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Modelling the effect of telegraph noise in the SIRS epidemic model using Markovian switching

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HIGHLIGHTS

- We introduce telegraph noise with Markov Switching into the SIRS epidemic model.
- Establish extinction and persistence conditions.
- SIR model a special case.
- Analytical results confirmed by simulation.

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ABSTRACT

We discuss the effect of introducing telegraph noise, which is an example of an environmental noise, into the susceptible–infectious–recovered–susceptible (SIRS) model by examining the model using a finite-state Markov Chain (MC). First we start with a two-state MC and show that there exists a unique nonnegative solution and establish the conditions for extinction and persistence. We then explain how the results can be generalised to a finite-state MC. The results for the SIR (Susceptible–Infectious–Removed) model with Markovian Switching (MS) are a special case. Numerical simulations are produced to confirm our theoretical results.

1. Introduction

The dynamics of population systems are often influenced by different types of environmental noise for example white noise or Brownian motion. The effects of white noise have already been considered by various authors (e.g. Refs. [1–4]). Environmental noise has the potential to have a huge impact on the population dynamics of a system. For example it has been shown that sufficiently large white noise can cause a population that would otherwise explode or tend to a unique endemic equilibrium to die out [2,5]. In this paper, we will focus on another type of environmental noise, namely telegraph noise (or burst noise). This consists of sudden instantaneous transitions between two or more sets of parameter values in the underlying model corresponding to two or more different environments or regimes (e.g. Refs. [6–8]). Switching between environments follows a finite state continuous time Markov Chain (MC) with state space $\mathbb{S} = \{1, 2, \dots, M\}$, where M is the number of different environments. Hence the switching times are memoryless and follow an exponential distribution.

There are already various papers which have looked at the effect of telegraph noise in a population system model. As an example, let us consider a Lotka–Volterra predator–prey model,

$$\dot{x}(t) = x(t)(a - by(t)),$$
$$\dot{y}(t) = y(t)(-c + dx(t)), \tag{1.1}$$

where x is the number of prey and y is the number of predators at time t. a, b, c and d are positive constants.

So if $r(t)$ is a continuous time Markov Chain with state space $\{1, 2\}$ this Lotka–Volterra model with Markov Switching becomes

$$\dot{x}(t) = x(t)(a_{r(t)} - b_{r(t)}y(t)),$$
$$\dot{y}(t) = y(t)(-c_{r(t)} + d_{r(t)}x(t)), \tag{1.2}$$

where a_i, b_i, c_i and d_i are positive constants.

This Lotka–Volterra predator–prey model has been looked at in various papers. For example Takeuchi et al. [9] studied the behaviour of this system. Note that under each environment the numbers of predators and prey follow a deterministic predator–prey equation, switching between environments according to telegraph noise. Takeuchi et al. have shown that if the two equilibrium states of the two subsystems differ, all positive trajectories of the system always exit from any compact set in $\mathbb{R}_+^2$ with probability one. On the other hand, if the two equilibrium states coincide, then the trajectory either exits from any compact set in $\mathbb{R}_+^2$ or converges to the common equilibrium point. These properties imply that the population system (1.1) under telegraph noise is neither permanent nor dissipative [9]. Du et al. [6] investigated the impact that telegraph noise has on the behaviour of Lotka–Volterra competition systems. The oscillatory behaviour of the solution to the systems with telegraph noise was observed. Li et al. [10] looked at a more generalised Lotka–Volterra model with n interacting species described by an n-dimensional system of ordinary differential equations. In their paper they looked at the effect that two different types of environmental noise have on the system. First of all they introduced white noise into the model in the form of Brownian motion. They then took a further step by considering telegraph noise using a finite-state MC. They obtained the existence conditions for the system to have global positive solutions as well as the conditions for the solutions to be ultimately bounded and permanent. Furthermore, they also established the extinction conditions.

In this paper, we want to look at the effect that telegraph noise has on the dynamics of an SIRS epidemic model. In the 1920s, Kermack and McKendrick [11] constructed the SIR and SIRS epidemic models to illustrate respectively diseases where there is a permanent acquired immunity such as measles [12] and where there is a temporary acquired immunity such as rubella. The SIR model is a special case of the SIRS model. We will explain later on in this paper that the results for our SIRS model also apply to the SIR model. There has been much research done on different aspects of both SIR and SIRS epidemic models. For example Hethcote [12] shows that the behaviour of each model is determined by a threshold parameter (the basic reproduction number R_0). If R_0 is less than or equal to one or no disease is initially present then the system tends to the unique disease-free equilibrium (DFE), but if R_0 exceeds one then the system tends to a unique endemic equilibrium.

Tornatore et al. [13] propose a stochastic SIR model with environmental white noise added into the disease transmission term with or without distributed time delay and study the stability of the DFE. The numerical simulation of the stochastic SIR model shows that the introduction of noise modifies the threshold of the system for an epidemic to occur and the threshold value is found. Lu [14] later extended their results into an SIRS model.

Yang et al. [15] introduce stochastic environmental noise into the death rates for SIR and SEIR (susceptible–exposed–infectious–removed) epidemic models with different death rates for different population classes. They investigated the dynamics of the models depending on the basic reproduction number R_0. The long-term behaviour of the two stochastic systems is studied. The authors mainly use stochastic Lyapunov functions to show that under certain conditions, the solutions are ergodic if $R_0 > 1$, and that they are exponentially stable when $R_0 \leq 1$. Finally they show numerically that the analytical results are true.

Zhao and Jiang [16] studied the dynamics of a stochastic SIRS epidemic model with saturated incidence. The disease transmission term is $\beta SI/(1 + \alpha I)$, where α is a constant, and there are deaths due to the disease. They introduce environmental white noise into the disease transmission parameter β in a similar way to Gray et al. [17]. They obtain a threshold value of the stochastic system which determines the extinction and persistence of the epidemic. They also show that large noise will suppress the epidemic.

O' Regan et al. [18] constructed a new Lyapunov function for a variety of deterministic SIR and SIRS models in epidemiology. They considered the SIR and SIRS models with proportional disease incidence and deaths due to the disease. They used this to establish the global stability of the endemic equilibrium states in these models. On the other hand Korobeinikov [19] constructed different Lyapunov functions for two-dimensional SIR and SIRS compartmental epidemic models with nonlinear transmission rate of a very general form $f(S, I)$ subject to a few biologically relevant conditions. The models included some with vertical and horizontal transmission. Korobeinikov shows that provided that the population size is constant and $f(S, I)$ is concave in I, the number of infectious individuals, then the positive endemic equilibrium state is globally stable.

Vargas de Leon [20] establishes the global stability conditions for classic deterministic SIS, SIR and SIRS epidemic models with constant recruitment, disease-induced death and standard incidence rate. He uses novel methods to construct

Lyapunov functions and shows that for the SIRS model the unique endemic equilibrium is globally stable under certain parameter conditions.

Liu and Stechlinski [21] consider pulse and constant control schemes for deterministic SIR epidemic models with seasonality in the contact rate. A constant treatment scheme is applied to the model. Easily verifiable conditions on the basic reproduction number of the infectious disease are established which ensure disease eradication under these constant control strategies. Later both pulse vaccination and pulse treatment models are applied to an SIR model with time-varying contact rate. Further, a vaccine failure model as well as a model with a reduced infective class are considered with pulse control schemes. Again conditions on the basic reproduction number are developed which ensure disease eradication.

Nasell [22] considers stochastic models of some endemic infections with demography. Approximations of quasi-stationary distributions and times to extinction are derived for stochastic versions of the SI (susceptible–infectious), SIS, SIR and SIRS epidemic models. The approximations are valid for sufficiently large population sizes. Conditions for validity of the approximations are given for each of the models. There are also conditions for validity of the corresponding deterministic model. It is noted that some deterministic models are unacceptable approximations of the stochastic models for a large range of realistic parameter values.

Chen and Li [23] introduced the effect of white noise into the SIR epidemic model and the time delayed SIR epidemic model. This was done by adding a separate independent Brownian motion term to each of the per capita susceptible and infectious death rates. They showed that the system has a positive global solution. They then linearised the stochastic delayed SIR model and studied the exponential mean square stability of the linearised system with and without delay.

A more recent paper written by Shrestha et al. [24] looked at a different aspect of the SIRS model. They developed a new dynamic message-passing (DMP) algorithm, namely rDMP for recurrent epidemic models such as the SIRS model on networks. They have shown that the rDMP algorithm provides a good approximation to the results obtained from Monte Carlo simulation, its accuracy is often better than the pair approximation and that rDMP is more user friendly.

The well-known SIS (Susceptible–Infectious–Susceptible) epidemic model [12,25] is used to model diseases which do not develop immunity once infected individuals recover, for example gonorrhea, Ref. [25], meningitis [12] and pneumococcus [26,27]. Inspired by the work done by Takeuchi et al. [9], Gray et al. [2] introduced the effect of telegraph noise into this model using a finite state MC. They established the conditions required for almost sure (a.s.) extinction and persistence for their solution to the stochastic SIS model with finite state Markovian Switching (MS).

Consequently motivated by Refs. [2,9], in this paper, we will extend the results given in Ref. [2] to a more complicated three-dimensional SIRS epidemic model as well as the SIR epidemic model. Note that Wei et al. [28] also looked at the stochastic SIR model under regime switching, but the two models and the results obtained differ. In Ref. [28] they considered an SIR epidemic model with a nonlinear incidence term different to ours and different per capita death rates for susceptible, infectious and removed individuals. Similarly to Ref. [10], Wei et al. simultaneously introduced the effect of white noise into the deterministic model in the form of Brownian motion as well as telegraph noise using a continuous time finite state space MC. The results that Wei et al. obtain are different to ours and although their model is more general some of their results are obtained under quite restrictive conditions, whereas our results are not. So their model and results are different. Our model follows the same basic idea as in Refs. [2,9]. We have a group of deterministic SIS models with different parameter values corresponding to different environmental regimes. The switching between regimes occurs according to a continuous time finite state space MC. As far as we know, although there have been various types of work done on the SIRS and SIR models, ours is the first paper that gives a detailed analysis on the effect that telegraph noise has on the SIRS model, and thus we have filled a gap in the existing literature.

The paper is organised as follows. In Section 2, we will introduce the MS SIRS epidemic model. A recap of some of the fundamental concepts of finite state MCs will also be given. In Section 3, the existence of a unique nonnegative solution will be proven. In Section 4, we will look at the conditions needed for extinction for the MS SIRS model. In Section 5, we will obtain the conditions needed for persistence. In Section 6, by using the Lyapunov Theorem, we examine the persistence conditions on the stochastic SIRS model. In Section 7, we will summarise our results and explain how the results for the SIR model are a special case of the SIRS model. Numerical simulations are produced throughout the paper to support our theoretical results.

2. MS SIRS epidemic model

Unless stated otherwise, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let us consider the following deterministic SIRS epidemic model:

$$
\begin{aligned}
\frac{dS(t)}{dt} &= -\beta I(t)S(t) + \mu N - \mu S(t) + \upsilon R(t), \\
\frac{dI(t)}{dt} &= \beta I(t)S(t) - (\mu + \gamma)I(t), \\
\frac{dR(t)}{dt} &= \gamma I(t) - \mu R(t) - \upsilon R(t),
\end{aligned}
\tag{2.1}
$$

where S, I and R denote respectively the number of susceptible, infectious and recovered individuals. N is the total size of the population, β is the disease transmission coefficient and $\beta = \lambda/N$ where λ is the disease contact rate, that is the rate at which susceptibles come into potentially infectious contact with infecteds. μ is the per capita birth and death rate and γ is the rate at which an infected becomes cured and thus moves to the recovery group. υ is the rate of loss of immunity. Those individuals who lose immunity immediately re-enter into the susceptible class.

Note that in this paper S, I and R denote respectively the absolute numbers of individuals in the population as opposed to the proportions. The total population size remains constant so if s, i and r denote the fractions of individuals in each of these categories they satisfy the differential equations:

$$\frac{ds(t)}{dt} = -\beta N i(t) s(t) + \mu - \mu s(t) + \upsilon r(t),$$

$$\frac{di(t)}{dt} = \beta N i(t) s(t) - (\mu + \gamma) i(t), \tag{2.2}$$

$$\frac{dr(t)}{dt} = \gamma i(t) - \mu r(t) - \upsilon r(t).$$

So the equations have the same functional form, the only differences are that the per capita disease transmission coefficient in the model using absolute numbers is replaced by the per capita disease contact rate in the model using proportions and the population input term changes from μN in the absolute numbers model to μ in the proportional model.

Whilst there are many mathematical epidemic models which use the proportions of individuals in each class there are also many papers that use numbers, for example Refs. [29–31,23,32,33,2,17,34–37] and many more. It is more natural to use absolute numbers rather than proportions as the differential equations are usually derived by considering the changes in absolute numbers and then converted to proportions.

As our results are concerned with persistence of solutions and lower and upper bounds for the lim supremum and lim infimum of the variables there is no qualitative difference between the results obtained for our model using absolute numbers and the results which would have been obtained from a model using proportions. It is straightforward to convert the results for our model with absolute numbers into those for the proportional model and vice-versa.

