Strathprints Institutional Repository

Mahonian STAT on words

Sergey KITAEV

Department of Computer and Information Sciences, University of Strathclyde
Livingstone Tower, 26 Richmond Street, Glasgow G1 1XH, UK
sergey.kitaev@cis.strath.ac.uk

Vincent VAJNOVSZKI

LE2I, Université de Bourgogne
BP 47870, 21078 Dijon Cedex, France
vvajnov@u-bourgogne.fr

August 19, 2014

Abstract

In 2000, Babson and Steingrímsson introduced the notion of what is now known as a permutation vincular pattern, and based on it they re-defined known Mahonian statistics and introduced new ones, proving or conjecturing their Mahonity. These conjectures were proved by Foata and Zeilberger in 2001, and by Foata and Randrianarivony in 2006.

In 2010, Burstein refined some of these results by giving a bijection between permutations with a fixed value for the major index and those with the same value for STAT, where STAT is one of the statistics defined and proved to be Mahonian in the 2000 Babson and Steingrímsson's paper. Several other statistics are preserved as well by Burstein's bijection.

At the Formal Power Series and Algebraic Combinatorics Conference (FPSAC) in 2010, Burstein asked whether his bijection has other interesting properties. In this paper, we not only show that Burstein's bijection preserves the Eulerian statistic ides, but also use this fact, along with the bijection itself, to prove Mahonity of the statistic STAT on words we introduce in this paper. The words statistic STAT introduced by us here addresses a natural question on existence of a Mahonian words analogue of STAT on permutations. While proving Mahonity of our STAT on words, we prove a more general joint equidistribution result involving two six-tuples of statistics on (dense) words, where Burstein's bijection plays an important role.

1 Introduction

In [1], the notion of what is now known as a *vincular pattern*[1] on permutations was introduced, and it was shown that almost all known *Mahonian* permutation statistics (that is, those statistics that are distributed as INV or as MAJ to be defined in Section 2) can be expressed as combinations of vincular patterns. The authors of [1] also introduced some new vincular pattern-based permutation statistics, showing that some of them are Mahonian and conjecturing that others are Mahonian as well. These conjectures were proved later in [4, 5], and recently, alternative proofs based on *Lehmer code transforms* were given in [8].

---

[1]Such patterns are called *generalized patterns* in [1].

Three statistics expressed in terms of vincular pattern combinations in [1] (namely, MAK, MAD and DEN) are known to be Mahonian not only on permutations, but also on words (see [3, Theorem 5]); more precisely, for any word v, the three statistics are distributed as INV on the set of rearrangements of the letters of v.

One of the statistics defined and shown to be Mahonian in [1] is STAT. Generalizing a result in [4], Burstein [2] shown the equidistribution of STAT and MAJ together with other statistics by means of an involution p on the set of permutations. At the Formal Power Series and Algebraic Combinatorics Conference (FPSAC) in 2010, Burstein asked whether p has other interesting properties.

In this paper, we not only show that p preserves the Eulerian statistic ides (which is not preserved, e.g. by the bijection Φ on words [3] mapping MAD to INV), but also use this fact, along with p itself, to prove Mahonity of the statistic STAT on words introduced in Subsection 2.2 (see relation (3)). The words statistic STAT introduced by us in this paper addresses a natural question on existence of a Mahonian words analogue of STAT on permutations. While proving Mahonity of our STAT on words, we prove a more general joint equidistribution result involving two six-tuples of statistics on (dense) words, where the bijection p plays an important role (see Theorems 1 and 2 in Section 5).

2 Preliminaries

We denote by $[n]$ the set $\{1, 2, \ldots, n\}$, by $\mathfrak{S}_n$ the set of permutations of $[n]$, and by $[q]^n$ the set of length n words over the alphabet $[q]$. Clearly $\mathfrak{S}_n \subset [q]^n$ for $q \geq n > 1$. A word v in $[q]^n$ is said to be *dense* if each letter in $[q]$ occurs at least once in v. Dense words are also called *multi-permutations*.

