{rcc} \oplus : \overline{\mathcal{N}}[x] \times \overline{\mathcal{N}}[x] & \rightarrow & \overline{\mathcal{N}}[x] \\ a(x) \oplus b(x) & = & \mathcal{Q}^{-1}(\sum_{i \geq 0}(a_i + b_i)x^i /_{\overline{R}}) \end{array}$$

which is not well defined, because when the identifications under the equivalence relation are taken we might have three different representatives in $\overline{\mathcal{N}}[x]$, as was pointed out in proposition 3.2.2. However this "operation" suggests the introduction of the notion of an inverse of a point in $\overline{\mathcal{N}}[x]$.

Definition 3.2.5 *Let $a(x)$ be an element of $\overline{\mathcal{N}}[x]$; an element $b(x)$ of $\overline{\mathcal{N}}[x]$ is an inverse of $a(x)$ if $(a(x)+b(x))\overline{R}0$, where $a(x)+b(x) = \sum_{i \geq 0}(a_i+b_i)x^i$ and $\overline{R}$ is taken on $\mathbf{N}^*_B\{x\}$.*

By proposition 3.2.2 a point may have more than one inverse but no more than three.

We will use the following convention for denoting the inverses:

- If the inverse of $a(x)$ is unique we denote it by $-a(x)$
- If $a(x)$ has more than one inverse we denote them as follows:

$$\begin{array}{ll} (-a(x))^{\underline{i}} & \underline{i} = (i_1 i_2 \dots i_r) \\ (-a(x))^{\underline{j}} & \underline{j} = (j_1 j_2 \dots j_{r'}) \\ (-a(x))^{\underline{k}} & \underline{k} = (k_1 k_2 \dots k_{r''}) \end{array}$$

such that $(-a(x))^{\underline{l}} \in \overline{\mathcal{N}}_{\underline{l}}[x]$ and the words $\underline{i}, \underline{j}, \underline{k}$ have minimum length, such that, allow us to distinguish that they are different.

As an example we consider $a(x) = 1$. A direct computation show that its inverses are:

$$\sum_{n\geq 0}(x^{3n+1} + x^{3n+2})$$
$$\sum_{n\geq 0}(x^{3n} + x^{3n+2})$$
$$\sum_{n\geq 0}(x^{3n} + x^{3n+1})$$

and they are denoted $(-1)^0$, $(-1)^{10}$ and $(-1)^{11}$ since they belong to $\overline{\mathcal{N}}_0[x]$, $\overline{\mathcal{N}}_{10}[x]$ and $\overline{\mathcal{N}}_{11}[x]$, respectively.

Lemma 3.2.1 *The 0-triple points of ω are:*

- $\delta_x((-1)^{\underline{i}})$ *for* $\underline{i} = 0, 10, 11$
- $\delta_x((-x)^{\underline{i}})$ *for* $\underline{i} = 10, 00, 010$
- $\delta_x((-(1+x))^{\underline{i}})$ *for* $\underline{i} = 0, 10, 11$
- $\delta_x(r^{\underline{i}}(x))$ *for* $\underline{i} = 0, 10, 11$ *where:*

$$\begin{aligned} r^0(x) &= x + \sum_{n\geq 1} x^{3n} \\ r^{10}(x) &= 1 + \sum_{n\geq 0} x^{3n+2} \\ r^{11}(x) &= 1 + \sum_{n\geq 0} x^{3n+1} \end{aligned}$$

- $\delta_x(s^{\underline{i}}(x))$ *for* $\underline{i} = 10, 00, 010$ *where:*

$$\begin{aligned} s^{10}(x) &= \sum_{n\geq 0} x^{3n} \\ s^{00}(x) &= \sum_{n\geq 0} x^{3n+2} \\ s^{010}(x) &= \sum_{n\geq 0} x^{3n+1} \end{aligned}$$

- $\delta_x(t^{\underline{i}}(x))$ *for* $\underline{i} = 0, 10, 11$

$$\begin{aligned} t^0(x) &= \sum_{n\geq 1} x^{3n} \\ t^{10}(x) &= 1 + \sum_{n\geq 1} x^{3n+1} \\ t^{11}(x) &= 1 + x + \sum_{n\geq 1} x^{3n+2} \end{aligned}$$

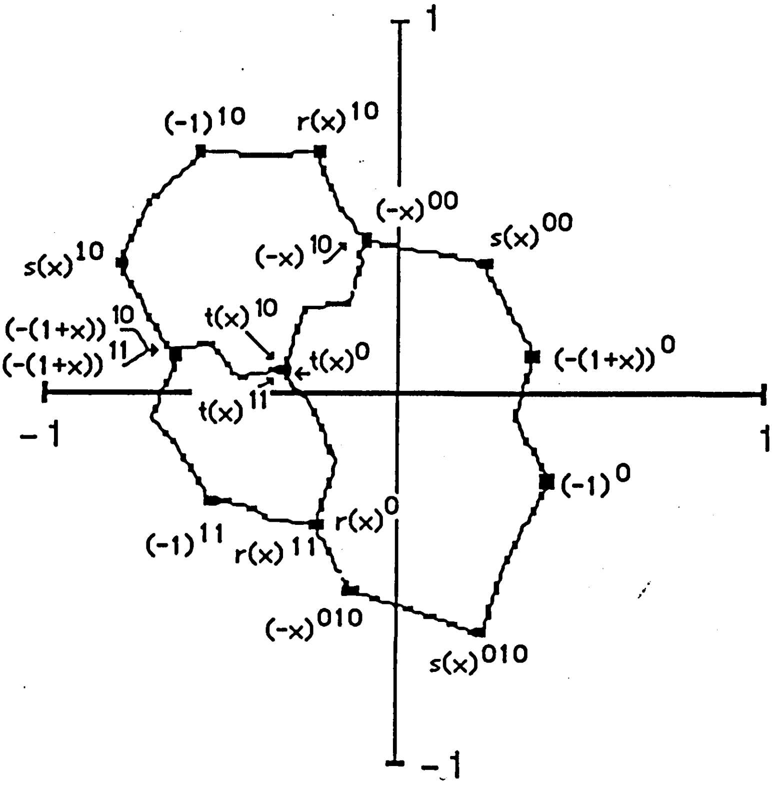


Figure 3.1: Identifications of the boundary of ω

and the identifications of the boundary of ω *are given by:*

- $\partial_{\hat{\delta}_x((-1)^{11}),\hat{\delta}_x(s^{010}(x))}$ *is identified with* $\partial_{\hat{\delta}_x((-1)^{10}),\hat{\delta}_x(s^{00}(x))}$
- $\partial_{\hat{\delta}_x((-1)^{11}),\hat{\delta}_x(s^{10}(x))}$ *is identified with* $\partial_{\hat{\delta}_x((-1)^{0}),\hat{\delta}_x(s^{00}(x))}$
- $\partial_{\hat{\delta}_x((-1)^{10}),\hat{\delta}_x(s^{10}(x))}$ *is identified with* $\partial_{\hat{\delta}_x((-1)^{0}),\hat{\delta}_x(s^{010}(x))}$

where $\partial_{p,q}$ *is the shortest segment of boundary between* p *and* q, *i.e. the segment of boundary that has smaller diameter.*

In the proof of this lemma the following proposition is required:

Proposition 3.2.4

$(\omega_{\underline{i}}+(n,m))\cap\omega_{\underline{i}} = \emptyset$ *for* $\underline{i} \in \{2,3,12,13\}$ *and any* $(n,m) \in \mathbf{Z}^2\backslash(0,0)$

Proof of Proposition 3.2.4: In [38] it is proved that $||p|| < 1/2$ for every $p \in \omega$ where $||\ ||$ is a suitable norm in the plane, with the property:

$$||Bp|| = \alpha^{1/2}||p|| \quad \text{for any} \quad p \in \omega.$$

therefore $\|B^j p\| \leq \frac{\alpha^{j/2}}{2} < \frac{1}{2}$ for $j \geq 2$.

The rectangle ω_{12} is disjoint from its translates under the lattice $\mathbf{Z}^2$, since $\omega_{12} = B^2\omega$. Similarly for ω_2, since

$$\omega_2 = TB^2\omega = B^2\omega + (\alpha, \alpha^2)$$

On the other hand $\omega_{13} = BTB^2\omega$, so $\|\omega_{13}\| \leq \alpha^{3/2}$; therefore it is disjoint from its translates under $\mathbf{Z}^2$. Similarly for ω_3, which is equal to $T\omega_{13}$

End of the proof of Proposition 3.2.4

Proof of Lemma 3.2.1: Now we are going to prove that these points are 0-triple points.

As has been shown before, the inverses of 1 are:

$$\begin{aligned}
(-1)^0 &= \sum_{n\geq 0}(x^{3n+1} + x^{3n+2}) \\
(-1)^{10} &= \sum_{n\geq 0}(x^{3n} + x^{3n+2}) \\
(-1)^{11} &= \sum_{n\geq 0}(x^{3n} + x^{3n+1})
\end{aligned}$$

according to remark 3.2.1, we have:

$$\begin{aligned}
\hat{\delta}_x((-1)^0) &= \hat{\delta}_x((-1)^{10}) + (1, -1) \\
\hat{\delta}_x((-1)^0) &= \hat{\delta}_x((-1)^{11}) + (1, 0)
\end{aligned}$$

So the images under $\hat{\delta}_x$ of the points $(-1)^{\underline{i}}$ are three different points in ω which differ by an vector of integer coordinates, therefore they have the same image in $\mathbf{T}^2$ and it is a 0-triple point.

On the other hand:

$$\begin{aligned}
\hat{\delta}_x(r^0(x)) &= \hat{\delta}_x(r^{10}(x)) + (0, -1) \\
\hat{\delta}_x(r^0(x)) &= \hat{\delta}_x(r^{11}(x))
\end{aligned}$$

and

$$\begin{aligned}
\hat{\delta}_x(s^{00}(x)) &= \hat{\delta}_x(s^{010}(x)) + (0, 1) \\
\hat{\delta}_x(s^{00}(x)) &= \hat{\delta}_x(s^{10}(x)) + (1, 0).
\end{aligned}$$

Similarly we have

$$\hat{\delta}_x(t^0(x)) = \hat{\delta}_x(t^{10}(x)) = \hat{\delta}_x(t^{11}(x))$$

which is the point where the rectangles ω_1, ω_2 and ω_3 intersect in the interior of ω.

An easy computation shows that the inverses of $1+x$ are:

$$\begin{aligned}
(-(1+x))^0 &= x^2 + \sum_{n\geq 1}(x^{3n+1} + x^{3n+2}) \\
(-(1+x))^{10} &= 1 + \sum_{n\geq 1}(x^{3n} + x^{3n+1}) \\
(-(1+x))^{11} &= 1 + x + \sum_{n\geq 1}(x^{3n} + x^{3n+2})
\end{aligned}$$

and

$$\begin{aligned}
\hat{\delta}_x((-(1+x))^0) &= \hat{\delta}_x((-(1+x))^{10}) + (1,0) \\
\hat{\delta}_x((-(1+x))^0) &= \hat{\delta}_x((-(1+x))^{11}) + (1,0)
\end{aligned}$$

Also, the inverses of x are:

$$\begin{aligned}
(-x)^{00} &= x^2 + \sum_{n\geq 1}(x^{3n} + x^{3n+2}) \\
(-x)^{010} &= x + \sum_{n\geq 1}(x^{3n} + x^{3n+1}) \\
(-x)^{10} &= 1 + x^2 + \sum_{n\geq 1}(x^{3n+1} + x^{3n+2})
\end{aligned}$$

and the relations between their images are:

$$\begin{aligned}
\hat{\delta}_x((-x)^{010}) &= \hat{\delta}_x((-x)^{10}) + (0,-1) \\
\hat{\delta}_x((-x)^{00}) &= \hat{\delta}_x((-x)^{10})
\end{aligned}$$

Therefore $\delta_x((-1)^{\underline{i}})$ for $\underline{i} = 0, 10, 11$, $\delta_x((-(1+x))^{\underline{i}})$ for $\underline{i} = 0, 10, 11$, $\delta_x((-x)^{\underline{i}})$ for $\underline{i} = 00, 010, 10$, $\delta_x(t^{\underline{i}}(x))$ for $\underline{i} = 0, 10, 11$ $\delta_x(r^{\underline{i}}(x))$ for $\underline{i} = 0, 10, 11$ and $\delta_x(s^{\underline{i}}(x))$ for $\underline{i} = 00, 010, 10$ are 0-triple points.

In order to show that these are the only 0-triple points, we are going to describe the identifications on the boundary ($\partial\omega_2$) of ω_2. In this way we shall find all the 0-triple points contained in this boundary. Later this analysis is done in the boundary of ω_3 and ω_1 ($\partial\omega_3$ and $\partial\omega_1$, respectively). In this way we shall have all the identifications on the boundary of the fundamental domain ω and all its 0-triple points.

