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Bate, Michael orcid.org/0000-0002-6513-2405, Martin, Benjamin and Roehrle, Gerhard (2016) On a question of Külshammer for representations of finite groups in reductive groups. Israel J. Math. pp. 463-470.

https://doi.org/10.1007/s11856-016-1337-2

ON A QUESTION OF KÜLSHAMMER FOR REPRESENTATIONS OF FINITE GROUPS IN REDUCTIVE GROUPS

MICHAEL BATE, BENJAMIN MARTIN, AND GERHARD RÖHRLE

To Burkhard Külshammer on his sixtieth birthday

Abstract. Let G be a simple algebraic group of type G_2 over an algebraically closed field of characteristic 2. We give an example of a finite group Γ with Sylow 2-subgroup Γ_2 and an infinite family of pairwise non-conjugate homomorphisms $\rho\colon \Gamma \to G$ whose restrictions to Γ_2 are all conjugate. This answers a question of Burkhard Külshammer from 1995. We also give an action of Γ on a connected unipotent group V such that the map of 1-cohomologies $\mathrm{H}^1(\Gamma, V) \to \mathrm{H}^1(\Gamma_p, V)$ induced by restriction of 1-cocycles has an infinite fibre.

1. Introduction

Let k be an algebraically closed field and let Γ be a finite group. By a *representation* of Γ in a linear algebraic group H over k, we mean a group homomorphism from Γ to H. We denote by $\mathrm{Hom}(\Gamma, H)$ the set of representations ρ of Γ in H; this has the natural structure of an affine variety over k (see, e.g., [11, II.2]). The group H acts on $\mathrm{Hom}(\Gamma, H)$ by conjugation and we call the orbits $H \cdot \rho$ *conjugacy classes*.

If either $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) = p > 0$ and $|\Gamma|$ is coprime to p, then every representation of Γ in $\mathrm{GL}_n(k)$ is completely reducible and $\mathrm{Hom}(\Gamma, \mathrm{GL}_n(k))$ is a finite union of conjugacy classes, by Maschke's Theorem. Now suppose that $\mathrm{char}(k) = p > 0$ and p divides $|\Gamma|$. It is no longer true that $\mathrm{Hom}(\Gamma, \mathrm{GL}_n(k))$ is a finite union of conjugacy classes—for example, this fails even for $n = 2$ and $\Gamma = C_p \times C_p$ (cf. the last paragraph of the proof of Theorem 1.2 below). Let Γ_p be a Sylow p-subgroup of Γ. It is natural to ask instead whether representations of Γ are controlled by their restrictions to Γ_p. Burkhard Külshammer raised the following question in 1995 in [5, Sec. 2] (see also [11, I.5]).

Question 1.1. Let G be a linear algebraic group and let $\sigma \in \mathrm{Hom}(\Gamma_p, G)$. Are there only finitely many conjugacy classes of representations $\rho \in \mathrm{Hom}(\Gamma, G)$ such that $\rho|_{\Gamma_p}$ is conjugate to σ?

Straightforward representation-theoretic arguments show that the answer is yes if $G = \mathrm{GL}_n(k)$ (see [5, Sec. 2]). On the other hand, an example of Cram with $p = 2$ shows that the answer is no in general if we allow G to be non-connected and non-reductive [4].

For the rest of this paper, we assume G is connected and reductive. Slodowy proved that the answer to Question 1.1 is yes under some extra hypotheses [11]; we briefly summarise his results. If one embeds G in some $\mathrm{GL}_n(k)$, then $\mathrm{Hom}(\Gamma, G)$ embeds in $\mathrm{Hom}(\Gamma, \mathrm{GL}_n(k))$. Given $\rho \in \mathrm{Hom}(\Gamma, G)$, the set $(\mathrm{GL}_n(k) \cdot \rho) \cap \mathrm{Hom}(\Gamma, G)$ splits into a union of G-conjugacy

2010 *Mathematics Subject Classification.* 20C20 (20G15).

