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Approximation of the Whole Pareto-Optimal Set for the Vector Optimization Problem

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Abstract

In multi objective optimization problems several objective functions have to be minimized simultaneously. In this work, we present a new computational method for the numerical solution of the linearly constrained, convex multi objective optimization problem. We propose some technique to find joint decreasing direction for unconstrained and linearly constrained case as well. Based on these results we introduce a method using subdivision technique to approximate the whole Pareto-optimal set of the linearly constrained, convex multi objective optimization problem. Finally, we illustrate computations of our algorithm by solving the Markowitz-model on real data.

1 Introduction

In the economics and financial literature the measure of risk was always a very interesting topic, and now days it may be even more important than ever. One of the first idea to take into consideration the risk in financial activities came from Harry Markowitz [23] who developed his famous model where the investors make portfolios from different securities, and try to maximize their profit and minimize their risk at the same time. In this model the profit was linear and the risk was defined as the variance of the securities. From mathematical programming point of view Markowitz-model can be formulated as linearly constrained optimization problem with two objective (linear profit and quadratic risk) functions.

In general case, the least risky portfolio won't be the most profitable one, which means we could not optimize the two objectives at the same time. Therefore we need to find portfolios, where one of the goal can not be improved without worsen the other. This kind of solutions are called *Pareto-optimal* or *Pareto-efficient* solutions [27].

Standard way to find a Pareto-optimal solution [20], of the Markowitz-model to make a convex combination of the objective functions, and solve the new problem with quadratic objective function and linear constraints [22]. Weighted sum of the objective functions, as a new objective function, simplifies the problem. The simplified problem's optimal solution is a Pareto-optimal solution of the original problem. The effect of the weights of objective functions, determine the computed Pareto-efficient solution of the original problem, but we have no control over this. The weights, have unpredictable effects on the computed Pareto-efficient solution in general. Weakness of this approach is that restrict the Pareto-efficient solution set to an element and it's local neighborhood. In this way we loose some information, like how much extra profit can be gained by accepting larger risk. Finding, or at least approximating, the whole Pareto-efficient solution set of the original, multi-objective problem, may lead to better understanding of the modeled practical problem [24].

For some unconstrained multi-objective optimization problems there are research papers [10],[11],[30],[12] discussing algorithms applicable for approximating Pareto-effecient solution set. However, many multi-objective optimization problems, naturally, have constraints [12],[13]. A simple example for constrained multi-objective optimization problem is the earlier mentioned Markowitz-model. In this paper we extend and generalize the algorithm of O. Schütze at. all. [10] for approximating Pareto-efficient set of linearly constrained convex multi-objective problem. J. Flige [12] had some theoretical results which are similar to our approach of finding joint decreasing direction.

In the next section the most important definitions and results of vector optimization problem, useful to our approach, has been summarized. In the third section we discuss some results about the unconstrained vector optimization problem. The method called subdivision technique [10], [11] was developed to approximate Pareto-efficient solution set of unconstrained vector optimization problems. The subdivision method use some results described in [30]. An important ingredient of all methods that can approximate the Pareto-optimal set of a convex vector optimization problem is the computation of a joint decreasing direction for all the objective functions. We show that using linear optimization results, a joint decreasing direction for an unconstrained vector optimization problem can be computed. In the fourth section, computation of a feasible joint decreasing direction for linearly constrained convex vector optimization problem is discussed. Section 5 contain an algorithm that is a generalization of subdivision method for linearly constrained convex vector optimization problem. In section 6 we show some numerical results obtained on a real data set (securities from Budapest Stock Exchange) for Markowitz-model. Finally, we summarize our results and list some idea for future research.

2 Basic definitions and results in vector optimization

In this section we discuss some notations, define vector (or multi objective) optimization problem and the concept of Pareto-optimal solutions. Furthermore, we state two well known results of vector optimization, that are playing important role in our approach.

We use the following notations throughout the paper: scalars and indices are denoted by lowercase Latin letters, column vectors by lowercase boldface Latin letters, matrices by capital Latin letters, and finally sets by capital calligraphic letters.

The vector of all one coordinates is denoted by

$$\mathbf{e}^T = (1, 1, \ldots, 1, \ldots, 1),$$

where ^{T} stands for the transpose of a (column) vector (or a matrix). Vector $\mathbf{e}_i$ is the ith unit vector.

We define the *simplex set*, as,

Definition 2.1 *Let $\mathcal{S}_k$ denote the simplex in the k dimensional vector space, and define it as follows:*

$$\mathcal{S}_k = \{\mathbf{w} \in \mathbb{R}^k : \mathbf{e}^T\mathbf{w} = 1, \quad \mathbf{w} \geq \mathbf{0}\}.$$

Let $\mathcal{F} \subseteq \mathbb{R}^n$ be a set and $F : \mathcal{F} \to \mathbb{R}^k$ is a function defined as $F(\mathbf{x}) = [f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_k(\mathbf{x})]^T$, where $f_i : \mathcal{F} \to \mathbb{R}$ is a coordinate function for all i. General vector optimization problem can be formulated as

$$\left.\begin{array}{c} \text{MIN } F(\mathbf{x}) \\ \mathbf{x} \in \mathcal{F} \end{array}\right\} \qquad (GVOP),$$

the MIN means that we try to minimize all the coordinates of the function F, simultaneously.

If the set $\mathcal{F}$ and the function F are convex then $(GVOP)$ is a convex vector optimization problem. Similarly to many cases of $(GVOP)$ models in the literature we assume that F is a differentiable function.

Usually, different objective functions of $(GVOP)$ describe conflicting goals, therefore such $\mathbf{x} \in \mathcal{F}$ that minimizes all objective functions at the same time is unlikely to exist. For this reason the following definitions naturally extends the concept of optimal solution for $(GVOP)$ settings.

Definition 2.2 *Let a $(GVOP)$ problem be given. We say that $\mathbf{x}^* \in \mathcal{F}$ is a*

1. *weakly Pareto-optimal solution if does not exist feasible solution $\mathbf{x} \in \mathcal{F}$ which satisfies the $F(\mathbf{x}) < F(\mathbf{x}^*)$ vector inequality;*

2. *Pareto-optimal solution if does not exist feasible solution $\mathbf{x} \in \mathcal{F}$ which satisfies the $F(\mathbf{x}) \leq F(\mathbf{x}^*)$, vector inequality and $F(\mathbf{x}) \neq F(\mathbf{x}^*)$.*

Furthermore, we call the set $\mathcal{F}^* \subseteq \mathcal{F}$, weakly Pareto-optimal set if every $x^* \in \mathcal{F}^*$ is a weakly Pareto-optimal solution of the $(GVOP)$.

