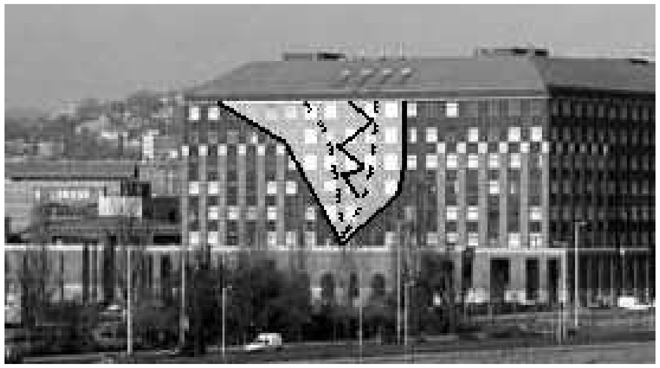
OPERATIONS RESEARCH
REPORT 2015-02



Strongly polynomial primal monotonic build-up simplex algorithm for maximal flow problems

Tibor Illés, Richárd Molnár-Szipai

September 2015

Eötvös Loránd University of Sciences
Department of Operations Research

ISSN 1215 - 5918

Strongly polynomial primal monotonic build-up simplex algorithm for maximal flow problems

Tibor Illés, Richárd Molnár-Szipai

Abstract

The maximum flow problem (MFP) is a fundamental model in operations research. The network simplex algorithm is one of the most efficient solution methods for MFP in practice. The theoretical properties of established pivot algorithms for MFP is less understood. Variants of the primal simplex and dual simplex methods for MFP have been proven strongly polynomial, but no similar result exists for other pivot algorithms like the monotonic build-up or the criss-cross simplex algorithm.

The monotonic build-up simplex algorithm (MBU SA) starts with a feasible solution, and fixes the dual feasibility one variable a time, temporarily losing primal feasibility. In the case of maximum flow problems, pivots in one such iteration are all dual degenerate, bar the last one. Using a labelling technique to break these ties we show a variant that solves the maximum flow problem in $2|V||A|^2$ pivots.

1 Introduction

The maximal flow problem is one of the basic models in network optimization that already has a wide range of applications from railway optimization [13], [18] to corresponding minimum cut problems in computer vision [12]. It is not surprising then, that it has been studied extensively, with numerous solution methods developed.

The first substantial results are due to Ford and Fulkerson [8], including the maximum flow – minimum cut theorem, and the idea of augmenting path

algorithms. Later Edmonds and Karp [7], and independently Dinic [6] proved that the shortest augmenting path algorithm is strongly polynomial.

Another family of algorithms use so-called preflows. A preflow is a flow except that each intermediate node are allowed to have more inflow than outflow (but not the other way around). The first such algorithm by Karzanov [15] used preflows only to solve the subproblems in Dinic's algorithm, so the conservation equations are restored at the end of each phase. Later preflow algorithms took a more holistic approach, where the preflow becomes a flow only at the final step (see e.g. Goldberg and Tarjan [9]). This phenomenon is similar to how primal feasibility is restored only at the last pivot of a dual simplex algorithm, while the augmenting path algorithms are more like the primal simplex algorithm in that they proceed through feasible solutions.

In fact, the maximum flow problem is a special linear programming problem, therefore it can be solved using pivot algorithms. Indeed, the description of the network simplex algorithm appears as far back as Dantzig's book on linear programming [5]. The first published strongly polynomial pivot algorithm for a network optimization problem is a dual simplex algorithm for the minimum cost flow problem due to Orlin [16], while the first primal simplex variant (specifically for the maximum flow problem) is by Goldfarb and Hao [10, 11]. However, pivot algorithms that traverse bases that are neither primal nor dual feasible received less attention, no such algorithm has been proven strongly polynomial so far. We show that the primal monotonic build-up simplex algorithm [2] has a strongly polynomial variant for the maximum flow problem.

The structure of the article is as follows: in the next section we describe the MBU SA for linear programming in general, and how it works on maximum flow problems in particular. In the third section we describe our variant in detail. In the fourth section we prove the polynomiality of this algorithm, and then we conclude the article with some remarks and possible future research directions.

