# Monopoles on $\mathbb{R}^5$

Rodrigo Pires dos Santos

Submitted in accordance with the requirements for the degree of

Doctor of Philosophy

The University of Leeds

School of Mathematics

September 2015

.

# Acknowledgements

I would like to thank Professor Roger Bielawski, who was my supervisor in my first year and proposed the topic for this thesis, for his guidance, patience and encouragement. I would also like to thank Professor Martin Speight for accepting to supervise me from the second year of my PhD. I must also express my gratitude to other PhD students and staff at the School of Mathematics for useful discussions about mathematics.

I am also grateful for my family, specially my mother, for their understanding, continuous support and patience during my time in England.

Completing this work would have been impossible without the wonderful friends that have been by my side. I would like to thank them all, specially those that came to study here from Brazil, my friends from the School of Mathematics and all my friends from St Mark's Residences. Moreover, special mention goes to my friends from Hyde Park Harriers, being part of such an amazing running club will be an experience that will be difficult to leave behind.

Finally, I shall thank the sponsors of my scholarship: Coordenação de Aperfeiçoamento de Pessoal de Nível Superior, under the grant number BEX:5705-10-0.

A minha familia

"From a drop of water a logician could infer the possibility of an Atlantic or a Niagara without having seen or heard of one or another. So all life is a great chain, the nature of which is known whenever we are shown a single link of it. Like all other arts, the Science of Deduction and Analysis is one which can only be acquired by long and patient study, nor is life long enough to allow any mortal to attain the highest possible perfection in it."

Sir Arthur Conan Doyle

*Sherlock Holmes - A Study in Scarlet*

# Abstract

This thesis is motived by the wish to understand the structure of the moduli space of monopoles on $\mathbb{R}^5$. Our approach to define monopoles is twistorial and we start by developing the twistor theory of $\mathbb{R}^5$, which is an analogue of the twistor theory for $\mathbb{R}^3$ developed by Hitchin. Using this, we describe a Hitchin-Ward transform for $\mathbb{R}^5$, giving monopoles for the group $SU(2)$. In order for us to construct monopoles we make use of spectral curves. Then, using those spectral curves we find a new system of equations, analogue to the Nahm's equations. Lastly, we prove that the geometry of the moduli space of solutions to this Nahm's equations carries a 2-symplectic structure.

# Contents

.

# Chapter 1

# Introduction

## 1.1 A brief introduction to monopoles on $\mathbb{R}^3$

In physics, a *magnetic monopole* is an elementary particle that behaves as an isolated magnet. In other words, it can can be regarded as the isolated pole of a magnet, either the north or the south. Even though those particles have not been found yet, Paul Dirac proved that the existence of a single monopole in the universe can explain the quantisation of the charge of electrically charged particles [Dir31]. Explicitly, the magnetic field of a point magnetic monopole of charge $k$ at the origin is given by

$$\mathbf{B} = \frac{k}{4\pi r^2}\hat{\mathbf{x}}.$$

It is important to highlight that the Dirac monopoles have a singularity at the origin, thus they are not topological solitons [1].

It was in 1974 that 't Hooft [Hoo74] and Polyakov [Pol74] found out that the non-abelian Yang-Mills-Higgs theory on $\mathbb{R}^3$ admits non singular magnetic monopole. The Yang-Mills-Higgs theory can be obtained as the Yang-Mills energy functional coupled with a scalar field, the *Higgs field*. Intuitively, the 't Hooft Polyakov monopole behaves like the Dirac monopole at large distances, but with the singularity in the centre smoothed out. In what follows, we shall give a succinct explanation on the development of methods to find monopoles on $\mathbb{R}^3$ when the gauge group is $SU(2)$.

A monopole on the Euclidean $\mathbb{R}^3$ consists of a pair $(\nabla, \phi)$ minimising, with finite energy, the Yang-Mills-Higgs energy functional

$$\mathcal{V} = \int_{\mathbb{R}^3} |F_\nabla|^2 + |\nabla\phi|^2,$$

where $\nabla$ is a $SU(2)$-connection on a complex vector bundle $E$ over $\mathbb{R}^3$ and $\phi$ is a skew-symmetric section of $End(E)$. One can show [AH88] that the pair $(\nabla, \phi)$ is a monopole if and only if it satisfies the Bogomolny equation

$$F_\nabla = *\nabla\phi,$$

In this case there exists an integer $k \geq 0$ such that

$$\mathcal{V} = 4\pi k.$$

[1] A topological soliton is a non-singular, static, finite energy solution of field equations. Those properties allows one to define a topological invariant of the solution, called *charge*.

Moreover, $\nabla$ and $\phi$ are subject to the following boundary conditions:

$$|\phi| = 1 - \frac{m}{2r} + O(r^{-2})$$
$$\frac{\partial|\phi|}{\partial\Omega} = O(r^{-2})$$
$$|\nabla\phi| = O(r^{-2}), \text{ as } r \to \infty,$$

where $\frac{\partial|\phi|}{\partial\Omega}$ is the angular derivative of $|\phi|$.

We shall now give an interpretation of the integer $m$ that appears in the boundary conditions. The first boundary condition says that, at large distances, $\phi$ does not vanish. Therefore, if we restrict the bundle $E$ to a large sphere $S_\infty$ at the infinity, it splits as a direct sum $E = L \oplus L^*$ of eigenspaces for $\phi$ and the degree of $L$ is $m$. Furthermore, integration over $S_\infty$ shows that the $m = k$. Equivalently, we can restrict $\phi$ to $S_\infty$ and obtain a map $\frac{\phi}{|\phi|} : S_\infty \to S^2 \subset \mathfrak{su}(2)$; this map is well-defined since $\phi$ does not vanish at large distances. Again, by integration on the sphere at the infinity, one can show that the degree of this map is $k$. From now on, we call the integer $k$ the topological charge of the monopole.

The problem now was to find solutions to the monopole equation. For $k = 1$, the exact solution was found by Prasard and Sommerfield [PS75]. The explicit Higgs field is given by:

$$\phi = \left(\coth r - \frac{1}{r}\right)\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Charge $2$ monopoles solutions were constructed by Ward via twistor methods. He relied on the idea that monopoles can be regarded as self-dual Yang-Mills connections on $\mathbb{R}^4$ that are invariant under translations in one direction. Since, by that time, it was known that every self-dual Yang-Mills connection corresponded to holomorphic bundles over $\mathbb{C}P^3$, Ward knew that monopoles should also correspond to holomorphic bundles or, equivalently, to holomorphic transition matrices. In [War81b], Ward constructed the axially symmetric monopole of charge $2$. For this, he used the Atiyah-Ward ansatz for self-dual connections on $\mathbb{R}^4$, described in [AW77]. Briefly, he considered a specific transition function for a holomorphic bundle on $\mathbb{C}P^3$ for which he knew how to construct back the corresponding self-dual connection on $\mathbb{R}^4$ explicitly. Then he proved it was a monopole on $\mathbb{R}^3$ satisfying the boundary conditions and, moreover, it was axially symmetric. As an extension of this result, Ward also constructed a seven parameter family (six parameters come from translations and rotations on $\mathbb{R}^3$) of charge $2$ monopoles [War81a].

In [PS81] Prasad and Sommerfield constructed an axially symmetric solution for each charge $k$. It is important to mention that their method differ from Ward's construction of charge $2$ monopoles and the regularity of those solutions was not proved in that paper.

Using the geometry if oriented lines in $\mathbb{R}^3$, Hitchin introduced in [Hit82] the dimensional reduction at the twistor level. Namely, he proved that a solution to the Bogomolny equations corresponds to a holomorphic bundle on $\mathbb{T}$, the total space of $\mathbb{T}P^1$; this type of result is known in the literature as the Hitchin-Ward correspondence. Hitchin's idea was to extend the Ward's twistor method to general charge $k$ monopoles. He proved that if a bundle $\tilde{E}$ on $\mathbb{T}$ is the bundle corresponding to a monopole, then $\tilde{E}$ is given by an extension of line bundles on $\mathbb{T}$. Moreover, he uses the properties of those holomorphic bundles to define a compact Riemann surface in $\mathbb{T}P^1$, the so called *spectral curve*. He also shows that the spectral curve determines the monopole. However, there was no proof that the solutions obtained in this way were non-singular.

Concomitantly, Nahm, using a type of generalised Fourier transform, proved that the Bogomolny equations can be reduced to a non-linear system of differential equations, called nowadays as Nahm's equations [Nah82]:

$$\dot{T}_1 = [T_2, T_3],$$

$$\dot{T}_2 = [T_3, T_1],$$

$$\dot{T}_3 = [T_1, T_2],$$

where $T_j$ is a $k \times k$ analytic matrix-valued function on the interval $(0, 2)$ with simple poles at $0$ and $2$. Furthermore, solutions to Nahm's equations satisfying these boundary conditions corresponds to non-singular monopoles. In a later work [Hit83], Hitchin proved an equivalence between solutions to Nahm's equations and spectral curves satisfying a cohomological condition. Using this result, Hitchin proved that the Prasad-Sommerfield monopoles were actually non-singular.

Another important problem in the study of monopoles is the description of the moduli space. It was proved by Taubes in [JT80] that the space of solutions of charge $k$ monopoles is a smooth manifold $\tilde{\mathcal{M}}_k$ of dimension $4k - 1$, however, we can add a phase factor to a monopole and obtain a circle bundle $\mathcal{M}_k$ over the moduli space whose dimension is $4k$, called the gauged moduli space.

The importance of Nahm's equations to the study of moduli space comes from the fact that $\mathcal{M}_k$ can be described as an infinite dimensional Hyperkähler quotient [HKLR87] and Nahm's equations can be interpreted as the moment map for this quotient. Moreover, using this technique, Atiyah and Hitchin described the metric on $\mathcal{M}_2$ explicitly.

More recently, Bielawski [Bie06] described a class of manifolds whose tangent space at every point decomposes as copies of irreducible representations of $SU(2)$ of dimension $k$ for some fixed $k$, the so called *generalised hypercomplex manifolds*, or GHC manifolds for short. Furthermore, these manifolds have a twistor interpretation similar to hyperkähler manifolds. He also endows some of the GHC manifolds with symplectic structures and performs GHC-symplectic quotients. It is important to highlight here that GHC manifolds are defined in such way that hyperkähler manifolds, and their hyperkähler quotients, are a special case.

Bielawski also proves a Hitchin-Ward correspondence for GHC manifolds. Namely, a holomorphic bundle on the twistor space of a GHC manifold $M$ corresponds to a pair $(\nabla, \Phi)$, where $\nabla$ is a connection on a complex vector bundle $E$ over $M$ and $\Phi$ is a section of $End(E) \otimes \mathbb{C}^{(k-2)}$ satisfying a flatness condition, this is the Bogomolny condition.

Our objective in this thesis is to present a construction of monopoles on $\mathbb{R}^5$. Our approach is to construct a Bogomolny pair $(\nabla, \Phi)$ from a spectral curve, satisfying some additional conditions, in the twistor space of $\mathbb{R}^5$. We then move towards the construction of spectral curves, which is similar to the construction in [Hit83], and we prove the equivalence between a new system of equations, analogue to Nahm's equations, and the spectral curves. The motivation to take this approach comes from the fact that the twistor space of $\mathbb{R}^5$ is similar to the twistor space of $\mathbb{R}^3$. Therefore, we expected some similarities with the spectral curves for monopoles on $\mathbb{R}^3$.

The main challenge faced was the fact that our monopoles are not defined in terms of a Yang-Mills-Higgs functional. This implies, for instance, that we do not have a topological definition of charge. We define the charge of a monopole on $\mathbb{R}^5$ to be the degree of the spectral curve associated. Moreover, this implies there is no natural way to define a $L^2$ metric on the moduli space and we should rely on our Nahm's equations to define the metric.

It is important to notice that, to our knowledge, this thesis is the second source in the literature where monopoles in $\mathbb{R}^5$ are mentioned, the first being [Bie06]. However, there are other recent work in higher dimensional monopoles. For instance, in [Oli14] Oliveira investigates monopoles on $6$ dimensional Calabi-Yau manifolds and $7$ dimensional $G_2$ manifolds.

## 1.2 Overview and statement of results

In **Chapter 2** we describe the twistor theory of $\mathbb{R}^5$. Using the results of [Bie06] on GHC manifolds we define $\mathbb{R}^5$ as the space of real sections of the bundle $\mathcal{O}(4)$ over $\mathbb{C}P^1$ or, equivalently, as the space of polynomials of degree $4$ in the variable $\xi$, the holomorphic coordinate for $\mathbb{C}P^1$, invariant under a real structure. Moreover, if we call $\mathbb{T}$ the total space of $\mathcal{O}(4)$, we describe the twistor correspondence between $\mathbb{T}$ and $\mathbb{R}^5$:

*Every point* $(x_0, x_1, x_2, x_3, x_4) \in \mathbb{R}^5$ *corresponds to the section* $p(\xi) = (x_0 + ix_4) + (x_1 + ix_3)\xi + x_2\xi^2 - (x_1 - ix_3)\xi^3 + (x_0 - ix_4)\xi^4$ *of* $\mathcal{O}(4)$. *Conversely, every point* $z \in \mathbb{T}$ *corresponds to an oriented* $3$*-dimensional affine subspace of* $\mathbb{R}^5$. *Those* $3$*-dimensional submanifolds are called* $\alpha$*-surfaces and are described explicitly in proposition 2.5.2.*

In **Chapter 3** we introduce the twistor description of $\mathbb{R}^5$ and prove the Hitchin-Ward correspondence for $\mathbb{R}^5$:

**Theorem 3.4.6.** Let $E$ be a $SU(2)$ bundle on $\mathbb{R}^5$. There is a 1-1 onto correspondence between $SU(2)$ Bogomolny pairs $(\nabla, \Phi)$ and holomorphic bundles $\tilde{E}$ on $\mathbb{T}$ satisfying:

(i) $\tilde{E}$ is trivial on real sections,

(ii) $\tilde{E}$ has a symplectic structure,

(iii) $\tilde{E}$ is equipped with a quaternionic structure $\sigma$ covering $\tau$, this is to say, $\sigma$ is an anti-holomorphic linear map

$$\sigma : \tilde{E}_z \to \tilde{E}_{\tau(z)},$$

such that $\sigma^2 = -id_{\tilde{E}_z}$.

It is important to highlight that in the proof of the above theorem we described explicitly the holomorphic structure for the bundle $\tilde{E}$ from the pair $(\nabla, \Phi)$. Moreover, the Hitchin-Ward correspondence stated in [Bie06] does not apply to the group to $SU(2)$.

We also use this to find the explicit line bundle $L_{(\frac{a}{2},b,\frac{c}{2})}$ on $\mathbb{T}$ corresponding to a trivial $U(1)$ pair $(d, (i\frac{a}{2}, ib, i\frac{c}{2}))$. Namely, if we set holomorphic local coordinates $\mu$ and $\lambda$ on the total space of $\mathcal{O}(4)$, we prove the following proposition:

**Poposition 3.5.1.** The bundle $L_{(\frac{a}{2},b,\frac{c}{2})}$ has transition function

$$g_{01}^{(a,b,c)} = exp\left(-a\mu\left(\frac{1}{\lambda} - \frac{1}{\lambda^3}\right) + b\frac{\mu}{\lambda^2} - ic\mu\left(\frac{1}{\lambda} + \frac{1}{\lambda^3}\right)\right).$$

We later discuss the global equations that are the equivalent, in $\mathbb{R}^5$, to the Bogomolny equations. Then, we bring the chapter to an end by discussing the fact that monopoles on $\mathbb{R}^5$ are dimensional reductions of self-dual connections on $\mathbb{R}^8$.

**Chapter 4** contains the construction of monopoles. We begin the chapter with a discussion on how a spectral curve gives rise to a pair $(\nabla, \Phi)$ on $\mathbb{R}^5$. Then, we use the methods developed by Hitchin in [Hit82] and construct a new system of differential equations, which is an analogue of Nahm's equations, from spectral curves. Furthermore, we prove an equivalence between solutions to this new Nahm's equations satisfying reality and boundary conditions and spectral curves.

We bring the thesis to an end with **Chapter 5**. In this chapter we discuss the geometry of the moduli space of solutions to our Nahm's equations. We first show that our Nahm's equations can be interpreted as the moment map for an infinite dimensional generalised symplectic manifold. Moreover, we prove that the moduli space of solutions to those equations is a $2$-*symplectic manifold*, it is a GHC manifold equipped with a generalised symplectic structure. To give an example of this construction we find the moduli space of solutions to our Nahm's equations with trivial boundary conditions and compute their twistor space explicitly. As special case of this we have the moduli space of charge $1$ monopoles in $\mathbb{R}^5$.

# Chapter 2

# $\mathbb{R}^5$ as a Generalised Hypercomplex Manifold

## 2.1 Overview

In sections (2.2) and (2.3) we review some of the material presented in [Bie06]. Then, we use these results to endow $\mathbb{R}^5$ with a generalised hypercomplex structure. More precisely, we define $\mathbb{R}^5$ as the space of real sections of the degree $4$ holomorphic line bundle over $\mathbb{C}P^1$. It is know, from the Borel-Weil theorem (see for e.g. [BE89]) that the space of sections of this bundle is the fourth-symmetric power of the defining representation of $SL(2,\mathbb{C})$, which is irreducible. Thus, we have given $\mathbb{R}^5$ an integrable generalised hypercomplex structure.

We shall also describe explicitly the $\alpha$-surfaces in $\mathbb{R}^5$. They turn out to be the analogue of the oriented lines in $\mathbb{R}^3$ as investigated by Hitchin in [Hit82]. By definition, the $\alpha$-surface $P_z$ for the point $z \in \mathbb{T}$, where $\mathbb{T}$ is the total space of the bundle $\mathcal{O}(4)$, consists of points of $\mathbb{R}^5$, regarded as sections of $\mathcal{O}(4)$, passing through $z$. The explicit description of the $\alpha$-surfaces is given in (2.5.2).

## 2.2 Generalised hypercomplex manifolds

**Definition 2.2.1.** Let $M$ be a smooth manifold. A *generalised almost complex manifold* is a smooth fibrewise action of $SU(2)$ in the tangent bundle such that each $T_xM$ decomposes as $V \otimes \mathbb{R}^n$, where $V$ is an irreducible representation of $SU(2)$. The complexified representation $V^{\mathbb{C}}$ is one or two copies of the $k^{\text{th}}$-symmetric power of the defining representation of $SL(2, \mathbb{C})$. We shall then call $M$ an *almost k-hypercomplex manifold.*

The dimension of a $k$-almost hypercomplex manifold is $m(k+1)$ where $m$ is even if $k$ is odd. Moreover, the structure group of such manifolds is the centraliser of $SU(2)$ in $GL(m(k+1), \mathbb{R})$, and this centraliser is $GL(m, \mathbb{R})$ is $k$ is even or $GL(\frac{m}{2}, \mathbb{H})$ if $k$ is odd. Now, let $E_M$ be the bundle on $M$ associated to the canonical representation of $GL(m, \mathbb{R})$ or $GL(\frac{m}{2}, \mathbb{H})$ and $H$ is the trivial bundle with fibre $S^k\mathbb{C}^2$. This gives an isomorphism $TM^{\mathbb{C}} = E_M \otimes H$.

One way to produce examples of those structures is to look into the space of sections of holomorphic bundles over $\mathbb{C}P^1$. This happens because irreducible representations of $SL(2,\mathbb{C})$ can be realised as sections of line bundles over $\mathbb{C}P^1$. A $\sigma$-bundle (or *real* bundle) over $\mathbb{C}P^1$ is a holomorphic bundle $E$ equipped with a anti-holomorphic involution $\sigma$ covering the antipodal map on $\mathbb{C}^1$, a *real section* of a $\sigma$-bundle is a section invariant under the involution $\sigma$. The map $\sigma$ will be called a *real structure*. Following these definitions, we can describe the irreducible representation of $SU(2)$ as real sections of a $\sigma$-bundle over $\mathbb{C}P^1$. Consequently, the tangent space of a generalised almost hypercomplex manifold is the space of real sections of a $\sigma$-bundle. In fact:

**Proposition 2.2.2** ( [Bie06] proposition 2.2)**.** Let $Z$ be a complex manifold fibering over $\mathbb{C}P^1$ equipped with an anti-holomorphic involution $\tau$ which covers the antipodal map on $\mathbb{C}P^1$. Suppose there exists a holomorphic real section of $Z \to \mathbb{C}P^1$ whose normal bundle is isomorphic to $\mathcal{O}(k) \otimes \mathbb{C}^n$ $(k > 0)$, then the space of such real sections is an almost $k$-hypercomplex manifold of dimension $n(k+1)$.

This proposition motivates the following definition:

**Definition 2.2.3.** An almost $k$-hypercomplex structure on a manifold $M$ is *integrable* if $M$, together with the $SU(2)$ action on its tangent bundle, can be described (locally) as the space of real sections of a complex manifold $Z$ fibering over $\mathbb{C}P^1$. We shall say that $M$ is a *generalised hypercomplex manifold* or GHC manifold for short. The space $Z$ is called the *twistor space* of $M$.

**Example 2.2.4.** Consider $S^3 \cong SU(2)$, where the diffeomorphism is given by defining

$SU(2)$ as the quaternions of unit length. Since $TSU(2) = SU(2) \times \mathfrak{su}(2)$, we have an action on the tangent bundle given by the adjoint action of $SU(2)$ on $\mathfrak{su}(2)$.

For the integrability, notice that $S^3$ can be understood as the space of real sections of $\mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(1))$. Namely, sections of $\mathcal{O}(1)$ are polynomials of degree 2 in $\xi$, where $\xi$ is a holomorphic coordinate for $\mathbb{C}P^1$. Moreover, $\mathcal{O}(1)$ can be endowed with a quaternionic structure, therefore $E = \mathcal{O}(1) \oplus \mathcal{O}(1) \cong \mathcal{O}(1) \otimes \mathbb{C}^2$ possesses a real structure. Now, $H^0(\mathbb{C}P^1, E) = \mathbb{C}^4$ and the space of real sections of $E$ is $\mathbb{R}^4$. Taking the projectivization $\mathbb{P}(E)$ of $E$, gives us the real sections of $\mathbb{P}(E) = S^3$. Therefore, $S^3$ is the space of real sections of a holomorphic bundle over $\mathbb{C}P^1$ and this proves the integrability.

**Example 2.2.5.** Let $H$ be the $k$-dimensional, for $k$ even, irreducible representation of $SL(2, \mathbb{C})$, then it acts irreducibly on the dual $H^*$. Let $B$ be a Borel subgroup of $SL(2, \mathbb{C})$, then $SL(2, \mathbb{C})/B \cong \mathbb{C}P^1$. For each $q \in \mathbb{C}P^1$, let $B_q$ be its corresponding Borel subgroup and $l_q$ be the highest weight vectors for $B_q$. This gives an injective map $\mathbb{C}P^1 \mapsto \mathbb{P}(H^*)$ and let $\tilde{L}_k$ be the bundle on $\mathbb{C}P^1$ given by the pullback of the tautological bundle on $\mathbb{P}(H^*)$. For $L_k = (\tilde{L}_k)^*$ we have:

**Theorem 2.2.6.** (Borel-Weil theorem) In the notation of the example above,

$$H^0(G/B, L_k) \cong H.$$

Since $k$ is even, we can endow $H$ with a real structure and then $H^{\mathbb{R}}$ is a GHC-manifold with twistor space $L_k$. We shall later describe explicitly the $\alpha$-surfaces when $k = 4$.

Let $M$ be a GHC manifold and consider the action of $SL(2,\mathbb{C})$ on the complexified cotangent bundle $T^*M^{\mathbb{C}}$. For each point $q \in \mathbb{C}P^1$ let $B_q$ be the corresponding Borel subgroup of $SL(2,\mathbb{C})$. Define the following:

i) $\mathcal{U}_q$ is the subbundle of $(T^*M)$ corresponding to the highest weight with respect to $B_q$,

ii) $\mathcal{K}_q$ is the subbundle of $TM^{\mathbb{C}}$ annihilated by $\mathcal{U}_q$ and

iii) $\mathcal{F}_q = \mathcal{K}_q \cap \overline{\mathcal{K}_q} \cap TM$ is a distribution on $M$.

We then have:

**Theorem 2.2.7.** ([Bie06] theorem 2.5) An almost $k$-hypercomplex structure on a manifold $M$ is integrable if and only if for every $q \in \mathbb{C}P^1$ the subbundle $\mathcal{K}_q$ is involutive for all $q \in \mathbb{C}P^1$, this is to say, $[\mathcal{K}_q, \mathcal{K}_q] \subset \mathcal{K}_q$.

We shall not prove the theorem above, however we shall see how it can be used to construct the twistor space of a GHC-manifold.

Define the *twistor distribution* $\mathcal{Z}$ of $M$ to be the distribution on $M \times \mathbb{C}P^1$ given by $\mathcal{Z}_{(m,q)} = ((\mathcal{F}_q)_m, 0)$. The theorem above says that this distribution is involutive and thus it defines a foliation of $M \times \mathbb{C}P^1$. Moreover, the leaf space $Z = (M \times \mathbb{C}P^1)/\mathcal{Z}$ is the twistor space of the GHC-manifold $M$. If the foliation is simple, then $Z$ is a complex manifold and the projection $\eta : M \times \mathbb{C}P^1 \to Z$ is a surjective submersion, in this case $M$ is called a *regular* GHC-manifold. The leaves of the foliation $\mathcal{Z}$ will be called $\alpha$-surfaces.

Let $M$ be a GHC-manifold, then it is given as the space of real sections of a fibration $Z \to \mathbb{C}P^1$, then $M$ has a natural complexification $M^{\mathbb{C}}$ given by the space of all sections of the fibration. Notice that the holomorphic tangent bundle $TM^{(1,0)}$ of $M^{\mathbb{C}}$ is then endowed with a holomorphic action of $SL(2,\mathbb{C})$ such that $TM^{(1,0)} = S^k\mathbb{C}^2 \otimes \mathbb{C}^n$. Furthermore, we can define the complexified bundles $\mathcal{U}_q, \mathcal{K}_q$ and $\mathcal{F}_q$, we also have theorem (2.2.7).

## 2.3 The twistor theory of GHC-manifolds

In this section we shall describe some distinguished bundles on a GHC-manifold M. But first we need some results regarding bundles on $\mathbb{C}P^1$ and representations of $SL(2,\mathbb{C})$. In what follows in this section $G = SL(2,\mathbb{C})$ and $B$ is the Borel subgroup of the upper diagonal matrices.

Let $L = \mathcal{O}(k)$ be the degree $k$ line bundle on $\mathbb{C}P^1$, for $k > 0$. Then the space of sections $H$ is an irreducible representation of $G$ from (2.2.6). Notice that the homogeneous bundle $\underline{H} = G \times_B H$ is trivial and that we have an equivariant map:

$$\underline{H} \to L,$$

which is given by evaluation. Namely, if $h \in H$ and $q \in G/B \cong \mathbb{C}P^1$ the map above sends $(h, q)$ to $h(q)$.

Now, define a bundle $K$ on $\mathbb{C}P^1$ given by the exact sequence of homogeneous bundles:

$$0 \to K \to \underline{H} \to L \to 0. \tag{2.1}$$

The cohomology exact sequence of the dual to the sequence (2.1) gives an exact sequence of $G$ representations:

$$0 \to H^* \xrightarrow{i} \hat{H} \xrightarrow{j} H' \to 0, \tag{2.2}$$

where $H' = H^1(L^*) \cong S^{k-2}\mathbb{C}^2$, $\hat{H} = H^0(K^*)$ and notice that $H^* \cong H^0(\underline{H^*})$, since $H$ is a trivial bundle. Moreover, the sequence (2.2) is split.