2.1 Statistics

A *statistic* on $[q]^n$ (and thus on $\mathfrak{S}_n$) is an association of an integer to each word in $[q]^n$. Classical examples of statistics are:

$$\text{INV}\, v = \operatorname{card}\{(i,j) \ : \ 1 \leq i < j \leq n, v_i > v_j\},$$

$$\text{MAJ}\, v = \sum_{\substack{1 \leq i < n \\ v_i > v_{i+1}}} i,$$

$$\text{des}\, v = \operatorname{card}\{i \ : \ 1 \leq i < n, v_i > v_{i+1}\},$$

where $v = v_1 v_2 \ldots v_n$ is a length n word. For example, $\text{INV}(31425) = 3$, $\text{MAJ}(3314452) = 8$, and $\text{des}(8416422) = 4$.

For a word v and a letter a in v, other than the largest one in v, let us denote by $next_v(a)$ the smallest letter in v larger than a. With this notation, we define

$$\text{ides}\, v = \operatorname{card}\{a : \text{ there are } i \text{ and } j, 1 \leq i < j \leq n, \text{ with } v_i = next_v(a) \text{ and } v_j = a\}.$$

Clearly, when v is a permutation, ides v is simply des v^{-1}, where v^{-1} is the inverse of v. For example, $\text{ides}(144625) = 2$, and the corresponding values for a are 2 and 5.

For a set of words S, two statistics ST and ST' have the same distribution (or are equidistributed) on S if, for any k,

$$\operatorname{card}\{v \in S : \mathsf{ST}\, v = k\} = \operatorname{card}\{v \in S : \mathsf{ST}'\, v = k\},$$

and it is well-known that INV and MAJ have the same distribution on both, the set of permutations and that of words.

A *multi-statistic* is simply a tuple of statistics.

2.2 Vincular patterns

Let $1 \le r \le q$ and $1 \le m \le n$, and let $v \in [r]^m$ be a dense word. One says that v occurs as a (classical) pattern in $w = w_1 w_2 \cdots w_n \in [q]^n$ if there is a sequence $1 \le i_1 < i_2 < \cdots < i_m \le n$ such that $w_{i_1} w_{i_2} \cdots w_{i_m}$ is order-isomorphic to v. For example, 1231 occurs as a pattern in 6214562, and the three occurrences of it are 2452, 2462 and 2562.

Vincular patterns were introduced in the context of permutations in [1] and they were extensively studied since then (see Chapter 7 in [6] for a comprehensive description of results on these patterns). Vincular patterns generalize classical patterns and they are defined as follows:

- Any pair of two adjacent letters may now be underlined, which means that the corresponding letters in the permutation must be adjacent[2]. For example, the pattern $2\underline{13}$ occurs in the permutation 425163 four times, namely, as the subsequences 425, 416, 216 and 516. Note that, the subsequences 426 and 213 are *not* occurrences of the pattern because their last two letters are not adjacent in the permutation.

- If a pattern begins (resp., ends) with a hook[3] then its occurrence is required to begin (resp., end) with the leftmost (resp., rightmost) letter in the permutation. For example, there are two occurrences of the pattern $\lfloor 2\underline{13}$ in the permutation 425163, which are the subsequences 425 and 416.

The notion of a vincular pattern is naturally extended to words. For example, in the word 6214562, 645 is an occurrence of the pattern $3\underline{12}$, and 262 is that of $1\underline{21}\rfloor$.

For a set of patterns $\{p_1, p_2, \ldots\}$ we denote by $(p_1 + p_2 + \ldots)$ the statistic giving the total number of occurrences of the patterns in a permutation. It follows from definitions that

$$\mathsf{MAJ}\, v = (1\underline{32} + 1\underline{21} + 2\underline{31} + 2\underline{21} + 3\underline{21} + \underline{21})\, v. \tag{1}$$

A vincular pattern of the form $u\underline{vx}$, with $\{u, v, x\} = \{1, 2, 3\}$, is determined by the relative order of u, v and x. For example, $2\underline{13}$ is determined by $v < u < x$, and $3\underline{21}$ by $x < v < u$.