In vector optimization our goal is to compute Pareto-optimal or weakly Pareto-optimal solutions. Literature contains several methods that finds one of the Pareto-optimal solutions, but sometimes it is interesting to find all of them, or at least as much of them as we can [25].

One of the frequently used method to compute a Pareto-optimal solution uses a weighted sum of the objective function as a single objective optimization problem. Let $\mathbf{w} \in \mathcal{S}_k$ be a given vector of weights. From a vector optimization problem, using a vector of weights, we can define the weighted optimization problem as follows

$$\left.\begin{array}{c} \min\ \mathbf{w}^T F(\mathbf{x}) \\ \mathbf{x} \in \mathcal{F} \end{array}\right\} \qquad (WOP).$$

We state, without proof, two well-known theorems that describes the relationship between $(GVOP)$ and (WOP). The first theorem shows, that the (WOP) can be used to find a Pareto-optimal solutions [25].

Theorem 2.3 *Let a $(GVOP)$ and the corresponding (WOP) for a $\mathbf{w} \in \mathcal{S}_k$ be given. Assume that the $\mathbf{x}^* \in \mathcal{F}$ be an optimal solution of the (WOP) problem, then $\mathbf{x}^*$ is a weak Pareto-optimal solution for the $(GVOP)$.*

This statement has an elementary, indirect proof. The next theorem needs a bit more complicated reasoning, but for convex case every Pareto-optimal solution of the $(GVOP)$ can be found with a (WOP) using the proper weights.

Theorem 2.4 *Let $(GVOP)$ be a convex vector optimization problem, and assume that $\mathbf{x}^* \in \mathcal{F}$ is a Pareto-optimal solution of the $(GVOP)$, then there is a $\mathbf{w} \in \mathcal{S}_k$ weight vector, and a (WOP) problem, for which $\mathbf{x}^*$ is an optimal solution.*

The method that will be described in the fifth section, decreases every coordinate function of F at the same time and always move form a feasible solution to another feasible solution, hence we introduce the following, useful definition.

Definition 2.5 *Let a $(GVOP)$, a feasible point $\mathbf{x} \in \mathcal{F}$ and a vector $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{v} \neq \mathbf{0}$ be given. Vector $\mathbf{v}$ is called*

1. *joint decreasing direction at point $\mathbf{x}$ if there exists $h_0 > 0$, for every $h \in (0, h_0]$ satisfying that $F(\mathbf{x} + h\mathbf{v}) < F(\mathbf{x})$;*

2. *feasible joint decreasing direction if it is a joint decreasing direction and there exists $h_1 > 0$, for every $h \in (0, h_1]$ satisfying that $\mathbf{x} + h\mathbf{v} \in \mathcal{F}$.*

Let the following unconstrained vector optimization problem

$$\text{MIN } F(x_1, x_2) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} x_1^2 + x_2^2 \\ (x_1 - 1)^2 + (x_2 - 1)^2 \end{pmatrix} \qquad (GVOP_1),$$

be given. Let a point and a direction have been chosen as $\mathbf{x}^T = (x_1, x_2) = (0, 1)$ and $\mathbf{v}^T = (1, -1)$. Now we show that $\mathbf{v}$ is a joint decreasing direction for the objective function F at point $\mathbf{x}$.

It is easy to show that

$$f_1(\mathbf{x} + h\,\mathbf{v}) = f_2(\mathbf{x} + h\,\mathbf{v}) = h^2 + (1 - h)^2 = 2\left(h - \frac{1}{2}\right)^2 + \frac{1}{2}$$

From the last form of the coordinate functions, it is easy to see that the coordinate functions are decreasing in the $[0; \frac{1}{2}]$ interval, therefore $\mathbf{v}$ is a joint decreasing direction with $h_0 = \frac{1}{2}$.

If we add a single constraint to our example we obtain a new problem

$$\left.\begin{aligned} \text{MIN } F(x_1, x_2) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} &= \begin{pmatrix} x_1^2 + x_2^2 \\ (x_1 - 1)^2 + (x_2 - 1)^2 \end{pmatrix} \\ x_1 \leq \frac{1}{3} \qquad & \end{aligned}\right\} \qquad (GVOP_2).$$

It is easy to see that, $\mathbf{v}$ is a feasible joint decreasing direction for problem $(GVOP_2)$, too, with $h_1 = \frac{1}{3}$.

Let us consider $(GVOP)$ with convex, differentiable objective function F and let us denote the Jacobian-matrix of F at point $\mathbf{x}$ by $J(\mathbf{x})$. Then $\mathbf{v} \in \mathbb{R}^n$ is a joint decreasing direction of function F at point $\mathbf{x}$ if and only if $[J(\mathbf{x})]\mathbf{v} < 0$.

3 Results for unconstrained vector optimization

In this section we review some results of unconstrained vector optimization, namely for $\mathcal{F} = \mathbb{R}^n$. We assume that F is a differentiable function. The unconstrained vector optimization problem is denoted by $(UVOP)$.

Before we show how can we found joint decreasing direction we need a criterion, to decide wether an $\mathbf{x}$ is a Pareto-optimal solution or not.

Definition 3.1 *Let $J(\mathbf{x}) \in \mathbb{R}^{k\times n}$ be the Jacobian-matrix of differentiable function F: $\mathbb{R}^n \to \mathbb{R}^k$ at $\mathbf{x} \in \mathbb{R}^n$ point. An $\mathbf{x}^*$ is called substationery point of F if there exist a $\mathbf{w} \in \mathcal{S}_k$ which fulfill the following equation:*

$$[J(\mathbf{x}^*)]^T\mathbf{w} = \mathbf{0}.$$

We are ready to discuss two models to find joint decreasing direction. The first model has been discussed in [30], as well and uses a quadratic programming problem formulation for computing joint decreasing direction. Later we show that joint decreasing direction can be computed in a simpler way using a special linear programming problem, too.

Let us define the following quadratic programming problem for every $\mathbf{x} \in \mathbb{R}^n$, with variable $\mathbf{w}$

$$\left.\min_{\mathbf{w}\in\mathcal{S}_k} \mathbf{w}^T \left(J(\mathbf{x})\,[J(\mathbf{x})]^T \right) \mathbf{w} \right\} \qquad (QOP(\mathbf{x})).$$

From the well known Weierstarss-theorem follows that this problem always has an optimal solution, since the feasible set is compact and the function

$$g : \mathcal{S}_k \to \mathbb{R}, \quad g(\mathbf{w}) = \mathbf{w}^T \left(J(\mathbf{x})\,[J(\mathbf{x})]^T \right) \mathbf{w}$$

is convex, quadratic, continuous function for any given $\mathbf{x} \in \mathbb{R}^n$.

