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Sign-changing solutions for fourth-order elliptic equations of Kirchhoff type with critical exponent

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Abstract. In this paper, we study the existence of ground state sign-changing solutions for the following fourth-order elliptic equations of Kirchhoff type with critical exponent. More precisely, we consider

$$\begin{cases} \Delta^2 u - \left(1 + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = \lambda f(x,u) + |u|^{2^{**}-2}u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ^2 is the biharmonic operator, $N = \{5,6,7\}$, $2^{**} = 2N/(N-4)$ is the Sobolev critical exponent and $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary and b, λ are some positive parameters. By using constraint variational method, topological degree theory and the quantitative deformation lemma, we prove the existence of ground state sign-changing solutions with precisely two nodal domains.

Keywords: Kirchhoff type problem, fourth-order elliptic equation, critical growth, sign-changing solution.

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1 Introduction and main results

In this paper, we are interested in the existence of least energy nodal solutions for the following Kirchhoff-type fourth-order Laplacian equations with critical growth:

$$\begin{cases} \Delta^2 u - \left(1 + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = \lambda f(x,u) + |u|^{2^{**}-2}u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Δ^2 is the biharmonic operator, $N = \{5,6,7\}$, $2^{**} = 2N/(N-4)$ is the Sobolev critical exponent, $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary, and b, λ are some positive parameters.

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Now we introduce the assumptions on the function f that will in full force throughout the paper. More precisely, we suppose that $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies the following conditions:

(f_1) $\lim_{t\to 0} \frac{f(x,t)}{|t|^3} = 0$;

(f_2) There exist $\theta \in (4, 2^{**})$ and $C > 0$ such that $|f(x,t)| \leq C(1 + |t|^{\theta-1})$ for all $t \in \mathbb{R}$;

(f_3) $\frac{f(x,t)}{|t|^3}$ is a strictly increasing function of $t \in \mathbb{R} \setminus \{0\}$.

A simple example of function satisfying the above assumptions (f_1)–(f_3) is $f(t) = t|t|^{\theta-2}$ for any $t \in \mathbb{R}$, where $\theta \in (4, 2^{**})$.

Our motivation for studying problem (1.1) is two-fold. On the one hand, there is a vast literature concerning the existence and multiplicity of solutions for the following Dirichlet problem of Kirchhoff type

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u) & x \in \Omega, \\ u = 0 & x \in \partial\Omega. \end{cases} \tag{1.2}$$

Problem (1.2) is a generalization of a model introduced by Kirchhoff. More precisely, Kirchhoff proposed a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.3}$$

where ρ, ρ_0, h, E, L are constants, which extends the classical d'Alembert's wave equation, by considering the effects of the changes in the length of the strings during the vibrations. The problem (1.2) is related to the stationary analogue of problem (1.3). Problem (1.2) received much attention only after Lions [17] proposed an abstract framework to the problem. For example, some important and interesting results can be found in [5,9,10,12–14,16,25,26,39]. We note that the results dealing with the problem (1.2) with critical nonlinearity are relatively scarce. The main difficulty in the study of these problems is due to the lack of compactness caused by the presence of the critical Sobolev exponent.

Recently, many researchers devoted themselves to the following fourth-order elliptic equations of Kirchhoff type

$$\begin{cases} \Delta^2 u - \left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u), & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega. \end{cases} \tag{1.4}$$

In fact, this is related to the following stationary analogue of the Kirchhoff-type equation:

$$u_{tt} + \Delta^2 u - \left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u), \quad x \in \Omega, \tag{1.5}$$

where $a, b > 0$. In [2,4], Eq. (1.5) was used to describe some phenomena appearing in different physical, engineering and other sciences for dimension $N \in \{1,2\}$, as a good approximation for describing nonlinear vibrations of beams or plates. Different approaches have been taken to deal with this problem under various hypotheses on the nonlinearity. For example, Ma in [21] considered the existence and multiplicity of positive solutions for the fourth-order equation by using the fixed point theorems in cones of ordered Banach spaces. By variational methods, Wang and An in [34] studied the following fourth-order equation of Kirchhoff type

$$\begin{cases} \Delta^2 u - M\left(\int_\Omega |\nabla u|^2 dx\right) \Delta u = f(x,u), & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega, \end{cases} \tag{1.6}$$

and obtained the existence and multiplicity of solutions, see [19,20,34] for more results. For $M(t) = \lambda(a + bt)$, Wang *et al.* in [35] proved the existence of solutions for problem (1.6) as λ small, by employing the mountain pass theorem and the truncation method. In [30], Song and Shi obtained the existence and multiplicity of solutions for problem (1.6) critical exponent in bounded domains by using the concentration-compactness principle and variational method. In [41], by variational methods together with the concentration-compactness principle, Zhao *et al.* investigated the existence and multiplicity of solutions for problem (1.6) with critical nonlinearity. In [15], by using the same method as in [41], Liang and Zhang obtained the existence and multiplicity of solutions for perturbed biharmonic equation of Kirchhoff type with critical nonlinearity in the whole space.