Key words and phrases. Modular representations of finite groups; reductive algebraic groups; conjugacy classes; nonabelian 1-cohomology.

classes; in the first part of his paper, Slodowy applies a beautiful geometric argument due to Richardson [8] to show that this union is finite when p is good for G, which allows one to deduce a positive answer to Question 1.1 for G from the positive answer for $\mathrm{GL}_n(k)$ [11, I.5, Thm. 3].

The second part of Slodowy's paper gives a different criterion for Question 1.1 to have positive answer: he shows that if $\sigma(\Gamma_p)$ has reduced centralizer in G then there are only finitely many conjugacy classes of representations $\rho \in \mathrm{Hom}(\Gamma, G)$ such that $\rho|_{\Gamma_p}$ is conjugate to σ [11, II.4, Cor. 1]. An important ingredient in this proof, which dates back to work of André Weil, is that one can interpret elements of the tangent space to $\mathrm{Hom}(\Gamma, G)$ at ρ as elements of the space of 1-cocycles $\mathrm{Z}^1(\Gamma, \mathfrak{g})$, where $\mathfrak{g}$ denotes the Lie algebra of G and Γ acts on $\mathfrak{g}$ by $\gamma \cdot X = \mathrm{Ad}(\rho(\gamma))(X)$. In fact, Slodowy proved a more general finiteness criterion in terms of the "inseparability defects" of ρ and $\rho|_{\Gamma_p}$ [11, II.4, Thm. 2][1]. The case of arbitrary connected reductive G was, however, still left open.

In this note we show that the answer to Question 1.1 is no in general for connected reductive G. We prove the following result.

Theorem 1.2. *Suppose G is a simple algebraic group of type G_2 and $\mathrm{char}(k) = 2$. Let $q > 3$ be odd, let D_{2q} denote the dihedral group of order $2q$, let $\Gamma = D_{2q} \times C_2 = \langle r, s, z \mid r^q = s^2 = z^2 = 1, srs^{-1} = r^{-1}, [r, z] = [s, z] = 1\rangle$ and let $\Gamma_2 = \langle s, z\rangle$ (a Sylow 2-subgroup of Γ). Then there exist representations $\rho_a \in \mathrm{Hom}(\Gamma, G)$ for all $a \in k$ such that the ρ_a are pairwise non-conjugate and the restrictions $\rho_a|_{\Gamma_2}$ are conjugate for all $a \in k$.*

In [2, Sec. 7] the authors and Tange constructed families of finite subgroups of $G = G_2$ in characteristic 2 with unusual properties (note, for example, that [2, Ex. 7.15] shows that Richardson's argument can fail in bad characteristic). Our proof of Theorem 1.2 involves a modification of this construction.

Our results can be interpreted in the language of nonabelian 1-cohomology (see Section 3). Let Γ act by group automorphisms on a unipotent group V. One can form the 1-cohomology $\mathrm{H}^1(\Gamma, V)$, and the inclusion of Γ_p in Γ gives a map Θ from $\mathrm{H}^1(\Gamma, V)$ to $\mathrm{H}^1(\Gamma_p, V)$ induced by restriction of 1-cocycles.

Theorem 1.3. *Let $p = 2$, let $q > 3$ be odd and let $\Gamma = D_{2q} \times C_2$. There is an action of Γ on a connected unipotent group V such that the map Θ has an infinite fibre.*

This is in sharp contrast to the case when V is abelian: standard results from abelian cohomology (cf. [3, III, Prop. 10.4]) show that if V is an abelian unipotent group (e.g., a finite-dimensional vector space over k) on which Γ acts by group automorphisms then Θ is injective. In fact, Slodowy uses precisely this result in the special case when V is the Γ-module $\mathfrak{g}$ on the way to proving [11, II.4, Thm. 2] (see [11, II.4, Lem.]).

Lond gave a different example with Θ having an infinite fibre [7, Ex. 4.1], using the example of Cram discussed above. In our case, the group V is the unipotent radical of a parabolic subgroup P of a simple group G of type G_2, and Γ acts on V by conjugation, via a homomorphism $\sigma\colon \Gamma \to P$. Theorem 1.3 follows quickly from the construction in Section 2 (see Section 3).

[1] This result actually holds for non-reductive G as well.