We consider the boundary of ω_2 as the union of different pieces of boundary:

$$\begin{aligned}\partial\omega_2 \;=\; & \partial_{\hat{\delta}_x((-1)^{10}),\hat{\delta}_x(r^{10}(x))} \cup \partial_{\hat{\delta}_x(r^{10}(x)),\hat{\delta}_x((-x)^{10})} \cup \\ & \partial_{\hat{\delta}_x((-x)^{10}),\hat{\delta}_x(t^{10}(x))} \cup \partial_{\hat{\delta}_x(t^{10}(x)),\hat{\delta}_x((-(1+x))^{10})} \cup \\ & \partial_{\hat{\delta}_x((-(1+x))^{10}),\hat{\delta}_x(s^{10}(x))} \cup \partial_{\hat{\delta}_x(s^{10}(x)),\hat{\delta}_x((-1)^{10})}\end{aligned}$$

Let $p \in \partial_{\hat{\delta}_x((-1)^{10}),\hat{\delta}_x(r^{10}(x))}$. Since $\hat{\delta}_x((-1)^{10})$ is identified to $\hat{\delta}_x((-1)^{11})$ and $\hat{\delta}_x(r^{10}(x))$ to $\hat{\delta}_x(r^{11}(x))$, p is identified to an unique point in ω_3, if not there would be two points in ω_3 that differ by an element of $\mathbf{Z}^2$ fact that contradicts proposition 3.2.4. Therefore $\partial_{\hat{\delta}_x((-1)^{10}),\hat{\delta}_x(r^{10}(x))}$ is identified to $\partial_{\hat{\delta}_x((-1)^{11}),\hat{\delta}_x(r^{11}(x))}$ and moreover there is not a 0-triple point in this piece of boundary, with the exception of $\hat{\delta}_x((-1)^{10})$ and $\hat{\delta}_x(r^{10}(x))$.

Similarly it is proved:

- $\partial_{\hat{\delta}_x((-(1+x))^{10}),\hat{\delta}_x(s^{10}(x))}$ is identified to $\partial_{\hat{\delta}_x((-(1+x))^{0}),\hat{\delta}_x(s^{010}(x))}$
- $\partial_{\hat{\delta}_x(s^{10}(x)),\hat{\delta}_x((-1)^{10})}$ is identified to $\partial_{\hat{\delta}_x(s^{011}(x),\hat{\delta}_x((-1)^{0})}$
- $\partial_{\hat{\delta}_x(r^{10}(x)),\hat{\delta}_x((-x)^{10})}$ is identified to $\partial_{\hat{\delta}_x(r^{0}(x)),\hat{\delta}_x((-x)^{010})}$

On the other hand $\partial_{\hat{\delta}_x((-x)^{10}),\hat{\delta}_x(t^{10}(x))}$ is identified to $\partial_{\hat{\delta}_x((-x)^{00}),\hat{\delta}_x(t^{0}(x))}$. Since $\hat{\delta}_x(t^0(x))$ and $\hat{\delta}_x(t^{10}(x))$ are in the interior of ω and $\hat{\delta}_x((-x)^{10}) = \hat{\delta}_x((-x)^{00})$, therefore

$$\partial_{\hat{\delta}_x((-x)^{10}),\hat{\delta}_x(t^{10}(x))} = \partial_{\hat{\delta}_x((-x)^{00}),\hat{\delta}_x(t^{0}(x))}$$

and this section of the $\partial\omega_2$ and $\partial\omega_1$ lies in the interior of ω, with the exception of the point $\hat{\delta}_x((-x)^{10})$ which is in the boundary (since $\hat{\delta}_x((-x)^{010}) = \hat{\delta}_x((-x)^{10}) + (0,-1)$).

Similarly $\partial_{\hat{\delta}_x(t^{10}(x)),\hat{\delta}_x((-(1+x))^{10})}$ is identified to $\partial_{\hat{\delta}_x(t^{11}(x)),\hat{\delta}_x((-(1+x))^{11})}$ and therefore

$$\partial_{\hat{\delta}_x(t^{10}(x)),\hat{\delta}_x((-(1+x))^{10})} = \partial_{\hat{\delta}_x(t^{11}(x)),\hat{\delta}_x((-(1+x))^{11})}$$

The identifications in ω_3 and ω_1 are obtained in a similar way.

End of the proof of Lemma 3.2.1

3.3 Boundary Transitions and Dimension

The partition of the symbolic space Ω into ω_1 ω_2 and ω_3 gives rise to the iterated system of maps (ISM)–studied in section 1.5–:

$$\begin{array}{rccc} \Pi : & \Omega & \longrightarrow & \Omega_1 \\ \sigma\Pi^2 : & \Omega & \longrightarrow & \Omega_2 \\ \sigma\Pi\sigma\Pi^2 : & \Omega & \longrightarrow & \Omega_3 \end{array}$$

which induces equivalent systems of maps in all the other geometrical realizations of the dynamical system associated to the substitution.

In this section we consider a map that are related to the inverses of the previous ISM:

$$\varphi \ : \ \Omega \longrightarrow \Omega$$

$$\varphi \ : \ \begin{cases} \Pi^{-1} & : \ \Omega_1 \longrightarrow \Omega \\ \Pi^{-1}\sigma^{-1} & : \ \Omega_2 \longrightarrow \Omega_1 \\ \Pi^{-1}\sigma^{-1} & : \ \Omega_3 \longrightarrow \Omega_2 \end{cases}$$

This map induces a system of maps in all the other geometrical realizations of $\sigma : \Omega \longrightarrow \Omega$. In particular, we are interested in the realization on ω, we denote this system of maps by Φ.

$$\Phi \ : \ \omega \longrightarrow \omega$$

$$\Phi \ : \ \begin{cases} \Phi_1 = B^{-1} & : \ \omega_1 \longrightarrow \omega \\ \Phi_2 = B^{-1}T^{-1} & : \ \omega_2 \longrightarrow \omega_1 \\ \Phi_3 = B^{-1}T^{-1} & : \ \omega_3 \longrightarrow \omega_2. \end{cases}$$

Observe that Φ is not a map since it is not defined uniquely in the boundary points of ω.

Lemma 3.3.1 *The boundary of ω is invariant under Φ and the induced system of maps on the boundary, as a subset of $\mathbf{T}^2$, can be represented by the transition matrix:*

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Proof: Each map Φ_i is a homeomorphism in the regions where it is defined, So Φ_i maps the boundary of ω_i into the boundary of ω_{i-1} for $i = 1, 2, 3$ (where ω_0 is ω). Therefore the extended boundary of ω, i.e. $\partial\omega_1 \cup \partial\omega_2 \cup \partial\omega_3$, is invariant under Φ_i.

Next we are going to compute the transitions of Φ in this extended boundary, and these transitions will show that the boundary of ω is invariant under Φ.

For finding the transitions it is sufficient to compute the images of the 0-triple points, since the extended boundary of ω is invariant under Φ. In order to do that we will work in the space $\overline{\mathcal{N}}[x]$, since the computations are easier, here the map equivalent to $\varphi : \Omega \longrightarrow \Omega$ is $\phi : \overline{\mathcal{N}}[x] \longrightarrow \overline{\mathcal{N}}[x]$

$$\phi : \begin{cases} \phi_1 : \overline{\mathcal{N}}_0[x] \longrightarrow \overline{\mathcal{N}}[x] & \phi_1(p(x)) = \frac{1}{x}p(x) \\ \phi_2 : \overline{\mathcal{N}}_{10}[x] \longrightarrow \overline{\mathcal{N}}_0[x] & \phi_2(p(x)) = \frac{1}{x}(p(x)-1) \\ \phi_3 : \overline{\mathcal{N}}_{11}[x] \longrightarrow \overline{\mathcal{N}}_{10}[x] & \phi_3(p(x)) = \frac{1}{x}(p(x)-1) \end{cases}$$

First we shall find the transitions of $\partial\omega \cap \partial\omega_3$ according to Φ_3. Observe:

$$\begin{aligned} \phi_3((-1)^{11}) &= \phi_3(\sum_{n\geq 0}(x^{3n} + x^{3n+1})) \\ &= \phi_3(1 + x + \sum_{n\geq 1}(x^{3n} + x^{3n+1})) \\ &= 1 + \sum_{n\geq 1}(x^{3n-1} + x^{3n}) \\ &= (-1)^{10} \end{aligned}$$

Similarly we get:

$$\begin{aligned} \phi_3(r^{11}(x)) &= s^{10}(x) \\ \phi_3(t^{11}(x)) &= t^{10}(x) \\ \phi_3((-(1+x))^{11}) &= (-x)^{10} \end{aligned}$$

Therefore:

$$\Phi_1(\partial_{\hat{\delta}_x((-1)^{11}),\hat{\delta}_x(r^{11}(x))}) = \partial_{\hat{\delta}_x((-1)^{10}),\hat{\delta}_x(s^{10}(x))}$$

$$\begin{aligned}
\Phi_1(\partial_{\hat{\delta}_x(r^{11}(x)),\hat{\delta}_x(t^{11}(x))}) &= \partial_{\hat{\delta}_x(s^{10}(x)),\hat{\delta}_x(t^{10}(x))}\\
\Phi_1(\partial_{\hat{\delta}_x(t^{11}(x)),\hat{\delta}_x((-(1+x))^{11})}) &= \partial_{\hat{\delta}_x(t^{10}(x)),\hat{\delta}_x((-x)^{10})}\\
\Phi_1(\partial_{\hat{\delta}_x((-(1+x))^{11}),\hat{\delta}_x((-1)^{11})}) &= \partial_{\hat{\delta}_x((-x)^{10}),\hat{\delta}_x((-1)^{10})}
\end{aligned}$$

But as far $\partial\omega \cap \partial\omega_3$ is concerned we have:

$$\begin{aligned}
\Phi_1(\partial_{\hat{\delta}_x((-1)^{11}),\hat{\delta}_x(r^{11}(x))}) &= \partial_{\hat{\delta}_x((-1)^{10}),\hat{\delta}_x(s^{10}(x))}\\
\Phi_1(\partial_{\hat{\delta}_x((-(1+x))^{11}),\hat{\delta}_x((-1)^{11})}) &= \partial_{\hat{\delta}_x((-x)^{10}),\hat{\delta}_x((-1)^{10})}
\end{aligned}$$

Doing similar computations we get the transitions of $\partial\omega \cap \partial\omega_2$ according to Φ_2:

$$\begin{aligned}
\Phi_2(\partial_{\hat{\delta}_x((-x)^{10}),\hat{\delta}_x(r^{10}(x))})) &= \partial_{\hat{\delta}_x((-x)^{010}),\hat{\delta}_x(s^{010}(x))}\\
\Phi_2(\partial_{\hat{\delta}_x(r^{10}(x)),\hat{\delta}_x((-1)^{10})}) &= \partial_{\hat{\delta}_x(s^{010}(x)),\hat{\delta}_x((-1)^{0})}\\
\Phi_2(\partial_{\hat{\delta}_x((-1)^{10}),\hat{\delta}_x(s^{10}(x))}) &= \partial_{\hat{\delta}_x((-1)^{0}),\hat{\delta}_x(s^{00}(x))}\\
\Phi_2(\partial_{\hat{\delta}_x(s^{10}(x)),\hat{\delta}_x((-(1+x))^{10})}) &= \partial_{\hat{\delta}_x(s^{00}(x)),\hat{\delta}_x((-x)^{00})}
\end{aligned}$$

Finally the transitions of $\partial\omega \cap \partial\omega_1$ according to Φ_1 are:

$$\begin{aligned}
\Phi_1(\partial_{\hat{\delta}_x(r^{0}(x)),\hat{\delta}_x((-x)^{010})}) &= \partial_{\hat{\delta}_x(r^{10}(x)),\hat{\delta}_x((-1)^{10})}\\
\Phi_1(\partial_{\hat{\delta}_x((-x)^{010}),\hat{\delta}_x(s^{010}(x))}) &= \partial_{\hat{\delta}_x((-1)^{10}),\hat{\delta}_x(s^{10}(x))}\\
\Phi_1(\partial_{\hat{\delta}_x(s^{010}(x)),\hat{\delta}_x((-1)^{0})}) &= \partial_{\hat{\delta}_x(s^{10}(x)),\hat{\delta}_x((-1)^{11})}\\
\Phi_1(\partial_{\hat{\delta}_x((-1)^{0}),\hat{\delta}_x((-(1+x))^{0})}) &= \partial_{\hat{\delta}_x((-1)^{11}),\hat{\delta}_x((-x)^{010})}\\
\Phi_1(\partial_{\hat{\delta}_x((-(1+x))^{0}),\hat{\delta}_x(s^{00}(x))}) &= \partial_{\hat{\delta}_x((-x)^{010}),\hat{\delta}_x(s^{010}(x))}\\
\Phi_1(\partial_{\hat{\delta}_x(s^{00}(x)),\hat{\delta}_x((-x)^{00})}) &= \partial_{\hat{\delta}_x(s^{010}(x)),\hat{\delta}_x((-1)^{0})}
\end{aligned}$$

This shows the invariance of the boundary under Φ:

$$\Phi(\partial\omega) \subset \partial\omega$$

All these transitions can be expressed in a 12x12 matrix — since we have considered 12 different rectangles (R_i) in the boundary in the computations of the transitions — according to the rule:

$$\mathcal{M}_{ij} = \begin{cases} 1 & \text{if } \Phi(\mathring{R}_j) \cap \mathring{R}_i \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

where $\overset{\circ}{R}_i$ is the interior of R_i as a subset of $\partial\omega$.