As a consequence of the long exact sequence in cohomology of (2.1) we have:

**Lemma 2.3.1.** The bundle $K^*$ decomposes as a direct sum of the bundles $\mathcal{O}(1)$.

**Remark 2.3.2.** The lemma above says that $\hat{H}$ can be given the structure of a quaternionic vector space.

For each $q \in \mathbb{C}P^1$ we notice that the line of highest weight vectors in $H$, denoted by $S_q$, is contained in $K$. Therefore, we can define a subbundle $S$ of $K$ whose fibre at $q$ is $S_q$.

We can consider the short exact sequence:

$$0 \to (K/S)^* \to K^* \to S^* \to 0. \tag{2.3}$$

The long exact sequence in cohomology starts as:

$$0 \to H^0((K/S)^*) \to \hat{H} \to H^0(S^*). \tag{2.4}$$

Now Borel-Weil theorem says that $H^*$ and $H^0(S^*)$ are isomorphic representations of $G$. Thus, we obtain a map $p : \hat{H} \to H^*$. It is proved in [Bie06] lemma 3.3 that $p$ is the left inverse for the map $i$ in (2.2).

We can now state and prove:

**Proposition 2.3.3.** We have an isomorphism of homogeneous bundles

$$(K/S)^* \cong G \times_B H'.$$

In particular, $H^0((K/S)^*) \cong H^1(\mathbb{C}P^1, L^*)$ and $K/S$ is trivial.

*Proof.* $H$ is isomorphic to $S^k\mathbb{C}^2$ as a representation of $G$ and we shall write the vectors of $H$ as $(v_0, v_1, \cdots, v_k)$ where the coordinates are relative to the weight decomposition with respect to $B$, where $v_0$ correspond to the minimal weight and $v_k$, the maximal weight. The fibre $K_{[1]}$ of $K$ at the point $[1] \in G/B \cong \mathbb{C}P^1$ is given by vectors of the form $(0, v_1, \cdots, v_k)$ and the fibre $S_{[1]}$, by $(0, \cdots, 0, v_k)$. The map $K_{[1]}/S_{[1]} \to S^{k-2}\mathbb{C}^2$ induced by

$$(0, v_1, \cdots, v_k) \mapsto (v_1, \cdots, v_{k-1})$$

is an isomorphism of $B$-modules. Since the bundles are homogeneous we have an isomorphism of bundles. □

We can now return our attentions to differential geometry. Let $M$ be a regular GHC-manifold and $Z$ its twistor space. Therefore, on the complexified case, we have the double fibration:

$$Z \overset{\eta}{\leftarrow} Y = M^{\mathbb{C}} \times \mathbb{C}P^1 \overset{p}{\rightarrow} M^{\mathbb{C}}. \tag{2.5}$$

**Definition 2.3.4.** The sheaf of *$\eta$-vertical holomorphic $l$-forms* $\Omega^l_\eta$ is defined by

$$\Omega^l_\eta = \Lambda^l(\Omega^1(Y)/\eta^*(\Omega^1(Z))).$$

**Proposition 2.3.5.** We have an isomorphism of sheaves $p_*(\Omega^1_\eta) \cong E^* \otimes \hat{H}$, where $\hat{H}$ is defined in the sequence (2.2).

*Proof.* Let $x \in M^{\mathbb{C}}$ and let $\mathbb{C}P^1_x$ be the fibre of $p$ over $x$. The $\eta$-normal bundle of $\mathbb{C}P^1_x$ in $Y$, this is to say, the normal bundle of $\mathbb{C}P^1_x$ along the fibres of $\eta$, is the bundle whose fibre at $(x, q) \in \mathbb{C}P^1_x$ is $\mathcal{K}_q$. From the definition of push-forward we have:

$$p_*(\Omega^1_\eta) = H^0(\mathbb{C}P^1_x, \mathcal{K}^*).$$

Now we have the decompositions $TM^{\mathbb{C}} = E_M \otimes H$ and $\mathcal{K} = \mathbb{C}^n \otimes K$, where $K$ is defined in (2.1). Since $H^0(\mathbb{C}P^1, K^*) = \hat{H}$, we have proved the proposition. □

**Proposition 2.3.6.** We have a splitting: $p_*(\Omega^1_\eta) \cong \Omega^1(M^{\mathbb{C}}) \oplus (E^* \otimes H')$.

**Proposition 2.3.7.** $p_*(\Omega^2_\eta) \cong (S^2E^* \otimes H_-) \oplus (\Lambda^2 E^* \otimes H_+)$, where $H_- = H^0(\mathbb{C}P^1, \Lambda^2 K^*)$ and $H_+ = H^0(\mathbb{C}P^1, S^2 K^*)$.

## 2.4 $\mathbb{R}^5$ as a GHC-manifold and its twistor theory

Example (2.2.5) defines $\mathbb{R}^5$ as a 2-GHC manifold. In this section we shall describe explicitly the twistor distribution for $\mathbb{R}^5$.

First we shall fix some notations that will be used throughout this thesis. Let $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ and put coordinates $\xi$ on $U = \mathbb{C} \subset \mathbb{C}P^1$ and $\xi'$ on $U' = (\mathbb{C} \setminus \{0\}) \cup \{\infty\}$ such that $\xi' = \dfrac{1}{\xi}$ on $U \cap U'$.

We can now fix holomorphic coordinates on $\mathcal{O}(k)$. Let $\pi : \mathcal{O}(k) \to \mathbb{C}P^1$ be the projection and define the open sets $U_0 = \pi^{-1}(U)$ and $U_1 = \pi^{-1}(U')$. Put coordinates $(\eta, \xi)$ in $U_0$ and $(\eta', \xi')$ on $U_1$ such that $\eta' = \eta/\xi^k$. Furthermore, from now on, whenever we refer to the total space of the bundle $\mathcal{O}(k)$, we shall name it $\mathbb{T}$.

Under these coordinates we can express a holomorphic section $p$ of $\mathcal{O}(k)$ as a polynomial of degree $k$ in $\xi$, namely $p(\xi) = a_0 + a_1\xi + \cdots + a_k\xi^k$. We can define an anti-holomorphic involution in the total space of $\mathcal{O}(k)$, in local coordinates, by $\tau(\eta, \xi) = (\overline{\eta}/\overline{\xi}^k, -1/\overline{\xi})$. Observe that $\tau$ covers the antipodal map in $\mathbb{C}P^1$ and therefore swaps the open sets $U_0$ and $U_1$. This map induces an involution in $H^0(\mathbb{C}P^1, \mathcal{O}(k))$, which will still be called by $\tau$, in the following way: If $p(\xi) = a_0 + a_1\xi + \cdots + a_k\xi^k$ is a holomorphic section of $\mathcal{O}(k)$, then $\tau(p) = b_0 + \cdots + b_k\xi^k$, where $b_j = (-1)^j\overline{a}_{k-j}$. For a point $(\eta, \lambda) \in \mathcal{O}(k)$ we can define the *$\alpha$-surface* $\Pi_{(\eta,\lambda)} = \{p(\xi) \in \mathbb{C}^5 | \, p(\lambda) = \eta\}$.

We can now concentrate on $\mathbb{R}^5$. A point $(x_0, x_1, x_2, x_3, x_4) \in \mathbb{R}^5$ corresponds to the section $p(\xi) = (x_0 + ix_4) + (x_1 + ix_3)\xi + x_2\xi^2 - (x_1 - ix_3)\xi^3 + (x_0 - ix_4)\xi^4 \in H^0(\mathbb{C}P^1, \mathcal{O}(4))$. Conversely, given a point $z \in Z$, we define the *real $\alpha$-surface* corresponding to $z$, denoted by $P_z$, to be the subspace in $\mathbb{R}^5$ consisting of real sections through $z$. Namely, we can consider $z \in U_0$ so that we can write $z = (\eta_0, \xi_0)$ in local coordinates, then we have $P_z = \{p \in H^0(\mathbb{C}P^1, \mathcal{O}(4)) | \; p(\xi_0) = \eta_0\}$.

We define $\mathbb{C}^5$ as the fourth symmetric power of the defining representation of $SL(2, \mathbb{C})$, therefore it can be described as the space of polynomials of degree $4$ in $\xi$ and the explicit action of $SL(2,\mathbb{C})$ on $\mathbb{C}^5$ is given by:

$$g \cdot p(\xi) = (c\xi + d)^4 \cdot p\left(\frac{a\xi + b}{c\xi + d}\right), \tag{2.6}$$

where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{C})$ and $p(\xi) \in \mathbb{C}^5$. We can understand this action as being induced by the action of $SL(2,\mathbb{C})$ in the total space of $\mathcal{O}(4)$ defined by

$$g \cdot (\eta, \xi) = \left(\frac{\eta}{(c\xi + d)^4}, \frac{a\xi + b}{c\xi + d}\right). \tag{2.7}$$

For the following proposition, we write an element $g \in SU(2) \subset SL(2,\mathbb{C})$ as $g = \left(\begin{smallmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{smallmatrix}\right)$.

**Proposition 2.4.1.** This action is compatible with the real structure $\tau$ in $\mathcal{O}(4)$, this is to say, $\tau g = g\tau$ for all $g \in SU(2)$.

*Proof.* The proof follows by direct computation using the action (2.7) and the definition of $\tau$. We have:

$$g \cdot \tau(\eta, \xi) = \left(\frac{\overline{\eta}}{(\overline{\alpha}\overline{\xi} + \overline{\beta})^4}, \frac{\overline{\beta\xi} - \overline{\alpha}}{(\overline{\alpha}\overline{\xi} + \overline{\beta})}\right) = \tau g \cdot (\eta, \xi).$$

□

For a point $\lambda \in U \subset \mathbb{C}P^1$ define $g_\lambda \in SU(2)$ by $g_\lambda = \frac{1}{\sqrt{1+\overline{\lambda}\lambda}} \begin{pmatrix} 1 & \lambda \\ -\overline{\lambda} & 1 \end{pmatrix}$ and notice that $g_\lambda$ is the unique, up to a $U(1)$ multiplication, element in $SU(2)$ such that $g_\lambda^{-1} \cdot (0,0) = (0,\lambda)$.

Now we shall explicitly describe the bundle $K$, which is defined in (2.1). First, we identify the tangent space $T_x\mathbb{C}^5$ at $x \in \mathbb{C}^5$ with $S^4\mathbb{C}^2$ and denote a vector in $T_x\mathbb{C}^5$ as a polynomial of degree $4$ in $\xi$. We then have:

**Proposition 2.4.2.** Let $\lambda \in U \subset \mathbb{C}P^1$, then the fibre $K_\lambda = Span_{\mathbb{C}}\{V_1^\lambda, V_2^\lambda, V_3^\lambda, V_4^\lambda\}$, where

- $V_1^\lambda = g_\lambda^{-1} \cdot \xi = \dfrac{1}{(1+\lambda\overline{\lambda})^2}(\overline{\lambda}\xi + 1)^3(\xi - \lambda);$
- $V_2^\lambda = g_\lambda^{-1} \cdot \xi^2 = \dfrac{1}{(1+\lambda\overline{\lambda})^2}(\overline{\lambda}\xi + 1)^2(\xi - \lambda)^2;$
- $V_3^\lambda = g_\lambda^{-1} \cdot \xi^3 = \dfrac{1}{(1+\lambda\overline{\lambda})^2}(\overline{\lambda}\xi + 1)(\xi - \lambda)^3;$
- $V_4^\lambda = g_\lambda^{-1} \cdot \xi^4 = \dfrac{1}{(1+\lambda\overline{\lambda})^2}(\xi - \lambda)^4.$

*Proof.* First remember that the fibre $K_\lambda$ is given by the holomorphic sections $p$ of $\mathcal{O}(4)$ such that $p(\lambda) = 0$. Then, notice that $K_0 = Span_{\mathbb{C}}\{\xi, \xi^2, \xi^3, \xi^4\}$. Since the group action is an endomorphism, the subspace of $H^0(\mathbb{C}P^1, \mathcal{O}(4))$ generated by the $V_k^\lambda$s is a basis for $K_\lambda$. This proves the proposition. □

**Remark 2.4.3.** It is important to highlight the use of the group action in the proof above. It will be important when we discuss aspects of the twistor theory of $\mathbb{R}^5$ that are invariant under the group action.

We have that in our case $\mathcal{K}_q = K_q$, for all $q \in \mathbb{C}P^1$. Therefore, applying the reality condition we have:

**Proposition 2.4.4.** The twistor distribution $\mathscr{F}$ on $\mathbb{R}^5 \times \mathbb{C}P^1$ is given by $\mathscr{F}_{(x,\lambda)} = Span_{\mathbb{R}}\{(v_1^\lambda, 0), (v_2^\lambda, 0), (v_3^\lambda, 0)\}$, where

$$v_1^\lambda = g_\lambda^{-1} \cdot (\xi - \xi^3) = \frac{(1+\overline{\lambda}\xi)^3(\xi - \lambda) - (1+\overline{\lambda}\xi)(\xi-\lambda)^3}{(1+\lambda\overline{\lambda})^2}, \tag{2.8}$$

$$v_2^\lambda = g_\lambda^{-1} \cdot \xi^2 = \frac{(1+\overline{\lambda}\xi)^2(\xi-\lambda)^2}{(1+\lambda\overline{\lambda})^2} \text{ and} \tag{2.9}$$

$$v_3^\lambda = g_\lambda^{-1} \cdot i(\xi + \xi^3) = i\left(\frac{(1+\overline{\lambda}\xi)^3(\xi - \lambda) + (1+\overline{\lambda}\xi)(\xi-\lambda)^3}{(1+\lambda\overline{\lambda})^2}\right). \tag{2.10}$$

Moreover, fixing an ordered frame $\{(v_1^\lambda, 0), (v_2^\lambda, 0), (v_3^\lambda, 0)\}$ for the twistor distribution gives an orientation for the vector space $\mathscr{F}_{(x,\lambda)}$.

**Remark 2.4.5.** We are describing $\mathbb{R}^5$ as the real form of the fourth symmetric power of the defining representation of $SU(2)$. Let $B_q$ be the Borel subgroup of $SU(2)$ corresponding to $q \in \mathbb{C}P^1$. If we consider the weight decomposition of $\mathbb{R}^5$ with respect to $B_q$, we must have that $v_1$, $v_2$ and $v_3$ are the weight-vectors corresponding to the weights $-2$, $0$ and $+2$ respectively. Therefore, the orientation mentioned in the proposition above is natural with respect to the $SU(2)$ action.

## 2.5 Invariant metric on $\mathbb{R}^5$, $\alpha$-surfaces and further properties

We begin this section by stating the following proposition:

**Proposition 2.5.1.** ([Muk03] page 27 proposition 1.25) Let $p(\xi) = a_0 + a_1\xi + a_2\xi^2 +$

$a_3\xi^3 + a_4\xi^4$ as a point in $T_x\mathbb{C}^5$. Define the quadratic form on $T_x\mathbb{C}^5$ by $N(p) = a_2^2 - 3a_1a_3 + 12a_0a_4$. Then, $N$ is $SL(2,\mathbb{C})$-invariant, this is to say, $N(g \cdot p) = N(p)$, for all $p \in T_x\mathbb{C}^5$ and $g \in SL(2,\mathbb{C})$.

We can apply the reality condition and restrict this form to the tangent space $T_x\mathbb{R}^5$ for $x \in \mathbb{R}^5$. For a tangent vector $p(\xi) = (x_0 + ix_4) + (x_1 + ix_3)\xi + x_2\xi^2 - (x_1 - ix_3)\xi^3 + (x_0 - ix_4)\xi^4 \in T_x\mathbb{R}^5$, we have

$$N(p) = x_2^2 + 3(x_1^2 + x_3^2) + 12(x_0^2 + x_4^2).$$

Thus, $N(p)$ is positive definite and defines an $SU(2)$-invariant metric $g$ on $\mathbb{R}^5$ by the polarisation formula. Moreover, we must have that $\{v_1^\lambda, v_2^\lambda, v_3^\lambda\}$, defined in proposition (2.4.4), is an orthogonal frame for the twistor distribution $\mathscr{F}$.

We now turn to the description of the leaves of the twistor foliation, the so called $\alpha$-surfaces. Let $z \in \mathbb{T}$, we define $\Pi_z$ to be the space of section of $\mathcal{O}(4)$ that contains $z$, in local coordinates, $\Pi_{(\eta,\xi)} = \{p \in \mathcal{O}(4) |\, p(\xi) = \eta\}$. Applying the reality structure, we define $P_z = \Pi_z \cap \tau(\Pi_z) \cap \mathbb{R}^5$.

**Proposition 2.5.2.** Let $(\eta, \lambda) \in U_0$, then

$$\Pi_{(\eta,\lambda)} = \left\{ \frac{1}{(1+\lambda\overline{\lambda})^4} \left[\eta(1+\overline{\lambda}\xi)^4 + a_1(1+\overline{\lambda}\xi)^3(\xi-\lambda) + a_2(1+\overline{\lambda}\xi)^2(\xi-\lambda)^2 + \right.\right.$$

$$\left.\left. + a_3(1+\overline{\lambda}\xi)(\xi-\lambda)^3 + a_4(\xi-\lambda)^4\right] |a_1,\, a_2,\, a_3,\, a_4 \in \mathbb{C} \right\}.$$

Applying the reality condition:

$$P_{(\eta,\lambda)} = \left\{ \frac{1}{(1+\lambda\overline{\lambda})^4} \left(\eta(1+\overline{\lambda}\xi)^4 + \overline{\eta}(\xi-\lambda)^4 + \right.\right.$$
$$x_1[(1+\overline{\lambda}\xi)^3(\xi-\lambda) - (1+\overline{\lambda}\xi)(\xi-\lambda)^3] +$$
$$\left.\left. x_2(1+\overline{\lambda}\xi)^2(\xi-\lambda)^2 - x_3[(1+\overline{\lambda}\xi)^3(\xi-\lambda) + (1+\overline{\lambda}\xi)(\xi-\lambda)^3]\right) | x_1,\, x_2,\, x_3 \in \mathbb{R} \right\}.$$

Now we shall concentrate on the tangent space to the $\alpha$-surfaces. We shall use the isomorphism $T\mathbb{R}^5 \cong T^*\mathbb{R}^5$ given by the above inner product and define what we shall call "natural forms" on $\Omega^{0,1}(\mathcal{O}(4))$.

The tangent space of the $\alpha$-surface $P_{(\eta,\lambda)}$, $\lambda \in \mathbb{C}P^1$, is generated by vectors $v_1^\lambda, v_2^\lambda, v_3^\lambda$, where

$$v_1^\lambda = \frac{1}{(1+\lambda\overline{\lambda})^2}[(1+\overline{\lambda}\xi)^3(\xi-\lambda) - (1+\overline{\lambda}\xi)(\xi-\lambda)^3], \tag{2.11}$$
$$v_2^\lambda = \frac{1}{(1+\lambda\overline{\lambda})^2}[(1+\overline{\lambda}\xi)^2(\xi-\lambda)^2], \tag{2.12}$$
$$v_3^\lambda = \frac{1}{(1+\lambda\overline{\lambda})^2}[(1+\overline{\lambda}\xi)^3(\xi-\lambda) + (1+\overline{\lambda}\xi)(\xi-\lambda)^3], \tag{2.13}$$

where $\lambda$ is the holomorphic coordinate for a point in $U_0 = \mathbb{C}P^1 \setminus \{\infty\}$.

Using the metric, we can find the dual to the basis above. Namely, we define $\omega_j^\lambda := g(v_j^\lambda, \cdot) \in \Omega^1 P_{(\eta,\lambda)}$. Using holomorphic coordinates $(a_0, a_1, a_2, a_3, a_4)$ for $\mathbb{C}^5$ we can write a frame for $(1,0)$-forms as $\{da_0, da_1, da_2, da_3, da_4\}$. Expanding the formulas for $v_j^\lambda$ above and calculating the $\omega_j^\lambda$ gives us:

$$\begin{aligned}
\omega_1^\lambda &= \tfrac{1}{3(1+\lambda\overline{\lambda})^2}[6\overline{f_0}da_0 + \tfrac{3}{2}\overline{f_1}da_1 + f_2da_2 + \tfrac{3}{2}\overline{f_3}da_3 + 6\overline{f_4}da_4],\\
\omega_2^\lambda &= \tfrac{1}{(1+\lambda\overline{\lambda})^2}[6\overline{\lambda}^2 da_0^2 - 3\overline{\lambda}(1-\lambda\overline{\lambda})da_1 +\\
&(1-4\lambda\overline{\lambda} + (\lambda\overline{\lambda})^2)da_2 + 3\lambda(1-\lambda\overline{\lambda})da_3 + 6\lambda^2 da_4],\\
\omega_3^\lambda &= \tfrac{i}{3(1+\lambda\overline{\lambda})^2}[6\overline{g_0}da_0 + \tfrac{3}{2}\overline{g_1}da_1 + \overline{g_2}da_2 + \tfrac{3}{2}\overline{g_3}da_3 + 6\overline{g_4}da_4],
\end{aligned}$$

where:

$$\begin{aligned}
f_0 &= (\lambda^3 - \lambda) = \overline{f_4},\\
f_1 &= (-3\overline{\lambda}\lambda + 1 - 3\lambda^2 + \overline{\lambda}\lambda^3) = -\overline{f_3},\\
f_2 &= (-3\overline{\lambda}^2\lambda + 3\overline{\lambda} - 3\overline{\lambda}\lambda^2 + 3\lambda) = \overline{f_2},\\
g_0 &= -(\lambda^3 + \lambda) = -\overline{g_4},\\
g_1 &= (-3\overline{\lambda}\lambda + 1 + 3\lambda^2 - \overline{\lambda}\lambda^3) = \overline{g_3},\\
g_2 &= (-3\overline{\lambda}^2\lambda + 3\overline{\lambda} + 3\overline{\lambda}\lambda^2 - 3\lambda) = -\overline{g_2}.
\end{aligned}$$

Observe that $(0, \omega_k^\lambda)$ defines a 1-form on $\mathbb{C}P^1 \times \mathbb{R}^5$. It will be denoted by the same symbol, $\omega_k^\lambda$.

Now we consider a section $s$ of $\eta : \mathbb{C}P^1 \times \mathbb{R}^5 \to \mathcal{O}(4)$, $\eta(q, m) = m(q)$, and shall find the pull back $\theta_k := s^*\omega_k$, notice that $\theta_k$ is independent of the section $s$. In the next chapter, we shall use $\theta_k^{0,1}$ to describe distinguished bundles on the total space of $\mathcal{O}(4)$ that correspond with a trivial $U(1)$ monopole data. Thus, this method allows us to define line bundles over $\mathcal{O}(4)$ with vanishing first Chern class.

We can choose an explicit section s of $\eta$:

$$\begin{aligned} s : \mathcal{O}(4) &\to \mathbb{C}P^1 \times \mathbb{R}^5, \\ (\mu, \lambda) &\mapsto \left(\lambda, \tfrac{1}{(1+\lambda\overline{\lambda})^2}(xv_0^\lambda + yv_4^\lambda)\right), \end{aligned} \tag{2.14}$$

where $\mu = x + iy$ and

$$v_0^\lambda = \frac{1}{(1+\lambda\overline{\lambda})^2}[(1+\overline{\lambda}\xi)^4 + (\xi - \lambda)^4]$$

and

$$v_4^\lambda = \frac{i}{(1+\lambda\overline{\lambda})^2}[(1+\overline{\lambda}\xi)^4 - (\xi - \lambda)^4].$$

The vector fields $v_0^\lambda$ and $v_4^\lambda$ on $\mathbb{R}^5$ correspond respectively to the maximal and minimal weights of $\mathbb{R}^5$, as a $SU(2)$ representation, with respect to the Borel subgroup $B_\lambda$.

We can now state the following:

**Proposition 2.5.3.** The $(0,1)$ parts of the *natural forms* are given by:

$$\begin{aligned} \theta_1^{0,1} &= \frac{3\mu}{(1+\lambda\overline{\lambda})^3} d\overline{\lambda}, \\ \theta_2^{0,1} &= 0, \\ \theta_3^{0,1} &= \frac{3i\mu}{(1+\lambda\overline{\lambda})^3} d\overline{\lambda}. \end{aligned}$$

**Remark 2.5.4.** Before we proceed with the proof of this result, we shall point out that the differential forms above shall be used in the description of distinguished line bundles on the total space of $\mathbb{T}$.

*Proof.* First write $\omega_j^\lambda = \sum_{j=0}^4 h_k^j(\lambda) da_k$, where the $h_k^j$s are given by the equations defining $\omega_j$s. The pullback by $s$ is given by:

$$s^*\omega_j = \sum_{j=0}^{4} h_k^j(s(\mu,\lambda))d(a_k(s(\mu,\lambda))),$$

where $a_k(s(\mu,\lambda))$ is the coordinate function and notice that $h_k^j(s(\mu,\lambda)) = h_k^j(\lambda)$.

Expanding $v_0^\lambda$ and $v_4^\lambda$ above we get

$$v_0^\lambda = \frac{1}{(1+\lambda\overline{\lambda})^2}\left[(1+\lambda^4)+4(\overline{\lambda}-\lambda^3)\xi+6(\lambda^2+\overline{\lambda}^2)\xi^2-4(\lambda-\overline{\lambda}^3)\xi^3+(1+\overline{\lambda}^4)\xi^4\right] \text{ and}$$

$$v_4^\lambda = \frac{i}{(1+\lambda\overline{\lambda})^2}\left[(1-\lambda^4)+4(\overline{\lambda}+\lambda^3)\xi+6(\overline{\lambda}^2-\lambda^2)\xi^2+4(\lambda+\overline{\lambda}^3)\xi^3+(\overline{\lambda}^4-1)\xi^4\right].$$

From the definition of $s$ we have:

- $x_0(s(\mu,\lambda)) = \frac{1}{(1+\lambda\overline{\lambda})^4}[x(1+\lambda^4)+iy(1-\lambda^4)] = \frac{1}{(1+\lambda\overline{\lambda})^4}[\mu+\overline{\mu}\lambda^4],$
- $x_1(s(\mu,\lambda)) = \frac{4}{(1+\lambda\overline{\lambda})^4}[x(\overline{\lambda}-\lambda^3)]+iy(\overline{\lambda}+\lambda^3) = \frac{4}{(1+\lambda\overline{\lambda})^4}[\mu\overline{\lambda}+\overline{\mu\lambda^3}],$
- $x_2(s(\mu,\lambda)) = \frac{6}{(1+\lambda\overline{\lambda})^4}[x(\lambda^2+\overline{\lambda}^2)+iy(\overline{\lambda}^2)-\lambda^2] = \frac{6}{(1+\lambda\overline{\lambda})^4}[\overline{\mu}\lambda^2+\mu\overline{\lambda}^2],$
- $x_3(s(\mu,\lambda)) = \frac{4}{(1+\lambda\overline{\lambda})^4}[-x(\lambda-\overline{\lambda}^3)+iy(\overline{\lambda}+\lambda)] = \frac{4}{(1+\lambda\overline{\lambda})^4}[-\overline{\mu}\lambda+\mu\overline{\lambda}^3]$ and
- $x_4(s(\mu,\lambda)) = \frac{1}{(1+\lambda\overline{\lambda})^4}[x(1+\lambda^4)+iy(\overline{\lambda}^4-1)] = \frac{1}{(1+\lambda\overline{\lambda})^4}[\overline{\mu}+\mu\overline{\lambda}^4].$

Since we are interested only in the $(0,1)$ part of the $s^*\omega_j$s, we shall compute:

$$dx_j(s(\mu,\lambda))^{0,1} = \frac{\partial x_j(s(\mu,\lambda))}{\partial\overline{\lambda}}d\overline{\lambda}+\frac{\partial x_j(s(\mu,\lambda))}{\partial\overline{\mu}}d\overline{\mu}.$$

Computing the derivatives:

$$\begin{cases} \dfrac{\partial x_0(s(\mu,\lambda))}{\partial \overline{\lambda}} = \dfrac{-4(\mu\lambda + \overline{\mu}\lambda^5)}{(1+\lambda\overline{\lambda})^5}, \\ \dfrac{\partial x_0(s(\mu,\lambda))}{\partial \overline{\mu}} = \dfrac{\lambda^4}{(1+\lambda\overline{\lambda})^4}. \end{cases}$$

$$\begin{cases} \dfrac{\partial x_1(s(\mu,\lambda))}{\partial \overline{\lambda}} = \dfrac{4[\mu(1+\lambda\overline{\lambda}) - 4\lambda(\mu\overline{\lambda} - \overline{\mu}\lambda^3)]}{(1+\lambda\overline{\lambda})^5}, \\ \dfrac{\partial x_1(s(\mu,\lambda))}{\partial \overline{\mu}} = \dfrac{-4\lambda^3}{(1+\lambda\overline{\lambda})^4}. \end{cases}$$

$$\begin{cases} \dfrac{\partial x_2(s(\mu,\lambda))}{\partial \overline{\lambda}} = \dfrac{12[\mu\overline{\lambda}(1+\lambda\overline{\lambda}) - 4\lambda(\overline{\mu}\lambda^2 + \mu\overline{\lambda}^2)]}{(1+\lambda\overline{\lambda})^5}, \\ \dfrac{\partial x_2(s(\mu,\lambda))}{\partial \overline{\mu}} = \dfrac{6\lambda^2}{(1+\lambda\overline{\lambda})^4}. \end{cases}$$

$$\begin{cases} \dfrac{\partial x_3(s(\mu,\lambda))}{\partial \overline{\lambda}} = \dfrac{4[3\overline{\lambda}^2\mu(1+\lambda\overline{\lambda}) - 4\lambda(-\overline{\mu}\lambda + \mu\overline{\lambda^3})]}{(1+\lambda\overline{\lambda})^5}, \\ \dfrac{\partial x_3(s(\mu,\lambda))}{\partial \overline{\mu}} = \dfrac{-4\lambda}{(1+\lambda\overline{\lambda})^4}. \end{cases}$$

$$\begin{cases} \dfrac{\partial x_4(s(\mu,\lambda))}{\partial \overline{\lambda}} = \dfrac{4\mu\overline{\lambda}^3[(1+\lambda\overline{\lambda}) - 4\lambda(\overline{\mu} + \mu\overline{\lambda}^4)]}{(1+\lambda\overline{\lambda})^5}, \\ \dfrac{\partial x_4(s(\mu,\lambda))}{\partial \overline{\mu}} = \dfrac{1}{(1+\lambda\overline{\lambda})^4}. \end{cases}$$

Substituting these into the equation for the pullback, we obtain the expressions stated in the proposition.

