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Word-representability of triangulations of grid-covered cylinder graphs

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Abstract. A graph $G = (V, E)$ is word-representable if there exists a word w over the alphabet V such that letters x and y, $x \neq y$, alternate in w if and only if $(x, y) \in E$. Halldórsson, Kitaev and Pyatkin have shown that a graph is word-representable if and only if it admits a so-called semi-transitive orientation. A corollary of this result is that any 3-colorable graph is word-representable.Masárová

Akrobotu, Kitaev and Masárová have shown that a triangulation of a grid graph is word-representable if and only if it is 3-colorable. This result does not hold for triangulations of grid-covered cylinder graphs; indeed, there are such word-representable graphs with chromatic number 4. In this paper we show that word-representability of triangulations of grid-covered cylinder graphs with three sectors (resp., more than three sectors) is characterized by avoiding a certain set of six minimal induced subgraphs (resp., wheel graphs W_5 and W_7).

Keywords: word-representability, semi-transitive orientation, triangulation, grid-covered cylinder graph, forbidden induced subgraph

1 Introduction

Let $G = (V, E)$ be a simple (i.e. without loops and multiple edges) undirected graph with the vertex set V and the edge set E. We say that G is *word-representable* if there exists a word w over the alphabet V such that letters x and y alternate in w if and only if $(x, y) \in E$ for any $x \neq y$. By definition, each letter in V must appear in w.

The notion of word-representable graphs has its roots in algebra, where a prototype of these graphs was used by Kitaev and Seif to study the growth of the free spectrum of the well known *Perkins semigroup* [11].

A number of results on word-representable graphs were obtained in the literature [1, 2, 3, 5, 6, 7, 8, 10, 12, 13]. In particular, Halldórsson, Kitaev and Pyatkin [8] have shown that a graph is word-representable if and only if it admits a *semi-transitive orientation* (to be defined in Section 2), which, among other important corollaries, implies that all 3-colorable graphs are word-representable. We refer to [9] for the state of the art in the theory of word-representable graphs.

Most relevant to our paper are [1, 2, 5], where *triangulations* and *subdivisions* of certain graphs are studied with respect to word-representability. In particular, Akrobotu, Kitaev and Masárová [1] proved that any triangulation of the graph G associated with a convex polyomino is word-representable if and only if G is 3-colorable. The method to prove this characterization theorem was essentially in showing that such a triangulation is 3-colorable if and only if it contains no wheel graph W_5 or W_7 as an induced subgraph (neither W_5 nor W_7 are word-representable).

In this paper we extend the results of Akrobotu, Kitaev and Masárová [1] to the case of grid-covered cylinder graphs, which is a cyclic version of rectangular grid graphs; see Subsection 2.2 for definitions. It turns out that in this case, some of the graphs in question with chromatic number 4 are actually word-representable; for example, see the underlying graph in Figure 3.7. Still, assuming that there are at least four sectors in a grid-covered cylinder graph, word-representable triangulations of such graphs are characterized by avoidance of W_5 and W_7 as induced subgraphs. On the other hand, we can also characterize word-representability of triangulations of grid-covered cylinder with three sectors as those avoiding the six graphs in Figure 4.11 as induced subgraphs. Moreover, we show that our characterization results in the case of more than three sectors hold even when some of cells (faces) of grid-covered cylinder graphs are not triangulated.

The paper is organized as follows. In Section 2 we will provide all necessary definitions and known results to be used. In particular, we will introduce the notion of a triangulation of a grid-covered cylinder graph, the main concern of this paper. Also, we will introduce the notion of a semi-transitive orientation, the main tool to prove our results. Further, we classify word-representable triangulations of the graphs in question depending on the number of sectors they have. Namely, in Sections 3 we will consider the case of grid-covered cylinder graphs with more than three sections, and in Section 4 we will consider the case of grid-covered cylinder graphs with three sections. Finally, in Section 5 we discuss a generalization of our results and state an open problem.

2 Definitions, notation, and known results

Suppose that w is a word and x and y are two distinct letters in w. We say that x and y *alternate* in w if the deletion of all other letters from the word w results in either $xyxy\cdots$ or $yxyx\cdots$.

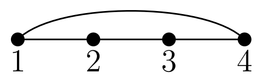

Figure 2.1: The graph represented by the word $w = 134231241$

A graph $G = (V, E)$ is *word-representable* if there exists a word w over the alphabet V such that letters x and y alternate in w if and only if $(x, y) \in E$ for each $x \neq y$. We say that w *represents* G, and such a word w is called a *word-representant* for G. For example, if the word $w = 134231241$ then the subword induced with letters 1 and 2 is 12121, hence the letters 1 and 2 alternate in w, and thus the respective vertices are connected in G. On the other hand, the letters 1 and 3 do not alternate in w, because removing all other letters we obtain 1331; thus, 1 and 3 are not connected in G. Figure 2.1 shows the graph represented by w.

2.1 Semi-transitive orientations

A directed graph (digraph) is *semi-transitive* if it is acyclic (that is, it contains no directed cycles), and for any directed path $v_1 \to v_2 \to \cdots \to v_k$ with $v_i \in V$ for all $i, 1 \leq i \leq k$, either

- there is no edge $v_1 \to v_k$, or
- the edge $v_1 \to v_k$ is present and there are edges $v_i \to v_j$ for all $1 \leq i < j \leq k$. That is, in this case, the (acyclic) subgraph induced by the vertices $v_1, \ldots, v_k$ is transitive.

