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The Dynamics and Equilibria of Day-to-day Assignment Models

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Abstract: Traffic network modelling is a field that has developed over a number of decades, largely from the economics of predicting equilibria across route travel choices, in consideration of the congestion levels on those routes. More recently, there has been a growing influence from the psychological and social science fields, leading to a greater interest in understanding behavioural mechanisms that underlie such travel choice decisions. The purpose of the present paper is to describe mathematical models which aim to reflect day-to-day dynamic adjustments in route choice behaviour in response to previous travel experiences. Particularly, the aim is to set these approaches in a common framework with the conventional economic equilibrium models. Starting from the analysis of economic equilibria under perturbations, the presentation moves onto deterministic dynamical system models and stochastic processes. Simple illustrative examples are used to introduce the modelling approaches. It is argued that while such dynamical approaches have appeal, in terms of the range of adaptive behavioural processes that can be incorporated, their estimation may not be trivial. In particular, the obvious solution technique (namely, explicit simulation of the dynamics) can lead to a rather complex problem of interpretation for the model-user, and that more "analytical" approximation techniques may be a better way forward.

1 Introduction: The Roots of Equilibrium

Network equilibrium models have a long history, both in the transport research literature and in practical studies of scheme assessment. In particular, the class of static equilibrium models (as described, for example, in [1]) have been seen to be versatile, with extensions able to represent perceptual variations across the driver population, mixed vehicle types, and some mixed mode operations.

In the 1980s, with a growing international interest in real-time information and control systems, serious reservations first began to be voiced about the assumptions of these approaches, and in particular their inability to deal with many of the dynamic aspects of traffic systems. Initially, the predominant focus was on efforts to generalise the equilibrium model to represent *within-day dynamics*: that is to say, the dependence of route choice on departure time, while accounting for the proper temporal and spatial evolution of congestion across the network. Subsequently, more fundamental questions began to be asked about the behavioural basis of the equilibrium paradigm, particularly in the sense of its usefulness for representing drivers' response to exogenous information. This led to a growing interest in *day-to-day dynamics*, whereby the evolution over days of travel choices and traffic congestion is linked through a learning model based on drivers' past experiences. The premise of equilibrium had been at the heart of transport modelling for many decades. The interest in day-to-day dynamics led to modelling approaches which, for the first time, constituted a serious alternative to the established approach.

The particular interest of the present paper is the study of such day-to-day dynamics. We do not consider within-day dynamic issues, although the day-to-day framework is highly suited to such problems (see, for example, [2]). Indeed, one of the great attractions of the day-to-day approach is its great flexibility, allowing a wide range of behavioural rules, levels of aggregation and types of traffic model to be integrated within the same modelling framework. The neglecting of within-day dynamic issues here is not intended to question their importance. Rather, it is our belief that it is essential to first gain the maximum understanding from simpler within-day static models, since (as we shall illustrate in the present paper) the behaviour of the resulting day-to-day models may already be rather complex.

Many general contrasts may readily be made between the day-to-day dynamic and traditional equilibrium approaches. For example, the estimation of equilibrium has been typically achieved through the solution of some optimization, complementarity or variational inequality problem, yet this makes the approach rather restrictive in terms of the generalisations that are feasible. The contrasting flexibility of the day-to-day dynamic approach has already been noted above. On the other hand, given a suitably constructed equilibrium solution algorithm, the interpretation of model outputs is typically straightforward, whereas in simulations of the day-to-day approach the interpretation stage is often the most challenging task.

However, perhaps the greatest contrast to be made is the fundamental basis of each approach. In traditional equilibrium models, the underlying hypothesis is the very notion of a market in equilibrium: a typically isolated, 'self consistent' state of the network which, if attained, would persist under certain rational rules of behaviour. In the day-to-day approach, the underlying belief is in the behavioural dynamics, namely how the behaviour on day n is affected by behaviour and the state of the network on days $n - 1$ and earlier. It is therefore tempting to regard the two approaches as competing philosophies. Yet the classical preliminary analyses of dynamical approaches involves examining the 'equilibrium' behaviour of the system. While the concept of equilibrium of a dynamical system does not relate directly to the traditional economic equilibria of traffic network models, there are many important links between the two approaches. The purpose of the present paper is to explore these links, as well as contrasting the notions of 'equilibrium' that arise.

The structure of the paper is as follows. In section 2, the notion of an economic equilibrium (the traditional one adopted in the traffic assignment literature) is defined. It is illustrated that for general problems, even such models cannot avoid dynamics, in the sense that an additional notion of 'stability' is needed to identify sensible equilibrium predictions. This leads to the introduction of explicit dynamical models, with the focus in section 3 on deterministic process models, for both continuous and discrete time. It is shown how convergent fixed points of such models may be related to economic equilibria, but that the issue of such convergence is critically related to behavioural and traffic parameters of the model. The paper culminates in section 4 with the most general kind of model, namely the stochastic process, which incorporates variability of the transient kind as well as variability "in equilibrium". However, it is seen that

this generality comes at a price, with many complex issues of interpretation of model outputs then arising. Finally, in section 5, paths for further research are discussed.

2   Equilibrium and Perturbations

The focus in this paper will be on the relationship between day-to-day models and the two most well-known concepts of economic equilibrium in transport networks. These latter two concepts are first defined below.

Suppose that:

- the inter-zonal (origin-destination) movements are labelled $1, 2, ..., W$, with the corresponding inter-zonal demand flows denoted $q_1, q_2, ..., q_W$

- the flow on link a is denoted v_a $(a = 1, 2, ..., A)$, with $\mathbf{v}$ the A-vector of network link flows

- the travel cost on link a is $t_a(\mathbf{v})$ $(a = 1, 2, ..., A)$, a function of the link flows

- the flows on the N acyclic routes across all inter-zonal movements are denoted $f_1, f_2, ..., f_N$, and are contained in an N-vector $\mathbf{f}$

- the induced travel cost on route r, as a function of the route flows $\mathbf{f}$, is denoted $c_r(\mathbf{f})$, held in an N-vector $\mathbf{c}(\mathbf{f})$, where $c_r(\mathbf{f}) = \sum_{a=1}^{A} \delta_{ar} t_a(\mathbf{V}(\mathbf{f}))$ $(r = 1, 2, ..., N)$, where $\mathbf{V}(\mathbf{f})$ has elements $V_a(\mathbf{f}) = \sum_{r=1}^{N} \delta_{ar} f_r$ $(a = 1, 2, ...A)$, and where δ_{ar} is a 0/1 indicator equal to 1 only if link a is part of route r

- $\mathcal{R}_w$ is the index set of acyclic routes corresponding to inter-zonal movement w, for $w = 1, 2, ..., W$.

Then define the following two equilibrium models:

Deterministic User Equilibrium (DUE) Following [3], for example, the flow vector $\mathbf{f}$ is a DUE if and only if it is both demand-feasible, i.e. $\sum_{r \in \mathcal{R}_w} f_r = q_w$ $(w = 1, 2, ..., W)$, and all used routes have minimum cost at the equilibrium flow levels:

$$f_r > 0 \Rightarrow c_r(\mathbf{f}) \leq c_s(\mathbf{f}) \quad (r, s \in \mathcal{R}_w; \ w = 1, 2, ...W) . \tag{1}$$

Stochastic User Equilibrium (SUE) Suppose additionally that $\mathbf{p}(\mathbf{u})$ is a function with elements $p_r(\mathbf{u})$ ($r = 1, 2, ...N$) denoting the proportion of drivers on each movement w who would choose route $r \in \mathcal{R}_w$ when the route costs are $\mathbf{u}$. Then, following [4], the flow $\mathbf{f}$ is said to be an SUE if and only if:

$$f_r = q_w p_r(\mathbf{c}(\mathbf{f})) \quad (r \in \mathcal{R}_w; \ w = 1, 2, ...W) \, . \tag{2}$$

Typical functional forms for the function $\mathbf{p}(.)$ are based on random utility theory, such as the logit and probit models [1].

Clearly any model is a simplification of reality, and its value should be judged on the basis of a comparison of its predictions with observed data. However, even before this step, this is a more basic question of plausibility of the model, in respect of whether it is well-defined. In particular, for any given network data:

- Do SUE/DUE solutions exist?

- Is there a unique SUE/DUE solution?

It transpires that provided no strict capacity constraints are additionally imposed, existence of SUE/DUE predictions can be guaranteed under rather mild conditions, primarily involving continuity of the involved functions $\mathbf{t}(.)$ and $\mathbf{p}(.)$ ([3]; [2]).

