Decomposing labeled interval orders as pairs of permutations

Anders Claesson Stuart A. Hannah

Computer and Information Sciences
University of Strathclyde
Glasgow, Scotland

{anders.claesson,stuart.a.hannah}@strath.ac.uk

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Abstract

We introduce ballot matrices, a signed combinatorial structure whose definition naturally follows from the generating function for labeled interval orders. A sign reversing involution on ballot matrices is defined. We show that matrices fixed under this involution are in bijection with labeled interval orders and that they decompose to a pair consisting of a permutation and an inversion table. To fully classify such pairs, results pertaining to the enumeration of permutations having a given set of ascent bottoms are given. This allows for a new formula for the number of labeled interval orders.

Keywords: ballot matrix, composition matrix, sign reversing involution, interval order, 2+2-free poset, Fishburn, ascent bottom

1 Introduction

Recent work has employed the use of sign reversing involutions in the study of unlabeled interval orders. Successes include taking structures related to unlabeled interval orders directly to their generating function [8, 10] and identifying statistical refinements [7].

In this paper we apply similar techniques to the labeled case. We introduce ballot matrices, a combinatorial structure consisting of signed, upper triangular, non-row empty matrices whose entries are ballots. The definition of such matrices follows naturally from the generating function of labeled interval orders. A bijection of Dukes et al. [5] is adapted to a surjection mapping ballot matrices to labeled interval orders and used to define an equivalence relation on ballot matrices. A sign reversing involution is then used to identify fixed points for which there is exactly one per equivalence class. The decomposition of any single fixed point into a pair consisting of a permutation and an inversion table is then provided. This allows for the main result of the paper, that the set of labeled interval orders on $[n]$ is in bijection with two separate sets. Firstly,

$$\{(\pi, \tau) \in \mathcal{S}_n \times \mathcal{S}_n : A(\tau) \subseteq D(\pi)\},$$

where $A(\tau)$ is the set of ascent bottoms of τ, and $D(\pi)$ is the set of descent positions of π. Secondly,

$$\{(\pi, \tau) \in \mathcal{S}_n \times \mathcal{S}_n : D(\pi) \subseteq A(\tau)\}.$$

As a consequence we derive a new formula for the number of labeled interval orders on $[n]$:

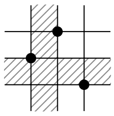
$$\sum_{\{s_1,\ldots,s_k\}\subseteq[n-1]} \left(\det \left[\binom{n-s_i}{s_{j+1}-s_i} \right] \cdot \prod_{r=1}^{k+1} r^{s_r - s_{r-1}} \right)$$

where $s_0 = 0$ and $s_{k+1} = n$.

1.1 Background

A poset P is said to be an *interval order* if each $z \in P$ can be assigned a closed interval $[l_z, r_z] \in \mathbb{R}$ such that $x <_P y$ if and only if $r_x < l_y$. Fishburn [6] demonstrates that interval orders are equivalently characterized as posets with no induced subposet isomorphic to the pair of disjoint two element chains, the so called $(2+2)$-free posets.

Bousquet-Mélou et al. [3] show that unlabeled interval orders are in bijection with ascent sequences (a subset of inversion tables), permutations avoiding the mesh pattern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and a class of fixed point free involutions with no neighbor nestings. Such involutions had previously been studied by Zagier [11] who determined their ordinary generating function to be

$$\sum_{m\geqslant 0} \prod_{i=1}^{m} (1 - (1-x)^i).$$

Levande [8] and Yan [10] independently employ the use of sign-reversing involutions to provide direct interpretations of structures related to unlabeled interval orders from Zagier's function.

To study labeled interval orders, Claesson et al. [4] introduce composition matrices. A *composition matrix* is an upper triangular matrix on some underlying set U whose entries are sets partitioning U satisfying that there are no rows or columns which contain only empty set partitions. They show that composition matrices have exponential generating function

$$\sum_{m\geqslant 0} \prod_{i=1}^{m} (1 - e^{-xi}),$$

again a function originally considered by Zagier [11]. They present a one-to-one correspondence between labeled interval orders and composition matrices via the Cartesian product of ascent sequences and set partitions.

Dukes et al. [5] give a direct bijection between composition matrices and interval orders, where the downsets of elements within the interval order are determined by hooks occurring below the diagonal of the matrix.

2 Terminology and preliminaries

Throughout this text, for non-negative integers a and b with $a < b$, let $[b]$ denote the set $\{1, \ldots, b\}$ and $[a, b]$ the set $\{a, \ldots, b\}$. This paper will feature three main combinatorial structures: permutations, inversion tables and ballots. In this section a summary is provided to remind the reader of relevant results pertaining to these structures and to set the notational convention that shall be followed.

2.1 Permutations

A permutation is a bijection on a finite set. A descent in a permutation $\pi = a_1a_2 \ldots a_n \in \mathcal{S}_n$ is a pair (a_i, a_{i+1}) where $a_i > a_{i+1}$. Following Stanley [9, Section 2.2] let $D(\pi) = \{i : a_i < a_{i+1}\} \subseteq [n-1]$ denote the set of descent positions and define

$$\begin{aligned} \boldsymbol{\alpha}_n(S) &= \{\pi \in \mathcal{S}_n : D(\pi) \subseteq S\}, \quad \alpha_n(S) = |\boldsymbol{\alpha}_n(S)|, \\ \boldsymbol{\beta}_n(S) &= \{\pi \in \mathcal{S}_n : D(\pi) = S\}, \quad \beta_n(S) = |\boldsymbol{\beta}_n(S)|. \end{aligned}$$

Let $S = \{s_1, s_2, \ldots, s_k\}$ and $1 \leqslant s_1 < s_2 < \cdots < s_k < n$. Also, let $s_0 = 0$ and $s_{k+1} = n$. Partitioning $[n]$ into blocks of cardinalities

$$s_1 - s_0,\ s_2 - s_1,\ \ldots,\ s_{k+1} - s_k$$

a permutation is formed by listing elements within the blocks in increasing order and concatenating the blocks. The only position in which a descent *can* occur is at the join between two blocks. Thus,

$$\alpha_n(S) = \binom{n}{s_1 - s_0, s_2 - s_1, \ldots, s_{k+1} - s_k}. \tag{1}$$

By the sieve principle we have that $\beta_n(S) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} \alpha_n(T)$. One can show [9, Example 2.2.4] that this leads to the formula

$$\beta_n(S) = \det \left[\binom{n - s_i}{s_{j+1} - s_i} \right],$$

where $(i, j) \in [0, k] \times [0, k]$.