We call such an orientation a *semi-transitive orientation*.

We can alternatively define semi-transitive orientations in terms of induced subgraphs. A *semi-cycle* is the directed acyclic graph obtained by reversing the direction of one arc of a directed cycle. An acyclic digraph is a *shortcut* if it is induced by the vertices of a semi-cycle and contains a pair of non-adjacent vertices. Thus, a digraph on the vertex set $\{v_1, \ldots, v_k\}$ is a shortcut if it contains a directed path $v_1 \to v_2 \to \cdots \to v_k$, the arc $v_1 \to v_k$, and it is missing an arc $v_i \to v_j$ for some $1 \leq i < j \leq k$; in particular, we must have $k \geq 4$, so that any shortcut is on at least four vertices. Slightly abusing the terminology, in this paper we refer to the arc $v_1 \to v_k$ in the last definition as a shortcut (a more appropriate name for this would be a *shortcut arc*). Figure 2.2 gives examples of shortcuts, where the edges $1 \to 4$, $2 \to 5$ and $3 \to 6$ are missing, and hence $1 \to 5$, $1 \to 6$ and $2 \to 6$ are shortcuts.

Thus, an orientation of a graph is semi-transitive if it is acyclic and contains no shortcuts. Halldórsson, Kitaev and Pyatkin [8] proved the following theorem that characterizes word-representable graphs in terms of graph orientations.

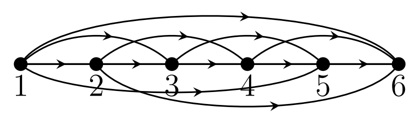

Figure 2.2: An example of a shortcut

Theorem 2.1. [6] *A graph is word-representable if and only if it admits a semi-transitive orientation.*

Thus, in this paper, to find out if a graph G in question is word-representable, we will be studying existence of a semi-transitive orientation of G.

An immediate corollary of Theorem 2.1 is the following result.

Theorem 2.2. [6] *Three-colorable graphs are word-representable.*

2.2 Grid-covered cylinder graphs

A *grid-covered cylinder*, *GCC* for brevity, is a 3-dimensional figure formed by drawing vertical lines and horizontal circles on the surface of a cylinder, each of which are parallel to the generating line and the upper face of the cylinder, respectively. A GCC can be thought of as the object obtained by gluing the left and right sides of a rectangular grid. See the left picture in Figure 2.3 for a schematic way to draw a GCC. The vertical lines and horizontal circles are called the *grid lines* by us.

Any GCC defines a graph whose set of vertices is given by intersection of the grid lines, and whose edges are parts of grid lines between the respective vertices. *Vertical edges* and *horizontal edges* are defined by vertical and horizontal grid lines, respectively. Such a graph is necessarily planar, and it is convenient to consider its edge-crossing-free embedding in the plane as shown schematically in the right picture in Figure 2.3, where by convention, the internal circle C_0 corresponds to the top face of the respective GCC.

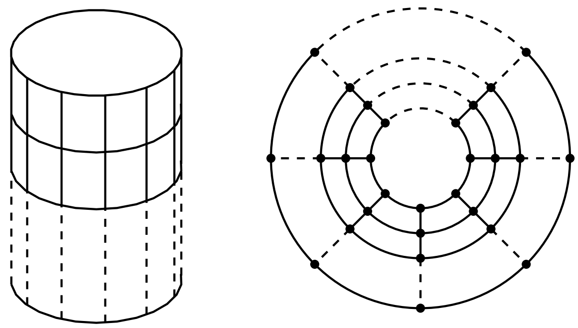

Figure 2.3: Grid-covered cylinder

We next introduce some notions/notation related to a GCC graph (abbreviated *GCCG*) $G_{m,n}$ defined by intersection of m vertical lines and $n+1$ horizontal circles. Let $C_0, C_1, \ldots, C_n$ denote the circles of $G_{m,n}$ in order from inside to outside. Further, let C_i, for $0 \leq i \leq n$, have m equally spaced vertices denoted by $v_{i0}, v_{i1}, \ldots, v_{i(m-1)}$ in the counter-clock-wise direction, so that for a fixed y and any x, vertices v_{xy} lie on the same vertical grid line labelled by V_y; see Figure 2.4 for an example of a proper labelling of a GCCG with four sectors.