We turn attention then to the question of uniqueness of such equilibrium predictions. Regarding the DUE model, it is simple to show (e.g. in a figure-of-eight network) that seeking uniqueness of *route* flows is not a sensible goal in general. In practically all realistic networks there exist linear combinations of the route flows that leave link flows and costs unchanged, while still satisfying the DUE conditions. Therefore, it is usual to refer to uniqueness questions in terms of the induced DUE *link* flows. While SUE route flows do not generally suffer from the same linear combination problem, for consistency we may also refer to uniqueness of SUE in the link flow domain. It transpires, then, that the discussion below for DUE has a parallel for SUE (see [5] for details), and so below we shall restrict attention to the simpler DUE case.

Regarding DUE link flows, then, Beckmann *et al.* [6] established uniqueness of the DUE solution for *separable problems* - that is, when the travel cost on a link depends, in a continuous and

strictly increasing way, on the flow on that link only. However, due to many factors such as junction interactions, this requirement is too restrictive for many practical cases[1]. The most general known sufficient condition for uniqueness is that the vector of cost functions be strictly monotone [3], though this still only permits mild interactions between links [7]. Indeed, simple examples are readily constructed to illustrate the potential for multiple equilibria under realistic model assumptions [8].

The question of uniqueness of DUE/SUE solutions is therefore not a simple one to classify, and is critically controlled by the form of the cost-flow relationships. Partly because of this potential for multiple equilibria, a third plausibility property (in addition to existence and uniqueness) is commonly considered, at least in the theoretical literature: *stability*. This term in fact covers a variety of subtly different conditions, but here we shall be concerned with *perturbation stability*. This property involves characterizing the "attractiveness" of an equilibrium solution following a perturbation to the flow levels, where the attractiveness is governed by certain reasonable, conservative rules regarding user behaviour. As is illustrated in the examples below, not all DUE are stable in this sense, and so the number of solutions can be reduced by restricting attention only to stable DUE.

Example 1 Consider a network consisting of a single inter-zonal movement, serving a demand of 2 flow units, connected by two parallel routes with cost functions:

$$c_1(\mathbf{f}) = 3f_1{}^n + 1 \qquad c_2(\mathbf{f}) = 2f_2 + 2 \tag{3}$$

where $n > 0$ is some positive constant. Note that the n has no significance in the analysis to follow, indeed any increasing function would suffice; the n is only included as a suggestion of generality. Regardless of the value of n, this separable problem has a unique DUE at $(f_1, f_2) = (1, 1)$. Furthermore, if from this solution, ϵ drivers ($0 < \epsilon \leq 1$) were to swap from route 1 to route 2, then (when the costs are updated) these drivers would now experience a cost that is greater than the (DUE) cost they experienced on their old route before they swapped. Hence, following any such perturbation, there would be an incentive to return back to their previous choice. A similar incentive would arise for perturbations about the DUE involving swaps the other way, from route 2 to route 1. Therefore, in view of both types of perturbation, the DUE

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[1]Incidentally, it is noted in passing that when within-day dynamic traffic interactions are additionally introduced, the resulting problem is inherently non-separable across time periods

solution can be said to be 'attractive' or 'stable'.

This latter attractiveness comment in the example above turns out to be a critical motivating factor in the work to follow in the present paper. It may be traced back to a definition of Dafermos and Sparrow [9], who referred to any DUE with such an attractiveness property (for any such pairwise route flow perturbation) as *user-optimized*. This property is interesting as it is suggestive of dynamical behaviour, namely it refers to how the traffic system may behave in disequilibrium. It transpires that for any separable problem (i.e. any network of arbitrary size with separable and increasing link cost functions), the DUE is user-optimized [10]. However, for non-separable problems, the same is not necessarily true.

Example 2 (Smith, [10]) Consider Example 1, but suppose instead that the cost functions have the form:

$$c_1(\mathbf{f}) = 3f_1 + f_2 + 1 \qquad c_2(\mathbf{f}) = 2f_1 + f_2 + 2. \tag{4}$$

The unique DUE is again $(f_1, f_2) = (1,1)$, but it is not user-optimized. For example, following a perturbation to $(f_1, f_2) = (\frac{3}{4}, \frac{5}{4})$, then the drivers who have switched to route 2 now experience a cost of $\frac{19}{4}$, which is less than the cost of 5 which they experienced on route 2 in equilibrium. So in this respect the equilibrium is not attractive, as these drivers do not appear to have an incentive to return to their old route. However, under this perturbation, the cost on route 1 changes to $\frac{18}{4}$, and so if drivers compare the costs of their new and old routes both in the perturbed situation, there *is* an incentive for drivers to switch back to route 1. Heydecker [11] referred to this alternative attractiveness property as "equilibrated".

Example 3 (Watling, [5]) Suppose now that the cost functions have the form:

$$c_1(\mathbf{f}) = f_1 + 3f_2 + 1 \qquad c_2(\mathbf{f}) = 2f_1 + f_2 + 2. \tag{5}$$

This problem has three separated DUE solutions, at $(f_1, f_2) = (2,0)$, $(1,1)$ and $(0,2)$. While none of these is either user-optimized or equilibrated, the two extremal equilibria satisfy these conditions for perturbations within an open neighbourhood of each equilibrium (i.e. provided the flow-swaps are not too large). Therefore, they may be classified as *locally* attractive. However, this is not the case for the DUE at $(1,1)$, even in a local sense. Therefore, it is difficult to justify (1,1) as a potential long-term persistent state of the system.

Summary The appeal of conventional equilibrium analysis is often claimed to be its very simplicity, in the sense that it does not concern itself with issues of dynamical adjustments and disequilibrium. '[Equilibrium] has the great attraction that it is not necessary to consider *how* or indeed *when* decisions are taken' [12]. This claim may be substantiated by the perturbation analysis of separable problems, since all such equilibria are automatically stable. However, for *non-separable* problems, the same is not true: a point definition of equilibrium is not on its own sufficient to identify sensible, attractive states. This is highly relevant as many practical problems are non-separable (e.g. interactions across links or across time periods). Although the focus above has been on DUE examples, analogous concepts and examples exist for SUE [5]. The perturbation-type approaches conventionally used to classify stability of traditional economic equilibria make implicit dynamical assumptions, meaning that the issue of dynamical adjustments is concealed rather than avoided. Therefore, it is claimed, explicit dynamical systems are more sensible as a foundation for network modelling, as we shall begin to consider in the following section.

3   Deterministic Dynamical Systems

Dynamical system models are concerned with describing the evolution over all time of some process. They may be broadly classified into deterministic and stochastic processes. In this section we consider the former class of approaches, characterised by the fact that the evolution of the system is assumed to be precisely determined once the initial conditions are specified. Smith [13], Friesz *et al.* [14] and Zhang and Nagurney [15] each proposed continuous time deterministic processes, in order to examine the evolution towards DUE. While these analyses provide some suggestive information on the links between the dynamical and equilibrium approaches, they suffer from two significant limitations:

1. They assume revisions to a route may be made in *continuous* time, whereas in reality repeated trips are subject to activity constraints that mean they are not continuously adjustable in time (i.e. over days). Therefore, a more appropriate representation would seem to be a discrete time system, with each discrete time epoch representing a lag

over which trips may be repeated, e.g. days or weekdays[2]. Critically, this allows an interpretation to be given to the *rate* of adjustment, which Horowitz [16] showed to be a critical factor in both the transient and long-term evolution of the system. On the other hand, a continuous time analysis only allows identification of equilibria that *could* be stable, given a sufficiently slow rate of adjustment (see [17] for further elaboration on this issue).

2. The study of dynamical approaches to DUE is inherently problematic, due its assumption of a homogeneous population. In particular, some additional modelling device is needed to disperse drivers among alternative routes for a given inter-zonal movement, which is not achieved naturally if they are all aiming to minimise the same travel cost by the same adjustment process. This dispersion across routes is particularly critical given that at the DUE state itself, these drivers need to be on minimal and equal cost routes. In the dynamical models cited above, this dispersion is effectively built into the process by defining the transient stage as something like a perturbed version of the DUE conditions. In terms of philosophy, then, these approaches could be said to be somewhat closer to the equilibrium paradigm, whereas in 'pure' dynamical models the behavioural adjustments are the defining feature.