An *extension* of a vincular pattern $u\underline{vx}$, $\{u, v, x\} = \{1, 2, 3\}$, is the combination of the vincular patterns obtained by replacing an order relation involving u (possibly both of them if there are two) by its (their) weak counterpart. For example,

- the unique extension of $1\underline{32}$ is $(1\underline{32} + 1\underline{21})$; and

- the three extensions of $2\underline{31}$ are:

---

[2] The original notation for vincular patterns uses dashes: the absence of a dash between two letters of a pattern means that these letters are adjacent in the permutation.

[3] In the original notation the role of hooks was played by square brackets.

$$(2\underline{31} + 1\underline{21}),$$
$$(2\underline{31} + 2\underline{21}), \text{ and}$$
$$(2\underline{31} + 1\underline{21} + 2\underline{21}).$$

An extension of a vincular pattern $\underline{uvx}$ is defined similarly, and an extension of $(p_1 + p_2 + \ldots)$ is the statistic obtained by extending some of p_i's.

With these notations, the definition of MAJ in (1) is an extension of MAJ defined on $\mathfrak{S}_n$:

$$\text{MAJ}\, v = (1\underline{32} + 2\underline{31} + 3\underline{21} + \underline{21})\, v. \tag{2}$$

The statistic STAT on permutations was introduced and shown to be Mahonian in [1], i.e. distributed as MAJ; STAT is defined as:

$$\text{STAT}\, \pi = (\underline{21}3 + \underline{13}2 + \underline{32}1 + \underline{21})\, \pi.$$

An extension of STAT to words, where repeated letters are allowed, is to extend:

- $\underline{21}3$ as $(\underline{21}3 + \underline{21}2)$, and

- $\underline{13}2$ as $(\underline{13}2 + \underline{12}1)$,

and thus to define the statistic STAT on words as:

$$\text{STAT}\, v = (\underline{21}3 + \underline{21}2 + \underline{13}2 + \underline{12}1 + \underline{32}1 + \underline{21})\, v. \tag{3}$$

In what follows, we will use this definition which seems to be sporadic and not any better than any other possible extension of STAT from permutations to words. However, a consequence of Theorem 2 is that this extension has the same distribution as MAJ on words, and experimental tests show that no other extension (in the sense specified above) does so.

3 The bijection p on $\mathfrak{S}_n$

Now we present the involution on $\mathfrak{S}_n$ introduced in [2] which maps a permutation with a given value for MAJ to one with the same value for STAT, and show that among other statistics, it preserves ides.

For three integers $a \le x \le b$, the *complement* of x with respect to the interval $\{a, a+1, \ldots, b\}$ is simply the integer $b - (x - a)$.

For a $\pi = \pi_1 \pi_2 \ldots \pi_n \in \mathfrak{S}_n$, let us define

- $\pi' \in \mathfrak{S}_n$ by $\pi'_1 = \pi_1$, and for $i \ge 2$, π'_i is the complement of π_i with respect to

    - $\{\pi_1 + 1, \pi_1 + 2, \ldots, n\}$, if $\pi_i > \pi_1$, and
    - $\{1, 2, \ldots, \pi_1 - 1\}$, if $\pi_i < \pi_1$;

- $\pi'' \in \mathfrak{S}_n$ by $\pi''_1 = \pi'_1 = \pi_1$ and $\pi''_i = \pi'_{n-i+2}$.

Clearly, the map $\pi \mapsto \pi''$ is a bijection on $\mathfrak{S}_n$. In fact, p is an involution, that is $p(p(\pi)) = \pi$. See Figure 1 for an example.