2. Proof of Theorem 1.2

Until the end of this section we take G to be a simple algebraic group of type G_2 and $\mathrm{char}(k)$ to be 2. We recall some notation from [2, Sec. 7]. The positive roots of G with respect to a fixed maximal torus T and a fixed Borel subgroup containing T are α (short), β (long), $\alpha+\beta$, $2\alpha+\beta$, $3\alpha+\beta$ and $\omega := 3\alpha+2\beta$. Given a root δ, we denote the corresponding root group by U_δ and coroot by $\delta^\vee$. We fix a group isomorphism $\kappa_\delta \colon k \to U_\delta$. We write G_δ for $\langle U_\delta \cup U_{-\delta}\rangle$ and we set $s_\delta = \kappa_\delta(1)\kappa_{-\delta}(1)\kappa_\delta(1)$; then s_δ represents the reflection corresponding to δ in the Weyl group of G (since $\mathrm{char}(k) = 2$, s_δ has order 2).

Fix $t \in \alpha^\vee(k^*)$ such that $|t| = q$. For $a \in k$, define $\rho_a \in \mathrm{Hom}(\Gamma, G)$ by

$$\rho_a(r) = t, \quad \rho_a(s) = s_\alpha\kappa_\omega(a), \quad \rho_a(z) = \kappa_\omega(1).$$

It is easily checked that this is well-defined (note that $[G_\alpha, G_\omega] = 1$). Set $u(x) = \kappa_\beta(x)\kappa_{3\alpha+\beta}(x)$ for $x \in k$. Then $u(x)$ commutes with U_ω and $u(x)s_\alpha u(x)^{-1} = s_\alpha\kappa_\omega(x^2)$ (see the first paragraph of [2, p. 4307]). It follows that $u(\sqrt{a})\cdot(\rho_0|_{\Gamma_2}) = \rho_a|_{\Gamma_2}$.

To complete the proof of Theorem 1.2, we now need to show that the ρ_a are pairwise non-conjugate. Let $a, b \in k$ and suppose $g\cdot\rho_a = \rho_b$ for some $g \in G$. Then $g \in C_G(t)$. It follows from [2, (7.1) and (7.2)] that $C_G(t) = TG_\omega$ (this is where we need our assumption that $q > 3$; cf. [2, (7.7)]). So write $g = hm$ with $h \in T$ and $m \in G_\omega$. We have $(hm)s_\alpha\kappa_\omega(a)(hm)^{-1} = s_\alpha\kappa_\omega(b)$, so $hs_\alpha h^{-1}(hm)\kappa_\omega(a)(hm)^{-1} = s_\alpha\kappa_\omega(b)$ since m commutes with s_α. Now $G_\alpha \cap G_\omega = 1$ (see the paragraph following [2, (7.8)]), so the condition $hs_\alpha h^{-1}(hm)\kappa_\omega(a)(hm)^{-1} = s_\alpha\kappa_\omega(b)$ forces h to commute with s_α, as $hs_\alpha h^{-1} \in G_\alpha$ and $(hm)\kappa_\omega(a)(hm)^{-1} \in G_\omega$. A simple calculation now shows that $h \in \ker(\alpha) \subseteq G_\omega$. Hence $g \in G_\omega$. But G_ω is a simple group of type A_1, so the pair $(\kappa_\omega(a), \kappa_\omega(1))$ is not G_ω-conjugate to the pair $(\kappa_\omega(b), \kappa_\omega(1))$ unless $a = b$. We conclude that ρ_a and ρ_b are not conjugate if $a \neq b$, as required.

Remarks 2.1. (i). Choose an embedding i of G in some $\mathrm{GL}_n(k)$. Then the representations $i\circ\rho_a$ of Γ in $\mathrm{GL}_n(k)$ fall into finitely many $\mathrm{GL}_n(k)$-conjugacy classes, since Question 1.1 has positive answer for $\mathrm{GL}_n(k)$. Hence there exists $a \in k$ such that $(\mathrm{GL}_n(k)\cdot\rho_a) \cap \mathrm{Hom}(\Gamma, G)$ is an infinite union of G-conjugacy classes. This gives another example of the phenomenon in [2, Ex. 7.15] discussed above.