On the other hand, many authors paid attention to finding sign-changing solutions for problem (1.2) or similar Kirchhoff-type equations, and consequently some interesting results have been obtained recently. For example, Zhang and Perera in [40] and Mao and Zhang in [23] used the method of invariant sets of descent flow to obtain the existence of a sign-changing solution of problem (1.2). In [7], Figueiredo and Nascimento studied the following Kirchhoff equation of type:

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2 dx\right) \Delta u = f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.7}$$

where Ω is a bounded domain in $\mathbb{R}^3$, M is a general C^1 class function, and f is a superlinear C^1 class function with subcritical growth. By using the minimization argument and a quantitative deformation lemma, the existence of a sign-changing solution for this Kirchhoff equation was obtained. In unbounded domains, Figueiredo and Santos Júnior in [8] studied a class of nonlocal Schrödinger–Kirchhoff problems involving only continuous functions. Using a minimization argument and a quantitative deformation lemma, they got a least energy sign-changing solution to Schrödinger–Kirchhoff problems. Moreover, the authors obtained that the problem has infinitely many nontrivial solutions when it presents symmetry.

It is worth mentioning that combining constraint variational methods and quantitative deformation lemma, Shuai in [29] proved that problem (1.2) has one least energy sign-changing solution u_b and the energy of u_b strictly larger than the ground state energy. Moreover, the author investigated the asymptotic behavior of u_b as the parameter $b \searrow 0$. Later, under some more weak assumptions on f (especially, Nehari type monotonicity condition been removed), with the aid of some new analytical skills and Non-Nehari manifold method, Tang and Cheng in [32] improved and generalized some results obtained in [29]. In [6], Deng *et al.* studied the following Kirchhoff-type problem:

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\right) \Delta u + V(x)u = f(x,u), \qquad x \in \mathbb{R}^3. \tag{1.8}$$

The authors obtained the existence of radial sign-changing solutions with prescribed numbers of nodal domains for Kirchhoff problem (1.8), by using a Nehari manifold and gluing solution pieces together, when $V(x) = V(|x|)$, $f(x,u) = f(|x|,u)$ and satisfies some assumtions.

Precisely, they proved the existence of a sign-changing solution, which changes signs exactly k times for any $k \in \mathbb{N}$. Moreover, they investigated the energy property and the asymptotic behavior of the sign-changing solution. By using a combination of the invariant set method and the Ljusternik-Schnirelman type minimax method, Sun *et al.* in [31] obtained infinitely many sign-changing solutions for Kirchhoff problem (1.8) when $f(x,u) = f(u)$ and f is odd in u. It is worth noticing that, in [31], the nonlinear term may not be 4-superlinear at infinity; In particular, it encloses the power-type nonlinearity $|u|^{p-2}u$ with $p \in (2,4]$. In [33], the authors obtained the existence of least energy sign-changing solutions of Kirchhoff-type equation with critical growth by using the constraint variational method and the quantitative deformation lemma. For more results on sign-changing solutions for Kirchhoff-type equations, we refer the reader to [6,11,18,22,36] and the references therein.

However, concerning the existence of sign-changing solutions for fourth-order elliptic equations of Kirchhoff type with critical exponent, to the best of our knowledge, so far there has been no paper in the literature where existence of sign-changing solutions to problem (1.1). Hence, a natural question is whether or not there exists sign-changing solutions of problem (1.1)? The goal of the present paper is to give an affirmative answer.

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth open domain, $E = H^2(\Omega) \cap H_0^1(\Omega)$ be the Hilbert space equipped with the inner product

$$\langle u, v\rangle_E = \int_\Omega (\Delta u \Delta v + \nabla u \nabla v) dx$$

and the deduced norm

$$\|u\|_E^2 = \int_\Omega (|\Delta u|^2 + |\nabla u|^2) dx.$$

It is well know that $\|u\|_E$ is equivalent to

$$\|u\| := \left(\int_\Omega |\Delta u|^2 dx\right)^{\frac{1}{2}}.$$

And there exists $\tau > 0$ such that

$$\|u\| \leq \|u\|_E \leq \tau \|u\|.$$

For the weak solution, we mean the one satisfies the following definition.