Also, in [2] is proved that, for any $\pi = \pi_1 \pi_2 \ldots \pi_n \in \mathfrak{S}_n$, the 5-tuple $(\mathsf{adj}, \mathsf{des}, \mathsf{F}, \mathsf{MAJ}, \mathsf{STAT})\, \pi$ is equal to $(\mathsf{adj}, \mathsf{des}, \mathsf{F}, \mathsf{STAT}, \mathsf{MAJ})\, p(\pi)$, where
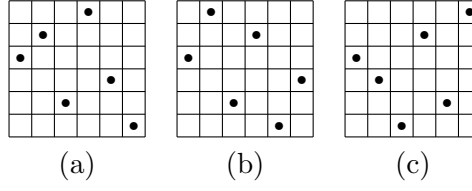
Figure 1: The permutations: (a) $\pi = 452631$, (b) $\pi' = 462513$, and (c) $\pi'' = p(\pi) = 431526$.

- $\mathsf{adj}\,\pi = \mathrm{card}\,\{i\ :\ 1 \le i \le n, \pi'_i = \pi'_{i+1} + 1\}$, where $\pi' = \pi 0$, and

- $\mathsf{F}\,\pi = \pi_1$.

Below, we will use the following result.

Lemma 1. *For any $\pi = \pi_1\pi_2\ldots\pi_n \in \mathfrak{S}_n$, $\mathsf{ides}\,\pi = \mathsf{ides}\,p(\pi)$.*

Proof. An integer a is an occurrence of an ides in π if there are $i < j$ such that $\pi_i = a + 1$ and $\pi_j = a$. Clearly, if $\pi_1 > 1$, then $\pi_1 - 1$ is an occurrence of an ides in both π and $\sigma = p(\pi)$. And $a \ne \pi_1 - 1$ is an occurrence of an ides in π if and only if so is the element in position $n - \pi^{-1}(a + 1) + 2$ in σ, where $\pi^{-1}(a + 1)$ is the position of the element $a + 1$ in π. $\qquad\square$

The following lemma, to be used later, follows directly from the proof of Lemma 1.

Lemma 2. *The number of ides in the interval $\{1, 2, \ldots, \pi_1 - 1\}$ is the same for π and $p(\pi)$.*

4 Interval partitions

In this section, we define the notions of interval partitions of sets, permutations and words. We also define the notion of a word expansion.

4.1 Interval partition of a set

An *interval partition of a set* $\{1, 2, \ldots, n\} = [n]$ is a partition of this set, where each part is an interval (i.e., a set consisting of successive integers), and the *size* of an interval partition is the number of its parts. For example, $\{\{1, 2, 3\}, \{4, 5\}, \{6, 7\}\}$ is an interval partition of size 3 of the set $[7]$.

For two interval partitions R and P of $[n]$, we say that R is a *refinement* of P, denoted by $R \subseteq P$, if each part of R is a weak subset of a part of P. In particular, P is a refinement of itself. For example, $R = \{\{1, 2\}, \{3\}, \{4, 5\}, \{6\}, \{7\}\}$ is a refinement of $P = \{\{1, 2, 3\}, \{4, 5\}, \{6, 7\}\}$.

Note that any refinement R of size $k + j$ of an interval partition P of size k can be encoded by an increasing sequence of j numbers. For the last example, R can be encoded by (2,4) because when creating the refinement, we scanned P from left to right and have broken parts in the second and forth possible places. For the same interval partition $P = \{\{1, 2, 3\}, \{4, 5\}, \{6, 7\}\}$, the encoding (1,3,4) would give the refinement $R = \{\{1\}, \{2, 3\}, \{4\}, \{5\}, \{6\}, \{7\}\}$. In general, for a partition of size k of $\{1, 2, \ldots, n\}$, we have $n - k$ possibilities to break a part and one possibility not to break anything. Thus, breaking parts, which gives refinements, can be encoded uniquely by a possibly empty subsequence of increasing integers in $\{1, 2, \ldots, n - k\}$.

Let I_n denote the set of all interval partitions of $[n]$, and for $P \in I_n$, we let

$$I_n|P = \{R \in I_n : R \subseteq P\}.$$

Now, for two same size interval partitions $P, S \in I_n$, we define a map

$$\psi_{P,S} : I_n|P \to I_n|S,$$

which sends a refinement R in $I_n|P$ to the refinement T in $I_n|S$ such that R and T have the same encodings. It is straightforward to see that $\psi_{P,S}$ is a bijection, and its inverse is $\psi_{S,P}$.