Next theorem is an already known statement (see [30], Theorem 2.1) for which we give a new and shorter proof. This shows that using the $(QOP(\mathbf{x}))$ problem we can find a joint decreasing direction of F or a certificate that $\mathbf{x}$ is a Pareto-optimal solution of the $(UVOP)$ problem.

Theorem 3.2 *Let an $(UVOP)$, a point $\mathbf{x} \in \mathbb{R}^n$ and the associated $(QOP(\mathbf{x}))$ be given. Let $\mathbf{w}^* \in \mathbb{R}^k$ denote the optimal solution of $(QOP(\mathbf{x}))$. We define vector $\mathbf{q} \in \mathbb{R}^n$ as $\mathbf{q} = [J(\mathbf{x})]^T \mathbf{w}^*$. If $\mathbf{q} = \mathbf{0}$, then $\mathbf{x}$ is a substationery point, otherwise $-\mathbf{q}$ is a joint decreasing direction for F at point $\mathbf{x}$.*

Proof. When $\mathbf{q} = \mathbf{0}$ then Definition 3.1 shows that $\mathbf{x}$ is substationery point. When $\mathbf{q} \neq \mathbf{0}$, we indirectly assume that $-\mathbf{q}$ is not a decreasing direction for i-th coordinate function, f_i of F. It means that $[\nabla f_i(\mathbf{x})]^T \mathbf{q} < 0$. Since $[\nabla f_i(\mathbf{x})]^T = \mathbf{e}_i^T J(\mathbf{x})$, so our indirect assumption means

$$[\nabla f_i(\mathbf{x})]^T \mathbf{q} = \mathbf{e}_i^T [J(\mathbf{x})][J(\mathbf{x})]^T \mathbf{w}^* < 0.$$

We show that $\mathbf{e}_i - \mathbf{w}^* \neq \mathbf{0}$ is a feasible decreasing direction of $g(\mathbf{w}^*)$ which contradict the optimality of $\mathbf{w}^*$. The $\mathbf{e}_i = \mathbf{w}^*$ can not be fulfilled because it contradicts the indirect assumption, and it is easy to see, that $\mathbf{e}_i$ is a feasible solution of $(QOP(\mathbf{x}))$ so $\mathbf{e}_i - \mathbf{w}^*$ is a feasible direction at $\mathbf{w}^*$.
Since

$$\nabla g(\mathbf{w}) = 2[J(\mathbf{x})][J(\mathbf{x})]^T \mathbf{w}$$

thus

$$[\nabla g(\mathbf{w}^*)]^T (\mathbf{e}_i - \mathbf{w}^*) = 2\mathbf{w}^{*T} [J(\mathbf{x})][J(\mathbf{x})]^T (\mathbf{e}_i - \mathbf{w}^*) =$$
$$2\mathbf{w}^{*T} [J(\mathbf{x})][J(\mathbf{x})]^T \mathbf{e}_i - 2\mathbf{w}^{*T} [J(\mathbf{x})][J(\mathbf{x})]^T \mathbf{w}^* < 0,$$

where the first term of the sum is negative, because of the indirect assumption, and the second term is not positive, because $[J(\mathbf{x})][J(\mathbf{x})]^T$ is a positive semidefinite matrix.

□

Previous result underline the importance of solving $(QOP(\mathbf{x}))$ problem efficiently. For solving smaller size linearly constrained convex quadratic problems pivot algorithms [1, 4, 5, 6, 7, 19] can be used. In case of larger size linearly constrained, convex quadratic problems, interior point algorithms can be used to solve the problem (see for instance [14, 18]).

Theorem 3.2 shows that joint decreasing direction can be computed as the convex combination of the gradient vectors of coordinate functions of F. Following the idea discussed above, we can formulate a linear programming problem such that any optimal solution of the linear program defines a joint decreasing direction. Some similar result can be found [12].

Let we define the linear optimization problem: in the following way:

$$\left.\begin{aligned} \max q_0 \\ [J(\mathbf{x})]\mathbf{q} + q_0\mathbf{e} \le \mathbf{0} \\ 0 \le q_0 \le 1 \end{aligned}\right\} \qquad (LP(\mathbf{x}))$$

Now we are ready to state and prove a theorem that discuss a connection between $(UVOP)$ and $(LP(\mathbf{x}))$.

Theorem 3.3 *Let a point* $\mathbf{x} \in \mathbb{R}^n$*, a* $(UVOP)$ *and an associated* $(LP(\mathbf{x}))$ *be given. Then the* $(LP(\mathbf{x}))$ *always has an optimal solution* $(\mathbf{q}^*, q_0^*)$*. There are two cases for the optimal value of the* $(LP(\mathbf{x}))$*, either* $q_0 = 0$ *thus* $\mathbf{x}$ *is a Pareto-optimal solution of the* $(UVOP)$*, or* $q_0 = 1$ *thus* $\mathbf{q}^*$ *is a joint decreasing direction for the function* F *at* $\mathbf{x}$*.*

Proof. It is easy to see that $\mathbf{q} = \mathbf{0}$, $q_0 = 0$ is a feasible solution of the $(LP(\mathbf{x}))$ and 1 is an upper bound of the objective function, which means $(LP(\mathbf{x}))$ should have an optimal solution.
Let we examine the case

$$\begin{aligned} [J(\mathbf{x})]\mathbf{q} + q_0\mathbf{e} \le \mathbf{0}, \\ q_0 > 0. \end{aligned} \tag{1}$$

If system (1) has a solution, than $\left(\frac{1}{q_0}\mathbf{q}, 1\right)$ is a solution of the system, so the optimal value of the objective function is 1. This mean that

$$[J(\mathbf{x})]\mathbf{q} \le -\mathbf{e}$$

so the $\mathbf{q}$ is a joint decreasing direction of function F.
If the system (1) has no solution then the optimal value of the objective function is 0, and from the Farkas-lemma [8, 9, 15, 19, 26, 29] we know that there exists a $\mathbf{w}$ which satisfies the following:

$$\begin{aligned} [J(\mathbf{x})]^T\mathbf{w} = \mathbf{0} \\ \mathbf{e}^T\mathbf{w} = 1 \\ \mathbf{w} \ge \mathbf{0}. \end{aligned} \tag{2}$$

It means, that if the optimal value of the $(LP(\mathbf{x}))$ is 0, than $\mathbf{x}$ is a substationery point.