Also note that our models include births and deaths. Again there is a dichotomy of models in the literature. Whilst there are some models that do not include births and deaths these are generally appropriate only for modelling relatively short epidemic outbreaks. If epidemics are modelled over a long period of time births and deaths must be included. Whilst there are some standard models that do not include births and deaths many standard epidemic models do [29,33,38,12,39]. As many of our results are about the long-term behaviour of the system we feel that it is appropriate to include population demographics, that is births and deaths, into the model.

We shall now recall some of the fundamental theories of MCs. Let $r(t)$, $t \geq 0$, be a right-continuous MC on the probability space taking values in finite state space $\mathbb{S} = \{1, 2, \ldots, M\}$ with generator $\Gamma = (\nu_{ij})_{M \times M}$ defined as

$$\mathbb{P}\{r(t+\delta) = j | r(t) = i\} = \begin{cases} \nu_{ij}\delta + o(\delta), & \text{if } i \neq j, \\ 1 + \nu_{ii}\delta + o(\delta), & \text{if } i = j, \end{cases} \tag{2.3}$$

where $\delta > 0$, $\nu_{ij} \geq 0$ is the transition rate from state i to j for $i \neq j$ and $\nu_{ii} = -\sum_{1 \leq j \leq M, j \neq i} \nu_{ij}$. Almost every sample path of $r(\cdot)$ is a right-continuous step function with a finite number of sample jumps in any finite subinterval of $\mathbb{R}_+ = [0, \infty)$ [40]. There is a sequence $\{\tau_k\}_{k \geq 0}$ of finite-valued $\mathcal{F}_t$-stopping times such that $0 = \tau_0 < \tau_1 < \cdots < \tau_k \to \infty$ a.s. and

$$r(t) = \sum_{k=0}^{\infty} r(\tau_k) \mathbf{1}_{[\tau_k, \tau_{k+1})}(t) \tag{2.4}$$

where $\mathbf{1}_A$ denotes the indicator function of set A. The switching is memoryless and if $r(\tau_k) = i$, the random variable $\tau_{k+1} - \tau_k$ will have an exponential distribution with parameter $-\nu_{ii}$. In addition, let us define $\mathbf{\Pi} = (\pi_1, \pi_2, \ldots, \pi_M)$ to be the unique stationary distribution of this MC. If $M = 2$

$$\pi_1 = \frac{\nu_{21}}{\nu_{12} + \nu_{21}} \quad \text{and} \quad \pi_2 = \frac{\nu_{12}}{\nu_{12} + \nu_{21}}. \tag{2.5}$$

Now we will introduce two-state MS into (2.1) which becomes

$$\frac{dS(t)}{dt} = -\beta_{r(t)} I(t) S(t) + \mu_{r(t)} N - \mu_{r(t)} S(t) + \upsilon_{r(t)} R(t),$$

$$\frac{dI(t)}{dt} = \beta_{r(t)} I(t) S(t) - (\mu_{r(t)} + \gamma_{r(t)}) I(t), \tag{2.6}$$

$$\frac{dR(t)}{dt} = \gamma_{r(t)} I(t) - \mu_{r(t)} R(t) - \upsilon_{r(t)} R(t),$$

where $r(t)$ is a right-continuous MC with state space $\mathbb{S} = \{1, 2\}$. We will focus on analysing this model.

3. Existence of unique nonnegative solution

Theorem 3.1. *For any given initial value $S(0) = S_0 \in (0, N)$, $I(0) = I_0 \in (0, N)$ and $R(0) = R_0 \in (0, N)$, there exists a unique and nonnegative solution for the MS SIRS model* (2.6) *for all t.*

Proof. The proof is straightforward and so we will not discuss this in detail here.

4. Extinction

In this section we will focus on discussing the conditions for extinction for our MS SIRS model (2.6). For the deterministic SIRS model, the criterion used to determine whether a disease will go extinct or persist is called the basic reproduction number $R_0^D = \frac{\beta N}{\mu + \gamma}$. This represents the expected number of secondary infections caused by an infected individual entering the DFE (e.g. Refs. [29,33,38,12]). If $R_0^D > 1$ then we expect that the disease will persist while $R_0^D \leq 1$ indicates that it will die out. We will use another type of threshold to determine whether the disease will die out or persist a.s., namely

$$T_0^S = \frac{\pi_1 \beta_1 N + \pi_2 \beta_2 N}{\pi_1(\mu_1 + \gamma_1) + \pi_2(\mu_2 + \gamma_2)}, \tag{4.1}$$

where

- (π_1, π_2) is the unique stationary distribution,
- β_1 is the disease transmission coefficient in state 1,
- β_2 is the disease transmission coefficient in state 2,
- μ_1 is the per capita birth and death rates in state 1,
- μ_2 is the per capita birth and death rates in state 2,
- γ_1 is the recovery rate in state 1,
- γ_2 is the recovery rate in state 2.

This notation is used by Gray et al. [2] to analyse extinction and persistence for their MS SIS model. By working with the same threshold we will extend their results to our more complex three-dimensional SIRS model (2.6). Let us recall that $r(t)$ is a MC with state space $\mathbb{S} = \{1, 2\}$. If $r(t) = 1$, then we are in state 1 and if $r(t) = 2$ then we are in state 2.

Proposition 4.1. *Let us define $\alpha_{r(t)} = \beta_{r(t)} N - \mu_{r(t)} - \gamma_{r(t)}$, then we have the following alternative ways of interpreting T_0^S:*

- $T_0^S < 1 \Leftrightarrow \pi_1 \alpha_1 + \pi_2 \alpha_2 < 0$,
- $T_0^S = 1 \Leftrightarrow \pi_1 \alpha_1 + \pi_2 \alpha_2 = 0$,
- $T_0^S > 1 \Leftrightarrow \pi_1 \alpha_1 + \pi_2 \alpha_2 > 0$.

Proof. The proof is straightforward.

Theorem 4.2. *If $T_0^S < 1$, then for any given initial value $(S_0, I_0, R_0) \in (0, N)^3$, the solution of the stochastic SIRS epidemic model* (2.6) *obeys*

(i) $\limsup_{t \to \infty} \frac{1}{t} \log(I(t)) \leq \alpha_1 \pi_1 + \alpha_2 \pi_2 < 0$ *a.s.,*
(ii) $\lim_{t \to \infty} R(t) = 0$ *a.s.,*
(iii) $\lim_{t \to \infty} S(t) = N$ *a.s.*

By Proposition 4.1, we hence conclude that $I(t)$ tends to zero exponentially and $R(t)$ tends to zero as $t \to \infty$, thus making $S(t)$ tend to N as $t \to \infty$ a.s. In other words, the disease will die out with probability one and the solution will tend to its DFE $(N, 0, 0)$.

Proof. (i) It is a straightforward modification of the proof of Theorem 4.2 in Ref. [2].

(ii) Suppose that $\limsup_{t \to \infty} R(t) > 0$ on a set Ω_1 where $\mathbb{P}(\Omega_1) = \delta$ for some $\delta > 0$. Then $I(t) \to 0$ as $t \to \infty$ on a set Ω_2 where $\mathbb{P}(\Omega_2) \geq 1 - \frac{\delta}{2}$. For $\omega \in \Omega_2$ then given $\epsilon > 0$ let us choose ϵ_1 small enough so that

$$\frac{\epsilon_1 \max(\gamma_1, \gamma_2)}{\min(\mu_1 + \upsilon_1, \mu_2 + \upsilon_2)} < \frac{\epsilon}{2}, \tag{4.2}$$

where υ_1 and υ_2 represent the rate of loss of immunity in state 1 and state 2 respectively. The other parameter values are defined as before.

$\exists t_0$ such that for $t \geq t_0$, $0 \leq I(t) \leq \epsilon_1$. By integrating the $R(t)$ equation in (2.6), we have that for $t \geq t_0$,

$$R(t) = R(t_0) e^{-Q(t)} + e^{-Q(t)} \int_{t_0}^{t} \gamma_r(s) I(s) e^{Q(s)} ds,$$

$$\leq N e^{-Q(t)} + \int_{t_0}^{t} \gamma_{r(s)} \epsilon_1 e^{-\int_s^t (\mu_{r(u)} + \upsilon_{r(u)}) du} ds, \tag{4.3}$$

where $Q(t) = \int_{t_0}^{t} (\mu_r(s) + \upsilon_r(s))\mathrm{d}s$. By choosing $t_1 \geq t_0$ such that for $t \geq t_1$, $Ne^{-Q(t)} \leq \frac{1}{2}\varepsilon$ we have that for $t \geq t_1$,

$$R(t) \leq \frac{\varepsilon}{2} + \varepsilon_1 \max(\gamma_1, \gamma_2) \int_{t_0}^{t} e^{-\min(\mu_1+\nu_1, \mu_2+\nu_2)(t-s)} \mathrm{d}s. \tag{4.4}$$

Using (4.2) we have that for $t \geq t_1$, $R(t) \leq \varepsilon$. Hence for $\omega \in \Omega_2$, $\limsup_{t\to\infty} R(t) = 0$. This is a contradiction. Hence as $t \to \infty$, $R(t) \to 0$ a.s.

Theorem 4.2(iii) is obvious by using the fact that $S + I + R = N$. $\square$

Note that if both $\alpha_1 < 0$ and $\alpha_2 < 0$, then clearly the corresponding $R_{0,i}^D$ values for both subsystems (state 1 and state 2) are less than one, thus both subsystems will die out. However, the readers may wonder what would happen if one subsystem, say state 1, has $\alpha_1 < 0$ while in state 2 $\alpha_2 > 0$? In other words, one subsystem will go extinct whilst the other will persist. It turns out that if the time it takes for the MC to switch from state 2 to state 1 is relatively faster than from state 1 to 2, so that $\pi_1\alpha_1 + \pi_2\alpha_2 < 0$, then the effect from state 1 will predominate, thus making the overall system die out.

Throughout the paper we shall use numerical simulations to illustrate our results. We shall assume that the unit of time is one day, and the population sizes are measured in units of one million. We now illustrate Theorem 4.2 using a numerical example.

Example 4.3. Let us define the parameters to be

$$\mu_1 = 0.65, \qquad \mu_2 = 0.10, \qquad \gamma_1 = 0.45, \qquad \gamma_2 = 0.25, \qquad \upsilon_1 = 0.15, \qquad \upsilon_2 = 0.75$$
$$\beta_1 = 0.002, \qquad \beta_2 = 0.005, \qquad \nu_{12} = 0.5, \qquad \nu_{21} = 0.8 \quad \text{and} \quad N = 100.$$

By using the definition of $\alpha_{r(t)}$ in Proposition (2.5) and (4.1), we deduce that $\alpha_1 = -0.90$, $\alpha_2 = 0.15$, $\pi_1 = 8/13$ and $\pi_2 = 5/13$. Thus $\pi_1\alpha_1 + \pi_2\alpha_2 = -0.4962 < 0$ to four d.p. Similarly, by using Theorem 4.2, we expect that for any initial value $(S(0), I(0), R(0)) \in (0, N)^3$, the solution to our stochastic SIRS model (2.6) satisfies

1. $\limsup_{t\to\infty} \frac{1}{t} \log(I(t)) \leq -0.4962 < 0$ a.s.,
2. $\lim_{t\to\infty} R(t) = 0$ a.s.,
3. $\lim_{t\to\infty} S(t) = N$ a.s.

In other words, the disease will die out a.s.

Again, the numerical simulations produced by using the Euler method support our results in Theorem 4.2, namely the disease dies out a.s. Note that in this case, $\alpha_1 < 0$ while $\alpha_2 > 0$. The biological meaning of this is that one subsystem will die out while the other subsystem will persist. Similarly, the numerical simulations were repeated numerous times with different parameter values and initial values and all support our results (see Fig. 1).

5. Persistence

Apart from extinction, the aspect of persistence of a disease is very important when analysing an epidemic model for a particular disease. As a result, in this section we will be looking at different types of conditions on persistence for the MS SIRS model (2.6) when $T_0^S > 1$. Note that there are two possible cases that could arise from the condition $T_0^S > 1$, i.e. $\pi_1\alpha_1 + \pi_2\alpha_2 > 0$, namely:

(a) Both α_1 and α_2 are positive. Without loss of generality, we will assume that $0 < \frac{\alpha_1}{\beta_1} \leq \frac{\alpha_2}{\beta_2}$.
(b) One of α_1 and α_2 is positive. Without loss of generality, we will assume that $\frac{\alpha_1}{\beta_1} \leq 0 < \frac{\alpha_2}{\beta_2}$.

First, we will examine in detail the persistence condition $T_0^S > 1$ by looking at the above two cases separately in order to give us a better understanding of the persistence results. Before we begin with the main theorems, we will look at another aspect of persistence which is given by using the uniform persistence theorem (e.g. Refs. [41,42]). We will prove that our solution $I(t)$ for our stochastic SIRS model (2.6) under either case for $T_0^S > 1$ is uniformly strong persistent. Recall that $\alpha_{r(t)} = \beta_{r(t)}N - \mu_{r(t)} - \gamma_{r(t)}$.