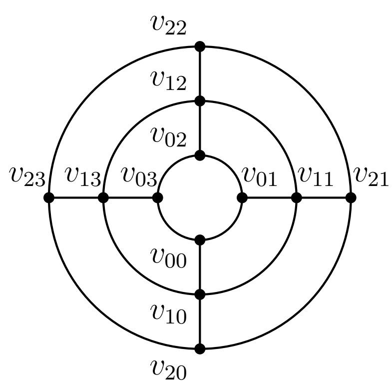

Figure 2.4: Labelling of a GCCG

We say that the vertices on a circle C_i are on the *ith level*. Also, we say that $G_{m,n}$ has n *layers* and m *sectors*. For $1 \leq i \leq n$, the ith layer L_i is the induced subgraph of $G_{m,n}$ formed by the vertices on C_{i-1} and C_i. For $1 \leq j \leq m-1$, the jth sector S_j is the induced subgraph formed by the vertices on the $(j-1)$th and jth vertical grid lines (i.e. V_{j-1} and V_j); the mth sector S_m is the induced subgraph formed by the vertices on the $(m-1)$th and 0th vertical grid lines (i.e. V_{m-1} and V_0). For example, referring to Figure 2.4, the layer L_2 is the induced subgraph formed by the vertices $\{v_{10}, v_{11}, v_{12}, v_{13}, v_{20}, v_{21}, v_{22}, v_{23}\}$, while the sector S_3 is the induced subgraph formed by the vertices $\{v_{02}, v_{12}, v_{22}, v_{03}, v_{13}, v_{23}\}$.

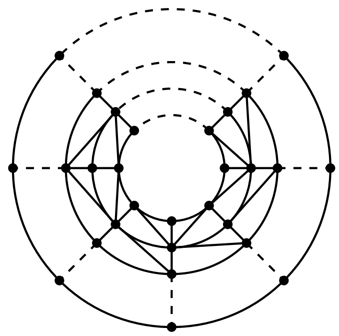

Figure 2.5: A triangulation of a GCCG

Intersections of grid lines define GCCG's *cells* all of which are 4-cycles. Note that in the case of $m = 4$, the vertices and edges on C_0 and C_m form 4-cycles, but we do not call these cells. Thus, by definition, $G_{m,n}$ has mn cells. A *triangulation* T of $G_{m,n}$ is the graph obtained from $G_{m,n}$ by triangulating *each* cell in it. The total number of (possibly isomorphic) triangulations of $G_{m,n}$ is 2^{mn}. The subdivision edges in T that are used to subdivide cells into triangles are called *diagonal edges*.

3 Word-representable triangulations of GCCGs with more than three sectors

For $n \geq 3$, a *wheel graph* W_n is obtained by adding to the *cycle graph* C_n an all-adjacent vertex (*apex*). It is known [9, 10] that for odd $n \geq 5$, W_n is not word-representable. In particular, W_5 and W_7 shown in bold in Figure 3.6 (each twice) are not word-representable.

In this section we will prove the following theorem.

Theorem 3.1. *A triangulation of a GCCG with more than three sectors is word-representable if and only if it contains no W_5 or W_7 as an induced subgraph.*

The proof of Theorem 3.1 will follow from Lemma 3.3 below, whose proof is based on Lemma 3.2 giving the structure of GCCGs with more than three sectors that contain no W_5 or W_7 as induced subgraphs.

Next, we introduce the notion of a *type* of a cell subdivision on layer L_i for $i \geq 2$. We say that a cell C defined by $v_{ij}v_{(i+1)j}v_{(i+1)(j+1)}v_{i(j+1)}$, where

$1 \leq i \leq n-1$ and $0 \leq j \leq m-2$, in a triangulation T of a GCCG $G_{m,n}$ is of *type A* if its diagonal edge has no vertex in common with the diagonal edge of the cell above defined by $v_{(i-1)j}v_{ij}v_{i(j+1)}v_{(i-1)(j+1)}$. We say that C is of *type B* otherwise. The type of a cell in sector S_m is defined in the same way. Note that the type of cells on L_1 is not defined.

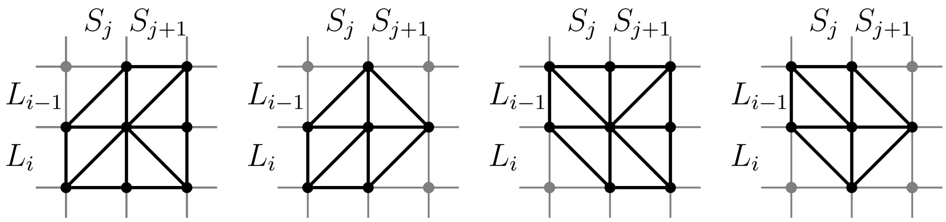

Figure 3.6: The bottom-left cells are of type A and the bottom-right cells are of type B

Lemma 3.2. *If a triangulation of a GCCG with more than three sectors contains no W_5 or W_7 as an induced subgraph, then each cell on layer L_i, for $i \geq 2$, must be of the same type.*

Proof. If two cells on a layer L_i, $i \geq 2$, are of different types, then there must be two adjacent cells on L_i of different types. Suppose that these cells are in the sectors S_j and S_{j+1}. Considering these cells together with two cells in the same sectors on the layer L_{i-1} we will meet either W_5 or W_7 as an induced subgraph, as shown in Figure 3.6 (where we assumed that the bottom-left cells are of type A; the cases when these are of type B are obtained from those in Figure 3.6 by reflection with respect to a vertical line). Contradiction. ∎

By Lemma 3.2, each cell on a layer L_i, for $i \geq 2$, is of the same type, and thus the notion of a *layer type* (starting from layer 2 upwards) is well defined as the type of the layer's cells.

Next, we describe an orientation O of a triangulation T of a GCCG $G_{m,n}$ with $m \geq 4$, which will be shown by us in Lemma 3.3 to be semi-transitive.