$\square$

Linear programming problem $(LP(\mathbf{x}))$ (and later on $(LPS(\mathbf{x})))$ can be solved by either pivot or interior point algorithms [16]. In case of applying pivot methods to solve linear programming problem, simplex algorithm is a natural choice [21, 26, 29]. A recent study on anti-cycling pivot rules for linear programming problem, contains a numerical study on different pivot algorithms [6]. Sometimes, if the problem is well structured and small, criss-cross algorithm of T. Terlaky can be used for solving linear programming problem, as well [15, 31]. More about interior point algorithms for linear programming problems can be learnt from [17, 21, 28].

4 Vector optimization with linear constraints

In this section we show how can we find feasible joint decreasing direction for a linearly constrained vector optimization problem. First we find the joint decreasing direction for a special problem, where we only have sign constraints on the variables. After this we generalize our results to general linearly constrained vector optimization problems. Our method can be considered as the generalization of the well known reduced gradient method to vector optimization problems. Some similar result can be found in [12], for feasible direction method of Zountendijk.

First we define vector optimization problem with sign constraints.

$$\left.\begin{array}{c} MIN\ F(\mathbf{x}) \\ \mathbf{x} \geq \mathbf{0} \end{array}\right\} \qquad (SVOP),$$

where F is a convex function. From Theorem 2.3 we know that $\mathbf{x}^* \geq \mathbf{0}$ is a Pareto-optimal solution if there exists a $\mathbf{w} \in \mathcal{S}_k$ vector such that $\mathbf{x}^*$ is an optimal solution of

$$\left.\begin{array}{c} \min\ \mathbf{w}^T F(\mathbf{x}) \\ \mathbf{x} \geq \mathbf{0} \end{array}\right\} \qquad (SWOP).$$

From KKT-theorem [21] we know that $\mathbf{x}^* \geq \mathbf{0}$ is an optimal solution of $(SWOP)$ if it satisfies the following system:

$$\begin{aligned} [J(\mathbf{x}^*)]^T \mathbf{w} &\geq \mathbf{0}, \\ \mathbf{w}^T [J(\mathbf{x}^*)] \mathbf{x}^* &= 0. \end{aligned} \tag{3}$$

Let the vector $\mathbf{x} \geq \mathbf{0}$ be given and we would like to decide wether it is an optimal solution of the (SWOP) problem or not. Let us define

$$I_+ = \{i : x_i > 0\},$$
$$I_0 = \{i : x_i = 0\}.$$

index sets that depends on the selected vector $\mathbf{x}$. Using the index set I_0, I_+ we partition the column vectors of matrix $J(\mathbf{x})$ into two parts. The two parts are denoted $J(\mathbf{x})_{I_0}$ and $J(\mathbf{x})_{I_+}$. Taking into consideration the partition, KKT conditions can be written in equivalent form as,

$$\begin{aligned} [J(\mathbf{x})]_{I_+}^T \mathbf{w} &= \mathbf{0}, \\ [J(\mathbf{x})]_{I_0}^T \mathbf{w} &\geq \mathbf{0}, \\ \mathbf{w} &\in \mathcal{S}_k. \end{aligned} \tag{4}$$

The inequality system (4) plays the same role for $(SVOP)$, as (2) for $(UVOP)$, namely $\mathbf{x}$ is a Pareto-optimal solution if (4) has a solution.

Now we can define a linear programming problem corresponding to $(SVOP)$ such that optimal solution of the linear programming problem either defines a joint decreasing direction or gives a certificate that the solution $\mathbf{x}$ is a Pareto-optimal solution of $(SVOP)$.

$$\left.\begin{array}{c} \max z \\ [J(\mathbf{x})]_{I_+}\mathbf{u} + [J(\mathbf{x})]_{I_0}\mathbf{v} + z\mathbf{e} \leq \mathbf{0} \\ \mathbf{v} \geq \mathbf{0} \\ 0 \leq z \leq 1 \end{array}\right\} \qquad (LPS(\mathbf{x}))$$

Now we are ready to prove the following theorem.

Theorem 4.1 *Let a $(SVOP)$, and an associated $(LPS(\mathbf{x}))$ be given where $\mathbf{x} \in \mathcal{F}$ is a feasible point. Then the $(LPS(\mathbf{x}))$ always has an optimal solution $(\mathbf{u}^*, \mathbf{v}^*, z^*)$. There are two cases for the optimal value of the $(LPS(\mathbf{x}))$, $z^* = 0$ which means that $\mathbf{x}$ is a Pareto-optimal solution of the $(SVOP)$, or $z^* = 1$ which means that $\mathbf{q}^T = (\mathbf{u}^*, \mathbf{v}^*)$ is a feasible joint decreasing direction of function F.*

Proof. It is easy to see that $\mathbf{u} = \mathbf{0}$, $\mathbf{v} = \mathbf{0}$, $z = 0$ is a feasible solution of the $(LPS(\mathbf{x}))$ and 1 is an upper bound of the objective function, therefore $(LPS(\mathbf{x}))$ has an optimal solution.

Let we examine the following system

$$\begin{aligned} [J(\mathbf{x})]_{I_+}\mathbf{u} + [J(\mathbf{x})]_{I_0}\mathbf{v} + z\mathbf{e} \leq \mathbf{0}, \\ \mathbf{v} \geq \mathbf{0} \\ z > 0. \end{aligned} \tag{5}$$

If system (5) has a solution, then $\left(\frac{1}{z}\mathbf{u}, \frac{1}{z}\mathbf{v}, 1\right)$ is an optimal solution of the $(LPS(\mathbf{x}))$ with optimal value 1. Thus the vector $\mathbf{q}^T = (\mathbf{u}, \mathbf{v})$ satisfies

$$[J(\mathbf{x})]\mathbf{q} \leq -\mathbf{e} < \mathbf{0},$$

so the $\mathbf{q}$ is a joint decreasing direction for function F at $\mathbf{x} \in \mathcal{F}$. Vector $\mathbf{q}$ is a feasible because $\mathbf{q}_{I_0} = \mathbf{v} \geq \mathbf{0}$.