Theorem 5.1 (*Uniform Strong Persistence*). *Suppose that* $I(0) > 0$.
Case (a): *If* $0 < \frac{\alpha_1}{\beta_1} \leq \frac{\alpha_2}{\beta_2}$, $\exists \varepsilon' > 0$ *independent of the initial conditions such that*

$$\liminf_{t\to\infty} I(t) \geq \varepsilon' > 0 \quad \text{a.s.} \tag{5.1}$$

In other words the MS SIRS model is a.s. uniformly persistent.
Case (b): *If* $\frac{\alpha_1}{\beta_1} < 0 < \frac{\alpha_2}{\beta_2}$, *given* $\delta_1 > 0$, $\exists \varepsilon' > 0$ *such that* $\forall t_1 > 0$, $I(t) \geq \varepsilon'$ *for some* $t \geq t_1$ *on a set* Ω_1 *where* $\mathbb{P}(\Omega_1) \geq 1 - \delta_1$. *To put this another way,*

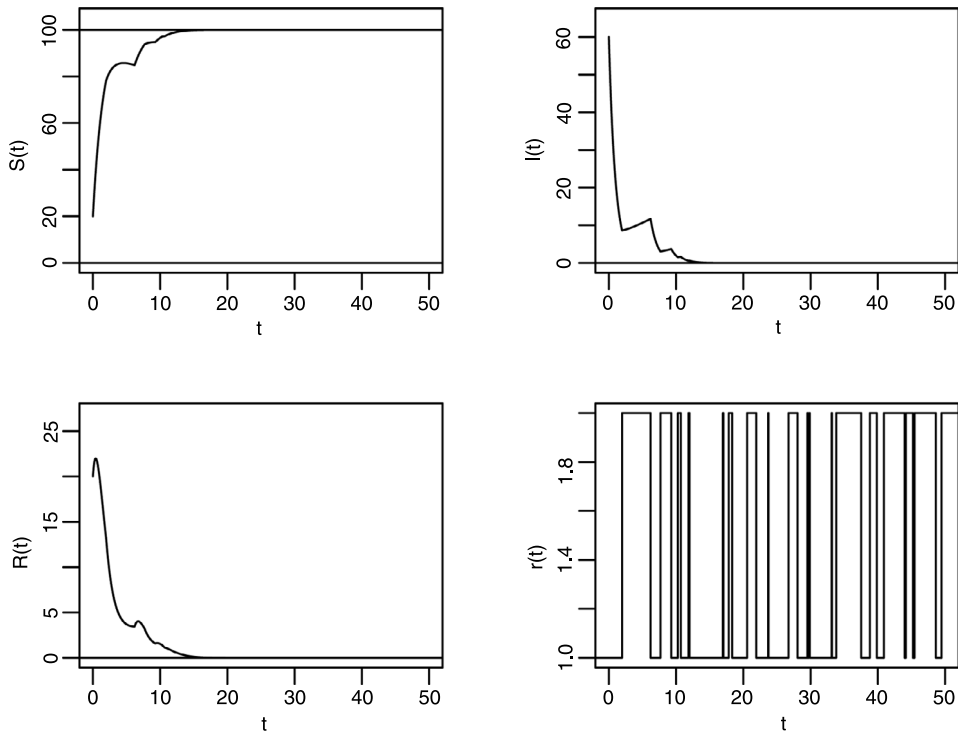$$\liminf_{t\to\infty} I(t) > 0 \quad \text{a.s.}$$

Fig. 1. Numerical simulation for our solution to (2.6) with $T_0^S < 1$ and the corresponding MC $r(t)$ using the parameter values given in Example 4.3 with initial values $S(0) = 20$, $I(0) = 60$, $R(0) = 20$ and $r(0) = 1$.

Proof. Case (a): Let us choose $\varepsilon > 0$ small enough such that

$$\varepsilon < \frac{\alpha_1}{\beta_1} \frac{\min(\mu_1 + \upsilon_1, \mu_2 + \upsilon_2)}{\min(\mu_1 + \upsilon_1, \mu_2 + \upsilon_2) + 2\max(\gamma_1, \gamma_2)}.$$

Suppose that $I(t) < \varepsilon$ for all $t \geq t_0$ and $I(0) > 0$. Then from the third equation in (2.6) for $t \geq t_0$

$$\frac{\mathrm{d}R(t)}{\mathrm{d}t} \leq \max(\gamma_1, \gamma_2)\varepsilon - \min(\mu_1 + \upsilon_1, \mu_2 + \upsilon_2)R(t). \tag{5.2}$$

By integrating (5.2), it is easy to obtain the following expression:

$$R(t) \leq Ne^{-\min(\mu_1 + \upsilon_1, \mu_2 + \upsilon_2)(t - t_0)} + \frac{\max(\gamma_1, \gamma_2)\varepsilon}{\min(\mu_1 + \upsilon_1, \mu_2 + \upsilon_2)}. \tag{5.3}$$

Let us choose $t_1 > t_0$ such that for $t \geq t_1$, we have

$$Ne^{-\min(\mu_1 + \upsilon_1, \mu_2 + \upsilon_2)(t - t_0)} \leq \frac{\max(\gamma_1, \gamma_2)\varepsilon}{\min(\mu_1 + \upsilon_1, \mu_2 + \upsilon_2)}. \tag{5.4}$$

By using (5.4), (5.3) becomes

$$R(t) \leq \frac{2\max(\gamma_1, \gamma_2)\varepsilon}{\min(\mu_1 + \upsilon_1, \mu_2 + \upsilon_2)}, \tag{5.5}$$

for $t \geq t_1$.

We have that

$$\frac{1}{I(t)} \frac{\mathrm{d}I(t)}{\mathrm{d}t} = \alpha_{r(t)} - \beta_{r(t)}(I(t) + R(t)),$$

$$\geq \alpha_{r(t)} - \beta_{r(t)}(\varepsilon + R(t)). \tag{5.6}$$

By substituting (5.5) into the above equation, we have that

$$\frac{1}{I(t)} \frac{\mathrm{d}I(t)}{\mathrm{d}t} \geq \min_{r=\{1,2\}} \left[\alpha_r - \beta_r\varepsilon \left(1 + \frac{2\max(\gamma_1, \gamma_2)}{\min(\mu_1 + \upsilon_1, \mu_2 + \upsilon_2)} \right) \right] = K_1 > 0. \tag{5.7}$$

This implies that $I(t)$ is an increasing function and it must eventually increase above ε. Moreover, from our argument we know that by time t_1, $R(t)$ must drop to a level at most $\frac{2\max(\gamma_1,\gamma_2)\varepsilon}{\min(\mu_1+\upsilon_1,\mu_2+\upsilon_2)}$, where

$$t_1 - t_0 = \begin{cases} \dfrac{-1}{\min(\mu_1+\upsilon_1,\mu_2+\upsilon_2)}\log\left(\dfrac{\max(\gamma_1,\gamma_2)\varepsilon}{N\min(\mu_1+\upsilon_1,\mu_2+\upsilon_2)}\right), & \text{if } \dfrac{\max(\gamma_1,\gamma_2)\varepsilon}{\min(\mu_1+\upsilon_1,\mu_2+\upsilon_2)} < N, \\ 0, & \text{if } \dfrac{\max(\gamma_1,\gamma_2)\varepsilon}{\min(\mu_1+\upsilon_1,\mu_2+\upsilon_2)} \geq N. \end{cases} \quad (5.8)$$

Furthermore, from the second equation of (2.6) $I(t_1) \geq I(0)\mathrm{e}^{-\max(\mu_1+\gamma_1,\mu_2+\gamma_2)t_1} > 0$. For $t \geq t_1$, $I(t) \geq I(t_1)\mathrm{e}^{K_1(t-t_1)} \geq I(0)\mathrm{e}^{-\max(\mu_1+\gamma_1,\mu_2+\gamma_2)t_1}\mathrm{e}^{K_1(t-t_1)}$, hence $I(t)$ must reach level ε by a time at most t_2 where

$$t_2 = t_1 + \frac{1}{K_1}\left[\log\left(\frac{\varepsilon}{I(0)}\right) + \max(\mu_1+\gamma_1,\mu_2+\gamma_2)t_1\right]. \quad (5.9)$$

As a result, we have shown that if $I(0) < \varepsilon$, then $I(t)$ will reach the level ε by at most time t_2. In other words, $I(t)$ will always be greater than ε at some time in the future provided that we start below it. However it is possible for $I(t)$ subsequently to go below ε again later. Consequently, we will now assume that $I(0) = \varepsilon$ and from the above if $I(t)$ does go below ε, it will eventually rise back up again by time at most

$$t_2' = t_1'\left[1 + \frac{1}{K_1}\max(\mu_1+\gamma_1,\mu_2+\gamma_2)\right], \quad (5.10)$$

where t_1' is defined by (5.8) with $t_0 = 0$.

In general, let us define t^* with $I(t^*) = \varepsilon$ to be the first time that $I(t)$ drops beneath ϵ. Then a similar argument as before will show that for $t \geq t^*$,

$$I(t) \geq \varepsilon\mathrm{e}^{-\max(\mu_1+\gamma_1,\mu_2+\gamma_2)t_2'} = \varepsilon' > 0. \quad (5.11)$$

So we have shown that our solution $I(t)$ to (2.6) is uniformly strong persistent in case (a).

Case (b): In this case, we have that $\alpha_1 < 0$, which indicates that $R_{0,1}^D < 1$ in state 1 while in state 2 we have $R_{0,2}^D > 1$. In other words, if we stay in state 1 long enough, $I(t)$ will tend to 0 thus making our solution $(S(t), I(t), R(t))$ for (2.6) tend towards the DFE $(N, 0, 0)$. As a result, unlike in case (a), the uniform strong persistence result will not hold for all the domain as there will be a region where it is possible for $I(t)$ to approach 0 arbitrarily closely with a small but non-zero probability. However, we can make the probability of that happening as small as we want.

Choose ε small enough so that

$$\pi_1\left[\alpha_1 - \beta_1\varepsilon\left(1 + \frac{2\max(\gamma_1,\gamma_2)}{\min(\mu_1+\upsilon_1,\mu_2+\upsilon_2)}\right)\right]$$
$$+ \pi_2\left[\alpha_2 - \beta_2\varepsilon\left(1 + \frac{2\max(\gamma_1,\gamma_2)}{\min(\mu_1+\upsilon_1,\mu_2+\upsilon_2)}\right)\right] > K_2 > 0. \quad (5.12)$$

Now suppose that $\liminf_{t\to\infty} I(t) = 0$ on a set Ω_1 where $\mathbb{P}(\Omega_1) = \delta_1 > 0$. By the ergodic theory of the MC, $\exists T$ independent of the initial state such that on a set Ω_2 where $\mathbb{P}(\Omega_2) \geq 1 - \frac{\delta}{2}$ for $t \geq T$,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\left\{\alpha_{r(s)} - \beta_{r(s)}\varepsilon\left(1 + \frac{2\max(\gamma_1,\gamma_2)}{\min(\mu_1+\upsilon_1,\mu_2+\upsilon_2)}\right)\right\}\mathrm{d}s$$
$$= \pi_1\left[\alpha_1 - \beta_1\varepsilon\left(1 + \frac{2\max(\gamma_1,\gamma_2)}{\min(\mu_1+\upsilon_1,\mu_2+\upsilon_2)}\right)\right]$$
$$+ \pi_2\left[\alpha_2 - \beta_2\varepsilon\left(1 + \frac{2\max(\gamma_1,\gamma_2)}{\min(\mu_1+\upsilon_1,\mu_2+\upsilon_2)}\right)\right] > K_2. \quad (5.13)$$

Consider any $\omega \in \Omega_1 \cap \Omega_2$. Suppose that $\exists t_0(\omega)$ such that $I(t) \leq \varepsilon$ for all $t \geq t_0(\omega)$. Similarly to case (a) we have that $R(t)$ falls beneath a level at most $\frac{2\max(\gamma_1,\gamma_2)\varepsilon}{\min(\mu_1+\upsilon_1,\mu_2+\upsilon_2)}$ from time $t_1(\omega) = t_0(\omega) + t_1'$ onwards. Again similarly to case (a), $I(t_1) > 0$. By integrating (5.6) and substituting $R(t)$ by its upper bound given by (5.5), we have that for $t \geq t_1(\omega)$,

$$\log\left(\frac{I(t)}{I(t_1)}\right) \geq \int_{t_1}^t\left[\alpha_{r(s)} - \beta_{r(s)}\varepsilon\left(1 + \frac{2\max(\gamma_1,\gamma_2)}{\min(\mu_1+\upsilon_1,\mu_2+\upsilon_2)}\right)\right]\mathrm{d}s. \quad (5.14)$$

Hence using (5.13) for $t \geq t_1(\omega) + T$,

$$\lim_{t\to\infty}\frac{1}{t-t_1(\omega)}\log\left(\frac{I(t)}{I(t_1(\omega))}\right) \geq K_2 > 0, \quad (5.15)$$

so for $t \geq t_1(\omega) + T$, $I(t) \geq I(t_1)e^{K_2(t-t_1)}$. In other words, from time $t_1 + T$ onwards, $I(t)$ is bounded below by an increasing unbounded function and thus we have a contradiction and $I(t)$ must rise above the level ε by a time at most $\max(t_2(\omega), t_1(\omega) + T)$ where $\varepsilon = I(t_1(\omega))e^{K_2(t_2(\omega)-t_1(\omega))}$.