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

where the rectangles R_i are:

$$\begin{aligned}
R_1 &= \partial_{\hat{\delta}_x((-1)^{11}),\hat{\delta}_x(r^{11}(x)} \\
R_2 &= \partial_{\hat{\delta}_x((-(1+x))^{11}),\hat{\delta}_x((-1)^{11})} \\
R_3 &= \partial_{\hat{\delta}_x(s^{10}(x)),\hat{\delta}_x((-(1+x))^{10})} \\
R_4 &= \partial_{\hat{\delta}_x((-1)^{10}),\hat{\delta}_x(s^{10}(x))} \\
R_5 &= \partial_{\hat{\delta}_x((-1)^{10}),\hat{\delta}_x(r^{10}(x))} \\
R_6 &= \partial_{\hat{\delta}_x(r^{10}(x)),\hat{\delta}_x((-x)^{10})} \\
R_7 &= \partial_{\hat{\delta}_x((-x)^{00}),\hat{\delta}_x(s^{00}(x))} \\
R_8 &= \partial_{\hat{\delta}_x(s^{00}(x)),\hat{\delta}_x((-(1+x))^{0})} \\
R_9 &= \partial_{\hat{\delta}_x((-(1+x))^{0}),\hat{\delta}_x((-1)^{0})} \\
R_{10} &= \partial_{\hat{\delta}_x((-1)^{0}),\hat{\delta}_x(s^{010}(x))} \\
R_{11} &= \partial_{\hat{\delta}_x((-x)^{010}),\hat{\delta}_x(s^{010}(x))} \\
R_{12} &= \partial_{\hat{\delta}_x(r^{0}(x)),\hat{\delta}_x((-x)^{010})}
\end{aligned}$$

using algebraic computational software, we found that the characteristic polynomial of $\mathcal{M}$ is

$$(-1+x)\, x^4 \left(-1+x+x^2+x^3\right) \left(-1-2\,x+x^4\right) \tag{3.1}$$

We can reduce the number of rectangles by merging the ones which have the same image under ϕ, and later considering the identifications, under $\mathbf{Z}^2$.

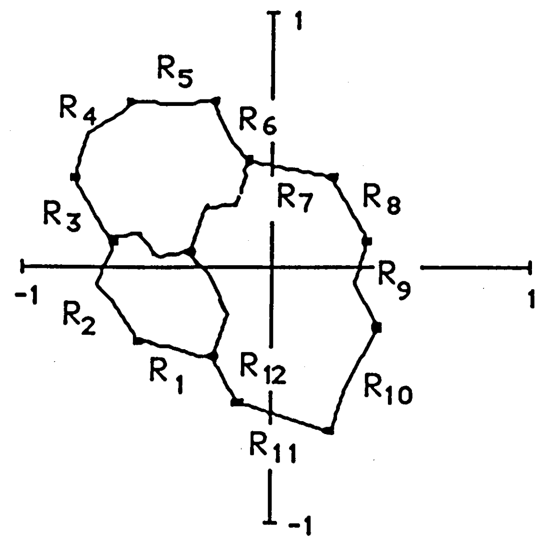


Figure 3.2: The partition of the boundary of ω as a subset of $\mathbf{R}^2$

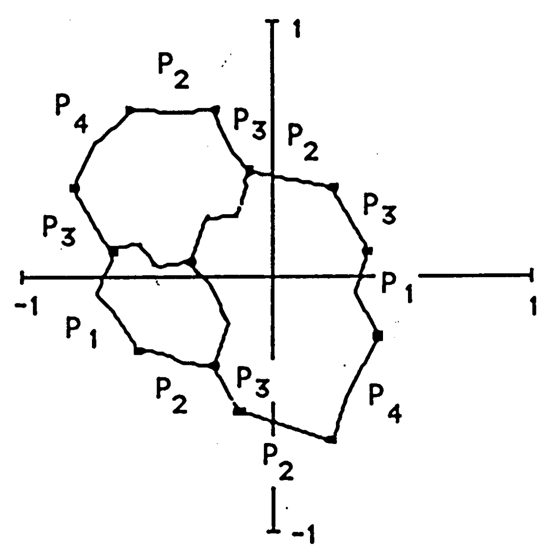


Figure 3.3: The partition of the boundary of ω as a subset of $\mathbf{T}^2$

We end up with 4 rectangles:

$$\begin{array}{lcll} P_1 & = & R_2 & R_9 \text{ is identified to } R_2 \\ P_2 & = & R_1 \cup R_7 & R_5 \text{ is identified to } R_1 \text{ and } R_{11} \text{ to } R_7 \\ P_3 & = & R_3 \cup R_6 & R_8 \text{ is identified to } R_3 \text{ and } R_{12} \text{ to } R_6 \\ P_4 & = & R_4 & R_{10} \text{ is identified to } R_4 \end{array}$$

and the transition matrix is

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

defined as:

$$M_{ij} = \begin{cases} 1 & \text{if } \Phi(\mathring{P}_j) \cap \mathring{P}_i \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

This operation of merging and identifying rectangles can be represented using matrices. If the matrices Q and Q' are

$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \text{ and } Q' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

then

$$Q\mathcal{M}Q' = M$$

Theorem 3.3.1 *The Hausdorff dimension of the boundary of ω is*

$$\frac{2\log\rho}{\log\lambda}$$

where ρ is the Perron-Frobenius eigenvalue of

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Proof: First observe that the Perron-Frobenius eigenvalues of M and $\mathcal{M}$ are the same, therefore we are going to work in the plane where the matrix $\mathcal{M}$ is the transition matrix for Φ. Consider the coverings of the boundary, as a subset of the plane, given by the partition $R = \{R_1, \dots, R_{12}\}$ — the rectangles of this partition are disjoint except for the end points — and its iteration under Φ^{-1}. We denote by $\vee_{i=0}^{k}\Phi^{-i}R$ the partition of the boundary given by cylinders of the form

$$R_{\underline{i}} = \mathring{R}_{i_0} \cap \Phi^{-1}(\mathring{R}_{i_1}) \cap \cdots \cap \Phi^{-k}(\mathring{R}_{i_k}), \quad \underline{i} = (i_0, \dots, i_k)$$

By definition the s-Hausdorff measure of $\partial\omega$ is:

$$\mathcal{H}_s(\partial\omega) = \lim_{\epsilon\to 0}\inf\{\sum_i |U_i|^s \mid \bigcup_i U_i \supset \partial\omega,\ |U_i| \le \epsilon\ \forall i\}.$$

therefore

$$\mathcal{H}_s(\partial\omega) \le \lim_{k\to\infty} \sum_{R_{\underline{i}}\in\vee_{j=0}^{k}\Phi^{-j}R} |R_{\underline{i}}|^s.$$

Since Φ is a composition of B^{-1} —which expands distance by $\lambda^{1/2}$— and T which is a piece exchange transformation on ω, exist constants $C_1, C_2 > 0$ such that

$$C_1\alpha^{k/2}||p|| \le ||\Phi^{-k}p|| \le C_2\alpha^{k/2}||p|| \quad \forall p \in \omega \tag{3.2}$$

So exists a constant $C' > 0$ such that

$$|R_{i_0,\dots,i_k}| \le C'\alpha^{k/2}$$

On the other hand the number of rectangles of the partition $\vee_{i=0}^{k}\Phi^{-i}R$ is equal to $\underline{1}^t\mathcal{M}^k\underline{1}$ where $\underline{1}^t = (1, \dots, 1)$

Proposition 3.3.1 ([23]) *Exists a constant $C_3 > 0$ such that*

$$\underline{1}^t \mathcal{M}^k \underline{1} \leq C_3 \rho^k \quad \textit{for all} \quad k > 0$$

where ρ is the Perron-Frobenius eigenvalue of the Matrix $\mathcal{M}$

Hence

$$\begin{aligned} \mathcal{H}_s(\partial\omega) &\leq \lim_{k\to\infty} \sum_{R_{\underline{i}} \in \vee_{j=0}^{k} \Phi^{-j} R} |R_{\underline{i}}|^s \\ &\leq \lim_{k\to\infty} C' \alpha^{\frac{sk}{2}} (\text{number of cylinders in } \vee_{j=0}^{k} \Phi^{-j} R) \\ &\leq \lim_{k\to\infty} C' \alpha^{\frac{sk}{2}} \rho^k. \end{aligned}$$

therefore the Hausdorff measure of $\partial\omega$ is smaller or equal to

$$s_0 = -\frac{2 \log \rho}{\log \alpha}$$

In order to prove the opposite inequality, we consider a measure μ on $\partial\omega$. We define μ on the cylinders of the partition $\vee_{j=0}^{k} \Phi^{-j} R$, which generate the Borel σ-algebra of the boundary of ω, by

$$\mu(R_{(i_0,\ldots,i_k)}) = \begin{cases} v_{i_0} \rho^k & \text{if } R_{(i_0,\ldots,i_k)} \neq \emptyset \\ 0 & \text{if } R_{(i_0,\ldots,i_k)} = \emptyset \end{cases}$$

where the v_i-s are the components of the normalize right positive eigenvector of $\mathcal{M}$ given by the Perron-Frobenius theorem. Clearly this measure is supported on $\partial\omega$.

Proposition 3.3.2 *There exists a constant $C_0 > 0$ such that*

$$\mu(R_{\underline{i}}) \leq C_0 |R_{\underline{i}}|^{s_0}$$

Proof of Proposition 3.3.2:

$$\mu(R_{(i_0,\ldots,i_k)}) = v_{i_0} \rho^k \leq \rho^k = \alpha^{\frac{(k)s_0}{2}}$$

On the other hand, due to the inequality 3.2 there exists a constant $C_4 > 0$ such that

$$|\Phi^{-1} R_i| \geq C_4 \alpha^{1/2} \text{ for all } R_i \in R$$

and since the partition R satisfies the Markov condition i.e. if $\Phi(\mathring{R}_i)\cap \mathring{R}_j \neq \emptyset$ then $R_j \subset \Phi(R_i)$; we have:

$$\begin{aligned} |R_{\underline{i}}| &= |R_{(i_0,\ldots,i_k)}| \\ &\geq C_4\alpha^{\frac{k}{2}} \end{aligned}$$

therefore

$$\mu(R_{\underline{i}}) \leq C_0|R_{\underline{i}}|^{s_0} \quad \text{for some} \quad C_0 > 0$$

End of the proof of proposition 3.3.2

Let U be a open subset of $\partial\omega$ and $R_{\underline{i}}$ with $\underline{i} \in J$ a covering of U by cylinders of $\bigcup_{k\geq 0} \bigvee_{i=0}^{k} \Phi^{-i}R$, therefore

$$0 < \mu(U) \leq \mu(\bigcup R_{\underline{i}}) \leq \sum \mu(R_{\underline{i}}) \leq C_o \sum |R_{\underline{i}}|^{s_0}$$

Since this inequality is true for any covering of U by cylinders of $\bigcup_{k\geq 0} \bigvee_{i=0}^{k}$ $\Phi^{-i}R$ and any other covering can be express in terms of this covering, we have:

$$0 < \mu(U) \leq \mathcal{H}_{s_0}(U)$$

hence the Hausdorff dimension of $\partial\omega$ is equal to s_0.

End of the proof of Theorem 3.3.1

The system of maps $\Phi : \omega \longrightarrow \omega$ induces a system of maps on the interval, using the Arnoux semiconjugacy $\hat{\xi} : \mathbf{I} \longrightarrow \omega$ (Chapter 0. This new system

$$\hat{\xi}^{-1}\Phi\hat{\xi} \;:\; \mathbf{I} \longrightarrow \mathbf{I}$$

is given on the interval by:

$$\hat{\xi}^{-1}\Phi\hat{\xi} = \begin{cases} \hat{\xi}^{-1}\Phi_1\hat{\xi} &= \quad h^{-1} \quad : \; I_1 \longrightarrow I \\ \hat{\xi}^{-1}\Phi_2\hat{\xi} &= \quad h^{-1}f^{-1} \; : \; I_2 \longrightarrow I_1 \\ \hat{\xi}^{-1}\Phi_3\hat{\xi} &= \quad h^{-1}f^{-1} \; : \; I_3 \longrightarrow I_2 \end{cases}$$

Since the boundary of ω is invariant under Φ, its preimage under Arnoux's map is an invariant set for the transformation $\hat{\xi}^{-1}\Phi\xi$.

Corollary 3.3.1 *The Hausdorff dimension of the preimage of the boundary of ω, under the Arnoux map ξ, is*

$$\frac{\log \rho}{\log \lambda}$$

Proof: The proof is the same as in theorem 3.3.1, but we consider the partition of $\hat{\xi}^{-1}(\partial\omega)$ given by $S = \{S_1, \ldots, S_{12}\}$ where $S_i = \hat{\xi}^{-1} R_i$ and $\vee_{j=0}^{k} \hat{\xi}^{-j} \Phi \xi(S)$.

Since each map of the system expands the distance by a factor of λ (while Φ expands the distance by $\lambda^{1/2}$) and each S_i is contained in the continuity component of f there exist constants C_1' and C_2' such that

$$C_1'\alpha \leq |\hat{\xi}\Phi^{-1}\hat{\xi}^{-1}(S_i)| \leq C_2'\alpha \quad \text{for any} \quad S_i \in S$$

and therefore

$$C_1'\alpha^k \leq |\hat{\xi}\Phi^{-k}\hat{\xi}^{-1}(S_{\underline{i}})| \leq C_2'\alpha^k \quad \text{for any} \quad S_{\underline{i}} \in \vee_{j=0}^{k} \hat{\xi}^{-j} \Phi \hat{\xi}(S)$$

hence the Hausdorff dimension of $\hat{\xi}^{-1}(\partial\omega)$ is half of the dimension of $\partial\omega$.