If the system (5) has no solution then the optimal value of the objective function is 0, and from a variant of the Farkas-lemma [8, 9, 15, 19, 29] we know that there exists a $\mathbf{w}$ which satisfies the following system of inequalities:

$$\begin{aligned} [J(\mathbf{x})]_{I_+}^T\mathbf{w} = \mathbf{0} \\ [J(\mathbf{x})]_{I_0}^T\mathbf{w} \geq \mathbf{0} \\ \mathbf{e}^T\mathbf{w} = 1 \\ \mathbf{w} \geq \mathbf{0}. \end{aligned} \tag{6}$$

It means, that if the optimal value of the $(LPS(\mathbf{x}))$ is 0, than $\mathbf{x}$ is a Pareto-optimal solution of the (SVOP).

□

We are ready to find feasible joint decreasing direction to a generalized linearly constrained vector optimization problem at a feasible solution $\tilde{\mathbf{x}}$. Let the matrix $A \in \mathbb{R}^{m\times n}$ and vector $\mathbf{b} \in \mathbb{R}^m$ be given, where $rank(A) = m$. Furthermore let us assume the following non degeneracy assumption (for details see [2]): any m columns of A are linearly independent and every basic solution is non degenerate. We have a vector optimization problem, with linear constraint, in the following form

$$\left.\begin{aligned} \text{MIN } F(\mathbf{x}) \\ A\mathbf{x} = \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{aligned}\right\} \qquad (LVOP).$$

Like in the reduced gradient method [2] we can partition the matrix A into two parts $A = [B, N]$, where B is a basic and N the non-basic part of the matrix. Similarly every $\mathbf{v} \in \mathbb{R}^n$ vector can be partitioned as, $\mathbf{v} = [\mathbf{v}_B, \mathbf{v}_N]$. We call $\mathbf{v}_B$ basic and $\mathbf{v}_N$ a nonbasic vector. We can chose the matrix B, so that the $\tilde{\mathbf{x}}_B > 0$ fulfill. While $A\mathbf{x} = \mathbf{b}$ holds, we know that

$$\begin{aligned} B\mathbf{x}_B + N\mathbf{x}_N &= \mathbf{b} \\ \mathbf{x}_B &= B^{-1}(\mathbf{b} - N\mathbf{x}_N) \end{aligned}$$

We can redefine function F in a reduced form as

$$F_N(\mathbf{x}_N) = F(\mathbf{x}_B, \mathbf{x}_N) = F(B^{-1}(\mathbf{b} - N\mathbf{x}_N), \mathbf{x}_N).$$

Let we define at point $\tilde{\mathbf{x}}$ the following sign constraint optimization problem

$$\left.\begin{array}{c} \text{MIN } F_N(\mathbf{x}_N) \\ \mathbf{x}_N \geq \mathbf{0} \end{array}\right\} \qquad (SVOP(\tilde{\mathbf{x}}))$$

Let $\mathbf{q}_N$ denote a feasible joint decreasing direction for $(SVOP(\tilde{\mathbf{x}}))$ at point $\tilde{\mathbf{x}}_N$, which can be found by applying Theorem 4.1. Let $\mathbf{q}_B = -B^{-1}N\mathbf{q}_N$, then we show that $\mathbf{q} = [\mathbf{q}_B, \mathbf{q}_N]$ is feasible joint decreasing direction for $(LVOP)$ at point $\tilde{\mathbf{x}}$. Let we notice that

$$A(\tilde{\mathbf{x}} + h\mathbf{q}) = A\tilde{\mathbf{x}} + h(B\mathbf{q}_B + N\mathbf{q}_N) = \mathbf{b} + \big(-B(B^{-1}N\mathbf{q}_N) + N\mathbf{q}_N\big) = \mathbf{b},$$

for every $h \in \mathbb{R}$. So $F_N(\tilde{\mathbf{x}}+h\mathbf{q}_N) = F(\tilde{\mathbf{x}}+h\mathbf{q})$ and while $\mathbf{q}_N$ is a feasible joint decreasing direction with an $h_1 > 0$ stepsize for $(SVOP(\tilde{\mathbf{x}}))$, then $\mathbf{q}$ is a joint decreasing direction for $(LVOP)$ and with $h_1 > 0$ step size the $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x}_N \geq \mathbf{0}$ conditions are satisfied. While $\tilde{\mathbf{x}}_B > 0$ there exists $h_2 > 0$, $\tilde{\mathbf{x}}_B + h_2\mathbf{q}_B$, so $\mathbf{q}$ is a feasible joint decreasing direction for $(LVOP)$ with a step-size

$$h_3 = \min(h_1, h_2) > 0. \tag{7}$$

5 The subdivision algorithm for constrained vector optimization problem

In this section we show, how can we build a subdivision method to approximate the Pareto-optimal set of a vector optimization problem with linear constraints. Our method is a generalization of the algorithm discussed in [10], where you can find some result about convergence of the subdivision technique. The original method can not handle linear constraints.

Our algorithm approximate $\mathcal{F}^*$ with small sets which contain Pareto-optimal solution. The smaller the sets, the better approximation of the $\mathcal{F}^*$, therefore we define the following measure of sets involved in approximation of $\mathcal{F}^*$.

Definition 5.1 *Let an $\mathcal{H} \subseteq \mathbb{R}^n$ set be given, the diameter of $\mathcal{H}$ define as*

$$diam(\mathcal{H}) = \sup_{x,y \in \mathcal{H}} ||x - y||.$$

Let $\mathbb{H}$ be a family of set which contain finite number of sets from $\mathbb{R}^n$, then the diameter of $\mathbb{H}$ is

$$diam(\mathbb{H}) = \max_{\mathcal{H} \in \mathbb{H}} diam(\mathcal{H}).$$

Let we assume, that the feasible set of our problem is bounded. Then there exists

$$\mathcal{H}_0 = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{l} \leq \mathbf{x} \leq \mathbf{u}\},$$

where $\mathbf{l}, \mathbf{u} \in \mathbb{R}^n$ are given vectors and

$$\mathcal{F} \subseteq \mathcal{H}_0 \cap \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}\}.$$

The input of our method is a matrix $A \in \mathbb{R}^{m \times n}$, a vector $\mathbf{b} \in \mathbb{R}^m$, a function F: $\mathbb{R}^n \to \mathbb{R}^k$, set $\mathcal{H}_0$ and a constant $\varepsilon > 0$. The matrix A and the vector $\mathbf{b}$ define our feasible set $\mathcal{F}$ of $(LVOP)$, function F is our objective function. The output of our algorithm is a family of sets $\mathbb{H}$, such that $diam(\mathbb{H}) < \varepsilon$ and $\forall \mathcal{H} \in \mathbb{H}$ contains Pareto-optimal solution. The algorithm uses some other variables and subroutines. The $\mathcal{SP}$, $\mathcal{FP}$ are finite-element sets of points from $\mathbb{R}^n$, $\mathcal{H}, \mathcal{G} \subseteq \mathcal{F}$, $\mathbb{H}', \mathbb{K}, \mathbb{K}'$ and $\mathbb{A}$ are family of sets like $\mathbb{H}$.