Starting at $\max(t_2(\omega), t_1(\omega) + T)$, $\exists\, t_3(\omega) > \max(t_2(\omega), t_1(\omega) + T)$ with $I(t_3(\omega)) = \epsilon$. Moreover arguing as previously every time that $I(t)$ drops beneath ε it must rise up again to this level by time at most t_4' where

$$t_4' = (t_1' + T)\left(1 + \frac{1}{K_2}\max(\mu_1 + \gamma_1, \mu_2 + \gamma_2)\right) > t_1'.$$

Then similarly to (a) we have that $\liminf_{t\to\infty} I(t) \geq \varepsilon'$ for some $\varepsilon' > 0$, contradicting $\omega \in \Omega_1$. This completes the proof of Theorem 5.1. $\quad\square$

Let us now look at more conditions on persistence for our MS SIRS model (2.6).

Theorem 5.2. *If $T_0^S > 1$, then for any given initial value $(S(0), I(0), R(0)) \in (0, N)^3$, then the solution $S(t)$ of the stochastic SIRS model has the properties that:*

(a) $\liminf_{t\to\infty} S(t) \leq N - \frac{\pi_1\alpha_1 + \pi_2\alpha_2}{\pi_1\beta_1 + \pi_2\beta_2}$ *a.s.,*

(b) $\limsup_{t\to\infty} S(t) \geq N - \frac{\pi_1\alpha_1 + \pi_2\alpha_2}{\pi_1\beta_1 + \pi_2\beta_2}$ *a.s.*

In other words, the number of susceptibles will reach the neighbourhood of the level $N - \frac{\pi_1\alpha_1 + \pi_2\alpha_2}{\pi_1\beta_1 + \pi_2\beta_2}$ infinitely many times a.s.

Proof. Case (a): Assume the statement given in Theorem 5.2(a) is not true, then $\exists\, \varepsilon > 0$ sufficiently small such that $\mathbb{P}(\Omega_1) > 0$ where

$$\Omega_1 = \left\{\omega \in \Omega : \lim_{t\to\infty} \inf S(t) > N - \frac{\pi_1\alpha_1 + \pi_2\alpha_2}{\pi_1\beta_1 + \pi_2\beta_2} + 2\varepsilon\right\}.$$

In addition, by the ergodic theory of the MC, we have that $\mathbb{P}(\Omega_2) = 1$ where for any $\omega \in \Omega_2$,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \left\{\alpha_{r(s)} - \beta_{r(s)}\left(\frac{\pi_1\alpha_1 + \pi_2\alpha_2}{\pi_1\beta_1 + \pi_2\beta_2} - \varepsilon\right)\right\} ds$$
$$= \pi_1\left\{\alpha_1 - \beta_1\left(\frac{\pi_1\alpha_1 + \pi_2\alpha_2}{\pi_1\beta_1 + \pi_2\beta_2} - \varepsilon\right)\right\} + \pi_2\left\{\alpha_2 - \beta_2\left(\frac{\pi_1\alpha_1 + \pi_2\alpha_2}{\pi_1\beta_1 + \pi_2\beta_2} - \varepsilon\right)\right\},$$
$$= (\pi_1\beta_1 + \pi_2\beta_2)\varepsilon. \tag{5.16}$$

Now consider any $\omega \in \Omega_1 \cap \Omega_2$. Then there is a positive number $T = T(\omega)$ such that

$$S(t) \geq N - \frac{\pi_1\alpha_1 + \pi_2\alpha_2}{\pi_1\beta_1 + \pi_2\beta_2} + \varepsilon, \quad \forall t \geq T(\omega), \tag{5.17}$$

which we can easily rearrange to get

$$I(t) + R(t) \leq \frac{\pi_1\alpha_1 + \pi_2\alpha_2}{\pi_1\beta_1 + \pi_2\beta_2} - \varepsilon, \quad \forall t \geq T(\omega). \tag{5.18}$$

By integrating (5.6) and using (5.18), we have that for all $t \geq T(\omega)$,

$$\log(I(t)) \geq \log(I(0)) + \int_0^T [\alpha_{r(s)} - \beta_{r(s)}(I(s) + R(s))]ds + \int_T^t \left[\alpha_{r(s)} - \beta_{r(s)}\left(\frac{\pi_1\alpha_1 + \pi_2\alpha_2}{\pi_1\beta_1 + \pi_2\beta_2} - \varepsilon\right)\right] ds.$$

Dividing both sides by t and letting $t \to \infty$, we could simplify the above expression to

$$\lim_{t\to\infty} \inf \frac{1}{t}\log(I(t)) \geq (\pi_1\beta_1 + \pi_2\beta_2)\varepsilon > 0 \tag{5.19}$$

by using (5.16). So $I(t) \to \infty$ as $t \to \infty$, which clearly contradicts (5.18). As a result, it is obvious that our assumption at the beginning is false and Theorem 5.2(a) follows.

Case (b): As above we will assume that there exists $\varepsilon > 0$ sufficiently small such that $\mathbb{P}(\Omega_3) > 0$ where

$$\Omega_3 = \left\{\omega \in \Omega : \lim_{t\to\infty} \sup S(t) < N - \frac{\pi_1\alpha_1 + \pi_2\alpha_2}{\pi_1\beta_1 + \pi_2\beta_2} - 2\varepsilon\right\}.$$

Consider any $\omega \in \Omega_2 \cap \Omega_3$. Then there is a positive number $T = T(\omega)$ such that

$$S(t) \leq N - \frac{\pi_1\alpha_1 + \pi_2\alpha_2}{\pi_1\beta_1 + \pi_2\beta_2} - \varepsilon, \quad \forall t \geq T(\omega). \tag{5.20}$$

The remainder of the proof follows the same lines as (a). $\quad\square$

As a result we have proved that the number of susceptibles will persist and it will reach the neighbourhood of the level $N - \frac{\pi_1\alpha_1+\pi_2\alpha_2}{\pi_1\beta_1+\pi_2\beta_2}$ infinitely many times a.s.

Before we look at the persistence theorem for $I(t)$ we need the following lemma:

Lemma 5.3. *Given $\varepsilon_1 > 0$,*

(a) *If $I(t) \geq \xi$ for $t \geq t_0$, $\exists t_1 \geq t_0$ such that for $t \geq t_1$,*

$$R(t) \geq \xi \min\left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2}\right)(1 - \varepsilon_1).$$

(b) *If $I(t) \leq \xi$ for $t \geq t_0$, $\exists t_1 \geq t_0$ such that for $t \geq t_1$,*

$$R(t) \leq \xi \max\left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2}\right)(1 + \varepsilon_1).$$

Proof. Case (a): Let us define a sequence of stopping times $t_0 = \tau_0 < \tau_1 < \cdots < \tau_m < t$ where τ_{m+1} is interpreted as t. Then for the case $I(t) \geq \xi$, the equation $\frac{dR(t)}{dt}$ for our stochastic SIRS model defined in (2.6) gives:

$$\frac{d}{dt}\left(R(t)e^{F(t)}\right) \geq \gamma_{r(t)}\xi e^{F(t)}, \tag{5.21}$$

where $\quad F(t) = \sum_{k=0}^{m}(\mu_{r(\tau_k)} + \upsilon_{r(\tau_k)})(\tau_{k+1} - \tau_k). \tag{5.22}$

By integrating Eq. (5.21), replacing the term $F(t)$ with (5.22) and some rearranging, we deduce that:

$$R(t)e^{F(t)} - R(t_0) \geq \int_{t_0}^{t} \gamma_{r(s)}\xi \exp\left[(\mu_{r(\tau_0)} + \upsilon_{r(\tau_0)})(\tau_1 - \tau_0) + \cdots + (\mu_{r(\tau_m')} + \upsilon_{r(\tau_m')})(s - \tau_m')\right] ds,$$

where $t_0 = \tau_0 < \tau_1 < \cdots < \tau_m' \leq s \ldots \leq \tau_m \leq t$,

$$= \sum_{k=0}^{m} \int_{\tau_k}^{\tau_{k+1}} \gamma_{r(s)}\xi \exp\left[(\mu_{r(\tau_0)} + \upsilon_{r(\tau_0)})(\tau_1 - \tau_0) + \cdots + (\mu_{r(\tau_k)} + \upsilon_{r(\tau_k)})(s - \tau_k)\right] ds,$$

$$= \frac{\gamma_{r(\tau_0)}}{\mu_{r(\tau_0)} + \upsilon_{r(\tau_0)}}\xi\left(e^{F(\tau_1)} - e^{F(\tau_0)}\right) + \cdots + \frac{\gamma_{r(\tau_m)}}{\mu_{r(\tau_m)} + \upsilon_{r(\tau_m)}}\xi\left(e^{F(t)} - e^{F(\tau_m)}\right),$$

where $e^{F(\tau_0)} = 1$,

$$\geq \xi \min\left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2}\right)(e^{F(t)} - 1). \tag{5.23}$$

As a result,

$$R(t) \geq R(t_0)e^{-F(t)} + \xi \min\left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2}\right)(1 - e^{-F(t)}). \tag{5.24}$$

Given $\varepsilon_1 > 0$ by choosing t large enough, we have that for $t \geq t_1$,

$$R(t) \geq \xi \min\left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2}\right)(1 - \varepsilon_1). \tag{5.25}$$

We have thus completed the proof for Lemma 5.3(a). The proof for case (b) follows similarly. □

Theorem 5.4. *If $T_0^S > 1$, then for any given initial value $(S(0), I(0), R(0)) \in (0, N)^3$, the solution $I(t)$ of the stochastic SIRS model has the properties that:*

(a) $\liminf_{t\to\infty} I(t) \leq \left(\frac{\pi_1\alpha_1+\pi_2\alpha_2}{\pi_1\beta_1+\pi_2\beta_2}\right) \frac{1}{1+\min\left(\frac{\gamma_1}{\mu_1+\upsilon_1}, \frac{\gamma_2}{\mu_2+\upsilon_2}\right)}$ *a.s.,*

(b) $\limsup_{t\to\infty} I(t) \geq \left(\frac{\pi_1\alpha_1+\pi_2\alpha_2}{\pi_1\beta_1+\pi_2\beta_2}\right) \frac{1}{1+\max\left(\frac{\gamma_1}{\mu_1+\upsilon_1}, \frac{\gamma_2}{\mu_2+\upsilon_2}\right)}$ *a.s.*

So given $\epsilon > 0$ the number of infectives will enter between the levels

$$\left\{\frac{\pi_1\alpha_1 + \pi_2\alpha_2}{\pi_1\beta_1 + \pi_2\beta_2} - \epsilon\right\} \frac{1}{1 + \max\left(\frac{\gamma_1}{\mu_1+\upsilon_1}, \frac{\gamma_2}{\mu_2+\upsilon_2}\right)} \quad and \quad \left\{\frac{\pi_1\alpha_1 + \pi_2\alpha_2}{\pi_1\beta_1 + \pi_2\beta_2} + \epsilon\right\} \frac{1}{1 + \min\left(\frac{\gamma_1}{\mu_1+\upsilon_1}, \frac{\gamma_2}{\mu_2+\upsilon_2}\right)}$$

infinitely often a.s.