□

We now finish this chapter with a result concerning the behaviour of the $\alpha$-surfaces with respect to the real structure on $\mathbb{T}$. More specifically, for $z \in \mathbb{T}$, we want to compare $P_z$ with $P_{\tau(z)}$, where $\tau$ is the real structure in $\mathbb{T}$. With this intention we shall state the following results whose proofs follow by straightforward calculations and shall not be done here.

**Lemma 2.5.5.** Let $\lambda \in U \cap U' = \mathbb{C}P^1 \backslash \{\infty, 0\}$ and write $\dfrac{\overline{\lambda}}{\lambda} = x + iy$. The change of basis matrix from the basis $\{v_0^\lambda, v_1^\lambda, v_2^\lambda, v_3^\lambda, v_4^\lambda\}$ to $\{v_0^{(-1/\overline{\lambda})}, v_1^{(-1/\overline{\lambda})}, v_2^{(-1/\overline{\lambda})}, v_3^{(-1/\overline{\lambda})}, v_4^{(-1/\overline{\lambda})}\}$ is given by

$$\begin{pmatrix} x^2 - y^2 & 0 & 0 & 0 & -2xy \\ 0 & x & 0 & -y & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -y & 0 & -x & 0 \\ -2xy & 0 & 0 & 0 & -(x^2 - y^2) \end{pmatrix}. \tag{2.15}$$

**Corollary 2.5.6.** Under the notation of the lemma above, the change of basis from $\{v_1^\lambda, v_2^\lambda, v_3^\lambda\}$ to $\{v_1^{(-1/\overline{\lambda})}, v_2^{(-1/\overline{\lambda})}, v_3^{(-1/\overline{\lambda})}\}$ is given by:

$$\begin{pmatrix} x & 0 & -y \\ 0 & 1 & 0 \\ -y & 0 & -x \end{pmatrix}. \tag{2.16}$$

In particular, the $\alpha$-surfaces corresponding to $z \in \mathbb{T}$ and $\tau(z)$ are the same 3-dimensional affine subspaces of $\mathbb{R}^5$ with reverse orientation.

We shall conclude this chapter by stating the twistor correspondence between $\mathbb{R}^5$ and $\mathbb{T}$:

*Every point* $(x_0, x_1, x_2, x_3, x_4) \in \mathbb{R}^5$ *corresponds to the section* $p(\xi) = (x_0 + ix_4) + (x_1 + ix_3)\xi + x_2\xi^2 - (x_1 - ix_3)\xi^3 + (x_0 - ix_4)\xi^4 \in H^0(\mathbb{C}P^1, \mathcal{O}(4))$*. Conversely, every point* $z \in \mathbb{T}$ *corresponds to an oriented* 3*-dimensional affine subspace of* $\mathbb{R}^5$ *given explicitly in local coordinates by proposition* (2.5.2) *and whose orientation is given by the choice of oriented basis* $\{v_1^\lambda, v_2^\lambda, v_3^\lambda\}$.

# Chapter 3

# Twistor Approach to Monopoles on $\mathbb{R}^5$

## 3.1 Overview

In this chapter we present a higher dimensional version of Hitchin's twistorial approach [Hit82] to monopoles. There, Hitchin describes the twistor space of $\mathbb{R}^3$ as the space of oriented lines and this turns out to be the holomorphic tangent bundle to $\mathcal{O}(2)$, which is the total space of the line bundle $\mathbb{C}P^1$. As we have seen in the last chapter, we can use the $SU(2)$ action on $\mathbb{R}^5$ to define oriented 3-dimensional affine subspaces, the analogue of oriented lines in $\mathbb{R}^3$. First, we shall recover further material from [Bie06] and then we shall prove a Hitchin-Ward correspondence for Bogomolny pairs in $\mathbb{R}^5$ for the group $SU(2)$.

The motivation to study monopoles in higher dimensions comes from the desire to understand what type of geometry the moduli space has. As we have seen in the introduction, we do not rely on an energy functional, therefore the concept of charge differs from what we know about the $\mathbb{R}^3$ case. Instead, as we shall see in the next chapter, we use the spectral curve to define this invariant of the monopole. Another consequence of this is the non-existence of a natural $L^2$ metric on the moduli space and we shall use Nahm's equations to define a metric.

## 3.2 Bogomolny pairs on GHC-manifolds

Let $M$ be a regular GHC-manifold whose twistor space is $Z$ and consider the double fibration for the complexified GHC-manifold:

$$Z \overset{\eta}{\leftarrow} Y = \mathbb{C}P^1 \times M^{\mathbb{C}} \overset{p}{\rightarrow} M^{\mathbb{C}}.$$

Also, let $\Omega^*_\eta$ be the sheaf on $Y$ of $\eta$-vertical holomorphic forms and define the relative differential operator $d_\eta$ to be the composition map:

$$\Omega^0(Y) \overset{d}{\rightarrow} \Omega^1(Y) \xrightarrow{proj.} \Omega^1_\eta;$$

observe that $d_\eta$ annihilates $\eta^*\Omega^0(Z)$.

We shall now state and prove the following lemma:

**Lemma 3.2.1.** Let $F$ be a holomorphic bundle on $Z$. Then $d_\eta$ extends to a *flat relative connection* on $\eta^*F$, this is to say, an operator

$$\nabla_\eta : \eta^*F \rightarrow \Omega^1 \otimes F,$$

satisfying the Leibniz rule

$$\nabla_\eta(fs) = f\nabla_\eta(s) + d_\eta f \otimes s.$$

Conversely, if $\eta$ has simply connected fibres, then the holomorphic bundles on $Y$ arising from pull-back of a bundle on $Z$ are those which admit a flat relative connection.

*Proof.* Suppose $F$ has rank $k$, let $U$ and $U'$ be open sets on $Y$ and $\{e_0, \cdots, e_k\}$ and $\{e'_0, \cdots, e'_k\}$ be local frames for $\eta^*F$ on $U$ and $U'$ respectively. Let $g_{ij}$ be the transition function of $\eta^*F$ from $U'$ to $U$, this is to say, $e_i = \sum_{j=0}^k g_{ij}e'_j$, such that $d_\eta(g_{ij}) = 0$.

Let $s = \sum_{i=0}^k f_i \otimes e_i$ be a local section for $\eta^*F$ on $U$ define $\nabla_\eta$ on this open set by:

$$\nabla_\eta(s) = \sum_{j=0}^k d_\eta(f_i) \otimes e_i.$$

If we define it similarly for other trivialisations, we shall prove it is well defined.

In fact, on $U \cap U'$ we can write $s = \sum_{i=0}^k f_i g_{ij} \otimes e'_i$, therefore we have:

$$\nabla_\eta(s) = \sum_{j=0}^k d_\eta(f_i)g_{ij} \otimes e'_i = \sum_{j=0}^k d_\eta(f_i) \otimes e_i,$$

since $d_\eta(g_{ij}) = 0$. Then, $\nabla_\eta$ is well-defined and clearly satisfies the Leibniz rule.

Conversely, Let $E$ be a bundle on $Y$ endowed with a flat relative connection $\nabla_\eta$. Since $\eta$ has simply connected fibres, we can trivialise $E$ with relative parallel section, this is to say, we can find local frames $\{e_0, \cdots, e_k\}$ for $E$ such that $\nabla_\eta(e_j) = 0$. Now it is easy to see that the transition functions $g$ for this trivialisation must satisfy $d_\eta(g) = 0$, this means that $g$ is constant along the fibres of $\eta$. Thus, each transition function factors as $g = \eta \circ h$, where $h$ is the transition function for a holomorphic bundle $F$ on $Z$.

□

Suppose now that a holomorphic bundle $F$ on $Z$ is trivial on each section of $Z$, then the pull-back $\eta^* F$ is trivial on each fibre of $p$. Therefore, $\hat{F} = p_* \eta^* F$ is a vector bundle on $M^{\mathbb{C}}$ with the same rank as $F$. Moreover, from the lemma above, the relative flat connection $\nabla_\eta$ can be pushed down via $p$ to an operator

$$D : \hat{F} \to p_* \Omega^1_\eta \otimes \hat{F},$$

satisfying the Leibinitz rule

$$D(fs) = fD(s) + p_* d_\eta(f) \otimes s.$$

We now use a fact from the last chapter that there exists a canonical isomorphism

$$(p_* \Omega^1_\eta)_x \cong H^0(\mathbb{C}P^1_x, \mathcal{K}^*),$$

where $\mathcal{K}_q$ is the subspace of $T_x M \times \mathbb{C}P^1$ given by the kernel of the highest weight 1-forms for each $q \in \mathbb{C}P^1$. This isomorphism allows us to define a canonical map

$$e_q : p_* \Omega^1_\eta \to \mathcal{K}^*_q, \tag{3.1}$$

given by evaluation at $q \in \mathbb{C}P^1$.

Restrict $\hat{F}$ and the operator $D$ to the submanifold $p(\eta^{-1}(z))$ of $M^{\mathbb{C}}$, where $z \in Z_q$ is a point in the fibre of $Z$ at $q \in CP^1$. Since $\nabla_\eta$ is relatively flat, $D$ is a flat connection restricted to this submanifold. Conversely, notice that if we have a bundle $\hat{F}$ with an operator $D$ that is flat on $p(\eta^{-1}(z))$ for all $z \in Z$, then we obtain a bundle $p^*(\hat{F})$ endowed with a relative flat connection $\nabla_\eta = p^*(D)$.

Now consider the splitting

$$p_*\Omega^1_\eta = \Omega^1(M^{\mathbb{C}}) \oplus (E^* \otimes H').$$

Moreover, it is proved in [Bie06] that we have $p_* d_\eta = d \oplus 0$ under the splitting above. This means that $D$ can be written as $D = \nabla \oplus \Phi$, where $\nabla$ is an actual connection and $\Phi$ is a section of $End(\hat{F}) \otimes (E^* \otimes H')$ and is called the *Higgs field*. Moreover, on each $\alpha$-surface $\Pi_z = p(\eta^{-1}(z))$ by the composition:

$$\hat{F} \xrightarrow{\nabla \oplus \Phi} (\hat{F} \otimes E^* \otimes H^*) \oplus (\hat{F} \otimes E^* \otimes H') = \hat{F} \otimes E^* \otimes \hat{H} \xrightarrow{e_q} \hat{F} \otimes \Omega^1(\Pi_z).$$

This motivates the following definition:

**Definition 3.2.2.** Let $M$ be a regular GHC-manifold and $\hat{F}$ a vector bundle on $M^{\mathbb{C}}$, a *Bogomolny pair* on $\hat{F}$ is a pair $(\nabla, \Phi)$, where $\nabla$ is connection on $\hat{F}$ and $\Phi$ is a section of $End(\hat{F}) \otimes (E^* \otimes H')$, such that the connection $\nabla \oplus \Phi$, as defined by the composition above, is flat on each $\alpha$-surface. Applying the reality condition gives Bogomolny pairs on $M$.

We have proved the Hitchin-Ward correspondence for GHC-manifolds:

**Theorem 3.2.3.** Let $M$ be a regular GHC manifold. There is a one to one and onto correspondence between Bogomolny pairs $(\nabla, \Phi)$ for a bundle $\hat{F}$ on $M^{\mathbb{C}}$ and holomorphic bundles $F$ on $Z$ that are trivial on sections. The correspondence remains true in the presence of a real structure.

**Remark 3.2.4.** The theorem above gives a Bogomolny pair for the group $SL(n, \mathbb{C})$, where $n$ is the rank of $F$. By considering bundles $F$ on $Z$ whose structure group reduces we have the above correspondence between those bundles and Bogomolny pairs $(\nabla, \Phi)$ for a gauge group $G \subset SL(n, \mathbb{C})$. The objective of this chapter is to describe Bogomolny pairs for the group $SU(2)$ when $M = \mathbb{R}^5$.

**Remark 3.2.5.** In [Hit82], Hitchin proves a Hitchin-Ward correspondence between solutions to the Bogomolny equation in $\mathbb{R}^3$ and holomorphic bundles on the total space of the holomorphic tangent bundle $\mathbb{T}_2$ to $CP^1$ that are trivial on real sections. Therefore, $(\nabla, \Phi)$ is a Bogomolny pair in $\mathbb{R}^3$ if and only if it satisfies the Bogomolny equation $F_\nabla = *D_\nabla \Phi$.

## 3.3 The map $e_q$ and the Higgs field

We shall now turn our attentions to the case where $M = \mathbb{R}^5$. In the last section we saw that the map $e_q$, given by equation (3.1), plays a very important role in the description of Bogomolny pairs. In this section we shall describe it in the case $M = \mathbb{R}^5$.

We know that $p_*(\Omega^1_\eta) = H^* \oplus H'$, therefore, $e_q$ is an equivariant map

$$e_q : (\mathbb{C}^5)^* \oplus \mathbb{C}^3 \to K_q^*.$$

In this section we shall describe the real version of this map:

$$e_q : (\mathbb{R}^5)^* \oplus H'_{\mathbb{R}} \to (K^*_{\mathbb{R}})_q. \tag{3.2}$$

According to [Bie06], under the splitting above, the map $e_q$ acts on $\mathbb{R}^5$ as a projection and on $H'_{\mathbb{R}}$ it is described in the sequence (2.2). Then, we shall move towards the description of

$$e_q : H'_{\mathbb{R}} \to K^*_q$$

and its real version.

First we shall decompose $\mathbb{C}^3$ in weights with respect to $\lambda \in \mathbb{C}P^1$. Similarly to the the $\mathbb{C}^5$ case, defining $\mathbb{C}^3$ as polynomials of degree 2 in the variable $\xi$ allows us to write the action of $SL(2,\mathbb{C})$ in $\mathbb{C}^3$ by:

$$g \cdot p(\xi) = (c\xi + d)^2 \cdot p\left(\frac{a\xi + b}{c\xi + d}\right),$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$ and $p(\xi) \in \mathbb{C}^3$. Moreover, for a point $\lambda \in U \subset \mathbb{C}P^1$ define $g_\lambda \in SU(2)$ by $g_\lambda = \frac{1}{\sqrt{1+\bar{\lambda}\lambda}} \begin{pmatrix} 1 & \lambda \\ -\bar{\lambda} & 1 \end{pmatrix}$. Thus, the weight decomposition of $\mathbb{C}^3$ with respect to $\lambda$ is:

$$\begin{cases} \alpha_1^\lambda = g_\lambda^{-1}\cdot 1 & = \dfrac{1}{(1+\lambda\overline{\lambda})}(\overline{\lambda}\xi+1)^2 \\ & = \dfrac{1}{(1+\overline{\lambda}\lambda)}(1+2\overline{\lambda}\xi+\overline{\lambda}^2\xi^2), \\ \alpha_2^\lambda = g_\lambda^{-1}\cdot \xi & = \dfrac{1}{(1+\lambda\overline{\lambda})}(\overline{\lambda}\xi+1)(\xi-\lambda) \\ & = \dfrac{1}{(1+\overline{\lambda}\lambda)}(-\lambda+(1-\lambda\overline{\lambda})\xi+\overline{\lambda}\xi^2), \\ \alpha_3^\lambda = g_\lambda^{-1}\cdot \xi^2 & = \dfrac{1}{(1+\lambda\overline{\lambda})}(\xi-\lambda)^2 \\ & = \dfrac{1}{(1+\overline{\lambda}\lambda)}(\lambda^2-2\lambda\xi+\xi^2). \end{cases} \tag{3.3}$$

Write $(H')^* = \mathbb{C}^3$, $G = SL(2,\mathbb{C})$ and $B$ the Borel subgroup of upper diagonal matrices. Then, the $\alpha_j^\lambda$ trivialise the homogeneous bundle $(G\times_B H')^*$ over $\mathbb{C}P^1$, where $\lambda$ is a local holomorphic coordinate for $q\in\mathbb{C}P^1$. Then, from proposition (2.3.3) we know that there is an isomorphism $(K/S)^*_\lambda \to (H')^*$. This isomorphism allows us to describe a global frame for $(K/S)^*_\lambda$:

$$\begin{cases} F_1^\lambda = \dfrac{1}{(1+\overline{\lambda}\lambda)}(W_1^\lambda + 2\overline{\lambda}W_2^\lambda + \overline{\lambda}^2 W_3^\lambda), \\ F_2^\lambda = \dfrac{1}{(1+\overline{\lambda}\lambda)}(-\lambda W_1^\lambda + (1-\lambda\overline{\lambda})W_2^\lambda + \overline{\lambda}W_3^\lambda), \\ F_3^\lambda = \dfrac{1}{(1+\overline{\lambda}\lambda)}(\lambda^2 W_1^\lambda - 2\lambda W_2^\lambda + W_3^\lambda), \end{cases} \tag{3.4}$$

where $W_1^\lambda = \omega_1^\lambda + i\omega_3^\lambda$, $W_2^\lambda = \omega_2^\lambda$ and $W_3^\lambda = \omega_1^\lambda - i\omega_3^\lambda$, where the $\omega_j^\lambda$ were defined in the last chapter.

**Remark 3.3.1.** Before proceeding, it is important to notice that $e_q^*: End(E)\otimes(H')^* \to (K/S)^*$ is given by

$$e_q(\phi_1,\phi_2,\phi_3) = \sum_{j=1}^{3}\phi_j F_j^q.$$

We can now apply the reality condition on $(\mathbb{C}^5)^*$ to explicitly describe a global frame for $(K/S)^*_{\mathbb{R}}$:

$$\begin{cases} h_1^\lambda & = F_1^\lambda - F_3^\lambda = \dfrac{1}{(1+\lambda\overline{\lambda})}[(1-\lambda^2)W_1^\lambda + 2(\lambda+\overline{\lambda})W_2^\lambda - (1-\overline{\lambda}^2)W_3^\lambda], \\ h_2^\lambda & = F_2^\lambda = \dfrac{1}{(1+\lambda\overline{\lambda})}[-\lambda W_1^\lambda + (1+\lambda\overline{\lambda})W_2^\lambda + \overline{\lambda}W_3^\lambda], \\ h_3^\lambda & = i(F_1^\lambda + F_3^\lambda) = \dfrac{i}{(1+\lambda\overline{\lambda})}[(1+\lambda^2)W_1^\lambda + 2(\lambda-\overline{\lambda})W_2^\lambda + (1+\overline{\lambda}^2)W_3^\lambda]. \end{cases} \tag{3.5}$$

The proposition below describes the map $e_q$ and follows from the discussion above and proposition (2.3.3):

**Proposition 3.3.2.** Let $E$ be a vector bundle over $\mathbb{R}^5$, $\nabla$ a connection on $E$ and $\Phi = (\phi_1, \phi_2, \phi_3)$ a section of $End(E) \otimes \mathbb{C}^3$. On the $\alpha$-surface $P_{(\lambda,\mu)}$ we have:

$$e_q(\nabla \oplus \Phi)|_{P_{(\lambda,\mu)}} = \nabla|_{P_{(\lambda,\mu)}} + \sum_{j=1}^{3} \phi_j h_j^\lambda.$$

We conclude this section by mentioning how the results of this section give a natural orientation for the $\alpha$-surfaces. A straightforward calculation proves the following lemma:

**Lemma 3.3.3.** Let $\lambda \in \mathbb{C}P^1$ and $-1/\overline{\lambda}$ be its antipodal, then $h_j^{-1/\overline{\lambda}} = -h_j^\lambda$, for $j = 1, 2, 3$.

The following corollary says that a choice of frame for the homogeneous bundle $(K/S)^*$ naturally defines an orientation on the $\alpha$-surfaces:

**Corollary 3.3.4.** Let $P_{(\lambda,\mu)}$ be an $\alpha$-surface. Define its orientation by the 3-form

$$\Xi_{(\lambda,\mu)} = h_1^\lambda \wedge h_2^\lambda \wedge h_3^\lambda.$$

Then, $P_{\tau(\lambda,\mu)}$ and $P_{(\lambda,\mu)}$ are the same submanifold of $\mathbb{R}^5$ with reverse orientation.

**Remark 3.3.5.** It is important to notice here that the way the $h_j^\lambda$ were defined makes them dual to the $v_j^\lambda$, in other words, $h_j^\lambda(v_j^\lambda) = 1$. Therefore, $\Xi_{(\lambda,\mu)}(v_1^\lambda, v_2^\lambda, v_3^\lambda) = 1$ and the orientation in $P_{(\lambda,\mu)}$ given by the choice of order of the triple $v_1^\lambda, v_2^\lambda, v_3^\lambda$ is the same as the one given by the 3-form in the corollary above.

## 3.4 SU(2)-Bogomolny pairs on $\mathbb{R}^5$

We begin this section by defining the *fundamental forms*:

**Definition 3.4.1.** Consider $h_j^\lambda$, for $j = 1, 2, 3$, as a 1-form on $\mathbb{C}P^1 \times \mathbb{R}^5$. Let $s$ be the section of $\eta : \mathbb{C}P^1 \times \mathbb{R}^5 \to \mathbb{T}$ defined in (2.14) . Define the *fundamental forms* on $\mathbb{T}$ by $\Psi_j = s^* h_j^\lambda$.

In our local coordinates we have the following lemma:

**Lemma 3.4.2.** In local coordinates for the open set $U_0 \subset \mathbb{T}$, the fundamental forms are given by:

$$\begin{cases} \Psi_1 = & 6\mu \dfrac{(1-\overline{\lambda}^2)}{(1-\overline{\lambda}\lambda)^4} d\overline{\lambda}, \\ \Psi_2 = & -6\mu \dfrac{\overline{\lambda}}{(1-\overline{\lambda}\lambda)^4} d\overline{\lambda}, \\ \Psi_3 = & -6i\mu \dfrac{(1+\overline{\lambda}^2)}{(1-\overline{\lambda}\lambda)^4} d\overline{\lambda}. \end{cases}$$

*Proof.* The result follows from proposition (2.5.3) and from substituting $h_j^\lambda$ in proposition (3.5). Moreover, notice that $W_1^\lambda = \omega_1^\lambda + i\omega_3^\lambda$ and $W_3^\lambda = -\omega_1^\lambda + i\omega_3^\lambda$, therefore $s^*W_1^\lambda = 0$ and $s^*W_3^\lambda = -6\frac{\mu}{(1-\overline{\lambda}\lambda)^4}d\overline{\lambda}$. □

**Remark 3.4.3.**

1. The fundamental forms will play an important role in the explicit description of the holomorphic structure of the bundle corresponding to a Bogomolny pair on $\mathbb{R}^5$.

2. It is important to notice that each $\Psi_j$ defines a cohomology class in $H^1(\mathbb{T}, \mathcal{O})$ and hence, by exponentiation, an element of the Picard group $Pic_0(\mathbb{T})$. The line bundles corresponding to this class shall be explicitly described in the next section.

**Definition 3.4.4.** Let $E$ be a $SU(2)$ vector bundle on $\mathbb{R}^5$, this is to say, $E$ has complex rank 2 and is equipped with a symplectic form and a quaternionic structure. We say that the pair $(\nabla, \Phi)$ on $E$ is a $SU(2)$ Bogomolny pair if

1. $\nabla$ and $\Phi = (\phi_1, \phi_2, \phi_3)$ preserve the symplectic form;

2. For every $\alpha$-surface $P_z$, the connection $\nabla \oplus \Phi$, given in (3.3.2), is flat.

We know that $P_z$ is a leaf of the integral distribution $\{v_1^q, v_2^q, v_3^q\}$. From the previous section we can choose coordinates $\{\chi_1^z, \chi_2^z, \chi_3^z\}$ such that $d\chi_k^z = h_k^q$. If $A$ is the connection 1-form for $\nabla$ on $P_z$, then we can write:

$$e^q(\nabla \oplus -i\Phi)|_{P_z} = \sum_{k=1}^{3}(A_k - i\phi_k)d\chi_k^z.$$ [1]

[1] The $-i$ here will become clear in the proof of theorem (3.4.6).

The zero curvature condition for this connection gives:

$$F_{kj} + i\nabla_k \phi_j - i\nabla_j \phi_k - [\phi_j, \phi_k] = 0, \tag{3.6}$$

where $F$ is the curvature 2-form for $\nabla$.

Before proceeding to the main result of this section we shall state the following lemma which compares the connections $e^q(\nabla \oplus \Phi)|_{P_z}$ and $e_{\tau(q)}(\nabla \oplus \Phi)|_{P_{\tau(z)}}$:

**Lemma 3.4.5.** $e_{\tau(q)}(\nabla \oplus \Phi)|_{P_{\tau(z)}} = \nabla - \phi_1 h_1^q - \phi_2 h_2^q - \phi_3 h_3^q$.

*Proof.* The proof is a straightforward calculation using lemma (3.3.3). □

**Theorem 3.4.6.** Let $E$ be a $SU(2)$ bundle on $\mathbb{R}^5$. There is a 1-1 onto correspondence between $SU(2)$ Bogomolny pairs $(\nabla, \Phi)$ and holomorphic bundles $\tilde{E}$ on $\mathbb{T}$ satisfying:

(i) $\tilde{E}$ is trivial on real sections,

(ii) $\tilde{E}$ has a symplectic structure,

(iii) $\tilde{E}$ is equipped with a quaternionic structure $\sigma$ covering $\tau$, this is to say, $\sigma$ is an anti-holomorphic linear map

$$\sigma : \tilde{E}_z \to \tilde{E}_{\tau(z)},$$

such that $\sigma^2 = -id_{\tilde{E}_z}$.

*Proof.* We shall prove the conditions to reduce the gauge group to $SU(2)$ and describe the holomorphic structure for the bundle $\tilde{E}$ explicitly.