Q.E.D.

Chapter 4

Relationships between the Π_n substitution dynamical systems.

4.1 Introduction

In the previous chapters we have been studying the family of substitutions:

$$\Pi_n : \begin{array}{ccc} 1 & \longrightarrow & 12 \\ 2 & \longrightarrow & 13 \\ \vdots & & \\ (n-1) & \longrightarrow & 1n \\ n & \longrightarrow & 1 \end{array}$$

and for each n we have shown different properties of the dynamical system associated to this substitution and of it various geometrical realization.

In this chapter we describe how the dynamics of the systems of this family, corresponding to lower dimensions – i.e. the parameter n in the definition of Π_n – are present in systems of higher dimensions. In particular we show that there is a subset of $\overline{\mathcal{N}}^n$, whose dynamics resembles the dynamics of $\overline{\mathcal{N}}^{n-1}$. We compute the Hausdorff and Billingsley dimensions, with respect to a natural metric and measure on $\overline{\mathcal{N}}^n$, of this subset. Also we study the realization of this subset in the interval.

4.2 Topological conjugacies

Let $\sigma : \Omega^n \longrightarrow \Omega^n$ be the dynamical system associated to Π_n as described in page 4. This system is topologically conjugate to $(+1)_n : \overline{\mathcal{N}}^n \longrightarrow \overline{\mathcal{N}}^n$ where

$$\overline{\mathcal{N}}^n = \{\underline{a} \in \{0,1\}^{\mathbb{N}^*} | \sum_{j=0}^{n-1} a_{i+j} < n \ \forall i\}$$

and the map $(+1)_n$ is the extension to $\overline{\mathcal{N}}^n$ of adding 1 on

$$\mathcal{N}^n = \{\underline{a} \in \overline{\mathcal{N}}^n | \exists N > 0 \text{ such that } \forall i \geq N \ a_i = 0\}$$

as it was described in page 17. This system is self-induced, i.e the diagram

$$\begin{array}{ccc} \overline{\mathcal{N}}^n & \xrightarrow{(+1)_n} & \overline{\mathcal{N}}^n \\ \tau \downarrow & & \downarrow \tau \\ \overline{\mathcal{N}}_0^n & \xrightarrow{\widetilde{(+1)}_n} & \overline{\mathcal{N}}_0^n \end{array} \tag{4.1}$$

commutes where

$$\overline{\mathcal{N}}_0^n = \{\underline{a} \in \overline{\mathcal{N}}^n | a_0 = 0\},$$
$$\begin{array}{cccc} \tau : & \overline{\mathcal{N}}^n & \longrightarrow & \overline{\mathcal{N}}_0^n \\ & (a_0 a_1 \ldots) & \rightarrow & (0 a_0 a_1 \ldots) \end{array}$$

and $\widetilde{(+1)}_n$ is the induced map of $(+1)_n$ in $\overline{\mathcal{N}}_0^n$.

Let C^n be the subset of $\overline{\mathcal{N}}^n$ in which n consecutive 0's are not allowed. i.e:

$$C^n = \{\underline{a} \in \overline{\mathcal{N}}^n | \sum_{j=0}^{n-1} a_{i+j} > 0 \ \forall i\}$$

Theorem 4.2.1 *There exists a continuous and surjective map $\psi_n : C^n \rightarrow \overline{\mathcal{N}}^{n-1}$ such that the following diagram commutes:*

$$\begin{array}{ccc} C^n & \xrightarrow{\sigma} & C^n \\ \psi_n \downarrow & & \downarrow \psi_n \\ \overline{\mathcal{N}}^{n-1} & \xrightarrow{\sigma} & \overline{\mathcal{N}}^{n-1} \end{array}$$

and also there exists a continuous map $g_n : \mathcal{C}^n \longrightarrow \mathcal{C}^n$ such that the diagram

$$
\begin{array}{ccc}
\mathcal{C}^n & \xrightarrow{\quad g_n \quad} & \mathcal{C}^n \\
\Big\downarrow \psi_n & & \Big\downarrow \psi_n \\
\overline{\mathcal{N}}^{n-1} & \xrightarrow{(+1)_n} & \overline{\mathcal{N}}^{n-1}
\end{array}
$$

commutes.

Proof: Let $\{a_{i_0} \dots a_{i_{n-2}}\}_{i=1}^{n-1}$ be the symbols of $\mathcal{C}^n$, which are all the elements of $\{0,1\}^{n-1}$ — since the non-allowed symbols of $\mathcal{C}^n$ have length n or greater — and $M(n)$ its transition matrix:

$$
M_{ij}(n) = \begin{cases} 1 & \text{if } a_{i_1} \dots a_{i_{n-2}} = a_{j_0} \dots a_{j_{n-3}} \\ & \text{and } 0 < a_{i_0} + \dots + a_{i_{n-2}} + a_{j_{n-2}} < n \\ 0 & \text{otherwise .} \end{cases}
$$

We re-arrange the entries of this matrix as follows: Let $a_{i_0} \dots a_{i_{n-2}}$ be any of the symbols of $\mathcal{C}^n$. Consider $\overline{a}_{i_0} \dots \overline{a}_{i_{n-2}}$ where

$$
\overline{a_i} = \begin{cases} 0 & \text{if } a_i = 1 \\ 1 & \text{if } a_i = 0. \end{cases}
$$

The transition matrix of $\mathcal{C}^n$ can be re-written in such way that it can be split into sub-blocks of size 2x2, so that each of them gives the transitions from $a_{i_0} \dots a_{i_{n-2}}$ and $\overline{a}_{i_0} \dots \overline{a}_{i_{n-2}}$ to $a_{j_0} \dots a_{j_{n-2}}$ and $\overline{a}_{j_0} \dots \overline{a}_{j_{n-2}}$.

Consider any of these 2x2 blocks. If there is at least one entry equal to 1, in this block, we can suppose that this entry represents the transition between $a_{i_0} \dots a_{i_{n-2}}$ and $a_{j_0} \dots a_{j_{n-2}}$. Then the transition between $\overline{a}_{i_0} \dots \overline{a}_{i_{n-2}}$ and $\overline{a}_{j_0} \dots \overline{a}_{j_{n-2}}$ is also allowed and the other two transitions are not allowed. Therefore the block is of the form

$$
\mathrm{Id}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
$$

On the other hand if there is not any 1 in the block, it is of the form

$$
O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
$$

We obtain a 2^{n-2}x2^{n-2} matrix K by collapsing each of these blocks to only one entry. If the block is of the form Id_2 or J_2 the corresponding entry in K is 1. It is 0 if it comes from a block of type O.

We introduce the maps $\varsigma : \{0,1\}^2 \to \{0,1\}$ defined by $\varsigma(00) = \varsigma(11) = 1$, $\varsigma(10) = \varsigma(01) = 0$ and

$$\begin{aligned} \Psi_n &: \{0,1\}^{n-1} \longrightarrow \{0,1\}^{n-2} \\ \Psi_n(a_0 \cdots a_{n-2}) &= \varsigma(a_0a_1)\varsigma(a_1a_2)\cdots\varsigma(a_{n-3}a_{n-2}). \end{aligned}$$

Observe that $\Psi_n(a_0 \cdots a_{n-2}) = \Psi_n(\overline{a}_0 \cdots \overline{a}_{n-2})$. So the map is two to one.

The matrix K is also a transition matrix, in fact: if the entry K_{ij} comes from the block corresponding to the transitions from $a_{i_0} \ldots a_{i_{n-2}}$, $\overline{a}_{i_0} \ldots \overline{a}_{i_{n-2}}$ to $a_{j_0} \ldots a_{j_{n-2}}$, $\overline{a}_{j_0} \ldots \overline{a}_{j_{n-2}}$ then k_{ij} gives the transitions from $\Psi_n(a_{i_0} \ldots a_{i_{n-2}})$ to $\Psi_n(a_{j_0} \ldots a_{j_{n-2}})$. In order to show that the matrix K is the transition matrix for $\overline{\mathcal{N}}^{n-1}$, it is sufficient to prove that the pattern 1_{n-1} – i.e. $\underbrace{1\cdots 1}_{n-1}$ – is not allowed and also if $b_{i_0} \ldots b_{i_{n-3}}$ and$b_{j_0} \ldots b_{j_{n-3}}$ are words of length $n-2$ in $\overline{\mathcal{N}}^{n-1}$ such that at least one of them is different from 1_{n-2} and $b_{i_1} \ldots b_{i_{n-3}} = b_{j_0} \ldots b_{j_{n-4}}$ then $K_{ij} = 1$. Consider the pre-image of 1_{n-2} under Ψ_n, which is 1_{n-1} or 0_{n-1}. Since the symbols 1_n and 0_n are not allowed in C^n, the block of $M(n)$ which expresses the transitions between 1_{n-1} , 0_{n-1} and themselves is of type O. So the transition, expressed in K, between 1_{n-2} and itself is not allowed. Consider $b_{i_0} \ldots b_{i_{n-3}}$ and $b_{j_0} \ldots b_{j_{n-3}}$ such that $b_{i_1} \ldots b_{i_{n-3}} = b_{j_0} \ldots b_{j_{n-4}}$ and at least one of them is different from 1_{n-2}. Let $a_{i_0} \ldots a_{i_{n-2}}$, $\overline{a}_{i_0} \ldots \overline{a}_{i_{n-2}}$ and $a_{j_0} \ldots a_{j_{n-2}}$, $\overline{a}_{j_0} \ldots \overline{a}_{j_{n-2}}$ be the pre-images of $b_{i_0} \ldots b_{i_{n-3}}$ and $b_{j_0} \ldots b_{j_{n-3}}$, respectively, under Ψ_n. Since $b_{i_1} \ldots b_{i_{n-3}} = b_{j_0} \ldots b_{j_{n-4}}$ we have

$$a_{i_1} \ldots a_{i_{n-2}} = a_{j_0} \ldots a_{j_{n-3}} \text{ (or } = \overline{a}_{j_0} \ldots \overline{a}_{j_{n-3}} \text{)}$$

and also $a_{i_0} \ldots a_{i_{n-1}}$ is allowed in C^n (the only way that it could not be allowed is in the case that it is equal to 0_{n-1} or 1_{n-1} which implies that $b_{i_0} \ldots b_{i_{n-3}}$ or $b_{j_0} \ldots b_{j_{n-3}}$ is equal to 1_{n-2}). Therefore the block of $M(n)$ that gives the transitions between $a_{i_0} \ldots a_{i_{n-2}}$, $\overline{a}_{i_0} \ldots \overline{a}_{i_{n-2}}$ and $a_{j_0} \ldots a_{j_{n-2}}$, $\overline{a}_{j_0} \ldots \overline{a}_{j_{n-2}}$ is of type Id_2 or J_2, so $K_{ij} = 1$. This proves that the matrix K describes the transitions in $\overline{\mathcal{N}}^{n-1}$.

Let ψ_n be the map:

$$\begin{aligned} \psi_n &: \; \mathcal{C}^n \longrightarrow \overline{\mathcal{N}}^{n-1} \\ \psi_n(a_0a_1\cdots) &= \Psi_n(a_0\ldots a_{n-2})\Psi_n(a_{n-2}\ldots a_{2n-4})\cdots \\ &= \varsigma(a_0a_1)\varsigma(a_1a_2)\varsigma(a_2a_3)\cdots \\ &= b_0b_1b_2 \end{aligned}$$

In order to show that the image of this map lies in $\overline{\mathcal{N}}^{n-1}$ take any subword of length $n-1$ of $\underline{b} = \psi_n(\underline{a})$, for some $\underline{a}$ in $\mathcal{C}^n$, say $b_{i-1}b_i\ldots b_{i+n-3}$. If $a_ja_{j+1}\ldots a_{j+n-2}$ is one of the pre-images of $b_i\ldots b_{i+n-3}$ under Ψ_n then one of the preimages of $b_{i-1}b_i\ldots b_{i+n-4}$ is $a_{j-1}a_ja_{j+1}\ldots a_{j+n-3}$. Since $\underline{a}$ is in $\mathcal{C}^n$ the transition from $a_{j-1}a_ja_{j+1}\ldots a_{j+n-3}$ to $a_ja_{j+1}\ldots a_{j+n-2}$ is allowed, therefore the transition between $b_{i-1}b_i\ldots b_{i+n-4}$ and $b_i\ldots b_{i+n-3}$ is allowed according to the matrix K which gives the transitions in $\overline{\mathcal{N}}^{n-1}$, so $\underline{b}$ is an element of $\overline{\mathcal{N}}^{n-1}$.