We therefore turn attention to discrete time systems, and in particular those that have a relationship with SUE[3]. Models of this kind were first considered for two-link networks by Horowitz [16] and Cantarella [18], and the analysis was more recently extended to general networks by Cantarella and Cascetta [2] and Watling [17]. Horowitz in particular considered a range of possible 'learning models', describing the way in which drivers may integrate accumulated past knowledge with their most recent experience. These included models where the weight given to the most recent experience may vary during the dynamical process, perhaps decreasing in proportion to accumulation of information, or decreasing at a higher rate to reflect habit/inertia.

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[2]Looking forward to the section on stochastic systems, if one also includes in such a discrete time system some probabilistic mechanism by which a "no-travel" option is selected, potentially varying across the population, then there is no requirement that all trips actually be made at the same regular time interval.

[3]In fact this approach is rather general, as it is well-known that DUE may be approximated to an arbitrary accuracy by an SUE model, by allowing the perceptual dispersion across drivers to become very small; this corresponds to $\theta \to \infty$ in the logit examples given later in the text. However, as we shall see, the perceptual dispersion itself has a critical role to play in the attractiveness of equilibrium.

Here, for illustration, we consider a rather simpler form of system, following [17]:

$$
\begin{aligned}
\mathbf{C}^{(n)} &= \beta \mathbf{c}(\mathbf{f}^{(n-1)}) + (1 - \beta)\mathbf{C}^{(n-1)} \\
f_r{}^{(n)} &= q_w p_r(\mathbf{C}^{(n)}) \quad (r \in \mathcal{R}_w;\ w = 1, 2, ...W). \quad (n = 1, 2, ...)
\end{aligned}
\tag{6}
$$

where $\mathbf{f}^{(n)}$ is the vector of day n route flows, $\mathbf{C}^{(n)}$ is the vector of mean perceived route costs at the start of day n, and $\beta > 0$ is a constant weight (independent of n). In this model, drivers' mean perceptions of travel cost are built up through an exponential-smoothing style of learning process, which involves a weighted combination of the perceived and actual costs on the previous day. Then, based on these mean perceived costs, drivers choose between the alternative routes on any one day according to $\mathbf{p}(.)$. Note that since $\mathbf{p}(.)$ is a vector of proportions (rather than probabilities), the process defined is *deterministic*. This is commonly a source of confusion with the SUE model which, in spite of its name, is essentially a deterministic model.

So-called *point equilibria* of such a system occur when $\left(\mathbf{f}^{(n)}, \mathbf{C}^{(n)}\right) = \left(\mathbf{f}^{(n-1)}, \mathbf{C}^{(n-1)}\right)$, and these are readily shown to be given by the SUE solutions. However, this does not guarantee that an SUE solution is in any sense 'attractive' or 'stable'. To study this issue, the first step is to examine the nature of the dynamical system in the neighbourhood of an equilibrium. Provided the system is sufficiently smooth to be approximated by a Taylor series expansion in the neighbourhood of the equilibrium in question, this may be achieved by examining the system Jacobian. Although the two-link example below avoids many of the mathematical complexities associated with such an analysis, it at least allows an illustration of a number of the key issues.

Example 4 The notation is simplified somewhat by considering a network with a single movement of demand 1 unit. Now, by substituting the second expression in the first in system (6), we obtain a system described purely in terms of the evolution of mean perceived route costs:

$$
\mathbf{C}^{(n)} = \beta \mathbf{c}\left(\mathbf{p}\left(\mathbf{C}^{(n-1)}\right)\right) + (1 - \beta)\mathbf{C}^{(n-1)} \quad (n = 1, 2, ...).
\tag{7}
$$

Let us suppose further that the network consists of just two parallel links/routes, with identical cost functions:

$$
c_r(\mathbf{f}) = a + b f_r{}^d \quad (r = 1, 2)
\tag{8}
$$

where a, $b > 0$ and $d > 0$ are link-independent parameters. Suppose that the route choice fractions at given route costs $\mathbf{u}$ are made according to a logit model with dispersion parameter

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$\theta > 0$, so that:

$$p_1(\mathbf{u}) = \frac{\exp(-\theta u_1)}{\exp(-\theta u_1) + \exp(-\theta u_2)} = [1 + \exp(\theta(u_1 - u_2))]^{-1} \tag{9}$$

$$p_2(\mathbf{u}) = [1 + \exp(-\theta(u_1 - u_2))]^{-1} \ . \tag{10}$$

In view of the fact that choices depend only on the *difference* in route cost, the system above may be further simplified to a one-dimensional state variable, $\tilde{C}^{(n)} = C_1{}^{(n)} - C_2{}^{(n)}$:

$$\tilde{C}^{(n)} = g\left(\tilde{C}^{(n-1)}\right) \qquad (n = 1, 2, ...) \tag{11}$$

where

$$g(x) = \beta b \left\{ (1 + \exp(\theta x))^{-d} - (1 + \exp(-\theta x))^{-d} \right\} + (1 - \beta)x \ . \tag{12}$$

Now by the symmetry of the network, the unique SUE clearly occurs at equal cost on the two alternative routes, i.e. at $\tilde{C} = 0$. This is also the only point equilibrium of the dynamical system. To investigate the attractiveness of this equilibrium, we linearise the system in the neighbourhood of $\tilde{C} = 0$. Noting that $g(0) = 0$ and differentiating g yields, after simplification, a locally linearised system of:

$$\tilde{C}^{(n)} \simeq g(0) + g'(0)\tilde{C}^{(n-1)} = \left[1 - \beta(2^{-d}db\theta + 1)\right]\tilde{C}^{(n-1)} \ . \tag{13}$$

Clearly, such a sequence will approach 0 from arbitrary starting conditions if and only if $\left|1 - \beta(2^{-d}db\theta + 1)\right| < 1$, i.e. if and only if $0 < \beta < \frac{2}{2^{-d}db\theta+1}$. Thus, increasing the steepness of the cost-flow functions (through increasing b or d) or decreasing the dispersion in perceived costs (through increasing θ) will tend to decrease the range of β values for which the equilibrium is attractive. This is a necessary and sufficient condition for the linearised system in a 'global' sense (i.e. for all initial conditions), but will also apply to the original non-linear system, though now only as a 'local' condition (within some unspecified neighbourhood of the equilibrium in question).

In particular, with $d = 1$, $\theta = 2$ and $b = 3$, this condition implies that the equilibrium is (locally) stable if $0 < \beta < \frac{1}{2}$, and unstable otherwise. This issue is readily appreciated by a numerical evaluation of system (11)/(12). For example, with $\beta = \frac{1}{4}$ and an initial cost difference

of $\tilde{C}^{(0)} = 5$, then to three decimal place accuracy the subsequent cost differences are: 3.000, 1.504, 0.448, 0.021, 0.000. That is to say, the equilibrium solution (at a cost difference of zero) is rapidly approached after only a small number of days. On the other hand, with $\beta = \frac{3}{4}$, then even when initiated with an extremely small (but non-zero) cost difference, the system rapidly appears to diverge. For example, with $\tilde{C}^{(0)} = 0.1$, the subsequent cost differences are: -0.199, 0.393, -0.743, 1.233, -1.590, 1.673, -1.679, 1.679, -1.679, ... etc. Indeed, it is relatively easily verified that, unless initialised at exactly $\tilde{C}^{(0)} = 0$, then the system is always ultimately attracted towards a periodic motion, whereby the cost difference alternates between 1.679 and -1.679. In terms of the original system (6), this relates to a "flip-flopping" flow behaviour, whereby the route flow split is alternately 78%/22% and 22%/78%.

The example above illustrates how the attractiveness of equilibrium is controlled by the rate of dynamical adjustment in drivers' learning (β in the model above), the steepness of the cost-flow relationships, and the degree of heterogeneity in drivers' route selections (θ in the model above). These are quite general properties, not specific to the example above. In general networks, analysis of the locally linearised system yields the necessary and sufficient condition that the (generally complex) eigen-values of the Jacobian, evaluated at equilibrium, are inside the unit circle. Fortunately, explicit computation of the eigen-values can be avoided for many problems (see [17] for the details).

In general, where a particular SUE is a point equilibrium of some dynamical system, there are many potential links between the SUE and dynamical approaches. It may be that this SUE is unstable in an analogous sense to that found in the perturbation Examples 2 and 3. The identification of equilibria that are intrinsically unstable, regardless of the rate of adjustment of the system (e.g. regardless of $\beta > 0$ above), may in fact be achieved by the more straightforward analysis of a continuous time approximation to the underlying discrete time system (e.g. see [17]). In such cases, where a given SUE is unstable, other SUE that are stable may exist. Alternatively (as in Example 4 above), it may be that a given SUE is stable for sufficiently slow rates of adjustment (i.e. sufficiently small β in the model above), but for larger values is attracted to a convergent behaviour that is itself dynamic, such as periodic or aperiodic motion.