Let $(\nabla, \Phi)$ be a $SU(2)$ Bogomolny pair on $E$ consider the double fibration:

$$\mathbb{T} \xleftarrow{\eta} Y = \mathbb{C}P^1 \times \mathbb{R}^5 \xrightarrow{p} \mathbb{R}^5.$$

Let $s$ be the section of $\eta$ as defined in (2.14). We already know from theorem (3.2.3) that $\tilde{E} = s^*(p^*E)$ is holomorphic and trivial on real sections of $\mathbb{T}$, however we shall describe this holomorphic structure explicitly:

Define the operator $\overline{\partial} : \Omega^0(\mathbb{T}, \tilde{E}) \to \Omega^{0,1}(\mathbb{T}, \tilde{E})$ by:

$$\overline{\partial} t = \left( (s^*\nabla)t - i \left[ \sum_{k=1}^{3} (s^*\phi_k) t \otimes \Psi_k \right] \right)^{0,1},$$

where $t$ is a section of $\tilde{E}$. We claim that $\overline{\partial}$ is a holomorphic structure on $\tilde{E}$.

We have to prove that $\overline{\partial}^2 = 0$. To simplify our notation, write

$$\hat{\nabla} = s^*\nabla - i\Omega,$$

where $\Omega = \sum_{k=1}^{3} s^*\phi_k \otimes \Psi_k$. Observe that $\Omega$ is a section of $\Omega^1 \otimes End(\tilde{E})$ and this makes $\hat{\nabla}$ a connection on $\tilde{E}$. Then, $\overline{\partial}^2 = F_{\hat{\nabla}}^{0,2}$, where $F_{\hat{\nabla}}$ is the curvature of $\hat{\nabla}$. We have:

$$\begin{aligned} F_{\hat{\nabla}} &= s^*F_{\nabla} - i(s^*\nabla(\Omega)) - \Omega \wedge \Omega \\ &= s^*F_{\nabla} + i \left[ \sum_{k=1}^{3} \Psi_k \wedge s^*(\nabla\phi_k) \right] - i \left[ \sum_{k=1}^{3} (s^*\phi_k) \otimes d\Psi_k \right] - \sum_{j<k} [s^*\phi_j, s^*\phi_k] \Psi_j \wedge \Psi_k. \end{aligned}$$

Now, $F_{\hat{\nabla}}^{0,2}$ vanishes from the zero curvature condition (3.6) on every $\alpha$-surface and $d\Psi_k^{0,2} = \overline{\partial}\Psi_k = 0$. This proves that $\overline{\partial}$ is a holomorphic structure on $\tilde{E}$.

Let $\omega$ be a symplectic structure on $E$. Since $\nabla$ and $-i\Phi$ preserve $\omega$, from the definition of $\overline{\partial}$ we must have that $ss^*\omega$ is also preserved by $\overline{\partial}$. Therefore, $\tilde{E}$ is endowed with a symplectic structure compatible with $\overline{\partial}$.

To describe the quaternionic structure, we shall use an alternative description for the fibres of $\tilde{E}$. Let $z \in \mathbb{T}$ and define:

$$\tilde{E}_z; \{t \in \Gamma(P_z, E)| \; e_q(\nabla \oplus \Phi)t = 0\}.$$

Now $E$ has a quaternionic structure $\sigma$ and let $t \in \tilde{E}_z$, then $t$ satisfies

$$\left(\nabla - i \sum_{k=1}^{3} \phi_k h_k^q\right) t = 0$$

Applying $\sigma$:

$$\left(\nabla + i \sum_{k=1}^{3} \phi_k h_k^q\right) \sigma(t) = 0.$$

Using lemma (3.4.5):

$$\left(\nabla - i \sum_{k=1}^{3} \phi_k h_k^{\tau(q)}\right) \sigma(t) = 0,$$

Thus, $t \in E_z$ implies $\sigma(t) \in E_{\tau(z)}$. Therefore, $\sigma : E_z \to E_{\tau(z)}$ is anti-holomorphic and satisfies $\sigma^2 = -id_{E_z}$.

For the converse, we just need to observe that both the symplectic structure $\eta^*\omega$ and the quaternionic structure $\eta^*\sigma$ on the bundle $\eta^*(E)$ are compatible with the flat relative connection $\nabla_\eta$ on $\eta^*(\tilde{E})$. Furthermore, both structures remain compatible with $D$ on $E = p_*(\eta^*\tilde{E})$ when they are pushed down to $\mathbb{R}^5$ via $p$ and, therefore $\nabla$ and $\Phi$ are both compatible with the quaternionic and symplectic structures on $E$.

$\square$

The theorem above is phrased for the group $SU(2)$, however minor modifications in the real structure leads to Bogomolny pairs for other groups. In this thesis, we are interested in the construction of Bogomolny pairs for the group $SU(2)$ only.

## 3.5 The bundles $L_{(a,b,c)}$

To illustrate the construction of the previous section we shall find the explicit transition functions for the bundles on $\mathbb{T}$ that correspond to a trivial $U(1)$ Bogomolny pair corresponding to the following data: $E = \mathbb{R}^5 \times \mathbb{C}$, $\nabla = d$ and $\Phi = (-ia, -ib, -ic)$, where $a, b, c$ are real numbers, not all vanishing.

Let $\tilde{L}$ be the trivial complex line bundle on $\mathbb{T}$. From theorem (3.4.6) we can endow $\tilde{L}$ with a holomorphic structure $\overline{\partial}$ given by:

$$\overline{\partial}(s) = \frac{\partial s}{\partial \overline{\lambda}} + \Omega(s),$$

where $\Omega = \sum_{j=1}^{3} -i\phi_j \Psi_j$.

Let $l$ be a smooth trivialisation for $\tilde{L}$, i.e. $l$ is a non-vanishing complex function on $\mathbb{T}$, a local section $s = fl$ is holomorphic if and only if $\overline{\partial}(fl) = 0$. But this means that:

$$\frac{\partial f}{\partial \overline{\mu}} = 0$$

and

$$\frac{\partial f}{\partial \overline{\lambda}} = f\beta,$$

where $\Omega = \beta d\overline{\lambda}$.

Suppose that $f = g \cdot exp(u)$, with $g$ holomorphic, then

$$\frac{\partial f}{\partial \overline{\lambda}} = \frac{\partial u}{\partial \overline{\lambda}} g.$$

Thus, if we want to trivialise $\tilde{L}$ in a given open set, we have to find a function $u$, regular on this open set, such that $\frac{\partial u}{\partial \overline{\lambda}} = \beta$. In this case, $f = g \cdot exp(u)$ will be the given trivialisation.

We shall investigate three separate cases:

- $\phi_1 = \frac{i}{2}$, $\phi_2 = 0$ and $\phi_3 = 0$.

  The bundle corresponding to this data will be denoted by $L_{(\frac{1}{2},0,0)}$ In this case, we must have $\Omega = \Psi_1 = 3\mu \frac{(1-\overline{\lambda}^2)}{(1-\overline{\lambda}\lambda)^4} d\overline{\lambda}$. Then

$$\beta_1 = 3\mu \frac{(1-\overline{\lambda}^2)}{(1-\overline{\lambda}\lambda)^4}.$$

  Define

$$\tilde{u}_1 = -\frac{\mu}{(1-\overline{\lambda}\lambda)^3}\left(\frac{1}{\lambda} + \overline{\lambda}^3\right)$$

  and observe that $\tilde{u}_1$ is singular at $\infty$ and at 0. Now, define $\tilde{g}_1 = \frac{\mu}{\lambda}$ and

$$u_1 = \tilde{u}_1 + \tilde{g}_1 = \frac{\mu}{(1-\overline{\lambda}\lambda)^3}\left(3\overline{\lambda} + \lambda\overline{\lambda}^2 + \lambda^2\overline{\lambda}^3\right).$$

  Then, since $u_1$ is regular at 0 and singular at $\infty$, $f_0 = exp(u_1)$ defines a trivialisation of $L_{(\frac{1}{2},0,0)}$ in the open set $U_0$.

Now define $\tilde{\tilde{g}}_1 = \frac{\mu}{\lambda^3}$. Write $g_1 = -\tilde{g}_1 + \tilde{\tilde{g}}_1$ We have:

$$u_1 + g = \frac{\mu}{(1-\overline{\lambda}\lambda)^3}\left(\frac{1}{\lambda^3} + \frac{\overline{\lambda}}{\lambda^2} + \frac{\overline{\lambda}^2}{\lambda}\right).$$

Since $u_1 + g_1$ is regular at $\infty$ and singular at $0$, $f_1 = exp(u_1 + g_1)$ is a trivialisation of $L_{(\frac{1}{2},0,0)}$ over $U_1$. On the intersection $U_0 \cap U_1$ we have $f_1 e^{g_1} = f_0$. Then the transition function for $L_{(\frac{1}{2},0,0)}$ is given by

$$g^1_{01} = exp\left(-\mu\left(\frac{1}{\lambda} - \frac{1}{\lambda^3}\right)\right). \tag{3.7}$$

- $\phi_1 = 0$, $\phi_2 = i$ and $\phi_3 = 0$.

We shall denote the bundle corresponding to this data by $L_{(0,1,0)}$ and in this case we have

$$\Omega = \Psi_2 = -6\mu\frac{\overline{\lambda}}{(1-\overline{\lambda}\lambda)^4}d\overline{\lambda}.$$

Define

$$u_2 = \frac{\mu}{(1-\overline{\lambda}\lambda)^3}\left(\frac{3\overline{\lambda}}{\lambda} + \frac{1}{\lambda^2}\right).$$

We have that $u_2$ is singular at $0$ but regular at $\infty$, therefore $f_1 = u_2$ trivialises $L_{(0,1,0)}$ on $U_1$. Now, for $g_2 = -\frac{\mu}{\lambda^2}$ we have:

$$u_2 + g_2 = -\frac{\mu}{(1-\overline{\lambda}\lambda)^3}\left(3\overline{\lambda} + \lambda\overline{\lambda}\right),$$

which is regular at $0$, but singular at $\infty$. Thus, $f_0 = u_2 + g_2$ trivialises $L_{(0,1,0)}$ on $U_0$. On the intersection we then have $f_0 = e^{\mu/\lambda^2} f_1$. Therefore, the transition function of $L_{(0,1,0)}$ is:

$$g^2_{01} = exp\left(\frac{\mu}{\lambda^2}\right). \tag{3.8}$$

- $\phi_1 = 0$, $\phi_2 = 0$ and $\phi_3 = \dfrac{i}{2}$

  This case is similar to the first one and we shall write the transition function for this bundle without proof:

$$g_{01}^3 = exp\left(-i\mu\left(\frac{1}{\lambda}+\frac{1}{\lambda^3}\right)\right). \tag{3.9}$$

Now we state:

**Proposition 3.5.1.** The bundle $L_{(\frac{a}{2},b,\frac{c}{2})}$ has transition function

$$g_{01}^{(a,b,c)} = exp\left(-a\mu\left(\frac{1}{\lambda}-\frac{1}{\lambda^3}\right)+b\frac{\mu}{\lambda^2}-ic\mu\left(\frac{1}{\lambda}+\frac{1}{\lambda^3}\right)\right). \tag{3.10}$$

Since the real structure in our local coordinates is given by

$$\tau(\lambda,\mu) = (-\overline{\mu}/\overline{\lambda}^4, -1/\overline{\lambda}),$$

noting that $\tau$ interchanges $U_0$ and $U_1$ gives us

$$\tau(g_{01}^{(a,b,c)}) = \left(\overline{g_{01}^{(a,b,c)}}\right)^{-1}.$$

Therefore we have an anti-holomorphic isomorphism

$$\sigma : L_{(\frac{a}{2},b,\frac{c}{2})} \cong \left(L_{(\frac{a}{2},b,\frac{c}{2})}\right)^*.$$

## 3.6 The monopole equations

In [MS92], the authors prove a Hitchin-Ward correspondence for $\mathbb{C}^{n+1}$, obtaining a system of differential equations called the *Bogomolny hierarchy*. In this section we shall use the language and the results of GHC-manifolds to give a new construction for the same Bogomolny hierarchy equations.

In this section, let $M = \mathbb{C}^{n+1}$ and $E$ a rank $k$ bundle over $M$. Identifies $M = S^n(\mathbb{C}^2)$, where $\mathbb{C}^2$ is the defining representation of $SL(2, \mathbb{C})$, this identify $M$ as the space of global holomorphic sections of the bundle $\mathcal{O}(n)$ on $\mathbb{C}P^1$. Moreover, we shall also identify the cotangent space at $x \in M$ with $S^n(\mathbb{C}^2)$, this means we shall also write a differential form at $x$ as a polynomial of degree $n$ in $\xi$; or in $\xi'$ if we are dealing in the local coordinates at $U'$.

Denote the total space of $\mathcal{O}(n)$ by $\mathbb{T}_n$ and remember that for each $z \in \mathbb{T}_n$ we can define the $\alpha$-surface $\Pi_z = \{x \in M |\ x(q) = z\}$, where $z$ is in the fibre at $q \in \mathbb{C}P^1$, and denote its tangent bundle by $\mathcal{K}_z$. Now note that $\mathcal{K}_z^*$ at $x \in \Pi_z$ can be identified with polynomials of degree $n$ vanishing at $\lambda$ for all $x$, where $\lambda$ is the coordinate corresponding to $q \in \mathbb{C}P^1$. We shall start with the following proposition:

**Proposition 3.6.1.** Let $\lambda \in U \subset \mathbb{C}P^1$, define the 1-forms in $M$:

$$V_j^{\lambda} = \xi^{j-1}(\xi - \lambda), \text{ for } 1 \leq j \leq n.$$

Likewise, for $\lambda' \in U' \subset \mathbb{C}P^1$:

$$\tilde{V}_j^{\lambda'} = -\xi'^{n-j}(\xi' - \lambda') \text{ for } 1 \leq j \leq n.$$

Then, $(K^*_{(\mu,\lambda)})_x = Span_{\mathbb{C}}\{V_j^{\lambda};\ 1 \leq j \leq n\}$. Furthermore, the subbundle of $K^*$ whose fibre at $\lambda \in \mathbb{C}P^1$ is the subspace of $K^*_\lambda$ spanned by $V_j^\lambda$ (respect. $\tilde{V}_j^{\lambda'}$) is isomorphic to

$\mathcal{O}(1)$. This means that $K^* \cong \oplus_{j=1}^{n}\mathcal{O}(1)$. This splitting of $K^*$ is the one given in lemma (2.3.1).

*Proof.* We just have to observe that on the intersection $U \cap U'$ we have for all $j,\ 1 \leq j \leq n$:

$$V_j^\lambda = \lambda \tilde{V}_j^\lambda.$$

This means that the transition function, from $U'$ to $U$, of the mentioned bundles is $\lambda$. Therefore those bundles are isomorphic to $\mathcal{O}(1)$. □

**Remark 3.6.2.** By taking the dual, we have defined an integrable distribution $\mathcal{F}$ on the tangent bundle of $\mathbb{C}P^1 \times M$. Now, we can endow the $\alpha$-surfaces $\Pi_{(\mu,\lambda)}$ with coordinates $\chi_j^\lambda$ such that $V_j^\lambda = d\chi_j^\lambda$. Notice also that any differential 1-form with values in $\mathcal{F}^*$ can be written as a section of $\oplus_{j=1}^{n}\mathcal{O}(1)$, or more explicitly as $\gamma_1 V_1^\lambda + \gamma_2 V_2^\lambda + \gamma_3 V_3^\lambda + \gamma_4 V_4^\lambda$, where $\gamma_k = A_k + \lambda B_{k-1}$.

Let $F$ be a holomorphic bundle on $\mathbb{T}_n$ trivial on holomorphic sections and $\eta : \mathbb{C}P^1 \times M \to \mathbb{T}_n$ be the projection. From lemma (3.2.1), there exists a flat relative connection $\nabla_\eta$ on $\eta^* F$. From the proposition above, $\nabla_\eta$ can be given in terms of a matrix of 1-forms $\gamma$ with values on $\mathcal{F}^*$, this is to say, $\nabla_\eta$ can be written as:

$$\nabla_\eta(s) = d_\eta(s) + \gamma^\lambda(s).$$

Using the remark above, we see that

$$\nabla_\eta(s) = (\sum_k L_k V_k^\lambda)(s),$$

where $L_k(s) = \partial_k - \lambda\partial_{k-1} + A_k + \lambda B_{k-1} = \Delta_k + \lambda D_{k-1}$, with $\Delta_k = \partial_k + A_k$ and $D_{k-1} = \partial_{k-1} - B_{k-1}$, where $\partial_k = \dfrac{\partial}{\partial a_k}$ with respect to the euclidean coordinates on $\mathbb{C}^n$. The condition that the curvature of $\nabla_\eta$ vanishes implies the following system of commutator equations:

$$\begin{cases} [\Delta_j, \Delta_k] = 0 \\ [D_{j-1}, D_{k-1}] = 0 \\ [\Delta_k, D_{j-1}] - [\Delta_j, D_{k-1}] = 0, \text{ for } 1 \leq k \leq n. \end{cases} \tag{3.11}$$

This is the same system of equations discussed in [MS92]. Solving these equations means that we are finding the flat relative connection $\nabla_\eta$. Pushing $\nabla_\eta$ forward gives us the data for a Bogomolny pair with Higgs fields given by $\phi_k = A_k - B_k$.

## 3.7 Bogomolny pairs as dimensional reduction of self-dual connections in $\mathbb{R}^8$

We conclude this chapter by relating the concepts of Bogomolny pairs and Self-duality. We start with a 1-hypercomplex manifold $M$ and a complex vector bundle $E$ on $M$. Since, there is no Higgs field for a Bogomolny pair, we can say that a connection $\nabla$ is *self-dual*, or *hyperholomorphic* [Ver96], if $\nabla$ restricted to the $\alpha$-surfaces is flat.

Remember that we have $TM^{\mathbb{C}} = E_M \otimes H$, then we have a decomposition $\Lambda^2 T^* M^{\mathbb{C}} = (S^2 E_M^* \otimes \Lambda^2 H) \oplus (\Lambda^2 E_M^* \otimes S^2 H)$. We can state results from [Bie06] and [Ver96]:

**Proposition 3.7.1.** With the notation as above, the following are equivalent:

(i) $\nabla$ is self-dual,

(ii) $F_\nabla$ lies in the component $(S^2E_M^* \otimes \Lambda^2 H)$ in the decomposition above,

(iii) $F_\nabla$ is $SU(2)$ invariant.

A connection $\nabla$ is called *Yang-Mills* if $\nabla$ is a minimal of the *Yang-Mills* functional:

$$\mathcal{Y}(\nabla) = \int_M |F_\nabla|^2 \text{vol}_g, \tag{3.12}$$

where $|F_\nabla|^2 = \text{Tr}(F_\nabla \wedge *F_\nabla)$ and $\text{vol}_g$ is the volume form on $M$ with respect to $g$.

**Remark 3.7.2.** It is proved in [Ver96] that if a connection $\nabla$ satisfies conditions (i), (ii) or (iii) of the proposition above, then it is Yang-Mills.

In order for us to explain the relations between self-dual connections and Bogomolny pairs, we shall first recover the following result from [Bie06]:

**Theorem 3.7.3.** Let $M$ be a $k$-hypercomplex manifold, then there exists a hypercomplex manifold $\tilde{M}$ with a projection $p : \tilde{M} \to M$ such that the pair $(\nabla, \Phi)$ on a bundle $F$ on $M$ is a monopole if and only if $p^*(\nabla \oplus \Phi)$ on the bundle $p^*F$ on $\tilde{M}$ is self-dual.

We shall not prove this result here, however we shall roughly see how $\tilde{M}$ is constructed. First, remember the representation $H'$ defined in equation (2.2). Now, consider the bundle $Z = (E_M \otimes H')^{\mathbb{R}}$ over $M$. It is then possible to define an integrable hypercomplex structure in the zero section of the fibration $p : Z \to M$ and this neighbourhood will be $\tilde{M}$. When $M$ is flat, lemma (2.3.1) gives an integrable hypercomplex structure for the total space of $Z$.

In our case, $M = \mathbb{R}^5$, $E$ is the trivial line bundle over $M$ and $H' \cong S^2(\mathbb{C}^2)$. Then $\tilde{M} = \mathbb{R}^5 \times \mathbb{R}^3$. The results above give that Bogomolny pairs in $\mathbb{R}^5$ are obtained from self-duality in $\mathbb{R}^8$. However, it is important to remark that a self-dual connection $\nabla$ in $\mathbb{R}^8$ does not have finite energy [JT80], this is to say, $\mathcal{Y}(\nabla)$ is not finite. This makes us to believe that the Yang-Mills-Higgs functional on $\mathbb{R}^5$, obtained from (3.12) by dimensional reduction, also does not have finite energy solutions.

# Chapter 4

# Spectral Curves and Nahm's equations

## 4.1 Overview

Our main objective now is to construct Bogomolny pairs on $\mathbb{R}^5$ and we take the spectral curve approach to reach this goal. We mainly follow the methods in [HM89] and [Hit83]. We begin by defining the spectral curve in $\mathbb{T}$ and explain how we obtain holomorphic bundles on $\mathbb{T}$ that correspond to Bogomolny pairs on $\mathbb{R}^5$ from the spectral curves.

Then, we impose extra conditions on the spectral curves. Namely, they are the analogue of the conditions satisfied by a spectral curve for a monopole on $\mathbb{R}^3$. Furthermore, those conditions also allow us to derive a system of Nahm's equations and find their boundary conditions. It turns out that there is an equivalence between solutions to our new Nahm's equations and the spectral curves in $\mathbb{T}$.

## 4.2 Spectral curves and Beauville's theorem

Let $S \subset \mathcal{O}(4)$ be a real compact algebraic curve in the linear system $|\mathcal{O}(4k)|$, this is to say, on the open set $U$, $S$ can be defined by the equation

$$P(\xi,\eta) = \eta^k + a_1(\xi)\eta^{k-1} + \cdots + a_{k-1}(\xi)\eta + a_k(\xi) = 0, \tag{4.1}$$

where $a_j(\xi)$ is a polynomial of degree $4j$ in $\xi$.

Next we shall discuss how those curves relate to holomorphic bundles on $\mathbb{T}$. In this section, we shall use $L = L_{(a,b,c)}$ for any non-zero $(a,b,c) \in \mathbb{R}^3$, where $L_{(a,b,c)}$ is defined in (3.5.1). Then, there exists a short exact sequence of sheaves:

$$0 \to \mathcal{O}(L^2(-4k)) \to \mathcal{O}(L^2) \to \mathcal{O}_S(L^2) \to 0. \tag{4.2}$$

This gives a long exact sequence on cohomology:

$$0 \to H^0(\mathbb{T}, L^2(-4k)) \to H^0(\mathbb{T}, L^2) \to H^0(S, L^2) \tag{4.3}$$

$$\to H^1(\mathbb{T}, L^2(-4k)) \to H^1(\mathbb{T}, L^2) \to H^1(S, L^2) \cdots \tag{4.4}$$

Assume further that $S$ is such that $L^2|_S$ is trivial. This implies that $H^1(S, L^2) = 0$ and (4.3) becomes:

$$0 \to H^0(S, L^2) \xrightarrow{\delta} H^1(\mathbb{T}, L^2(-4k)) \xrightarrow{\otimes\psi} H^1(\mathbb{T}, L^2) \to 0, \tag{4.5}$$

where $\psi \in H^0(\mathbb{T}, \mathcal{O}(4k))$ is the section defining $S$.

Choose a trivialisation $s$ of $L^2$ over $S$, this is to say, $s$ is a non-zero element in $H^0(S, L^2)$. Define the bundle $\tilde{E}$ over $\mathbb{T}$ by the cohomology class $\delta(s)$. This means $\tilde{E}$ is given as an extension:

$$0 \to L(-2k) \xrightarrow{\alpha} \tilde{E} \xrightarrow{\beta} L^*(2k) \to 0.$$

We then have the following:

**Proposition 4.2.1.** $\tilde{E}$ satisfies the following conditions:

(i) $\tilde{E}$ has a symplectic structure,

(ii) $\tilde{E}$ is equipped with a quaternionic structure $\sigma$ covering $\tau$, this is to say, $\sigma$ is an anti-holomorphic linear map

$$\sigma : \tilde{E}_z \to \tilde{E}_{\tau(z)},$$

such that $\sigma^2 = -id_{\tilde{E}_z}$.

*Proof.* The properties (i) is straightforward from the definition of $\tilde{E}$. For (ii), let $\sigma : L \to L^*$ be the anti-holomorphic isomorphism. We can define a bundle $\sigma(\tilde{E})$ on $\mathbb{T}$ via the extension:

$$0 \to L^*(-2k) \xrightarrow{\alpha'} \sigma(\tilde{E}) \xrightarrow{\beta'} L(2k) \to 0.$$

Now, we can extend the antiholomorphic isomorphism $L \cong L^*$ to an antiholomorphic isomorphism $\tilde{E} \cong \sigma(\tilde{E})$. □

We shall need the following facts about these spectral curves [AHH90]

**Proposition 4.2.2.** The cohomology group $H^1(\mathbb{T}, \mathcal{O}_\mathbb{T})$ is generated by $\eta^i/\xi^j, 0 < i \leq k-1, 0 < j < ki$.

Noticing that $exp : H^1(\mathbb{T}, \mathcal{O}_\mathbb{T}) \to Pic_0(S)$ is an isomorphism, the bundles with vanishing degree over $S$ are generated by $exp(\eta^i/\xi^j)$.

**Proposition 4.2.3.** The natural map $H^1(\mathbb{T}, \mathcal{O}_\mathbb{T}) \to H^1(S, \mathcal{O}_S)$ is surjective, which means that $H^1(S, \mathcal{O}_S)$ is generated by $\eta^i/\xi^j, 0 < i \leq k-1, 0 < j < ki$.

Similar to the remark above, if $S$ is smooth the proposition above gives degree zero line bundles on $S$. In this chapter I will assume the curves are smooth, the non-smooth case is essentially done in [AHH90].

A bit of notation: Let $\pi : \mathbb{T} \to \mathbb{C}P^1$, then $\mathcal{O}_\mathbb{T}(l)$ denotes the pull-back of $\mathcal{O}(l)$ by $\pi$. Also, if $F$ is a sheaf on $\mathbb{T}$ we denote by $F(l)$ the sheaf $F \otimes \mathcal{O}_\mathbb{T}(l)$

**Definition 4.2.4.** The theta divisor $\Theta$ in $S$ is the set of line bundles of degree $g-1$ that have a non-zero global section. The affine Jacobian $J^{g-1}$ is the set of line bundles of degree $g-1$ on $S$.

**Theorem 4.2.5** (Beauville [Bea90])**.** There is a 1-1 correspondence between $J^{g-1}\setminus\Theta$ and $Gl(k, \mathbb{C})$-conjugacy classes of $gl(k, \mathbb{C})$-valued polynomials $A(\xi) = \sum_{j=0}^{4} A_j\xi^j$ such that $A(\xi)$ is regular for every $\xi$ and the characteristic polynomial of $A(\xi)$ is (4.1).

We shall now give the idea of this construction with enough details that will be necessary when we see the boundary conditions. In order to do this, we need the following lemma:

**Lemma 4.2.6.** Let $E$ be an invertible sheaf on $\mathbb{T}$ whose degree is $g-1$ and such that $H^0(S,E)=0$, then $H^0(S,E(1)) \cong \mathbb{C}^k$.

*Proof.* Let $\xi_0 \in \mathbb{C}P^1$ and denote by $D_{\xi_0}$ the divisor corresponding to the meromorphic function $(\xi-\xi_0)$ on $S$, this means that as a set $D_{\xi_0}$ consists of the points of $S$ in the fibre $E_{\xi_0}$, which is a set of $k$ points, counted with multiplicities.