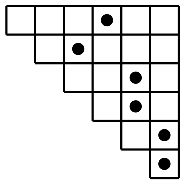


Figure 1: Inversion table 231100

2.2 Inversion tables

Given a permutation $\pi = a_1 a_2 \dots a_n$, an inversion in π is a pair (a_i, a_j) where $a_i > a_j$ and $i < j$. An inversion table is an encoding of a permutation where the ith value is the number of inversions in which i is involved as the smaller element. The set of inversion tables of length n will be denoted InvTab_n:

$$\mathrm{InvTab}_n = \{\, b_1 b_2 \dots b_n : b_i \in [0, n-i] \,\}.$$

An inversion table may be viewed diagrammatically. To make clear the relationship between inversion tables and n by n upper triangular matrices containing exactly one entry per row we shall break convention and view an inversion table as right aligned, decreasing rows where an entry in row i at column j corresponds to the inversion table with ith entry $n - j$. An example is shown in Figure 1.

Define Dent to be the function taking an inversion table to the set of distinct entries it contains. For example, $\mathrm{Dent}(430200) = \{0, 2, 3, 4\}$. We further say that $a \in [n-1]$ is *missing* from a length n inversion table if a is not in its set of distinct entries. For instance, 1 and 5 are both missing from 430200.

2.3 Ballots

A *ballot*, alternatively known as an ordered set partition, is a collection of pairwise disjoint non-empty sets (referred to as blocks) where the blocks are assigned some total ordering. Adopting a symbolic (or species) approach, let L be the construction taking a set U to the set of linear orders built upon U. Also, let E_+ be the non-empty set construction. That is, $E_+[U] = \{U\}$ if U is non-empty, and $E_+[\emptyset] = \emptyset$. Then define Bal, the construction of ballots, to be the composition $L(E_+)$:

$$\mathrm{Bal} = L(E_+) = \sum_{k \geqslant 0} (E_+)^k.$$

Consider *signed ballots*, as above but where each ballot is assigned to be either positive or negative. A positive ballot contains an even number of blocks and a negative ballot contains an odd number of blocks. For any species F, let $-1 \cdot F = -F$ be as F but with the sign of each object negated. Using E^{-1} to refer to signed ballots—the notation stemming from its role as the symbolic multiplicative inverse of set—we have

$$E^{-1} = L(-E_+) = \sum_{k \geqslant 0} (-1)^k (E_+)^k.$$

It follows that signed ballots have exponential generating function

$$\frac{1}{1+(e^x-1)} = e^{-x} = \sum_{n\geqslant 0}(-1)^n\frac{x^n}{n!}. \tag{2}$$

See, for example, Bergeron et al. [1, Section 2.5].

We use the notation $(E^{-1})^+$ to refer to the subset of signed ballots which are positive and $(E^{-1})^-$ to refer to the subset which are negative.

3 Ballot matrices and interval orders

Equation (2) implies that the number of ballots constructed on some set U with an even number of blocks differ from the number of ballots of U with an odd number of blocks by 1. To be precise

$$|(E^{-1})^+[U]| - |(E^{-1})^-[U]| = (-1)^{|U|}.$$

An involution on ballots witnesses this fact. In the above equation the sign of a ballot with k blocks is $(-1)^k$. Note that we can change the sign of a ballot with $|U| \geqslant 2$ by splitting a non-singleton block into two blocks or by merging two blocks. Let $\omega = B_1 \dots B_k$ be a ballot in $\mathrm{Bal}[U]$. That is, each B_i is non-empty and U is the disjoint union of the sets B_1 through B_k.

Take any linear order on U. Let $x = \min U$ be smallest element of U. If $x \in B_i$ and B_i contains at least two elements, then delete x from B_i and create a new block $\{x\}$ to the immediate right of B_i. For example,

$$\omega = \{2,5\}\{1,4,6\}\{3\} \;\mapsto\; \{2,5\}\{4,6\}\{1\}\{3\} = \xi.$$

If $B_i = \{x\}$ and $i > 1$ then delete this block from ω and add x to B_{i-1}. With ω and ξ as in the example above, we have $\xi \mapsto \omega$. If $B_1 = \{x\}$ then proceed with the next smallest element of U and the ballot $B_2B_3 \dots B_k$. For example,

$$\{1\}\{2\}\{5\}\{4,6\}\{3\} \mapsto \{1\}\{2\}\{5\}\{3,4,6\}.$$

For $U = \{u_1, u_2, \dots, u_n\}$ and $u_1 < u_2 < \dots < u_n$ the single fixed point under this sign reversing involution is $\{u_1\}\{u_2\} \dots \{u_n\}$.

3.1 Ballot Matrices

The exponential generating function for the number of labeled interval orders was shown by Claesson et al. [5] to be a function originally studied by Zagier [11],

$$\sum_{m\geqslant 0}\prod_{i=1}^{m}(1-e^{-xi}) = \sum_{m\geqslant 0}(-1)^m\prod_{i=1}^{m}(e^{-xi}-1).$$

It it thus natural to consider the signed combinatorial structure

$$\sum_{m\geqslant 0}(-1)^m\prod_{i=1}^{m}\Big((E^{-1})^i-1\Big).$$

An $((E^{-1})^i-1)$-structure is a non-empty sequence of i pairwise disjoint ballots. As such, a $(-1)^m\prod_{i=1}^{m}\Big((E^{-1})^i-1\Big)$-structure is an upper triangular $m\times m$ matrix of pairwise disjoint ballots such that each row is non-empty.

The sign of the matrix is the product of the signs of the ballot entries and the signs of the rows. If A is such a matrix and the total number of blocks of all ballots in A is ℓ, then the sign of A is $(-1)^{\ell+m}$. We shall call such matrices *Ballot matrices* and use the notation BalMat for the construction with BalMat^+ and BalMat^- the positive and negative parts respectively. As an example, for $U=\{1,2\}$ we have

$$\mathrm{BalMat}^+[U]=\left\{\begin{bmatrix}\{1,2\}\end{bmatrix},\begin{bmatrix}\emptyset & \{1\}\\ & \{2\}\end{bmatrix},\begin{bmatrix}\emptyset & \{2\}\\ & \{1\}\end{bmatrix},\begin{bmatrix}\{2\} & \emptyset\\ & \{1\}\end{bmatrix},\begin{bmatrix}\{1\} & \emptyset\\ & \{2\}\end{bmatrix}\right\}$$

and

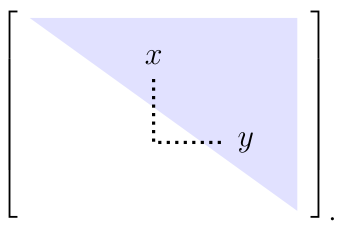
$$\mathrm{BalMat}^-[U]=\left\{\begin{bmatrix}\{1\}\{2\}\end{bmatrix},\begin{bmatrix}\{2\}\{1\}\end{bmatrix}\right\}.$$

We note the similarity between ballot matrices and the composition matrices of Claesson et al. [4]. The entries of composition matrices are sets, which may be viewed as either as ballots with a single block or as ballots where each element is contained within its own singleton block and the blocks are ordered according to the order on U. Therefore composition matrices are a subset of ballot matrices. For our purposes we wish to define an involution whose fixed points are either all positive or all negative for any given U. However for both interpretations of composition matrices as ballot matrices the sign is not consistent, there exist both positive and negative composition matrices when $|U|\geqslant 2$, and hence they are not suitable candidates for the fixed points of our involution.