- Orient the horizontal edges of T as follows: for $0 \leq x \leq n$ and $0 \leq y \leq m-3$, $v_{x0} \rightarrow v_{x(m-1)}$, $v_{x(m-1)} \rightarrow v_{x(m-2)}$, and $v_{xy} \rightarrow v_{x(y+1)}$. Thus, all horizontal edges in the same sector receive the same orientation. In fact, we could pick any semi-transitive orientation of the cycle graph on C_0 and make the orientation of any other horizontal edge h be the same as the orientation of the edge on C_0 belonging to h's sector. However, we fixed a particular orientation, which is easy to deal with.

- Each diagonal edge d is oriented in the same direction as the horizontal edges of the cell d belongs to. Thus, each horizontal or diagonal edge in a sector has the same orientation, which makes the *orientation of a sector* to be a well-defined notion.

- Finally, orient vertical edges as follows: $v_{0y} \to v_{1y}$ for $0 \le y \le m-1$. More generally, for $1 \le x \le n-1$ and $0 \le y \le m-1$, a vertical edge $v_{xy}v_{(x+1)y}$ has the same orientation as the edge $v_{(x-1)y}v_{xy}$ if the layer L_{x+1} is of type A, and it is oriented in the opposite direction if L_{x+1} is of type B. Thus, we can orient all vertical edges, layer by layer, starting from layer L_2 and following our rules, so that all vertical edges on the same layer will be oriented in the same direction.

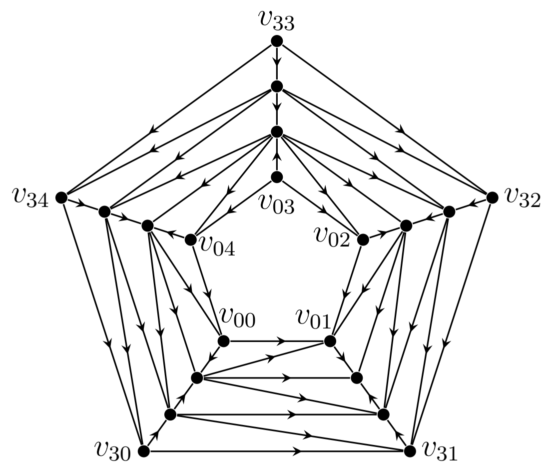

Figure 3.7: The semi-transitive orientation O on $T_{5,4}$

In what follows, when we refer to C_0, we mean the cycle graph induced by the vertices on C_0.

Lemma 3.3. *The orientation O is semi-transitive.*

Proof. First note that O is acyclic. Indeed, any cycle must involve horizontal or diagonal edges, and since all such edges in a sector have the same direction, existence of a cycle in O would force all sectors be oriented in the same direction contradicting the definition of O.

Suppose now that there is a shortcut edge $v_1 \to v_k$, which is defined by a directed path $P = v_1 \to v_2 \to \cdots \to v_k$, for $k \ge 4$. Taking into account that each horizontal or diagonal edge in a sector has the same direction, no more than one (horizontal or diagonal) edge from each sector can be present in P unless P goes around the entire cylinder (that is, visits each vertical line), which is not possible by the definition of O (in particular, in such a situation C_0 would be forced to be a directed cycle).

Further, note that each directed path in T induces a directed path on C_0 because each sector has the same orientation. In particular, steps on a vertical line correspond to no step on C_0. Clearly, P cannot be located on a single vertical line. Thus, there are only two cases to consider, namely, when P visits at least three distinct vertical lines, and when it visits two distinct vertical lines.

- **Case 1.** P involves vertices from at least three (consecutive) different vertical lines, say V_x, V_{x+1} and V_{x+2} for some x taken mod m, in order P visits the lines; we also assume that v_1 is on V_x. However, once P reaches V_{x+2}, the vertices on V_{x+1} are not reachable for P giving no possibility for a shortcut unless P goes around the cylinder. But in the latter case, C_0 will be forced to have a shortcut or a cycle contradicting the definition of O.

- **Case 2.** Only two (consecutive) vertical lines are involved in P. By symmetry, only three subcases are possible for a shortcut, which are shown in Figure 3.8.

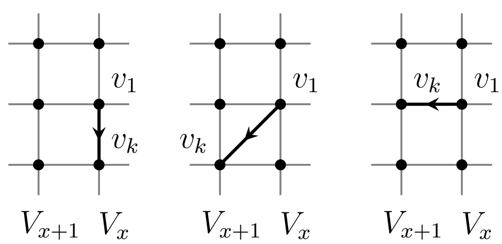

Figure 3.8: Three possibilities for Case 2

Subcase 2.1. Consider the leftmost picture in Figure 3.8. The orientations of edges between vertical lines V_x and V_{x+1} ensure that v_2 cannot be on V_{x+1}. But then the presence of the edge $v_1 \to v_2$ shows that the cell containing $\{v_1, v_k, y\}$ in Figure 3.9 is of type B, so that only two situations are possible here, both presented in Figure 3.9. Note that in any case, $v_2 \to v_k$ cannot be an edge. But then, because of the orientation of the sector defined by the lines V_x and V_{x+1}, the path P can never reach v_k. Contradiction.