Now consider the exact sequence of sheaves; [GH94] page 139.

$$0 \to \mathcal{O}_S(E) \to \mathcal{O}_S(E(1)) \to \mathcal{O}_{D_{\xi_0}}(E(1)) \to 0.$$

From Riemann-Roch, the hypothesis $H^0(S,E)=0$ implies that $H^1(S,E)=0$. Taking the exact sequence on cohomology and noticing that $H^0(D_{\xi_0},E)=\mathbb{C}^k$, since $D_{\xi_0}s$ are a set of $k$ points counted with multiplicity, gives the required isomorphism.

□

For $\xi \in U$, define a map $Z : H^0(D_\xi, E(1)) \to H^0(D_\xi, E(1))$ given by multiplication by $\eta$. We define the linear map

$$A(\xi) : H^0(S,E(1)) \to H^0(S,E(1))$$

by the commutative diagram:

$$\begin{array}{ccc} H^0(S,E(1)) & \longrightarrow & H^0(D_\xi,E(1)) \\ {\scriptstyle A(\xi)}\downarrow & & \downarrow{\scriptstyle Z} \\ H^0(S,E(1)) & \longrightarrow & H^0(D_\xi,E(1)), \end{array}$$

where the horizontal maps are the isomorphism given in the lemma (4.2.6).

Conversely, let $A(\xi) = \sum_{j=0}^{4} A_j \xi^j$ be a regular matricial polynomial and define a sheaf $E(1)$ over $\mathbb{T}$ via the exact sequence:

$$0 \to \mathcal{O}(-4)_{\mathbb{T}}^{\oplus k} \xrightarrow{\eta - A(\xi)} \mathcal{O}_{\mathbb{T}}^{\oplus k} \to E(1) \to 0. \tag{4.6}$$

$E(1)$ is supported on $S$ and since $A(\xi)$ has 1-dimensional kernel, $E(1)$ is a line bundle of degree $g-1$, where $g$ is the genus of $S$.

## 4.3 Monopoles

In this section, we shall use the following definition:

**Definition 4.3.1.** A spectral curve is a compact algebraic curve $S$ in $\mathbb{T}$ satisfying:

(i) $S$ is a compact algebraic real curve in the linear system $\mathcal{O}(4k)$, therefore it is given by an equation of the type

$$P(\xi,\eta) = \eta^k + a_1(\xi)\eta^{k-1} + \cdots + a_{k-1}(\xi)\eta + a_k(\xi) = 0, \tag{4.7}$$

where $a_j(\xi)$ is a polynomial of degree $4j$ in $\xi$.

(ii) $S$ has no multiple components and $L(2k-2)$ is real.

(iii) The line bundle $L$ has order 2 on $S$.

(iv) $H^0(S, L^z(2k-3)) = 0$ for $z \in (0,2)$.

**Remark 4.3.2.** A notice about the notation. By $L$ in the definition above we mean any of the distinguished line bundles $L_{(a,b,c)}$ described in the last chapter. Furthermore, whenever we omit the subscript $(a,b,c)$ we mean one of those bundles, unless explicitly stated.

Using the results from the last section, we are able to make the following definition:

**Definition 4.3.3.** A $SU(2)$ Bogomolny pair $(\nabla, \Phi)$ on $\mathbb{R}^5$ is a $SU(2)$-*monopole* if the corresponding holomorphic bundle $\tilde{E}$ on $\mathbb{T}$ is defined via a spectral curve $S$ satisfying the conditions of definition (4.3.1). We say the *algebraic charge* of the monopole is $k$ if the curve $S$ corresponding to the monopole has degree $k$.

We shall call the algebraic charge shortly by charge, however we must bear in mind that we do not have yet a topological definition of charge for monopoles in $\mathbb{R}^5$.

**Remark 4.3.4.** Remember that when we constructed the bundle over $\mathbb{T}$ from the curve $S$, we chose a trivialisation $s$ of $L^2$ over $S$, which is a complex valued holomorphic function on $S$, since $L^2$ is a trivial line bundle. Now we can normalise $s$ into $\tilde{s} = \frac{s}{|s|}$ and $s$ and $\tilde{s}$ determine the same bundle $\tilde{E}$ on $\mathbb{T}$. Thus, the trivialisation $s$ of $L^2$ on $S$ can be chosen to satisfy $|s| = 1$, this means that the spectral curve determines the monopole up to a $U(1)$ factor.

**Example 4.3.5.** A spectral curve $S$ for $k = 1$ is given by the equation $\eta + a_1(\xi) = 0$, where $a_1$ is a polynomial of degree 4. Imposing the condition that $S$ is real gives $\overline{a}_1(\xi) = \overline{\xi}^4 a_1(-\frac{1}{\overline{\xi}})$, but this condition says that $S$ is a real section $P$ of $\mathbb{T}$ over $\mathbb{C}P^1$. Moreover, conditions $(ii)$ and $(iii)$ are clearly satisfied, remember that $L$ is trivial on real sections of $\mathbb{T}$ since it corresponds to a bogomolny pair on $\mathbb{R}^5$. For condition $(iv)$, notice that on $P$, we have $L^z(2k-3) = L^z(-1) \cong \mathcal{O}(-1)$. Then, we conclude that the spectral curves for charge 1 monopoles correspond to real sections of $\mathbb{T}$.

Now we can state:

**Proposition 4.3.6.** The moduli space of charge 1 monopoles in $\mathbb{R}^5$ is $\mathcal{M}_1 = \mathbb{R}^5 \times S^1$.

The circle $S^1$ factor comes from the freedom of choice of the trivialisation of $L^2$ over $S$. Notice also that this result holds for spectral curves $S$ with respect to any bundle $L_{(a,b,c)}$.

We can also conjecture the dimension of the moduli space of higher charges. Namely, the spectral curve is determined by $(k+1)(2k+1)-1$ parameters, however it must satisfy some constraints also. The number of constraints are counted by the genus of the curve. Let $\mathcal{M}_k$ be the moduli space of charge $k$. If $\mathcal{M}_k$ is non empty, we must have:

$$\dim\mathcal{M}_k = (k+1)(2k+1) - 1 - (k-1)(2k-1) + 1 = 6k.$$

Observe that we added 1 because of the $U(1)$ freedom.

**Remark 4.3.7.** Our main motivation to make this definition is that the conditions in (4.3.1) are similar to the conditions for the spectral curves for monopoles in $\mathbb{R}^3$ [Hit83].

Furthermore, condition (iv) above allows us to define a flow of endomorphisms for a bundle over the interval $(0,2)$ whose fibre at $z \in (0,2)$ is $H^0(S, L^z(2k-2))$.

## 4.4 From linear flows to Nahm's equations

Also, in the last section, we saw that in order for us to construct the monopole data we need a antiholomorphic isomorphism $L = L^*$ on $S$ and this implies that $L^2$ is trivial on $S$. This condition and the condition (iii) of (4.3.1) above implies that the element $g_{(a,b,c)} \in H^1(S, \mathcal{O})$ is a lattice point in $H^1(S, \mathbb{Z})$. Thus, the straight line between $0$ and $g_{(a,b,c)}$ defines a morphism, which we will refer as a flow:

$$h : S^1 \to H^1(S, \mathcal{O})/H^1(S, \mathbb{Z}) \cong Pic^0(S)$$

$$exp(i\pi z) \mapsto exp(izg_{(a,b,c)}),\ z \in [0,2].$$

Let $Pic^0(S)$ be the group of degree $0$ line bundles on $S$ and $J^{g-1}(S)$, the Jacobian of line bundles of degree $g-1$, where $g$ is the genus of $S$. We can identify $Pic^0(S)$ with $J^{g-1}(S)$ by doing $F \to F(2k-3)$, since $deg(F(2k-3)) = k(2k-3) = 2k^2 - 3k = (k-1)(2k-1) - 1$. Now, $h$ can be considered a flow in the Jacobian and the condition (iv) in (4.3.1) says that, for $z \in (0,2)$, $h(z)$ is not in the theta divisor, the line bundles in $J^{g-1}(S)$ with a non-vanishing holomorphic section.

These properties will allow us to derive the Nahm's equations satisfying the appropriate boundary conditions. However, the boundary conditions will be treated in the next section.

**Lemma 4.4.1.** For $z \in (0,2)$ we have $dimH^0(S, L^z(2k-2)) = k$.

*Proof.* To prove the lemma above, we shall compactify $\mathbb{T}$ to an algebraic surface $\tilde{\mathbb{T}}$ defined as the total space of the $\mathbb{C}P^1$-bundle associated to $\mathcal{O}(4)$[1], this is to say

$$\tilde{\mathbb{T}} = \mathbb{P}(\mathcal{O}(4) \oplus \mathcal{O}) = \mathbb{P}(\mathcal{O}(2) \oplus \mathcal{O}(-2)).$$

We shall now define some distinctive divisor classes and their intersection number on $\tilde{\mathbb{T}}$. Let $E_0$ be the image of the section $(0,1)$ of $\mathcal{O}(4) \oplus \mathcal{O}$. Let $\sigma$ be a section of $\mathcal{O}(4)$ and notice that away from the zeros of $\sigma$, $(\sigma, 0)$ gives a curve in $\tilde{\mathbb{T}}$, now, let $E_\infty$ be the projective closure of this curve ($E_\infty$ does not depend on the choice $\sigma$). Finally, let $C$ be any fibre of the bundle map $\pi : \tilde{\mathbb{T}} \to \mathbb{C}P^1$ and notice that the bundle associated to the divisor $C$ is $\pi^*(\mathcal{O}(1))$. We then have the intersection numbers ([GH94] pages 517-518):

$$E_0 \cdot E_0 = 4,$$

$$C \cdot C = 0,$$

$$E_0 \cdot C = 1.$$

Furthermore, the divisor class of the canonical bundle of $\tilde{\mathbb{T}}$ is given by ([GH94] page 519)

$$K = -2E_0 + 2C.$$

Last but not least, we shall note that the curve $S$ in $\tilde{\mathbb{T}}$ is defined by a divisor of a section $\tilde{\psi} \in H^0(\tilde{\mathbb{T}}, kE_0)$.

[1]this surface is also know as the $n$-Hirzebruch surface.

Now the following sequence is exact[2]

$$0 \to \mathcal{O}_{\tilde{\mathbb{T}}}([lC - kE_0]) \to \mathcal{O}_{\tilde{\mathbb{T}}}([lC]) \to \mathcal{O}_S([lC]) \to 0.$$

Therefore, the Euler characteristics of the bundles above must satisfy

$$\chi([lC]|_S) = \chi([lC]) - \chi([lC - kE_0]).$$

We can now use the Riemann-Roch theorem for $\tilde{\mathbb{T}}$ (as in [GH94] page 472) and the relations above to compute the Euler characteristics:

$$\begin{aligned}\chi([lC]|_S) =& \frac{1}{2}([lC]\cdot[lC] - [lC]\cdot K) - \frac{1}{2}([lC - kE_0]\cdot[lC - kE_0] - [lC - E_0]\cdot K)\\ =& \frac{1}{2}(2lC\cdot E_0 + 2lkC\cdot E_0 - k^2 E_0\cdot E_0 - 2lC\cdot E_0 + 2kE_0\cdot E_0 - 2kE_0\cdot C)\\ =& lk - 2k^2 + 3k\\ =& k(l-2k) + 3k.\end{aligned}$$

Reminding that $[lC] = \mathcal{O}(l)$ and using the invariance under deformation, for the line bundle $L^z_{(a,b,c)}$, with $z \in (0,2)$, we have

$$dimH^0(S, L^z_{(a,b,c)}(l)) - dimH^1(S, L^z_{(a,b,c)}(l)) = k(l-2k) + 3k. \tag{4.8}$$

In particular, for $l = 2k-3$, $dimH^0(S, L^z_{(a,b,c)}(2k-3)) = dimH^1(S, L^z_{(a,b,c)}(2k-3))$. Thus, from condition (iv) in (4.3.1), we have

$$H^1((S, L^z_{(a,b,c)}(2k-3)) = 0, \text{ for } z \in (0,2).$$

[2]Let $D$ be a divisor, then $[D]$ denotes the line bundle defined by $D$.

Since $S$ does not have multiple components we can find a fibre $F$ of $\mathbb{T}$ that intersects $S$ in $k$ distinct points. Therefore, we have the exact sequence

$$0 \to \mathcal{O}_S(L^z_{(a,b,c)}(l)) \to \mathcal{O}_S(L^z_{(a,b,c)}(l+1)) \to \mathcal{O}_{S\cap F}(L^z_{(a,b,c)}(l+1)) \to 0. \tag{4.9}$$

The exact sequence on cohomology for this sequence tells us that

$$H^1(S, L^z_{(a,b,c)}(l)) \to H^1(S, L^z_{(a,b,c)}(l+1)),$$

is surjective, since $H^1(S \cap F, L^z_{(a,b,c)}(l+1)) = 0$. Put $l = 2k-3$, then (4.9) says that $H^1(S\cap F, L^z_{(a,b,c)}(2k-2)) = 0$. Using (4.8) we conclude that $H^0(S\cap F, L^z_{(a,b,c)}(2k-2)) = k$ and we have proved the lemma. □

We can now define a bundle $V$ on $\mathbb{C}$ in the following way: Let $W$ be the bundle over $\mathbb{C} \times S$ whose fibre at $(z,p)$ is $L^z(2k-2)_p$ and $P_1 : \mathbb{C} \times S \to \mathbb{C}$ be the projection in the first coordinate, define $V = (P_1)_* W$.[3] From the proposition above, we know that $V$ has rank $k$ and, moreover, the fibre at $z \in (0,2)$ is $V_z = H^0(S, L^z(2k-2))$.

Now we shall state the following lemma whose proof is similar to the proof of proposition (4.5) in [Hit83].

**Lemma 4.4.2.** If $l < 4k$, then any section $s \in H^0(S, \mathcal{O}(l))$ can be written uniquely as:

$$s = \sum_{j=0}^{[l/4]} \eta^j \pi^*(c_j),$$

where $c_j \in H^0(\mathbb{C}P^1, l-4j)$.

[3] V is a locally free sheaf since the direct image sheaf $(P_1)_* W$ over $\mathbb{C}$ is torsion free.

Observe that this lemma implies that at $z = 0$ the bundle $L^z(2k-2)$ has more sections than for $z \in (0,2)$. This means that the fibre $V_0$ is not just $H^0(S, L^z(2k-2))$ and we shall treat this case later. In this section we shall consider the behaviour of the bundle $V$ on the interval $(0,2)$ and the endpoints will be studied in the next section.

From Beauville's theorem we have that, for $z \in (0,2)$, each line bundle $L^z(2k-3)$ corresponds to a conjugacy class of a regular matricial polynomial $A(\xi, z) = \sum_{j=0}^{j=4} A_j(z)\xi^j$. Moreover, $A(\xi, z)$ can be seen, from its construction, as a linear map $A(\xi, z) : H^0(S, L^z(2k-2)) \to H^0(S, L^z(2k-2))$, this is to say, $A(\xi, z) : V_z \to V_z$. However, we want to define actual matrices, and so far we only have an equivalence class of matrices, in other words, we have endomorphisms of $V_z$. The objective now is to use the endomorphisms $A_j(z)$ to define a connection for $V$ on the interval $(0,2)$. Then we shall trivialise $V$ by parallel constant section with respect to this connection.

From now on, we shall consider the bundle $L$ to be given by the transition function $exp(\eta/\xi^2)$. The corresponding Nahm's equations for the other bundles will be stated without proof, since it is done by performing the same calculations.

Let $s(z)$ be a local holomorphic section of $V$, we can write it as a pair of holomorphic functions $f_0 : S \cap U_0 \times \mathbb{C}^* \to \mathbb{C}^k$ and $f_1 : S \cap U_1 \times \mathbb{C}^* \to \mathbb{C}^k$ satisfying $f_0 = exp(z\eta/\xi^2)\xi^{2k-2} f_1$ on $U_0 \cap U_1$. We now follow the construction on [Hit83] page 169:

Differentiating with respect to $z$:

$$\frac{\partial f_0}{\partial z} = \frac{\eta}{\xi^2} e^{z\eta/\xi^2} \xi^{2k-2} f_1 + e^{z\eta/\xi^2} \xi^{2k-2} \frac{\partial f_1}{\partial z}.$$

From the definition of $A$, we have:

$$(\eta - A_0 - A_1\xi - A_2\xi^2 - A_3\xi^3 - A_4\xi^4)s = 0,$$

or

$$\frac{\eta}{\xi^2} s = (A_0\xi^{-2} + A_1\xi^{-1} + \frac{1}{2}A_2)s + (\frac{1}{2}A_2 + A_3\xi + A_4\xi^2)s.$$

This implies that on $U \cap U'$ we have

$$\begin{aligned}
&\frac{\partial f_0}{\partial z} - (\frac{1}{2}A_2 + A_3\xi + A_4\xi^2)s \\
=&\frac{\partial f_0}{\partial z} - \frac{\eta}{\xi^2} f_0 + (A_0\xi^{-2} + A_1\xi^{-1} + \frac{1}{2}A_2)s \\
=&e^{z\eta/\xi^2}\xi^{2k-2}\frac{\partial f_1}{\partial z} + e^{z\eta/\xi^2}\xi^{2k-2}(A_0\xi^{-2} + A_1\xi^{-1} + \frac{1}{2}A_2)s \\
=&e^{z\eta/\xi^2}\xi^{2k-2}\left[\frac{\partial f_1}{\partial z} + (A_0\xi^{-2} + A_1\xi^{-1} + \frac{1}{2}A_2)s\right].
\end{aligned}$$

The lines above tell us that we can define a connection on $V$, over $(0, 2)$, whose covariant derivative on $U$ is given by:

$$\nabla_z s = \frac{\partial f_0}{\partial z} - (\frac{1}{2}A_2 + A_3\xi + A_4\xi^2)s.$$

We shall use this to define a frame $(s_1, \cdots, s_k)$ of covariant sections for $V$.

Let $A_+ = \frac{1}{2}A_2 + A_3\xi + A_4\xi^2$, then we can write

$$\frac{\partial s}{\partial z} - A_+ s = 0.$$

Taking the derivative of

$$(\eta - A)s = 0,$$

with respect to $z$, we have

$$(\eta - A)\frac{\partial s}{\partial z} - \frac{\partial A}{\partial z}s = 0.$$

Thus,

$$-(\eta - A)A_+ s - \frac{\partial A}{\partial z}s = -\eta A_+ s + AA_+ s - \frac{\partial A}{\partial z}s = 0,$$

hence

$$\left([A, A_+] - \frac{\partial A}{\partial z}\right)s = 0.$$

Observe that this equation is independent of $\eta$.

Now let $F$ be a fibre of $\mathbb{T}$ such that $F \cap S = \{x_1, \cdots, x_k\}$ with the $x_j$ all distinct. Therefore, we have an exact sequence

$$0 \to \mathcal{O}_S L^z(2k-3) \to \mathcal{O}_S L^z(2k-2) \to \mathcal{O}_{F\cap S} \to 0.$$

Using the fact that $H^0(S, L^z(2k-2)) = H^1(S, L^z(2k-2)) = 0$, the exact cohomology sequence says that the restriction map $H^0(S, L^z(2k-2)) \to H^0(F \cap S, \mathcal{O})$ is an isomorphism. Thus, we can find a frame $s_1, \cdots, s_k$ for $H^0(S, L^z(2k-2))$ such that $s_i(x_j) = \delta_{ij}$.

We then have that $B = [A, A_+] - \frac{\partial A}{\partial z}$ satisfies $\sum_j B_{ij}s_j(x_l) = 0$ for all $i, l$. But this says that $B_{ij} = 0$. Since the condition on $F$ is generic, we must have $B_{ij} = 0$ for all the fibres.

Thus, we must have

$$\frac{\partial A}{\partial z} = [A, A_+].$$

We therefore have the Nahm's equations

$$\begin{aligned}
\dot{A}_0 &= \frac{1}{2}[A_0, A_2] \\
\dot{A}_1 &= [A_0, A_3] + \frac{1}{2}[A_1, A_2] \\
\dot{A}_2 &= [A_1, A_3] + [A_0, A_4] \\
\dot{A}_3 &= [A_1, A_4] + \frac{1}{2}[A_2, A_3] \\
\dot{A}_4 &= \frac{1}{2}[A_2, A_4],
\end{aligned}$$

for $z \in (0, 2)$ and $\dot{A}_j = \frac{\partial A_j}{\partial z}$.

Let $A_0 = T_1 + iT_2$, $A_1 = T_3 + iT_4$, $A_2 = 2iT_5$, $A_3 = T_3 - iT_4$ and $A_4 = -T_1 + iT_2$, then the equations above become

$$\begin{aligned}
\dot{T}_1 &= [T_5, T_2] \\
\dot{T}_2 &= [T_1, T_5] \\
\dot{T}_3 &= [T_1, T_3] + [T_2, T_4] + [T_5, T_4] \\
\dot{T}_4 &= -[T_1, T_4] + [T_2, T_3] - [T_5, T_3] \\
\dot{T}_5 &= [T_1, T_2] + [T_4, T_3].
\end{aligned}$$

Now we can state

**Proposition 4.4.3.** Let $S$ be a spectral curve in $\mathbb{T}$ satisfying the conditions in (4.3.1) for the line bundle $L_{(0,1,0)}$, the line bundle whose transition function is $e^{\eta/\xi^2}$ on $\mathbb{T}$. If $A(\xi,z) = (T_1 + iT_2) + (T_3 + iT_4)\xi + 2iT_5\xi^2 + (T_3 - iT_4)\xi^3 + (-T_1 + iT_2)\xi^4$, then the $T_j$s satisfy the equations above. Moreover, relative to the bundle $L_{(a,b,c)}$, the bundle on $\mathbb{T}$ whose transition function is $e^{(a(\eta/\xi - \eta/\xi^3) + b\eta/\xi^2 + ci(\eta/\xi + \eta/\xi^3))}$, the $T_j$s. must satisfy:

$$\dot{T}_1 = -a[T_2, T_4] + b[T_5, T_2] - c[T_2, T_3]$$

$$\dot{T}_2 = a[T_1, T_4] + b[T_1, T_5] + c[T_1, T_3]$$

$$\dot{T}_3 = -2a[T_2, T_5] + b([T_1, T_3] + [T_2, T_4] + [T_5, T_4]) - c(2[T_1, T_5] + 2[T_1, T_2] + [T_4, T_3])$$

$$\dot{T}_4 = a(2[T_1, T_5] - 2[T_1, T_2] - [T_4, T_3]) + b(-[T_1, T_4] + [T_2, T_3] - [T_5, T_3]) - 2c[T_2, T_5]$$

$$\dot{T}_5 = a([T_3, T_5] + [T_1, T_4] + [T_2, T_3]) + b([T_1, T_2] + [T_4, T_3]) +$$
$$+ c(-[T_4, T_5] + [T_1, T_3] + [T_2, T_4]),$$

for $z \in (0, 2)$.

Before proceeding to the next section, we shall give an alternative way of describing the endomorphisms $A_j(z)$ that will be useful later. Let $S$ be a spectral curve and consider the map:

$$m : H^0(S, \mathcal{O}(4)) \otimes H^0(S, L^z(2k-2)) \to H^0(S, L^z(2k+2)), \tag{4.10}$$

and denote by $K_z$ its kernel at $z$, then we have the following proposition:

**Proposition 4.4.4.** The map $h : K_z \to V_z$ given by

$$h(\eta \otimes t_0 + 1 \otimes s_0 + \xi \otimes s_1 + \xi^2 \otimes s_2 + \xi^3 \otimes s_3 + \xi^4 \otimes s_4) \mapsto t_0$$

is an isomorphism for every $z \in (0, 2)$.

*Proof.* We start with an embedding of $\mathbb{T}$ as a quartic $\mathbb{C}P^5$:

$$\{(z_0, z_1, z_2, z_3, z_4, z_5)|\ z_0 = \eta, z_1 = 1, z_2 = \xi^1, z_3 = \xi^2, z_4 = \xi^3, z_5 = \xi^4\}.$$

Now consider the Euler sequence on $\mathbb{C}P^5$:

$$0 \to \Omega^1_{\mathbb{C}P^5}(H) \to \mathbb{C}^6 \to H \to 0,$$

where $H$ is the hyperplane bundle on $\mathbb{C}P^5$, $\Omega^1_{\mathbb{C}P^5}$ is the cotangent bundle of $\mathbb{C}P^5$ and $\mathbb{C}^6 \cong H^0(S, \mathcal{O}(4))$ is the trivial bundle. Restrict this to $S$ and twist with $L^z(2k-2)$ to obtain:

$$0 \to \Omega^1_{\mathbb{C}P^5} L^z(2k+2) \to H^0(S, \mathcal{O}(4)) \otimes L^z(2k-2) \to L^z(2k+2) \to 0.$$

The long exact sequence on cohomology of this sequence gives:

$$0 \to H^0(S, \Omega^1_{\mathbb{C}P^5} L^z(2k+2)) \to$$

$$H^0(S, \mathcal{O}(2)) \otimes H^0(S, L^z(2k-2)) \xrightarrow{m} H^0(S, L^z(2k+2)) \to H^1(S, \Omega^1_{\mathbb{C}P^5} L^z(2k+2)) \cdots$$

We shall next identify the kernel of $m$ and prove that it is onto.

First we shall trivialise the bundle $\Omega^1_{\mathbb{C}P^5}$ over $\mathbb{T}$. Consider the open set complement to the line $z_1 \neq 0$ in $\mathbb{T}$, we can trivialise it by:

$$\omega_1 = d\left(\frac{z_0}{z_1}\right);\ \omega_2 = d\left(\frac{z_2}{z_1}\right);\ \omega_3 = d\left(\frac{z_3}{z_1}\right);\ \omega_4 = d\left(\frac{z_4}{z_1}\right);\ \omega_5 = d\left(\frac{z_5}{z_1}\right).$$

On the open set $z_5 \neq 0$:

$$\tilde{\omega}_1 = d\left(\frac{z_0}{z_5}\right);\ \tilde{\omega}_2 = d\left(\frac{z_2}{z_5}\right);\ \tilde{\omega}_3 = d\left(\frac{z_3}{z_5}\right);\ \tilde{\omega}_4 = d\left(\frac{z_4}{z_5}\right);\ \tilde{\omega}_5 = d\left(\frac{z_5}{z_5}\right).$$

Then we have on the intersection:

$$\begin{aligned}
\tilde{\omega}_1 = d\left(\frac{z_0}{z_1}\cdot\frac{z_1}{z_5}\right) &= \omega_1\xi^{-4} + \eta\tilde{\omega}_5,\\
\tilde{\omega}_2 = d\left(\frac{z_2}{z_1}\cdot\frac{z_1}{z_5}\right) &= \omega_2\xi^{-4} + \xi\tilde{\omega}_5,\\
\tilde{\omega}_3 = d\left(\frac{z_3}{z_1}\cdot\frac{z_1}{z_5}\right) &= \omega_3\xi^{-4} + \xi^2\tilde{\omega}_5,\\
\tilde{\omega}_4 = d\left(\frac{z_4}{z_1}\cdot\frac{z_1}{z_5}\right) &= \omega_4\xi^{-4} + \xi^3\tilde{\omega}_5,\\
\tilde{\omega}_5 &= -\xi^{-8}\omega_5.
\end{aligned}$$

This means that $\Omega^1_{\mathbb{C}P^1}$ over $\mathbb{T}$ has transition function:

$$\begin{pmatrix} \xi^{-4} & 0 & 0 & 0 & 0 \\ 0 & \xi^{-4} & 0 & 0 & 0 \\ 0 & 0 & \xi^{-4} & 0 & 0 \\ 0 & 0 & 0 & \xi^{-4} & 0 \\ -\eta\xi^{-8} & -\xi^{-7} & -\xi^{-6} & -\xi^{-5} & -\xi^{-8} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \\ \omega_5 \end{pmatrix} = \begin{pmatrix} \tilde{\omega}_1 \\ \tilde{\omega}_2 \\ \tilde{\omega}_3 \\ \tilde{\omega}_4 \\ \tilde{\omega}_5 \end{pmatrix}.$$

Now we can consider a map of sheaves on $\mathbb{T}$ given by the composition:

$$f : \Omega^1_{\mathbb{C}P^5} \to \Omega^1_{\mathbb{T}} \to \Omega^1_F,$$

where $F$ is a fibre of $\mathbb{T}$ and notice that $\Omega^1_F \cong \mathcal{O}(-4)$ and $f$ is onto, since it is a projection.