2 Preliminaries

The reader is expected to have a basic understanding of the simplex method of linear programming (e.g. basic solutions, primal and dual feasibility), and of the maximum flow problem. For reference, see [1, 5, 8].

Consider a linear programming problem in the following form:

$$\begin{aligned}\max \mathbf{c}^T\mathbf{x}\\ A\mathbf{x} = \mathbf{b}\\ \mathbf{x} \geq \mathbf{0}\end{aligned}$$

Where $A \in \mathbb{R}^{n\times m}$, $\mathbf{x}, \mathbf{c} \in \mathbb{R}^m$, $\mathbf{b} \in \mathbb{R}^n$. Without loss of generality we may assume that A has full row rank. Let B be an $n \times n$ invertible submatrix of A, I_B the set of column indices of B, and I_N the remaining column indices. Then the solution $\mathbf{x}_B = B^{-1}\mathbf{b}$, $\mathbf{x}_N = \mathbf{0}$ is a so-called basic solution, with the corresponding simplex tableau:

$$\begin{array}{|c|c|}\hline \overline{A} & \overline{\mathbf{b}} \\ \hline \overline{\mathbf{c}}^T & -z_0 \\ \hline \end{array} := \begin{array}{|c|c|}\hline B^{-1}A & B^{-1}\mathbf{b} \\ \hline \mathbf{c}^T - \mathbf{c}_B^T B^{-1}A & -\mathbf{c}_B^T B^{-1}\mathbf{b} \\ \hline \end{array}$$

Using the notations $\overline{a}_{i,j}$, $\overline{b}_i$ and $\overline{c}_j$ for the elements of the current tableau, the *primal MBU SA* is as follows:

1. Start with a primal feasible basic solution
2. Choose a dual infeasible variable x_{p^*}. We refer to x_{p^*} as the driving variable [2]. If there are none, stop, we have an optimal solution.
3. Select the leaving variable using a minimum ratio test on feasible basic variables:
$$q = \arg\min\left\{\frac{\overline{b}_q}{\overline{a}_{q,p^*}} : \overline{a}_{q,p^*} > 0, \overline{b}_q \geq 0\right\}$$
Let $\vartheta_1 = |\overline{c}_{p^*}|/\overline{a}_{q,p^*}$.
4. Choose the entering variable using a minimum ratio test on dual feasible nonbasic variables:
$$p = \arg\min\left\{\frac{\overline{c}_p}{|\overline{a}_{q,p}|} : \overline{c}_p \geq 0, \overline{a}_{q,p} < 0\right\}$$
Let $\vartheta_2 = \overline{c}_p/|\overline{a}_{q,p}|$.
5. Variable x_q leaves the basis. If $\vartheta_2 < \vartheta_1$, then x_p enters the basis and go to step 3, otherwise x_{p^*} enters the basis, and go to step 2.

The name of the algorithm comes from the property that dual feasible variables do not lose their feasibility (due to step 4), and thus the set of these variables monotonically build up. The fact that we might select the variable x_p instead of x_{p^*} on whose column we took the minimum ratio test means that we can lose primal feasibility. However, when x_{p^*} finally enters the basis, primal feasibility is restored (see [2] for details).

Let $G = (V, E)$ be a connected directed graph with two distinguished nodes s and t, the source and the sink, respectively. Given lower and upper bounds on the arcs and obeying conservation of flow at intermediate nodes, we wish to maximize the amount of flow from s to t. Introducing an arc from t to s, the maximum flow problem can be stated as the following linear program:

$$\begin{aligned}&\max x_{t,s}\\ \forall v \in V: \quad &\sum_{(w,v)\in E} x_{w,v} - \sum_{(v,w)\in E} x_{v,w} = 0\\ \forall e \in E: \quad &l_e \leq x_e \leq u_e\end{aligned}$$

Throughout the article we are using the following notations:

- v, w and z for nodes of a graph
- e for an arc of a graph
- p and q for the entering and leaving arc of a graph
- $p^* = (g, h)$ as the arc corresponding to the driving variable

Now let us break down how the primal monotonic build-up simplex algorithm works on maximum flow problems step by step.

