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Almost sure stability of the Euler–Maruyama method with random variable stepsize for stochastic differential equations

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Abstract

In this paper, the Euler–Maruyama (EM) method with random variable stepsize is studied to reproduce the almost sure stability of the true solutions of stochastic differential equations. Since the choice of the time step is based on the current state of the solution, the time variable is proved to be a stopping time. Then the semimartingale convergence theory is employed to obtain the almost sure stability of the random variable stepsize EM solution. To our best knowledge, this is the first paper to apply the random variable stepsize (with clear proof of the stopping time) to the analysis of the almost sure stability of the EM method.

Key words: stopping time, almost sure stability, Euler–Maruyama, variable stepsize, semimartingale convergence theory.

1 Introduction

This paper is devoted to the analysis of the numerical reproduction of the almost sure stability for stochastic differential equations (SDEs) by using the well-known semimartingale convergence theory. Almost sure stability of solutions to SDEs has been widely studied (see for example, Chapter 5.8 in [9], Chapter 4.3 in [12], and the references therein). The ability to reproduce the almost sure stability is one important characteristic of numerical methods. Many papers have

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studied the numerical reproduction of the almost sure stability by adopting the semimartingale convergence theory, for example [2, 14, 19, 20, 23, 24, 25] and the references therein. However, in most of the papers the stepsize is either fixed or nonrandom variable.

The classical explicit methods, such as the Euler-Maruyama method, may reproduce the almost sure stability of SDEs with the global Lipschitz coefficients, but the requirements on the time stepsize are very restrictive. For SDEs with non-global Lipschitz coefficient, the Euler-Maruyama method may not preserve this properties with any stepsize (see for example Lemma 3.1 in [8]). To tackle this, the methods with implicit structure are often employed as the alternatives [13, 19, 23]. Compared with the classical explicit methods, those implicit methods can reproduce larger ranges of SDEs with less restrictions on the stepsize. Nevertheless, the implicit methods may require additional computational costs to solve nonlinear equation system at each iteration.

Bearing those points above in mind, the random variable stepsize is introduced to embed into the classic Euler-Maruyama (EM) method in this paper. Our key contribution is that we prove the time variable is a stopping time. Moreover, the stopping time is essential for the application of the semimartingale convergence theory in our approach. Benefiting from the random variable stepsize, the sufficient conditions for the almost sure stability of the EM method obtained in this paper are much weaker than those established in [14] and [23]. To our best knowledge, this is the first paper to apply the random variable stepsize (with clear proof of the stopping time) to the analysis of the almost sure stability of the EM method.

It should be noted that the technique of adjusting the size of each step has been broadly used in the multi-stage methods (see for example [3, 4, 18], and references therein). Due to the application of the local error control technique, some steps could be rejected then smaller steps may be retreated. Since the stepsize in those methods is dependent on the state of the solution, it is indeed a random variable. However, the current stepsize may be decided after future information available and this indicates the time variable can not be a stopping time [15]. In fact, not like the case in this paper the stopping time is not necessary for those methods [6].

The Euler-type methods with the random variable stepsize, were also considered in different aspects, for instance in [5] to reproduce the finite time explosion of SDEs, in [11] to study convergence and ergodicity, and in [16] to optimise the error constant.

We also mention here that there are lots of other approaches to study the almost sure stability of the numerical methods for SDEs, for example by the local error control, by directly applying the the strong law of large numbers, and by the Chebyshev inequality and the Borel-Cantelli lemma the almost sure stability can be derived from the moment exponential stability. We refer to some of the works [8, 10, 13, 17, 21] and references therein.

This paper is constructed as follows. Section 2 is devoted to the mathematical notation and some preparation for the main result. In Section 3 we present our main result, Theorem 3.1, in which we demonstrate the strategy of choosing the stepsize, give the proof of the stopping time and conclude the almost sure stability of the EM method with random variable stepsize. Section 4 sees the computer simulations of the proposed method. In Section 5, alternative sufficient conditions for the numerical almost sure stability are proposed, which enable the EM method with random variable stepsize to cover wider range of SDEs. Proofs in the last section are only briefed as the same techniques to those in Theorem 3.1 are employed.

2 Preliminary

Throughout this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ which is increasing and right continuous, with $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets. Let $B(t) = (B_1(t), ..., B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space, where T denotes the transpose of a vector or a matrix. Let $|\cdot|$ denote both the Euclidean vector norm and the Frobenius matrix norm. The inner product of x, y in $\mathbb{R}^n$ is denoted by $\langle x, y\rangle$. Denote $\max(a, b)$ and $\min(a, b)$ by $a \vee b$ and $a \wedge b$, respectively. Denote the smallest integer larger than a real number x by $\lceil x \rceil$. $\mathbb{R}_+$ denotes the set of all nonnegative real numbers. $\mathbb{N}$ denotes the set of all nonnegative integers. $\mathbb{Z}$ denotes the set of all integers. $\mathbb{Q}$ denotes the set of all rational numbers.

In this paper, we investigate the numerical methods for the n-dimensional SDE

$$dx(t) = f(x(t))dt + g(x(t))dB(t), \quad x(0) \in \mathbb{R}^n, \tag{2.1}$$

where f: $\mathbb{R}^n \to \mathbb{R}^n$ and g: $\mathbb{R}^n \to \mathbb{R}^{n\times m}$. The following two conditions are imposed on the drift and diffusion coefficients. For every integer $R \geq 1$, there exists a positive constant $C(R)$ such that, for all $x, y \in \mathbb{R}^n$ with $|x| \vee |y| \leq R$,

$$|f(x) - f(y)|^2 \vee |g(x) - g(y)|^2 \leq C(R)|x - y|^2. \tag{2.2}$$

And $\forall x \in \mathbb{R}^n$

$$-z(x) := 2\langle x, f(x)\rangle + |g(x)|^2 \leq 0. \tag{2.3}$$

From (2.3), we can see that the coercivity condition holds automatically. Therefore under (2.2) and (2.3), there exists a unique solution to (2.1) for any given initial value $x(0) \in \mathbb{R}^n$ (see, for example Theorem 2.3.5 in [12]). The theorem for the almost sure asymptotic stability for the SDE (2.1) is presented as follows.

Theorem 2.1 *Let (2.2) and (2.3) hold. Assume $z(x) = 0$ if and only if $x = 0$, then for any initial value $x(0) \in \mathbb{R}^n$*

$$\lim_{t \to \infty} x(t) = 0 \quad a.s.$$

We refer to the stochastic version of the LaSalle theorem in [22] for the proof of this theorem.

Lemma 2.2 *Assume $z(x)$, defined by (2.3), is zero if and only if $x = 0$. Then both $f(x) = 0$ and $g(x) = 0$ if $x = 0$, and $f(x) \neq 0$ if $x \neq 0$.*

Proof. We first prove $f(x) \neq 0$ if $x \neq 0$. Assume $f(\bar{x}) = 0$ for some $\bar{x} \neq 0$, then by (2.3) we have $-z(\bar{x}) = |g(\bar{x})|^2 \geq 0$. But this contradicts that $-z(x) < 0$ for $x \neq 0$.

We now prove $f(x) = 0$ if $x = 0$. Assume $f(0) \neq 0$, that is $f(0) = (f_1(0), ..., f_n(0))^T \neq 0$. Without loss of generality, we assume $f_1(0) < 0$. Due to the continuity of $f(x)$, for some sufficiently small $\varepsilon > 0$ we have $f_1(x) < 0$ for some vector x, where the first entry lies in $(-\varepsilon, \varepsilon)$ and all the rest are zeros. Then given $\bar{x} = (-\varepsilon/2, 0, ..., 0)^T$, we have $\langle \bar{x}, f(\bar{x}) \rangle > 0$. But this contradicts to $-z(\bar{x}) < 0$.

Suppose $x = 0$, by (2.3) it is easy to see that $|g(0)|^2 = -z(0) = 0$, i.e. $g(0) = 0$. ∎

The next lemma is a discrete version of the semimartingale convergence theorem. We refer the readers to Lemma 4 in [1] for the proof.

Lemma 2.3 *Let $\{A_i\}$ and $\{B_i\}$ be two nonnegative $\mathcal{F}_i$-measurable processes for $i = 0, 1, 2, ...$ with $A_0 = B_0 = 0$ a.s. and $\{M_i\}$ be $\mathcal{F}_i$-measurable local martingale for $i = 0, 1, 2, ...$ with $M_0 = 0$. If a nonnegative stochastic process $\{Z_i\}_{i=0,1,...}$ can be decomposed as $Z_i = Z_0 + A_i - B_i + M_i$, then*

$$\left\{ \lim_{i \to \infty} A_i < \infty \right\} \subseteq \left\{ \lim_{i \to \infty} B_i < \infty \right\} \cap \left\{ \lim_{i \to \infty} Z_i \quad \text{exists and is finite} \right\} \quad a.s.$$

3 The EM method with random variable stepsize

In this section, we present our main results about the variable stepsize EM method. To keep the proof simple and clear we specify the choice of the stepsize in the proof, but readers should notice that there are other choices. We emphasise here that there are two important properties of the variable stepsize that the sum of the steps is a stopping time and divergent. The feature of stopping time is essential to the proof of the local martingale term in Theorem 3.1, and the divergence guarantees the time is able to tend to infinity.

The first main result is that the variable stepsize method can reproduce the stability of the SDE shown in Theorem 2.1.