In order to prove that ψ_n is surjective, take any element $\underline{b}$ of $\overline{\mathcal{N}}^{n-1}$. Consider $b_0\ldots b_{n-3}$ and its two preimages under Ψ_n: $a_0\ldots a_{n-2}$ and $\overline{a}_0\ldots\overline{a}_{n-2}$. On the other hand the preimages of $b_1\ldots b_{n-2}$ are $a_1\ldots a_{n-1}$ and $\overline{a}_1\ldots\overline{a}_{n-1}$, since the transition between $b_0\ldots b_{n-3}$ and $b_1\ldots b_{n-2}$ is allowed, we get that $a_0\ldots a_{n-1}$ and $\overline{a}_0\ldots\overline{a}_{n-1}$ are allowed words in $\overline{\mathcal{N}}^{n-1}$, so by induction we construct the sequences $\underline{a}$ and $\underline{\overline{a}}$ which are the preimages of $\underline{b}$ under ψ_n. This proves that ψ_n is surjective and two to one.

The set $\mathcal{C}^n$ admits the natural partition $\mathcal{C}^n = \mathcal{C}_0^n \cup \mathcal{C}_1^n$ where $\mathcal{C}_i^n = \{\underline{a} \in \mathcal{C}^n | a_0 = i\}$ for $i = 0, 1$. Observe that the previous argument proves that the map ψ_n is bijective in each $\mathcal{C}_i^n$.

The continuity of ψ_n is straight-forward. Also, from the construction of this map it follows that

$$\psi_n(\sigma(\underline{a})) = \sigma(\psi_n(\underline{a})) \quad \text{for all} \quad \underline{a} \in \mathcal{C}^n.$$

On the other hand the map $(+1)_{n-1} : \overline{\mathcal{N}}^{n-1} \to \overline{\mathcal{N}}^{n-1}$ lifts continuously to $g_n^0 : \mathcal{C}_0^n \to \mathcal{C}_0^n$ and $g_n^1 : \mathcal{C}_1^n \to \mathcal{C}_1^n$ but since $\mathcal{C}_0^n$ and $\mathcal{C}_1^n$ are disjoint we obtain $g_n : \mathcal{C}^n \to \mathcal{C}^n$ which is continuous and

$$\psi_n(g_n(\underline{a})) = (+1)_{n-1}(\psi_n(\underline{a})) \quad \text{for all } \underline{a} \in \mathcal{C}^n.$$

Q.E.D.

Corollary 4.2.1 *Define an involution*

$$\begin{array}{rccc} \phi & : & \{0,1\}^{\mathbf{N}^*} & \longrightarrow & \{0,1\}^{\mathbf{N}^*} \\ & & (a_0a_1\ldots) & \longrightarrow & (\overline{a_0a_1\ldots}) \end{array}.$$

The set C^n is invariant under ϕ and $\hat{g_n} : {C^n}/_{\phi} \to {C^n}/_{\phi}$ is topologically conjugate to $(+1)_{n-1} : \overline{\mathcal{N}}^{n-1} \to \overline{\mathcal{N}}^{n-1}$ where $\hat{g_n}$ is the map corresponding to g_n under the projection which maps C^n into ${C^n}/_{\phi}$.

Proof: In order to prove the invariance of C^n under ϕ, it is sufficient to show the transitions in C^n are invariant under the operation that changes 0's to 1's and vice versa, i.e. if $a_{i_0}\ldots a_{i_{n-1}}$ and $a_{j_0}\ldots a_{j_{n-1}}$ are symbols such that $a_{i_1}\ldots a_{i_{n-1}} = a_{j_0}\ldots a_{j_{n-2}}$ and $a_{i_0}\ldots a_{i_{n-1}}a_{j_{n-1}}$ is an allowed word in C^n then $\overline{a}_{i_0}\ldots\overline{a}_{i_{n-1}}\overline{a}_{j_{n-1}}$ is also an allowed word in C^n. This is true because $\overline{a}_{i_1}\ldots\overline{a}_{i_{n-1}} = \overline{a}_{j_0}\ldots\overline{a}_{j_{n-2}}$ and the only case where $\overline{a}_{i_0}\ldots\overline{a}_{i_{n-1}}\overline{a}_{j_{n-1}}$ is not allowed is when is equal to 0_{n+1} or 1_{n+1}, which implies that $a_{i_0}\ldots a_{i_{n-1}}a_{j_{n-1}}$ is equal to 1_{n+1} or 0_{n+1}, respectively, contradicting the transition between $a_{i_0}\ldots a_{i_{n-1}}$ and $a_{j_0}\ldots a_{j_{n-1}}$.

Since ϕ is a homeomorphism between C_0^n and C_1^n, we have that ${C^n}/_{\phi}$ is homeomorphic to C_0^n and to C_1^n. According to theorem 4.2.1, $\psi_n|_{C_i^n} : C_i^n \to: \overline{\mathcal{N}}^{n-1}$ is continuous and bijective, for $i = 0, 1$; since both spaces are compact we have that this map is a homeomorphism. Therefore $g_n^i : C_i^n \to C_i^n$ is topologically conjugate to $(+1)_{n-1} : \overline{\mathcal{N}}^{n-1} \to \overline{\mathcal{N}}^{n-1}$, hence $\hat{g_n} : {C^n}/_{\phi} \to {C^n}/_{\phi}$ is conjugate to $(+1)_{n-1} : \overline{\mathcal{N}}^{n-1} \to \overline{\mathcal{N}}^{n-1}$.

Q.E.D.

4.3 Metric relations

In this section we shall show that the dynamics on C^n resembles the dynamics on $\overline{\mathcal{N}}^{n-1}$ from the metric point of view, i.e. the shift map $\sigma|_{C^n}$ expands the Hausdorff measure – of its dimension – by the same amount as $\sigma|_{\overline{\mathcal{N}}^{n-1}}$. In particular the Hausdorff and Billingsley dimensions of C^n are computed, for a natural metric and measure on $\overline{\mathcal{N}}^n$.

We consider $\overline{\mathcal{N}}^n$ with the metric d_n defined as:

$$d_n(\underline{a}, \underline{a}') = \lambda_n^{-\min\{i|a_i \neq a_i'\}}$$

where $\underline{a}, \underline{a}' \in \overline{\mathcal{N}}^n$ and λ_n is the Pisot number of the polynomial $x^n - x^{n-1} - \cdots - x - 1$ ([6]). This metric is compatible with the product topology on $\overline{\mathcal{N}}^n$.

In order to define a measure on $\overline{\mathcal{N}}^n$, we introduce cylinders of the form:

$$P_{a_{i_0}\dots a_{i_m}} = \{\underline{b} \in \overline{\mathcal{N}}^n | b_j = a_{i_j} \;\; 0 \le j \le m\}.$$

The measure ν_n is defined on these cylinders as

$$\nu_n(P_{a_{i_0}\dots a_{i_m}}) = \vartheta_{a_{i_0}\dots a_{i_{2^{n-1}}}} \lambda_n^{-(m+1-2^{n-1})} \quad \text{for } m \ge 2^{n-1}$$

where $\vartheta_{a_{i_0}\dots a_{i_{2^{n-1}}}}$ are the components of the normalized positive right eigenvector of the transition matrix that defines $\overline{\mathcal{N}}^n$, given by the Perron-Frobenius theorem.

In theorem 4.3.1 we compute the Hausdorff dimension of any subset $\mathcal{S}$ of $\overline{\mathcal{N}}^n$, defined by a transition matrix S. Let S be an sxs-matrix with 0's and 1's as coefficients:

$$\mathcal{S} = \{\underline{a} \in \overline{\mathcal{N}}^n | S_{a_i \dots a_{i+k-1} \; a_{i+1}\dots a_{i+k}} = 1 \;\; \forall i\}$$

where k is such that there are s symbols of length k: $a_{i_0} \dots a_{i_{k-1}}$.

Theorem 4.3.1 *If $\mathcal{S} \subset \overline{\mathcal{N}}^n$ is defined by a transition matrix S then the Hausdorff dimension of $\mathcal{S}$ is $\frac{\log \rho}{\log \lambda_n}$, where ρ is the Perron-Frobenius eigenvalue of S. Moreover the ν_n-Billingsley dimension of $\mathcal{S}$ is equal to its Hausdorff dimension.*

Proof: Consider the covering of $\mathcal{S}$ given by cylinders of the form:

$$R_{a_{i_0}\dots a_{i_m}} = \{\underline{b} \in \overline{\mathcal{N}}^n | b_j = a_{i_j} \;\; 0 \le j \le m\}$$

where $a_{i_0} \dots a_{i_m}$ is an allowed word in $\mathcal{S}$ and $m \ge k$. We denote this covering by $\mathcal{R}_m$ and $\mathcal{R} = \cup_{m \ge k} \mathcal{R}_m$. Observe that the diameter of each cylinder of $\mathcal{R}_m$ is $\lambda_n^{-(m+1)}$.

Since S is the transition matrix that define $\mathcal{S}$, we obtain:

$$\begin{aligned} R_{a_{i_0}\dots a_{i_m}} &= \sigma^{m-k+1}(R_{a_{i_0}\dots a_{i_{k-1}}}) \cap \sigma^{m-k}(R_{a_{i_1}\dots a_{i_{k-2}}}) \cap \cdots \\ &\quad \cdots \cap \sigma(R_{a_{i_{m-k+1}}\dots a_{i_{m-1}}}) \cap R_{a_{i_{m-k}}\dots a_{i_m}} \end{aligned}$$

therefore the number of cylinders in $\mathcal{R}_m$ is given by

$$\underline{1}(s)^t S^{m-k+1}\underline{1}(s)$$

where $\underline{1}(s)^t = \underbrace{(1,\ldots,1)}_{s}$.

Hence, if we denote the diameter of any set by "| |", we get:

$$\begin{aligned}
\mathcal{H}_\tau(\mathcal{S}) &= \liminf_{\epsilon\to 0}\{\sum |U_i|^\tau \mid \cup_i U_i \supset \mathcal{S} \text{ and } |U_i| < \epsilon\} \\
&\le \lim_{m\to\infty} \sum_{\mathcal{R}_m} |R_{a_{i_0}\ldots a_{i_m}}|^\tau \\
&\le \lim_{m\to\infty} \lambda_n^{-(m+1)\tau}(\#\text{ of cylinders in} \mathcal{R}_m) \\
&= \lim_{m\to\infty} \lambda_n^{-(m+1)\tau}\underline{1}(s)^t S^{m-k+1}\underline{1}(s))
\end{aligned}$$

Since ρ is the Perron-Frobenius eigenvalue of S, there exists a constant $C > 0$ such that

$$\underline{1}(s)^t S^l \underline{1}(s) \le C\rho^l \quad \text{for all } l > 0$$

So:

$$\mathcal{H}_\tau(\mathcal{S}) \le \lim_{m\to\infty}(C\rho^{-k})\lambda_n^{-(m+1)\tau}\rho^{m+1}$$

therefore the Hausdorff dimension of $\mathcal{S}$ is smaller or equal to $\tau_0 = \frac{\log\rho}{\log\lambda_n}$. In order to prove the opposite inequality we consider the measure μ on $\overline{\mathcal{N}}^n$. We define it on cylinders of the form $R_{a_{i_0}\ldots a_{i_m}}$

$$\mu(R_{a_{i_0}\ldots a_{i_m}}) = v_{a_{i_0}\ldots a_{i_{k-1}}}\rho^{-(m+1-k)}$$

where the $v_{a_{i_0}\ldots a_{i_{k-1}}}$'s are the components of the normalized positive right eigenvector of S given by the Perron-Frobenius theorem. Clearly this measure is supported on $\mathcal{S}$.

Since $|R_{a_{i_0}\ldots a_{i_m}}| = \lambda_n^{-(m+1)}$, we get:

$$\begin{aligned}
\mu(R_{a_{i_0}\ldots a_{i_m}}) &= v_{a_{i_0}\ldots a_{i_{k-1}}}\rho^{-(m+1-k)} = (v_{a_{i_0}\ldots a_{i_{k-1}}}\rho^k)|R_{a_{i_0}\ldots a_{i_m}}|^{\tau_0} \\
&\le C'|R_{a_{i_0}\ldots a_{i_m}}|^{\tau_0}
\end{aligned}$$

where C' is a constant independent of the cylinder.

Let U be an subset of $\mathcal{S}$ of positive μ-measure and $\cup_{i\in J}R_i$ any covering of U where R_i are elements of $\mathcal{R} = \cup_{m\ge k}\mathcal{R}_m$ and J a family of indexes; so

$$0 < \mu(U) \le \mu(\cup_{i\in J}R_i) \le \sum_i \mu(R_i) \le C'\sum_i |R_i|^{\tau_0}.$$

Since this inequality is true for any covering of U chosen from $\mathcal{R}$, and this family of sets generates the Borel σ-algebra of $\mathcal{S}$, we obtain:

$$0 < \mu(U) \leq \mathcal{H}_{\tau_0}(U)$$

therefore the Hausdorff dimension of $\mathcal{S}$ is

$$\tau_0 = \frac{\log \rho}{\log \lambda_n}.$$

The computation of the ν_n-Billingsley dimension of $\mathcal{S}$ follows the same lines as before. If $\mathcal{B}_\tau(\mathcal{S})$ denotes the ν_n-Billingsley measure of dimension τ we have:

$$\begin{aligned}
\mathcal{B}_\tau(\mathcal{S}) &= \lim_{\epsilon \to 0} \inf\{\sum \nu_n(U_i)^\tau \mid \cup_i U_i \supset \mathcal{S} \text{ and } \nu_n(U_i) < \epsilon\} \\
&\leq \lim_{m\to\infty} \sum_{\mathcal{R}_m} \nu_n(R_{a_{i_0}\dots a_{i_m}})^\tau \\
&= \lim_{m\to\infty} \lambda_n^{-(m+1+2^{n-1})\tau} \underline{1}(s)^t S^{m-k+1} \underline{1}(s)) \\
&\leq \lim_{m\to\infty} C\lambda_n^{-(m+1+2^{n-1})\tau} \rho^{m+1-k}
\end{aligned}$$

therefore the ν_n-Billingsley dimension of $\mathcal{S}$ is smaller or equal than $\tau_0 = \log \rho / \log \lambda_n$. And the opposite inequality follows from

$$\mu(R_{a_{i_0}\dots a_{i_m}}) \leq C'' \nu_n(R_{a_{i_0}\dots a_{i_m}})^{\tau_0} \quad \text{for a constant } C'' > 0.$$

End of the proof of Theorem 4.3.1

In order to compute the Hausdorff dimension of $\mathcal{C}^n$ and $\overline{\mathcal{N}}^n$, as a corollary of theorem 4.3.1, we need to prove some properties of the transition matrices of these two spaces, which deal with their eigenvalues.