All of these properties may, however, be 'local' in the sense of applying to a specific range of initial conditions. In the case of a point equilibrium, determined to be locally stable, an estimate may subsequently be made of the range of initial conditions for which the stability is valid, by returning to the original non-linear system. This range, termed a domain of attraction, may be estimated by the construction of a so-called Lyapunov function: in brief, a *scalar* function that decreases as equilibrium is approached. Although the construction of such functions in not in itself trivial, there do exist systematic procedures that result in Lyapunov functions for at least a fairly wide range of initial conditions, if not the entire state-space.

Example 5 The example here was first considered by Cascetta [19]. It consists of a single movement with two parallel routes serving a demand of 10 units, with a logit choice rule (9) with $\theta = 0.3$, and cost functions:

$$c_1(\mathbf{f}) = 0.7f_1 + 7 \tag{14}$$

$$c_2(\mathbf{f}) = \begin{cases} -8.464797f_2 + 31.9296 & (f_2 < 3.132) \\ \frac{2}{3}f_2 + \frac{10}{3} & (f_2 \geq 3.132) \end{cases} \tag{15}$$

A significant feature of this example is that the cost function on link/route 2 has both decreasing and increasing parts. This feature may seem unusual, but there are circumstances where it may indeed arise in practice. Morlok [20] describes such a case, arising from a responsive public transport service, where an increase in service frequency that is prompted by an increase in demand, can lead to a local decrease in user costs. A similar feature arise in responsive traffic signal systems, which apportion junction green-times according to some strategy based on the flows on the junction approaches. Perhaps most significantly, however, this should be considered as an example of a problem that violates the assumption of separable, increasing link cost functions. In this respect, it is not significant that the example considered violates the 'increasing' requirement, as opposed to the separability requirement. Indeed, qualitatively similar system behaviour may arise in strongly non-separable problems, arising, for example, from junction interactions or mixed mode operations [8].

The example above may be shown to have three separated SUE solutions, at flow levels on route 1 of approximately $f_1 = 3.60$, 8.40 and 9.95 (and corresponding cost differences of $c_1 - c_2 = 1.92$, -5.54 and -17.53). Now let us consider dynamical system (6), and for illustration

fix attention on a particular value of the learning weight, $\beta = 0.1$. Adopting the same kind of analysis as that used in Example 4, and linearising in turn in the neighbourhood of each SUE, it may be shown that the SUE at $f_1 = 3.60$ and 9.95 are locally stable, and the SUE at $f_1 = 8.40$ is locally unstable. Therefore, we know that at least if we initialise system (6) "sufficiently close to" either of the locally stable SUE, then the system will ultimately converge to that SUE. However, this analysis does not tell us what "sufficiently close to" means in practice: i.e. given some initial condition, which of the SUE (if any) will be attained? For example, let us suppose that the system is initialised at flow levels of $(f_1, f_2) = (6, 4)$, which equates to an initial cost difference of $\tilde{C}^{(0)} = 5.2$.

Writing our system in the form (11), then:

$$g(x) = \beta \left(c_1(f_1(x), f_2(x)) - c_2(f_1(x), f_2(x)) \right) + (1 - \beta)x \tag{16}$$

where

$$f_1(x) = 10 \left(1 + \exp(\theta x) \right)^{-1} \qquad f_2(x) = 10 - f_1(x) . \tag{17}$$

Now, locally-valid Lyapunov functions may be deduced from each of the linearisations of the stable equilibria. In this simple case, these are effectively just $V_{\text{SUE1}}(x) = (x - 1.92)^2$ and $V_{\text{SUE2}}(x) = (x + 17.53)^2$ for the SUE at cost differences of 1.92 and -17.53 respectively. These may then be applied to the underlying non-linear system. If x^* is the equilibrium in question, and if S is a bounded subset of the state-space containing x^*, then if for all $x \in S$ ($x \neq x^*$), $V(g(x)) - V(x) < 0$ then S is a *domain of attraction* for x^*. That is to say, given any initialisation of the system within S, the system will converge to the equilibrium x^*.

Figure 1 illustrates the Lyapunov difference $V_{\text{SUE1}}(g(x)) - V_{\text{SUE1}}(x)$ in the neighbourhood of the SUE at a cost difference $x = 1.92$. Note that a $x = 1.92$, the difference is exactly zero, since by definition of a point equilibrium $g(1.92) = 1.92$. For $x < 1.92$, the difference is negative until around $x = -0.18$, and for $x > 1.92$ the difference is always negative (continuing off the right-hand edge of the graph). The subspace $-0.18 < x < \infty$ is therefore a domain of attraction for the SUE at $x = 1.92$. Translated into the flow-space, this implies that if the initial flow on route 1 is in the range $0 < f_1^{(0)} < 5.13$, then the system will ultimately converge to the SUE at $f_1 = 3.60$. Figure 2 illustrates the corresponding graph for $V_{\text{SUE2}}(g(x)) - V_{\text{SUE2}}(x)$, and it

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is similarly seen that $-\infty < x < -5.54$ is a domain of attraction for the SUE at $x = -17.53$. (Equivalently in the flow-space, $8.40 < f_1^{(0)} < 10$ is a domain of attraction for the SUE at $f_1 = 9.95$.) Certainly this is a maximal domain, since it stretches to the unstable equilibrium where $g(-5.54) = -5.54$, and so this must certainly act as a "dividing line". In contrast, in Figure 1, we have not attained this dividing line, and so have been unable to determine the convergent behaviour of the system for $-5.54 < x < -0.18$ (i.e. for $5.13 < f_1^{(0)} < 8.40$). This should not be confused, however, with evidence of non-convergent behaviour. It is undoubtedly due to a break-down in the local linearisation implicit in the definition of V_{SUE1}, likely caused by the discontinuity in the derivative of $c_2(\mathbf{f})$.

FIGURE 1 HERE

FIGURE 2 HERE

Typically, as in the example above, a Lyapunov analysis will be successful in classifying the convergent behaviour of a system for a rather wide range of initial conditions. In networks of a general size, one again is able to obtain a locally-valid, (scalar) Lyapunov function of quadratic form $V(\mathbf{x}) = (\mathbf{x} - \mathbf{x}^*)^T \mathbf{P}(\mathbf{x} - \mathbf{x}^*)$. Determining $\mathbf{P}$ now requires a little more algebraic effort, however, as one must solve the matrix equation $\mathbf{A}^T \mathbf{P} \mathbf{A} - \mathbf{P} = -\mathbf{Q}$, where $\mathbf{A}$ is the locally-evaluated system Jacobian, and $\mathbf{Q}$ is an arbitrary matrix of positive definite form. For the details the reader is referred to [17].

Summary Explicit discrete time, dynamical systems may be specified which link naturally to conventional models of SUE, in the sense that the SUE are point equilibria of the dynamical system. A key component of such systems is a day-to-day learning model, describing how drivers assimilate previous perceptions of travel cost with their experiences, in order to form updated perceptions. The link to SUE allows examination of the question of whether certain dynamical adjustments in behaviour converge to equilibrium, which can be achieved by a linearisation of the dynamics in the neighbourhood of equilibrium. In contrast to the perturbation analyses of section 2, it is seen that with discrete time dynamical models, convergence to a point equilibrium may be crucially affected by the rate of adjustment. This rate is in turn affected by various parameters of the model, such as those that control the steepness of the cost-flow curves, the dispersion in driver preferences, and the nature of the learning process. When a point equilibrium is not attained, there a number of other possible "convergent" behaviours, e.g.

periodic motion. Lyapunov functions may be used to extend the local analysis of convergence, to a wider analysis across the state space of initial conditions.

4   Stochastic Dynamical Systems

4.1   Self-Consistent Stochastic Equilibria

Even in circumstances most conducive to the formation and retention of equilibrium in a real world traffic network, it is hard to believe that the traffic flow pattern (during some specific time period within the day) would remain absolutely unchanged from day to day. Rather, one would expect some kind of haphazard variation resulting from the idiosyncrasies of driver behaviour. A natural way to account for these fluctuations is by modelling within a stochastic framework. Equilibrium stochastic assignment models can be defined at a microscopic (individual traveller) level in terms of route choice probabilities, represented as functions of route costs. This implies that macroscopic properties (e.g. flows on particular road links) will be characterised by probability distributions, rather than by fixed points.