vector_optimization_solver$(A, \mathbf{b}, F, \mathcal{H}_0, \varepsilon)$

1. $\mathbb{H} = \{\mathcal{H}_0\}$
2. **While** $diam(\mathbb{H}) \geq \varepsilon$ **do**
    (a) $\mathbb{H}'$=Newsets($\mathbb{H}$)
    (b) $\mathcal{SP} = \emptyset$

(c) **While** $\mathbb{H} \neq \emptyset$ **do**

   i. $\mathcal{H} \in \mathbb{H}$

   ii. $\mathcal{SP} = \mathcal{SP} \cup \text{Startpoint}(\mathcal{H})$

   iii. $\mathbb{H} = \mathbb{H} \setminus \{\mathcal{H}\}$

   End While

(d) $\mathcal{FP}$ =Points($\mathcal{SP}, A, \mathbf{b}, F$)

(e) **While** $\mathbb{H}' \neq \emptyset$ **do**

   i. $\mathcal{H} \in \mathbb{H}'$

   ii. **If** $\mathcal{H} \cap \mathcal{FP} \neq \emptyset$ **then** $\mathbb{H} = \mathbb{H} \cup \{\mathcal{H}\}$ **End If**

   iii. $\mathbb{H}' = \mathbb{H}' \setminus \{\mathcal{H}\}$

   End While

End While

3. Output($\mathbb{H}$)

Our algorithm in the first step defines the family of sets $\mathbb{H}$, which contains only $\mathcal{H}_0$ set. The cycle in step 2 runs while the diameter of $\mathbb{H}$ is not small enough. The algorithm reach this goal in finite number of iteration, because as you will see in subroutine Newset($\mathbb{H}$) the diameter of $\mathbb{H}$ tends to zero. Nevertheless we show that after every execution of the cycle the family of sets $\mathbb{H}$ contains sets $\mathcal{H}$ which has Pareto-optimal solutions. At the beginning it is trivial, because $\mathbb{H}$ contain the whole feasible set.

In step 2(a) we define a family of sets $\mathbb{H}'$ using the subroutine Newset($\mathbb{H}$). The sets from $\mathbb{H}'$ are smaller than sets form $\mathbb{H}$ and cover the same set. Therefore the result of this subroutine has two important properties:

1. $\cup_{\mathcal{H}\in\mathbb{H}'} (\mathcal{H} \cap \mathcal{F}) = \cup_{\mathcal{H}\in\mathbb{H}} (\mathcal{H} \cap \mathcal{F})$,

2. $diam(\mathbb{H}') = \frac{1}{K} diam(\mathbb{H})$,

where $K > 1$ is a constant.

The steps in cycle 2, from step 2(b), deletes the sets from $\mathbb{H}'$ which does not contain Pareto-optimal point. Step 2(b) makes set $\mathcal{SP}$ empty. The cycle in step 2(c) produces finite number of random starting points in set $\mathcal{H} \cap \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}\}$ for every $\mathcal{H} \in \mathbb{H}$ using subroutine Startpoint($\mathcal{H}$), and put the generated point into the set SP.

The main step of our algorithm 2(d) is the subroutine Points($\mathcal{SP}, A, \mathbf{b}, F$) that produce a set $\mathcal{FP}$ which contain Pareto-optimal points. This subroutine use our results from section 4.

In cycle 2(e) we keep every set form $\mathbb{H}'$ which contains Pareto-optimal solution, and add those to $\mathbb{H}$. Finally, we check the length of the diameter of $\mathbb{H}$ and repeat the cycle until the diameter is larger than the accuracy parameter ϵ.

Subroutine Points uses a version of reduced gradient method for computing Pareto-optimal solutions or joint decreasing directions, discussed in section 4.

Points($\mathcal{SP}$, A, $\mathbf{b}$, F)

1. **While** $\mathcal{SP} \neq \emptyset$ **do**
   (a) $\mathbf{s} \in \mathcal{SP}$
   (b) $\mathbf{x} = \mathbf{s}$, $z = 1$
   (c) **While** $z = 1$ **do**
      i. $(B, N) = A$
      ii. $(\mathbf{x}_B, \mathbf{x}_N) = \mathbf{x}$
      iii. $(\mathbf{q}, z)$=Solve($LPS(\mathbf{x}_N)$)
      iv. **If** $z = 1$ **then**
         A. h_3=stepsize(F, B, N, $\mathbf{b}$, $\mathbf{x}_N$, $\mathbf{q}$)
         B. $\mathbf{x}_N = \mathbf{x}_N + h_3\mathbf{q}$
         C. $\mathbf{x}_B = \mathbf{b} - B^{-1}N\mathbf{x}_N$
         D. $\mathbf{x} = (\mathbf{x}_B, \mathbf{x}_N)$
         End If
      End While
   (d) $\mathcal{SP} = \mathcal{SP} \setminus \{\mathbf{s}\}$
   (e) $\mathcal{FP} = \mathcal{FP} \cup \{\mathbf{x}\}$
   End While
2. Output($\mathcal{FP}$)

This subroutine works until it does not find a Pareto-optimal point from every starting points. The cycle 1(c) runs until it finds a Pareto-optimal point. As we see in section

4 it happens when $z = 0$. In line 1(c)i the matrix A is partitioned into a basis B and a non basic part N. The same partition is made with $\mathbf{x}$ according to 1(c)ii, and we choose the basis such that $\mathbf{x}_B > 0$ is satisfied. The $LCP(\mathbf{x}_N)$ is solved in step 1(c)iii. If the variable $z = 0$ than $\mathbf{x}$ is a Pareto-optimal solution and we select a new starting point from SP, unless SP is empty. Otherwise $\mathbf{q}$ is a feasible joint decreasing direction for the reduced function F_N. In step 1(c)ivB we compute step-size h_3 which was defined in (7), and a new feasible solution $\mathbf{x}$ is computed.