Proposition 4.3.1 *The characteristic polynomial of the matrix*

$$A(n) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ & & & \ddots & & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{pmatrix}.$$

is $x^n - x^{n-1} - \cdots - x - 1$, *for every* $n \geq 2$.

Proof: We shall prove this by induction. For $n = 2$, the proposition is true since

$$A(2) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

and its characteristic polynomial is $x^2 - x - 1$.

Observe that $A(n)$ has the structure

$$A(n) = \begin{pmatrix} & & & & & 1 \\ & & A(n-1) & & & 0 \\ & & & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

We denote by Id_n the nxn-identity matrix. Developing the determinant of $A(n) - x\ \mathrm{Id}_n$ through its last row we obtain:

$$\mathrm{Det}(A(n) - x\ \mathrm{Id}_n) = (-x)\ \mathrm{Det}(A(n-1) - x\ \mathrm{Id}_{n-1}) - \Delta_{n-1}$$

where

$$\Delta_{n-1} = \mathrm{Det} \begin{pmatrix} 1-x & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 1 & -x & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & -x & 0 & \cdots & 0 & 0 & 0 & 0 \\ & & & & \ddots & & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -x & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{pmatrix}$$

but

$$\Delta_{n-1} = -\Delta_{n-2} = (-1)^2\Delta_{n-3} = \cdots (-1)^{n-4}\Delta_3$$

and $\Delta_3 = 1$, therefore

$$\mathrm{Det}(A(n) - x\ \mathrm{Id}_n) = (-x)\ \mathrm{Det}(A(n-1) - x\ \mathrm{Id}_{n-1}) + (-1)^3.$$

If n is even $\mathrm{Det}(A(n-1) - x\ \mathrm{Id}_{n-1}) = -(x^{n-1} - x^{n-2} - \cdots - x - 1)$, hence

$$\begin{aligned} \mathrm{Det}(A(n) - x\ \mathrm{Id}_n) &= x(x^{n-1} - x^{n-2} - \cdots - x - 1) - 1 \\ &= x^n - x^{n-1} - \cdots - x - 1 \end{aligned}$$

and if n is odd $\mathrm{Det}(A(n-1) - x\ \mathrm{Id}_{n-1}) = (x^{n-1} - x^{n-2} - \cdots - x - 1)$, so

$$\begin{aligned} \mathrm{Det}(A(n) - x\ \mathrm{Id}_n) &= (-x)(x^{n-1} - x^{n-2} - \cdots - x - 1) + 1 \\ &= -(x^n - x^{n-1} - \cdots - x - 1) \end{aligned}$$

Q.E.D.

Proposition 4.3.2 *Let $N(n)$ be the transition matrix of $\overline{\mathcal{N}}^n$, with $n \geq 2$. There exist matrices $P(n)$ and $Q(n)$, with 0's and 1's as coefficients, such that*

$$P(n)N(n)Q(n) = A(n)$$

Proof: We prove this by induction on n. In the case $n = 3$, we order the symbols of $\overline{\mathcal{N}}^3$ of length 2 as follows: 00, 01, 10, 11 and the transition matrix of $\overline{\mathcal{N}}^3$, according to this order of the symbols is

$$N(3) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We define $P(3)$ and $Q(3)$ as

$$P(3) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad Q(3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and when the multiplication $P(3)N(3)Q(3)$ is done we obtain that is equal to $A(3)$. Observe that the equality also holds if we consider

$$Q(3) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Suppose that the proposition is true for $n-1$ where $P(n-1)$ is:

$$P(n-1) = \begin{pmatrix} \overbrace{1\cdots1}^{2^{n-3}} & \overbrace{0\cdots0}^{2^{n-4}} & \cdots & 0\,0 & 0 & 0 \\ 0\cdots0 & 1\cdots1 & \cdots & 0\,0 & 0 & 0 \\ & & \ddots & & & \\ 0\cdots0 & 0\cdots0 & \cdots & 1\,1 & 0 & 0 \\ 0\cdots0 & 0\cdots0 & \cdots & 0\,0 & 1 & 0 \\ 0\cdots0 & 0\cdots0 & \cdots & 0\,0 & 0 & 1 \end{pmatrix}$$

and the transpose of $Q(n-1)$ is

$$\begin{pmatrix} \overbrace{10\cdots0}^{2^{n-3}} & \overbrace{00\cdots0}^{2^{n-4}} & \cdots & 00 & 0 & 0 \\ 00\cdots0 & 10\cdots0 & \cdots & 00 & 0 & 0 \\ & & \ddots & & & \\ 00\cdots0 & 00\cdots0 & \cdots & 10 & 0 & 0 \\ 00\cdots0 & 00\cdots0 & \cdots & 00 & 1 & 0 \\ 00\cdots0 & 00\cdots0 & \cdots & 00 & 0 & 1 \end{pmatrix}$$

Consider the case of dimension n, the ordering of the symbols of length $n-1$ of $\overline{\mathcal{N}}^n$, is given by the lexicographical order. Since $\{a_{i_0}\dots a_{i_{n-2}}\}_{i=1}^{n-1}$ are the lexicographically ordered symbols of length $n-2$ of $\overline{\mathcal{N}}^{n-1}$, $a_{j_0}=0$ for $1\le j\le 2^{n-3}$, $a_{j_1}=1$ for $1\le j\le 2^{n-4}$ and in particular 0_{n-2} is the first symbol and 1_{n-2} the last one.

So according to this ordering of the symbols of $\overline{\mathcal{N}}^n$, we obtain that the transition matrix is $N(n)=(B_{ij})_{1\le i,j\le 2^{n-2}}$ where B_{ij} is a 2x2- block expressing the transitions of $a_{i_0}\dots a_{i_{n-2}}0$, $a_{i_0}\dots a_{i_{n-2}}1$ and $a_{j_0}\dots a_{j_{n-2}}0$, $a_{j_0}\dots a_{j_{n-2}}1$. The block B_{ij} is either:

$$O=\begin{pmatrix}0&0\\0&0\end{pmatrix},\ \Upsilon=\begin{pmatrix}0&0\\1&0\end{pmatrix},\ \overline{\Xi}=\begin{pmatrix}1&1\\0&0\end{pmatrix}\ \text{or}\ \underline{\Xi}=\begin{pmatrix}0&0\\1&1\end{pmatrix}$$

in each case the corresponding entry – the i,j-th entry – of $N(n-1)$ is $0,0,1$ or 1 respectively. In fact, if $(N(n-1))_{ij}=1$ then $a_{i_1}\dots a_{i_{n-2}}=a_{j_0}\dots a_{j_{n-3}}$, if $a_{j_{n-2}}=0$ then $a_{i_1}\dots a_{i_{n-2}}0$ is equal to $a_{j_0}\dots a_{j_{n-3}}a_{j_{n-2}}$, so $B_{ij}=\overline{\Xi}$; and if $a_{j_{n-2}}=1$ we get $B_{ij}=\underline{\Xi}$. In the case $N(n-1))_{ij}=0$ then $a_{i_0}\dots a_{i_{n-2}}=a_{j_0}\dots a_{j_{n-2}}=1_{n-2}$ or $a_{i_1}\dots a_{i_{n-2}}\ne a_{j_0}\dots a_{j_{n-3}}$. In the latter case we obtain that $B_{ij}=O$ and in the former, the transition from $a_{i_1}\dots a_{i_{n-2}}1$ to $a_{j_0}\dots a_{j_{n-2}}0$ is allowed and the other three transitions are not.

We define the matrices $P(n)$ and $Q(n)$ as follows:

$$P(n)=\begin{pmatrix}\tilde P_{11} & \cdots & \tilde P_{1\,2^{n-2}} & 0 & 0\\ & \vdots & & \vdots & \vdots\\ \tilde P_{n-2\,1} & \cdots & \tilde P_{n-2\,2^{n-2}} & 0 & 0\\ 0 & \cdots & 0 & 1 & 0\\ 0 & \cdots & 0 & 0 & 1\end{pmatrix}\quad \text{where } \tilde P_{ij}=(P_{ij}(n-1),P_{ij}(n-1))$$

$$Q(n) = \begin{pmatrix} \tilde{Q}_{11} & \cdots & \tilde{Q}_{1\,n-2} & 0 & 0 \\ & \vdots & & \vdots & \vdots \\ \tilde{Q}_{2^{n-2}\,1} & \cdots & \tilde{Q}_{2^{n-2}\,n-2} & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \quad \text{where } \tilde{Q}_{ij} = \begin{pmatrix} Q_{ij}(n-1) \\ 0 \end{pmatrix}.$$

Since the matrix $A(n)$ has the structure

$$A(n) = \begin{pmatrix} & & A(n-1) & & & 1 \\ & & & & & 0 \\ & & & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

we have that the i,j-th entries of the matrix $A(n)$, for $1 \leq i, j \leq n-1$, are:

$$\begin{aligned} A_{ij}(n) &= A_{ij}(n-1) \\ &= \textstyle\sum_{l=1}^{2^{n-2}} \sum_{r=1}^{2^{n-2}} P_{ir}(n-1) N_{rl}(n-1) Q_{lj}(n-1) \\ &= \textstyle\sum_{l=1}^{2^{n-2}} \sum_{r=1}^{2^{n-2}} \tilde{P}_{ir} B_{rl} \tilde{Q}_{lj} \\ &= \textstyle\sum_{l,r} \; (P_{ir}(n) \quad P_{ir+1}(n)) \begin{pmatrix} N(n)_{2r-1\,2l-1} & N(n)_{2r-1\,2l} \\ N(n)_{2r\,2l-1} & N(n)_{2r\,2l} \end{pmatrix} \begin{pmatrix} Q_{lj}(n) \\ Q_{l+1\,j} \end{pmatrix} \\ &= \textstyle\sum_{s=1}^{2^{n-1}} \sum_{t=1}^{2^{n-1}} P_{is}(n) N_{st}(n) Q_{tj}(n) \end{aligned}$$

Next we shall show

$$A_{1n}(n) = \sum_{l=1}^{2^{n-1}} \sum_{r=1}^{2^{n-1}} P_{1r}(n) N_{rl}(n) Q_{ln}(n).$$

According to our construction of $P(n)$ and $Q(n)$, we have $P_{1r}(n) = P_{(n-1)\,r} = 0$ for $r = 2^{n-1}, 2^{n-1}-1$ and $Q_{s\,n}(n) = 0$ for all $s \leq 2^{n-1}-1$ and $Q_{2^{n-1}\,n}(n) = 1$; So:

$$\sum_{l=1}^{2^{n-1}} \sum_{r=1}^{2^{n-1}} P_{1r}(n) N_{rl}(n) Q_{ln}(n) = \sum_{r=1}^{2^{n-2}} P_{1r}(n) N_{r\,2^{n-1}}(n)$$

regrouping the terms in this sum, we obtain that it is equal to:

$$\sum_{s=1}^{2^{n-2}} \tilde{P}_{1s}(n) B_{s\,2^{n-1}} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and by the construction of the $\tilde{P}$ and $\tilde{Q}$, we have:

$$\sum_{l=1}^{2^{n-1}}\sum_{r=1}^{2^{n-1}} P_{1r}(n)N_{rl}(n)Q_{ln}(n) = \sum_{s=1}^{2^{n-2}} P_{1\,s}(n-1)N_{s\,2^{n-2}}(n-1).$$

According to our ordering of the symbols of $\overline{\mathcal{N}}^{n-1}$, the entries $N_{s\,2^{n-2}}(n-1)$, express the transitions between $\{a_{i_0}\ldots a_{i_{n-2}}\}_{i=1}^{n-1}$ and 1_{n-2}. Since 1_{n-1} is not allowed in $\overline{\mathcal{N}}^{n-1}$, the only allowed transition expressed by $N_{s\,2^{n-2}}(n-1)$, occurs when $a_{i_0}\ldots a_{i_{n-2}} = 0\underbrace{1\ldots1}_{n-2}$. Let s' be the corresponding integer, i.e. $N_{s'\,2^{n-2}}(n-1) = 1$. On the other hand, $s' \leq 2^{n-3}$, since $a_{i_0} = 0$, so $P_{1\,s'}(n-1) = 1$. Therefore:

$$\sum_{r,l=1}^{2^{n-1}} P_{1r}(n)N_{rl}(n)Q_{ln}(n) = 1$$

which is the value of $A_{1n}(n)$.