Theorem 3.1 *Let (2.2) and (2.3) hold. Assume $z(x) = 0$ if and only if $x = 0$, and*

$$\liminf_{|x|\to 0} \frac{z(x)}{|f(x)|^2} > 0. \tag{3.1}$$

Define the EM method with variable stepsize as

$$Y_{i+1} = Y_i + f(Y_i)\Delta t_i + g(Y_i)\Delta B_i, \quad Y_0 = x(0), \quad i \geq 0, \tag{3.2}$$

where $\Delta B_i = B(t_i) - B(t_{i-1})$ with $t_i = \sum_{k=0}^{i} \Delta t_k$ for $i = 0, 1, 2...$ and $t_{-1} = 0$, Δt_i is chosen to be 2^{-n_i} with $n_i = \lceil 1 - \log_2(z(Y_i)/|f(Y_i)|^2) \rceil$ for $|Y_i| \neq 0$ and 2^{-2} for $|Y_i| = 0$. Then t_i is an $\{\mathcal{F}_t\}$-stopping time for each $i = 0, 1, 2...$, and the sequence of time steps obeys $\sum_{i=0}^{\infty} \Delta t_i = \infty$ a.s. Moreover, for any initial value $Y_0 \in \mathbb{R}^n$

$$\lim_{i\to\infty} Y_i = 0 \quad a.s.$$

Proof. Taking square on both sides of (3.2), we have

$$\begin{aligned} |Y_{i+1}|^2 &= |Y_i|^2 + 2\langle Y_i, f(Y_i)\Delta t_i + g(Y_i)\Delta B_i\rangle + |f(Y_i)\Delta t_i + g(Y_i)\Delta B_i|^2 \\ &= |Y_i|^2 + \Delta t_i(2\langle Y_i, f(Y_i)\rangle + |g(Y_i)|^2 + |f(Y_i)|^2\Delta t_i) + \Delta m_i, \end{aligned} \tag{3.3}$$

where $\Delta m_i = 2\langle Y_i, g(Y_i)\Delta B_i\rangle + 2\langle f(Y_i)\Delta t_i, g(Y_i)\Delta B_i\rangle + |g(Y_i)|^2(|\Delta B_i|^2 - \Delta t_i)$.

The proof is divided into three parts. Firstly, we demonstrate the strategy of choosing the stepsize Δt_i in each time step and show that t_i is an $\{\mathcal{F}_t\}$-stopping time for every $i = 0, 1, ...$ Then we prove that $m_i = \sum_{k=0}^{i} \Delta m_k$ is a local martingale for $i = 0, 1, ...$ At last, we give the proof of the divergence of the sequence of the timesteps and conclude the almost sure stability.

Step 1

Since (2.3), in each step we can choose sufficiently small and rational stepsize Δt_i such that

$$-U(Y_i, \Delta t_i) := -z(Y_i) + |f(Y_i)|^2\Delta t_i \leq 0. \tag{3.4}$$

For example, when $Y_i \neq 0$ (by Lemma 2.2 we know $f(Y_i) \neq 0$) we could choose $\Delta t_i = 2^{-n_i}$ with $n_i = \lceil 1 - \log_2(z(Y_i)/|f(Y_i)|^2) \rceil$. Then it is obvious that $\Delta t_i \leq z(Y_i)/(2|f(Y_i)|^2)$, thus the inequality (3.4) holds. When $Y_i = 0$ (i.e. $z(Y_i) = 0$ and $f(Y_i) = 0$), any choice of Δt_i will satisfy (3.4) and we simply choose, for example $\Delta t_i = 2^{-2}$. From the iteration (3.2), we know that if at some time point the solution becomes zero, the solution afterwards will stay at zero. Hence in this case the stepsize is fixed and the almost sure stability follows naturally. In the following, we focus on the case when $\Delta t_i = 2^{-n_i}$ with $n_i = \lceil 1 - \log_2(z(Y_i)/|f(Y_i)|^2) \rceil$. We emphasise here that the requirement that each Δt_i is a rational number is key to the following proof that $t_i = \sum_{k=0}^{i} \Delta t_k = \sum_{k=0}^{i} 2^{-n_k}$ is an $\{\mathcal{F}_t\}$-stopping time for every $i = 0, 1, \ldots$.

Assume t_i is an $\{\mathcal{F}_t\}$-stopping time for some $i \geq 0$, i.e. $\{t_i \leq t\} \in \mathcal{F}_t$ for any $t \geq 0$. Note that Y_{i+1} is $\mathcal{F}_{t_i}$-measurable. Because the choice of Δt_{i+1} is dependent on Y_{i+1} we have that Δt_{i+1} is $\mathcal{F}_{t_i}$-measurable. Then we need to show $t_{i+1} = t_i + \Delta t_{i+1}$ is an $\{\mathcal{F}_t\}$-stopping time, that is to show $\{t_i + \Delta t_{i+1} \leq t\} \in \mathcal{F}_t$ for any $t \geq 0$. For any $s \in \mathbb{Z}$ and any $j \in \mathbb{N}$ with $j2^s \in [0, t]$, we have $\{t_i \leq j2^s\} \in \mathcal{F}_{j2^s} \subseteq \mathcal{F}_t$, and $\{\Delta t_{i+1} \leq t - j2^s\} \in \mathcal{F}_{t_i} \subset \mathcal{F}$. Thus we have $\{t_i \leq j2^s\} \cap \{\Delta t_{i+1} \leq t - j2^s\} \in \mathcal{F}_t$ (see for example [12]). As both $\mathbb{Z}$ and $\mathbb{N}$ are countable sets, we have that for any $t \geq 0$ [7]

$$\{t_i + \Delta t_{i+1} \leq t\} = \bigcup_{\{0 \leq j2^s \leq t, s \in \mathbb{Z}, j \in \mathbb{N}\}} (\{t_i \leq j2^s\} \cap \{\Delta t_{i+1} \leq t - j2^s\}) \in \mathcal{F}_t.$$

Thus we have proved that t_{i+1} is an $\{\mathcal{F}_t\}$-stopping time. Since Δt_0 is dependent on the given initial value Y_0, we have Δt_0 and Y_0 are $\mathcal{F}_{t_{-1}}$-measurable (recalling $t_{-1} = 0$). By induction we conclude that t_i is an $\{\mathcal{F}_t\}$-stopping time for each $i = 0, 1, \ldots$. Substituting (3.4) into (3.3), we obtain

$$|Y_{i+1}|^2 = |Y_i|^2 - U(Y_i, \Delta t_i)\Delta t_i + \Delta m_i.$$

Then taking sum on i we have

$$|Y_{i+1}|^2 = |Y_0|^2 - \sum_{k=0}^{i} U(Y_k, \Delta t_k)\Delta t_k + m_i, \tag{3.5}$$

where $m_i = \sum_{k=0}^{i} \Delta m_k$.

Step 2

Due to (3.2) and the definition of t_i, it is clear that Y_i is $\mathcal{F}_{t_{i-1}}$-measurable for $i = 0, 1, \ldots$. We define another filtration $\{\mathcal{G}_i\}_{i=-1,0,1,\ldots}$ by $\mathcal{G}_i = \mathcal{F}_{t_i}$ for $i = -1, 0, 1, \ldots$. So Y_i is $\mathcal{G}_{i-1}$-measurable and m_i is $\mathcal{G}_i$-measurable. We are going to prove that $\{m_i\}_{i \geq 0}$ is a $\{\mathcal{G}_i\}$-local martingale. Choosing

R s.t. $|x(0)| < R$, we define a stopping time

$$\rho_R = \inf\{i \geq 0, |Y_i| > R\}.$$

Clearly, $\rho_R \to \infty$ a.s. when $R \to \infty$. It is easy to see that ρ_R is a $\{\mathcal{G}_{i-1}\}$-stopping time i.e. $\{\rho_R \leq i\} \in \mathcal{G}_{i-1}$. This indicates $\{\rho_R - 1 \leq i\} \in \mathcal{G}_i$. Denoting $\tau_R = \rho_R - 1$, we have τ_R is a $\{\mathcal{G}_i\}$-stopping time. By the definition of ρ_R, we have that $|Y_{i\wedge(\rho_R-1)}| \leq R$ a.s. and $|Y_{i\wedge\tau_R}| \leq R$ a.s. for all $i \geq 0$.