Step 1: Start from a primal feasible basic solution. The basic variables correspond to the arcs of a spanning tree T containing (t, s), with the non-basic arcs having flow values of either the lower or the upper bound. The basic variables are then uniquely determined by the conservation equations. If the lower bounds are zero, then $\mathbf{x} = \mathbf{0}$ is such a feasible basic solution with an arbitrary spanning tree. Otherwise finding such a starting solution is not trivial, one can do so by transforming the network and solving another (zero lower bounds) maximum flow problem (see e.g. [1] section 6.2), which corresponds to solving the first phase of a two phase linear programming problem.

Another way is to use the feasibility MBUSA [14] (which is a specialization of [4]). Having zero or nonzero lower bounds do not affect the algorithm otherwise.

Step 2: Choose a dual infeasible variable. Let $T^i \subset E$ be the spanning tree before the ith pivot. Dropping (t,s) from T^i disconnects the spanning tree, with one subset containing s, and the other one containing t. These vertex sets are denoted by S^i and Z^i respectively. The reduced cost $\overline{\mathbf{c}}_e$ of a nonbasic arc e is:

$$\overline{\mathbf{c}}_e = \begin{cases} 1 & \text{if } e : S^i \to Z^i \text{ and } x_e = u_e \text{ or } e : Z^i \to S^i \text{ and } x_e = l_e \\ -1 & \text{if } e : S^i \to Z^i \text{ and } x_e = l_e \text{ or } e : Z^i \to S^i \text{ and } x_e = u_e \\ 0 & \text{otherwise: } e : S^i \to S^i \text{ or } e : Z^i \to Z^i \end{cases}$$

Without loss of generality we can assume that the *driving variable* $p^* = (g,h) \in E$ is on its lower bound, i.e. $g \in S^i$ and $h \in Z^i$.

Step 3: Choose the leaving variable with a primal ratio test. $T^i \cup (g,h)$ contains a unique cycle C^i, consider it directed according to (g,h). Then the leaving arc $q \in C^i$ is an arc where

$$\delta = \min \left\{ \begin{array}{ll} u_q - x_q : & \text{q is forward, } x_q \leq u_q \text{; or} \\ x_q - l_q : & \text{q is backward, } x_q \geq l_q \end{array} \right\}$$

takes its value. (We are examining how much we could augment along the cycle C_i, not counting arcs that are already infeasible in the appropriate direction.) Note that such an arc q is uniquely determined if the bounds are sufficiently diverse, so we will choose arbitrarily, should a tie occur.

Step 4: Choose the entering variable with a dual minimum ratio test. The simplex tableau, along with the reduced costs is totally unimodular, so this ratio test can result in either a 0, or a 1 quotient. As $\vartheta_1 = 1$ the only instance when we do not let p^* enter the basis is if we find an arc p with ratio 0, i.e. with reduced cost 0, and $a_{q,p} = -1$. As we've seen, $\overline{\mathbf{c}}_p = 0$ means that p is either $S^i \to S^i$ or $Z^i \to Z^i$, and $\vartheta_2 < \vartheta_1$ means that performing the pivot (g,h) for q would make p dual infeasible. In the case of $q \in S^i$, dropping q from the tree would disconnect s and g, suitable entering variables would either be arcs from the subtree of s to the subtree of g on their lower bounds, or arcs the other way around on their upper bounds.

3 Description of the algorithm

To describe the algorithm we will consider the subtrees S^i and Z^i to be rooted at g and h respectively. Picturing the tree with the root at the top, and using

the usual notions of "parent" and "child" node, T^i_v will denote the subtree in T^i spanning the node v and its descendants. Using this image, we will also refer to the nodes of a basic arc as the "upper" and "lower" nodes. For a basic arc $q \in T^i$ we will use the notation T^i_q for the subtree "below" q, that is, T^i_v, where v is the lower node of q (we will use this notation only in the context of the leaving variable q).