(ii). It follows from Slodowy's result [11, II.4, Thm. 2] discussed above that ρ_a has greater inseparability defect than $\rho_a|_{\Gamma_2}$ for at least one $a \in k$. In fact, it can be shown using the calculations in [2, Sec. 7] that if $a \neq 0$ then ρ_a has inseparability defect 1 and $\rho_a|_{\Gamma_2}$ has inseparability defect 5. This answers a question of Slodowy [11, II.4, Rem. 2].

We do not know of any analogous examples in odd characteristic; recall from the discussion in Section 1 that if such an example exists then p must be bad for G. Our construction is closely related to the construction of a certain triple (G, M, H) in [2, Sec. 7], where $G = G_2$, M is a reductive subgroup of G and H is a finite subgroup of M. We guess that further examples can be obtained from other triples (G, M, H) with similar properties, but we leave this for future work. The mechanism for producing these triples works only in characteristic 2 (see the paragraph following [15, Rem. 1.6]). Uchiyama found triples (G, M, H) for G of type E_7 [15, Sec. 3], and showed that the construction fails for several cases involving groups of rank at most 6, including A_3, A_4, B_3 and E_6 [14, Thm. 3.1.1, Ch. 4].

It seems an interesting problem to find examples like that of Cram [4] but in odd characteristic, where we allow G to be non-reductive.

3. Nonabelian 1-cohomology

Another approach to Külshammer's problem is via the 1-cohomology of the unipotent radical $R_u(P)$, where P is a proper parabolic subgroup of G. Here is a brief explanation. Recall that a closed subgroup M of G is said to be *G-completely reducible* if whenever M is contained in a parabolic subgroup P of G, M is contained in some Levi subgroup of P [10], [9]. As a special case, we say that M is *G-irreducible* if M is not contained in any proper parabolic subgroup of G at all. We say that $\rho \in \mathrm{Hom}(\Gamma, G)$ is G-completely reducible (resp., G-irreducible) if its image is.

Although in general $\mathrm{Hom}(\Gamma, G)$ is an infinite union of conjugacy classes for reductive G, it was proved in [1, Cor. 3.8] that there are only finitely many conjugacy classes of representations that are G-completely reducible. This generalizes the classical result that a finite group admits only finitely many completely reducible n-dimensional representations in any characteristic. Moreover, it follows from [1, Cor. 3.7] that the conjugacy classes of G-completely reducible representations of Γ in G are precisely the conjugacy classes that are Zariski-closed subsets of $\mathrm{Hom}(\Gamma, G)$. Given $\rho \in \mathrm{Hom}(\Gamma, G)$, choose a minimal parabolic subgroup P of G with $\rho(\Gamma) \subseteq P$. Let L be a Levi subgroup of P and let $\pi \colon P \to L$ be the canonical projection. It follows from [1, Cor. 3.5] that $\sigma := \pi \circ \rho \in \mathrm{Hom}(\Gamma, L)$ is L-irreducible and G-completely reducible. Conversely, given G-irreducible $\sigma \in \mathrm{Hom}(\Gamma, G)$, we can consider the set C_σ of all $\rho \in \mathrm{Hom}(\Gamma, P)$ such that $\pi \circ \rho = \sigma$. By the result described in the first sentence of this paragraph, there are only finitely many possibilities for (P, L, σ) up to G-conjugacy. Hence if $C \subseteq \mathrm{Hom}(\Gamma, G)$ is an infinite union of G-conjugacy classes then for some triple (P, L, σ), C_σ must meet infinitely many G-conjugacy classes in C. Thus we have reduced the "global" problem of considering all representations into G to the "local" problem of considering all representations into a fixed proper parabolic subgroup P.