Then, we must have that $ker(f)$ is the bundle on $\mathbb{T}$ whose transition function is given by:

$$\begin{pmatrix} \xi^{-4} & 0 & 0 & 0 \\ 0 & \xi^{-4} & 0 & 0 \\ 0 & 0 & \xi^{-4} & 0 \\ -\xi^{-7} & -\xi^{-6} & -\xi^{-5} & -\xi^{-8} \end{pmatrix}.$$

But this is isomorphic to the pull back on $\mathbb{T}$ of the bundle $\mathcal{O}(-5)^{\oplus 4}$ on $\mathbb{C}P^1$. Then

$$ker(f) \cong \mathcal{O}(-5)^{\oplus 4}.$$

This gives an exact sequence:

$$0 \to \mathcal{O}(-5)^{\oplus 4} \to \Omega^1_{\mathbb{C}P^1} \xrightarrow{f} \mathcal{O}(-4) \to 0.$$

Tensoring it by $L^z(2k+2)$ and restricting to $S$ we have an exact sequence of sheaves on $S$:

$$0 \to L^z(2k-3)^{\oplus 4} \to \Omega^1_{\mathbb{C}P^1} L^z(2k+2) \xrightarrow{f} L^z(2k-2) \to 0.$$

Taking the long exact sequence in cohomology for this sequence gives us:

$$\begin{aligned} 0 \to & H^0(S, L^z(2k-3)^{\oplus 4}) \to H^0(S, \Omega^1_{\mathbb{C}P^1} L^z(2k+2)) \xrightarrow{f} H^0(S, L^z(2k-2)) \to \\ \to & H^1(S, L^z(2k-3)^{\oplus 4}) \to H^1(S, \Omega^1_{\mathbb{C}P^1} L^z(2k+2)) \to H^1(S, L^z(2k-2)) \cdots \end{aligned}$$

Remember that we proved that:

$$H^1(S, L^z(2k-3)) = H^0(S, L^z(2k-3)) = H^1(S, L^z(2k-2)) = 0.$$

Then, we can see that $f$ is the map $h$ from the proposition. Moreover,

$$f : H^0(S, \Omega^1_{\mathbb{C}P^1} L^z(2k+2)) \to H^0(S, L^z(2k-2))$$

is an isomorphism and $H^1(S, \Omega^1_{\mathbb{C}P^1} L^z(2k+2)) = 0$. This implies that $m$ is onto and its kernel is isomorphic to $H^0(S, L^z(2k-2))$ via $f$. □

An immediate consequence of this proposition is that there exist endomorphisms $A_j(z) \in$ End$V_z$ such that :

$$(\eta - A_0 - A_1\xi - A_2\xi^2 - A_3\xi^3 - A_4\xi^4)s = 0.$$

The uniqueness of Beauville's theorem tells us that these endomorphism are the same ones obtained via Beauville's theorem.

## 4.5 Boundary conditions for the Nahm's equations

The results in this section correspond to the Nahm's equations relative to the line bundle whose transition function is $e^{\eta/\xi^2}$.

**Definition 4.5.1.** Let $p(\xi, \eta)$ be the polynomial defining the spectral curve $S$, this is to say, $S = \{(\xi, \eta) |\ p(\xi, \eta) = 0\}$, we shall use the following notation in this section:

a) Define $M = \mathbb{C} \times S$ and $P : M \to \mathbb{C}$ is the projection in the first coordinate.

b) $\tilde{M} = \{(z,\xi,\eta) \in \mathbb{C} \times \mathbb{T} |\ \tilde{p}(z,\xi,\eta) = 0\}$, where $\tilde{p}(z,\xi,\eta) = z^k p\left(\xi, \frac{\eta}{z}\right)$ and $\tilde{P}$ : $\tilde{M} \to \mathbb{C}$ is the projection in the first coordinate.

c) For fixed $z \in \mathbb{C}$ we define the curve $zS$, it is $S$ shrunk by a factor $z$, to be the curve defined by $\tilde{p}(z,\xi,\eta)$.

d) $\tilde{V} = \tilde{P}_*(X|_{\tilde{M}})$, where $X$ is the bundle on $\mathbb{T}$ whose fibre at $(z,\eta,\xi)$ is $L^z(2k-2)_{(\eta,\xi)}$.

e) Define $\mathcal{L}$ over $\mathbb{C} \times \mathbb{T}$ to be the bundle such that $\mathcal{L}_{\{z\}\times\mathbb{T}} = L^z$.

f) Similarly, we have $X = P_*(\mathcal{L}(2k+2))$ and $\tilde{X} = \tilde{P}_*(\mathcal{L}(2k+2))$

g) Bundles on $\mathbb{T}$, their lifts to $\mathbb{C} \times \mathbb{T}$ and their restrictions to $M$ and $\tilde{M}$ will be denoted by the same letter.

**Remark 4.5.2.** i) If we denote the zero section of $\mathcal{O}(4)$ by $F$, we then notice that $\tilde{P}^{-1}(0) = F^{(k-1)}$, the $(k-1)^{\text{th}}$ formal neighbourhood of $F$ in the total space of $\mathcal{O}(4)$.

ii) $V = P_*(X|_M)$ is the bundle defined in the previous section.

### 4.5.1 The fibre of $V$ at $0$

**Lemma 4.5.3.** Define a map $\rho : L(k)|_{\tilde{M}} \to \mathcal{L}(k)_M$ in the following way: Let $s$ be a section of $L(k)$ on $\tilde{M}$ such that on the trivialisation $U_i$ it is represented by $\tilde{f}_i(z,\eta,\xi)$.

Define $\rho(s)$ to be the section of $\mathcal{L}(k)$ on $M$ represented by $f_i(z,\eta,\xi) = \tilde{f}_i(z, z\eta, \xi)$. Then, $\rho$ is a well defined map of bundles and it is an isomorphism for $z \neq 0$.

*Proof.* We just need to verify that $f_0 = \exp(z\eta/\xi^2) f_1$, but this is true since $\tilde{f}_0 = \xi^k \exp(\eta/\xi^2)\tilde{f}_1$. It is immediate that $\rho$ is an isomorphism. □

**Corollary 4.5.4.** Taking the direct images in the lemma above, there is a map of sheaves over $\mathbb{C}$

$$\rho : \tilde{V} \to V$$

which is an isomorphism for $z \neq 0$.

Consider now the evaluation map:

$$\tilde{ev}_z : \tilde{V}_z \to H^0(\tilde{P}^{-1}(z), L(2k-2)).$$

It is an isomorphism for $z \neq 0$.

For the next result, we shall use the following notation: $\Gamma_m \subset \mathcal{O}(2m)$ consist of sections $s$ of the form $s = \sum_{j=0}^{m} a_j \xi^{2j}$ and denote by $L \otimes \Gamma \subset L(2m)$ the set of sections of the form $\sum_{jk} \alpha_j \otimes s_k$, with $\alpha_j$ a section of $L$ and $s_k \in \Gamma_m$.

The first result in this section is:

**Proposition 4.5.5.** Let $V_0 \subset H^0(S, \mathcal{O}(2k-2))$ be the fibre of $V$ at $z = 0$ and $\Gamma_{k-1} \subset H^0(\mathbb{C}P^1, \mathcal{O}(2k-2))$ be the sections of the form $p(\xi) = c_{2k-2}\xi^{2k-2} + c_{2k-4}\xi^{2k-4} + \cdots + c_2\xi^2 + c_0$. Then, $V_0 \cong \Gamma_{k-1}$.

An extension of a section of $\mathcal{O}(2k-2)$ to a section of $X$ to the $m^{\text{th}}$ formal neighbourhood consists of the following data:

$$s = s_0 + zs_1 + \cdots + z^m s_m, \quad s_i \in H^0(U_0, \mathcal{O}),$$

$$s' = s'_0 + zs'_1 + \cdots + z^m s'_m, \quad s'_i \in H^0(U_1, \mathcal{O}),$$

such that $s = \xi^{2k-2}(e^{\eta/x_i^2})s' \text{mod} z^{m+1}$ on $U_0 \cap U_1$. From lemma (4.5.3) we have that we can change $z$ to $z\eta$ near $z = 0$. This means the extension above can be written as:

$$p = p_0 + zp_1 + \cdots + z^m p_m, \quad p_i \in H^0(U_0, \mathcal{O}(2k-2-4i)),$$

$$p' = p'_0 + zp'_1 + \cdots + z^m p'_m, \quad p'_i \in H^0(U_1, \mathcal{O}(2k-2-4i)),$$

such that $p = (e^{\eta/x_i^2})p' \text{mod} \eta^{m+1}$. We can now state and prove the following:

**Lemma 4.5.6.** Every section in $L \otimes \Gamma_m$ on $Z \subset \mathbb{T}$ can be extended uniquely to the $m^{\text{th}}$ formal neighbourhood, but no section can be extended to the $(m+1)^{\text{th}}$ neighbourhood.

*Proof.* A section of $L(2m)$ on the m$^{\text{th}}$ neighbourhood consists of local section $p_i \in H^0(U_0, \mathcal{O}(2m-4i))$ and $p'_i \in H^0(U_1, \mathcal{O}(2m-4i))$, such that

$$p_0 + \eta p_1 + \cdots + \eta^m p_m = e^{\eta/\xi^2}(p'_0 + \eta p'_1 + \cdots + \eta^m p'_m) \text{mod} \eta^{m+1}.$$

We are therefore looking for functions $p_i$ on $U_0$ and $p'_i$ on $U_1$ such that on the intersection $U_0 \cap U_1$ we have:

$$\begin{pmatrix} \xi^{2m} & 0 & 0 & \dots & 0 \\ \xi^{2m-2} & \xi^{2m-4} & 0 & \dots & 0 \\ \frac{1}{2}\xi^{2m-4} & \xi^{2m-6} & \xi^{2m-8} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{m!} & \frac{1}{(m-1)!}\xi^{-2} & \dots & \dots & \xi^{-2m} \end{pmatrix} \begin{pmatrix} p'_0 \\ p'_1 \\ \vdots \\ \vdots \\ p'_m \end{pmatrix} = \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ \vdots \\ p_m \end{pmatrix}. \tag{4.11}$$

Now for $l$ even and such that $0 \leq l \leq 2m$,

$$\begin{pmatrix} p'_0 \\ p'_1 \\ \vdots \\ \\ \vdots \\ p'_m \end{pmatrix} = \begin{pmatrix} c_0\xi^{-2m+l} \\ c_1\xi^{-2m+l+2} \\ \vdots \\ c_{(\frac{2m-l}{2})} \\ \vdots \\ 0 \end{pmatrix} \tag{4.12}$$

solves (4.11) if

$$\sum_{i=0}^{(\frac{2m-l}{2})} \frac{c_i}{(n-i)!} = 0, \tag{4.13}$$

where $\left(\frac{l}{2}+1\right) \leq n \leq m$.

From [Hit83] page 173, there exists a unique solution of (4.13), and for this solution we have $c_0$ and $c_{(\frac{2m-l}{2})}$ are both non-vanishing. This implies that (4.12) trivialises a rank-$m+1$ bundle $E_m \to \mathbb{C}P^1$ whose transition function is given by the matrix in (4.11).

From the exact sequence

$$0 \to E_{m-1}(-2) \to E_m \xrightarrow{p_0} \mathcal{O}(2m) \to 0, \tag{4.14}$$

we have the following long exact sequence in cohomology:

$$0 \to H^0(\mathbb{C}P^1, E_{m-1}(-2)) \to H^0(\mathbb{C}P^1, E_m) \xrightarrow{p_0} H^0(\mathbb{C}P^1, \mathcal{O}(2m)) \to \tag{4.15}$$

$$\to H^1(\mathbb{C}P^1, E_{m-1}(-2)) \to \dots \tag{4.16}$$

We can deduce from general sheaf cohomology theory that $H^1(\mathbb{C}P^1, E_{m-1}(-2)) = 0$. Therefore $H^0(\mathbb{C}P^1, E_m) \xrightarrow{p_0} H^0(\mathbb{C}P^1, \mathcal{O}(2m))$ is injective. It remains to find the image of the map $p_0$, in cohomology, above.

Since $l$ is even, we can write $l = 2j$, for $0 \leq j \leq m$. Define $v_j$ by the equation (4.12) and notice that $\{v_0, \cdots, v_m\}$ is a global frame for $E_m$. Thus, for $\alpha \in H^0(\mathbb{C}P^1, E_m)$, we can write $\alpha = \sum_{j=0}^m \alpha_j v_j$ and we have:

$$p_0(\alpha) = \sum_{j=0}^{m} \alpha_j \xi^{2j} \in H^0(\mathbb{C}P^1, \mathcal{O}(2m)). \tag{4.17}$$

Using our notation, this means that the image of $p_0$ is $\Gamma_m$. This implies that sections of the form (4.17) can be extended uniquely to $E_m$ and hence to the $m^{\text{th}}$ formal neighbourhood.

An extension of sections given by (4.17) on the $(m+1)^{\text{th}}$ neighbourhood is given by the pull-back to $E_{m+1}(-2)$ in the exact sequence:

$$0 \to E_m(-4) \to E_{m+1}(-2) \xrightarrow{p_0} \mathcal{O}(2m) \to 0. \tag{4.18}$$

However, in this case $H^0(\mathbb{C}P^1, E_{m+1}(-2)) = 0$ and no extension exists.

As a concern of notation, if $s$ is a section in $\Gamma_m$, its formal extension in $L^{(m)}(2m)$ will be denoted by $\overline{s}$. □

Before we proceed we shall state the following lemma, whose proof is similar to the proof of lemma (5.2) in [Hit83]:

**Lemma 4.5.7.** Every element $c \in H^1(S, O(2k-2))$ can be written uniquely in the form:

$$c = \sum_{i=[k+1/2]}^{2k-2} \eta^i \pi^* c_i,$$

where $c_i \in H^1(\mathbb{C}P^1, O(2k-2-4i))$.

*Proof of proposition* (4.5.5). Let us start with the exact sequence:

$$0 \to \mathcal{O}(-2m-4) \to L^{(m+1)}(2m) \to L^{(m)}(2m) \to 0.$$

Form its exact sequence in cohomology we have a map

$$\delta : H^0(\mathbb{C}P^1, L^{(m)}(2m)) \to H^1(\mathbb{C}P^1, O(-2m-4)).$$

Since $H^0(\mathbb{C}P^1, L^{(m+1)}(2m)) = 0$ and $H^0(\mathbb{C}P^1, L^{(m)}(2m)) = \Gamma_m$ from lemma (4.5.6), we can define an injective map

$$h : \Gamma_m \to H^1(\mathbb{C}P^1, \mathcal{O}(-2m-4)),$$

defined by $hs = \delta\overline{s}$.

Let $s \in \Gamma_{k-1}$ and take the extension of $\pi^* s \in H^0(S, \mathcal{O}(2k-2))$ to the order $k-1$, as in lemma (4.5.6), and consider it to be a section of $L^z(2k-2)$ over $\mathbb{C} \times S$. The obstruction to extending to the order $k$ is the element

$$c = \eta^k \pi^* hs \in H^1(S, \mathcal{O}(2k-2)).$$

Now, since $S$ satisfies $\eta^k + a_1\eta^{k-1} + \cdots + a_0 = 0$, we must have

$$c = -\sum a_i \eta^{k-i} \pi^* hs.$$

Then we can write the above as:

$$c = -\sum \eta^{k-j} \pi^* h_j,$$

where $h_j \in H^1(\mathbb{C}P^1, \mathcal{O}(4j-2k-2))$ and also each $h_j$ must be in the image of $h$. Therefore, for each $j$ we can find a unique section $s_i \in \Gamma_{k-1-2j}$ such that $\eta^{k-j}\pi^* h_j$ is the obstruction to extend $\pi^* s_j \in H^0(S, \mathcal{O}(2k-2-4j))$ to the order $(k-2j)$ as a section of $L^z(2k-2-4j)$. This is the obstruction to extending $z^{2j}\eta^j\pi^* s_j$ from the order $(k-1)$ to the order $k$. Therefore, if $\overline{s}$ denotes a formal extension, we have that

$$s^1 = \overline{s} - z^2\eta\overline{s}_1 - z^4\eta^2\overline{s}_2 - \cdots - z^2 l\eta^l\overline{s}_l$$

extends to the order $k$ in $z$. Now, we can consider an extension of $s^1$ whose obstruction is $c' \in H^1(S, \mathcal{O}(2k-2))$. We can proceed as above we shall add modifications of order $z^3$. Then, every coefficient of $z^n$ requires a finite number of modifications and we have a power series in $z$. Now we can use a result in [Har97] (proposition II 9.6) to prove that a convergent extension exists. We have then proved that $\pi^*(\Gamma_{k-1}) \subset V_0$. Since both vector spaces have dimension $k$, we have proved the proposition. □

**Remark 4.5.8.** An important remark here is that since $\Gamma_m$ is not a natural irreducible representation of $SL(2,\mathbb{C})$, the maps in cohomology in the proof above are interpreted only as maps of abelian groups and not as maps between irreducible representations. Therefore, the fibre of $V$ at $z = 0$ does not have a natural $SL(2,\mathbb{C})$ representation structure. This is an important difference between the $\mathbb{R}^3$ case.

### 4.5.2 The behaviour of the matrix $A$ at $0$

After having established the fibre of $V$ at $0$, we can move toward the description of the behaviour of the matrix $A(z,\xi)$ at $0$. Namely, we shall prove that $A(z,\xi)$ has a pole at $0$.

As before, we shall work with $\tilde{M}$ instead of $M$. Remember that in corollary (4.5.4) we defined a map $\rho : \tilde{V} \to V$, which is an isomorphism away from $z = 0$. Also, remember that $X = P_*(\mathcal{L}(2k+2))$ and $\tilde{X} = \tilde{P}_*(\mathcal{L}(2k+2))$. Then, we can state the following lemma:

**Lemma 4.5.9.** The diagram:

$$\begin{array}{ccc} \tilde{V} & \xrightarrow{\tilde{F}} & \tilde{X} \\ \big\downarrow{\scriptstyle\rho} & & \big\downarrow{\scriptstyle\rho} \\ V & \xrightarrow{F} & X \end{array}$$

is commutative if either $F = z\eta$ and $\tilde{F} = \eta$ or $F = \tilde{F} = A(\xi,\eta)$.

The proof of this lemma is direct from lemma (4.5.3) and corollary (4.5.4). We now have the following:

**Corollary 4.5.10.** Define $B(\xi,\eta) = zA(\xi\eta)$, then $(\eta - A(\xi,\eta))V = 0$ if and only if $(\eta - B(\xi,\eta))\tilde{V} = 0$.

We shall now study the behaviour of $B$ at $z = 0$ and use this corollary to deduce the corresponding behaviour of $A$. To start with this, we shall use Beauville's construction of $B$.

We start with the commutative diagram:

$$\begin{array}{ccc}
\tilde{V}_z & \xrightarrow{restr_{z,q}} & H^0(zS \cap T_q, L(2k-2)) \cong \mathbb{C}^{k-1} \\
\Big\downarrow B(\xi,\eta) & & \Big\downarrow \times\eta \\
\tilde{V}_z & \xrightarrow{restr_{z,q}} & H^0(zS \cap T_q, L(2k-2)) \cong \mathbb{C}^{k-1},
\end{array}$$

where $q \in \mathbb{C}P^1$, $T_q$ is the fibre of $\mathbb{T}$ over $q$ and

$$restr_{z,q} : H^0(zS, L(2k-2)) \to H^0(zS \cap T_q, L(2k-2))$$

is the natural restriction map. Moreover, as in the construction of $A$, the cohomologies in the diagram above can be interpreted as polynomials in $\eta$ of degree $k-1$.

Observe that $restr_{z,q}$ is an isomorphism for all $z \neq 0$ and its limit $restr_{0,q}$ is also an isomorphism. Now, let $\tilde{e}_0, \cdots, \tilde{e}_{k-1}$ be a local frame for $\tilde{V}$, in a neighbourhood of $0$, such that $restr_{0,q}(\tilde{e}_j) = \eta^j$.

Then $B$ is well-defined and continuous at $z = 0$ and, if $\xi_0$ correspond to the point $q \in \mathbb{C}P^1$,

$$B(0,\xi_0)(\tilde{e}_j) = \tilde{e}_{k+1}.$$

Since $B = zA$, we must have that $A$ has simple poles at $z = 0$ and the next objective will be the description of the residues of $A$ at $0$.

Now we shall use the alternative description of $A$ given in (4.4.4) to find the residues of $A$. This means we shall investigate the behaviour of the kernel $K_z$ of the product map

$$m : H^0(S, \mathcal{O}(4)) \otimes H^0(S, L^z(2k-2)) \to H^0(S, L^z(2k+2))$$

as $z \to 0$. We start by noticing that, under the embedding $\mathbb{T} \subset \mathbb{C}P^5$, finding $K_0$ is equivalent to finding which sections of $H^0(S, \Omega^1_{\mathbb{C}P^5}(2k+2))$ extend to $H^0(S, L^z\Omega^1_{\mathbb{C}P^5}(2k+2))$. Since $dim K_z = k$ for $z \in (0,2)$, we should have a $k$-dimensional subspace $K_0$ that extends. Next, we shall describe $K_0$.

Let $\{1, \xi^2, \cdots, \xi^{2k}\}$ be a basis for $\Gamma_{k-1}$ and define the linear operators $B_0, B_1$ and $B_2$ in $\Gamma_{k-1}$ by the the matrices:

$$X_0 = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ -(k-1) & 0 & 0 & \dots & 0 \\ 0 & -(k-2) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & -1 & 0 \end{pmatrix}, \tag{4.19}$$

$$X_2 = \begin{pmatrix} k & 0 & 0 & \dots & 0 \\ 0 & (k-2) & 0 & \dots & 0 \\ 0 & 0 & (k-4) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & -k \end{pmatrix} \text{ and } \tag{4.20}$$

$$X_4 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & (k-1) \\ 0 & \cdots & \cdots & 0 & 0 \end{pmatrix} \tag{4.21}$$

we can now state the following result:

**Proposition 4.5.11.** Every element $s \in K_0$ can be written uniquely in the form

$$s = \pi^*(1 \otimes X_0\hat{s} + \xi^2 \otimes X_2\hat{s} + \xi^4 \otimes X_4\hat{s}),$$

where $\hat{s} \in \Gamma_{k-1}$.

*Proof.* The idea of the proof of this proposition is to work on the $(k-1)^{\text{th}}$ order neighbourhood first. We shall find a basis for the fibre of the bundle $V$ at $0$ in the formal neighbourhood in the language of the lemma (4.5.6), this is to say, we have to solve (4.11).

In what follows, we shall find

$$P'^j = p'^j_0 + p'^j_1(z\eta) + \cdots + p'^j_{(k-1)}(z\eta)^{(k-1)}$$

and

$$P^j = p^j_0 + p^j_1(z\eta) + \cdots + p^j_{(k-1)}(z\eta)^{(k-1)}$$

satisfying (4.11), for $0 \leq j \leq (k-1)$.

In what follows we shall use $m = k-1$ for simplicity.

Fix $j$ and and define on the open set $U_0$:

$$p'^j_l = \begin{cases} (-1)^l \dfrac{(m-l)!}{(m-l-j)!} \dfrac{(m-j)!}{m!} \dfrac{1}{l!} \xi^{-2(m-j-l)} \text{ for } 0 \leq l \leq (m-j), \\ 0 \text{ otherwise.} \end{cases} \tag{4.22}$$

And on $U_1$:

$$p^j_l = \begin{cases} \dfrac{(m-l)!}{(j-l)!} \dfrac{j!}{m!} \dfrac{1}{l!} \xi^{2(j-l)} \text{ for } 0 \leq l \leq j, \\ 0 \text{ otherwise.} \end{cases} \tag{4.23}$$

We now need to check this data satisfies (4.11). Let

$$\beta_b = \left( \frac{1}{b!} \xi^{(2m-2b)}, \frac{1}{(b-1)!} \xi^{(2m-2b-2)}, \cdots, \xi^{(2m-4b)}, 0, \cdots, 0 \right)$$

be the $b^{\text{th}}$ line of the matrix (4.11). We need to prove that

$$\beta_b \cdot P'^j = p^j_b.$$

$$
\begin{aligned}
\beta_b \cdot P^j &= \sum_{l=0}^{\min\{b,m-j\}} (-1)^l \frac{(m-l)!(m-j)!}{(b-l)!(m-j-l)!m!l!} \xi^{(2j-2b)} \\
&= \frac{(m-b)!}{m!} \left[ \sum_{l=0}^{\min\{b,m-j\}} (-1)^l \binom{m-l}{b-l} \binom{m-j}{l} \right] \xi^{(2j-2b)} \\
&= \frac{(m-b)!}{m!} \binom{j}{b} \xi^{(2j-2b)} \\
&= \frac{(m-b)!j!}{(j-b)!b!m!} \xi^{(2j-2b)} = p_b^j.
\end{aligned}
$$

Where we used the identity:

$$
\sum_{l=0}^{\min\{b,m-j\}} (-1)^l \binom{m-j}{l} \binom{m-l}{b-l} = \binom{j}{b}.
$$

Then, we have that $P^j$ gives a basis for $V_0$ in the $(k-1)^{\text{th}}$ neighbourhood.

Now we shall describe the kernel of the multiplication map

$$
m : H^0(F^{(k-1)}, \mathcal{O}(4)) \otimes H^0(F^{(k-1)}, L(2k-2)) \to H^0(F^{(k-1)}, L(2k+2)).
$$

First, notice that we have $H^0(F^{(k-1)}, \mathcal{O}(4)) \cong H^0(\mathbb{T}, \mathcal{O}(4)) = Span_{\mathbb{C}}\{1, \xi, \xi^2, \xi^3, \xi^4, \eta\}$.

Now, a direct computation shows that the kernel of $m$ is generated by the elements of the form:

$$
\omega_j^z = [z\eta \otimes P^j] - (m-j)[1 \otimes P^{(j+1)}] + (m-2j)[\xi^2 \otimes P^j] + j[\xi^4 \otimes P^{(j-1)}],
$$

for $0 \leq j \leq (k-1)$. In other words, this says that we can find sections $t_0, s_0, s_2, s_4 \in \Gamma_{k-1}$ such that

$$
z\eta\bar{t}_0 + \bar{s}_0 + \bar{s}_2\xi^2 + \bar{s}_4\xi^4 = 0 \ \text{mod} z^k,
$$

where $\bar{t}_0$ and $\bar{s}_j$ represent the canonical extensions of $t_0$ and $s_j$ respectively. Moreover, we have proved above that we can actually take $t_0 = s$ and $s_j = X_j(s)$, for $j = 0, 2, 4$, for $s \in \Gamma_{k-1}$.

Now, the canonical extension is of order $(k-1)$ and we proceed as in the proof of proposition (4.5.5) to extend to higher orders and produce a formal extension. We can use again a result in [Har97] (proposition II 9.6) to prove that the obstruction to extend to higher orders are removable and, therefore we can produce an actual extension. □

**Remark 4.5.12.** It is important to highlight how we found the solutions (4.22) and (4.23) to (4.11). We solved (4.11) explicitly, from $k = 2$ up to $k = 6$, using the constraints (4.13) and then we obtained a pattern for the solution for general $k$. In the proof written here, we just used this general form of the solution and proved it actually solves (4.11).

We can now use this to prove our main result:

**Theorem 4.5.13.** Let $S$ be a curve in $\mathbb{T}$ satisfying the conditions in definition (4.3.1). Then the matrices $T_i$ obtained in proposition (4.4.3) satisfy the following conditions:

1. $T_3$ and $T_4$ are analytic on the whole interval $[0, 2]$;
2. $T_1$, $T_2$ and $T_5$ have simple poles at 0 and 2, but are otherwise analytic;
3. The residues of $T_1$, $T_2$ and $T_5$ at $z = 0$ and $z = 2$ define an irreducible representation of $\mathfrak{su}(1, 1)$.