Daganoz and Sheffi [4] recognised some of the advantages of representing driver route choice within a stochastic framework, using such an approach to motivate their definition of SUE. However, the microscopic random effects in their model did not propagate through to the macroscopic level. Instead, probability became replaced by proportion, via a limiting argument as the travel demand becomes infinite, and a fixed point equilibrium solution resulted. With such comments in mind, there have been a number of recent attempts to modify Daganzo and Sheffi's SUE in order to provide a probability distribution at the macroscopic level: specifically, on the space of all feasible link flows.

In one such modification, Hazelton [21] highlighted some of the difficulties inherent in developing stochastic equilibrium models in the kind of 'self-consistent' process-free way in which DUE is defined. The crux of the problem is as follows. An equilibrium probability distribution on the set of route choices should (in a DUE type manner) be defined in terms of route travel costs, but these are in turn dependent on route flows and hence route choices. This is a logical short circuit, since in general the probability of an event cannot depend on whether or not

that event has taken place. Hazelton found a way of overcoming this difficulty, leading to his so called Conditional SUE. However, Conditional SUE lacks some of the elegant simplicity of other equilibrium assignment methods.

An alternative modification to SUE, proposed by Watling [22], was based on the premise that drivers experiences represent a large sample of independent, identically distributed (i.i.d.) travel cost variables, resulting from i.i.d. flow variables. This allowed the derivation of fixed point conditions on moments of the equilibrium flow probability distribution. In the so-called second order SUE model, this equates to equilibrating the mean flow vector and covariance matrix. A disadvantage of this approach is, however, that it is dependent on driver experiences being very large and uncorrelated; it is does not appear to be possible to accommodate more general learning models within such a framework.

In fact, the problem in defining general stochastic equilibrium models can be completely removed by introducing a time dimension. There is obviously no logical difficulty in having route choice probabilities on day n depending on costs generated by route flows on day $n - 1$ and earlier. This suggests that it is far more natural to think in terms of stochastic assignment processes evolving in time rather than look at 'self consistent' equilibria directly. Nonetheless, equilibrium remains an important concept for stochastic assignment models, as we shall see.

4.2 Stochastic Process Traffic Assignment Model

One such class of dynamic stochastic assignment model was introduced by Cascetta [19]. He proposed modelling the day-to-day evolution of a traffic system as a type of Markov process. In such models the route choice probabilities at day n are functions of the route (or equivalently link) costs on days $n - 1, n - 2, \ldots, n - m$ for some finite integer m, known as the *memory length*. In fact, this can be generalised to cases where route flows on day n depend on both route costs *and* route flows on days in the finite past (see [2], [23] and [24]). However, for illustrative purposes we shall here restrict attention to the simpler case, where there is a dependence only on the past costs.

It is then usual to define the route choice probabilities implicitly as follows. A typical driver

has his or her own perceived disutility for each route, composed of a measured disutility (a deterministic function of route costs during the previous m days) and a traveller-specific random variable: viz.,

$$\breve{u}_{ir}^{(n)} = u_r^{(n-1)} + \eta_{ir}^{(n)} \tag{18}$$

where $u_r^{(n-1)}$ is the measured disutility for route r and $\eta_{ir}^{(n)}$ is a person-specific random variable relating to driver i's perception of the attractiveness of route r. The dependence structure of these random terms deserves explicit mention. It is assumed that $\eta_{i_1 r}^{(n)}$ is idependent of $\eta_{i_2 r}^{(n)}$ (i.e. no dependence between drivers), but $\eta_{ir_1}^{(n)}$ and $\eta_{ir_2}^{(n)}$ may be dependent. (Such inter-route dependence is natural for two routes which share a large number of common links. This type of correlation structure can be generated in practice by defining η as the sum of link based random variables.) For any given driver we assume day-to-day independence; i.e. $\eta_{ir}^{(n_1)}$ is independent of $\eta_{ir}^{(n_2)}$. The temporal independence assumption is somewhat unrealistic, in that one might expect that a given traveller's personal preference for a particular route would persist from day to day. (Such a habitual preference would be represented by a sequence of η's with positive correlation for each route.) This lack of habitual behaviour in Markov process assignment models tends to lead to a rather greater day-to-day variation in flow patterns than occurs in reality. We do not pursue this issue further in this paper, but note that [23] and [24] describe an extension to the basic Markov assignment model for overcoming this problem.

A common form for the measured disutility is as a linear filter of past route costs:

$$\mathbf{u}^{(n-1)} = \sum_{j=1}^{m} w_j \mathbf{c}(\mathbf{f}^{(n-j)}). \tag{19}$$

Many forms for the weights $w_1, w_2, \ldots, w_{n-m}$ are clearly possible (e.g. see [16]), but typically they are assumed to be a decreasing sequence summing to unity. On day n driver i will then take feasible route r with minimum personal disutility $\breve{u}_{ir}^{(n)}$, implying a vector of probabilities $\mathbf{p}(\mathbf{u})$ on the set of feasible routes. For example, if $\eta_{ir}^{(n)}$ is Gumbel distributed, then these implied probabilities can be written down in closed form, giving a logit route choice model. On the other hand, if the random components are multivariate Normal, then a probit model arises; while $\mathbf{p}(\mathbf{u})$ is not then available in closed form, the general approach is equally valid. Whatever the distribution of $\eta_{ir}^{(n)}$, the number of drivers taking each possible route for inter-zonal movement k on day n is distributed conditionally as

$$\mathbf{f}_{[k]}^{(n)} | \mathbf{u}^{(n-1)} \sim \text{Multinomial}\left(q_k, \mathbf{p}_{[k]}(\mathbf{u}^{(n-1)}) \right), \tag{20}$$

where $\mathbf{p}_{[k]}$ is the set of route choice probabilities and $\mathbf{f}_{[k]}$ the set of flows for routes in $\mathcal{R}_k$.

Example 6 Consider a network with a single inter-zonal movement serviced by two non-overlapping routes. Suppose that there is a travel demand of $q = 2$ drivers so that the system has three states (corresponding to 0, 1 and 2 drivers respectively using the first route). Let the route cost functions be

$$c_r(f_r) = 10 + 5f_r \qquad\qquad r = 1, 2.$$

Suppose that drivers have a memory length of $m = 1$ day so that $u_r = c_r(f_r)$, and that a logit route choice model is used:

$$p_r(u_r) = \frac{e^{-0.1c_r}}{e^{-0.1c_1} + e^{-0.1c_2}} = \frac{1}{1 + e^{f_r - 1}}.$$

Then the *conditional* probability distribution for the number of drivers taking route 1 on day n is binomial, defined by

$$P(f_1^{(n)} = j | f_1^{(n-1)} = i) = \frac{2!}{j!(2-j)!}\alpha_i^j (1 - \alpha_i)^{2-j} \qquad\qquad i, j \in \{0, 1, 2\}$$

where simple calculations show that

$$\alpha_0 = p_1(10) = 0.73 \quad \alpha_1 = p_1(15) = 0.50 \quad \alpha_2 = p_1(20) = 0.27 \ .$$

These probabilities can be neatly set out in a transition matrix $\mathbf{M}$, whose (i, j)th element is $M_{ij} = P(f_1^{(n)} = j | f_1^{(n-1)} = i)$. We then have

$$\mathbf{M} = \left(\begin{array}{ccc} 0.07 & 0.40 & 0.53 \\ 0.25 & 0.50 & 0.25 \\ 0.53 & 0.40 & 0.07 \end{array} \right).$$

This transition matrix describes the dynamics of the system, namely defining the probabilities that govern how, given the state of the system on day $n - 1$, it evolves into its state on day n. More generally, when $m > 1$, we will need to consider transitions from the sequence of days $n - 1, n - 2, ..., n - m$ into day n, but the general approach presented here is still quite valid.

While stochastic assignment models were motivated above by the need to represent haphazard day-to-day variation, Markov models also incorporate systematic variation, akin to that seen in the deterministic assignment processes described in section 3. In other words, day-to-day variation in Markov models can be decomposed into: (i) a trend, whereby routes with high measured disutility on day $n - 1$ have low probability of being chosen by any given traveller on

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day n, and (ii) haphazard fluctuation about the trend, described by (20). In Example 6, for instance, if route 1 is used by both travellers on day $n - 1$ and is therefore expensive, there is a trend towards route 2 on day n. Nonetheless, random binomial variation can result in both travellers remaining on route 1, although this event has a probability of only 0.07.