We claim that $t_{i\wedge\tau_R}$ and $t_{(i-1)\wedge\tau_R}$ are $\{\mathcal{F}_t\}$-stopping times. For $t_{i\wedge\tau_R}$ we have for any $t \geq 0$

$$\{t_{i\wedge\tau_R} \leq t\} = \{\{t_i \leq t\} \cap \{\tau_R \geq i\}\} \cup \{\{t_{\tau_R} \leq t\} \cap \{\tau_R < i\}\}.$$

Since $\{\tau_R \geq i\} \in \mathcal{G}_{i-1} \subset \mathcal{G}_i = \mathcal{F}_{t_i}$, we have $\{t_i \leq t\} \cap \{\tau_R \geq i\} \in \mathcal{F}_t$. And

$$\{t_{\tau_R} \leq t\} \cap \{\tau_R < i\} = \bigcup_{j=0}^{i-1}(\{\{t_j \leq t\} \cap \{\tau_R = j\}),$$

because $\{\tau_R = j\} \in \mathcal{F}_{t_i}$ for $j = 0, 1, ...i-1$ we have $\{t_{\tau_R} \leq t\} \cap \{\tau_R < i\} \in \mathcal{F}_t$. Hence $\{t_{i\wedge\tau_R} \leq t\} \in \mathcal{F}_t$. Similarly for $t_{(i-1)\wedge\tau_R}$, we have

$$\{t_{(i-1)\wedge\tau_R} \leq t\} = \{\{t_{i-1} \leq t\} \cap \{\tau_R \geq i-1\}\} \cup \{\{t_{\tau_R} \leq t\} \cap \{\tau_R < i-1\}\}.$$

Since $\{\tau_R \geq i-1\} \in \mathcal{G}_{i-2} \subset \mathcal{F}_{t_i}$, we have $\{t_{i-1} \leq t\} \cap \{\tau_R \geq i-1\} \in \mathcal{F}_t$. And

$$\{t_{\tau_R} \leq t\} \cap \{\tau_R < i-1\} = \bigcup_{j=0}^{i-2}(\{\{t_j \leq t\} \cap \{\tau_R = j\}),$$

we have $\{\{t_{\tau_R} \leq t\} \cap \{\tau_R < i-1\}\} \in \mathcal{F}_t$ due to $\{\tau_R = j\} \in \mathcal{G}_j \subset \mathcal{F}_{t_i}$ for $j = 0, 1, ...i-2$. Thus $\{t_{(i-1)\wedge\tau_R} \leq t\} \in \mathcal{F}_t$.

Due to the iteration (3.2) and the fact that $|Y_{k\wedge\tau_R}| = |Y_{\tau_R}|$ for any $k \geq \tau_R$, we define the Brownian motion increment with the stopping time by $\Delta B_{i\wedge\tau_R} = B(t_{i\wedge\tau_R}) - B(t_{(i-1)\wedge\tau_R})$ and the time step with the stopping time by $\Delta t_{i\wedge\tau_R} = t_{i\wedge\tau_R} - t_{(i-1)\wedge\tau_R}$. Since $\tau_R \to \infty$ a.s. when $R \to \infty$, those two definitions can reproduce the original ones we used in the statement of the theorem. Thus they are valid. In addition, we have

$$m_{i\wedge\tau_R} = \sum_{k=0}^{i\wedge\tau_R} \Delta m_k = \sum_{k=0}^{i} \Delta m_{k\wedge\tau_R} \quad \text{and} \quad m_{i\wedge\tau_R} = m_{(i-1)\wedge\tau_R} + \Delta m_{i\wedge\tau_R}.$$

From condition (2.2) and Lemma 2.2, for $|x| \leq R$ there exists a constant $c(R)$ dependent on R such that $|f(x)| \vee |g(x)| \leq c(R)$. By the elementary inequality, we have

$$
\begin{aligned}
|m_{i\wedge\tau_R}| &= \left|\sum_{k=0}^{i\wedge\tau_R} \Delta m_k\right| \\
&\leq \sum_{k=0}^{i} |\Delta m_{k\wedge\tau_R}| \\
&\leq \sum_{k=0}^{i} (2|Y_{k\wedge\tau_R}||g(Y_{k\wedge\tau_R})||\Delta B_{k\wedge\tau_R}| + 2|f(Y_{k\wedge\tau_R})||g(Y_{k\wedge\tau_R})|\Delta t_{k\wedge\tau_R}|\Delta B_{k\wedge\tau_R}| \\
&\quad +|g(Y_{k\wedge\tau_R})|^2||\Delta B_{k\wedge\tau_R}|^2 - \Delta t_{k\wedge\tau_R}|) \\
&\leq \sum_{k=0}^{i} (c_1(R)|\Delta B_{k\wedge\tau_R}| + c_2(R)|\Delta B_{k\wedge\tau_R}|^2), \qquad (3.6)
\end{aligned}
$$

where $c_1(R)$ and $c_2(R)$ are constants dependent on R only. Hence we have

$$
\mathbb{E}|m_{i\wedge\tau_R}| \leq \sum_{k=0}^{i} (c_1(R)\mathbb{E}|\Delta B_{k\wedge\tau_R}| + c_2(R)\mathbb{E}|\Delta B_{k\wedge\tau_R}|^2) < \infty.
$$

Also we have

$$
\mathbb{E}(m_{i\wedge\tau_R}|\mathcal{G}_{i-1}) = \mathbb{E}(m_{(i-1)\wedge\tau_R} + \Delta m_{i\wedge\tau_R}|\mathcal{G}_{i-1}) = m_{(i-1)\wedge\tau_R} + \mathbb{E}(\Delta m_{i\wedge\tau_R}|\mathcal{G}_{i-1}). \qquad (3.7)
$$

Because $\{\tau_R > i-1\} \in \mathcal{G}_{i-1}$ and ΔB_i is independent of $\mathcal{G}_{i-1}$, we have

$$
\begin{aligned}
& \mathbb{E}(\Delta B_{i\wedge\tau_R}|\mathcal{G}_{i-1}) \\
= & \mathbb{E}[(B(t_i) - B(t_{i-1}))\mathbf{1}_{\{\tau_R > i-1\}}|\mathcal{G}_{i-1}] + \mathbb{E}[(B(t_{\tau_R}) - B(t_{\tau_R}))\mathbf{1}_{\{\tau_R \leq i-1\}}|\mathcal{G}_{i-1}] \\
= & \mathbf{1}_{\{\tau_R > i-1\}}\mathbb{E}[B(t_i) - B(t_{i-1})] \\
= & 0,
\end{aligned}
$$

$$
\begin{aligned}
& \mathbb{E}(|\Delta B_{i\wedge\tau_R}|^2|\mathcal{G}_{i-1}) \\
= & \mathbb{E}[|B(t_i) - B(t_{i-1})|^2\mathbf{1}_{\{\tau_R > i-1\}}|\mathcal{G}_{i-1}] + \mathbb{E}[|B(t_{\tau_R}) - B(t_{\tau_R})|^2\mathbf{1}_{\{\tau_R \leq i-1\}}|\mathcal{G}_{i-1}] \\
= & \mathbf{1}_{\{\tau_R > i-1\}}\mathbb{E}[|B(t_i) - B(t_{i-1})|^2] \\
= & \mathbf{1}_{\{\tau_R > i-1\}}(t_i - t_{i-1}),
\end{aligned}
$$

and

$$
\begin{aligned}
& \mathbb{E}(\Delta t_{i\wedge\tau_R}|\mathcal{G}_{i-1}) \\
= & \Delta t_{i\wedge\tau_R} \\
= & \mathbf{1}_{\{\tau_R > i-1\}}(t_i - t_{i-1}) + \mathbf{1}_{\{\tau_R \leq i-1\}}(t_{\tau_R} - t_{\tau_R}) \\
= & \mathbf{1}_{\{\tau_R > i-1\}}(t_i - t_{i-1}).
\end{aligned}
$$

Hence

$$
\begin{aligned}
& \mathbb{E}(\Delta m_{i\wedge\tau_R}|\mathcal{G}_{i-1}) \\
= \; & \mathbb{E}(2\langle Y_{i\wedge\tau_R}, g(Y_{i\wedge\tau_R})\Delta B_{i\wedge\tau_R}\rangle + 2\langle f(Y_{i\wedge\tau_R})\Delta t_{i\wedge\tau_R}, g(Y_{i\wedge\tau_R})\Delta B_{i\wedge\tau_R}\rangle \\
& + |g(Y_{i\wedge\tau_R})|^2(|\Delta B_{i\wedge\tau_R}|^2 - \Delta t_{i\wedge\tau_R})|\mathcal{G}_{i-1}) \\
= \; & 2\langle Y_{i\wedge\tau_R}, g(Y_{i\wedge\tau_R})\rangle\mathbb{E}(\Delta B_{i\wedge\tau_R}|\mathcal{G}_{i-1}) + 2\langle f(Y_{i\wedge\tau_R}), g(Y_{i\wedge\tau_R})\rangle\Delta t_{i\wedge\tau_R}\mathbb{E}(\Delta B_{i\wedge\tau_R}|\mathcal{G}_{i-1}) \\
& + |g(Y_{i\wedge\tau_R})|^2(\mathbb{E}(|\Delta B_{i\wedge\tau_R}|^2|\mathcal{G}_{i-1}) - \mathbb{E}(\Delta t_{i\wedge\tau_R}|\mathcal{G}_{i-1})) \\
= \; & 0. \qquad (3.8)
\end{aligned}
$$

Combining (3.7) and (3.8), we achieve the required

$$\mathbb{E}(m_{i\wedge\tau_R}|\mathcal{G}_{i-1}) = m_{(i-1)\wedge\tau_R}.$$

This means that $\{m_{i\wedge\tau_R}\}_{i\geq 0}$ is a $\{\mathcal{G}_i\}$-martingale. Recalling that $\tau_R \to \infty$ a.s. when $R \to \infty$, we see that $\{m_i\}_{i\geq 0}$ is a $\{\mathcal{G}_i\}$-local martingale.

Step 3

Therefore from (3.5) and Lemma 2.3, we have

$$\lim_{i\to\infty} |Y_i|^2 < \infty \quad \text{a.s.} \qquad (3.9)$$

and

$$\sum_{k=0}^{\infty} U(Y_k, \Delta t_i)\Delta t_k < \infty \quad \text{a.s.} \qquad (3.10)$$

From (3.10), we have $\lim_{i\to\infty} U(Y_i, \Delta t_i)\Delta t_i = 0$ a.s. We next show the time step Δt_i will never tend to zero as i goes to infinity, that is $\liminf_{i\to\infty} \Delta t_i > 0$ a.s.