Example 1. If $R = \{\{1\}, \{2,3\}, \{4\}, \{5,6\}\}$, $P = \{\{1\}, \{2,3\}, \{4,5,6\}\}$ and $S = \{\{1,2\}, \{3\}, \{4,5,6\}\}$, then $T = \psi_{P,S}(R) = \{\{1,2\}, \{3\}, \{4\}, \{5,6\}\}$.

4.2 Interval partition of permutations

The *interval partition of a permutation* $\pi \in \mathfrak{S}_n$, denoted $\mathsf{ppart}(\pi)$, is the interval partition of $[n]$ defined by: a and $a+1$ belong to the same part of $\mathsf{ppart}(\pi)$ if and only if a occurs to the left of $a+1$ in π. Thus, the partition of a permutation is given by its maximal increasing subpermutations of consecutive elements. For example, if $\sigma = 14235$ and $\pi = 45123$, then $\mathsf{ppart}(\pi) = \mathsf{ppart}(\sigma) = \{\{1,2,3\}, \{4,5\}\}$.

Since an ides in π is a value a such that $a+1$ occurs to the left of a in π, it follows that the size of $\mathsf{ppart}(\pi)$ is equal to $\mathsf{ides}\,\pi + 1$, and the next corollary is a consequence of Lemma 1.

Corollary 1. *For any $\pi \in \mathfrak{S}_n$, the interval partitions of π and that of $p(\pi)$ have the same size.*

4.3 Interval partition of words

The *interval partition of a word* $v \in [q]^n$, denoted by $\mathsf{wpart}(v)$, is the interval partition

$$\{p_1, p_2, \ldots, p_q\}$$

of $[n]$ where the cardinality of each part p_i is equal to the number of occurrences of the symbol $i \in [q]$ in v, and empty parts, if any, are omitted. Formally, p_i is given by

$$p_i = \{a+1, a+2, \ldots, b\},$$

with

$$a = |v|_1 + |v|_2 + \cdots + |v|_{i-1}, \text{ and } b = a + |v|_i,$$

and the number of occurrences of each letter in v determines $\mathsf{wpart}(v)$. For example, if $v = 12112$ and $w = 33111$, then $\mathsf{wpart}(v) = \mathsf{wpart}(w) = \{\{1,2,3\}, \{4,5\}\}$. In particular, when v is a permutation in $\mathfrak{S}_n$, $\mathsf{wpart}(v) = \{\{1\}, \{2\}, \ldots, \{n\}\}$. See also Figure 2 for other examples.

4.4 Words expansion

For $v \in [q]^n$, the *expansion* of v, denoted $\mathsf{exp}(v)$, is the unique permutation $\pi \in \mathfrak{S}_n$ with $\pi_i < \pi_j$ if and only if either $v_i < v_j$, or $v_i = v_j$ and $i < j$. In particular, if v is a permutation, then $\mathsf{exp}(v) = v$. For example, $\mathsf{exp}(12112) = 14235$ and $\mathsf{exp}(22111) = 45123$. We refer to Figure 2 for some other examples. The following fact is easy to check.
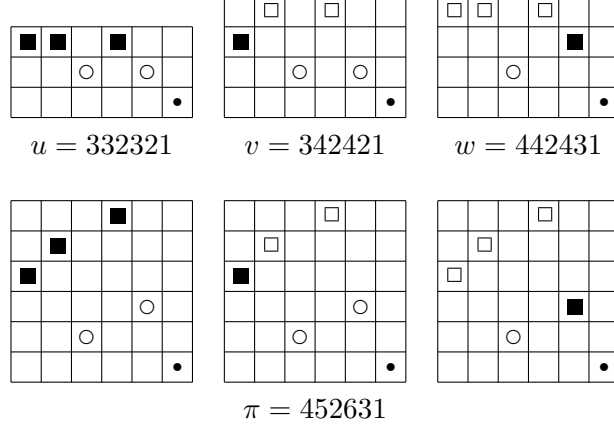
Figure 2: The permutation $\pi = 452631 \in \mathfrak{S}_6$ in Figure 1(a) is the expansion of each of u, v and w. We have that $\mathsf{wpart}(u) = \{\{1\}, \{2,3\}, \{4,5,6\}\}$, $\mathsf{wpart}(v) = \{\{1\}, \{2,3\}, \{4\}, \{5,6\}\}$ and $\mathsf{wpart}(w) = \{\{1\}, \{2\}, \{3\}, \{4,5,6\}\}$; also, $\mathsf{ppart}(\pi) = \{\{1\}, \{2,3\}, \{4,5,6\}\}$.