Subcase 2.2. Note that $v_1 \to v_2$ cannot be a horizontal edge. Since $k \geq 4$, we consider four possible situations presented in Figure 3.10.

(1) Suppose that $v_1 \to v_2$ is as shown in the rightmost picture in Figure 3.10. Since the cell defined by $\{v_1, a, v_k, c\}$ is of type B,

Figure 3.9: Possibilities for Subcase 2.1

the edges av_2 and av_k will receive opposite directions, so that $v_1 \to v_k$ cannot be a shortcut in this situation.

(2) If $v_1 \to v_2$ is an edge as shown in the second picture in Figure 3.10, then we have also an edge $a \to b$, and because of that, $v_2 \to a$ must also be an edge (otherwise, P has no possibility to reach eventually v_k). However, if $a \to b$ is an edge, the cell defined by $\{c, v_1, a, v_k\}$ is of type A, and thus $v_k \to a$ is an edge, and P cannot be a shortcut (the vertices v_1, v_2, a and v_k do not define a shortcut).

(3) Suppose that $v_1 \to v_2$ and $v_2 \to c$ are edges as shown in the third picture in Figure 3.10. Then the cell defined by $\{v_2, v_k, b, c\}$ is of type A (which is reflected in Figure 3.10) and clearly P will not reach v_k. Contradiction.

(4) Finally, suppose that $v_1 \to v_2$ and $c \to v_2$ are edges as shown in the fourth picture in Figure 3.10. Then the cell defined by $\{v_2, v_k, b, c\}$ is of type B (which is reflected in Figure 3.10) and clearly P will not reach v_k. Contradiction.

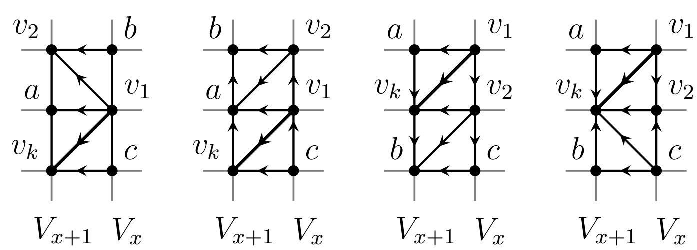

Figure 3.10: Possibilities for Subcase 2.2

Subcase 2.3. For the rightmost picture in Figure 3.8, we omit our arguments since they are very similar to the arguments in Subcases 2.1 and 2.2.

The lemma is proved. ∎

4 Word-representable triangulations of GC-CGs with three sectors

The goal of this section is to prove the following theorem.

Theorem 4.1. *A triangulation of a GCCG with three sectors is word-representable if and only if it contains no graph in Figure* 4.11 *as an induced subgraph.*

Our proof is organized as follows. In Subsection 4.1 we will provide all six minimum non-word-representable graphs that can appear in triangulations of GCCGs with three sectors (see Figure 4.11) and give an explicit proof that one of these graphs is non-word-representable. Then, in Subsection 4.2, we will give an inductive argument showing that avoidance of the six graphs in Figure 4.11 is a sufficient condition for a GCCG with three sectors to be word-representable. Note that the graphs in Figure 4.11 were obtained by an exhaustive computer search on graphs on up to eight vertices. However, our argument in Subsection 4.2 will show that no other non-word-representable induced subgraphs can be found among all triangulations of GCCGs with three sectors.

4.1 Non-word-representability of the graphs in Figure 4.11

Non-word-representability of the graphs in Figure 4.11 can be checked using existing software [4]. However, there is a way to check this fact by hand using the branching approach, which is rather space consuming, and thus we will demonstrate this approach only on one example, the second graph in Figure 4.11; the remaining cases can be checked similarly.

Note that for any of the partial orientations of the 3- or 4-cycles given in Figure 4.12, there is a unique way of completing these orientations, also shown in Figure 4.12, so that oriented cycles and shortcuts are avoided. This stays true in the context of triangulated 4-cycles.

Below, we use the following terminology introduced in [1]. *Complete $XYW(Z)$* refers to completing the orientations on a cycle $XYW(Z)$ according to the respective cases in Figure 4.12. Instances in which it is not possible to uniquely determine orientations of any additional edges in a partially oriented graph are referred to as *Branching XY*. Here, one picks a new, still

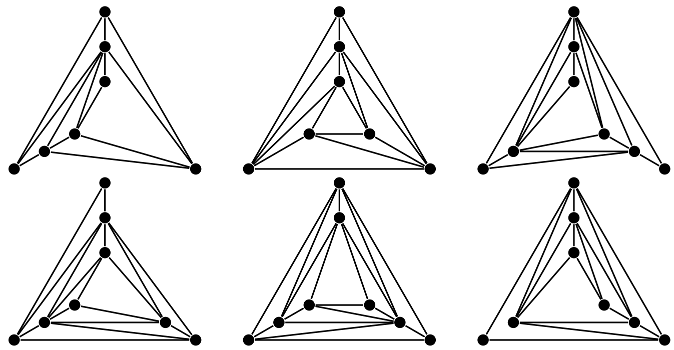

Figure 4.11: All minimal non-word-representable induced sugraphs in triangulations of GCCG's with three sectors

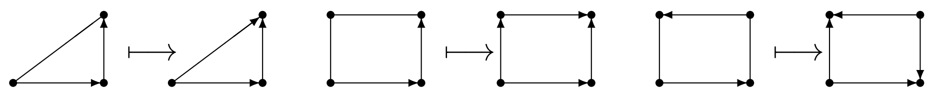

Figure 4.12: Unique semi-transitive completions of the partial orientations of a 3-cycle or a 4-cycle

non-oriented edge XY of the graph and assigns the orientation $X \to Y$, while, at the same time, one makes a copy of the graph, respectively, with its partial orientations and assigns orientation $Y \to X$ to the edge XY. The new copy is named and examined later on. Our terminology and relevant abbreviations are summarized in Table 4.1.