According to (3.9) for almost all $\omega \in \Omega$, there exists $C(\omega) \in \mathbb{R}_+$ such that $\lim_{i\to\infty} |Y_i(\omega)| = C(\omega)$. Fix any such ω, write $C(\omega) = C$ and $Y_i(\omega) = Y_i$. Consider two cases:

(i) For the case when $C \neq 0$, there exists a sufficiently large integer i_1^* such that for all $i > i_1^*$, $0.5C < |Y_i| < 1.5C$. This indicates either $0.5C < Y_i < 1.5C$ or $-1.5C < Y_i < -0.5C$. Because that $z(x) = 0$ and $f(x) = 0$ if and only if $x = 0$, in both of the two intervals we have $z(Y_i) \neq 0$ and $f(Y_i) \neq 0$. Furthermore, due to the continuity of $z(x)$ and $f(x)$, we have

$$\min_{0.5C\leq|x|\leq 1.5C} \frac{z(x)}{|f(x)|^2} = \eta > 0.$$

So for any $i > i_1^*$, we have

$$\frac{z(Y_i)}{|f(Y_i)|^2} \geq \eta > 0$$

then

$$1 - \log_2(z(Y_i)/|f(Y_i)|^2) \leq 1 - \log_2(\eta).$$

Recalling the choice of the stepsize, we see

$$n_i = \lceil 1 - \log_2(z(Y_i)/|f(Y_i)|^2) \rceil \leq \lceil 1 - \log_2(\eta) \rceil$$

then

$$\Delta t_i = 2^{-n_i} \geq 2^{-\lceil 1 - \log_2(\eta) \rceil} > 0.$$

(ii) For the case when $C = 0$, suppose the limit of (3.1) be $D > 0$. There exists a constant $\delta = \delta(D) > 0$ such that $|z(x)/|f(x)|^2 - D| < 0.5D$ for all $|x| \in (0, \delta)$. Also, there exists an integer i_2^* such that for all $i > i_2^*$, $|Y_i| \in (0, \delta)$, which indicates $|z(Y_i)/|f(Y_i)|^2 - D| < 0.5D$. So for any $i > i_2^*$, we have

$$1 - \log_2(1.5D) < 1 - \log_2(z(Y_i)/|f(Y_i)|^2) < 1 - \log_2(0.5D).$$

Recalling the choice of the stepsize, we see

$$\Delta t_i = 2^{-n_i} > 2^{-\lceil 1 - \log_2(0.5D) \rceil} > 0.$$

Thus Δt_i will never tend to 0 as i tends to infinity. Hence we have $\sum_{i=0}^{\infty} \Delta t_i = \infty$ a.s.

Now we have $\lim_{i\to\infty} U(Y_i, \Delta t_i) = 0$ a.s. Due to (3.4) and the choice of Δt_i that $\Delta t_i \leq z(Y_i)/(2|f(Y_i)|^2)$, we have

$$U(Y_i, \Delta t_i) = z(Y_i) - |f(Y_i)|^2 \Delta t_i \geq 0.5 z(Y_i) \geq 0.$$

Therefore $\lim_{i\to\infty} z(Y_i) = 0$ a.s. Given the condition "$z(x) = 0 \Leftrightarrow x = 0$", we obtain that $\lim_{i\to\infty} Y_i = 0$ a.s. Hence the proof is complete. ∎

We have three comments on the proof.

- The conditions in Theorem 3.1 for the EM method with variable stepsize is weaker than the condition for the EM method with fixed stepsize (i.e. when $\theta = 0$) stated in Theorem 5.3 of [14]. For example, a scalar SDE $dx(t) = (-x^3(t) - x(t))dt + x^2(t)dB(t)$ satisfies the conditions in Theorem 3.1, but not in Theorem 5.3 of [14].

- When conducting computer simulation, the stepsize is naturally rational number as computers can only deal with finite number of decimals. Thus we may simply set each stepsize to be $\alpha z(Y_i)/(|f(Y_i)|^2)$ for any rational number $\alpha \in (0, 1)$. We generalise the choice of stepsize in Theorem 3.1 in the next theorem.

- The condition (2.3) is the restriction on the relation between the drift and the diffusion coefficients, which is also required for the almost sure stability of the underlying SDEs. The condition (3.1) is solely required for the numerical methods. From the proof, we can see that (3.1) guarantees the sum of the sequence of stepsizes tends to infinity. This is essential as we are discussing asymptotic behaviour of the numerical solution.

Theorem 3.2 *Let (2.2) and (2.3) hold. Assume $z(x) = 0$ if and only if $x = 0$, and (3.1). For the EM method with variable stepsize (3.2), Δt_i is chosen to be rational number satisfying $\Delta t_i = \alpha z(Y_i)/(|f(Y_i)|^2)$ with $\alpha \in (0,1)$ for $|Y_i| \neq 0$, and any nonzero rational number for $|Y_i| = 0$. Then t_i is an $\{\mathcal{F}_t\}$-stopping time for each $i = 0, 1, 2...$, and the sequence of time steps obeys $\sum_{i=0}^{\infty} \Delta t_i = \infty$ a.s. Moreover, for any initial value $Y_0 \in \mathbb{R}^n$*

$$\lim_{i\to\infty} Y_i = 0 \quad a.s.$$

Most part of the proof of Theorem 3.2 is similar to the proof of Theorem 3.1, and the only different part is the proof of the stopping time as follows.

Assume t_i is an $\{\mathcal{F}_t\}$-stopping time for some $i \geq 0$, i.e. $\{t_i \leq t\} \in \mathcal{F}_t$ for any $t \geq 0$. Note that Y_{i+1} is $\mathcal{F}_{t_i}$-measurable, because the choice of Δt_{i+1} is dependent on Y_{i+1} we have that Δt_{i+1} is $\mathcal{F}_{t_i}$-measurable. Then we need to show $t_{i+1} = t_i + \Delta t_{i+1}$ is an $\{\mathcal{F}_t\}$-stopping time, that is to show $\{t_i + \Delta t_{i+1} \leq t\} \in \mathcal{F}_t$ for any $t \geq 0$. For any rational number $s \in [0, t]$, we have $\{t_i \leq s\} \in \mathcal{F}_s \subseteq \mathcal{F}_t$, and $\{\Delta t_{i+1} \leq t-s\} \in \mathcal{F}_{t_i} \subseteq \mathcal{F}$. Thus we have $\{t_i \leq s\} \cap \{\Delta t_{i+1} \leq t-s\} \in \mathcal{F}_t$ (see for example [12]). As the set of all rational number $s \in [0, t]$ is a countable set, we have that for any $t \geq 0$ [7]

$$\{t_i + \Delta t_{i+1} \leq t\} = \bigcup_{\{0 \leq s \leq t, s \in \mathbb{Q}\}} (\{t_i \leq s\} \cap \{\Delta t_{i+1} \leq t - s\}) \in \mathcal{F}_t.$$

Thus we have proved that t_{i+1} is an $\{\mathcal{F}_t\}$-stopping time. Since Δt_0 is dependent on the given initial value Y_0, we have Δt_0 and Y_0 are $\mathcal{F}_{t_{-1}}$-measurable (recalling $t_{-1} = 0$). By induction we conclude that t_i is an $\{\mathcal{F}_t\}$-stopping time for each $i = 0, 1, ...$

4 Examples

We first consider a scalar SDE

$$dx(t) = (-x^3(t) - x(t))dt + x^2(t)dB(t) \tag{4.1}$$

with a given initial value $x(0) = 1$. It is easy to verify that for any $x \in \mathbb{R}$ with $x \neq 0$

$$-z(x) := 2\langle x, f(x)\rangle + g^2(x) = -2x^2 - x^4 < 0.$$

It is clear that "$z(x) = 0 \Leftrightarrow x = 0$", by Theorem 2.1 we have the solution of the underlying SDE is asymptotically almost surely stable. Moreover,

$$\liminf_{|x|\to 0} \frac{z(x)}{|f(x)|^2} = \liminf_{|x|\to 0} \frac{2x^2 + x^4}{x^2 + 2x^4 + x^6} = 2 > 0.$$

Choose the stepsize, for example $\Delta t_i = 0.98 z(Y_i)/|f(Y_i)|^2$ in each step, from Theorem 3.2 we obtain the variable stepsize EM solution is asymptotically almost surely stable as well. Set $Y_0 = 1$, we simulated 1000 time steps of one path of the variable stepsize EM solution. The left plot on Figure 1 is the solution path, from which we can see that the oscillation decays and the solution tends zero as time increases. This is in line with the theoretical result. The plot on the right of Figure 1 is the size of each time step. It is clear that with the solution approaching the origin the stepsize tends to 1.96 and this is due to the limit 2 and the choice of factor 0.98. In addition, the plot also shows that the stepsize does not need to tend to zero, thus we have $\sum_{i=0}^{\infty} \Delta t_i = \infty$ a.s.