Dukes et al. [5] provide a direct bijection between composition matrices and labeled interval orders. We adapt their mapping to define a surjection taking ballot matrices to labeled interval orders as follows.

Definition 1. Let $A\in\mathrm{BalMat}[U]$, and let x and y be elements of U. Further, let ω and ξ be the ballot entries (i,j) and (i',j') of A such that x is contained in the underlying set of ω and y is contained in the underlying set of ξ. Define the poset $P(A)$ by declaring that $x<y$ in P if $j<i'$.

In other words, $x<y$ in P if the "hook" from x to y passing through (i',j) goes below the diagonal:



$$\begin{bmatrix} \emptyset & \{6\} & \emptyset & \{4,5\} \\ & \emptyset & \{3\} & \emptyset \\ & & \{1\} & \emptyset \\ & & & \{2\} \end{bmatrix}$$

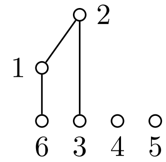


Figure 2: A ballot matrix and its corresponding poset

Equivalently, the strict downset of y is the union of columns 1 through $i'-1$. Figure 2 shows an example of a ballot matrix and its corresponding poset.

Given a poset P, the downset of $x \in P$ is the set of elements smaller than x:

$$D(x) = \{y \in P : y < x\}.$$

It is a well known that a poset is an interval order if and only if there is a linear ordering by inclusion on the downsets of each element $\{D(x) : x \in P\}$ (see, for example, Bogart [2]). As the mapping states that the strict downset of y is the union of columns 1 through $i'-1$ there is a linear ordering on downsets and hence every poset which is mapped to must be an interval order.

In addition, composition matrices are a subset of ballot matrices and as Dukes et al. [5] show that for composition matrices the mapping is a bijection it follows that the adapted mapping is a surjection.

If we declare that two ballot matrices in $\mathrm{BalMat}[U]$ are equivalent if they determine the same interval order, then, by definition, there are as many equivalence classes as there are interval orders on U. In the next section we define as sign reversing involution that respects this equivalence relation.

4 The involution

We now define the involution on ballot matrices. We begin by applying the ballot involution componentwise to entries of BalMat.

Choose a canonical linear order of the entries of the matrix; for instance, order the entries (ballots) with respect to their minimum element, or order them lexicographically with respect to their position (i, j) in the matrix. Then apply the ballot involution to the first entry that is not fixed, if such an element exists, and denote this operation ψ. A matrix is a fixed point under this sign reversing involution if and only if each entry of the matrix is fixed, and thus of the form

$$\{a_1\}\{a_2\}\ldots\{a_j\} \quad \text{with} \quad a_1 < a_2 < \cdots < a_j.$$

Note that if A is a $k \times k$ matrix fixed under ψ, then the sign of A is $(-1)^{n+k}$, where $n = |U|$. We shall define a sign reversing involution φ on the fixed points of ψ.

Let $A \in \mathrm{BalMat}[U]$ be a matrix fixed under ψ. Let $x \in U$ and assume that that x is on row i and column j of A. We say that x is a *pivot* element of A if row i contains at least two elements of U and x is the smallest element on row i, or the following three conditions are met:

1. column i is empty;
2. $\{x\}$ is the only non-empty ballot on its row;
3. x is smaller than the minimum element of row $i+1$ of A.

As an illustration, the pivot elements of the matrix

$$\begin{bmatrix} \emptyset & \{4\} & \emptyset & \emptyset & \emptyset \\ & \{6\}\{8\} & \emptyset & \{3\}\{7\} & \emptyset \\ & & \emptyset & \{2\} & \emptyset \\ & & & \{9\} & \{5\} \\ & & & & \{1\} \end{bmatrix}$$

are 2, 3 and 5.

If the set of pivot elements of A is empty, then let $\varphi(A) = A$. Otherwise, let x be the smallest pivot element of A, and assume that x belongs to the (i, j) entry of A.

1. If there is more than one element on row i, then remove x from row i and make a new row immediately above row i with the block $\{x\}$ in column j and the rest of the entries empty. Also insert a new empty column i, pushing the existing columns one step to the right.

2. If column i is empty, $\{x\}$ is the only non-empty ballot on its row, and x is smaller than the minimum element of row $i+1$, then remove column i and merge row i with row $i+1$ by inserting the singleton block x at the front of the ballot in position $(i+1, j)$.

Applying φ to the example matrix above we get

$$\begin{bmatrix} \emptyset & \{4\} & \emptyset & \emptyset \\ & \{6\}\{8\} & \{3\}\{7\} & \emptyset \\ & & \{2\}\{9\} & \{5\} \\ & & & \{1\} \end{bmatrix}.$$

Note that the smallest pivot element of this matrix is still 2, and applying φ to it would bring back the original matrix.

Our main involution $\eta : \mathrm{BalMat}[U] \to \mathrm{BalMat}[U]$ is then defined as the composition of ψ and φ in the following sense:

$$\eta(A) = \begin{cases} \varphi(A) & \text{if } \psi(A) = A, \\ \psi(A) & \text{if } \psi(A) \neq A. \end{cases}$$

It is clear that η is sign reversing. That any fixed point of η has positive sign will be seen in Section 5.

Proposition 2. *The involution η preserves the interval order in the following sense. Let $A \in \mathrm{BalMat}[U]$. Let P and Q be the interval orders corresponding to A and $\eta(A)$, respectively. Then $P = Q$.*

Proof. If A is a fixed point of η, equality is immediate. Further, the block structure of the elements of A is immaterial to the definition of the poset. Thus, if $\psi(A) \neq A$ and $\eta(A) = \psi(A)$, then equality is immediate. For the remainder of the proof assume that $\eta(A) = \varphi(A) \neq A$.