Next we study the structure of C_σ for fixed (P, L, σ). Let $V = R_u(P)$. Given $\rho \in C_\sigma$, there is a unique function $\theta_\rho \colon \Gamma \to V$ defined by $\rho(\gamma) = \theta_\rho(\gamma)\sigma(\gamma)$. It is easily checked that θ_ρ satisfies the 1-cocycle relation $\theta_\rho(\gamma_1\gamma_2) = \theta_\rho(\gamma_1)(\gamma_1 \cdot \theta_\rho(\gamma_2))$, where Γ acts on V by $\gamma \cdot v = \sigma(\gamma) v \sigma(\gamma)^{-1}$. The converse is also true, so we have a bijection between C_σ and the space of 1-cocycles $\mathrm{Z}^1(\Gamma, \sigma, V)$. A simple calculation shows that $\rho, \mu \in C_\sigma$ are V-conjugate if and only if the images $\overline{\theta_\rho}$ of θ_ρ and $\overline{\theta_\mu}$ of θ_μ in $\mathrm{H}^1(\Gamma, \sigma, V)$ are equal. Thus we have an interpretation of V-conjugacy classes in C_σ in terms of 1-cohomology (cf. the proof of [11, I.5, Lem. 1]).

This idea has been used in a slightly different context to study embeddings of reductive algebraic groups inside simple algebraic groups [6], [12], [13], [7]. In our case we have an extra ingredient arising from restriction of representations. The restriction map from $\mathrm{Hom}(\Gamma, G)$ to $\mathrm{Hom}(\Gamma_p, G)$ maps C_σ to $C_{\sigma|_{\Gamma_p}}$. Restriction of cocycles gives a map from $\mathrm{Z}^1(\Gamma, \sigma, V)$ to $\mathrm{Z}^1(\Gamma_p, \sigma|_{\Gamma_p}, V)$ which is compatible with the correspondence between representations and 1-cocycles, and this descends to give a map Θ from $\mathrm{H}^1(\Gamma, \sigma, V)$ to $\mathrm{H}^1(\Gamma_p, \sigma|_{\Gamma_p}, V)$. See [7, Ch. 3–4] for a fuller explanation.

Now we recast our example in this language. Let G, k, Γ, Γ_2 and the ρ_a be as in Section 2. Set $P = P_\alpha$, $L = L_\alpha$ and $V = R_u(P_\alpha)$, and define $\sigma \in \mathrm{Hom}(\Gamma, L)$ by $\sigma(r) = t$, $\sigma(s) = s_\alpha$ and $\sigma(z) = 1$. Then σ is L-irreducible and every ρ_a belongs to C_σ. Let $\theta_a \in \mathrm{Z}^1(\Gamma, \sigma, V)$ and $\theta_a' \in \mathrm{Z}^1(\Gamma_2, \sigma|_{\Gamma_2}, V)$ be the 1-cocycles corresponding to ρ_a and $\rho_a|_{\Gamma_2}$, respectively. The calculations in Section 2 show that the $\rho_a|_{\Gamma_2}$ are pairwise V-conjugate, so the 1-cohomology classes $\overline{\theta'_a} \in \mathrm{H}^1(\Gamma_2, \sigma|_{\Gamma_2}, V)$ are equal for all $a \in k$. In contrast, no two of the ρ_a are V-conjugate (since no two are G-conjugate), so the 1-cohomology classes $\overline{\theta_a} \in \mathrm{H}^1(\Gamma, \sigma, V)$ are all different. Thus we have an example where the map Θ from $\mathrm{H}^1(\Gamma, \sigma, V)$ to $\mathrm{H}^1(\Gamma_2, \sigma|_{\Gamma_2}, V)$ has an infinite fibre (cf. [7, Ex. 4.1]).

We do not know of any analogous examples in odd characteristic; cf. the discussion at the end of Section 2.

Acknowledgments: The authors acknowledge the financial support of EPSRC Grant EP/L005328/1, Marsden Grants UOC1009 and UOA1021, and the DFG-priority programme SPP1388 "Representation Theory". We are grateful to the referee for helpful suggestions.

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Department of Mathematics, University of York, York YO10 5DD, United Kingdom
E-mail address: `michael.bate@york.ac.uk`

Department of Mathematics, University of Aberdeen, King's College, Fraser Noble Building, Aberdeen AB24 3UE, United Kingdom
E-mail address: `b.martin@abdn.ac.uk`

Fakultät für Mathematik, Ruhr-Universität Bochum, D-44780 Bochum, Germany
E-mail address: `gerhard.roehrle@rub.de`