Proof. Case (a): Suppose that the assertion is false. Then there exists $\varepsilon > 0$ such that $\mathbb{P}(\Omega_5) > 0$ where

$$\Omega_5 = \left\{ \omega \in \Omega : \liminf_{t \to \infty} I(t) > \left(\frac{\pi_1 \alpha_1 + \pi_2 \alpha_2}{\pi_1 \beta_1 + \pi_2 \beta_2} \right) \frac{1}{1 + \min \left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2} \right)} + 2\varepsilon \right\}.$$

Now by considering any $\omega \in \Omega_5$, there is a positive number $T = T(\omega)$ such that

$$I(t) \geq \left(\frac{\pi_1 \alpha_1 + \pi_2 \alpha_2}{\pi_1 \beta_1 + \pi_2 \beta_2} \right) \frac{1}{1 + \min \left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2} \right)} + \varepsilon, \tag{5.26}$$

for all $t \geq T(\omega)$. From Lemma 5.3(i), given $\varepsilon_1 > 0$ and $I(t) \geq \xi + \varepsilon$, $\exists T_1(\omega) \geq T(\omega)$ such that for $t \geq T_1(\omega) \geq T(\omega)$,

$$R(t) \geq (\xi + \varepsilon) \min \left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2} \right) (1 - \varepsilon_1), \tag{5.27}$$

where $\xi = \left(\frac{\pi_1 \alpha_1 + \pi_2 \alpha_2}{\pi_1 \beta_1 + \pi_2 \beta_2} \right) \frac{1}{1 + \min \left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2} \right)}$. By using $S(t) + I(t) + R(t) = N$, (5.27) becomes

$$S(t) \leq N - (\xi + \varepsilon) \left[1 + \min \left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2} \right) (1 - \varepsilon_1) \right], \tag{5.28}$$

whence

$$\limsup_{t \to \infty} S(t) \leq N - (\xi + \varepsilon) \left[1 + \min \left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2} \right) (1 - \varepsilon_1) \right]. \tag{5.29}$$

Now let $\varepsilon_1 \to 0$. By using Theorem 5.2 we arrive at the following contradiction

$$0 \leq -\varepsilon \left[1 + \min \left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2} \right) \right], \tag{5.30}$$

and we must therefore have

$$\liminf_{t \to \infty} I(t) \leq \left(\frac{\pi_1 \alpha_1 + \pi_2 \alpha_2}{\pi_1 \beta_1 + \pi_2 \beta_2} \right) \frac{1}{1 + \min \left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2} \right)} \quad \text{a.s.} \tag{5.31}$$

Case (b): Similarly, we will assume that there exists $\varepsilon > 0$ sufficiently small such that $\mathbb{P}(\Omega_6) > 0$ where

$$\Omega_6 = \left\{ \omega \in \Omega : \limsup_{t \to \infty} I(t) < \left(\frac{\pi_1 \alpha_1 + \pi_2 \alpha_2}{\pi_1 \beta_1 + \pi_2 \beta_2} \right) \frac{1}{1 + \max \left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2} \right)} - 2\varepsilon \right\}.$$

By using a similar method as in Case (a), but this time using Lemma 5.3(b), the result follows easily. $\square$

Theorem 5.5. *If $T_0^S > 1$, then for any given initial value $(S(0), I(0), R(0)) \in (0, N)^3$, then the solution $R(t)$ of the stochastic SIRS model* (2.6) *has the properties that:*

(a) $\liminf_{t \to \infty} R(t) > 0$ *a.s.,*

(b) $\limsup_{t \to \infty} R(t) < \frac{N \max(\gamma_1, \gamma_2)}{\max(\gamma_1, \gamma_2) + \min(\mu_1 + \upsilon_1, \mu_2 + \upsilon_2)} < N$ *a.s.*

In other words, the limiting value of the number of recovered individuals will be strictly positive and will not ultimately exceed $\frac{N \max(\gamma_1, \gamma_2)}{\max(\gamma_1, \gamma_2) + \min(\mu_1 + \upsilon_1, \mu_2 + \upsilon_2)}$ a.s.

Proof. Case (a): If $\liminf_{t \to \infty} R(t) = 0$ on a set Ω_1 where $\mathbb{P}(\Omega_1) \geq \delta > 0$ then by Theorem 5.1, $\exists \varepsilon > 0$ and t_0 such that $I(t) \geq \varepsilon > 0$ for $t \geq t_0$ on a set Ω_2 where $\mathbb{P}(\Omega_2) \geq 1 - \frac{\delta}{2} > 0$. By Lemma 5.3 $\exists \varepsilon' > 0$ and $t_1 > t_0$ such that for $t \geq t_1$, $R(t) \geq \varepsilon' > 0$ on Ω_2. This is a contradiction proving the result.

Case (b): Let us choose

$$\xi = \frac{N \max(\gamma_1, \gamma_2)}{\max(\gamma_1, \gamma_2) + \min(\mu_1 + \upsilon_1, \mu_2 + \upsilon_2)} < N.$$

From (2.6),

$$\frac{dR(t)}{dt} \leq \max(\gamma_1, \gamma_2)(N - R) - \min(\mu_1 + \upsilon_1, \mu_2 + \upsilon_2)R(t), \tag{5.32}$$

$$= N \max(\gamma_1, \gamma_2) - [\max(\gamma_1, \gamma_2) + \min(\mu_1 + \upsilon_1, \mu_2 + \upsilon_2)]R(t). \tag{5.33}$$

The result of Theorem 5.5(b) follows. $\square$

We will continue to investigate persistence by looking at the two cases that could possibly arise from $T_0^S > 1$.

Theorem 5.6. *Assume that $T_0^S > 1$ and let $I(0) \in (0, N)$ be arbitrary. If $\frac{\alpha_1}{\beta_1} \leq 0 < \frac{\alpha_2}{\beta_2}$, then the following statements hold a.s.:*

(i) $\liminf_{t \to \infty} S(t) \geq N - \frac{\alpha_2}{\beta_2} \left(1 + \max \left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2} \right) \right)$.

(ii) $\limsup_{t \to \infty} I(t) \leq \frac{\alpha_2}{\beta_2}$.

(iii) $\limsup_{t \to \infty} R(t) \leq \frac{\alpha_2}{\beta_2} \max \left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2} \right)$.

Proof. Note that $I(t) > 0$ for all t. Suppose that $\limsup_{t \to \infty} I(t) > \frac{\alpha_2}{\beta_2}$. Then using Theorem 5.4(a), $\exists t_1$ and t_2 with $t_1 < t_2$, such that $\frac{\alpha_2}{\beta_2} < I(t_1) < I(t_2)$ and $I(t)$ is strictly monotonic increasing in $[t_1, t_2]$. Let us now choose $t_3 \in (t_1, t_2)$, not a jump point of the MC such that $\frac{dI(t)}{dt} > 0$. For $r(t) = 1$ and $r(t) = 2$, from (2.6):

$$\frac{1}{I(t_3)} \frac{dI(t)}{dt} \bigg|_{t_3} = \alpha_i - \beta_i(I(t_3) + R(t_3)) < 0, \quad \text{for } i = 1, 2. \tag{5.34}$$

Clearly, for both states we have that $\frac{dI(t)}{dt} < 0$ which is a contradiction. Theorem 5.6(ii) follows. Subsequently, by using Lemma 5.3(b), we have that

$$\limsup_{t \to \infty} R(t) \leq \frac{\alpha_2}{\beta_2} \max \left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2} \right), \tag{5.35}$$

whence by using the fact that $S(t) + I(t) + R(t) = N$, we obtain the desired result that

$$\liminf_{t \to \infty} S(t) \geq N - \frac{\alpha_2}{\beta_2} \left(1 + \max \left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2} \right) \right). \quad \square$$

Example 5.7. Let us now define the parameters to be

$$\mu_1 = 0.65, \qquad \mu_2 = 0.40, \qquad \gamma_1 = 0.45, \qquad \gamma_2 = 0.20, \qquad \upsilon_1 = 0.15, \qquad \upsilon_2 = 0.75$$
$$\beta_1 = 0.009, \qquad \beta_2 = 0.012, \qquad \nu_{12} = 0.5, \qquad \nu_{21} = 0.8 \quad \text{and} \quad N = 100.$$

We see that $\alpha_1 = -0.2$, $\alpha_2 = 0.60$, $\pi_1 = 8/13$ and $\pi_2 = 5/13$, where clearly $\pi_1 \alpha_1 + \pi_2 \alpha_2 = 0.1077 > 0$ to four d.p. From Theorem 5.6, we expect that for any initial value $(S(0), I(0), R(0)) \in (0, N)^3$, the solution to (2.6) satisfies:

(i) $\liminf_{t \to \infty} S(t) \geq N - \frac{\alpha_2}{\beta_2} \left(1 + \max \left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2} \right) \right) = 21.875$,

(ii) $\limsup_{t \to \infty} I(t) \leq \frac{\alpha_2}{\beta_2} = 50$,

(iii) $\limsup_{t \to \infty} R(t) \leq \frac{\alpha_2}{\beta_2} \max \left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2} \right) = 28.125$,

to three d.p.

Again, the numerical simulations generated by the Euler method illustrated in Fig. 2 support our results in Theorem 5.6.

Theorem 5.8. (a) *Assume that $T_0^S > 1$ and let $I(0) \in (0, N)$ be arbitrary. If $0 < \frac{\alpha_1}{\beta_1} \leq \frac{\alpha_2}{\beta_2}$, then the following statements hold a.s.:*

(i) $\liminf_{t \to \infty} S(t) \geq N - \frac{\alpha_2}{\beta_2} \left(1 + \max \left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2} \right) \right)$.

(ii) $\limsup_{t \to \infty} I(t) \leq \frac{\alpha_2}{\beta_2}$.

(iii) $\limsup_{t \to \infty} R(t) \leq \frac{\alpha_2}{\beta_2} \max \left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2} \right)$.

(b) *If $I(0) > 0$ under the same conditions the following statements hold a.s.:*

(i) $\limsup_{t \to \infty} S(t) \leq N - \left(\frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{\beta_2} \max \left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2} \right) \right) \times \left(1 + \min \left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2} \right) \right)$.

(ii) $\liminf_{t \to \infty} I(t) \geq \frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{\beta_2} \max \left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2} \right)$.

(iii) $\liminf_{t \to \infty} R(t) \geq \left(\frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{\beta_2} \max \left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2} \right) \right) \times \min \left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2} \right)$.
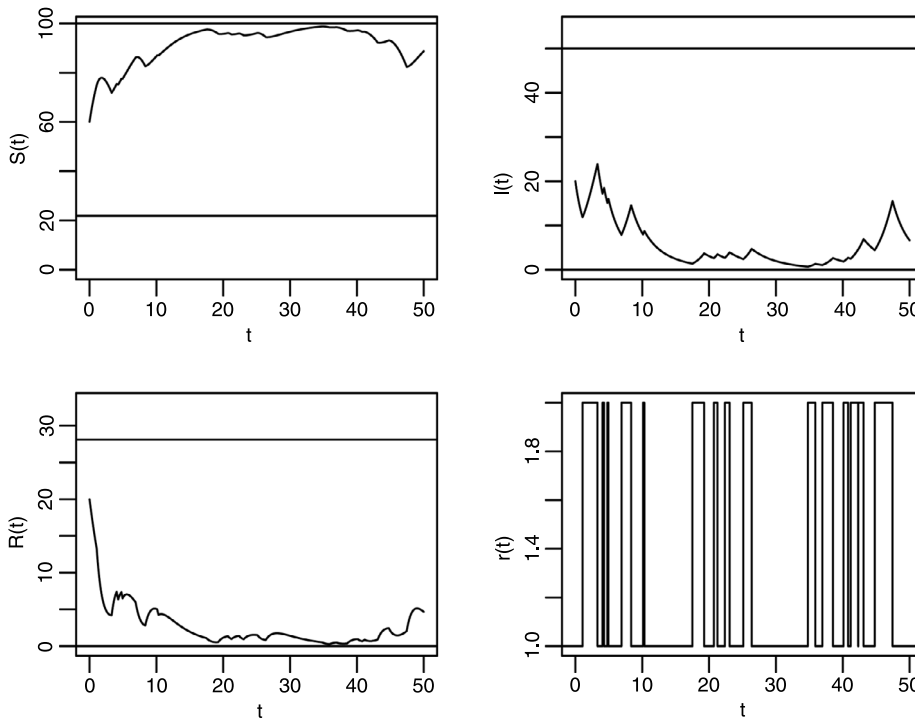
Fig. 2. Numerical simulation for our solution to (2.6) $T_0^S > 1$ and its corresponding MC $r(t)$ using the parameter values given in Example 5.7 with initial values $S(0) = 60$, $I(0) = 20$, $R(0) = 20$ and $r(0) = 1$.

Proof. The proof for case (a) follows as in Theorem 5.6. In order to prove Theorem 5.8(b), without loss of generality we may assume that

$$\frac{\alpha_1}{\beta_1} > \frac{\alpha_2}{\beta_2} \max\left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2}\right).$$

Suppose that Theorem 5.8(bii) is false and choose $\varepsilon > 0$ such that

$$\liminf_{t \to \infty} I(t) < \frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{\beta_2} \max\left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2}\right) - \varepsilon \tag{5.36}$$

on a set Ω_1 where $\mathbb{P}(\Omega_1) = \delta_1 > 0$. Moreover by Theorems 5.4(b) and 5.8(aii)

$$\limsup_{t \to \infty} I(t) \geq \left(\frac{\pi_1 \alpha_1 + \pi_2 \alpha_2}{\pi_1 \beta_1 + \pi_2 \beta_2}\right) \frac{1}{1 + \max\left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2}\right)}$$

$$\text{and} \quad \limsup_{t \to \infty} R(t) \leq \frac{\alpha_2}{\beta_2} \max\left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2}\right),$$

on a set Ω_2 where $\mathbb{P}(\Omega_2) = 1$.