The proof that the involution preserves the interval order is equivalent to saying that the strict downset of each element is preserved. This follows from a case analysis. Recall that the strict downset of x at position (i, j) in the matrix is the union of columns 1 through $i - 1$.

Let $B = \eta(A)$. The involution has two possibilities. If the minimal pivot element x at position (i, j) in A is not the only element on its row, then B is formed by initially inserting a new empty row above row i and a new empty column before column i. The pivot element x is moved to the new row maintaining its column and hence its strict downset is unchanged.

We now demonstrate that the insertion of the new empty row at position i and new empty column at position i preserves hooks below the diagonal. For $y \neq x$ at position (i', j') in A there are three possibilities.

1. The element y is above the newly inserted row and to the left of the new column, i.e. y remains at position (i', j') in B with $i' < i$ and $j' < i$. Then the new column is inserted to the right of the columns which form the strict downset of y and hence the downset is unchanged.

2. The element y is to the right of the newly inserted column and above the inserted row, i.e. y is at position $(i', j' + 1)$ in B with $i' < i < j'$. Again as $i' < i$ the new column is inserted to the right of the columns which form the strict downset of y and the downset is unchanged.

3. The element y is below the newly inserted row and to the right but of the new column, i.e. y is at position $(i' + 1, j' + 1)$ in B with $i < i'$ and $i < j'$. As $i < i'$, the number of columns which form the downset of y is increased by 1. The newly inserted column i is empty and therefore contributes no new entries. As $i < i'$ the previous rightmost column $i' - 1$ is shifted one place to the right to column $i' + 1$ in the new matrix. The downset of y in B is therefore the union of elements 1 through i' and hence the downset is unchanged.

Note that x remains the pivot element in the newly constructed matrix B, the only non-empty ballot on its row, and with column i empty. Therefore showing that the second possibility of the involution preserves posets follows from taking the reverse of the above cases.

As the strict downsets are equal the posets are equal. □

5 Fixed points

A fixed point under the sign reversing involution η on BalMat is an $n \times n$ matrix with no pivot elements, equivalently a matrix such that

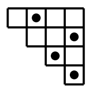
1. there is exactly one element per row;
2. if $a < b$, with a on row i, and b on row $i+1$, then column i is non-empty.

Note that the total number of blocks in such a matrix is n—each element is in its own block—and thus it has sign $(-1)^{2n} = 1$, positive.

Further, matrices which satisfy these conditions can be decomposed to a pair consisting of a permutation and an inversion table: As there is exactly one element per row, a permutation $\pi = a_1 \dots a_n$ can be read setting each a_i the value held in row i. As the matrix is also upper triangular, the position of the element in a row specifies an inversion table $b_1 b_2 \dots b_n$ where each b_i is n minus the column in which the entry in row i occurs.

As an example, consider the matrix below. It decomposes into the permutation 4132 together with the inversion table 2010:

$$\begin{bmatrix} \emptyset & \{4\} & \emptyset & \emptyset \\ & \emptyset & \emptyset & \{1\} \\ & & \{3\} & \emptyset \\ & & & \{2\} \end{bmatrix} \simeq \left(4132, \right.$$



$$\left. \right) \simeq (\,4132,\ 2010\,).$$

Take the equivalence class on ballot matrices where two matrices are equivalent if they correspond to the same interval order. We wish to show that there is exactly one fixed point under η per equivalence class. For this purpose and to make explicit the link to previous work we provide a bijection between composition matrices and ballot matrices.

For the following, take the structure of the entries of a composition matrix to be ballots where each element is contained within a singleton block and the blocks are ordered according to the order on the underlying set.

Given an $m \times m$ ballot matrix $A \in \mathrm{BalMat}[U]$, let u_i be the smallest element on the ith row of A, and define $G(A) = U \setminus \{u_1, u_2, \dots, u_m\}$.

Assuming that $G(A)$ is non-empty, define ρ to be the following operation. Take $x = \min G(A)$ at position (i, j) in A. Insert a new row containing only empty ballots above row i and a new column containing only empty ballots to the left of column i. Move x to create a singleton ballot in the new row preserving its column. Note that $|G(\rho(A))| = |G(A)| - 1$. An example with $G(A) = \{4, 6\}$ is given below:

$$\begin{bmatrix} \{2,6\} & \emptyset & \emptyset & \emptyset \\ & \{3\} & \emptyset & \mathbf{\{4\}} \\ & & \emptyset & \{1\} \\ & & & \{5\} \end{bmatrix} \xmapsto{\rho} \begin{bmatrix} \{2,6\} & \boldsymbol{\emptyset} & \emptyset & \emptyset & \emptyset \\ & \boldsymbol{\emptyset} & \boldsymbol{\emptyset} & \boldsymbol{\emptyset} & \mathbf{\{4\}} \\ & & \{3\} & \emptyset & \emptyset \\ & & & \emptyset & \{1\} \\ & & & & \{5\} \end{bmatrix}.$$

The inverse operation will be denoted ρ^{-1}. To state it explicitly, let $A \in \mathrm{BalMat}[U]$ be a $m \times m$ ballot matrix, and let u_i be the smallest element on the ith row of A, as before. Then take $H(A)$ to be the subset of $\{u_1, u_2, \ldots, u_{m-1}\}$ consisting of those u_i such that the following three conditions hold: column i is empty; u_i is the sole element on row i; and $u_i > u_{i+1}$.

Assuming that $H(A)$ is non-empty, define ρ^{-1} to be the following operation. Take $x = \max H(A)$ at position (i, j) in A. Append x in a singleton block at the end of the ballot in position $(i+1, j)$, then remove row and column i.

Proposition 3. *There is a bijection between composition matrices and ballot matrices fixed under η. As a result there is a unique ballot matrix fixed under η per equivalence class.*

Proof. We first show that successive application of the mapping ρ gives an injection from composition matrices into ballot matrices fixed under η.

The same argument as in Proposition 2 shows that ρ preserves the interval order.

Take a composition matrix. Let A be the matrix returned after repeated application of ρ until the set of elements $G(A)$ is empty. We claim A is a ballot matrix fixed under η.

From definition we know that $G(A)$ is empty. Therefore there is exactly one element per row. The other requirement to be a fixed point under η is that if $a < b$ with a on row i and b on row $i+1$ then column i must be non-empty. As composition matrices have the property that all columns are non-empty and ρ only introduces an empty column i when $a > b$ with a on row i, this requirement is met.

Repeated application of ρ is therefore a mapping between composition matrices and ballot matrices fixed under η with injectivity following from the preservation of interval order.