We will use the concept of a *pseudo-augmenting path* (PAP) as defined in [11]: a pseudo-augmenting path from v to w with respect to a basic solution $\mathbf{x}$, and spanning tree T is a directed path from v to w that can use (v_1, v_2) if

- $(v_1, v_2) \in E \backslash T^i$ and $x_{v_1,v_2} = l_{v_1,v_2}$,
- or $(v_2, v_1) \in E \backslash T^i$ and $x_{v_2,v_1} = u_{v_2,v_1}$,
- or if either $(v_1, v_2) \in T^i$ or $(v_2, v_1) \in T^i$.

Note that nonbasic arcs are used the same way as with classical augmenting path algorithms, the addition of using basic arcs in any direction makes the notion compatible with the basis structure.

The *label* $d^i(v)$ of vertex v before pivot i with respect to the current driving variable (g,h) and basis structure T^i is the length of the shortest PAP from h to v within T^i_h if $v \in T^i_h$, and the length of the shortest PAP from v to g within T^i_g if $v \in T^i_g$.

Using these labels to choose the entering variable results in the following variant of the *primal monotonic build-up simplex algorithm*:

0. Start with a primal feasible basic solution $\mathbf{x}$.
1. Let (g,h) be an arbitrary dual infeasible arc. If no such arc exists, then the current solution is optimal.
2. Let q be an arc limiting further augmentation along the cycle in $T^i \cup (g,h)$.
3. If there is a possible entering arc between T^i_q and the rest of T^i_g or T^i_h (whichever q is in)
    - then let p be such an arc with minimal label in $E \backslash T^i_q$, and perform a pivot with p entering and q leaving the basis. Go to step 2.
    - else let (g,h) enter the basis with q leaving. Go to step 1.

4 Proving Strong Polynomiality

First, we need to note that it is sufficient to establish the polynomiality of making a single (g, h) dual infeasible arc feasible. This statement relies on the primal MBU SA having the property that a dual feasible variable never becomes infeasible during the algorithm, and so the number of outer cycles is bound by the number of dual infeasible variables in the initial basic solution.

To bound the number of pivots needed to get the driving variable into the basis, we use a few features of the labelling technique. First, we prove that the label of every node is monotonically non-decreasing ("monotonicity lemma"). This tells us that the algorithm is progressing in a certain sense. The lemma's appropriate version appears in both [11] and [3]. As the label of a node represents the length of a shortest path, it is bounded by the number of vertices. After this lemma we show that not only the labels do not decrease, but strict increases must happen regularly. Therefore an upper bound on the number of pivots can be derived.

LEMMA [. Monotonicity lemma] Assume that a MFP is solved by Algorithm 3. For any $v \in V$ and iteration i during a single outer cycle $d^{i+1}(v) \geq d^i(v)$ holds.

Proof: Assume indirectly that there exists z and i such that $d^{i+1}(z) < d^i(z)$. We can assume that z is a counterexample with minimal $d^{i+1}(z)$ label.

As $d^{i+1}(z) < d^i(z)$, the shortest PAP from z to g after the ith pivot must use an arc in a direction that was not available before. Let us take a look at how the arcs change with respect to labelling:

- arcs that are in the basis both before and after the pivot ($T_g^i \cap T_g^{i+1}$) can be used in both directions for calculating $d^i(z)$ and $d^{i+1}(z)$,
- arcs not in either basis ($E \backslash (T_g^i \cup T_g^{i+1})$) didn't have their flow value changed, so they can be used in the same directions both before and after the pivot,
- the entering arc p was usable in one direction before entering the basis, and is usable in both directions after the pivot,
- conversely, the leaving arc q was usable in both directions before the pivot, and in only one direction after it.

Let $v \in T_q^i$ and $w \notin T_q^i$ be the two vertices of p, the only new possibility for labelling after the pivot is using p in the $w \to v$ direction, so the new shortest PAP from z must use that.

As $v \in T_q^i$, this PAP must leave T_q^i after using p via some p' arc, with $v' \in T_q^i$ and $w' \notin T_q^i$ its two vertices. Note that $p' \neq q$, because after leaving the basis q can only be used from its vertex not in T_q^i. Therefore $p' \notin T^i$, and could have been used to leave T_q^i, so it was a candidate for entering the basis at the ith pivot. However, we chose p over p', so $d^i(w) \leq d^i(w')$ must hold.