*Proof.* Remember that from corollary (4.5.10) the endomorphisms $B_j$, defined by $B_j = zA_j$, are analytic on the whole interval $[0, 2]$. Moreover, the proposition above tells us that $B_1$ and $B_3$ vanish at $z = 0$ and:

$$\lim_{z\to 0} zB_j(s) = X_j(s),$$

for $j = 0, 2, 4$.

This means the endomorphisms $A_1$ and $A_3$ are analytic on the whole interval $[0, 2]$ and the endomorphisms $A_0, A_2$ and $A_4$ have simple poles at $0$ whose residues are given by $X_0, X_2$ and $X_4$ respectively. We now shall extend this to the matrices that appear on the Nahm's equations. We have that the covariant derivative in $V$ is defined by:

$$\nabla_z s = \frac{\partial f_0}{\partial z} - \left(\frac{1}{2}A_2 s + \xi A_3 s + \xi^2 A_4 s\right).$$

From the above and the definition of $X_j$( equations (4.19), (4.20) and (4.21)) we have:

$$\frac{1}{2}A_2 + \xi A_3 + \xi^2 A_4 = \frac{(k-1)}{2z} \times \mathbb{I} + D,$$

where $D$ is analytic in the whole interval $[0, 2]$ and $\mathbb{I}$ is the $k \times k$ identity matrix. Since the residue of the connection is a scalar, we can use the same argument in [Hit83] page 179 to conclude that the matrices $A_j$ have the same residue as the corresponding endomorphisms.

If we write $A_0 = T_1 + iT_2$, $A_1 = T_3 + iT_4$, $A_2 = 2iT_5$, $A_3 = T_3 - iT_4$ and $A_4 = -T_1 + iT_2$, then we have that $T_3$ and $T_4$ are analytic on the whole interval $[0, 2]$ and the residues of $T_1$, $T_2$ and $T_5$ at $0$ define an irreducible representation of $\mathfrak{su}(1, 1)$. The condition that $L^2$ is trivial on $S$ says that the behaviour of the residues at $z = 2$ is the same as to $z = 0$. $\square$

### 4.5.3 Reality conditions

In this subsection we shall prove the following:

**Proposition 4.5.14.** The matrices $T_j$ satisfy, for $t \in (0, 2)$:

(i) $T_j(t) = -T_j^*(t)$,

(ii) $T_j(t) = -\overline{T}_j(2 - t)$.

*Proof.* To prove (i), we start by defining a hermitian structure on the bundle $V$ over $(0, 2)$, whose fibre at $z \in (0, 2)$ is $H^0(S, L^z(2k - 2))$.

The reality of $S$ defines an antilinear isomorphism:

$$\sigma : H^0(S, L^z(2k - 2)) \to H^0(S, L^{-z}(2k - 2))$$

$$s \mapsto s^*.$$

Now let $s, t \in H^0(S, L^z(2k - 2))$ and $st^* \in H^0(S, \mathcal{O}(4k - 4))$. From lemma (4.4.2) we can write this section uniquely as:

$$st^* = c_0\eta^{k-1} + c_1\eta^{k-2} + \cdots + c_{k-1},$$

with $c_j \in H^0(\mathbb{C}P^1, \mathcal{O}(4j))$. We can now define a hermitian inner product by:

$$\langle s, t\rangle = c_0.$$

Using the same argument in [Hit83] pages 180-181 we have that this inner product is non-zero and preserves the connection we used to trivialise $V$.

From the definition of $A_j$ we have:

$$\eta s + A_0 s + \xi A_1 s + \xi^2 A_2 s + \xi^3 A_3 s + \xi^4 A_4 s = 0 \in H^0(S, L^z(2k+2)). \tag{4.24}$$

Applying the reality structure $\sigma$ on $S$ we have:

$$\eta\sigma(s) + \xi^4\sigma(A_0 s) - \xi^3\sigma(A_1 s) + \xi^2\sigma(A_2 s) - \xi\sigma(A_3 s) + \sigma(A_4 s) = 0. \tag{4.25}$$

On the other hand we have:

$$\eta\sigma(s) + A_0\sigma(s) + \xi A_1\sigma(s) + \xi^2 A_2\sigma(s) + \xi^3 A_3\sigma(s) + \xi^4 A_4\sigma(s) = 0. \tag{4.26}$$

From the equations (4.25) and (4.26) we deduce that:

$$\begin{aligned} \sigma A_0 &= A_4\sigma, \\ \sigma A_1 &= -A_3\sigma \text{ and} \\ \sigma A_2 &= A_2\sigma. \end{aligned} \tag{4.27}$$

We now consider the inner product $\langle A_j s, t\rangle$. From (4.24) we have:

$$(\eta s)t^* + (A_0 s)t^* + \xi(A_1 s)t^* + \xi^2(A_2 s)t^* + \xi^3(A_3 s)t^* + \xi^4(A_4 s)t^* = 0, \tag{4.28}$$

and

$$(\eta t)s^* + (A_0 t)s^* + \xi(A_1 t)s^* + \xi^2(A_2 t)s^* + \xi^3(A_3 t)s^* + \xi^4(A_4 t)s^* = 0. \tag{4.29}$$

Applying the reality condition in the last equation gives:

$$(\eta t^*)s + \xi^4(A_0 t^*)s - \xi^3(A_1 t^*)s + \xi^2(A_2 t^*)s - \xi(A_3 t^*)s + (A_4 t^*)s = 0. \tag{4.30}$$

Subtracting (4.30) from (4.28) gives us:

$$\begin{aligned}\{(A_0 s)t^* - (A_4 t^*)s\} + \xi\{(A_1 s)t^* + (A_3 t^*)s\} + \xi^2\{(A_2 s)t^* - (A_2 t^*)\}+ \\ \xi^3\{(A_3 s)t^* + (A_1 t^*)s\} + \xi^4\{(A_4 s)t^* - (A_0 t^*)s\} = 0.\end{aligned} \tag{4.31}$$

Now we can deduce that:

$$\begin{aligned}\langle A_0 s, t\rangle &= \langle s, A_4 t\rangle, \\ \langle A_1 s, t\rangle &= -\langle s, A_3 t\rangle, \\ \langle A_2 s, t\rangle &= \langle s, A_2 t\rangle,\end{aligned} \tag{4.32}$$

If we write $A_0 = T_1 + iT_2$, $A_1 = T_3 + iT_4$, $A_2 = 2iT_5$, $A_3 = T_3 - iT_4$ and $A_4 = -T_1 + iT_2$ then each $T_j$ is skew-hermitian.

To prove (ii), we shall use the trivialisation of $L^2$ over $S$, $a \in H^0(S, L^2)$. We can use $a$ and the real structure $\sigma$ to define the antilinear map:

$$\sigma' : H^0(S, L^z(2k-2)) \to H^0(S, L^{2-z}(2k-2)),$$

given by $\sigma' = a\sigma$. Notice that $\sigma'^2(s) = cs$ for a positive constant $c$, and it is compatible with the connection used to trivialise $V$, then, after normalisation by $|c|$, $\sigma'$ defines a real structure. Therefore, we can trivialise $V$ with sections which are real with respect to $\sigma'$, we then obtain matrices $T_j$ satisfying (ii). □

### 4.5.4 From Nahm's equations to spectral curves

The goal of this subsection is to prove the following theorem:

**Proposition 4.5.15.** Let $T_j : (0,2) \to \mathfrak{g}l(k),\ j = 1,2,3,4,5$ satisfy the Nahm's equations, where $T_j$ satisfy:

(1) $T_j(t) = -T_j^*(t)$, this is to say, each $T_j$ is skew-adjoint;

(2) $T_j(t) = -\overline{T}_j(2-t)$;

(3) The $T_j$ have simple poles at $z = 0$ and $z = 2$ whose residues satisfy the conditions of theorem (4.5.13).

Then, the curve $S$ defined by $det(\eta - A) = 0$, where $A = A_0 + \xi A_1 + \xi^2 A_2 + \xi^3 A_3 + \xi^4 A_4$, with $A_0 = T_1 + iT_2$, $A_1 = T_3 + iT_4$, $A_2 = 2iT_5$, $A_3 = T_3 - iT_4$ and $A_4 = -T_1 + iT_2$, satisfy:

i) $S$ is real,

ii) the bundle $L(2k-2)$ is real,

iii) $L^2$ is trivial on $S$,

iv) $H^0(S, L^z(2k-3)) = 0$ for $z \in (0,2)$.

*Proof.* For part i) notice that, since $T_i$ are skew-adjoint, we have $\xi^4\overline{A(-\frac{1}{\bar{\xi}^{-1}})} = A(\xi)t$. Therefore $det(\eta - A(\xi)) = det\left(\eta - \overline{A(-\frac{1}{\bar{\xi}^{-1}})}\right)$ and $S$ is real.

We now start to invert the procedure we used to construct the $A(\xi, t)$. Namely, using Beauville's theorem, we obtain a flow of line bundles $K_t$ on $S$. More explicitly, given the matrix $A(\xi, t)$ we have that

$$K_t = coker(\eta - A(\xi, t)),$$

where

$$(\eta - A(\xi, t)) : \mathcal{O}(-4)^{\oplus k} \to \mathcal{O}^{\oplus k}.$$

However, it is easier to consider the dual approach. This means we are going to find the dual flow:

$$K_t^* = ker(\eta - A(\xi, t))^t,$$

where

$$(\eta - A(\xi, t))^t : \mathcal{O}^{\oplus k} \to \mathcal{O}(4)^{\oplus k}.$$

First we shall prove that $K_t^* = K_{t_0}^* \otimes L^{t-t_0}$. We start with a section $s$ of $K_{t_0}^*$ and it can be represented by $u$ in the open set $\{\xi \neq \infty\}$ and by $v$ on $\{\xi \neq 0\}$. Moreover, let $g(t_0)$ be the transition function of $K_{t_0}^*$ such that $u = g(t_0)v$. Observe that on $\{\xi \neq \infty\}$ we must have:

$$(\eta - A(\xi, t_0))^t u = 0$$

and on $\{\xi \neq 0\}$:

$$(1/\xi^4)(\eta - A(\xi, t_0))^t v = 0.$$

Let $A_+ = \frac{1}{2}A_2 + A_3\xi + A_4\xi^2$ and we shall vary $t$. To begin with, we impose that $u$ satisfies:

$$\frac{\partial u}{\partial t} = A_+^t u.$$

We can use Nahm's equation to prove that

$$\frac{\partial}{\partial t}(\eta - A)^t u = A_+(\eta - A)^t u.$$

Now, the initial condition for this differential equation is given by $(\eta - A)^t u = 0$. Thus, we have $(\eta - A)^t u = 0$ for all $t$.

On the other open set we can impose

$$\frac{\partial v}{\partial t} = -\left(A/\xi^2 - A_+\right)^t v$$

and prove that

$$(1/\xi^4)(\eta - A)^t v = 0$$

for all $t$. Now we have:

$$A_+^t = \frac{\partial u}{\partial t} = \frac{\partial gv}{\partial t} = \frac{\partial g}{\partial t} v - g\left(A/\xi^2 - A_+\right)^t v.$$

This implies that

$$\frac{\eta}{\xi^2} u = \frac{\partial g}{\partial t} g^{-1} u.$$

The solution of this equation can be written in terms of $g(t_0)$ as $g(\eta, \xi, t) = e^{t\eta/\xi^2} \cdot g(t_0)$. Therefore, we proved that $K_t^* = K_{t_0}^* \otimes L^{t-t_0}$.

We now move towards the description of $K_0$ and we shall use the boundary behaviour of the matrices $A_i$ to prove that $K_0 \cong \mathcal{O}(2k-2)$.

Near $t = 0$ we can write for $t > 0$ $A(\xi, t) = \dfrac{\alpha(\xi, t)}{t}$, with $\alpha(\xi, t)$ analytic near $t = 0$. Also, denote $a(\xi, t) = \alpha(\xi, t)^t$. Write $\alpha(0, \xi) = a(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 + a_4\xi^4$.

From our hypothesis, $a_j$ satisfy the conditions in theorem (4.5.13). This means that $a_1 = a_3 = 0$ and $a_0, a_2$ and $a_4$ define an irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$.

Let $\Gamma_{k-1}$ be the subspace of $\mathbb{C}^{2k-1}$ consisting of polynomials of the form $p(\xi) = \sum_{i=0}^{2k-2} \xi^{2i}$ and we shall also denote by $\Gamma_{k-1}$ its image in $S^2(\mathbb{C}^k)$ under $\phi$. The matrices $a_j$ act on $\Gamma_{k-1}$ by multiplication. Moreover, we can choose a basis $e_i$ for $\Gamma_{k-1}$ such that $ker[a(\xi)] = (\xi^{2k-2}, \cdots, \xi^{2j}, \cdots, 1)$. Notice that in this basis, $a_0(e_i) = e_{i+1}$.

We shall next compute a section of $ker(\eta - A)^t$, first observe that

$$(\eta - A)^t(\eta - A)^t_{adj} = det(\eta - A) \times \mathbb{I},$$

where $adj$ is the formal adjoint and $\mathbb{I}$ is the identity matrix. This means that on $S$, $Im(\eta - A)^t_{adj} \subset K^*_t$. However, since $(\eta - A)$ is regular $(\eta - A)^t_{adj}$ has rank one and the inclusion becomes an equality.

Now, we shall compute a section of $(\eta - A)^t_{adj}$. Observe that at $\xi = 0$, we have that the image of $(\eta - A)^t_{adj}$ has a finite limit, because of the choice of basis above. In the general case, $Im(\eta - A)^t_{adj} \subset K^*_t$ will consist of a polynomial of degree $2k-2$ in $\xi$ and therefore, $K_0 \cong \mathcal{O}(2k-2)$. This means that $K_t = L^t(2k-2)$.

Notice that, since the behaviour of the matrices $T_j$ at $t = 2$ are the same as at $t = 0$, we also have $K_2 \cong \mathcal{O}(2k-2)$. This implies that $L^2 = 0$. Lastly, from Beauville's theorem we must have $K_t(-1) \in J^{g-1}_S$, this is to say, $H^0(S, L(2k-3))$. $\square$

# Chapter 5

# Moduli space of solution to Nahm's equations

## 5.1 $k$-symplectic manifolds and $k$-symplectic reduction

In this chapter, we shall discuss some properties of the moduli space of solutions to our Nahm's equations and we shall see they are what is called 2-symplectic manifolds. We begin with this definition:

**Definition 5.1.1.** Let $M$ be a regular generalised $k$-hypercomplex manifold and $V$ be a $k$ dimensional irreducible representation of $SU(2)$ such that $T_xM^{\mathbb{C}} = V^{\mathbb{C}} \otimes \mathbb{C}^n$ for all $x \in M$. Remember that $V^{\mathbb{C}} \cong S^k(\mathbb{C}^2)$ and let $V^{[2]}$ be the irreducible real representation of $SU(2)$ such that $(V^{[2]})^{\mathbb{C}} = S^{2k}(\mathbb{C}^2)$. A *k-symplectic structure* on $M$ is a closed $SU(2)$-

invariant 2-form $\omega$ with values in $V^{[2]}$.

Recall the decomposition $T_M^{\mathbb{C}} = E \otimes H$ from the first chapter, where $H = S^k(\mathbb{C}^2)$ is the trivial bundle. Then, the bundle of 2-forms decomposes as

$$\Lambda^2 T_M^{\mathbb{C}} = (\Lambda^2 E \otimes S^2 H) \oplus (S^2 E \otimes \Lambda^2 H).$$

Notice that $\Lambda^2 E \otimes S^2 H$ is the only term in the decomposition above containing $S^{2k}(\mathbb{C}^2)$, then a symplectic structure corresponds to a nondegenerate 2-form $\omega_E$ on $E$ such that the map

$$\Lambda^2 TM^{\mathbb{C}} \to \Lambda^2 E \otimes S^{2k}(\mathbb{C}^2) \xrightarrow{\omega_E} S^{2k}(\mathbb{C}^2)$$

defines a closed $V^{[2]}$-valued 2-form. In practical terms, a $k$-symplectic form $\omega$ can be written as $\omega_E \otimes \alpha$, where $\alpha \in S^{2k}(\mathbb{C}^2) \subset S^2 H$. This means that if $\alpha_j$, for $0 \leq j \leq 2k$, is a basis for $S^{2k}(\mathbb{C}^2)$, then we have $(2k+1)$ closed 2-forms, namely $\omega_E \otimes \alpha_j$, on $M$ giving the $k$-symplectic structure.

Remember that $M$ has a twistor space $Z$ which is a complex manifold fibrering over $\mathbb{C}P^1$ and is endowed with a real structure $\tau$ covering the antipodal map on $\mathbb{C}P^1$. Then $M$, together with the $SU(2)$ action on the tangent bundle, is given by the space of real holomorphic sections of $Z$ whose normal bundle is $\mathcal{O}(k) \otimes C^n$.

Now, from the definition above, a $k$-symplectic structure on $M$ is a $SU(2)$-invariant and closed 2-form $\omega$ on $M$ with values on $S^{2k}(\mathbb{C}^2)$, so we can write $\omega$ as:

$$\omega = \omega_0 + \omega_1 \xi + \cdots + \omega_{2k} \xi^{2k}. \tag{5.1}$$

This means, in terms of the twistor space, that $\omega$ defines a complex symplectic form for each fibre of the fibration $p : Z \to \mathbb{C}P^1$. Globally, $\omega$ is a section of the bundle $\Lambda^2 T_F^*(2k)$ over $Z$, where $T_F = \mathrm{Ker}(dp)$, the tangent space along the fibres.

Suppose now that $M$ admits a proper and free action of a Lie group $G$ and that the action is compatible with the GHC structure and the symplectic form. Moreover, suppose the action is Hamiltonian, i.e. there exists a moment map

$$\mu : M \to V^{[2]} \otimes \mathfrak{g}^*,$$

where $\mathfrak{g}$ is the Lie algebra of $G$. By moment map, we mean that $\mu$ satisfies:

$$d(\langle \mu, \rho \rangle) = \mathfrak{i}_{\rho^*}\omega, \tag{5.2}$$

where $\langle \mu, \rho \rangle$ is the function from $M$ to $\mathbb{R}$ defined by $\langle \mu, \rho \rangle(x) = (\mu(x))(\rho)$, for every $\rho \in \mathfrak{g}$, $\rho^*$ is the vector field on $M$ given by:

$$\rho^*_x = \frac{d}{dt}(exp(t\rho) \cdot x)|_{t=0}$$

and $\mathfrak{i}_{\rho^*}\omega$ is the contraction of $\omega$ by the vector field $\rho^*$. This permits us to define GHC quotients. Namely, if $r \in V^{[2]} \otimes \mathfrak{g}^*$ is a regular value of $\mu$, we can define the reduced GHC manifold as $\hat{M} = \mu^{-1}(r)/G$ [Bie06].

From the twistor point of view, the action of $G$ on $M$ can be regarded as an action of the complexified group $G^{\mathbb{C}}$. Now, we can construct a space $\hat{Z}$ from $Z$ by taking a holomorphic symplectic quotient along the fibres of $Z$ and $\hat{Z}$ is the twistor space of $\hat{M}$.

**Example 5.1.2** (*Hyperkähler manifolds* [HKLR87])**.** A Hyperkähler manifold is a $4n$ dimensional Riemannian manifold $M$ endowed with a metric $g$ and three covariantly constant complex structures **I,J,K** satisfying the algebraic quaternion relation. The twistor space of $M$ is $Z = \mathbb{C}P^1 \times M$ endowed with the following complex structure: Identify $\mathbb{C}P^1$ with $S^2$ and let $(a, b, c)$ be a point in $S^2 \subset \mathbb{R}^3$, define the complex structure $\underline{\mathbf{I}}$ at the point $((a, b, c), x) \in Z$ by $\underline{\mathbf{I}} = a\mathbf{I} + b\mathbf{J} + c\mathbf{K}$. It is proved in [HKLR87] that a Hyperkähler structure on $M$ is equivalent to the existence of a section $\omega$ of $\Lambda^2 T^*_F(2)$ over $Z$. Thus, the $k$-symplectic quotient discussed above generalises the well-known Hyperkähler quotient.

**Example 5.1.3** ($\mathbb{R}^6$ *as a* 2*-symplectic manifold*)**.** We start by defining $\mathbb{R}^6$ as two copies of the real form of the irreducible $SL(2, \mathbb{C})$-representation $S^2(\mathbb{C}^2)$. More explicitly, a point $p$ in $\mathbb{C}^6$ is a pair of polynomials of degree 2,

$$p = ((z_0 + z_1\xi + z_2\xi^2), (z_3 + z_4\xi + z_5\xi^2)). \tag{5.3}$$

This turns $\mathbb{R}^6$ into a 2-hypercomplex manifold whose twistor space is $\mathcal{O}(2) \oplus \mathcal{O}(2)$.

Now, the complexified tangent bundle of $\mathbb{R}^6$ decomposes as $(T\mathbb{R}^6)^{\mathbb{C}} = \mathbb{C}^2 \otimes S^2(\mathbb{C}^2)$. In the notation above, we have $E = \mathbb{C}^2$ and $H = S^2(\mathbb{C}^2)$. Furthermore, a 2-symplectic structure $\omega$ is given by $\omega = \omega_E \otimes \alpha$, where $\omega_E$ a 2-form in $\Lambda^2 E$ and $\alpha \in S^4(\mathbb{C}^2) \subset S^2 H$.

We can fix a frame $e_0 = (1, 0), e_1 = (0, 1)$ for $E = \mathbb{C}^2$ and define the alternating 2-form $\omega_E$ on $E$ to be given by the matrix, with respect to this basis:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{5.4}$$

Since $\Lambda^2 E$ has dimension $1$ this is the unique, up to a scalar, 2-form on $E$.

Next we shall identify $S^4(\mathbb{C}^2)$ in $S^2(H) = S^2(S^2\mathbb{C}^2)$. Let $Z_0 = 1, Z_1 = \xi, Z_2 = \xi^2$ be a frame for $H$, from representation theory [FH99], we have that $S^4(\mathbb{C}^2) \subset S^2(H)$ is generated by $h_0 = Z_0^2$, $h_1 = Z_0 Z_1$, $h_2 = Z_1^2 + \frac{1}{2} Z_0 Z_2$, $h_3 = Z_1 Z_2$ and $h_4 = Z_2^2$. Then, the $h_k$ are symmetric bilinear forms on $H$ and have the matrix representation:

$$h_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.5}$$

$$h_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.6}$$

$$h_2 = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}, \tag{5.7}$$

$$h_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{5.8}$$

$$h_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.9}$$

Now consider the basis $e_j \otimes Z_k$, for $j = 0,1$ and $k = 0,1,2$, for $(T\mathbb{R}^6)^{\mathbb{C}}$. With respect to this basis we have the following 2-forms on $\Lambda^2(T\mathbb{R}^6)^{\mathbb{C}}$ given by $\omega_k = \omega_E \otimes h_k$. We can put coordinates $z_l$, $l = 0,1,2,3,4,5$ on $\mathbb{C}^6$ relative to the frame above, this is to say, we can consider $e_j \otimes Z_k$ as vector fields in $\mathbb{C}^6$ and $z_l$ is the coordinate system such that $\partial_{z_0} = e_0 \otimes Z_0$, $\partial_{z_1} = e_0 \otimes Z_1$, $\partial_{z_2} = e_0 \otimes Z_2$, $\partial_{z_3} = e_1 \otimes Z_0$, $\partial_{z_4} = e_1 \otimes Z_1$ and $\partial_{z_5} = e_1 \otimes Z_2$. Moreover, it is important to notice that these coordinates are the same ones introduced in (5.3).

We can now use the matrices for $\omega_E$ and $h_k$ given above to write the $\omega_k$ explicitly in these coordinates:

$$\begin{aligned} \omega_0 &= dz_0 \wedge dz_3, \\ \omega_1 &= dz_0 \wedge dz_4 + dz_1 \wedge dz_3, \\ \omega_2 &= dz_0 \wedge dz_5 + 4dz_1 \wedge dz_4 + dz_2 \wedge dz_3, \\ \omega_3 &= dz_1 \wedge dz_5 + dz_2 \wedge dz_4, \\ \omega_4 &= dz_2 \wedge dz_5. \end{aligned} \tag{5.10}$$

We now have a real structure on $\mathbb{C}^6$ which can be defined via the involution in the twistor space by $((\eta_1, \eta_2), \xi) \to ((\overline{\eta}_1/\overline{\xi}^2, -1/\overline{\xi}), (\overline{\eta}_2/\overline{\xi}^2, -1/\overline{\xi}))$, where $((\eta_1, \eta_2), \xi)$ is a local holomorphic coordinate system for $\mathcal{O}(2) \oplus \mathcal{O}(2)$. Our objective now is to investigate how this involution extends to a real structure in $\Lambda^2 E \otimes S^4(\mathbb{C}^2)$.

One way of doing this is by noticing that the coordinates we used in $\mathbb{C}^6$ come from the following short exact sequence of bundles on $\mathbb{C}P^1$ given by the class $\left[\frac{1}{\xi^2}\right] \in H^1(\mathbb{C}P^1, \mathcal{O}(-4))$:

$$0 \to \mathcal{O} \to \mathcal{O}(2) \oplus \mathcal{O}(2) \to \mathcal{O}(4) \to 0. \tag{5.11}$$

This yields an exact sequence in cohomology:

$$0 \to H^0(\mathbb{C}P^1, \mathcal{O}) \to H^0(\mathbb{C}P^1, \mathcal{O}(2) \oplus \mathcal{O}(2)) \xrightarrow{\gamma} H^0(\mathbb{C}P^1, \mathcal{O}(4)) \to 0,$$

where the map $\gamma$ is explicitly given by:

$$(p_0, p_1) \mapsto p_0 + \xi^2 p_1,$$

where $p_0 = z_0 + z_1\xi + z_2\xi^2$ and $p_1 = z_3 + z_4\xi + z_5\xi^2$. Moreover, the kernel of $\gamma$ consists of sections $(p_0, p_1)$ such that $p_0 = a\xi^2$ and $p_1 = -a$, for $a \in \mathbb{C}$.

Now, every $\omega \in H^0(\mathbb{C}P^1, \mathcal{O}(4)) = S^4(\mathbb{C}^2)$ can be written as $\omega = p_0 + \xi^2 p_1 = z_0 + z_1\xi + (z_2 + z_3)\xi^2 + z_4\xi^3 + z_5\xi^4$ and then the involution can be induced to $S^4(\mathbb{C}^2)$ as:

$$\begin{cases} z_0 \mapsto & \overline{z_5} \\ z_1 \mapsto & -\overline{z_4} \\ z_2 \mapsto & \overline{z_3}. \end{cases}$$

Therefore, $\omega \in S^4(\mathbb{C}^2)$ is real if and only if

$$\begin{cases} z_0 = & \overline{z_5} \\ z_1 = & -\overline{z_4} \\ z_2 = & \overline{z_3}. \end{cases}$$

Applying this reality condition to equation (5.10), yields 5 complex 2-forms on $\mathbb{R}^6$:

$$\begin{aligned} \omega_0 &= dz_0 \wedge d\overline{z}_2, \\ \omega_1 &= -dz_0 \wedge d\overline{z}_1 + dz_1 \wedge d\overline{z}_2, \\ \omega_2 &= dz_0 \wedge d\overline{z}_0 - 4dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2, \\ \omega_3 &= -d\overline{z}_0 \wedge dz_1 + dz_1 \wedge d\overline{z}_2, \\ \omega_4 &= -d\overline{z}_0 \wedge dz_2. \end{aligned} \tag{5.12}$$

Writing $z_0 = -t_1 - it_2$, $z_1 = t_3 + it_4$ and $z_2 = t_0 - it_5$[1] and $\alpha_0 = \frac{1}{2}(\omega_0 - \omega_4), \alpha_1 = \frac{1}{2}(\omega_1 + \omega_3), \alpha_2 = \frac{i}{2}\omega_2, \alpha_3 = \frac{i}{2}(\omega_1 - \omega_3)$ and $\alpha_4 = \frac{i}{2}(\omega_0 + \omega_4)$ gives us real 2-forms on $\mathbb{R}^6$ giving a 2-symplectic structure on $\mathbb{R}^6$:

$$\begin{aligned} \alpha_0 &= -dt_0 \wedge dt_1 + dt_5 \wedge dt_2, \\ \alpha_1 &= dt_0 \wedge dt_3 - dt_5 \wedge dt_4 + dt_3 \wedge dt_1 + dt_4 \wedge dt_2, \\ \alpha_2 &= -dt_0 \wedge dt_5 - dt_3 \wedge dt_4 + dt_1 \wedge dt_2, \\ \alpha_3 &= dt_0 \wedge dt_4 + dt_0 \wedge dt_3 + dt_3 \wedge dt_2 - dt_4 \wedge dt_1, \\ \alpha_4 &= -dt_0 \wedge dt_2 - dt_5 \wedge dt_1. \end{aligned} \tag{5.13}$$

[1]The reason for this specific choice of coordinates will become clear when we interpret the Nahm's equations as an infinite dimensional moment map.