As ρ preserves the interval order, the reverse operation ρ^{-1} also preserves the interval order.

Take a fixed point matrix. Let A be the matrix returned after repeated application of ρ^{-1} until the set of elements $H(A)$ is empty. We claim A is a composition matrix.

Composition matrices are neither row nor column empty. Non-row empty is a property of fixed point ballot matrices and ρ^{-1} does not introduce any empty columns. If a fixed point matrix contains an empty column i then from definition there is an $a > b$ with a and b on rows i and $i+1$ respectively. However as $G(A)$ is empty it follows that all empty columns are removed.

Hence all fixed point matrices can be mapped to a composition matrices with the interval order preserved by repeated application of ρ^{-1}, giving surjectivity. $\square$

Let $\mathrm{BalMat}^{\eta}[U]$ denote the set of fixed points under η. Writing simply x for the ballot $\{x\}$, the complete list of matrices in $\mathrm{BalMat}^{\eta}[3]$ is given in Figure 3.

6 Permutations from ascent bottoms

In order to examine the fixed points under η we shall consider how to characterize the pairs resulting from their decomposition to a permutation and an inversion table. For this purpose, this section is concerned with counting the number of permutations whose

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\begin{bmatrix} 3 & \emptyset & \emptyset \\ & 1 & \emptyset \\ & & 2 \end{bmatrix}
\begin{bmatrix} \emptyset & 3 & \emptyset \\ & 1 & \emptyset \\ & & 2 \end{bmatrix}
\begin{bmatrix} \emptyset & 3 & \emptyset \\ & \emptyset & 1 \\ & & 2 \end{bmatrix}
\begin{bmatrix} \emptyset & \emptyset & 3 \\ & 1 & \emptyset \\ & & 2 \end{bmatrix}
$$

$$
\begin{bmatrix} 2 & \emptyset & \emptyset \\ & 3 & \emptyset \\ & & 1 \end{bmatrix}
\begin{bmatrix} 2 & \emptyset & \emptyset \\ & \emptyset & 3 \\ & & 1 \end{bmatrix}
\begin{bmatrix} 2 & \emptyset & \emptyset \\ & 1 & \emptyset \\ & & 3 \end{bmatrix}
\begin{bmatrix} \emptyset & 2 & \emptyset \\ & 1 & \emptyset \\ & & 3 \end{bmatrix}
\begin{bmatrix} \emptyset & 2 & \emptyset \\ & \emptyset & 1 \\ & & 3 \end{bmatrix}
$$

$$
\begin{bmatrix} \emptyset & \emptyset & 2 \\ & 1 & \emptyset \\ & & 3 \end{bmatrix}
\begin{bmatrix} 1 & \emptyset & \emptyset \\ & 3 & \emptyset \\ & & 2 \end{bmatrix}
\begin{bmatrix} 1 & \emptyset & \emptyset \\ & \emptyset & 3 \\ & & 2 \end{bmatrix}
\begin{bmatrix} 1 & \emptyset & \emptyset \\ & 2 & \emptyset \\ & & 3 \end{bmatrix}
$$

Figure 3: Complete list of matrices in $\mathrm{BalMat}^{\eta}[3]$

set of ascent bottoms is equal to some given set. Bijections between such permutations and two different sets of inversion tables are provided. We make repeated use of the sieve principle and our presentation follows that of Stanley [9, Section 2.2].

Recall the definitions of $\boldsymbol{\alpha}_n(S)$ and $\boldsymbol{\beta}_n(S)$:

$$
\begin{aligned}
\boldsymbol{\alpha}_n(S) &= \{\tau \in \mathcal{S}_n : D(\tau) \subseteq S\}, \quad \alpha_n(S) = |\boldsymbol{\alpha}_n(S)|, \\
\boldsymbol{\beta}_n(S) &= \{\tau \in \mathcal{S}_n : D(\tau) = S\}, \quad \beta_n(S) = |\boldsymbol{\beta}_n(S)|.
\end{aligned}
$$

In an analogous fashion, for $\pi = a_1 a_2 \dots a_n \in \mathcal{S}_n$, let

$$A(\pi) = \{a_i : i \in [n-1], a_i < a_{i+1}\}$$

be the set of ascent bottoms of π. Let

$$
\begin{aligned}
\boldsymbol{\kappa}_n(S) &= \{\pi \in \mathcal{S}_n : A(\pi) \subseteq S\}, \quad \kappa_n(S) = |\boldsymbol{\kappa}_n(S)|, \\
\boldsymbol{\lambda}_n(S) &= \{\pi \in \mathcal{S}_n : A(\pi) = S\}, \quad \lambda_n(S) = |\boldsymbol{\lambda}_n(S)|.
\end{aligned}
$$

Note that by definition $\kappa_n(S) = \sum_{T \subseteq S} \lambda_n(T)$, and by the sieve principle, $\lambda_n(S) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} \kappa_n(T)$.

The following set of sequences will be convenient as an intermediate structure for later proofs.

Definition 4. For fixed n, let $S = \{s_1, \dots, s_k\}$ with $1 \leqslant s_1 < \dots < s_k < n$ be given. Also, set $s_0 = 0$ and $s_{k+1} = n$. Define the Cartesian product

$$\mathcal{C}_n(S) = [0,k]^{s_{k+1}-s_k} \times \cdots \times [0,1]^{s_2 - s_1} \times [0,0]^{s_1 - s_0}.$$

We shall call an element of $\mathcal{C}_n(S)$ a *construction choice*.

As example, for $n = 8$ and $S = \{3, 5, 6, 7\}$ we have $s_1 - s_0 = 3$, $s_2 - s_1 = 2$, and $s_3 - s_2 = s_4 - s_3 = s_5 - s_4 = 1$. Thus

$$\mathcal{C}_n(S) = [0, 4] \times [0, 3] \times [0, 2] \times [0, 1] \times [0, 1] \times [0, 0] \times [0, 0] \times [0, 0].$$

An example of a construction choice in $\mathcal{C}_n(S)$ is 42001000, we shall use this as a running example throughout the remainder of this section.

Proposition 5. *For fixed n, let $S = \{s_1, \dots, s_k\}$ and $1 \leqslant s_1 < \cdots < s_k < n$ be given. Then $\boldsymbol{\kappa}_n(S)$ is in bijection with $\mathcal{C}_n(S)$.*

Proof. Take a construction choice $c_1 c_2 \dots c_n \in \mathcal{C}_n(S)$. We will use this to construct a permutation by insertion of entries at active sites. Start with the empty permutation. This has a single active site, labeled zero. Reading the construction choice in reverse order, insert elements into the permutation beginning with the minimal element. That is, c_i is the choice of active site for the insertion of $n + 1 - i$ into the permutation.