For $\omega \in \Omega_1 \cap \Omega_2$, $\exists t_4(\omega)$ such that for $t \geq t_4$,

$$R(t) < \frac{\alpha_2}{\beta_2} \max\left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2}\right) + \varepsilon.$$

$$\text{Also} \quad \limsup_{t \to \infty} I(t) > \frac{\alpha_1}{\beta_1} \left(1 - \max\left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2}\right)\right),$$

$$> \frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{\beta_2} \max\left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2}\right).$$
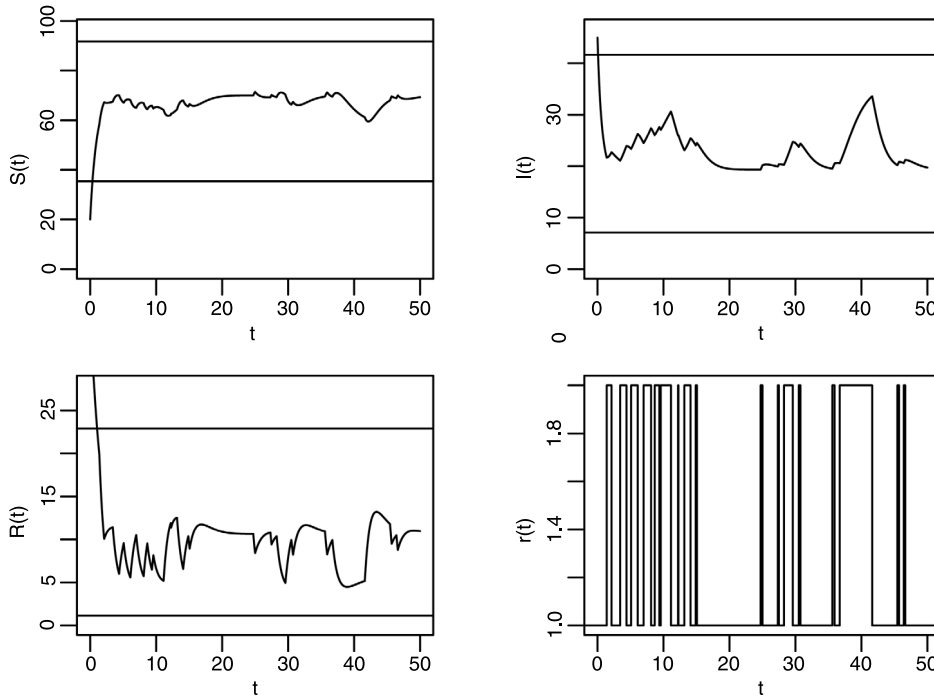
Fig. 3. Numerical simulation for our solution $(S(t), I(t), R(t))$ to (2.6) $T_0^S > 1$ and its corresponding MC $r(t)$ using the parameter values given in Example 5.9 with initial values $S(0) = 20, I(0) = 45, R(0) = 35$ and $r(0) = 1$.

Hence from (5.36) there must exist some t_5 and t_6 where $t_4 < t_5 < t_6$ such that

$$I(t_6) < I(t_5) < \frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{\beta_2} \max\left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2}\right) - \varepsilon,$$

$$\leq \frac{\alpha_2}{\beta_2}\left(1 - \max\left(\frac{\gamma_1}{\mu_1 + \upsilon_1}, \frac{\gamma_2}{\mu_2 + \upsilon_2}\right)\right) - \varepsilon, \tag{5.37}$$

and $I(t)$ is strictly monotonic decreasing in $[t_5, t_6]$.

Let us now choose $t_7 \in (t_5, t_6)$, not a jump point of the MC, such that $\frac{dI(t)}{dt}\big|_{t_7} < 0$. Similar to the proof for Theorem 5.6, it is easy to show that $\frac{dI(t)}{dt}\big|_{t_7} > 0$ which again is a contradiction proving Theorem 5.8(bii). Again, by using Lemma 5.3 and that $S(t) + I(t) + R(t) = N$, we obtained the required results (bi)–(biii). □

Therefore for the case $0 < \frac{\alpha_1}{\beta_2} \leq \frac{\alpha_2}{\beta_2}$, we have obtained both an upper and lower bound for our solution (S, I, R) for (2.6), which is a better result than in the case $\frac{\alpha_1}{\beta_1} \leq 0 < \frac{\alpha_2}{\beta_2}$.

Example 5.9. Let us define the parameter values to be

$$\mu_1 = 0.85, \qquad \mu_2 = 0.50, \qquad \gamma_1 = 0.55, \qquad \gamma_2 = 0.20, \qquad \upsilon_1 = 0.15, \qquad \upsilon_2 = 0.75$$
$$\beta_1 = 0.02, \qquad \beta_2 = 0.012, \qquad v_{12} = 0.5, \qquad v_{21} = 0.8 \quad \text{and} \quad N = 100.$$

By using the definition of $\alpha_{r(t)}$ defined in Proposition (2.5) and (4.1), we deduce that $\alpha_1 = 0.6, \alpha_2 = 0.5, \pi_1 = 8/13$ and $\pi_2 = 5/13$, where clearly $\pi_1\alpha_1 + \pi_2\alpha_2 = 0.5615 > 0$ to four d.p. By substituting the appropriate parameter values into Theorem 5.8, we would expect that for any initial value $(S(0), I(0), R(0)) \in (0, N)^3$,

(a) $35.4167 \leq \liminf_{t\to\infty} S(t) \leq \limsup_{t\to\infty} S(t) \leq 91.7833$,
(b) $7.0833 \leq \liminf_{t\to\infty} I(t) \leq \limsup_{t\to\infty} I(t) \leq 41.6667$,
(c) $1.1333 \leq \liminf_{t\to\infty} R(t) \leq \limsup_{t\to\infty} R(t) \leq 22.9167$,

to four d.p. a.s. This implies that regardless of whatever the initial values, the solution $(S(t), I(t), R(t))$ asymptotically approaches the appropriate region above.

Once again, we could conclude from Fig. 3 that the numerical simulations support our results proved in Theorem 5.8. The numerical simulations were repeated many times with various initial values and the same conclusion was obtained.

In the next section, we will continue to investigate persistence, but we will be using Lyapunov stability (e.g. Refs. [18,35,20]) as well as Theorem 5.1, to obtain results on the convergence of the solution (S, I, R) to its corresponding endemic and disease-free equilibria in each of state 1 and state 2 under the persistence conditions $0 < \frac{\alpha_1}{\beta_1} \leq \frac{\alpha_2}{\beta_2}$ and $\frac{\alpha_1}{\beta_1} \leq 0 < \frac{\alpha_2}{\beta_2}$.

6. Lyapunov stability

When analysing the behaviour of a dynamical system, one of the significant aspects would be the stability of the solution. There are various types of stability, but the most important one is the stability of a solution near its equilibrium point, in other words will the solution converge to its equilibrium point or will it diverge? This aspect of stability could be discussed by using a Lyapunov Theorem, which is what we shall look at here. By using the results from Theorem 5.1, we have obtained some results which further enhance our understanding. It is easy to see that the DFE is $(N, 0, 0)$ while the endemic equilibria for state 1 and 2 are:

$$S_i^* = \frac{N}{R_{0,i}}, \tag{6.1}$$

$$I_i^* = \frac{\mu_i + \upsilon_i}{\mu_i + \upsilon_i + \gamma_i}\left(1 - \frac{1}{R_{0,i}}\right)N = \frac{\mu_i + \upsilon_i}{\mu_i + \upsilon_i + \gamma_i}\left(\frac{\alpha_i}{\beta_i}\right), \tag{6.2}$$

$$R_i^* = \frac{\gamma_i}{\mu_i + \upsilon_i + \gamma_i}\left(1 - \frac{1}{R_{0,i}}\right)N = \frac{\gamma_i}{\mu_i + \upsilon_i + \gamma_i}\left(\frac{\alpha_i}{\beta_i}\right), \tag{6.3}$$

where $R_{0,i}^D = \frac{\beta_i N}{\mu_i + \gamma_i}$ is the basic reproduction number when the MC is in state i for $i = 1, 2$.

Theorem 6.1. *Assume that $T_0^S > 1$ and $0 < \frac{\alpha_1}{\beta_1} \leq \frac{\alpha_2}{\beta_2}$ and let $(S(0), I(0), R(0)) \in (0, N)^3$ be arbitrary and let the switching times of the MC be $0 = \tau_0 < \tau_1 < \cdots < \tau_k$ where $\tau_k \to \infty$ as $k \to \infty$. Define the Lyapunov function to be:*

$$V_i(\mathbf{x}) = I - I_i^* - I_i^* \log\left(\frac{I}{I_i^*}\right) + \frac{\beta_i}{2\gamma_i}(R - R_i^*)^2, \tag{6.4}$$

where $\mathbf{x} = (S(t), I(t), R(t))$, for $i = 1, 2$.
  Note that by considering the Taylor series expansion about $I = I_i^*$ for ϵ small enough, say $\varepsilon \leq \varepsilon_1$ then

$$\frac{1}{4I_i^*}(I - I_i^*)^2 \leq I - I_i^* - I_i^* \log\left(\frac{I}{I_i^*}\right) \leq \frac{(I - I_i^*)^2}{I_i^*}, \tag{6.5}$$

in $(I_i^* - \varepsilon, I_i^* + \varepsilon)$, for $i = 1, 2$.
  For any $\varepsilon \leq \varepsilon_1$ sufficiently small, the Lyapunov function (6.4) for our MS SIRS model has the properties that:

$$\mathbb{P}\left\{\liminf_{t \to \infty} V_1(t) < \frac{\varepsilon^2 \beta_1}{2(\mu_1 + \upsilon_1)}\left(1 + \frac{2(\mu_1 + \upsilon_1 + \gamma_1)}{\alpha_1}\right)\right\} \geq e^{-\upsilon_{12} T_1(\varepsilon)}, \tag{6.6}$$

and

$$\mathbb{P}\left\{\liminf_{t \to \infty} V_2(t) < \frac{\varepsilon^2 \beta_2}{2(\mu_2 + \upsilon_2)}\left(1 + \frac{2(\mu_2 + \upsilon_2 + \gamma_2)}{\alpha_2}\right)\right\} \geq e^{-\upsilon_{21} T_2(\varepsilon)}, \tag{6.7}$$

where $T_1(\varepsilon) = \frac{W}{\varepsilon^2 \beta_1} > 0$ and $T_2(\varepsilon) = \frac{W}{\varepsilon^2 \beta_2} > 0$ for some constant W.

Proof. By differentiating the Lyapunov function (6.4), we have that

$$\frac{dV_i}{dt} = (\beta_i S - \mu_i - \gamma_i)(I - I_i^*) + \left(\frac{\beta_i}{\gamma_i}\right)(R - R_i^*)(\gamma_i(I - I_i^*) - (\mu_i + \upsilon_i)(R - R_i^*)). \tag{6.8}$$

Now by substituting $(\mu_i + \gamma_i) = \beta_i(N - I_i^* - R_i^*)$ and $S = N - I - R$, consequently, (6.8) becomes

$$\frac{dV_i}{dt} = -\beta_i(I - I_i^*)^2 - \frac{(\mu_i + \upsilon_i)\beta_i}{\gamma_i}(R - R_i^*)^2 < 0. \tag{6.9}$$

Thus, $V_i(\mathbf{x}) \geq 0$ and $\dot{V}_i(\mathbf{x}) \leq 0$ with equality if and only if $I = I_i^*$ and $R = R_i^*$. If there is no switching then $V_i(\mathbf{x})$ is a Lyapunov function and the endemic equilibria (S_i^*, I_i^*, R_i^*) given by (6.1)–(6.3) are globally asymptotically stable, i.e. $S \to S_i^*$, $I \to I_i^*$ and $R \to R_i^*$ as $t \to \infty$, whatever the initial condition.

We shall now prove the results with switching time involved. The proof will split into two parts, corresponding to the Lyapunov functions for state 1 and state 2. First we will show that the result holds in state 1. By Theorem 5.1, $\exists t_1, W < \infty$ such that for $t \geq t_1$,

$$V_i(\mathbf{x}) = I - I_i^* - I_i^* \log\left(\frac{I}{I_i^*}\right) + \frac{\beta_i}{2\gamma_i}(R - R_i^*)^2 \leq W < \infty, \tag{6.10}$$

and $\max(V_1(t), V_2(t)) \leq W$.