4.3 Equilibrium Distributions for Markov Assignment Models

Consider a road network that is subjected to a major change – the closure of an important road link or the opening of a new tunnel, for example. One might well expect major fluctuations in the traffic pattern over some initial period of days, gradually settling down to some kind of equilibrium. Both the transitional and equilibrium behaviour of the traffic system would be of considerable interest to a transport planner, and both can be modelled using Markov assignment processes. It can be shown that, under quite general conditions – which are certainly satisfied if the random terms η_{ir} can take values on an infinite interval – a Markov assignment process will converge to a unique *equilibrium probability distribution*. This equilibrium distribution, π say, satisfies the stationary condition that $\mathbf{f}^{(n-1)} \sim \pi$ implies $\mathbf{f}^{(n)} \sim \pi$ (where $\sim$ denotes 'is distributed as'). A point that is worth emphasising here is that the present notion of equilibrium is not inconsistent with persistent variations in traffic conditions. We are effectively requiring that the variations settle down to some typical long-term pattern.

Example 7 The equilibrium distribution for the process in Example 6 satisfies the standard equilibrium condition

$$\pi^T = \pi^T \mathbf{M},$$

where superscript T denotes a transpose. The solution in this case is

$$\pi = (0.28, 0.44, 0.28)^T.$$

It is interesting to consider how this equilibrium distribution would differ were the logit dispersion parameter, set at $\theta = 0.1$ in Example 6, altered. For example, if this parameter were $\theta = 0.0001$, indicating great insensitivity to changes in travel cost, the equilibrium distribution would be

$$\pi = (0.25, 0.50, 0.25)^T.$$

This is (to 3 decimal places) identical to the equilibrium distribution that would be obtained if travellers chose their routes completely at random, without taking route costs into account. On the other hand, were θ set at 10, indicating great sensitivity to changes in cost, then the equilibrium distribution would be

$$\pi = (0.50, 0.00, 0.50)^T.$$

The bimodal appearance of this last distribution contrasts sharply with the unimodality of the first two equilibrium distributions. It is worth noting obvious parallels here with the stability analysis of deterministic dynamical systems, described in section 3 (see particularly Example 4), where θ was again seen to play a major role. Indeed, as $\theta \to \infty$, the stochastic process approaches a deterministic system; for the $\theta = 10$ case above, simulations of the stochastic process would exhibit a near-periodic behaviour (alternating between states 0 and 2 on successive days).

Equilibrium distributions therefore arise naturally from the application of dynamic stochastic assignment models. A host of properties of the assignment process can be obtained from these distributions – equilibrium mean link flows, equilibrium variances of link flows, and even equilibrium correlations between flows on different links. However, equilibrium distributions do not tell the whole story, in the sense that many assignment processes with markedly different dynamics may share the same equilibrium distribution.

Example 8 Consider Example 6, but with route costs

$$c(f_r) = 10 - 5f_r \qquad\qquad r = 1, 2.$$

Although this is an artificial example, it is noted that realistic cases exist where such decreasing cost functions can occur (see the comments made in the introduction to Example 5). It is easy to show that the transition matrix for this new process is

$$\mathbf{M} = \left(\begin{array}{ccc} 0.53 & 0.40 & 0.07 \\ 0.25 & 0.50 & 0.25 \\ 0.07 & 0.40 & 0.53 \end{array} \right).$$

In this case travellers tend to prefer the route which was the more heavily used on the previous day. This is in contrast to Example 6 where drivers tended to choose the more lightly used route on the previous day. A simulation of the flow on link 1 over a sequence of 50 days illustrates the

qualitative difference that exists in the dynamics of Example 6 and the present case, as shown in Figures 3 and 4.

<div align="center">FIGURE 3 HERE</div>

<div align="center">FIGURE 4 HERE</div>

Despite the different dynamics, the equilibrium distribution in the current Example is

$$\pi = (0.28, 0.44, 0.28)^T,$$

identical to that from Example 6.

The fact a stochastic assignment process is not uniquely characterised by its stationary distribution has analogies in the deterministic setting, where it is possible to define a host of different dynamical processes all converging to the same fixed point. However, in general when the memory length $m > 1$, there is a subtle distinction in the sense that the difference may be not only in the transitional phase of the process, but may also affect equilibrium properties. This is manifested in autocorrelated flows over sequences of days, which persist even in equilibrium. In this more general setting, the *marginal* equilibrium probability distribution of the network flows on any one day (while identical over days in equilibrium) only tells part of the story.

4.4   Computing Properties of Markov Assignment Models

A theoretical derivation of the properties of Markovian assignment models is usually impracticable for 'real sized' examples. Fortunately these processes are easily simulated. However, simulation results must be interpreted with care. Consider the problem of modelling the evolution of road traffic flow during some specific time period on a sequence of five days. A single simulation run will be subject to the haphazard binomial variation inherent in Markovian models, and will therefore provide only one possible scenario. To obtain a more complete picture it is necessary to implement many parallel simulations. From these parallel simulations one can compute properties of interest, such as the mean predicted link flows on day 5 (as the mean of the simulations at that time point) or 95% prediction intervals (obtained from the 'central' 95% of observed simulations).

When interest centres on the long-term behaviour of a road traffic system, the equilibrium

<div align="center">23</div>

distribution of a Markovian assignment model is an important tool. However, computation of this distribution is far from straightforward. In principle its properties can be derived to an arbitrary degree of accuracy by a sufficiently long run of simulated flows. In practice there are many difficulties, including the following.

1. The simulation process will only generate traffic flows from the equilibrium distribution once the simulation has converged to stochastic equilibrium. This means that a number of days at the beginning of the simulation must be discarded as a 'burn-in' period. How long should this burn-in period be?

2. The simulations will exhibit serial correlation because of the Markovian structure. This should be taken into account when deriving estimates of equilibrium properties.

Both these issues in relation to general Markov processes have received considerable attention in the statistical literature because of the recent popularity of Markov chain Monte Carlo (MCMC) methods of Bayesian inference. See [25], for example. A number of methodologies for addressing the first issue have been suggested (see [26] for instance), but it is still regarded as a difficult problem. Even when one is certain that convergence to stochastic equilibrium has been achieved, the second issue must be addressed. The serial correlation in the simulated output will mean that the precision of estimators will not necessarily be well represented by 'text book' standard errors (assuming independent and identically distributed data). For example, suppose that one is estimating the mean flow on a particular link for which the equilibrium link flow variance is σ^2. Then the standard error in estimating the mean flow from N simulated days would be $\sigma/\sqrt{N}$ (the 'text book' value) if the simulations were independent, but will actually be considerable higher than this if the serial correlation in the data is positive. This implies that the requisite number of simulated days to provide a specific precision in estimation may be significantly larger than one might expect (based upon text book results for independent data). See [27] for further comments.

In fact, there are particular cases relating to the traffic assignment application where not only lack of precision is a problem, but also bias in equilibrium estimates may occur over long simulation runs. A particular case occurs when in a deterministic setting there would be

multiple attractors (i.e. multiple stable point equilibria), such as Example 5 considered in section 3. As shown in [8], when applied in a stochastic process setting with a memory length $m = 1$, the true equilibrium distribution may be shown to be bimodal, with peaks focused in the vicinity of the two stable SUE solutions. A typical Monte Carlo simulation of such a process is illustrated in Figure 5. The equilibrium behaviour is seen to be characterised by stable periods in the vicinity of a stable SUE, with occasional transitions between SUE. These transitions between apparently stable regimes are an essential part of the process, and are what allow the bimodal distribution to be estimated. However, as m is increased such transitions occur much more rarely, so that (for example) with $m = 10$ simulations over many thousands of days are likely to see no transitions. Instead any single simulation will typically remain within the vicinity of the initial condition for extremely long periods. However, this does not give a true reflection of the equilibrium distribution, which is still bimodal in nature.