Similarly it is proved that

$$A_{ij}(n) = \sum_{l,r=1}^{2^{n-1}} P_{ir}(n)N_{rl}(n)Q_{lj}(n)$$

with $2 \leq i \leq n-1$, $j = n$ and $i = n$, $1 \leq j \leq n$.

Q.E.D.

Remark 4.3.1 *Proposition 4.3.2 is true for any matrix $Q(n)$ of the form:*

$$Q(n)^t = \begin{pmatrix} \overbrace{0\cdots010\cdots0}^{2^{n-2}} & \overbrace{0\cdots0\cdots0}^{2^{n-3}} & \cdots & \overbrace{00}^{2} & 0 & 0 \\ 0\cdots0 & 0\cdots010\cdots0 & \cdots & 00 & 0 & 0 \\ & & \vdots & & & \\ 0\cdots0 & 0\cdots0 & \cdots & 10 & 0 & 0 \\ 0\cdots0 & 0\cdots0 & \cdots & 00 & 1 & 0 \\ 0\cdots0 & 0\cdots0 & \cdots & 00 & 0 & 1 \end{pmatrix}$$

Proposition 4.3.3 *There exist constants $C_1, C_2 > 0$ such that*

$$C_1\lambda_n^k \leq \underline{1}(2^{n-1})^t\, N(n)^k\, \underline{1}(2^{n-1}) \leq C_2\lambda_n^k$$

where $\underline{1}(2^{n-1})^t = \underbrace{(1,\ldots,1)}_{2^{n-1}}$. In particular λ_n is the Perron-Frobenius eigenvalue of $N(n)$.

Proof: We subdivide the matrix $N(n)$ in sub-blocks,$\{D_{ij}\}_{i,j=1}^{n}$ – many of them are not square blocks – such that each D_{ij} is mapped to the i,j-th entry of $A(n)$, i.e. $A_{ij}(n)$; when the matrices $P(n)$ and $Q(n)$ of proposition 4.3.2 are apply to $N(n)$, i.e.

$$\begin{pmatrix} \hat{P}_{11} & \cdots & \hat{P}_{1n} \\ & \vdots & \\ \hat{P}_{n1} & \cdots & \hat{P}_{nn} \end{pmatrix} \begin{pmatrix} D_{11} & \cdots & D_{1n} \\ & \vdots & \\ D_{n1} & \cdots & D_{nn} \end{pmatrix} \begin{pmatrix} \hat{Q}_{11} & \cdots & \hat{Q}_{1n} \\ & \vdots & \\ \hat{Q}_{n1} & \cdots & \hat{Q}_{nn} \end{pmatrix} = A(n)$$

where $\hat{P}_{ii} = \underbrace{(1,\ldots,1)}_{2^{n-1-i}}$, $\hat{P}_{ij} = \underbrace{(0,\ldots,0)}_{2^{n-1-i}}$ for $i \neq j$ and $\hat{Q}^t_{ii} = \underbrace{(1,0,\ldots,0)}_{2^{n-1-j}}$, $\hat{Q}^t_{ij} = \underbrace{(0,\ldots,0)}_{2^{n-1-i}}$ for $i \neq j$. According to this construction:

$$\begin{aligned} A_{ij}(n) &= \hat{P}_{ii} D_{ij} \hat{Q}_{jj} \\ &= (1,\ldots,1) \begin{pmatrix} d_{i_1 j_1} & \cdots & d_{i_1 j_s} \\ & \vdots & \\ d_{i_\tau j_1} & \cdots & d_{i_t j_s} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ &= \textstyle\sum_{l=1}^{\tau} d_{i_1 j_l} \end{aligned}$$

where $s = 2^{n-1-j}$ and $\tau = 2^{n-1-i}$. Similarly it can be done for $N^k(n)$, we subdivide it in blocks $\{D^k_{ij}\}_{i,j=1}^{n}$ such that $D^k_{ij} = \sum_{l=1}^{n} D^{k-1}_{il} D_{lj}$ and $A^k_{ij}(n) = \hat{P}_{ii} D^k_{ij} \hat{Q}_{jj}$.

In the following lines, we prove by induction on k the equality

$$\underline{1}(2^{n-1-i})^t \, D^k_{ij} \, \underline{1}(2^{n-1-j}) = 2^{n-1-j} A^k_{ij}(n). \tag{4.2}$$

When $k = 1$:

$\underline{1}(2^{n-1-i})^t \, D_{ij} \, \underline{1}(2^{n-1-j}) =$

$$\begin{aligned} &= (1,\ldots,1) D_{ij} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + (1,\ldots,1) D_{ij} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + (1,\ldots,1) D_{ij} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \\ &= \sum_{p=1}^{2^{n-1-j}} (\sum_{l=1}^{2^{n-1-i}} d_{i_p j_l}) \end{aligned}$$

According to remark 4.3.1, $\sum_{l=1}^{2^{n-1-i}} d_{i_p j_l} = \sum_{l=1}^{2^{n-1-i}} d_{i_1 j_l}$ for $1 \leq p \leq s$. On the other hand $A_{ij}(n) = \sum d_{ij_l}$, hence

$$\underline{1}(2^{n-1-i})^t D_{ij} \underline{1}(2^{n-1-j}) = 2^{n-1-j} A_{ij}(n).$$

Consider

$$\begin{aligned}
A_{ij}^k(n) &= \sum_{l=1}^{n} A_{il}^{k-1}(n) A_{lj}(n) \\
&= \sum_{l=1}^{n} \left(\frac{1}{2^{(n-1-l)}}\, \underline{1}(2^{n-1-i})^t\, D_{il}^{k-1}\, \underline{1}(2^{n-1-l})\right) \left(\frac{1}{2^{(n-1-j)}}\, \underline{1}(2^{n-1-l})^t\, D_{lj}\, \underline{1}(2^{n-1-j})\right) \\
&= \frac{1}{2^{n-1-j}} \sum_{l=1}^{n} \frac{1}{2^{n-1-l}}\, \underline{1}(2^{n-1-i})^t\, D_{il}^{k-1}\, \left(\underline{1}(2^{n-1-l})\underline{1}(2^{n-1-l})^t\right) D_{lj}\, \underline{1}(2^{n-1-j}) \\
&= \frac{1}{2^{n-1-j}} \sum_{l=1}^{n} \frac{1}{2^{n-1-l}}\, 2^{n-1-l}\, \underline{1}(2^{n-1-i})^t\, D_{il}^{k-1}\, D_{lj}\, \underline{1}(2^{n-1-j}) \\
&= \frac{1}{2^{n-1-j}} \sum_{l=1}^{n} \underline{1}(2^{n-1-i})^t\, D_{il}^{k-1} D_{lj}\, \underline{1}(2^{n-1-j}) \\
&= \frac{1}{2^{n-1-j}}\, \underline{1}(2^{n-1-i})^t\, D_{ij}^k\, \underline{1}(2^{n-1-j})
\end{aligned}$$

Which proves the equality (4.2).

Therefore

$$\frac{1}{2}\left(\sum_{i,j=1}^{n} A_{ij}^k(n)\right) \leq \sum_{i=1}^{n}\sum_{j=1}^{n} 2^{n-1-j} A_{ij}^k(n) = \underline{1}(2^{n-1})^t N(n)^k \underline{1}(2^{n-1}) \leq 2^{n-1} \sum_{i,j=1}^{n} A_{ij}^k(n).$$

Since λ_n is the Perron-Frobenius eigenvalue of $A(n)$ – proposition 4.3.1 –, there exist constants $C, C' > 0$ such that

$$C\lambda_n^k \leq \underline{1}(n)^t A(n)^k \underline{1}(n) \leq C'\lambda_n^k$$

So:

$$\frac{C}{2}\lambda_n^k \leq \underline{1}(2^{n-1})^t N(n)^k \underline{1}(2^{n-1}) \leq (C' 2^{n-1})\lambda_n^k$$

Q.E.D.

Proposition 4.3.4 *There exist constants $C, C' > 0$ such that:*

$$C\lambda_{n-1}^k \leq \underline{1}(2^{n-1})^t M(n)^k \underline{1}(2^{n-1}) \leq C'\lambda_{n-1}^k \text{ for all } k > 0.$$

In particular λ_{n-1} is The Perron-Frobenius eigenvalue of $M(n)$.

Proof of proposition 4.3.4: The 2^{n-2}x2^{n-2}-matrix $N(n-1)$ defines the transitions of $\overline{\mathcal{N}}^{n-1}$. We have seen in theorem 4.2.1 that this matrix is equivalent to the matrix K, constructed there, So we can suppose that $N(n-1) = K$. In the proof of this theorem, we have seen that each entry of $K = (K_{ij})_{ij}$ comes from a 2x2 block B_{ij} of $M(n)$ – which is 2^{n-1}x2^{n-1} –, with the properties:

- if $K_{ij} = 0$ then

$$B_{ij} = O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

- if $K_{ij} = 1$ then $B_{ij} = \mathrm{Id}_2$ or $B_{ij} = J_2$, where

$$\mathrm{Id}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

From proposition 4.3.3 there exist constants $C_1, C_2 > 0$ such that

$$C_1\lambda_{n-1}^l \leq \underline{1}(2^{n-2})^t K^l \underline{1}(2^{n-2}) \leq C_2\lambda_{n-1}^l$$

for $l > 1$, where $\underline{1}(2^{n-2})^t \doteq \underbrace{(1, \ldots, 1)}_{2^{n-2}}$.

If we denote by K_{ij}^l the i,j-th entry of the matrix K_{ij}^l and by B_{ij}^l the i,j-th 2x2-block of $M^l(n)$, defined by: $B_{ij}^l = \sum_{p=1}^{2^{n-2}} B_{ip}^{l-1} B_{pj}$. So:

$$\underline{1}(2^{n-1})^t M^l(n) \underline{1}(2^{n-1}) = \sum_{i,j=1}^{2^{n-2}} \underline{1}(2)^t B_{ij}^l \underline{1}(2)$$

Therefore, in order to prove the proposition it is sufficient to show that the following inequality holds:

$$\underline{1}(2)^t B_{ij}^l \underline{1}(2) = 2K_{ij}^l.$$

We shall prove it using induction on l. When $l = 1$ it is clear that the inequality holds. Consider $K_{ij}^{l} = \sum_{p=1}^{2^{n-2}} K_{ip}^{l-1} K_{pj}$; if $K_{pj} = 1$ for some p then $B_{pj} = \mathrm{Id}_2$ or J_2. In the former case:

$$\underline{1}(2)^t B_{ip}^{l-1} B_{pj} \underline{1}(2) = \underline{1}(2)^t B_{ip}^{l-1} \underline{1}(2) = 2K_{ip}^{l-1}$$

and in the latter case $B_{ip}^{l-1} B_{pj}$ is the block obtained from B_{ip}^{l-1} permuting its columns. Hence

$$\underline{1}(2)^t B_{ip}^{l-1} B_{pj} \underline{1}(2) = \underline{1}(2)^t B_{ip}^{l-1} \underline{1}(2) = 2K_{ip}^{l-1}.$$

and if $K_{pj} = 0$ then $B_{pj} = 0$ and the corresponding term does not contribute to the sum.

Therefore

$$\begin{aligned} \underline{1}(2)^t B_{ij}^{l} \underline{1}(2) &= \textstyle\sum_{p=1}^{2^{n-2}} \underline{1}(2)^t B_{ip}^{l-1} B_{pj} \underline{1}(2) \\ &= \textstyle\sum_{p=1}^{2^{n-2}} 2K_{ip}^{l-1} K_{pj} \\ &= 2K_{ij}^{l}. \end{aligned}$$

Q.E.D.

Corollary 4.3.1 *The Hausdorff dimension of $\mathcal{C}^n$, as a subset of $\overline{\mathcal{N}}^n$ with the metric d_n, is*

$$s_n = \frac{\log \lambda_{n-1}}{\log \lambda_n}.$$

Moreover:

$$\mathcal{H}_{s_n}(\sigma(V)) = \lambda_{n-1} \mathcal{H}_{s_n}(V) \quad \textit{for} \quad V \subset \mathcal{C}^n$$

where $\mathcal{H}_{s_n}$ is the Hausdorff measure in dimension s_n.

Proof: According to proposition 4.3.4 the Perron-Frobenius eigenvalue of $M(n)$ – the transition matrix of $\mathcal{C}^n$ – is λ_{n-1}; therefore the Hausdorff dimension of $\mathcal{C}^n$ is $\frac{\log \lambda_{n-1}}{\log \lambda_n}$. On the other hand, for any subset V of $\mathcal{C}^n$, σ expands its diameter by λ_n, so

$$\mathcal{H}_{s_n}(\sigma(V)) = \lambda_n^{s_n} \mathcal{H}_{s_n}(V) = \lambda_{n-1} \mathcal{H}_{s_n}(V)$$

Q.E.D.