A new active site is created when an element of S is introduced into the permutation. The active sites are labeled according to the order in which they are inserted. That is, assuming entries of S are numerically ordered then the active site to the right of s_i in the permutation is labeled i. Note that a consequence of this is that s_i is an ascent bottom if and only if i is contained within the construction choice. As a larger element is inserted at each step this ensures that the only place where an ascent can take place is after an entry of in the permutation which is contained within S. Therefore only elements of S can be ascent bottoms.

It is easy to see how to reverse this procedure and thus it provides the claimed bijection. $\square$

Example 6. For $n = 8$ and $S = \{3, 5, 6, 7\}$ the construction process for the permutation with construction choice 42001000 is as follows. Note the new active site created when an element of S is inserted.

${}_0$	
${}_0 1$	Insert 1 at site 0
${}_0 21$	Insert 2 at site 0
${}_0 3_1 21$	Insert 3 at site 0, contained in S
${}_0 3_1 421$	Insert 4 at site 1
${}_0 5_2 3_1 421$	Insert 5 at site 0, contained in S
${}_0 6_3 5_2 3_1 421$	Insert 6 at site 0, contained in S
${}_0 6_3 5_2 7_4 3_1 421$	Insert 7 at site 2, contained in S
${}_0 6_3 5_2 7_4 8 3_1 421$	Insert 8 at site 4

So the resulting permutation is $\pi = 65783421$, with $A(\pi) = \{3, 5, 7\}$.

Corollary 7. *For fixed n, let $S = \{s_1, \ldots, s_k\}$ with $1 \leqslant s_1 < \cdots < s_k < n$ be given. Then*

$$\kappa_n(S) = \prod_{r=1}^{k+1} r^{s_r - s_{r-1}},$$

where $s_0 = 0$ and $s_{k+1} = n$.

Proof. By Proposition 5 we have that $\kappa_n(S)$ is the cardinality of $\mathcal{C}_n(S)$, from which the formula immediately follows. $\square$

We shall now show that construction choices in $\mathcal{C}_n(S)$, and thus permutations in $\boldsymbol{\kappa}_n(S)$, are in bijection with two different sets of inversion tables. Namely

$$\Big\{\upsilon \in \mathrm{InvTab}_n : \mathrm{Dent}(\upsilon) \subseteq \{0, s_1, s_2, \ldots, s_k\}\Big\}$$

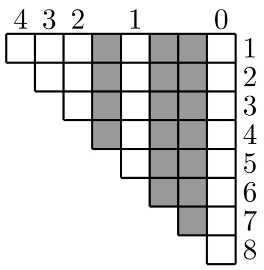
and

$$\Big\{\upsilon \in \mathrm{InvTab}_n : [n-1] \setminus \mathrm{Dent}(\upsilon) \subseteq \{n - s_1, \ldots, n - s_k\}\Big\}.$$

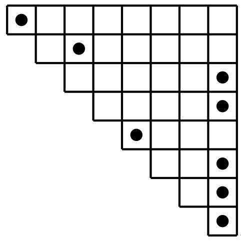
Proposition 8. *For fixed n, let $S = \{s_1, \ldots, s_k\}$ with $1 \leqslant s_1 < \cdots < s_k < n$ be given. Then there is a bijection between $\boldsymbol{\kappa}_n(S)$ and inversion tables whose entries are a subset of $\{0\} \cup S$,*

$$\Big\{\upsilon \in \mathrm{InvTab}_n : \mathrm{Dent}(\upsilon) \subseteq \{0, s_1, s_2, \ldots, s_k\}\Big\}.$$

Proof. Again we shall use the construction choice. Entries contained within the inversion table are a subset of S. Therefore elements which are in $[n-1]$ but *not* in S, that is, elements of $[n-1] \setminus S$, cannot be contained in the inversion table. These entries are therefore forbidden. Label the remaining possible entries right to left from $[0, k]$. In this context it is convenient to use our diagrammatic representation of an inversion table. As an example, let $n = 8$ and $S = \{3, 5, 6, 7\}$. As $[n-1] \setminus S = \{1, 2, 4\}$, the columns $8-1$, $8-2$, and $8-4$ are forbidden (dark, below). Labeling those which remain right-to-left with $[0, 4]$ yields



Given a construction choice $c_1 c_2 \ldots c_n \in \mathcal{C}_n(S)$, assign the entry on row i to be in the column labeled c_i. Note that as a consequence s_i is contained in the inversion table if and only if i is contained within the construction choice. To consider the range of construction choices which are valid, we also note that there are $k+1$ allowed columns for the first $s_k - s_{k-1}$ rows, k choices for the next $s_{k-1} - s_{k-2}$ rows, and so on. This agrees with the definition of $\mathcal{C}_n(n)$. Taking our example construction choice of 42001000 yields the inversion table $\upsilon = 75003000$ where $\mathrm{Dent}(\upsilon) = \{0, 3, 5, 7\}$:



.

$\square$

Applying the sieve principle to the set of inversion tables from Proposition 8 we arrive at the following result.

Corollary 9. *There is a bijection between $\boldsymbol{\lambda}_n(S)$ and inversion tables whose entries are exactly those in $\{0\} \cup S$,*

$$\Big\{ v \in \mathrm{InvTab}_n : \mathrm{Dent}(v) = \{0, s_1, \dots, s_k\} \Big\}.$$

To prove the bijection between $\boldsymbol{\kappa}_n(S)$ and the second set of inversion tables, consideration of a set of ballots is useful. The proof of Proposition 10 below shows one way to make a ballot in $\mathrm{Bal}[n]$ (short for $\mathrm{Bal}[[n]]$) from a given construction choice.