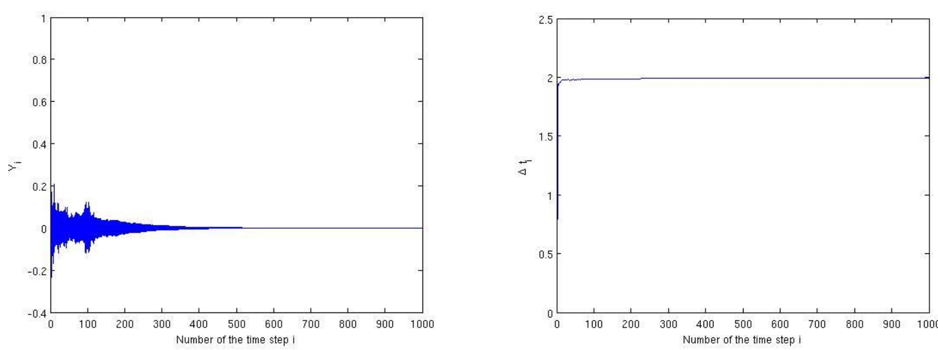

Figure 1: Left: One simulation path, Right: The stepsize of each time step

Now we consider a two-dimensional case

$$dx(t) = \text{diag}(x_1(t), x_2(t)) \left((b + A\text{diag}(x_1(t), x_2(t))x(t))\, dt + \sigma dB(t) \right), \tag{4.2}$$

where $\text{diag}(x_1(t), x_2(t))$ denotes diagonal matrix with nonzero entries $x_1(t)$ and $x_2(t)$ on the diagonal, $x(t) = (x_1(t), x_2(t))^T$, $b = (b_1, b_2)^T$, $A = (a_{ij})_{i,j\in\{1,2\}}$, $\sigma = (\sigma_{ij})_{i,j\in\{1,2\}}$ and $B(t) = (B_1(t), B_2(t))^T$.

We set $b = (-1, -2)^T, a_{11} = a_{22} = -1, a_{12} = -2, a_{21} = 1, \sigma_{11} = \sigma_{12} = 0.5, \sigma_{21} = 1, \sigma_{22} =$

-1. It is easy to verify that for any $x \in \mathbb{R}^2$ and $x \neq 0$

$$
\begin{aligned}
& 2\langle x, f(x)\rangle + g^2(x) \\
= \;& (2b_1 + \sigma_{11}^2 + \sigma_{12}^2)x_1^2 + (2b_2 + \sigma_{21}^2 + \sigma_{22}^2)x_2^2 + (a_{12} + a_{21})x_1^2x_2^2 + a_{11}x_1^4 + a_{22}x_2^4 < 0.
\end{aligned}
$$

From Theorem 2.1, we know the SDE solution is almost surely stable. In addition, by the elementary inequality $ab \leq a^2 + b^2$ we have

$$
\begin{aligned}
& \liminf_{|x|\to 0} \frac{z(x)}{|f(x)|^2} \\
= \;& \liminf_{|x|\to 0} \frac{1.5x_1^2 + 2x_2^2 + x_1^2x_2^2 + x_1^4 + x_2^4}{x_1^2 + 2x_1^4 + x_1^6 + 2x_1^2x_2^4 + 5x_1^4x_2^2 + 4x_2^2 + 4x_2^4 + x_2^6} \\
\geq \;& \liminf_{|x|\to 0} \frac{|x|^2}{4|x|^2 + 6.5|x|^4 + |x|^6 + 2.5|x|^8} \\
= \;& \frac{1}{4} > 0.
\end{aligned}
$$

By choosing the stepsize, for example $\Delta t_i = 0.1z(Y_i)/|f(Y_i)|^2$ in each step, we have from Theorem 3.1 that the variable stepsize EM solution is almost surely stable as well.

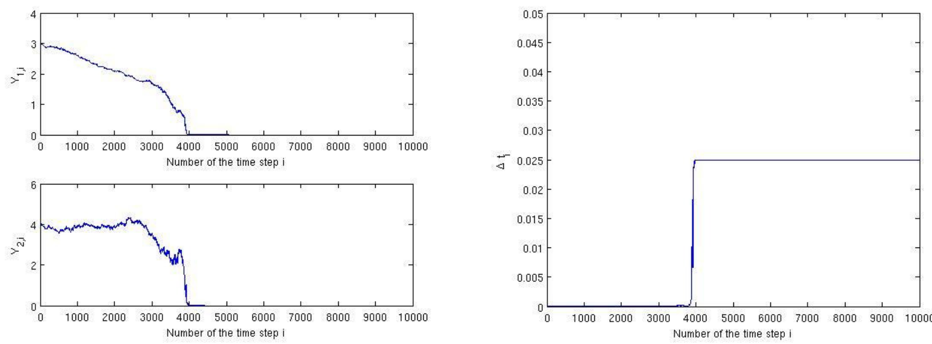

Figure 2: Left: One simulation path of $Y_{1,\cdot}$ and $Y_{2,\cdot}$, Right: The stepsize of each time step.

We simulated 10000 time steps and plotted the two solution paths on the left of Figure 2. It can be seen that as time increases both the solutions tend to zero. And from the plot on the right of Figure 2 the size of the time step approaches to 0.025 as the solutions go to zeros, which shows the stepsize will not tend to zero. Hence both the simulations of the one-dimensional and the multi-dimensional cases are in line with the theoretical result.

5 Other sufficient conditions

In this section, we propose some other sufficient conditions which can cover some SDEs that are not included in Section 3.

Another condition that can be regarded as an extension to (2.3) is to assume there exists a symmetric positive-definite $n \times n$ matrix Q such that for $\forall x \in \mathbb{R}^n$

$$-\bar{z}(x) := 2x^T Q f(x) + trace(g^T(x) Q g(x)) \leq 0. \tag{5.1}$$

It is clear to see that when Q is an identity matrix, (2.3) is recovered. Thanks to the stochastic version of the LaSalle theorem in [22], we have that the underlying solution of (2.1) is almost surely asymptotically stable if (2.2) and (5.1) hold, and $\bar{z}(x) = 0$ if and only if $x = 0$. In addition, it is obvious that given the condition that $\bar{z}(x) = 0$ if and only if $x = 0$ the results in Lemma 2.2 still hold for $f(x)$ and $g(x)$. Denote the smallest and largest eigenvalue of Q by $\lambda_{\min}(Q)$ and $\lambda_{\max}(Q)$ respectively. Now we are ready to present the following theorem.

Theorem 5.1 *Let (2.2) and (5.1) hold. Assume $\bar{z}(x) = 0$ if and only if $x = 0$, and*

$$\liminf_{|x| \to 0} \frac{\bar{z}(x)}{|f(x)|^2} > 0.$$

For the EM method with variable stepsize (3.2), Δt_i is chosen to be rational number satisfying $\Delta t_i = \alpha \bar{z}(Y_i)/(\lambda_{\max}(Q)|f(Y_i)|^2)$ with $\alpha \in (0,1)$ for $|Y_i| \neq 0$, and any nonzero rational number for $|Y_i| = 0$. Then t_i is an $\{\mathcal{F}_t\}$-stopping time for each $i = 0, 1, 2...$, and the sequence of time steps obeys $\sum_{i=0}^{\infty} \Delta t_i = \infty$ a.s. Moreover, for any initial value $Y_0 \in \mathbb{R}^n$

$$\lim_{i \to \infty} Y_i = 0 \quad a.s.$$

Proof. Since Q is a symmetric positive-definite $n \times n$ matrix, it is clear that for any $i \geq 0$

$$\lambda_{\min}(Q)|Y_i|^2 \leq Y_i^T Q Y_i \leq \lambda_{\max}(Q)|Y_i|^2$$

and

$$\lambda_{\min}(Q)|f(Y_i)|^2 \leq f^T(Y_i) Q f(Y_i) \leq \lambda_{\max}(Q)|f(Y_i)|^2.$$

From (3.2) we have

$$Y_{i+1}^T Q Y_{i+1} = Y_i^T Q Y_i + \Delta t_i [2Y_i^T Q f(Y_i) + trace(g^T(Y_i) Q g(Y_i)) + f^T(Y_i) Q f(Y_i) \Delta t_i] + \Delta m_i,$$

where

$$\Delta m_i = 2Y_i^T Q g(Y_i) \Delta B_i + 2f^T(Y_i) Q g(Y_i) \Delta B_i + (g(Y_i)\Delta B_i)^T Q (g(Y_i)\Delta B_i) - trace(g^T(Y_i) Q g(Y_i)) \Delta t_i.$$

Then the proof can be completed by adapting the same procedure used in Theorem 3.1. ∎

We see condition (5.1) as a generalisation of (2.3) as we can recover (2.3) by choosing Q to be identity matrix in (5.1).

To keep the notations simple in the next theorem, we investigate the SDEs with the scalar Brownian motion

$$dx(t) = f(x(t))dt + g(x(t))dB(t), \quad x(0) \in \mathbb{R}^n,$$

where f: $\mathbb{R}^n \to \mathbb{R}^n$, g: $\mathbb{R}^n \to \mathbb{R}^n$ and $B(t)$ is a scalar Brownian motion. We still assume condition (2.2), but replace condition (2.3) by the following condition: there exists a constant $p \in (0, 2)$ such that

$$-v := \sup_{x\in\mathbb{R}^n, x\neq 0} \left(\frac{2\langle x, f(x)\rangle + |g(x)|^2}{|x|^2} + (p-2)\frac{\langle x, g(x)\rangle^2}{|x|^4} \right) < 0. \tag{5.2}$$

Also we assume $f(0) = 0$ and $g(0) = 0$.