Abbreviation	**Operation**
B	Branch
NC	Obtain a new partially oriented copy
C	Complete
MC	Move to a copy
S	Obtain a shortcut

Table 4.1: List of used operations and their abbreviations

Name A the first copy of the second graph in Figure 4.11 with the single edge orientated as $1 \to 3$, and carry out the following operations, where the partially oriented graphs A – L are given in Figure 4.13. We will show that in case of any acyclic orientation of the graph, shortcuts are unavoidable.

- B 37 (NC B), C 137, C 1376, B 65 (NC C), C 1654, C 1764, C 4576, C 3752, C 2564, C 2457, C 123, S 1324;

- MC C, C 567, C 5672, B 45 (NC D), C 1654, C456, C 1452, C 452, C 2573, S 4132;
- MC D, C 1654, C 1452, C 2754, C 2413, S 5237;
- MC B, B 16 (NC E), C 6137, B 17 (NC F), C 1764, B 45 (NC G), C 1452, C 6145, C 765, C 7652, C 2573, C 2564, S 1423;
- MC G, C 546, C 5417, C 6457, C 5732, C 1452, C 2564, S 1423;
- MC F, C 7165, B 25 (NC H), C 2567, C 273, C 2754, C 1654, C 2541, C 456, S 4271;
- MC H, C 2573, C 132, C 732, C 1324, C 4125, C 4576, S 1642;
- MC E, C 7316, B 12 (NC I), C 6125, C 7652, C 6524, B 17 (NC J), C 1764, C 4125, C 5467, C 2573, S 1423;
- MC J, C 1764, C 4125, C 5467, C 3752, S 4132;
- MC I, C 213, B 14 (NC K), C 214, C 614, C 6145, C 7652, C 7325, C 7561, S 2714;
- MC K, C 4132, B 25 (NC L), C 5214, C 6145, C 7652, C 1427, C 1754, S 2573;
- MC L, C 4125, C 6145, C 7652, C 5237, C 7541, C 4576, S 6427.

4.2 An inductive argument proving Theorem 4.1

To show that avoidance of the six graphs in Figure 4.11 as induced subgraphs is a sufficient condition for a GCCG with three sectors to be word-representable, we use the following approach.

Each triangulation T of a GCCG having i layers and no graph in Figure 4.11 as an induced subgraph is obtained from a triangulation T_0 of a GCCG having $i-1$ layers and no graph in Figure 4.11 as an induced subgraph by adding a new external layer L_i. Since the graphs in Figure 4.11 involve vertices from three layers (that is, four levels), to obtain all possible T, we need to control the two external layers (layers L_{i-1} and L_{i-2}) in T_0. Further, if we assume existence of a semi-transitive orientation of T_0, which induces a semi-transitive orientation on L_{i-1} and L_{i-2}, we could try to extend such an orientation to a semi-transitive orientation of T (making sure that no cycles or shortcuts emerge).