Proposition 10. *For fixed n, let $S = \{s_1, \dots, s_k\}$ and $1 \leqslant s_1 < \cdots < s_k < n$ be given. Then $\mathcal{C}_n(S)$ is in bijection with the set of ballots*

$$\Big\{ B_1 \dots B_{k+1} \in \mathrm{Bal}[n] : \{\min B_1, \dots, \min B_{k+1}\} = \{1, s_1 + 1, \dots, s_k + 1\} \Big\}.$$

Proof. We will show how to construct a ballot from a given construction choice $c_1 c_2 \dots c_n$. Take $k+1$ empty blocks. At any point in the following construction each block will be considered either open or closed, and the open blocks will be numbered $0, 1, \dots, k$, from left to right. Initially all blocks are open. For i equal to $1, 2, \dots, n$, in that order, let $a = n + 1 - i$ and insert a into the c_ith open block. If $a \in \{1, s_1 + 1, \dots, s_k + 1\}$ then also close the block a is inserted into. This way a is guaranteed end up as the minimal element of its block. It is easy to see how to reverse this procedure and thus it provides the claimed bijection. $\square$

Example 11. For $n = 8$ and $S = \{3, 5, 6, 7\}$ consider the construction of a ballot whose minimal block elements are $\{1, 4, 6, 7, 8\}$ with construction choice 42001000. Initially we have 5 empty blocks labeled from $[0, 4]$. Note that when a minimal block element is inserted, that block is no longer open and the remaining blocks are relabeled.

$\{\}_0\{\}_1\{\}_2\{\}_3\{\}_4$	
$\{\}_0\{\}_1\{\}_2\{\}_3\{8\}_4$	8 inserted in block 4, is minimal entry
$\{\}_0\{\}_1\{7\}_2\{\}_3\{8\}$	7 inserted in block 2, is minimal entry
$\{6\}_0\{\}_1\{7\}\{\}_2\{8\}$	6 inserted in block 0, is minimal entry
$\{6\}\{5\}_0\{7\}\{\}_1\{8\}$	5 inserted in block 0

$\{6\}\{5\}_0\{7\}\{4\}\{8\}$	4 inserted in block 1, is minimal entry
$\{6\}\{3,5\}_0\{7\}\{4\}\{8\}$	3 inserted in block 0
$\{6\}\{2,3,5\}_0\{7\}\{4\}\{8\}$	2 inserted in block 0
$\{6\}\{1,2,3,5\}_0\{7\}\{4\}\{8\}$	1 inserted in block 0, is minimal entry

Therefore the final ballot is $\{6\}\{1,2,3,5\}\{7\}\{4\}\{8\}$.

Proposition 12. *There is a bijection between $\boldsymbol{\kappa}_n(S)$ and inversion tables whose missing elements are a subset of $n - s_1$, $n - s_2$, ..., $n - s_k$,*

$$\Big\{\upsilon \in \mathrm{InvTab}_n : [n-1] \setminus \mathrm{Dent}(\upsilon) \subseteq \{n - s_1, \dots, n - s_k\}\Big\}.$$

Or, equivalently,

$$\Big\{\upsilon \in \mathrm{InvTab}_n : [0, n-1] \setminus \{n - s_1, \dots, n - s_k\} \subseteq \mathrm{Dent}(\upsilon)\Big\}.$$

Proof. As seen in the proof of Equation (1) from Section 2, a ballot can be taken to a permutation by writing the entries within a block in decreasing order and concatenating the blocks. By this method only the minimal element in a block may be an ascent bottom in the permutation, with the exception of the final block whose minimal element is the last element in the permutation.

Hence, for a fixed n and S, the ballot construction gives a bijection between permutations whose set of ascent bottoms is a subset of S and permutations whose set of ascent bottoms plus the last element is a subset of $\{1\} \cup \{s_1 + 1, \dots, s_k + 1\}$. Let $\pi = a_1 \dots a_n$ be any such permutation. We shall denote the set of ascent bottoms plus the final element of π as $T = \{t_1, t_2, \dots, t_j\}$:

$$A(\pi) \cup \{a_n\} = T \subseteq \{1\} \cup \{s_1 + 1, \dots, s_k + 1\}.$$

An element in a permutation can either be an ascent bottom, a descent top, or the final element. Taking the complement of a permutation takes an ascent bottom t_i to a descent top $n + 1 - t_i$. Letting π^c denote the complement of π, it follows that for π^c the set of descent tops and final element is

$$\{n + 1 - t_1, n + 1 - t_2, \dots, n + 1 - t_j\} \subseteq \{n\} \cup \{n - s_1, \dots, n - s_k\},$$

which contains at least the element n. The set of ascent bottoms in π^c contains everything which is not a descent top or the final element.

$$A(\pi^c) = [n] \setminus \{n + 1 - t_1, \dots, n + 1 - t_j\}.$$

As $T \subseteq \{1\} \cup \{s_1 + 1, \dots, s_k + 1\}$, it follows that

$$[n-1] \setminus \{n - s_1, \dots, n - s_k\} \subseteq A(\pi^c).$$

From Corollary 9 we have that π^c corresponds to an inversion table whose entries are exactly those in $\{0\} \cup A(\pi^c)$, thus giving a unique inversion table satisfying

$$[0, n-1] \setminus \{n - s_1, \dots, n - s_k\} \subseteq \mathrm{Dent}(\upsilon).$$

This concludes the proof. $\square$

Example 13. As in previous examples, let $n = 8$, $S = \{3, 5, 6, 7\}$ and consider the construction choice 42001000. From Example 6 the permutation in $\boldsymbol{\kappa}_n(S)$ that is given by the construction choice is $\pi = 65783421$. We wish to find the inversion table υ corresponding to π satisfying

$$[0, 7] \setminus \{8-3, 8-5, 8-6, 8-7\} = \{0, 4, 6, 7\} \subseteq \mathrm{Dent}(\upsilon).$$

From Example 11 the ballot given by the construction choice is $\{6\}\{1, 2, 3, 5\}\{7\}\{4\}\{8\}$. Writing the elements within a block in decreasing order and concatenating the blocks gives the permutation $\tau = 65321748$ with set of ascent bottoms $\{1, 4\}$ and final element $\{8\}$ where

$$\{1, 4, 8\} \subset \{1, s_1 + 1, \ldots, s_k + 1\} = \{1, 3 + 1, 5 + 1, 6 + 1, 7 + 1\}.$$

The complement of τ is $\tau^c = 34678251$ and has set of descent tops $\{9-4, 9-1\} = \{5, 8\}$ and final element $9 - 8 = 1$. Every other entry in τ^c is an ascent bottom:

$$A(\tau^c) = \{2, 3, 4, 6, 7\}.$$

Taking $S' = A(\tau^c)$, it follows from Proposition 5 that the construction choice uniquely specifying $\tau^c \in \boldsymbol{\kappa}_n(S')$ is 54312000. Applying Proposition 8 and Corollary 9, we can show that τ^c corresponds to the inversion table 76423000, which, by construction, has set of distinct entries

$$\mathrm{Dent}(76423000) = \{0, 2, 3, 4, 6, 7\} = \{0\} \cup A(\tau^c).$$

Thus we have constructed υ satisfying $\{0, 4, 6, 7\} \subseteq \{0, 2, 3, 4, 6, 7\} = \mathrm{Dent}(\upsilon)$.