FIGURE 5 HERE

Such difficulties with the interpretation of stochastic process simulations are not restricted to (what might be claimed to be) pathological cases with multiple point equilibria. More generally in the authors' experience very long simulation runs (e.g. tens of thousands of days) can be required to obtain estimates of equilibrium properties of acceptable precision. The appeal of simulation, in terms of the simplicity of replicating dynamical process, needs therefore to be balanced against the difficulty of reliably estimating equilibrium properties from simulation. An alternative to simulation is to apply approximation methods. Davis and Nihan [28] were able to elegantly demonstrate that the evolution of a Markov assignment model with very large demand could be well approximated by a particular Gaussian multivariate autoregressive process. In principle the stationary mean vector and covariance matrix of this process can be used as summary statistics for the equilibrium distribution. However, the covariance matrix is only available as the solution to a high dimensional fixed point problem which will itself be difficult to solve for large networks. Hazelton and Watling [29] were able to obtain a much simpler expression for the equilibrium covariance matrix for Markov processes employing linear learning filters with exponentially decreasing weights. Nonetheless, there remains considerable scope for further research into such approximation methods for equilibrium analysis.

Summary The problem of representing both dynamical adjustments and general variability

may be addressed through a stochastic process model, in which the travel choices on day n depend probabilistically on experiences in day $n-1$ and earlier. In such a model, the dynamics are described by a transition matrix of probabilities, and "equilibrium" is now concerned with an equilibrium probability distribution of the possible network flow states. While simulating such processes is in principle straightforward, estimating reliable characteristics from them is not necessarily so simple, particularly for the model-user. The scope exists for further exploration of approximation methods that avoid the need for simulation.

5 Conclusion

The assignment models discussed have been developed over a period of almost 50 years, and have been motivated by a variety of precepts for driver behaviour. Nonetheless, they are inter-related through various limiting results. Starting with Markov assignment models – arguably the most intricate of the models considered – it can be shown that the mean equilibrium flow of any such model (satisfying certain regularity conditions) will converge to SUE as the demand becomes large. Furthermore, the day-to-day evolution of the flow pattern in a Markov assignment process will converge to a deterministic dynamical model as drivers perceived costs become increasingly homogeneous (for example, as the dispersion parameter tends to infinity in a logit based model). Economic equilibria, such as SUE, may in turn be linked to deterministic dynamical systems, as locally stable fixed points of the system. The range of initial conditions ("domain of attraction") for which an SUE is a convergent limit of such dynamics may subsequently be estimated by an analysis of the Lyapunov kind.

This work suggests that much can be learnt from the relationships between these apparently competing approaches. One particular example that is worth highlighting is the rapidly increasing interest in computer micro-simulation approaches, across the full range of transport modelling. Such approaches are often regarded simply as a competing ideology to the algebraic-style equilibrium models, with apparently little potential for cross-fertilisation of ideas. This is entirely misleading. For the first part, micro-simulation is effectively only a solution technique for estimating some (often poorly defined) underlying characteristic. Linking such approaches to the theory of Markov processes therefore provides the potential for a more rigorous, scien-

tific foundation for their application. Furthermore, by clearly separating the concepts of "the model" and "the solution method", it allows us to question whether simulation is necessarily the most appropriate solution technique for the model concerned. The rapidly growing interest in flexible models of activity dynamics is typically used to imply a need for simulation methods, yet it is our belief that (given sufficient research attention) approximation methods may be derived to allow properties of such models to be estimated without simulation, therefore avoiding the inherent difficulties of interpreting complex simulation results.

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Figure Captions

FIGURE 1: Lyapunov difference for Example 5, in neighbourhood of equilibrium at $x = 1.92$.

FIGURE 2: Lyapunov difference for Example 5, in neighbourhood of equilibrium at $x = -17.53$.

FIGURE 3: Simulated flows on route 1 using Example 6 cost functions.

FIGURE 4: Simulated flows on route 1 using Example 8 cost functions.