Define a stopping time

$$\sigma_1 = \inf\{t \geq t_1 : r(t) = 1\}.$$

Clearly, $\mathbb{P}(\sigma_1 < \infty) = 1$, and by the right-continuity of the MC, $r(\sigma_1) = 1$. Define

$$T_1'(\varepsilon) = \frac{V_1(\sigma_1)}{\varepsilon^2 \beta_1} < \infty, \tag{6.11}$$

and note that

$$T_1'(\varepsilon) = \frac{V_1(\sigma_1)}{\varepsilon^2 \beta_1} \leq T_1(\varepsilon) = \frac{W}{\varepsilon^2 \beta_1} \quad \text{a.s.} \tag{6.12}$$

The probability that the MC will not jump to state 2 before $\sigma_1 + T_1'(\varepsilon)$ is

$$\mathbb{P}(\Omega_1) = e^{-\nu_{12} T_1'(\varepsilon)},$$

where $\Omega_1 = \{\omega : r(\sigma_1 + t) = 1, \text{ for all } t \in [0, T_1'(\varepsilon)]\}$. Consider any $\omega \in \Omega_1$ on $[\sigma_1, \sigma_1 + T_1'(\varepsilon)]$ and suppose that

$$-\beta_1(I - I_1^*)^2 - \frac{(\mu_1 + \upsilon_1)\beta_1}{\gamma_1}(R - R_1^*)^2 \leq -\varepsilon^2 \beta_1, \tag{6.13}$$

in this region which by rearranging implies that

$$(I - I_1^*)^2 + \frac{(\mu_1 + \upsilon_1)}{\gamma_1}(R - R_1^*)^2 \geq \varepsilon^2 > 0, \tag{6.14}$$

for $t \in [\sigma_1, \sigma_1 + T_1'(\varepsilon)]$. As a result, for $t \in [\sigma_1, \sigma_1 + T_1'(\varepsilon)]$, (6.9) becomes

$$\frac{dV_i}{dt} \leq -\varepsilon^2 \beta_1. \tag{6.15}$$

Thus, after integrating:

$$0 \leq V_1(\sigma_1 + T_1'(\varepsilon)) \leq V_1(\sigma_1) - \varepsilon^2 \beta_1(T_1'(\varepsilon)), \tag{6.16}$$

from which by substituting $T_1'(\varepsilon)$ by its definition in (6.11),

$$V_1(\sigma_1 + T_1'(\varepsilon)) = 0. \tag{6.17}$$

However, if we recall the Lyapunov function given by (6.4), it is equal to zero if and only if $I(\sigma_1 + T_1'(\varepsilon)) = I_1^*$ and $R(\sigma_1 + T_1'(\varepsilon)) = R_1^*$. This clearly contradicts (6.14) for $t \in [\sigma_1, \sigma_1 + T_1'(\varepsilon)]$. Thus, we must have instead

$$\beta_1(I - I_1^*)^2 + \frac{(\mu_1 + \upsilon_1)\beta_1}{\gamma_1}(R - R_1^*)^2 < \varepsilon^2 \beta_1, \tag{6.18}$$

for some $s \in [\sigma_1, \sigma_1 + T_1'(\varepsilon)]$. Note that at time s,

$$\frac{(I - I_1^*)^2}{I_1^*} < \frac{\varepsilon^2}{I_1^*}, \quad \text{and} \quad (R - R_1^*)^2 < \frac{\varepsilon^2 \gamma_1}{\mu_1 + \upsilon_1}. \tag{6.19}$$

Therefore, if $\varepsilon \leq \varepsilon_1$, then by using (6.5)

$$0 \leq I - I_1^* - I_1^* \log\left(\frac{I}{I_1^*}\right) \leq \frac{(I - I_1^*)^2}{I^*} \leq \frac{\varepsilon^2}{I_1^*}. \tag{6.20}$$

By using (6.19) and (6.20), the Lyapunov function (6.4) at time s is bounded above by

$$V_1(s) < \varepsilon^2 \left(\frac{1}{I_1^*} + \frac{\beta_1}{2(\mu_1 + \upsilon_1)}\right). \tag{6.21}$$

Recall from (6.2) that $I_1^* = \frac{\mu_1 + \upsilon_1}{\mu_1 + \upsilon_1 + \gamma_1}\left(\frac{\alpha_1}{\beta_1}\right)$ hence (6.21) becomes

$$V_1(s) < \varepsilon^2 \left[\frac{(\mu_1 + \upsilon_1 + \gamma_1)\beta_1}{(\mu_1 + \upsilon_1)\alpha_1} + \frac{\beta_1}{2(\mu_1 + \upsilon_1)} \right]. \tag{6.22}$$

Consequently, if $T \geq 0$,

$$\mathbb{P}\left\{ \inf_{T \leq t < \infty} V_1(t) < \frac{\varepsilon^2 \beta_1}{2(\mu_1 + \upsilon_1)}\left(\frac{2(\mu_1 + \upsilon_1 + \gamma_1)}{\alpha_1} + 1\right) \right\} \geq \mathbb{P}(\Omega_1) = e^{-\nu_{12}T_1'(\varepsilon)},$$

$$\geq e^{-\nu_{12}T_1(\varepsilon)}, \tag{6.23}$$

where $T_1(\varepsilon) = \frac{W}{\varepsilon^2 \beta_1}$ defined as before.

Note that

$$\left(\liminf_{t \to \infty} V_1(t) < \frac{\varepsilon^2 \beta_1}{2(\mu_1 + \upsilon_1)}\left(\frac{2(\mu_1 + \upsilon_1 + \gamma_1)}{\alpha_1} + 1\right) \right)$$

$$= \bigcap_{0 < T < \infty} \left(\inf_{T \leq t < \infty} V_1(t) < \frac{\varepsilon^2 \beta_1}{2(\mu_1 + \upsilon_1)}\left(\frac{2(\mu_1 + \upsilon_1 + \gamma_1)}{\alpha_1} + 1\right) \right). \tag{6.24}$$

By letting $T \to \infty$ in (6.23), we have obtained (6.6). (6.7) is proven similarly. $\square$

Theorem 6.1 shows that our solution $(S(t), I(t), R(t))$ can approach either endemic equilibria (S_i^*, I_i^*, R_i^*) arbitrarily closely with strictly positive probability.

Corollary 6.2. *If $\varepsilon \leq \varepsilon_1$, then*

$$\mathbb{P}\left\{ \liminf_{t \to \infty} \max\{|S - S_1^*|, |I - I_1^*|, |R - R_1^*|\} < \varepsilon \left(\sqrt{4 + \frac{2\alpha_1}{\mu_1 + \upsilon_1 + \gamma_1}} + \sqrt{\frac{\gamma_1}{\mu_1 + \upsilon_1}\left(\frac{2(\mu_1 + \upsilon_1 + \gamma_1)}{\alpha_1} + 1\right)} \right) \right\}$$

$$\geq e^{-\nu_{12}T_1(\varepsilon)}, \tag{6.25}$$

and

$$\mathbb{P}\left\{ \liminf_{t \to \infty} \max\{|S - S_2^*|, |I - I_2^*|, |R - R_2^*|\} < \varepsilon \left(\sqrt{4 + \frac{2\alpha_2}{\mu_2 + \upsilon_2 + \gamma_2}} + \sqrt{\frac{\gamma_2}{\mu_2 + \upsilon_2}\left(\frac{2(\mu_2 + \upsilon_2 + \gamma_2)}{\alpha_2} + 1\right)} \right) \right\}$$

$$\geq e^{-\nu_{21}T_2(\varepsilon)}, \tag{6.26}$$

where $T_1(\varepsilon) = \frac{W}{\varepsilon^2 \beta_1}$, $T_2(\varepsilon) = \frac{W}{\varepsilon^2 \beta_2}$, and ε_1 is defined as in Theorem 6.1. Recall that $\alpha_i = \beta_i N - \mu_i \gamma_i$.

Proof. (i) We shall begin by looking at state 1. Recall from (6.5) that for $I \in (I_i^* - \varepsilon, I_i^* + \varepsilon)$ and $\varepsilon < \varepsilon_1$

$$\frac{1}{4I_i^*}(I - I_i^*)^2 \leq I_i - I_i^* - I_i^* \log\left(\frac{I}{I_i^*}\right) \leq \frac{(I - I_i^*)^2}{I_i^*}, \tag{6.27}$$

which implies that if (6.18) holds for $t \in [\sigma_1, \sigma_1 + T_1'(\varepsilon)]$ then for some $s \in [\sigma_1, \sigma_1 + T_1'(\varepsilon)]$,

$$\frac{1}{4I_1^*}(I - I_1^*)^2 \leq V_1(s) \leq \frac{\varepsilon^2 \beta_1}{2(\mu_1 + \upsilon_1)}\left(\frac{2(\mu_1 + \upsilon_1 + \gamma_1)}{\alpha_1} + 1\right). \tag{6.28}$$

By rearranging the above expression, and taking the square root we deduce that

$$|I - I_1^*| \leq \varepsilon \sqrt{4 + \frac{2\alpha_1}{\mu_1 + \upsilon_1 + \gamma_1}}. \tag{6.29}$$

Recall again from (6.4) that

$$V_1(s) = I - I_1^* - I_1^* \log\left(\frac{I}{I_1^*}\right) + \frac{\beta_1}{2\gamma_1}(R - R_1^*)^2, \tag{6.30}$$

which by using (6.28) and some simple rearrangement we have that

$$|R(s) - R_1^*| \leq \varepsilon \sqrt{\frac{\gamma_1}{\mu_1 + \upsilon_1}\left(\frac{2(\mu_1 + \upsilon_1 + \gamma_1)}{\alpha_1} + 1\right)}. \tag{6.31}$$

By using $S(s) = N - I(s) - R(s)$ and $S_1^* = N - I_1^* - R_1^*$, then

$$|S(s) - S_1^*| \leq \varepsilon \left(\sqrt{4 + \frac{2\alpha_1}{\mu_1 + \upsilon_1 + \gamma_1}} + \sqrt{\frac{\gamma_1}{\mu_1 + \upsilon_1} \left(\frac{2(\mu_1 + \upsilon_1 + \gamma_1)}{\alpha_1} + 1 \right)} \right). \tag{6.32}$$

Arguing as in the proof of Theorem 6.1 it is easy to see that (6.25) holds.

(ii) The proof for state 2 follows similarly. □

Corollary 6.2 shows similarly to Theorem 6.1, but using the Euclidean metric instead of the metric induced by the Lyapunov function, that the solution $(S(t), I(t), R(t))$ can approach either endemic equilibrium (S_i^*, I_i^*, R_i^*) arbitrarily closely with strictly positive probability.

In Theorem 6.1 and Corollary 6.2 we have been focusing on analysing the persistence condition where $0 < \frac{\alpha_1}{\beta_1} \leq \frac{\alpha_2}{\beta_2}$ by using Lyapunov stability. We will now complete the results on persistence by obtaining results on the convergence of the solution (S, I, R) to its corresponding disease-free and endemic equilibria under the condition $\frac{\alpha_1}{\beta_1} \leq 0 < \frac{\alpha_2}{\beta_2}$.

Theorem 6.3. *Assume that $T_0^S > 1$ (namely $\pi_1 \alpha_1 + \pi_2 \alpha_2 > 0$) and $\frac{\alpha_1}{\beta_1} \leq 0 < \frac{\alpha_2}{\beta_2}$. Let $(S_0, I_0, R_0) \in (0, N)$ be arbitrary. Then the solution to (2.6) has the properties that*

(i) *If $\varepsilon > 0$, then*

$$\mathbb{P} \left(\lim_{t \to \infty} \inf \max(|N - S|, |I|, |R|) \leq \varepsilon \left(1 + \frac{2\gamma_1}{\mu_1 + \upsilon_1} \right) \right) \geq e^{-\upsilon_{12} T_1(\varepsilon)}, \tag{6.33}$$

where $T_1(\varepsilon) = \bar{t}_1(\varepsilon) + \bar{t}_2(\varepsilon)$ and $\bar{t}_1(\varepsilon)$ and $\bar{t}_2(\varepsilon)$ are defined as:

$$\bar{t}_1(\varepsilon) = \begin{cases} \frac{1}{\beta_1 \varepsilon} \log \left(\frac{N}{\varepsilon} \right), & \text{if } N \geq \varepsilon, \\ 0, & \text{if } N < \varepsilon. \end{cases} \quad \text{and} \quad \bar{t}_2(\varepsilon) = \begin{cases} \frac{-1}{\mu_1 + \upsilon_1} \log \left(\frac{\gamma_1 \varepsilon}{(\mu_1 + \upsilon_1) N} \right), & \text{if } \frac{(\mu_1 + \upsilon_1) N}{\gamma_1} \geq \varepsilon, \\ 0, & \text{if } \frac{(\mu_1 + \upsilon_1) N}{\gamma_1} < \varepsilon, \end{cases} \tag{6.34}$$

respectively.

(ii) *If $\varepsilon > 0$ is small enough such that*

$$\pi_1 \left[\alpha_1 - \beta_1 2\varepsilon \left(1 + \frac{2 \max(\gamma_1, \gamma_2)}{\min(\mu_1 + \upsilon_1, \mu_2 + \upsilon_2)} \right) \right] + \pi_2 \left[\alpha_2 - \beta_2 2\varepsilon \left(1 + \frac{2 \max(\gamma_1, \gamma_2)}{\min(\mu_1 + \upsilon_1, \mu_2 + \upsilon_2)} \right) \right] > 0, \tag{6.35}$$

then $\mathbb{P} \left\{ \lim_{t \to \infty} \inf V_2(t) \leq \frac{\varepsilon^2 \beta_2}{2(\mu_2 + \upsilon_2)} \left(1 + \frac{2(\mu_2 + \upsilon_2 + \gamma_2)}{\alpha_2} \right) \right\} \geq e^{-\upsilon_{21} T_2(\varepsilon)},$ \tag{6.36}

where $T_2(\varepsilon) = \frac{W(\varepsilon)}{\beta_2 \varepsilon^2}$ and $W(\varepsilon) = \max \left\{ N - I_2^* - I_2^* \log \left(\frac{N}{I_2^*} \right), \left| \varepsilon - I_2^* - I_2^* \log \left(\frac{\varepsilon}{I_2^*} \right) \right| \right\} + \frac{\beta_2}{2\gamma_2} N^2 < \infty.$ $V_i(\mathbf{x})$ denotes the Lyapunov function which is defined as in (6.4) for $i = 1, 2$.