It is important to notice that the 2-hypercomplex structure on $\mathbb{R}^6$ was defined by means of the exact sequence (5.11). Remember this exact sequence was obtained from the class $\left[\frac{1}{\xi^2}\right] \in H^1(\mathbb{C}P^1, \mathcal{O}(-4))$, then we can obtain other 2-hypercomplex structures by considering extensions corresponding to other real elements in $H^1(\mathbb{C}P^1, \mathcal{O}(-4))^2$, thus this group parametrizes 2-hypercomplex structures on $\mathbb{R}^6$.

A bit more explicitly, we shall now consider another extension class $\left[\frac{1}{\xi} - \frac{1}{\xi^3}\right] \in H^1(\mathbb{C}P^1, \mathcal{O}(-4))$.

$$0 \to \mathcal{O} \to \mathcal{O}(1) \oplus \mathcal{O}(3) \to \mathcal{O}(4) \to 0. \tag{5.14}$$

This yields an exact sequence in cohomology:

$$0 \to H^0(\mathbb{C}P^1, \mathcal{O}) \to H^0(\mathbb{C}P^1, \mathcal{O}(1) \oplus \mathcal{O}(3)) \xrightarrow{\gamma} H^0(\mathbb{C}P^1, \mathcal{O}(4)) \to 0,$$

where the map $\gamma$ is given explicitly by:

$$\gamma(p_0, p_1) = p_0(1 - \xi^3) + p_1(\xi - 1),$$

with $p_0 = z_0 + z_1\xi + z_2\xi^2$ and $p_1 = z_3 + z_4\xi + z_5\xi^2$. Then, we can write explicitly

$$\gamma(p_0, p_1) = (z_0 - z_2) + (z_1 + z_2 - z_3)\xi + (z_3 - z_4)\xi^2 + (z_4 - z_0 - z_5)\xi^3 + (z_5 - z_3)\xi^4.$$

Using the real structure induced to $S^4(\mathbb{C}^2)$, as we did above, gives us the following reality condition:

[2] The real structure on $H^1(\mathbb{C}P^1, \mathcal{O}(-4))$ is the one induced from $H^0(\mathbb{C}P^1, \mathcal{O}(2))$ by the non-degenerated pairing $H^1(\mathbb{C}P^1, \mathcal{O}(-4)) \times H^0(\mathbb{C}P^1, \mathcal{O}(2)) \to \mathbb{C}$

$$\begin{cases} z_0 = & \overline{z_5} \\ z_1 = & \overline{z_2} \\ z_3 = & -\overline{z_4}. \end{cases}$$

Proceeding like we did in the other case, we have the following 2-symplectic structure on $\mathbb{R}^6$:

$$\begin{aligned}
\alpha_0 &= -dt_0 \wedge dt_1 + dt_5 \wedge dt_2, \\
\alpha_1 &= dt_0 \wedge dt_3 - dt_5 \wedge dt_4 + dt_3 \wedge dt_1 + dt_4 \wedge dt_2, \\
\alpha_2 &= -dt_0 \wedge dt_5 - dt_3 \wedge dt_4 + dt_1 \wedge dt_2, \\
\alpha_3 &= dt_0 \wedge dt_4 + dt_0 \wedge dt_3 + dt_3 \wedge dt_2 - dt_4 \wedge dt_1, \\
\alpha_4 &= -dt_0 \wedge dt_2 - dt_5 \wedge dt_1.
\end{aligned} \tag{5.15}$$

**Remark 5.1.4.** The inclusion $S^4(\mathbb{C}^2) \subset S^2(S^2\mathbb{C}^2)$ is $SL(2,\mathbb{C})$ covariant with isotropy $B$, the Borel subgroup of upper-triangular matrices, therefore we have that such inclusions of representations are parametrized by $\mathbb{C}P^1 = SL(2,\mathbb{C})/B$, therefore there is a $\mathbb{C}P^1$ of 2-symplectic structures for each 2-hypercomplex structure on $\mathbb{R}^6$.

## 5.2 Nahm's equations as a moment map

Let $G$ be a compact semisimple Lie group whose Lie algebra is $\mathfrak{g}$ and $\mathcal{A}$ be the space of $\mathfrak{g}$-valued functions $T_k$, $k = 0, 1, 2, 3, 4, 5$ on the interval $[0, 2]$. We can interpret $\mathcal{A}$ as the space of real sections of the bundle $C^\infty([0,2],\mathfrak{g}^{\mathbb{C}})\otimes(\mathcal{O}(2)\oplus\mathcal{O}(2))$. Furthermore, from the discussion in the last section, $\mathcal{A}$ has a symplectic 2-hypercomplex structure. Explicitly, the 2-symplectic 2-forms on $\mathcal{A}$ are given by:

$$\alpha_1 = \int_0^2 dT_0\wedge dT_1 - dT_5\wedge dT_2, \tag{5.16}$$

$$\alpha_2 = \int_0^2 dT_0\wedge dT_3 - dT_5\wedge dT_4 + dT_3\wedge dT_1 + dT_4\wedge dT_2, \tag{5.17}$$

$$\alpha_3 = \int_0^2 dT_0\wedge dT_5 + dT_3\wedge dT_4 - dT_1\wedge dT_2, \tag{5.18}$$

$$\alpha_4 = \int_0^2 dT_0\wedge dT_4 + dT_0\wedge dT_3 + dT_3\wedge dT_2 - dT_4\wedge dT_1, \tag{5.19}$$

$$\alpha_5 = \int_0^2 dT_0\wedge dT_2 + dT_5\wedge dT_1, \tag{5.20}$$

where the form $\omega_{ij} = \int_0^2 dT_i\wedge dT_j$ is given by:

$$\omega_{ij}((t_0,t_1,t_2,t_3,t_4),(t_0',t_1',t_2',t_3',t_4')) = \int_0^2 \langle t_i,t_j'\rangle - \langle t_j,t_i'\rangle,$$

with $t_j$ and $t_j'$ are tangent vector to $\mathcal{A}$ and $\langle\cdot,\cdot\rangle$ is the Killing form on $\mathfrak{g}$.

The group of gauge transformations, $\mathcal{G} = \{g : [0,2]\to G \mid g \text{ is smooth}\}$, acts on $\mathcal{A}$ as:

$$T_0 \mapsto gT_0g^{-1} - \dot{g}g^{-1},$$

$$T_j \mapsto gT_jg^{-1}, \text{ for } i = 1,2,3,4,5.$$

For the next proposition, we shall consider the Lie group $\mathcal{G}_0 = \{g\in\mathcal{G}\mid g(0) = g(2) = id\in G\}$, then its lie algebra is denoted by $\mathfrak{G}_0$ given by maps $\rho : [0,2]\to\mathfrak{g}$, such that $\rho(0) = \rho(2) = 0$. We can now state the following result:

**Proposition 5.2.1.** Consider the map:

$$\mu = (\mu_1, \mu_2, \mu_3, \mu_4, \mu_5) : \mathcal{A} \to \mathbb{C}^5 \otimes \mathfrak{G}_0^*,$$

given by

$$\mu_1 = \dot{T}_1 + [T_0, T_1] - [T_5, T_2]$$

$$\mu_2 = \dot{T}_2 + [T_0, T_2] - [T_1, T_5]$$

$$\mu_3 = \dot{T}_3 + [T_0, T_3] - [T_1, T_3] - [T_2, T_4] - [T_5, T_4]$$

$$\mu_4 = \dot{T}_4 + [T_0, T_4] + [T_1, T_4] - [T_2, T_3] + [T_5, T_3]$$

$$\mu_5 = \dot{T}_5 + [T_0, T_5] - [T_1, T_2] - [T_4, T_3].$$

Then $\mu$ is the moment map for the action of $\mathcal{G}_0$

*Proof.* Let $m = (T_0, T_1, T_2, T_3, T_4, T_5) \in \mathcal{A}$ and in this notation, the action of $\mathcal{G}$ on $\mathcal{A}$ is given by:

$$m \mapsto (gT_0g^{-1} - \dot{g}g^{-1}, gT_1g^{-1}, gT_2g^{-1}, gT_3g^{-1}, gT_4g^{-1}, gT_5g^{-1}).$$

Let $\rho \in \mathfrak{G}_0$ and consider the vector field $\rho^*|_m = \dfrac{d}{dt}(exp(t\rho) \cdot m)$. From the description of the action above we have:

$$\rho^*|_m = ([\rho, T_0] - \dot{\rho}, [\rho, T_1], [\rho, T_2], [\rho, T_3], [\rho, T_4], [\rho, T_5])$$

Now, let $\underline{t} = (t_0, t_1, t_2, t_3, t_4, t_5)$ be a tangent vector at $m \in \mathcal{A}$, then

$$\begin{aligned}
\alpha_1(\rho^*, \underline{t}) &= \int_0^2 \langle [\rho, T_0] - \dot{\rho}, t_1 \rangle - \langle [\rho, T_1], t_0 \rangle - \langle [\rho, T_5], t_2 \rangle + \langle [\rho, T_2], t_5 \rangle \\
&= \int_0^2 -\langle \dot{\rho}, t_1 \rangle + \langle [\rho, T_0], t_1 \rangle - \langle [\rho, T_1], t_0 \rangle - \langle [\rho, T_5], t_2 \rangle + \langle [\rho, T_2], t_5 \rangle \\
&=^{3} - \langle \rho, t_1 \rangle|_0^2 + \int_0^2 \langle \rho, \dot{t}_1 \rangle + \langle [\rho, T_0], t_1 \rangle - \langle [\rho, T_1], t_0 \rangle - \langle [\rho, T_5], t_2 \rangle + \langle [\rho, T_2], t_5 \rangle \\
&= \int_0^2 \langle \rho, \dot{t}_1 + [T_0, t_1] + [t_0, T_1] - [T_5, t_2] - [t_5, T_2] \rangle \\
&= \int_0^2 \langle \rho, d\mu_1(\underline{t}) \rangle,
\end{aligned}$$

where $\mu_1 = \dot{T}_1 + [T_0, T_1] - [T_5, T_2]$. The proof for the other $\mu_k$ is similar to this one. □

**Remark 5.2.2.** It is important to notice here that, from the discussion in the first section, the quotient $\mu^{-1}(0)/\mathcal{G}_0$ is a 2-symplectic manifold. Moreover, $\mu^{-1} = (0)$ is the space of solutions to the following system if equations:

$$\begin{aligned}
\dot{T}_1 &+ [T_0, T_1] - [T_5, T_2] = 0 \\
\dot{T}_2 &+ [T_0, T_2] - [T_1, T_5] = 0 \\
\dot{T}_3 &+ [T_0, T_3] - [T_1, T_3] - [T_2, T_4] - [T_5, T_4] = 0 \\
\dot{T}_4 &+ [T_0, T_4] + [T_1, T_4] - [T_2, T_3] + [T_5, T_3] = 0 \\
\dot{T}_5 &+ [T_0, T_5] - [T_1, T_2] - [T_4, T_3] = 0.
\end{aligned} \tag{5.21}$$

Now notice that we can always choose a gauge $g_0$ such that $T_0 = 0$ and we obtain our Nahm's equations corresponding to the line bundle with transition function $e^{\eta/\xi^2}$. Therefore, solutions to Nahm's equations modulo the action of $\mathcal{G}_0$ is a 2-symplectic

[3] We used integration by parts.

manifold. We can also change the behaviour of the $T_j$ at the endpoints of $[0, 2]$, we can add poles for instance, and the corresponding quotient will give other 2-symplectic manifolds.

## 5.3 The moduli space of solutions to Nahm's equations with trivial boundary conditions

Given $(T_0, T_1, T_2, T_3, T_4, T_5) \in \mathcal{A}$, define the following matrix valued functions:

$$\begin{aligned} \alpha &= T_0 + iT_5, \\ \beta &= T_1 + iT_2, \\ \gamma &= T_3 + iT_4. \end{aligned}$$

Observe that the equations (5.21) are equivalent to the following system of equations:

$$\begin{aligned} &\dot{\beta} = [\beta, \alpha], \\ &\dot{\gamma} = [\gamma, \alpha] - [\beta, \gamma^*], \\ &(\alpha \dot{+} \alpha^*) = [\alpha^*, \alpha] + [\beta, \beta^*] + [\gamma^*, \gamma]. \end{aligned} \tag{5.22}$$

Let $\hat{\mathcal{A}} = \{(\alpha, \beta, \gamma) |\ \alpha, \beta, \gamma : [0, 2] \to \mathfrak{g} \text{ is smooth}\}$. Observe that the action of $\mathcal{G}_0$ extends to an action of the complexified group $\mathcal{G}_0^{\mathbb{C}}$, namely:

$$\begin{aligned} \alpha &\mapsto g\alpha g^{-1} - \dot{g}g^{-1}, \\ \beta &\mapsto g\beta g^{-1}, \\ \gamma &\mapsto g\gamma g^{-1}. \end{aligned}$$

The system of equations above was already considered in [Bie06] and here we shall state a result concerning the moduli space of solutions to the equations above and we shall compute its twistor space.

We shall denote the moduli space of solutions to (5.22) with trivial boundary conditions by $\hat{\mathcal{M}}$. To describe $\hat{\mathcal{M}}$, fix a gauge $g_0$ such that $\alpha = 0$ and identify $(\alpha, \beta, \gamma) \mapsto (g_0, \beta(0), \gamma(0))$. We can state this as:

**Proposition 5.3.1** ([Bie06])**.** $\hat{\mathcal{M}}$ is diffeomorphic to $G^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}}$.

Next we shall find the twistor space of this 2-symplectic manifold by using method analogue to [Kro04].

**Theorem 5.3.2.** The twistor space of $\hat{\mathcal{M}}$ is the fibre bundle $\mathcal{Z}$ over $\mathbb{C}P^1$ with fibre $G^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}}$ and transition function

$$G^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}} \times U \to G^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}} \times U'$$

$$(g, \alpha, \xi) \mapsto (g \cdot exp(2\alpha/\xi^2), \alpha/\xi^4, 1/\xi).$$

*Proof.* Let $\mathcal{E} = (\mathcal{O}(2) \oplus \mathcal{O}(2)) \otimes \mathcal{C}$ be the twistor space of $\mathcal{A} \otimes \mathcal{C}$, where $\mathcal{C}$ is the space of smooth $\mathfrak{g}$-valued functions on $[0, 2]$. Also, fix a trivialisation for $\mathcal{E}$:

$$\mathcal{E}|_U = \mathcal{C} \times \mathcal{C} \times U, \text{ with coordinates } (\mu, \eta, \xi),$$

$$\mathcal{E}|_U' = \mathcal{C} \times \mathcal{C} \times U', \text{ with coordinates } (\mu', \eta', \xi').$$

such that on the intersection $U \cap U'$ we have $\mu' = \mu/\xi^2$, $\eta' = \eta/\xi^2$ and $\xi' = 1/\xi$. Now the action of $\mathcal{G}$ can be extended to a fibrewise action of $\mathcal{G}^{\mathbb{C}}$ on $\mathcal{E}$. This action is explicitly given by:

$$\begin{aligned}
&\mu \mapsto g\mu g^{-1} + \xi^2 \dot{g} g^{-1}, \\
&\eta \mapsto g\eta g^{-1} + \dot{g} g^{-1} \text{ on } U \text{ and} \\
&\mu' \mapsto g\mu' g^{-1} + \dot{g} g^{-1}, \\
&\eta' \mapsto g\eta' g^{-1} + \xi'^2 \dot{g} g^{-1} \text{ on } U'.
\end{aligned}$$

We can also define a complex symplectic form on the fibres of $\mathcal{E}$ by:

$$\omega_{\mathcal{E}}((a,b),(a',b')) = \int_0^2 \langle a, b' \rangle - \langle b, a' \rangle.$$

We now state that the action of $\mathcal{G}_0^{\mathbb{C}}$ on the fibres of $\mathcal{E}$ is Hamiltonian with moment map given by:

$$\begin{aligned}
\mathfrak{m} &= -\dot{\mu} + \xi^2 \dot{\eta} + [\eta, \mu] \text{ on } U, \\
\mathfrak{m}' &= \dot{\eta}' - \xi'^2 \dot{\eta}' + [\eta', \mu'] \text{ on } U'.
\end{aligned}$$

The proof that this is indeed a moment map is similar to (5.2.1) and we shall not prove it here. For $\xi \in U$ the solution to $\mathfrak{m} = 0$ is:

$$\begin{aligned}
\eta &= \dot{g} g^{-1}, \\
\mu &= g\alpha g^{-1} + \xi^2 \dot{g} g^{-1},
\end{aligned}$$

for $g : [0,2] \to G^{\mathbb{C}}$ with $g(0) = 1$ and $\alpha \in \mathfrak{g}^{\mathbb{C}}$. Then, the $\mathcal{G}^{\mathbb{C}}$-orbit of this solution is given by the pair $(g(2), \alpha) \in G^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}}$. This gives a trivialisation $\mathcal{Z}|_U \to G^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}} \times U$.

We can do the same for a point $\xi' \in U'$, a solution to $\mathfrak{m}' = 0$ is given by:

$$\eta' = g'\alpha'(g')^{-1} - \xi'^2\dot{g}(g')^{-1},$$
$$\mu' = -\dot{g}'(g')^{-1},$$

then $(g'(2), \alpha')$ is a trivialisation $\mathcal{Z}|_U' \to G^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}} \times U'$.

Now, we must have $\eta' = \eta/\xi^2$ and $\mu' = \mu/\xi^2$. Imposing this conditions for the solution above we can see that we must have:

$$g'(t) = g(t) \cdot exp(2t\alpha/\xi^2),$$
$$\alpha' = \alpha/\xi^4.$$

For $t = 2$ we have the transition function given in the statement of the theorem. $\square$

It is important to highlight here that the case where $G = U(1)$ correspond to the case of charge 1 monopoles on $\mathbb{R}^5$. Therefore, we have:

**Theorem 5.3.3.** The moduli space of charge 1 monopoles on $\mathbb{R}^5$ is a 2-symplectic manifold diffeomorphic to $S^1 \times \mathbb{R}^5$ and its twistor space is the total space of the $U(1)$-principal bundle associated to the line bundle $L^2$ over $\mathbb{T}$.

## 5.4 Open issue: The moduli space of solutions to Nahm's equations with modified boundary conditions

In this section we present explicit solutions to our Nahm's equations. The solutions presented here do not correspond to monopoles since they do not satisfy the appropriate reality conditions. However, it should still be interesting to study these spaces. We could obtain 2-hypercomplex structures and new hypercomplex structures, since there exists a hypercomplex manifold that fibres over every GHC manifold.

We shall define an ansatz to solve Nahm's equations. First, define the matrices:

$$t_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$t_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix},$$

$$t_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$t_4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$t_5 = \frac{1}{2}\begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}.$$

Now, define $T_j = f_j t_j$, for $j = 1, 2, 3, 4, 5$, where $f_j : (0, 2) \to \mathbb{R}$ is analytic and has simple poles at $0$ and $2$. We can substitute this ansatz in our Nahm's equation and obtain the following system of non-linear ordinary differential equations:

$$\dot{f}_1 = f_5 f_2,$$

$$\dot{f}_2 = f_5 f_1,$$

$$\dot{f}_3 = f_5 f_4,$$

$$\dot{f}_4 = f_5 f_3,$$

$$\dot{f}_5 = f_1 f_2 + f_3 f_4,$$

This system can be solve in terms of the Jacobi elliptic functions $dn_k$, $cn_k$ and $sn_k$:

$$f_1 = -\frac{1}{\sqrt{2}} \frac{K cn_k(K(s+1))}{sn_k(K(s+1))}$$
$$f_2 = -\frac{1}{\sqrt{2}} \frac{K dn_k(K(s+1))}{sn_k(K(s+1))}$$
$$f_3 = -\frac{1}{\sqrt{2}} \frac{K cn_k(K(s+1))}{sn_k(K(s+1))}$$
$$f_4 = -\frac{1}{\sqrt{2}} \frac{K dn_k(K(s+1))}{sn_k(K(s+1))}$$
$$f_5 = -\frac{K}{sn_k(K(s+1))},$$

where

$$K(k) = \int_0^{\pi/2} \frac{dt}{1 - k^2 \text{sin} t}.$$

Observe that the solution $T_1, T_2, T_3, T_4, T_5$ have poles at $0$ and $2$ whose residues are respectively $\frac{1}{\sqrt{2}}t_1$, $\frac{1}{\sqrt{2}}t_2$, $\frac{1}{\sqrt{2}}t_3$, $\frac{1}{\sqrt{2}}t_4$, $t_5$. We can generalise this ansatz and obtain solutions to the Nahm's equations of rank $k \geq 4$. It would be interesting to identify the moduli space of solutions to our Nahm's equations satisfying:

(i) $T_1$ and $T_2$ are Hermitian matrices,

(ii) $T_3$, $T_4$ and $T_5$ are skew-Hermitian and

(iii) $T_1, T_2, T_3, T_4, T_5$ have poles at $0$ and $2$ whose residues are respectively $\frac{1}{\sqrt{2}}t_1, \frac{1}{\sqrt{2}}t_2, \frac{1}{\sqrt{2}}t_3, \frac{1}{\sqrt{2}}t_4, t_5$.

A similar question is investigated for the canonical Nahm's equations in [Dan93].

Furthermore, we can also generalise Nahm's equations to higher degrees, this is to say, we can construct monopoles in $\mathbb{R}^{2n+1}$, for $n \geq 3$, and we should obtain equations that we can use to construct $n$-symplectic manifolds.

# Bibliography

[AH88] M. Atyiah and N. Hitchin. *The Geometry and Dynamics of Magnetic Monopoles*. Princeton University Press, 1988.

[AHH90] M. R. Adams, J. Harnad, and J. Hurtubise. Isospectral Hamiltonian flows in finite and infinite dimensions. II. Integration of flows. *Comm. Math. Phys.*, 134:555–585, 1990.

[AW77] M.F. Atyiah and R. S. Ward. Instantons and algebraic geometry. *Commun Math Phys*, 55:117–124, 1977.

[BE89] R. J. Baston and M. G. Eastwood. *The Penrose transform: its interaction with representation theory*. (Oxford Mathematical Monograph, 1989.

[Bea90] A. Beauville. Jacobiennes des courbes spectrales et systemes hamiltoniens completement integrables. *Acta Math.*, 164:211235, 1990.

[Bie06] R. Bielawski. Manifolds with an $SU(2)$ action on the tangent bundle. *Trans of the Am. Math. Soc.*, 358:3997 4019, 2006.

[Bie09] R. Bielawski. Line bundles on spectral curves and the generalised legendre transform construction of Hyperkahler metrics. *J.Geom.Phys.*, 59:374–390, 2009.

[Dan93] A. Dancer. Nahm's equations and hyperkahler geometry. *Comm. Math. Phys.*, 158:545–568, 1993.

[Dir31] P. A. M. Dirac. Quantized singularities in the electromagnetic field. *Proc. R. Soc. London*, 133:60–72, 1931.

[Don84] S. Donaldson. Nahm's equations and the classification of monopoles. *Comm. Math. Phys.*, 96:387–407, 1984.

[FH99] W. Fulton and J. Harris. *Representation Theory: A First Course*. Springer GTM, 1999.

[GH94] P. Griffiths and J. Harris. *Principles of Algebraic Geometry*. Wiley-Interscience, 1994.

[Har97] R. Hartshorne. *Algebraic Geometry*. Springer GTM, 1997.

[Hit82] N. Hitchin. Monopoles and Geodesics. *Comm. Math. Phys.*, 83:579–602, 1982.

[Hit83] N. Hitchin. On the construction of monopoles. *Comm. Math. Phys.*, 89:145–190, 1983.

[HKLR87] N. Hitchin, A. Karlhede, U. Lindstrom, and M. Rocek. Hyperkähler metrics and supersymmetry. *Comm. Math. Phys.*, 108:535–589, 1987.

[HM89] J. Hurtubise and M. K. Murray. On the construction of monopoles for the classical groups. *Comm. Math. Phys.*, 122:35–89, 1989.

[Hoo74] G.'t Hooft. Magnetic monopoles in unified gauge theories. *Nuclear Physics B*, 79:276–284, 1974.

[Hur85] J. Hurtubise. The asymptotic higgs field of a monopole. *Comm. Math. Phys.*, 97:381–389, 1985.

[JT80] A. Jaffe and C. Taubes. *Vortices and monopoles*. Birkhauser, 1980.

[Kro04] P. Kronheimer. A hyperkahler structure on the cotangent bundle of a complex Lie group. *arXiv:math/0409253v1*, 0:1–11, 2004.

[MS92] L. Mason and G. A. J. Sparling. Twistor correspondences for the soliton hierarchies. *Journal of Geom. and Phys.*, 8:243–271, 1992.

[MS04] N. Manton and P. Sutcliffe. *Topological solitons*. Cambridge Monographs on Math. Phys., 2004.

[Muk03] S. Mukai. *An Introduction to Invariants and Moduli*. Cambridge University Press, 2003.

[Nah82] W. Nahm. The construction of all self-dual multimonopoles by the ADHM method. *Monopoles in Quantum Field Theory, Singapore, World*, 1982.

[Nak93] H. Nakajima. Monopoles and nahm's equations. *In: Einstein metrics and Yang-Mills connections, Lecture Notes in Pure and Appl. Math.*, 145:0, 1993.

[Nas06] O. Nash. *Differential Geometry of Monopoles Moduli Spaces*. PhD Thesis, 2006.

[Oli14] G. Oliveira. *Higher dimensional monopoles*. PhD Thesis, 2014.

[Pol74] A. M. Polyakov. Particle spectrum in quantum field theory. *JEPT letters*, 20:194, 1974.

[PS75] M.K. Prasad and C. M. Sommerfield. Exact solutions for the't Hooft monopole and the Julia-Zee dyon. *Phys.Rev.Lett.*, 43:760, 1975.

[PS81] M.K. Prasad and C. M. Sommerfield. Yang-Mills-Higgs monopole solutions of arbitrary topological charge. *Commun Math Phys*, 80:137–149, 1981.

[SM96] M. Singer and M. Murray. Spectral curves of non-integral hyperbolic monopoles. *Nonlinearity*, 9:973997, 1996.

[Ver96] M. Verbitsky. Hyperholomorphic bundles. *Journ. of Alg. Geom.*, 5:633–669, 1996.

[War81a] R. S. Ward. Two Yang-Mills-Higgs monopoles close together. *Phys. Lett. B*, 102:136–139, 1981.

[War81b] R. S. Ward. A Yang-Mills-Higgs monopole of charge 2. *Commun Math Phys*, 79:317–325, 1981.