Fact 1. If v is a dense word in $[q]^n$ and $\pi = \mathsf{exp}(v)$, then $\mathsf{wpart}(v)$ is a refinement of $\mathsf{ppart}(\pi)$.

Actually, exp is a function from $[q]^n$ to $\mathfrak{S}_n$, which is surjective if $q \geq n$, but not injective (again, see Figure 2). However, one can see that the following fact holds.

Fact 2. The dense word v is uniquely determined from $\mathsf{exp}(v)$ and $\mathsf{wpart}(v)$.

If P is a refinement of $\mathsf{ppart}(\pi)$, we denote by $\mathsf{flat}_P(\pi)$ the unique word v with $\mathsf{exp}(v) = \pi$ and $\mathsf{wpart}(v) = P$, and so $\mathsf{exp}(\mathsf{flat}_P(\pi)) = \pi$. Also, we will use the following fact which follows from the definitions of MAJ and STAT given in relations (1) and (3).

Fact 3. For any word v, we have $\mathsf{MAJ}\, v = \mathsf{MAJ}\,(\mathsf{exp}(v))$ and $\mathsf{STAT}\, v = \mathsf{STAT}\,(\mathsf{exp}(v))$.

5   Extension of p to words

In this section, we show that the statistic STAT on words defined by us in Section 2 is equidistributed with the statistic MAJ on words, and thus our STAT is Mahonian. In fact, we show a more general result on joint equidistribution of six statistics on words: see Theorem 1 for the case of dense words, and Theorem 2 for the case of arbitrary words.

To this end, we extend the bijection p from permutations to words, which is roughly done by the following three steps: For a word v, we apply the expansion operation (exp defined above) in order to obtain a permutation π, then apply the bijection p on permutations to obtain $\sigma = p(\pi)$, and, finally, apply the inverse of the expansion operation to σ. The resulting word w is the image of v by the extension of p to words. The main difficulty consists in the third step, since with no additional constraints, the expansion operation is not invertible. The main ingredient to overcome this, is the bijection ψ defined in Section 4.1, which works due to a consequence of Lemma 1 expressed in Corollary 1.

5.1 Dense words

Here we will extend the bijection $p : \mathfrak{S}_n \to \mathfrak{S}_n$ to length n dense words over $[q]$. For a dense word v we construct a dense word w, and show that the transformation $v \mapsto w$ is a bijection which preserving certain properties of p.

Let v be a dense word in $[q]^n$, $R = \mathsf{wpart}(v)$, and π and σ be the permutations defined by:

- $\pi = \mathsf{exp}(v)$ with $P = \mathsf{ppart}(\pi)$, and

- $\sigma = p(\pi)$ with $S = \mathsf{ppart}(\sigma)$.

Now let $w = \mathsf{flat}_T(\sigma)$ with $T = \psi_{P,S}(R)$.

Clearly, when v is a permutation, then $w = \sigma = p(\pi)$, and so the restriction of the mapping $v \mapsto w$ to permutations is equal to p, and by a slight abuse of notation we denote this mapping by p.