Under (2.2) and (5.2), the true solution of SDE (2.1) is almost surely asymptotically stable [22]. Now we study the numerical solution.

Theorem 5.2 *Let (2.2) and (5.2) hold. Assume*

$$\limsup_{|x|\to 0} \frac{|f(x)|}{|x|} < \infty, \tag{5.3}$$

and

$$\limsup_{|x|\to 0} \frac{|g(x)|}{|x|} < \infty. \tag{5.4}$$

Define the EM method with variable stepsize as

$$Y_{i+1} = Y_i + f(Y_i)\Delta t_i + g(Y_i)\Delta B_i, \quad Y_0 = x(0), \quad i \geq 0, \tag{5.5}$$

where $\Delta B_i = B(t_i) - B(t_{i-1})$ *with* $t_i = \sum_{k=0}^{i} \Delta t_k$ *for* $i = 0, 1, 2...$ *and* $t_{-1} = 0$*. For* $Y_i \neq 0$, Δt_i *is chosen to be rational number satisfying* $\Delta t_i \leq (p/12)\min_{\{j=1,2,3,4,5\}}\{(v/A_j(Y_i))^{(1/j)}\}$*, where* $\{A_j\}_{j=1,2,3,4,5}$ *are defined in the proof. For* $Y_i = 0$, Δt_i *is chosen to be any nonzero rational number. Then* t_i *is an* $\{\mathcal{F}_t\}$*-stopping time for each* $i = 0, 1, 2...$*, and the sequence of time steps obeys* $\sum_{i=0}^{\infty} \Delta t_i = \infty$ *a.s. Moreover, for any initial value* $Y_0 \in \mathbb{R}^n$

$$\lim_{i\to\infty} Y_i = 0 \quad a.s.$$

The proof of this theorem is tedious but nontrivial. Therefore, we put it in Appendix.

Because of the extra negative term in the condition (5.2), $2\langle x, f(x)\rangle + |g(x)|^2$ is not necessarily less than 0 for all nonzero x. Therefore Theorem 5.2 does cover some SDEs that can not be covered by Theorem 3.1. But it should be noted that Theorem 3.1 is not fully included in Theorem 5.2. For example a scalar SDE with $f(x) = -0.5x^3 - x^5$ and $g(x) = x^2$. We check the conditions (2.3) and (3.1) that for any $x \in \mathbb{R}^n$ with $x \neq 0$

$$2\langle x, f(x)\rangle + g^2(x) = -2x^6 < 0 \quad \text{and} \quad \liminf_{|x|\to 0} \frac{z(x)}{|f(x)|^2} = \frac{2}{0.25} > 0,$$

i.e. all the conditions in Theorem 3.1 hold. To check the condition (5.2) in Theorem 5.2, we have

$$\frac{2\langle x, f(x)\rangle + |g(x)|^2}{|x|^2} + (p-2)\frac{\langle x, g(x)\rangle^2}{|x|^4} = -x^4 + (p-2)x^2.$$

But for any $p \in (0, 2)$, we can not find a $v > 0$ to satisfy (5.2).

6 Conclusions

In this paper, we investigate the Euler–Maruyama method with random variable stepsize and successfully reproduce the almost sure stability of the true solution using this method with the semimartingale convergence theorem. Conditions we impose on the drift and diffusion coefficients for the random variable stepsize method are much weaker than those for the fixed or nonrandom variable stepsize methods. Our key contribution also goes to the proof that the time variable is a stopping time, and only when this is true the rest of our proof is proper.

Considering that the random variable stepsize method works well for the stability, it is interesting to investigate other asymptotic properties of this method. Other numerical methods with random variable stepsize, such as the stochastic θ-method, are also worth to investigate.

The order-of-convergence is also essential for numerical methods. We have been working on the order of convergence of this newly developed Euler-Maruyama method with random variable stepsize, but due to the page limit here we will report the results in a follow-up paper.

Appendix

Proof of Theorem 5.2

From the first line of (3.3), we have that for the p given in (5.2) and $Y_i \neq 0$

$$|Y_{i+1}|^p = |Y_i|^p \left(1 + \frac{2\langle Y_i, f(Y_i)\Delta t_i + g(Y_i)\Delta B_i\rangle + |f(Y_i)\Delta t_i + g(Y_i)\Delta B_i|^2}{|Y_i|^2}\right)^{p/2}.$$

When $Y_i = 0$ (i.e. $f(Y_i) = 0$ and $g(Y_i) = 0$) for some $i > 0$, due to the iteration (5.5) the solution will stay at zero afterwards. In this case Δt_i could be set to be any nonzero rational number.

In the following we focus on the case that $Y_i \neq 0$ for all $i \geq 0$. Let

$$\zeta = \frac{2\langle Y_i, f(Y_i)\Delta t_i + g(Y_i)\Delta B_i\rangle + |f(Y_i)\Delta t_i + g(Y_i)\Delta B_i|^2}{|Y_i|^2},$$

and by the fundamental inequality that for any $\zeta \geq -1$

$$(1+\zeta)^{p/2} \leq 1 + \frac{p}{2}\zeta + \frac{p(p-2)}{8}\zeta^2 + \frac{p(p-2)(p-4)}{2^3 \times 3!}\zeta^3,$$

we have

$$|Y_{i+1}|^p \leq |Y_i|^p \left(1 + \frac{p}{2}\zeta + \frac{p(p-2)}{8}\zeta^2 + \frac{p(p-2)(p-4)}{2^3 \times 3!}\zeta^3\right). \tag{6.1}$$

We compute

$$\begin{aligned}\zeta \;=\;& \frac{1}{|Y_i|^2}(\Delta t_i(2\langle Y_i, f(Y_i)\rangle + |g(Y_i)|^2) + \Delta t_i^2 |f(Y_i)|^2 \\ & +2\langle Y_i, g(Y_i)\rangle \Delta B_i + 2f^T(Y_i)g(Y_i)\Delta t_i \Delta B_i + |g(Y_i)|^2(\Delta B_i^2 - \Delta t_i)),\end{aligned}$$

$$\begin{aligned}&\zeta^2 = \\ &\frac{1}{|Y_i|^4}(\Delta t_i(4\langle Y_i, g(Y_i)\rangle^2) \\ &+\Delta t_i^2(4\langle Y_i, f(Y_i)\rangle^2 + |g(Y_i)|^4 + 4\langle Y_i, f(Y_i)\rangle |g(Y_i)|^2 + 8\langle Y_i, g(Y_i)\rangle f^T(Y_i)g(Y_i)) \\ &+\Delta t_i^3(6|f(Y_i)|^2|g(Y_i)|^2 + 4\langle Y_i, f(Y_i)\rangle |f(Y_i)|^2) + \Delta t_i^4 |f(Y_i)|^4 \\ &+4\langle Y_i, g(Y_i)\rangle^2(\Delta B_i^2 - \Delta t_i) + |g(Y_i)|^4(\Delta B_i^4 - \Delta t_i^2) + 4\langle Y_i, f(Y_i)\rangle |g(Y_i)|^2 \Delta t_i(\Delta B_i^2 - \Delta t_i) \\ &+8\langle Y_i, g(Y_i)\rangle f^T(Y_i)g(Y_i)\Delta t_i(\Delta B_i^2 - \Delta t_i) + 6|f(Y_i)|^2|g(Y_i)|^2\Delta t_i^2(\Delta B_i^2 - \Delta t_i) \\ &+8\langle Y_i, f(Y_i)\rangle\langle Y_i, g(Y_i)\rangle \Delta t_i \Delta B_i + 4|f(Y_i)|^2 f^T(Y_i)g(Y_i)\Delta t_i^3 \Delta B_i + 4f^T(Y_i)g(Y_i)|g(Y_i)|^2 \Delta t_i \Delta B_i^3 \\ &+8\langle Y_i, f(Y_i)\rangle f^T(Y_i)g(Y_i)\Delta t_i^2 \Delta B_i + 4\langle Y_i, g(Y_i)\rangle |f(Y_i)|^2 \Delta t_i^2 \Delta B_i + 4\langle Y_i, g(Y_i)\rangle |g(Y_i)|^2 \Delta B_i^3),\end{aligned}$$