Therefore the shortest PAP from z to g first uses the shortest PAP from z to w, uses (w, v), uses the shortest PAP from v to v', uses (v', w'), and finally the shortest PAP from w' to z. Denoting the shortest PAP from v_1 to v_2 before pivot k by $d^k(v_1, v_2)$, $d^{i+1}(z)$ can be written as:

$$\begin{aligned} d^{i+1}(z) &= d^{i+1}(z, w) + 1 + d^{i+1}(v, v') + 1 + d^{i+1}(w') \\ &\geq d^i(z, w) + d^i(w') + 2 \geq d^i(z, w) + d^i(w) + 2 \geq d^i(z) + 2. \end{aligned}$$

Where we used:

- $d^{i+1}(z, w) \geq d^i(w, z)$, as the shortest PAP from z to w can not use p in the $w \to v$ direction.
- $d^{i+1}(v', v) \geq 0$.
- $d^{i+1}(w') \geq d^i(w')$, as otherwise w' would be a counterexample to the lemma, with $d^{i+1}(w') < d^{i+1}(z)$, contradicting the minimality of z.
- $d^i(w') \geq d^i(w)$ from the choice of p as the entering variable (see above).
- $d^i(w) + d^i(w, z) \geq d^i(z)$ is a triangle inequality for PAPs.

We assumed indirectly that $d^{i+1}(z) < d^i(z)$, but we concluded $d^{i+1}(z) \geq d^i(z) + 2$, a contradiction. ■

To proceed, we will show that for every arc $p = (p_v, p_w)$ the sum of its labels $d(p_v) + d(p_w)$ must increase between subsequent enterings into the basis ("Main lemma"). This approach is similar to that in [3], but due to not having a dual feasible basis, the acquired inequalities are somewhat weaker.

The proof of this lemma is split into two cases according to whether p leaves the basis having the same direction that it had when entering or not. First we will prove an inequality that helps us in the first case.

LEMMA [. Subtree lemma] Assume that a MFP is solved by Algorithm 3. If (v, w) entered the basis at the ith pivot, with w being the upper vertex, and this remains true throughout, even after the jth pivot, then for all $z \in T_v^{j+1}$: $d^{j+1}(z) \geq d^i(w) + 1$.

Proof: Case $j = i$. Note that $T_v^{i+1} = T_q^i$. Take a shortest reverse pseudo-augmenting path from g to z. As $z \in T_v^{i+1}$, this path must contain an arc leading into T_q^i. This arc can not be q, as it left the basis on the wrong bound for that.

If it is p, then $d^{i+1}(z) \geq d^{i+1}(w) + 1 \geq d^i(w) + 1$.

Otherwise that arc could have entered the basis at pivot i, but we chose p instead, so $d^i(w') \geq d^i(w)$ for its $w' \notin T_q^i$ vertex. Then $d^{i+1}(z) \geq d^{i+1}(w') + 1 \geq d^i(w') + 1 \geq d^i(w) + 1$.

This finishes the proof for $j = i$.

For $j > i$ we use induction, so let us assume that the lemma is true for $j - 1$, and let $z \in T_v^{j+1}$.

If $z \in T_v^j$ as well, then monotonicity and the induction hypothesis gives $d^{j+1}(z) \geq d^j(z) \geq d^i(w) + 1$.

Otherwise, z entered T_v during the jth pivot. Let the entering arc of that pivot be p with $p_v \in T_v^j$ and $p_w \notin T_v^j$ vertices. Then $d^{j+1}(z) \geq d^j(p_v) + 1$ using the $i = j$ case of this lemma for p, and $d^j(p_v) \geq d^i(w) + 1$ by the induction hypothesis, giving $d^{j+1}(z) \geq d^i(w) + 2$. This completes the proof. ■

The next lemma shows that if p changes direction since entering the basis, then a strict increase in his labels must already have happened.

LEMMA [. Reversal lemma] Assume that a MFP is solved by Algorithm 3. If (v, w) entered the basis at the ith pivot, with w being the upper vertex, this remains true throughout, but changes to v being upside with the jth pivot, then $d^{j+1}(w) \geq d^i(w) + 1$.