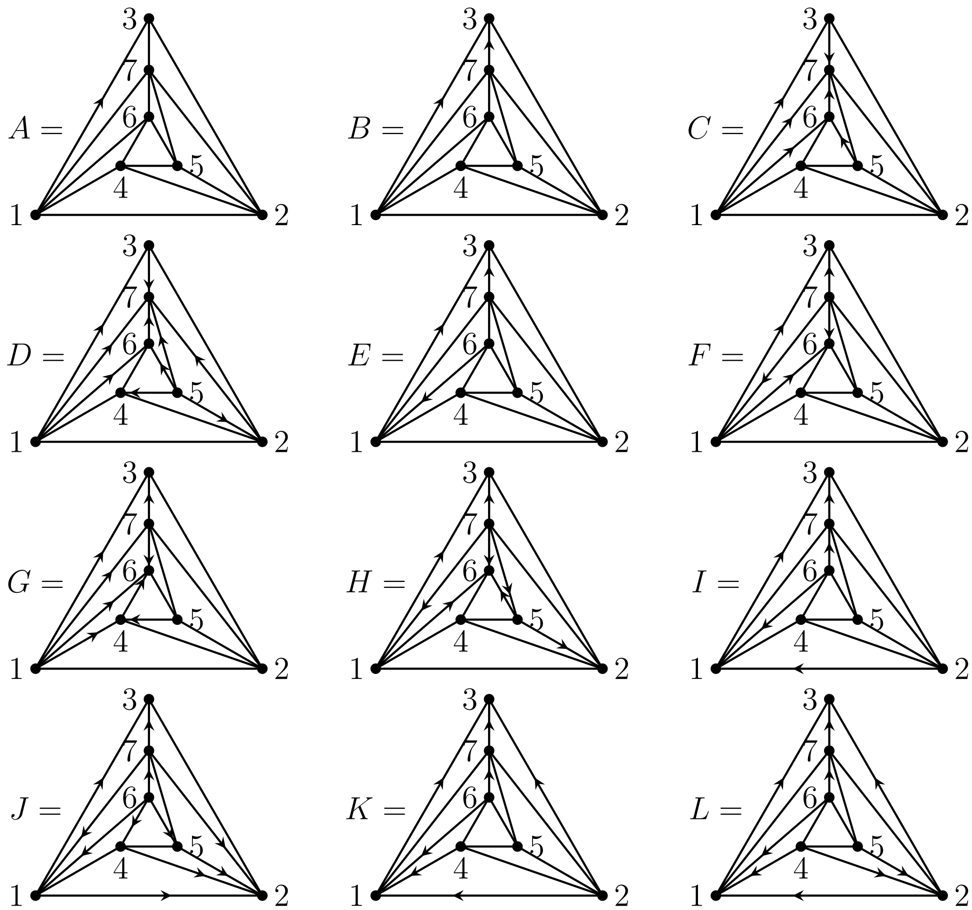

Figure 4.13: Partial orientations of the second graph in Figure 4.11

The next step is to compare the directed graphs induced by the layers $\{L_{i-1}, L_i\}$ and $\{L_{i-2}, L_{i-1}\}$. If these are the same directed graphs, then an inductive argument can be applied to extending the graph by new layers and proving that in each case a semi-transitive orientation exists giving word-representability by Theorem 2.1. On the other hand, if the graphs induced by the layers $\{L_{i-1}, L_i\}$ and $\{L_{i-2}, L_{i-1}\}$ are different, we need to replace the oriented layers $\{L_{i-2}, L_{i-1}\}$ by $\{L_{i-1}, L_i\}$ and repeat the procedure described above again. Namely, we need to extend the graph by another layer L_{i+1}, then try to extend the existing semi-transitive orientation to this layer, and compare two external layers $\{L_i, L_{i+1}\}$ with already considered orientations of two external layers with a hope to meet the same directed graph.

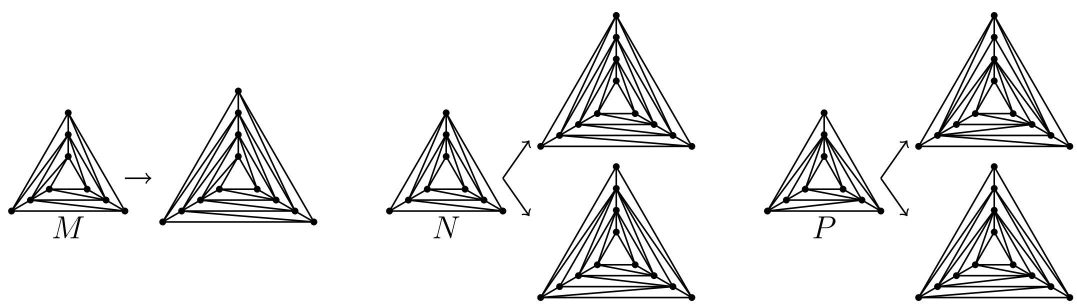

Figure 4.14: The extensions of the graphs M, N and P

The base for our inductive proof is the three graphs, M, N and P in Figure 4.14, which are the only non-isomorphic triangulations of the GC-CG on nine vertices containing no graph in Figure 4.11 as an induced subgraph. Each of these graphs can be semi-transitively oriented as shown in Figure 4.15. Note that we provide two semi-transitive orientations for the graphs N and P, which is essential in our inductive argument. Also, note that each triangulation of a GCCG with three sectors on less than nine vertices can be oriented semi-transitively (just remove the external layer in the graphs in Figure 4.15).

Further, we note that there are only five ways in total in which the graphs M, N and P can be extended by one more (external) layer if the graphs in Figure 4.11 are to be avoided as an induced subgraph. The extensions are shown in 4.14.

We now make an inductive hypothesis that each triangulation of a GCCG with three sectors and n layers having no graph in Figure 4.11 as an induced subgraph can be oriented semi-transitive so that two external layers form the same directed graph as one of the directed graphs in Figure 4.15.

We will next prove the statement for $n + 1$ layers by adding to all of the graphs in Figure 4.15 one more layer, in all possible ways, and extending

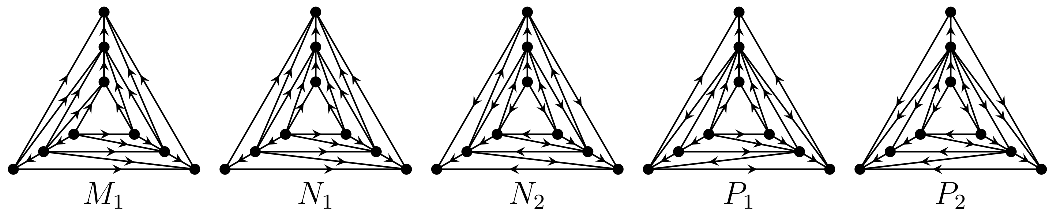

Figure 4.15: Semi-transitive orientations of the graphs M, N and P

the orientation of the resulting partially oriented graph to a semi-transitive orientation. It is important to note that in each case below, it will follow from our way to extend orientations that no cycle or shortcut will be possible involving vertices on newly added level C_{n+1} *and* vertices on levels C_{n-3}, $C_{n-4}, \dots, C_0$.