6 Markowitz-model and computational results

Let us illustrate our method by solving the Markowitz-model to find the most profitable and less risky portfolio. The standard way of solving the model is to find one of the Pareto-optimal solution with an associated (WOP) [23], [25]. The question is whether such single Pareto-optimal solution is what we need for practical purposes. Naturally, if we would like to make extra profit, we should accept larger risk. Therefore, a single Pareto-optimal solution does not contain enough information for making practical decision. If we produce or approximate Pareto-optimal solution set then we can make our decision based on more valuable information.

The analytical description of the whole Pareto-optimal set for the Markowitz-model is known [32]. Thus as a test problem, Markowitz-model has the following advantage: it is possible to derive its Pareto-optimal solution set in analytical way (for further details see, Vörös J. [32]), therefore the result of our subdivision algorithm can be compared with the analytical description of the Pareto-optimal solution set.

Now we are ready to formulate the original Markowitz-model. Let we assume, that we have to select from n different securities. Let x_i denote how much percentage we spend from our budget on security i $(i = 1, 2, \ldots, n)$. Therefore, our decision space is the n-dimensional unit simplex, S_n.

Let $\mathbf{a} \in \mathbb{R}^n$ denote the expected return of the securities, $C \in \mathbb{R}^{n\times n}$, denote the covariant matrix of the securities return. It is known, that the expected return of our Portfolio is equal to $\mathbf{a}^T\mathbf{x}$. One of our goal is to maximize the expected return.

Much harder to measure risk of the portfolio, but in this model it is equal to the variance of the securities return, namely by $\mathbf{x}^T C\mathbf{x}$. Our second goal is to minimize this value. Now we are ready to formulate our model

$$\left.\begin{array}{c} MIN \quad \begin{pmatrix} -\mathbf{a}^T\mathbf{x} \\ \mathbf{x}^T C\mathbf{x} \end{pmatrix} \\ \mathbf{x} \in \mathcal{S}_n \end{array}\right\} \quad (MM).$$

For computational purposes we used data from Budapest Stock Exchange [3], from spot market, A category shares, daily prices from 01. 09. 2010. to 01. 09. 2011 has been collected. Let $P_{i,d}$ denote the daily price of the i-th share on date d, then the i-th coordinate of the vector $\mathbf{a}$ is equal to $(P_{i,01.09.2011.} - P_{i,01.09.2010.})/P_{i,01.09.2010.}$. Thus we only work with the relative return from the price change and do not deal with shares dividend. We compute the daily return of the shares for every day (d) from 01. 09. 2010. to 31. 08. 2011. as $(P_{i,d} - P_{i,d+1})/P_{i,d}$, and C is the covariant matrix of this daily return. To illustrate our method we use three shares ($i =$ MOL, MTELEKOM, OTP) that are usually selected into portfolios because these shares correspond to large and stable Hungarian companies. We used the following data:

$$\mathbf{a} = \begin{pmatrix} -0,1906 \\ -0,2556 \\ -0,1665 \end{pmatrix}$$

$$C = 10^{-5} \begin{pmatrix} 27,1024 & 7,5655 & 17,1768 \\ 7,5655 & 16,4816 & 8,1816 \\ 17,1768 & 8,1816 & 34,2139 \end{pmatrix}$$

The input data of the vector_optimization_solver are: matrix $A = \mathbf{e} \in \mathbb{R}^3$, $\mathbf{b} = 1$ since we have a single constraint in our model, and the objective function $F(\mathbf{x}) = \begin{pmatrix} -\mathbf{a}^T\mathbf{x} \\ \mathbf{x}^T C\mathbf{x} \end{pmatrix}$. Let $\mathcal{H}_0 = \{\mathbf{0} \leq \mathbf{x} \leq \mathbf{e}\}$, $\epsilon = \frac{1}{2^6}$ and $K = 2$.

At the beginning of the algorithm in step 1 the family of set $\mathbb{H}$ has been defined (see Fig. 1).

At the first iteration of step 2 the method Newset define $\mathbb{H}'$ in two steps. First, it cuts the set $\mathcal{H}_0$ into eight equal pieces as you can see on Fig. 2:

After that we delete all those sets from $\mathbb{H}'$ that does not contain any point from the feasible set of the problem. Thus the $\mathbb{H}'$ is shown on Fig. 3. The main part of the algorithm starts at step 2(c). Two hundred random points are generated from the unit simplex (set $\mathcal{SP}$). For each generated point either a joint decreasing direction is computed and after that a corresponding Pareto-optimal solution has been identified