Example 2. Let $v = 342421$ as in Figure 2, with $R = \mathsf{wpart}(v) = \{\{1\}, \{2, 3\}, \{4\}, \{5, 6\}\}$. Then

- $\pi = \mathsf{exp}(v) = 452631$ (see also Figure 1(a)) and $P = \mathsf{ppart}(\pi) = \{\{1\}, \{2, 3\}, \{4, 5, 6\}\}$;

- $\sigma = p(\pi) = 431526$ is the permutation in Figure 1(c), and $S = \mathsf{ppart}(\sigma) = \{\{1, 2\}, \{3\}, \{4, 5, 6\}\}$.

With the previous values, $T = \psi_{P,S}(R) = \{\{1, 2\}, \{3\}, \{4\}, \{5, 6\}\}$ (see Example 1) and $p(v) = w = \mathsf{flat}_T(\sigma) = 321414$. It is routine to check that $\mathsf{STAT}\, v = \mathsf{MAJ}\, w = 7$, and $\mathsf{STAT}\, w = \mathsf{MAJ}\, v = 11$. Notice that, w is not a rearrangement of v.

The following theorem is a words counterpart of Theorem 2.1 in [2] endowed with ides statistic. In that theorem, adj is extended to dense words as follows: for a word v, $\mathsf{adj}\, v$ is the number of positions i, $1 \leq i \leq n$, in the word $v' = v0$ such that $v'_i = v'_{i+1} + 1$, and i is the leftmost position where the letter v'_i occurs in v', while $i + 1$ is the rightmost position where the letter v'_{i+1} occurs in v'.

Theorem 1. *The function p is a bijection from length n dense words over $[q]$ into itself, and the 6-tuple $(\mathsf{adj}, \mathsf{des}, \mathsf{ides}, \mathsf{F}, \mathsf{MAJ}, \mathsf{STAT})\, v$ is equal to $(\mathsf{adj}, \mathsf{des}, \mathsf{ides}, \mathsf{F}, \mathsf{STAT}, \mathsf{MAJ})\, p(v)$, for any dense word $v \in [q]^n$.*

Proof. First, since $p : \mathfrak{S}_n \to \mathfrak{S}_n$ is an involution, and the inverse of $\psi_{P,S} : I_n|P \to I_n|S$ is $\psi_{S,P} : I_n|S \to I_n|P$, it follows that $p(p(v)) = v$ for any dense words in $[q]^n$, and so p is an involution (and thus a bijection).

Let now v be a dense word in $[q]^n$, $w = p(v)$, $\pi = \mathsf{exp}(v)$ and $\sigma = p(\pi)$, as in the definition of the transformation p on words. It follows that $\sigma = \mathsf{exp}(w)$, and by Fact 3, that $\mathsf{STAT}\, v = \mathsf{STAT}\, \pi = \mathsf{MAJ}\, \sigma = \mathsf{MAJ}\, w$. Also, since p is an involution, we have $\mathsf{MAJ}\, v = \mathsf{STAT}\, w$.

In the word v, v_i is a descent if and only if π_i is a descent in π, and analogously for w and σ. Since p preserves the number of descents on permutations, so it does on words.

Similarly, in the word v, v_i is an occurrence of an ides if and only if π_i is an occurrence of an ides in π, and analogously for w and σ. By Lemma 1, p preserves the number of ides's on permutations, and thus so does on words.

The proof is similar for $\mathsf{adj}\, v = \mathsf{adj}\, p(v)$.

By the definition of ppart and the construction of p, $\mathsf{ppart}\, \pi$ and $\mathsf{ppart}\, \sigma$ are both refinements of $\{\{1, 2, \ldots, \pi_1 - 1\}, \{\pi_1, \ldots, n\}\}$. In addition, by Lemma 2, the number of 'sub-parts' of