M has a unique extension by an extra layer, and the orientation of M_1 can be extended to that shown in Figure 4.16. Note that the two external layers of the graph in Figure 4.16 form M_1, as desired.

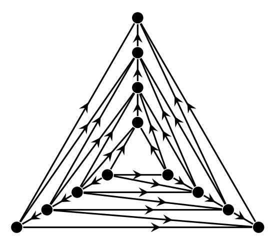

Figure 4.16: The extension of M_1

N_1 has two possible extensions by an extra layer, but both of them can be extended to semi-transitive orientation: see N_{11} and N_{12} in Figure 4.17. Two external layers of N_{11} and N_{12} form N_1 and P_1, respectively, as desired.

N_2 has two possible extensions by an extra layer, but both of them can be extended to semi-transitive orientation: see N_{21} and N_{22} in Figure 4.17. Two external layers of N_{21} and N_{22} form N_2 and P_2, respectively, as desired.

P_1 has two possible extensions by an extra layer, but both of them can be extended to semi-transitive orientation: see P_{11} and P_{12} in Figure 4.18. Two external layers of P_{11} and P_{12} form P_2 and N_2, respectively, as desired.

Finally, P_2 has two possible extensions by an extra layer, but both of them can be extended to semi-transitive orientation: see P_{21} and P_{22} in Figure 4.18. Two external layers of P_{21} and P_{22} form P_1 and N_1, respectively, as desired.

We are done.

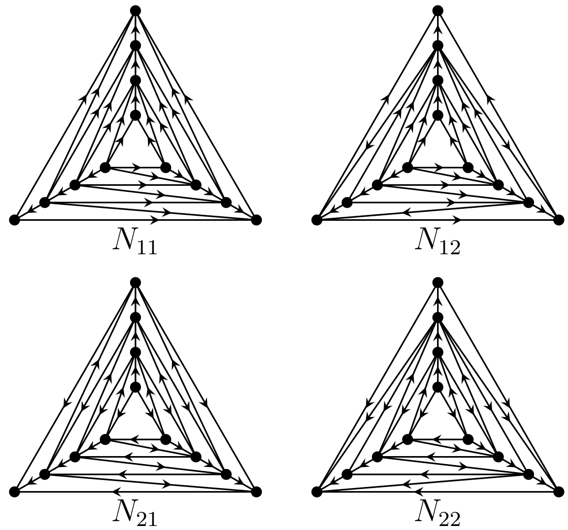

Figure 4.17: The extensions of N_1 (top row) and of N_2 (bottom row)

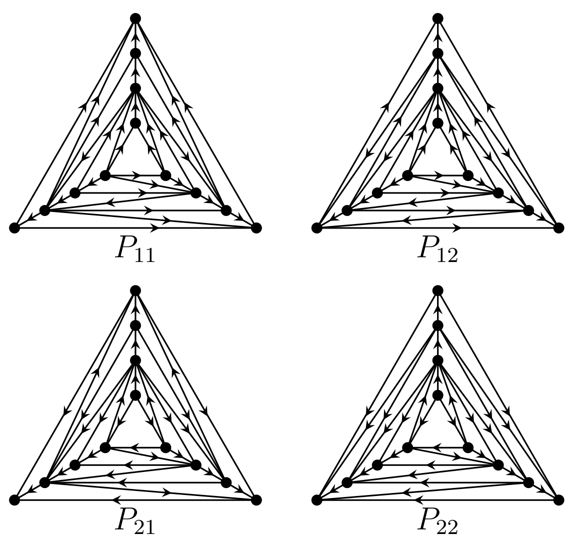

Figure 4.18: The extensions of P_1 (top row) and of P_2 (bottom row)

5 Concluding remarks

It is easy to see that any triangulation of a GCCG with more than three sectors contains no K_4 as an induced subgraph. Using this fact, it is not difficult to see that the orientation O defined in Section 3 contains no *transitively* oriented induced subgraphs on four or more vertices. That means that removing any directed edge in such an orientation, the resulting graph will avoid shortcuts (and directed cycles). As a particular case of this observation, when only diagonal edges can be removed, we have the following generalization of Theorem 3.1.

Theorem 5.1. *Given a GCCG G with more than three sectors, triangulate some of cells of G to obtain a graph T. Then T is word-representable if and only if it contains no W_5 or W_7 as an induced subgraph.*

Unfortunately, such a generalization for Theorem 4.1 does not follow directly from our proofs. Indeed, for example, the graph N_{11} in Figure 4.17 contains a *transitively* oriented copy of K_4 (in the middle of the graph), and removing the rightmost edge in the K_4 we will obtain a shortcut. Thus, we leave it as an open problem to decide whether avoidance of the graphs in Figure 4.11 characterize triangulations of selected cells in a GCCG with three sectors, and if not then to find such a characterization.

Finally, the results in this paper can be seen as a step towards characterizing word-representable planar graphs, which remains a challenging open problem.

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