7 Decomposition of fixed points

Recall that matrices fixed under the involution η satisfy the properties

1. there is exactly one element per row;
2. if $a < b$, with a on row i, and b on row $i + 1$, then column i is non-empty.

Also recall that a fixed point matrix can be viewed as a pair consisting of a permutation and an inversion table.

For $A \in \mathrm{BalMat}^\eta[U]$ where $n = |U|$, let $\pi(A) = a_1 \ldots a_n$ be the permutation defined by setting a_i the value held in the unique nonzero element of row i of A. Let an equivalence relation $\sim$ on $\mathrm{BalMat}^\eta[U]$ be defined by $A \sim B$ if $\pi(A) = \pi(B)$.

Proposition 14. *For $\pi \in \mathcal{S}_n$, the equivalence class $[\pi]_\sim$ is determined by the descent set $S = \{s_1, s_2, \ldots, s_n\} = D(\pi)$ of π alone. In fact, fixed point matrices in $[\pi]_\sim$ can be viewed as pairs consisting of the permutation π and an inversion table whose set of missing entries is a subset of $\{n - s_1, n - s_2, \ldots, n - s_k\}$.*

Proof. It is a defining property of a fixed point matrix that if $a < b$, with a on row i, and b on row $i+1$, then column i is required to be non-empty. This is equivalent to saying that when the matrix is decomposed into a permutation and inversion table, that $n-i$ is an entry contained within the inversion table.

So, if $a > b$ then we have a descent in the associated permutation and therefore column i may or may not be empty. It follows that $n-i$ may or may not be contained in the inversion table.

Therefore, for $\pi \in \mathcal{S}_n$, if the set of descent positions is $D(\pi) = S = \{s_1, s_2, \dots, s_k\}$, then the set of inversion tables with which π can be paired are exactly those where the set of missing entries is a subset of $\{n-s_1, n-s_2, \dots, n-s_k\}$. $\square$

Theorem 15. *Labeled interval orders on $[n]$ are in bijection with the set*

$$\sum_{S \subseteq [n-1]} \boldsymbol{\beta}_n(S) \times \boldsymbol{\kappa}_n(S).$$

This set may be alternatively written as

$$\{(\pi, \tau) \in \mathcal{S}_n \times \mathcal{S}_n : A(\tau) \subseteq D(\pi)\}.$$

Proof. The adapted surjection of Dukes et al. is a bijection between labeled interval orders and fixed point ballot matrices. This is given by the equivalence class on ballot matrices according to interval order and Proposition 3 which shows that there is a unique fixed point per equivalence class.

A fixed point matrix can be decomposed into a permutation π and an inversion table. If $D(\pi) = \{s_1, s_2, \dots s_k\}$ Proposition 14 gives that the set of inversion tables with which π can be paired are those whose set of missing elements is a subset of $\{n-s_1, n-s_2, \dots, n-s_k\}$. We know from Proposition 12 that such inversion tables are in bijection with permutations in $\boldsymbol{\kappa}_n(D(\pi))$. $\square$

Corollary 16. *The number of labeled interval orders on $[n]$ is given by the formula*

$$\sum_{\{s_1,\dots,s_k\} \subseteq [n-1]} \left(\det \left[\binom{n-s_i}{s_{j+1}-s_i} \right] \cdot \prod_{r=1}^{k+1} r^{s_r - s_{r-1}} \right),$$

in which $s_0 = 0$ and $s_{k+1} = n$.

Proof. This follows from the formula for β_n, see Stanley [9, Example 2.2.4], and the formula for κ_n given by Corollary 7. $\square$

In the above we have taken the permutation to be fixed and considered the set of inversion tables in the equivalence class under $\sim$. It is equally natural to instead take the inversion table as fixed.

As before, for $A \in \mathrm{BalMat}^{\eta}[U]$, let $\upsilon(A) = b_1 b_2 \dots b_n$ be the inversion table from the decomposition of a ballot matrix fixed under η defined by setting b_i to $n-j$ where j is the column of the only non-empty ballot entry on row i of A.

Let the equivalence relation $\approx$ on $\mathrm{BalMat}^{\eta}[U]$ be defined by $A \approx B$ if $\upsilon(A) = \upsilon(B)$.

Proposition 17. *For $\upsilon \in \mathrm{InvTab}_n$, the equivalence class $[\upsilon]_\approx$ is determined by* $\mathrm{Dent}(\upsilon)$ *alone. In fact, fixed point matrices in $[\upsilon]_\approx$ can be viewed as pairs consisting of the inversion table υ and a permutation whose descent set is a subset of* $\mathrm{Dent}(\upsilon) \setminus \{0\}$.

Proof. This proof is similar to that of Proposition 14. Define $S = \{s_1, s_2, \dots, s_k\}$ to be the set of distinct entries in υ with the exception of 0.

$$S = \mathrm{Dent}(\upsilon) \setminus \{0\}.$$

From the definition of the decomposition, matrices in $[\upsilon]_\approx$ satisfy that columns $n - s_1$, $n - s_2$, ..., $n - s_k$ are non-empty.

Recall that, for a ballot matrix fixed under η, if there is an ascent at position i, $a_i < a_{i+1}$, then column i must be non-empty. If there is a descent, then it may or may not be non-empty. Therefore the set of ascent positions in the associated permutation must be a subset of $n - s_1, n - s_2, \dots, n - s_k$. Trivially, reversing such a permutation yields a permutation whose descent set is a subset of $s_1, s_2, \dots, s_k$.

Therefore, for any given inversion table where the distinct entries is $\{0\} \cup S$, the set of permutations which can be associated are trivially in bijection with those where the descent set is a subset of S. $\square$

Theorem 18. *Labeled interval orders on $[n]$ are in bijection with the set*

$$\sum_{S \subseteq [n-1]} \boldsymbol{\alpha}_n(S) \times \boldsymbol{\lambda}_n(S).$$

This set may be alternatively written as

$$\{(\pi, \tau) \in \mathcal{S}_n \times \mathcal{S}_n : D(\pi) \subseteq A(\tau)\}.$$

Proof. Corollary 9 gives that permutations in $\boldsymbol{\lambda}_n(S)$ are in bijection with inversion tables with set of distinct elements $\{0\} \cup S$. Proposition 17 states that the permutations with which an inversion table υ can be paired are those with their descent set a subset of $\mathrm{Dent}(\upsilon) \setminus \{0\}$. From definition, such permutations are those contained within $\boldsymbol{\alpha}_n(S)$. $\square$

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