Proof. (i) Suppose that $\varepsilon > 0$. Define a stopping time such that

$$\sigma_1 = \inf \{ t \geq 0 : r(t) = 1 \}.$$

Clearly, $\mathbb{P}(\sigma_1 < \infty) = 1$ and by the right-continuity of the MC, $r(\sigma_1) = 1$. The probability that the MC will not jump to state 2 before $\sigma_1 + T_1(\varepsilon)$ is

$$\mathbb{P}(\Omega_1) = e^{-\upsilon_{12} T_1(\varepsilon)},$$

where $\Omega_1 = \{ \omega : r(\sigma_1 + t) = 1, \text{ for all } t \in [0, T_1(\varepsilon)] \}$. Consider any $\omega \in \Omega_1$ on $[0, T_1(\varepsilon)]$, it is easy to see that

$$\frac{dI(t)}{dt} \leq -\beta_1 I(t)^2 \leq -\beta_1 \varepsilon I(t),$$

provided $I \geq \varepsilon > 0$, which after integration becomes

$$I(\sigma_1 + t) \leq I(\sigma_1) e^{-\beta_1 \varepsilon t} \leq N e^{-\beta_1 \varepsilon t}. \tag{6.37}$$

If $N \geq \varepsilon$ then (6.37) shows that by time $\bar{t}_1(\varepsilon)$, $I(t)$ must drop to a level at most ε where

$$\bar{t}_1(\varepsilon) = \frac{1}{\beta_1 \varepsilon} \log \left(\frac{N}{\varepsilon} \right). \tag{6.38}$$

On the other hand if $N < \varepsilon$ then $I(0) < N < \varepsilon$. Arguing as in Theorem 5.1, we know that for $t \geq \bar{t}_1(\varepsilon) + \bar{t}_2(\varepsilon)$,

$$R(\sigma_1 + t) \leq \frac{2\gamma_1 \varepsilon}{\mu_1 + \upsilon_1}, \tag{6.39}$$

$$\text{where} \quad \bar{t}_2(\varepsilon) = \begin{cases} \dfrac{-1}{\mu_1 + \upsilon_1} \log\left(\dfrac{\gamma_1 \varepsilon}{N(\mu_1 + \upsilon_1)}\right), & \text{if } \varepsilon \leq \dfrac{(\mu_1 + \upsilon_1)N}{\gamma_1}, \\ 0, & \text{if } \varepsilon > \dfrac{(\mu_1 + \upsilon_1)N}{\gamma_1}. \end{cases} \tag{6.40}$$

Hence for $t \geq \bar{t}_1(\varepsilon) + \bar{t}_2(\varepsilon)$, we have that

$$|N - S(\sigma_1 + t)| = I(\sigma_1 + t) + R(\sigma_1 + t) \leq \varepsilon\left(1 + \frac{2\gamma_1}{\mu_1 + \upsilon_1}\right). \tag{6.41}$$

Thus we could see from (6.41) that

$$\max\left[N - S(\sigma_1 + \bar{t}_1(\varepsilon) + \bar{t}_2(\varepsilon)), I(\sigma_1 + \bar{t}_1(\varepsilon) + \bar{t}_2(\varepsilon)), R(\sigma_1 + \bar{t}_1(\varepsilon) + \bar{t}_2(\varepsilon))\right] \leq \varepsilon\left(1 + \frac{2\gamma_1}{\mu_1 + \upsilon_1}\right). \tag{6.42}$$

As this is true for each $\omega \in \Omega_1$, we have that

$$\mathbb{P}\left\{\max\left[|N - S(\sigma_1 + \bar{t}_1(\varepsilon) + \bar{t}_2(\varepsilon))|, I(\sigma_1 + \bar{t}_1(\varepsilon) + \bar{t}_2(\varepsilon)), R(\sigma_1 + \bar{t}_1(\varepsilon) + \bar{t}_2(\varepsilon))\right]\right.$$

$$\left. \leq \varepsilon\left(1 + \frac{2\gamma_1}{\mu_1 + \upsilon_1}\right)\right\} \geq e^{-\upsilon_{12}T_1(\varepsilon)}, \tag{6.43}$$

where $T_1(\varepsilon) = \bar{t}_1(\varepsilon) + \bar{t}_2(\varepsilon)$. Consequently, if $T \geq 0$, then

$$\mathbb{P}\left\{\inf_{T \leq t < \infty} \max\left(|N - S(t)|, I(t), R(t)\right) \leq \varepsilon\left(1 + \frac{2\gamma_1}{\mu_1 + \upsilon_1}\right)\right\} \geq e^{-\upsilon_{12}T_1(\varepsilon)}. \tag{6.44}$$

Theorem 6.3(i) follows by arguing as in the proof of Theorem 6.1.

(ii) Recall that (6.35) holds, which is inequality (5.12) with ε replaced by 2ε. Note also that when $r(t) = 1$, $R_{0,1}^D \leq 1$ and also $\frac{dI_1}{dt} < 0$. Hence, if Ω denotes the whole sample space, given $\omega \in \Omega$ and $t_3(\omega) > 0$, for $t \geq t_3(\omega)$, $I(t)$ must rise up and over the level ε at some time $t_4(\omega) > t_3(\omega)$. So $\exists t_5(\omega) > t_3(\omega)$ with $I(t_5(\omega)) = \varepsilon$ and $r(t_5(\omega)) = 2$. Also $V_2(t_5(\omega)) \leq W(\varepsilon)$ where $V_2(t)$ denotes the Lyapunov function in state 2 given by (6.4) in Theorem 6.1 and $W(\varepsilon)$ is a constant.

Now arguing as in the proof of Theorem 6.1 define a new stopping time

$$t_5(\omega) = \inf\{t \geq 0 : r(t_5(\omega)) = 2, \ I(t_5(\omega)) = \varepsilon\}$$

we will have the required result namely,

$$\mathbb{P}\left\{\liminf_{t \to \infty} V_2(t) < \frac{\varepsilon^2 \beta_2}{2(\mu_2 + \upsilon_2)}\left(\frac{2(\mu_2 + \upsilon_2 + \gamma_2)}{\alpha_2} + 1\right)\right\} \geq e^{-\upsilon_{21}T_2(\varepsilon)}, \tag{6.45}$$

where $T_2(\varepsilon) = \frac{W(\varepsilon)}{\varepsilon^2 \beta_2}$ and $W(\varepsilon) = \max\left\{N - I_2^* - I_2^* \log\left(\frac{N}{I_2^*}\right), \left|\varepsilon - I_2^* - I_2^* \log\left(\frac{\varepsilon}{I_2^*}\right)\right|\right\} + \frac{\beta_2}{2\gamma_2}N^2 < \infty$, which could be easily derived from (6.10). $\square$

In this theorem, we have obtained interesting probabilistic results on the convergence of the solution $(S(t), I(t), R(t))$ of the stochastic SIRS model (2.6) to its corresponding disease-free and endemic equilibria.

Corollary 6.4. If $\varepsilon < \varepsilon_1$, then:

$$\mathbb{P}\left\{\liminf_{t \to \infty} \max\{|S - S_2^*|, |I - I_2^*|, |R - R_2^*|\}\varepsilon\left(\sqrt{4 + \frac{2\alpha_2}{\mu_2 + \upsilon_2 + \gamma_2}} + \sqrt{\frac{\gamma_2}{\mu_2 + \gamma_2}\left(\frac{2(\mu_2 + \upsilon_2 + \gamma_2)}{\alpha_2} + 1\right)}\right)\right\}$$

$$\geq e^{-\upsilon_{21}T_2(\varepsilon)}, \tag{6.46}$$

where $T_2(\varepsilon) = \frac{W(\varepsilon)}{\varepsilon^2 \beta_2}$ and ε_1 is defined as in Theorem 6.1.

Proof. Similar to the proof for Corollary 6.2. $\square$

6.1. $T_0^S = 1$ case

So far, we have looked into great detail on the dynamic behaviour of (2.6) under the thresholds $T_0^S < 1$ and $T_0^S > 1$. The reader may ask what about the case when $T_0^S = 1$? We are unable to prove analytically the behaviour of our solution $(S(t), I(t), R(t))$ in this situation. However numerical simulations indicated that the disease would always ultimately die out whatever the initial conditions.

7. Summary and discussion

There are many environmental factors that could affect the behaviour of a population system such as the availability of food and temperature [43]. As discussed in the introduction some previous work on ecological models with telegraph noise has been done by Takeuchi et al. [9] who obtained results on boundedness or convergence to the equilibrium of trajectories of the Lotka–Volterra model with telegraph noise. Li et al. [10] have obtained results on the behaviour of an *n*-dimensional Lotka–Volterra model under regime switching. Motivated by Refs. [2,9] we have examined the effect of environmental noise on a more complicated model, the SIRS model, by using the concept of MS to include telegraph noise to give the MS SIRS model (2.6). In our model we have linked several deterministic models in different environmental regimes using a MC. We have obtained the conditions needed for almost surely extinction and persistence using the threshold T_0^S which was also used in Ref. [2]. In Theorem 4.2, we showed that if $T_0^S < 1$ then the disease will go extinct almost surely. On the other hand if $T_0^S > 1$, then the disease will persist almost surely (Theorems 5.2, 5.4 and 5.5). In Theorems 5.6 and 5.8 we obtained two sets of persistence conditions for the two possible cases in which $T_0^S > 1$, namely $\frac{\alpha_1}{\beta_1} \le 0 < \frac{\alpha_2}{\beta_2}$ and $0 < \frac{\alpha_1}{\beta_1} \le \frac{\alpha_2}{\beta_2}$. Furthermore by using the uniform strong persistence result for $I(t)$, (Theorem 5.1) and the Lyapunov stability theorem, we managed to obtain probabilistic results on convergence of our solution to the disease-free and endemic equilibria in Section 6. Numerical simulations were produced to support and illustrate our theoretical results.

Note that it is easy to see that the two-state MS Susceptible–Infectious–Removed (SIR) model is a special case of the MS SIRS model (2.6). In fact, we could easily derive the corresponding extinction and persistence results for the MS SIR model by setting $v_{r(t)}$, in (2.6) to zero. Furthermore, the results obtained for the two-state MC can be easily extended into a finite state MC with state space $\mathbb{S} = \{1, 2, \ldots, M\}$, similarly to the corresponding extension in Ref. [2].

Remember that we have chosen to model the absolute numbers of individuals in each category as opposed to the proportions. However the results which we have obtained have all been concerned with persistence of solutions and upper and lower limits for the lim supremum and lim infimum of the variables. The results do not depend on our choice to model the absolute numbers of individuals in the population rather than the proportions. There is no essential difference between the models in the two formats. It is straightforward to convert the results for the model in one format into the results for the model with the other format.

Note also that we chose to include births and deaths into the model because we felt that this was appropriate as we were considering modelling the disease over a long timescale. If we exclude births and deaths from the model it will be appropriate only for short term disease outbreaks. For the SIRS model ($v_1 > 0$ or $v_2 > 0$) all of the results go through (just set $\mu_1 = \mu_2 = 0$). Our proofs break down if $\mu_1 = \mu_2 = 0$, so that the population is closed with no births and deaths, and either or both of v_1 and v_2 are zero, so that at least one of the models is the closed population SIR model. If $\mu_1 = \mu_2 = v_1 = v_2 = 0$ then it is clear that Theorem 4.2 is not true as if $T_0^S < 1$ in this case $I(t) \to 0$ but $S(t)$ decreases monotonically to a limiting value $S^* < N$ whilst $R(t)$ increases monotonically to a limiting value $R^* > 0$.

If we consider the SIRS epidemic model without MS then provided that there is some mechanism for generating new susceptibles, whether that is through births and deaths in the population, or through immune individuals losing immunity and returning to the susceptible class, then the qualitative behaviour of the population is the same. There is a threshold value $R_0 = \frac{\beta N}{\mu + \gamma}$. If $R_0 \le 1$ then the disease ultimately dies out. If $R_0 > 1$ then there is a unique endemic equilibrium which the system ultimately approaches [12].

On the other hand for the SIR model in a closed population, with no births and deaths where immunity is permanent then there is a qualitative difference in the behaviour. Here $R_0 = \frac{\beta N}{\gamma}$. The initial infective replacement number is defined as the expected number of cases generated by each separate infective individual at the start of the epidemic. So this is

$$R_0 \frac{S(0)}{N} = \frac{\beta S(0)}{\gamma}.$$

If this number exceeds unity then the number of infectious individuals rises up to a limit and then drops down to zero. If this number is less than or equal to one then the number of infectious decreases to zero. The disease transmission ceases because the infective replacement number (the expected number of cases generated by a single infective individual) is less than one when the number of susceptibles is small, however the limiting number of susceptibles is strictly positive [12].

So even without MS there is a fundamental qualitative difference between an SIRS epidemic model that has a mechanism for introducing new susceptibles and one that does not. The results in this paper are true provided that all of the individual SIRS models corresponding to different states of the MC have some mechanism for generating new susceptibles, whether that is births and deaths, or immunity being temporary. The behaviour of SIRS models with MS where at least one of the

individual SIRS models is a closed population (without births and deaths) SIR model remains an interesting open question for further study.

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