and

$$
\begin{aligned}
&\zeta^3 = \\
&\frac{1}{|Y_i|^6}(\Delta t_i^2(24\langle Y_i, f(Y_i)\rangle\langle Y_i, g(Y_i)\rangle^2 + 12\langle Y_i, g(Y_i)\rangle^2|g(Y_i)|^2) \\
&+\Delta t_i^3(8\langle Y_i, f(Y_i)\rangle^3 + 12\langle Y_i, f(Y_i)\rangle^2|g(Y_i)|^2 + 48\langle Y_i, f(Y_i)\rangle\langle Y_i, g(Y_i)\rangle f^T(Y_i)g(Y_i) \\
&+12\langle Y_i, g(Y_i)\rangle^2|f(Y_i)|^2 + 6\langle Y_i, f(Y_i)\rangle|g(Y_i)|^4 + |g(Y_i)|^6 + 24\langle Y_i, g(Y_i)\rangle|f(Y_i)||g(Y_i)|^3) \\
&+\Delta t_i^4(12\langle Y_i, f(Y_i)\rangle^2|f(Y_i)|^2 + 36\langle Y_i, f(Y_i)\rangle|f(Y_i)|^2|g(Y_i)|^2 + 15|f(Y_i)|^2|g(Y_i)|^4 \\
&+24\langle Y_i, g(Y_i)\rangle|f(Y_i)|^3|g(Y_i)|) \\
&+\Delta t_i^5(6\langle Y_i, f(Y_i)\rangle|f(Y_i)|^4 + 15|f(Y_i)|^4|g(Y_i)|^2) \\
&+\Delta t_i^6(|f(Y_i)|^6) \\
&+24\langle Y_i, f(Y_i)\rangle^2\langle Y_i, g(Y_i)\rangle\Delta t_i^2\Delta B_i + 24\langle Y_i, f(Y_i)\rangle\langle Y_i, g(Y_i)\rangle^2\Delta t_i(\Delta B_i^2 - \Delta t_i) \\
&+8\langle Y_i, g(Y_i)\rangle^3\Delta B_i^3 + 24\langle Y_i, f(Y_i)\rangle^2 f^T(Y_i)g(Y_i)\Delta t_i^3\Delta B_i \\
&+12\langle Y_i, f(Y_i)\rangle^2|g(Y_i)|^2\Delta t_i^2(\Delta B_i^2 - \Delta t_i) \\
&+24\langle Y_i, f(Y_i)\rangle\langle Y_i, g(Y_i)\rangle|f(Y_i)|^2\Delta t_i^3\Delta B_i \\
&+48\langle Y_i, f(Y_i)\rangle\langle Y_i, g(Y_i)\rangle f^T(Y_i)g(Y_i)\Delta t_i^2(\Delta B_i^2 - \Delta t_i) \\
&+24\langle Y_i, f(Y_i)\rangle\langle Y_i, g(Y_i)\rangle|g(Y_i)|^2\Delta t_i\Delta B_i^3 + 12\langle Y_i, g(Y_i)\rangle^2|f(Y_i)|^2\Delta t_i^2(\Delta B_i^2 - \Delta t_i) \\
&+24\langle Y_i, g(Y_i)\rangle^2 f^T(Y_i)g(Y_i)\Delta t_i\Delta B_i^3 + 12\langle Y_i, g(Y_i)\rangle^2|g(Y_i)|^2(\Delta B_i^4 - \Delta t_i^2) \\
&+24\langle Y_i, f(Y_i)\rangle|f(Y_i)|^2 f^T(Y_i)g(Y_i)\Delta t_i^4\Delta B_i + 36\langle Y_i, f(Y_i)\rangle|f(Y_i)|^2|g(Y_i)|^2\Delta t_i^3(\Delta B_i^2 - \Delta t_i) \\
&+24\langle Y_i, f(Y_i)\rangle f^T(Y_i)g(Y_i)|g(Y_i)|^2\Delta t_i^2\Delta B_i^3 + 6\langle Y_i, f(Y_i)\rangle|g(Y_i)|^4\Delta t_i(\Delta B_i^4 - \Delta t_i^2) \\
&+6|f(Y_i)|^4 f^T(Y_i)g(Y_i)\Delta t_i^5\Delta B_i + 15|f(Y_i)|^4|g(Y_i)|^2\Delta t_i^4(\Delta B_i^2 - \Delta t_i) \\
&+20|f(Y_i)|^2 f^T(Y_i)g(Y_i)|g(Y_i)|^2\Delta t_i^3\Delta B_i^3 \\
&+15|f(Y_i)|^2|g(Y_i)|^4\Delta t_i^2(\Delta B_i^4 - \Delta t_i^2) + 6f^T(Y_i)g(Y_i)|g(Y_i)|^4\Delta t_i\Delta B_i^5 + |g(Y_i)|^6(\Delta B_i^6 - \Delta t_i^3) \\
&+6\langle Y_i, g(Y_i)\rangle|f(Y_i)|^4\Delta t_i^4\Delta B_i + 24\langle Y_i, g(Y_i)\rangle|f(Y_i)|^3|g(Y_i)|\Delta t_i^3(\Delta B_i^2 - \Delta t_i) \\
&+36\langle Y_i, g(Y_i)\rangle|f(Y_i)|^2|g(Y_i)|^2\Delta t_i^2\Delta B_i^3 + 24\langle Y_i, g(Y_i)\rangle|f(Y_i)||g(Y_i)|^3\Delta t_i(\Delta B_i^4 - \Delta t_i^2) \\
&+6\langle Y_i, g(Y_i)\rangle|g(Y_i)|^4\Delta B_i^5).
\end{aligned}
$$

Then we can rearrange (6.1) into

$$|Y_{i+1}|^p \leq |Y_i|^p - |Y_i|^p\Delta t_i U_1(\Delta t_i, Y_i) + \Delta m_i, \tag{6.2}$$

where

$$
\begin{aligned}
& -U_1(\Delta t_i, Y_i) := \\
& \frac{p}{2}\bigg(\frac{2\langle Y_i, f(Y_i)\rangle + |g(Y_i)|^2}{|Y_i|^2} + \frac{p-2}{4}\frac{4\langle Y_i, g(Y_i)\rangle^2}{|Y_i|^4}\bigg) \\
+ \quad & \Delta t_i\bigg(\frac{p}{2}\frac{|f(Y_i)|^2}{|Y_i|^2} \\
& + \frac{p(p-2)}{8}\frac{4\langle Y_i, f(Y_i)\rangle^2 + |g(Y_i)|^4 + 4\langle Y_i, f(Y_i)\rangle|g(Y_i)|^2 + 8\langle Y_i, g(Y_i)\rangle f^T(Y_i)g(Y_i)}{|Y_i|^4} \\
& + \frac{p(p-2)(p-4)}{2^3\times 3!}\frac{24\langle Y_i, f(Y_i)\rangle\langle Y_i, g(Y_i)\rangle^2 + 12\langle Y_i, g(Y_i)\rangle^2|g(Y_i)|^2}{|Y_i|^6}\bigg) \\
+ \quad & \Delta t_i^2\bigg(\frac{p(p-2)}{8}\frac{6|f(Y_i)|^2|g(Y_i)|^2 + 4\langle Y_i, f(Y_i)\rangle|f(Y_i)|^2}{|Y_i|^4} + \frac{p(p-2)(p-4)}{2^3\times 3!}\times \\
& \bigg(\frac{8\langle Y_i, f(Y_i)\rangle^3 + 12\langle Y_i, f(Y_i)\rangle^2|g(Y_i)|^2 + 48\langle Y_i, f(Y_i)\rangle\langle Y_i, g(Y_i)\rangle f^T(Y_i)g(Y_i)}{|Y_i|^6} \\
& + \frac{12\langle Y_i, g(Y_i)\rangle^2|f(Y_i)|^2 + 6\langle Y_i, f(Y_i)\rangle|g(Y_i)|^4 + |g(Y_i)|^6 + 24\langle Y_i, g(Y_i)\rangle|f(Y_i)||g(Y_i)|^3}{|Y_i|^6}\bigg)\bigg) \\
+ \quad & \Delta t_i^3\bigg(\frac{p(p-2)}{8}\frac{|f(Y_i)|^4}{|Y_i|^4} + \frac{p(p-2)(p-4)}{2^3\times 3!}\times \\
& \frac{12\langle Y_i, f(Y_i)\rangle^2|f(Y_i)|^2 + 36\langle Y_i, f(Y_i)\rangle|f(Y_i)|^2|g(Y_i)|^2}{|Y_i|^6} \\
& + \frac{15|f(Y_i)|^2|g(Y_i)|^4 + 24\langle Y_i, g(Y_i)\rangle|f(Y_i)|^3|g(Y_i)|}{|Y_i|^6}\bigg) \\
+ \quad & \Delta t_i^4\bigg(\frac{p(p-2)(p-4)}{2^3\times 3!}\frac{6\langle Y_i, f(Y_i)\rangle|f(Y_i)|^4 + 15|f(Y_i)|^4|g(Y_i)|^2}{|Y_i|^6}\bigg) \\
+ \quad & \Delta t_i^5\bigg(\frac{p(p-2)(p-4)}{2^3\times 3!}\frac{|f(Y_i)|^6}{|Y_i|^6}\bigg),
\end{aligned}
$$