Proof: We claim that the following inequalities hold:

$$d^{j+1}(w) \geq d^j(p_w) + 1 \geq d^j(p_v) \geq d^i(w) + 1$$

Let the entering arc at pivot j be p with $p_w \notin T_v^j$ and $p_v \in T_v^j$ vertices. The leaving arc q must be on the path in the spanning tree from w to g for (v, w) to change directions. This also means that $w \in T_{p_w}^{j+1}$, so by the subtree lemma we have $d^{j+1}(w) \geq d^j(p_w) + 1$.

As p was a candidate for entering, it could be used for labelling from the p_w end, which means $d^j(p_v) \leq d^j(p_w) + 1$.

Finally, $p_v \in T_v^j$, so using the subtree lemma we get $d^j(p_v) \geq d^i(w) + 1$. ■

Now we are ready to state our main lemma, describing the growth behavior of the labels.

LEMMA [. Main lemma] Assume that a MFP is solved by Algorithm 3. If (v, w) entered the basis at pivot i, left it at pivot j, and entered it again at pivot j, then $d^{k+1}(v) + d^{k+1}(w) \geq d^i(v) + d^i(w) + 2$ holds.

Proof: Without loss of generality we might assume that w is the upper vertex of (v, w) in T_g^{i+1}.

Case a: w is the upper vertex in T_g^j as well. We claim that

$$d^{k+1}(v) + d^{k+1}(w) \geq 2d^k(v) + 1 \geq 2d^{i+1}(v) + 1 \geq d^i(v) + d^i(w) + 2$$

After leaving a base (v, w) can be used for labelling only from its v end, therefore v will be the upper vertex after pivot k. According to the subtree lemma $d^{k+1}(w) \geq d^k(v) + 1$, and using $d^{k+1}(v) \geq d^k(v)$ we get the first inequality.

The second inequality is the monotonicity lemma.

In the third inequality we bound one of the $d^{i+1}(v)$ with the subtree lemma: $d^{i+1}(v) \geq d^i(w) + 1$, and the other one with monotonicity: $d^{i+1}(v) \geq d^i(v)$.

Case b: v is the upper vertex in T_g^j. We explain

$$d^{k+1}(v) + d^{k+1}(w) \geq 2d^k(w) + 1 \geq 2d^{l+1}(w) + 1 \geq 2d^i(w) + 3 \geq d^i(v) + d^w + 2$$

where l is the first pivot when the direction of (v, w) changes in the spanning tree ($i < l < j < k$).

As v is the upper vertex when leaving the basis, w is the upper vertex after pivot k, and we get the first inequality by using the subtree lemma and monotonicity similar to the previous case.

The second inequality is the monotonicity lemma.

The third inequality is the reversal lemma: $d^{l+1}(w) \geq d^i(w) + 1$.

In the fourth inequality we use that before pivot i we could use (v, w) for labelling from the side of w, therefore $d^i(v) \leq d^i(w) + 1$. ■

Finally, we deduce the strong polinomiality of the algorithm from the previous lemma.

Theorem 4.1. *Algorithm 3 solves a MFP in at most* $2nm^2$ *pivots.*

Proof: The number of dual infeasible arcs at the start of the algorithm is less than m. As the primal MBU simplex algorithm does not create new dual infeasible arcs, the inner cycle can happen at most m times. Let us then examine the number of pivots it takes to "fix" an infeasible arc.

For any (v, w) arc we have $1 \leq d(v) + d(w) \leq 2n - 5$ (we have 2 vertices with label 0, so the maximum label is $n - 2$, and there can be at most one such vertex). By the monotonicity lemma $d(v) + d(w)$ is not decreasing, and by the previous lemma it increases by at least 2 if it enters the basis twice. Therefore (v, w) can enter the basis at most $2n - 5$ times. As every pivot has an entering arc, we can thus have at most $2nm$ pivots, even counting the final pivot that lets the dual infeasible arc enter the basis. ■

5 Conclusions and further directions

Building upon the techniques used for proving the polynomiality of certain variants of the primal and dual simplex algorithms [11, 3] on the maximum flow problem, we have shown that the primal MBU simplex algorithm also has such a strongly polynomial variant. This variant has an interesting structure: the algorithm makes at most m dual nondegenerate steps, each two separated by at most $2nm$ dual degenerate steps. The corresponding flow becomes primal feasible after every nondegenerate step, but this property may not hold in between them.