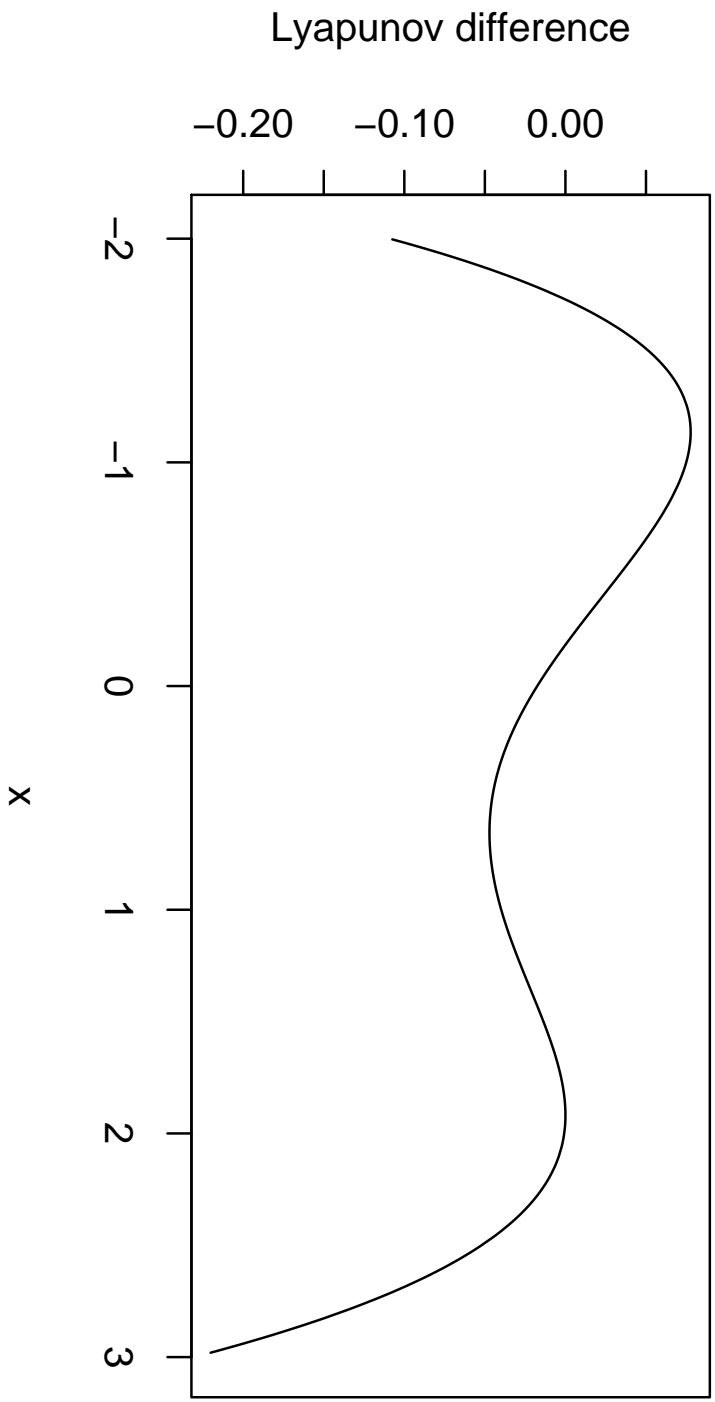FIGURE 5: Simulated flows on route 1 for Example 5, based on stochastic process model with $m = 1$.
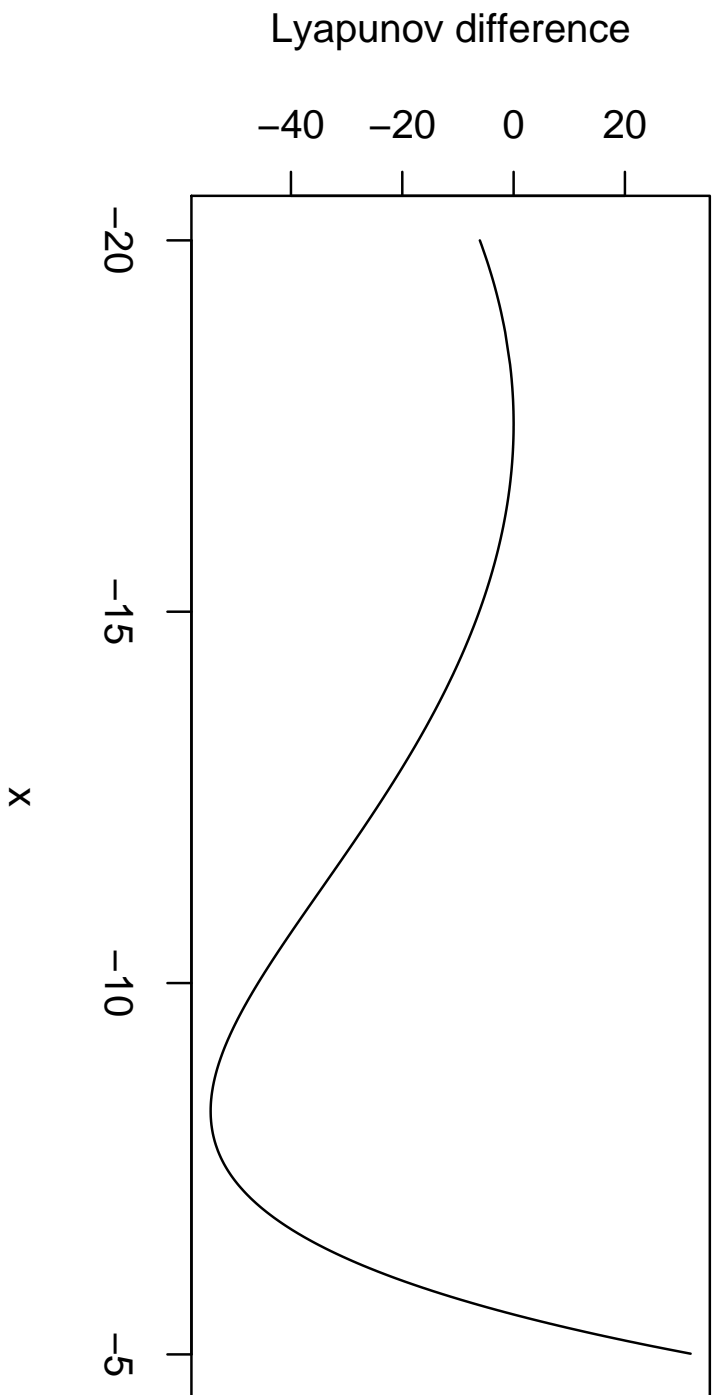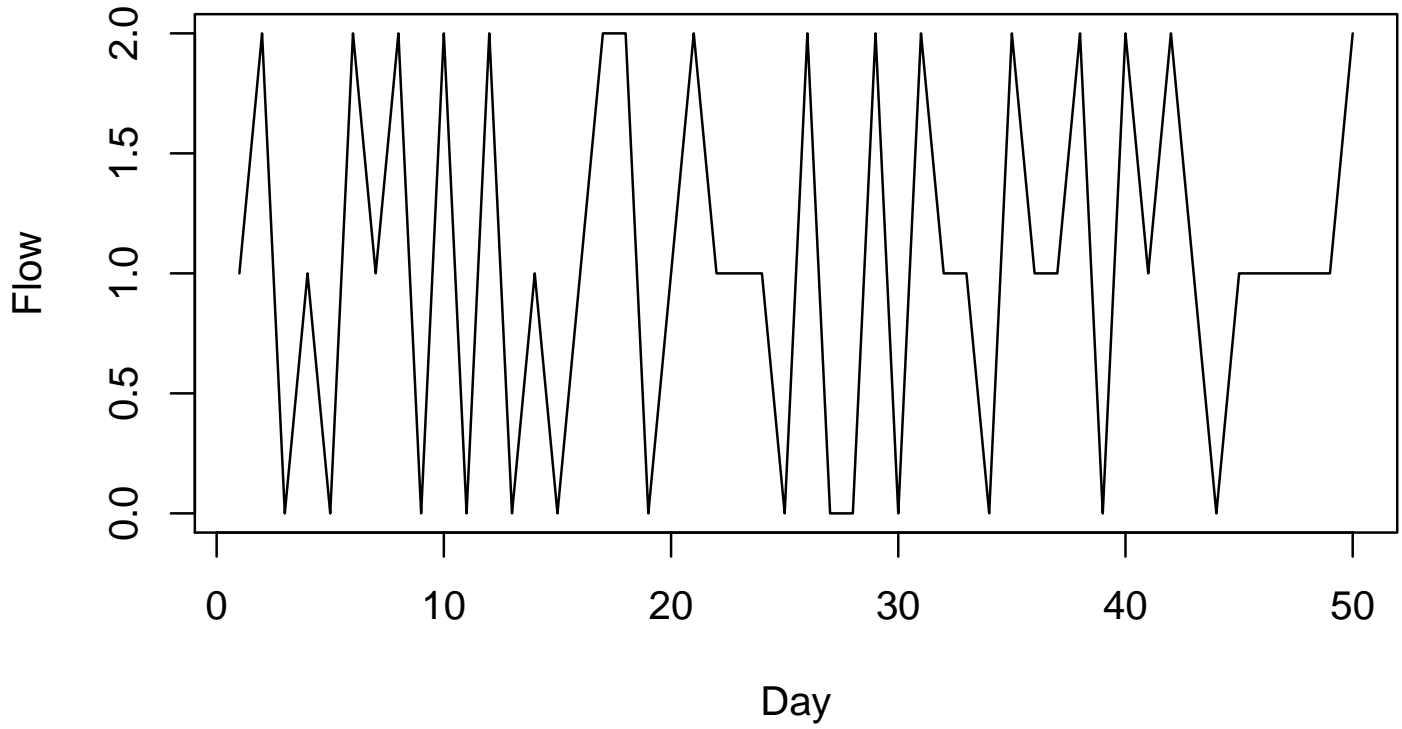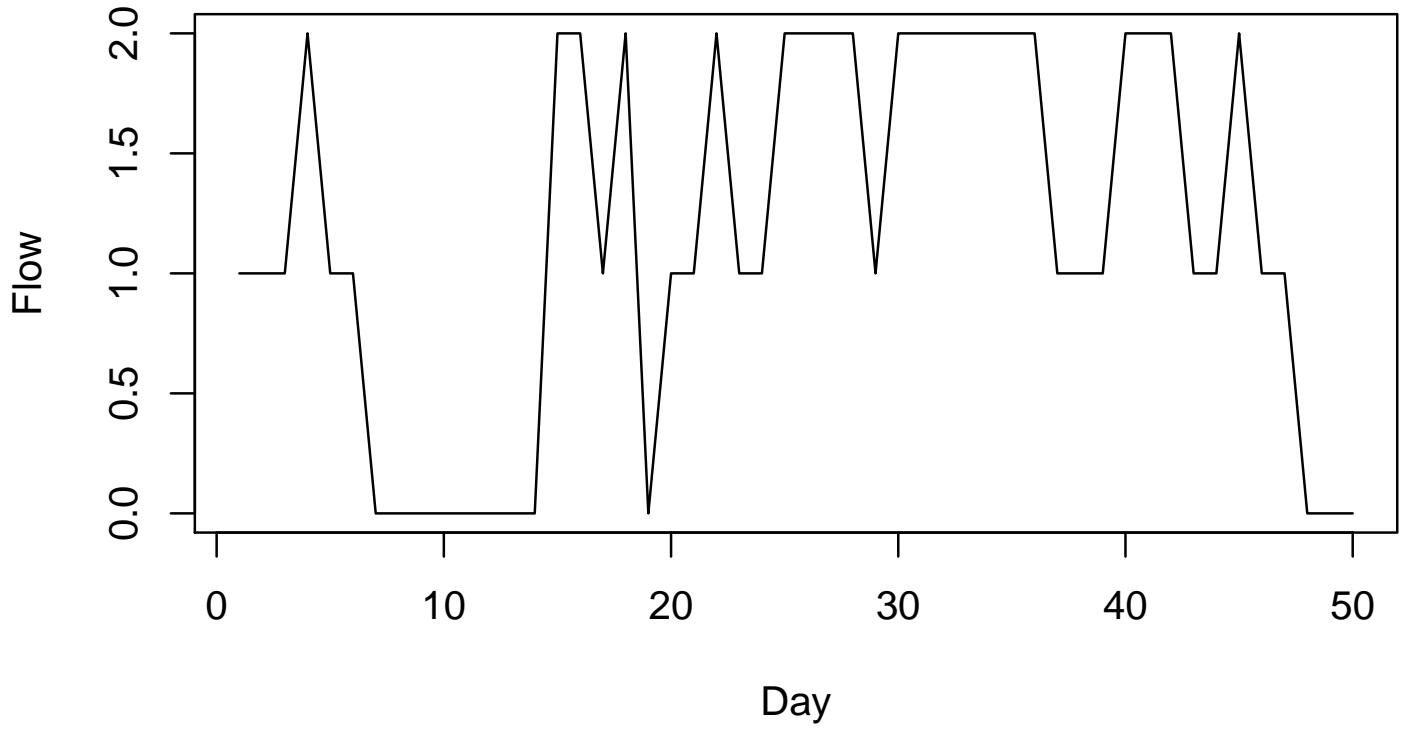
Fig 1

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Fig 2

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Fig 3

Fig 4

Fig 5