Corollary 4.3.2 *The Hausdorff dimension of $\overline{\mathcal{N}}^n$ is equal to 1.*

Proof: By proposition 4.3.3 the Perron-Frobenius eigenvalue of $N(n)$ – the transition matrix of $\overline{\mathcal{N}}^n$ – is equal to λ_n. So, according to theorem 4.3.1 the Hausdorff dimension of $\overline{\mathcal{N}}^n$ is equal to 1.

Q.E.D.

4.4 Geometrical realizations of $\mathcal{C}^n$

In this section we shall study some properties of the geometrical realizations, in particular in the circle, of the set $\mathcal{C}^n$.

In section 1.3 we introduced the set of formal power series associated to $\overline{\mathcal{N}}^n$:

$$\overline{\mathcal{N}}^n[x] = \{\sum_{i\geq 0} a_i x^i \mid \underline{a} = (a_0, a_1, \ldots) \in \overline{\mathcal{N}}^n\}$$

$$X : \begin{cases} \overline{\mathcal{N}} & \rightarrow & \overline{\mathcal{N}}[x] \\ \underline{a} = (a_0, a_1, \ldots) & \rightarrow & \sum_{i\geq 0} a_i x^i \end{cases}$$

We consider a metric and a measure in $\overline{\mathcal{N}}^n[x]$ that make the map X an isometry and a measure preserving map. Also in section 1.3 was introduced the map

$$\chi : \mathbf{S^1} \longrightarrow \overline{\mathcal{N}}^n[x].$$

In this section we consider the map $\overline{\chi} : \mathbf{S^1} \rightarrow \overline{\mathcal{N}}^n$ define by $\overline{\chi} = X^{-1} \circ \chi$. In order to prove that $\overline{\chi}$ is a measure preserving map between the Lebesgue measure of $\mathbf{S^1}$and the ν_n-measure – defined in section 4.3 –, we need to consider the concept of the standard partition defined in section 1.5. This partition arises from the self-similarity of the interval exchange map f on the circle and the addition by 1, i.e. $(+1)_n$ in $\overline{\mathcal{N}}^n$. This self-similarity is expressed by the commutative diagrams:

$$\begin{array}{ccc} \overline{\mathcal{N}}^n & \xrightarrow{(+1)_n} & \overline{\mathcal{N}}^n \\ \tau \downarrow & & \downarrow \tau \\ \overline{\mathcal{N}}^n_0 & \xrightarrow{\widetilde{(+1)}_n} & \overline{\mathcal{N}}^n_0 \end{array} \qquad\qquad \begin{array}{ccc} \mathbf{I} & \xrightarrow{f} & \mathbf{I} \\ h \downarrow & & \downarrow h \\ \mathbf{I}_1 & \xrightarrow{\tilde{f}} & \mathbf{I}_1 \end{array}$$

In section 1.5, it was shown that any cylinder of the standard partition in the interval, can be expressed as $O_{f,h}(m)\mathbf{I}$ for a integer compatible with the partition (ICP) m. This cylinder is mapped under $\overline{\chi}$ to a cylinder of the standard partition of $\overline{\mathcal{N}}^n$.

Proposition 4.4.1 *The map $\overline{\chi} : \mathbf{S^1} \rightarrow \overline{\mathcal{N}}^n$ is a measure preserving map between the Lebesgue measure of $\mathbf{S^1}$and the ν_n-measure on $\overline{\mathcal{N}}^n$.*

Proof: Since the standard partition on the circle (respectively $\overline{\mathcal{N}}^n$), generates the Borel σ-algebra, it is sufficient to show that $\overline{\chi}$ preserves measure for the cylinders of these partitions. Any cylinder of the standard partition of $\overline{\mathcal{N}}^n$ is the image, under $\overline{\chi}$ of a cylinder of the standard partition of $\mathbf{S}^1$, i.e. if m is an ICP then $O_{(+1),\tau}(m)\overline{\mathcal{N}}^n$ is equal to $\overline{\chi}(O_{f,h}(m)\mathrm{I})$, where $O_{(+1),\tau}(m)$ is defined as

$$O_{(+1),\tau}(m) = \tau^{i_0}(+1)\tau^{i_1-i_0}(+1)\cdots(+1)\tau^{i_{l-1}-i_{l-2}}(+1)\tau^{i_{l-1}-i_l}$$

if $m = g_{i_0} + \cdots g_{i_l}$. In section 1.4 it was shown that m can be written as:

$$m = g_{i_0} \diamond (1 + g_{i_1-i_0} \diamond (1 + \cdots + g_{i_{l-1}-i_{l-2}} \diamond (1 + g_{i_l-i_{l-1}})\cdots)).$$

Since the maps τ and $(+1)\tau$ contract the ν_n-measure by a factor of λ_n^{-1}, we obtain:

$$\nu_n(O_{(+1),\tau}(m)\overline{\mathcal{N}}^n = \lambda_n^{-i_l}$$

On the other hand h contracts the distance by a factor λ_n^{-1} and f is an interval exchange map, therefore the Lebesgue measure of $O_{f,h}(m)\mathbf{S}^1$ is $\lambda_n^{-i_l}$.

Q.E.D.

Denote by K_n the geometrical realization of $\mathcal{C}^n$ on the circle, i.e. $K_n = \chi_n^{-1}(\mathcal{C}^n)$.

Corollary 4.4.1 *The Hausdorff dimension of K_n is equal to $\frac{\log \lambda_{n-1}}{\log \lambda_n}$.*

Proof: According to theorem 4.3.1 the ν_n-Billingsley dimension of $\mathcal{C}^n$ is $\frac{\log \lambda_{n-1}}{\log \lambda_n}$. Since the map $\overline{\chi} : (\mathbf{S}^1, L) \to (\overline{\mathcal{N}}^n, \nu_n)$ is measure preserving, the L-Billingsley dimension of K_n is equal to the ν_n-Billingsley dimension of its image under $\overline{\chi}$. On the other hand, the L-Billingsley dimension of any subset of $\mathbf{S}^1$coincides with its Hausdorff dimension.

Q.E.D.

Bibliography

[1] J. P. Allouche, *Automates finis en théorie des nombres*, Expo. Math **5** (1987), 239-266.

[2] P. Arnoux, *Un invariant pour les échanges d'intervalles et les flots sur les surfaces*, Thèse de 3^{e} cycle, Université de Reims, 1981.

[3] P. Arnoux, *Un exemple de semi-conjugaison entre un échange d'intervalles et une rotation sur le tore*, Bull. Soc. Math. France, **116** (1988), 489-500.

[4] P. Arnoux, *Some remarks about Fibonacci multiplication*, Appl. Math. Lett. **2** (1989), 319-320.

[5] P. Arnoux and G. Rauzy, *Représentation géométrique de suites de complexité* $2n+1$, Bull. Soc. Math. France, **119**, (1991), 199-215.

[6] P. Arnoux and J.C. Yoccoz, *Construction de difféomorphismes pseudo-Anosov*, C.R.A.S. **292** (1981), 75-78.

[7] T.J. Bedford, *Crinkly curves, Markov partitions and Dimension*, Ph.D. Thesis, University of Warwick, 1984.

[8] P. Billingsley, *Ergodic theory and information*, Wiley, 1965, New York.

[9] R. Bowen, *Markov partition are not smooth*, Proc. Amer. Math. Soc., **71** (1978), 130-132.

[10] L. Carleson, *Selected problems on exceptional sets*, Van Nostrand, 1967, Princeton.

[11] A. J. Casson and S. A. Bleiler, *Automorphisms of Surfaces after Nielsen and Thurston*, Cambridge University Pres, 1988, Cambridge.

[12] J.L. Chabert, *Un demi-siècle de fractales: 1870-1920*, Historia Mathematica, **17** (1990), 339-365.

[13] E.M. Coven and G.A. Hedlund, *Sequences with minimal block growth*, Math. Systems Theory **7** (1983), 283-309.

[14] F.M. Dekking, *Substitutions, branching processes and fractal sets* in J. Bélair and S. Dubuc eds., *Fractal Geometry and Analysis*, Kluwer Academic Publishers, 1991, Dordrecht.

[15] V. Drobot and J. Turner, *Hausdorff dimension and Perron-Frobenius theory*, Illinois Journal of Mathematics, **33** (1989), 1-9.

[16] S. Dulucq and D. Gouyou-Beauchamps, *Sur les facteurs des suites de Sturm*, Theoretical Computer Science, **71** (1990), 381-400.

[17] G.A. Edgar, *Measure, topology and fractal geometry*, Springer-Verlag, 1990, New York.

[18] K. Falconer, *The geometry of fractal sets*, Cambridge University Press, 1985, Cambridge.

[19] K. Falconer, *Fractal Geometry -Mathematical foundations and applications*, John Willey & Sons, 1990, Chichester.

[20] A. Fathi, F. Laudenbach, V. Poénaru, - *Travaux de Thurston sur les surfaces*, Astérisque 66-67 (1979).

[21] A.S. Fraenkel, H. Porta and K.B. Stolarsky *Some arithmetical semigroups*, Pre-print.

[22] C. Frougny, *Representation of numbers and finite automata*, Math. Systems Theory, **25** (1992), 37-60.

[23] F.R. Gantmacher, *Matrix Theory*, vol 2, Chelsea, 1959, New York.

[24] G.A. Hedlund, *Sturmian minimal sets*, Amer. J. Math. **66** (1944), 605-620.

[25] G.A. Hedlund and M. Morse, *Symbolic dynamics*, Amer. J. Math. **60** (1938) 815-866.

[26] G.A. Hedlund and M. Morse, *Symbolic dynamics, part II: Sturmian trajectories*, Amer. J. Math. **62** (1940) 1-42.

[27] D. Hilbert, *Über die stetige Abbildung einer Linie auf ein Flächenstück*, Mathematische Annalen **38** (1891), 459-460. *Gesammelte Abhandlungen* Vol III, 1-2. , Springer-Verlag, Berlin, 1935.

[28] S. Ito and M. Kimura, *On the Rauzy fractal*, Japan J. Indust. Appl. Math., **8**, (1991), 461-486.

[29] S. Ito, *Weyl automorphisms, substitutions and fractals* in *Stability theory and related topics in dynamical systems (Nagoya, 1988)*, 60-72, World Sci. Adv. ser. Dyn. Syst., 6, World Sci. Publishing, Teaneck, NJ, 1989.

[30] J.P. Kahane, *Mesures et dimensions* in *Turbulence and Navier-Stokes equation* (Proc. Conf. Univ. Paris-Sud, Orsay, 1975), Lectures Notes in Mathematics, vol. 565, Springer-Verlag, 1976, Berlin.

[31] D.E. Knuth, *Fibonacci multiplication*, Appl. Math. Lett.**1** (1988), 57-60.

[32] H. Lebesgue, *Leçons sur l'integration*, 1928, Paris.

[33] M. Morse, *Recurrent geodesics on a surface of negative curvature*, Trans. Amer. Math. Soc., **22**, (1921), 719-721.

[34] G. Peano, *Sur une courbe qui remplit toute une aire plane*, Mathemmatische Annalen **36** (1890), 157-160.

[35] G. Peano, *La Curva di Peano Nel 'Formulario Mathematico'*, Formulario Mathematico, vol 5, 1908, Frates Bocca, Turin, pgs. 239-240.

[36] M. Queffelec, *Substitution Dynamical Systems -Spectral Analysis*, Lecture Notes in Mathematics, Vol 1294,Springer-Verlag, 1987, Berlin.

[37] G. Rauzy, *Une généralisation du développement en fraction continue*, Sèminaire Delange-Pisot-Poitou (Théorie des nombres) (1976-1977) **15**.

[38] G. Rauzy, *Nombres algébriques et substitutions*, Bull. Soc. Math. France **110** (1982), 147-178.

[39] G. Rauzy, *Suites à termes dans un alphabet fini*, Sèminaire de théorie des nombres de Bordeaux **25** (1982-1983), 1-16.

[40] G. Rauzy, *Rotations sur les groupes, nombres algébriques et substitutions*, pre-print.

[41] G. Rauzy, *Mots infinis en arithmètique* in *Automata on Infinite Words*, Lecture Notes in Computer Science **192**, Springer, Berlin, 1985, 165-171.

[42] G. Rauzy, *Sequences Defined by iterated morphisms*, in *Sequences*(Napoles/Positano, 1988), Lectures Notes in Computer Science, Springer,New York, 1990, 275-286.

[43] C. Series, *The Geometry of Markoff numbers*, Math. Intelligencer **7** (1985), 20-29.

[44] I. J. Schoenberg, *On the Peano curve of Lebesgue*, Bull. of the Amer. Math. Soc., **44**, (1938), 519.

[45] A. Thue, *Über unendliche Zeinchenreihen*, Norske vid. Selsk. Skr. I. Mat. Nat. Kl., Christiana, **7**, (1906), 1-22.

[46] A. Thue, *Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen*, Norske vid. Selsk. Skr. I. Mat. Nat. Kl., Christiana,**1**, (1912), 1-67.

[47] W.P. Thurston, *Groups, Tilings and Finite State Automata*, Lecture notes, Coll. Lect. Amer. Math. Soc. (1989), Boulder, Colorado.

[48] E. Zeckendorf, *Représentation des nombres naturals par une somme des nombres de Fibonacci ou de nombres de Lucas*, Bull. Soc. Roy. Sci. Liège **3-4**, 1972, 179-182.