It remains an open problem if similar results can be reached with other non-primal, non-dual pivot algorithms, such as the dual MBU simplex algorithm, exterior point simplex algorithms [17], or criss-cross type algorithms [19].

References

[1] R. K. Ahuja, T. L. Magnanti, and J. B. Orlin. *Network Flows*. Prentice Hall, New Jersey, 1993.

[2] K. M. Anstreicher and T. Terlaky. A monotonic build-up simplex algorithm for linear programming. *Operations Research*, 42:556–561, 1994.

[3] R. D. Armstrong, W. Chen, D. Goldfarb, and Z. Jin. Strongly polynomial dual simplex methods for the maximum flow problem. *Mathematical Programming*, 80:17–33, 1998.

[4] F. Bilen, Zs. Csizmadia, and T. Illés. Anstreicher–terlaky type monotonic simplex algorithms for linear feasibility problems. *Optimisation Methods and Software*, 22(4):679–695, 2007.

[5] G. B. Dantzig. *Linear Programming and Extensions*. Princeton University Press, Princeton, NJ., 1963.

[6] E. A. Dinic. Algorithm for solution of a problem of maximum flow in networks with power estimation. *Soviet Math. Doklady*, (11):1277–1280, 1970.

[7] J. Edmonds and R. M. Karp. Theoretical improvements in algorithmic efficiency for network flow problems. *Journal of ACM*, 19:248–264, 1972.

[8] L. R. Ford and D. R. Fulkerson. *Flows in Networks*. Princeton University Press, Princeton, NJ., 1962.

[9] A. V. Goldberg and R. E. Tarjan. A new approach to the maximum flow problem. In *Proceedings of the 18th ACM Symposium on the Theory of Computing*, pages 136–146, 1986.

[10] D. Goldfarb and J. Hao. A primal simplex algorithm that solves the maximum flow problem in at most nm pivots and $\mathcal{O}(n^2m)$ time. *Mathematical Programming*, 47:353–365, 1990.

[11] D. Goldfarb and J. Hao. On strongly polynomial variants of the network simplex algorithm for the maximum flow problem. *Operations Research Letters*, 10:383–387, 1991.

[12] D. M. Greig, B. T. Porteous, and A. H. Seheult. Exact maximum a posteriori estimation for binary images. *Journal of the Royal Statistical Society Series B*, 51:271–279, 1989.

[13] T. Illés, M. Makai, and Zs. Vaik. Combinatorial optimization model for railway engine assignment problem. In L. G. Kroon and R. H. Möhring, editors, *Proceedings of the 5th Workshop on Algorithmic Methods and Models for Optimization of Railways*, 2006.

[14] T. Illés and R. Molnár-Szipai. On strongly polynomial variants of the MBU-simplex algorithm for a maximum flow problem with non-zero lower bounds. *Optimization*, 63:39–47, 2014.

[15] A. V. Karzanov. Nakhozhdenie maksimal'nogo potoka v seti metodom predpotokov ("determining the maximal flow in a network by the method of preflows"). *Doklady Akademii Nauk SSSR*, 215(1):49–52, 1974.

[16] J. B. Orlin. Genuinely polynomial simplex and non-simplex algorithms for the minimum cost flow problem. Technical report, Technical Report 1615-84, Sloan School of Management, MIT, Cambridge, MA., 1984.

[17] K. Paparizzos, N. Samaras, and A. Sifaleras. Exterior point simplex-type algorithms for linear and network optimization problems. *Annals of Operations Research*, 229(1):607–633, 2015.

[18] F. Piu and M. G. Speranza. The locomotive assignment problem: a survey on optimization models. *International Transactions in Operational Research*, 21(3):327–352, 2014.

[19] T. Terlaky and S. Zhang. Pivot rules for linear programming: a survey on recent theoretical developments. *Annals of Operations Research*, 46(1):203–233, 1993.