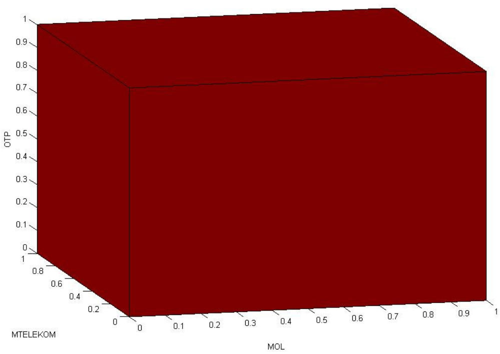

Figure 1

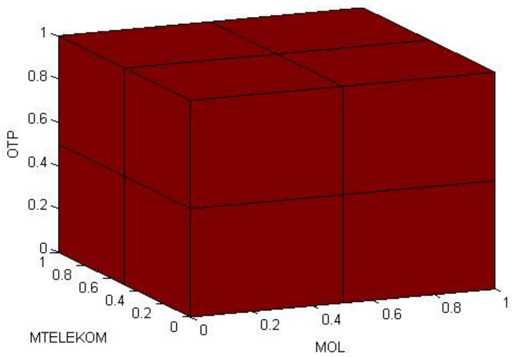

Figure 2

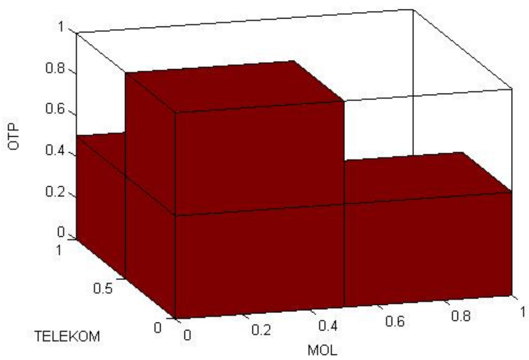

Figure 3

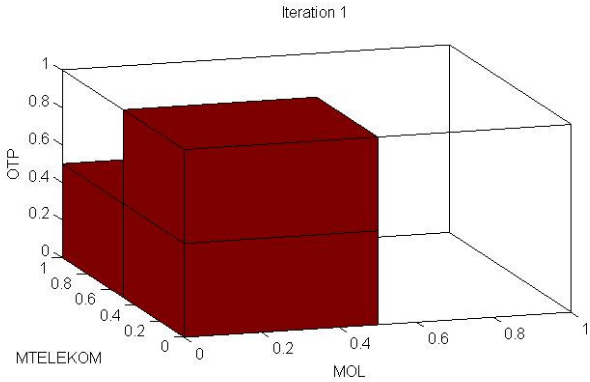

Figure 4

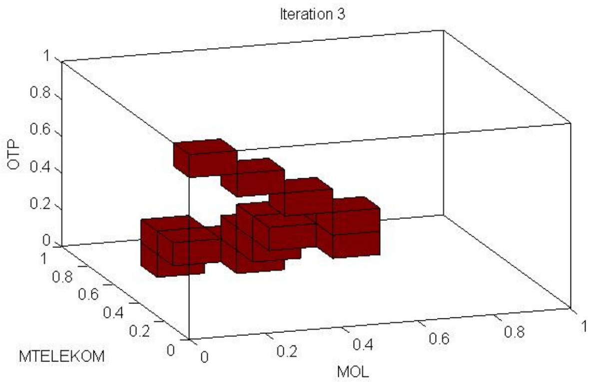

Figure 5

through some iteration or it has been shown that the generated point itself is a Parteo-optimal solution of the problem. After we obtained 200 Pareto optimal solutions in set $\mathcal{FP}$ at step $2(d)$ we delete those boxes that does not contain any point from $\mathcal{FP}$ at step $2(e)$. The result of the first iteration can be seen on Fig. 4. From the original eight boxes remained three. For these three boxes the procedure has been repeated in the second iteration. The results of iteration 3, 5 and 7 are illustrated on Fig. 5, Fig. 6, and Fig. 7, respectively.

These figures illustrate the flow of our computations. Finally to illustrate the convergence of our method the whole Pareto-optimal set was determined [32], and compared of the result of the fifth iteration on Fig. 8.

Summarizing our computations on the table 1 where I stands for the iteration number; B_{in} and B_{out} denotes the number of boxes at the beginning and at the end of iteration, respectively. Furthermore, $T(s)$ is the time of the I^{th} iteration, while d is the diameter of the set $\mathbb{H}$.

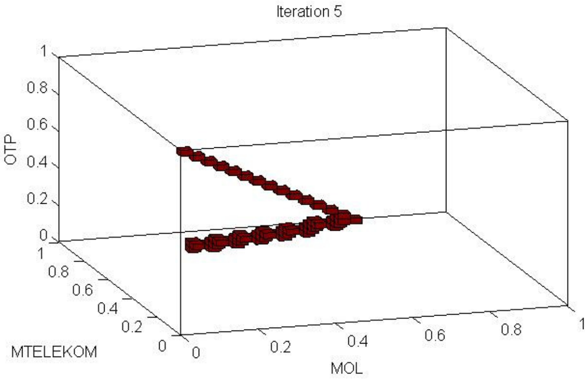

Figure 6

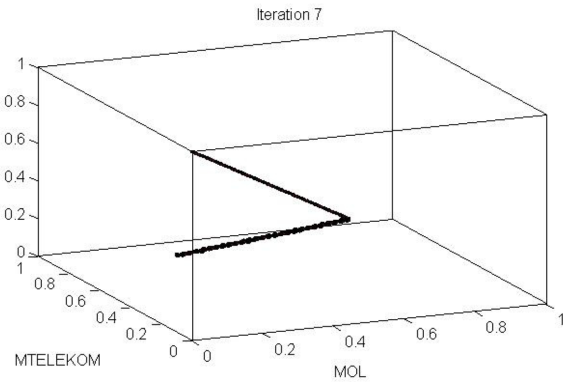

Figure 7

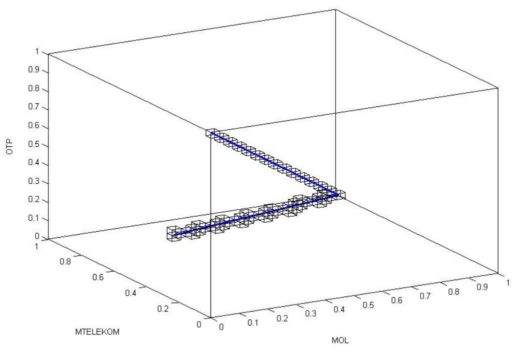

Figure 8

I	B_{in}	$T(s)$	B_{out}	d
1	1	8	3	2^{-3}
2	24	29	7	2^{-6}
3	56	70	15	2^{-9}
4	120	166	29	2^{-12}
5	232	312	56	2^{-15}
6	448	696	110	2^{-18}
7	880	1429	228	2^{-21}

Table 1: Computational results for Markowitz-model using subdivision method.

The total computational time, for our MATLAB implementation using a laptop with the following characteristics (processor: Intel(r) Core(TM) i3 [3.3 GHZ], RAM Memory: 4096 MB), took 2710 seconds for the subdivision algorithm for the given Markovitz-model to approximate the whole Pareto-optimal solution set with the accuracy $\varepsilon = 2.4\ 10^{-8}$.

Analyzing our approximation of the Pareto-optimal solution set, we can conclude that our option is to buy OTP shares only. From data, it can be understood, that this share has the biggest return (smallest loss in the financial crisis), so this solution represent the strategy, when someone does not care the risk only the return. From that point a line start which represents strategies related to portfolios based on OTP and MOL shares. Clearly, there exists a breaking point where new line starts. From the braking point the line lies in the interior of the simplex suggesting a portfolio based on all three selected shares.

7 Final remarks

In this paper we introduced the feasible joint decreasing direction for constrained vector optimization problems. We gave a new and elementary proof of a theorem from Schultz at al. [30] for finding joint decreasing direction for unconstrained multi objective problems. Based on our proof we developed a new method for finding joint decreasing direction for linearly constrained, convex vector optimization problems and defined a new, generalized subdivision algorithm, which outputs a numerical approximation of the whole Pareto-optimal set. Computational behavior of our method has been illustrated by numerical solving the Markowitz-model for a given data set.

The original subdivision technique [10] could not handle constraints and works only for convex functions. Some more general results for joint decreasing direction can be found in [12]. Based on our approach and ideas described by Flieg in [12, 13] further generalization of the subdivision method is possible for $(GVOP)$ with convex, compact set $\mathcal{F}$.

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