and

$$
\begin{aligned}
&\Delta m_i = \\
&|Y_i|^p(\frac{1}{|Y_i|^2}(2\langle Y_i, g(Y_i)\rangle \Delta B_i + 2f^T(Y_i)g(Y_i)\Delta t_i \Delta B_i + |g(Y_i)|^2(\Delta B_i^2 - \Delta t_i)) \\
&+\frac{1}{|Y_i|^4}(4\langle Y_i, g(Y_i)\rangle^2(\Delta B_i^2 - \Delta t_i) \\
&+|g(Y_i)|^4(\Delta B_i^4 - \Delta t_i^2) + 4\langle Y_i, f(Y_i)\rangle|g(Y_i)|^2\Delta t_i(\Delta B_i^2 - \Delta t_i) \\
&+8\langle Y_i, g(Y_i)\rangle f^T(Y_i)g(Y_i)\Delta t_i(\Delta B_i^2 - \Delta t_i) + 6|f(Y_i)|^2|g(Y_i)|^2\Delta t_i^2(\Delta B_i^2 - \Delta t_i) \\
&+8\langle Y_i, f(Y_i)\rangle\langle Y_i, g(Y_i)\rangle\Delta t_i\Delta B_i + 4|f(Y_i)|^2 f^T(Y_i)g(Y_i)\Delta t_i^3\Delta B_i + 4f^T(Y_i)g(Y_i)|g(Y_i)|^2\Delta t_i\Delta B_i^3 \\
&+8\langle Y_i, f(Y_i)\rangle f^T(Y_i)g(Y_i)\Delta t_i^2\Delta B_i + 4\langle Y_i, g(Y_i)\rangle|f(Y_i)|^2\Delta t_i^2\Delta B_i + 4\langle Y_i, g(Y_i)\rangle|g(Y_i)|^2\Delta B_i^3) \\
&+\frac{1}{|Y_i|^6}(24\langle Y_i, f(Y_i)\rangle^2\langle Y_i, g(Y_i)\rangle\Delta t_i^2\Delta B_i + 24\langle Y_i, f(Y_i)\rangle\langle Y_i, g(Y_i)\rangle^2\Delta t_i(\Delta B_i^2 - \Delta t_i) \\
&+8\langle Y_i, g(Y_i)\rangle^3\Delta B_i^3 + 24\langle Y_i, f(Y_i)\rangle^2 f^T(Y_i)g(Y_i)\Delta t_i^3\Delta B_i \\
&+12\langle Y_i, f(Y_i)\rangle^2|g(Y_i)|^2\Delta t_i^2(\Delta B_i^2 - \Delta t_i) \\
&+24\langle Y_i, f(Y_i)\rangle\langle Y_i, g(Y_i)\rangle|f(Y_i)|^2\Delta t_i^3\Delta B_i \\
&+48\langle Y_i, f(Y_i)\rangle\langle Y_i, g(Y_i)\rangle f^T(Y_i)g(Y_i)\Delta t_i^2(\Delta B_i^2 - \Delta t_i) \\
&+24\langle Y_i, f(Y_i)\rangle\langle Y_i, g(Y_i)\rangle|g(Y_i)|^2\Delta t_i\Delta B_i^3 + 12\langle Y_i, g(Y_i)\rangle^2|f(Y_i)|^2\Delta t_i^2(\Delta B_i^2 - \Delta t_i) \\
&+24\langle Y_i, g(Y_i)\rangle^2 f^T(Y_i)g(Y_i)\Delta t_i\Delta B_i^3 + 12\langle Y_i, g(Y_i)\rangle^2|g(Y_i)|^2(\Delta B_i^4 - \Delta t_i^2) \\
&+24\langle Y_i, f(Y_i)\rangle|f(Y_i)|^2 f^T(Y_i)g(Y_i)\Delta t_i^4\Delta B_i + 36\langle Y_i, f(Y_i)\rangle|f(Y_i)|^2|g(Y_i)|^2\Delta t_i^3(\Delta B_i^2 - \Delta t_i) \\
&+24\langle Y_i, f(Y_i)\rangle f^T(Y_i)g(Y_i)|g(Y_i)|^2\Delta t_i^2\Delta B_i^3 + 6\langle Y_i, f(Y_i)\rangle|g(Y_i)|^4\Delta t_i(\Delta B_i^4 - \Delta t_i^2) \\
&+6|f(Y_i)|^4 f^T(Y_i)g(Y_i)\Delta t_i^5\Delta B_i + 15|f(Y_i)|^4|g(Y_i)|^2\Delta t_i^4(\Delta B_i^2 - \Delta t_i) \\
&+20|f(Y_i)|^2 f^T(Y_i)g(Y_i)|g(Y_i)|^2\Delta t_i^3\Delta B_i^3 \\
&+15|f(Y_i)|^2|g(Y_i)|^4\Delta t_i^2(\Delta B_i^4 - \Delta t_i^2) + 6f^T(Y_i)g(Y_i)|g(Y_i)|^4\Delta t_i\Delta B_i^5 + |g(Y_i)|^6(\Delta B_i^6 - \Delta t_i^3) \\
&+6\langle Y_i, g(Y_i)\rangle|f(Y_i)|^4\Delta t_i^4\Delta B_i + 24\langle Y_i, g(Y_i)\rangle|f(Y_i)|^3|g(Y_i)|\Delta t_i^3(\Delta B_i^2 - \Delta t_i) \\
&+36\langle Y_i, g(Y_i)\rangle|f(Y_i)|^2|g(Y_i)|^2\Delta t_i^2\Delta B_i^3 + 24\langle Y_i, g(Y_i)\rangle|f(Y_i)||g(Y_i)|^3\Delta t_i(\Delta B_i^4 - \Delta t_i^2) \\
&+6\langle Y_i, g(Y_i)\rangle|g(Y_i)|^4\Delta B_i^5)).
\end{aligned}
$$

In each step, we need to choose Δt_i such that $U_1(\Delta t_i, Y_i) < 0$. To do this, we could choose Δt_i such that

$$
-U_2(\Delta t_i, Y_i) := -\frac{p}{2}v + A_1(Y_i)\Delta t_i + A_2(Y_i)\Delta t_i^2 + A_3(Y_i)\Delta t_i^3 + A_4(Y_i)\Delta t_i^4 + A_5(Y_i)\Delta t_i^5 < 0,
$$

where

$$A_1(Y_i) = \frac{p}{2}\frac{|f(Y_i)|^2}{|Y_i|^2} + \frac{p(2-p)}{8}\frac{4|Y_i||g(Y_i)|^3 + 8|Y_i||f(Y_i)|^2|g(Y_i)|}{|Y_i|^4} + \frac{p(p-2)(p-4)}{2^3\times 3!}\frac{24|Y_i|^3|f(Y_i)||g(Y_i)|^2 + 12|Y_i|^2|g(Y_i)|^4}{|Y_i|^6},$$

$$A_2(Y_i) = \frac{p(2-p)}{8}\frac{4|Y_i||f(Y_i)|^3}{|Y_i|^4} + \frac{p(p-2)(p-4)}{2^3\times 3!}\times \Big(\frac{8|Y_i|^3|f(Y_i)|^3 + 12|Y_i|^2|f(Y_i)|^2|g(Y_i)|^2 + 48|Y_i|^2|f(Y_i)|^2|g(Y_i)|^2}{|Y_i|^6} + \frac{12|Y_i|^2|f(Y_i)|^2|g(Y_i)|^2 + 6|Y_i||f(Y_i)||g(Y_i)|^4 + |g(Y_i)|^6 + 24|Y_i||f(Y_i)||g(Y_i)|^4}{|Y_i|^6}\Big),$$

$$A_3(Y_i) = \frac{p(p-2)(p-4)}{2^3\times 3!}\times \frac{12|Y_i|^2|f(Y_i)|^4 + 36|Y_i||f(Y_i)|^3|g(Y_i)|^2 + 15|f(Y_i)|^2|g(Y_i)|^4 + 24|Y_i|f(Y_i)|^3|g(Y_i)|^2}{|Y_i|^6},$$

$$A_4(Y_i) = \frac{p(p-2)(p-4)}{2^3\times 3!}\frac{6|Y_i||f(Y_i)|^5 + 15|f(Y_i)|^4|g(Y_i)|^2}{|Y_i|^6},$$

and

$$A_5(Y_i) = \frac{p(p-2)(p-4)}{2^3\times 3!}\frac{|f(Y_i)|^6}{|Y_i|^6}.$$

By the elementary inequality $\langle a, b\rangle \leq |a||b|$, it is clear that $-U_1(\Delta t_i, Y_i) < -U_2(\Delta t_i, Y_i)$ a.s. We choose rational number Δt_i such that

$$\Delta t_i \leq \frac{p}{12}\min_{\{A_j(Y_i)\neq 0, j=1,2,3,4,5\}}\{(v/A_j(Y_i))^{(1/j)}\}.$$

Apply the same techniques used in Theorem 3.1, we can prove that t_i is an $\{\mathcal{F}_t\}$-stopping time for each $i = 0, 1, ...$ and $\{m_i = \sum_{k=0}^{i}\Delta m_k\}_{i\geq 0}$ is a $\mathcal{G}_i$-local martingale. Now from (6.2), we have

$$|Y_{i+1}|^p \leq |Y_0|^p - \sum_{k=0}^{i}\Delta t_k|Y_k|^p U_1(\Delta t_k, Y_k) + m_i.$$

By Lemma 2.3, we conclude

$$\lim_{i\to\infty}|Y_i|^p < \infty \quad \text{a.s.} \quad \text{and} \quad \sum_{k=0}^{i}\Delta t_k|Y_k|^p U_1(\Delta t_k, Y_k) < \infty \quad \text{a.s.}$$

Hence we have $\lim_{i\to\infty}\Delta t_i|Y_i|^p U_1(\Delta t_i, Y_i) = 0$ a.s. For almost all $\omega \in \Omega$, there exists $C(\omega) \in \mathbb{R}_+$ such that $\lim_{i\to\infty}|Y_i(\omega)| = C(\omega)$. Fix any such ω, write $C(\omega) = C$ and $Y_i(\omega) = Y_i$. Due to the choice of Δt_i, we have $U_1 > pv/12 > 0$. Since (5.3) and (5.4), applying the same techniques employed in Theorem 3.1 we have $\liminf_{i\to\infty} v/A_j(Y_i) > 0$ for each $j = 1, 2, 3, 4, 5$. That is to say there is no requirement that Δt_i vanishes as i increases, thus $\sum_{i=0}^{\infty}\Delta t_i = \infty$ a.s. Hence we can only have $\lim_{i\to\infty}|Y_i|^p = 0$. The proof is complete.

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