$\{1, 2, \ldots, \pi_1 - 1\}$ of $\mathsf{ppart}\,\pi$ and of $\mathsf{ppart}\,\sigma$ are the same. Since $\mathsf{wpart}\,v$ is a refinement of $\mathsf{ppart}\,\pi$ having the same encoding as the refinement $\mathsf{wpart}\,w$ of $\mathsf{ppart}\,\sigma$, it follows that the number of 'sub-parts' of $\{1, 2, \ldots, \pi_1 - 1\}$ of $\mathsf{wpart}\,v$ and of $\mathsf{wpart}\,w$ are the same, and so $v_1 = w_1$, that is $\mathsf{F}\,v = \mathsf{F}\,w$. $\qquad\square$

5.2 General words

For a word $v = v_1 v_2 \ldots v_n \in [q]^n$, we let $\mathsf{red}(v)$ denote the word obtained from v in which the ith smallest letter in v is substituted by i. For example, $\mathsf{red}(162414) = 142313$.

Clearly, the function red produces a dense word and it is a bijection between the set of words over the alphabet from which v is constructed and the set of dense words of the same length as that of v. Thus, to find the pre-image of $\mathsf{red}(v)$, we need to know the alphabet from which v is constructed.

Since red and p are bijections and red preserves the order on $[q]$, we have the following generalization of Theorem 1, where by a slight abuse of notation, we denote by p the function $\mathsf{red}^{-1} \circ p \circ \mathsf{red}$, where red^{-1} uses the alphabet of the input word. Also, in the following theorem, we slightly abuse notation to denote by adj the composition $\mathsf{adj} \circ \mathsf{red}$. That is, to calculate the value of statistic adj on a given word $v \in [q]^n$, one should first turn v into the dense word $\mathsf{red}(v)$, and then calculate the value of adj using the definition state right before Theorem 1.

Theorem 2. *The function p is a bijection from $[q]^n$ into itself, and the 6-tuple*

$$(\mathsf{adj}, \mathsf{des}, \mathsf{ides}, \mathsf{F}, \mathsf{MAJ}, \mathsf{STAT})\,v$$

is the same as $(\mathsf{adj}, \mathsf{des}, \mathsf{ides}, \mathsf{F}, \mathsf{STAT}, \mathsf{MAJ})\,p(v)$, *for any word $v \in [q]^n$.*

6 Final remarks

It is worth mentioning that our bijection p does not preserve the number of occurrences of letters, while our computer experiments made us believe that such a bijection exists, and we invite the reader to find it. Also, it would be of interest to explore the property of being a Mahonian statistic on words for other Mahonian statistics on permutations defined in [1].

Finally, a C implementations of the bijection p is on the web site of the second author [9].

Acknowledgments

References

[1] E. Babson, E. Steingrímsson, Generalized permutation petterns and a classification of Mahonian statistics, *Sém. Lothar. Combin.* (electronic), **44** (2000).

[2] A. Burstein, On joint distribution of adjacencies, descents and some Mahonian statistics, *Discrete Math. Theor. Comp. Sci.*, proc. AN, 2010, 601-612. (Also *FPSAC* 2010, San Francisco, USA.)

[3] R.J. Clarke, E. Steingrímsson, J. Zeng, New Euler-Mahonian Statistics on Permutations and Words, *Adv. Appl. Math.*, **18** (1997), 237–270;

[4] D. Foata, D. Zeilberger, Babson-Steingrímsson statistics are indeed Mahonian (and sometimes even Euler-Mahonian), *Adv. Appl. Math.*, **27** (2001), 390–404.

[5] D. Foata, A. Randrianarivony, Two oiseau decompositions of permutations and their application to Eulerian calculus, *European J. Combin.*, **27** (2006), 342–363.

[6] S. Kitaev, Patterns in permutations and words, Springer-Verlag, 2011.

[7] D.H. Lehmer, Teaching combinatorial tricks to a computer, in *Proc. Sympos. Appl. Math.*, **10** (1960), Amer. Math. Soc., 179–193.

[8] V. Vajnovszki, Lehmer code transforms and Mahonian statistics on permutations, *Discrete Math.*, **313** (2013), 581–589.

[9] V. Vajnovszki, `v.vincent.u-bourgogne.fr/OABS/publi.html`.