Definition 1.1. We say that $u \in E$ is a (weak) solution of problem (1.1) if

$$\int_\Omega (\Delta u \cdot \Delta v + \nabla u \nabla v)\, dx + b\left(\int_\Omega |\nabla u|^2 dx\right) \int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega \left(|u|^{2^{**}-2} u v + \lambda f(x,u) v\right) \mathrm{d}x \quad (1.9)$$

for any $v \in E$.

The corresponding energy functional $I_b^\lambda : E \to \mathbb{R}$ to problem (1.1) is defined by

$$\begin{aligned} I_b^\lambda(u) = &\frac{1}{2}\int_\Omega (|\Delta u|^2 + |\nabla u|^2)\, dx + \frac{b}{4}\left(\int_\Omega |\nabla u|^2 dx\right)^2 \\ &- \lambda \int_\Omega F(x,u) dx - \frac{1}{2^{**}} \int_\Omega |u|^{2^{**}} dx. \end{aligned} \quad (1.10)$$

It is easy to see that I_b^λ belongs to $C^1(E,\mathbb{R})$ and the critical points of I_b^λ are the solutions of (1.1). Furthermore, if we write $u^+(x) = \max\{u(x), 0\}$ and $u^-(x) = \min\{u(x), 0\}$ for $u \in E$, then every solution $u \in E$ of problem (1.1) with the property that $u^\pm \neq 0$ is a sign-changing solution of problem (1.1).

Our goal in this paper is then to seek for the least energy sign-changing solutions of problem (1.1). As well known, there are some very interesting studies, which studied the existence and multiplicity of sign-changing solutions for the following problem:

$$-\Delta u + V(x)u = f(x,u), \quad x \in \Omega, \tag{1.11}$$

where Ω is an open subset of $\mathbb{R}^N$. However, these methods of seeking sign-changing solutions heavily rely on the following decompositions:

$$J(u) = J(u^+) + J(u^-), \tag{1.12}$$

$$\langle J'(u), u^+\rangle = \langle J'(u^+), u^+\rangle, \langle J'(u), u^-\rangle = \langle J'(u^-), u^-\rangle, \tag{1.13}$$

where J is the energy functional of (1.11) given by

$$J(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + V(x)u^2)dx - \int_\Omega F(x,u)dx.$$

However, if $b > 0$, the energy functional I_b^λ does not possess the same decompositions as (1.12) and (1.13). In fact, a straightforward computation yields that

$$I_b^\lambda(u) > I_b^\lambda(u^+) + I_b^\lambda(u^-),$$

$$\langle (I_b^\lambda)'(u), u^+\rangle > \langle (I_b^\lambda)'(u^+), u^+\rangle \quad \text{and} \quad \langle (I_b^\lambda)'(u), u^-\rangle > \langle (I_b^\lambda)'(u^-), u^-\rangle$$

for $u^\pm \neq 0$. Therefore, the classical methods to obtain sign-changing solutions for the local problem (1.11) do not seem applicable to problem (1.1). In this paper, we follow the approach in [3] by defining the following constrained set

$$\mathcal{M}_b^\lambda = \Big\{u \in E, u^\pm \neq 0 \text{ and } \langle (I_b^\lambda)'(u), u^+\rangle = \langle (I_b^\lambda)'(u), u^-\rangle = 0\Big\} \tag{1.14}$$

and considering a minimization problem of I_b^λ on $\mathcal{M}_b^\lambda$. Indeed, by using the parametric method and implicit theorem, Shuai in [29] proved $\mathcal{M}_b^\lambda \neq \emptyset$ in the absence of the nonlocal term. However, the nonlocal term in problem (1.1), consisting of the biharmonic operator and the nonlocal term will cause some difficulties. Roughly speaking, compared to the general Kirchhoff type problem (1.2), decompositions (1.12) and (1.13) corresponding to I_b^λ are much more complicated. This results in some technical difficulties during the proof of the nonempty of $\mathcal{M}_b^\lambda$. Moreover, we find that the parametric method and implicit theorem are not applicable for problem (1.1) due to the complexity of the nonlocal term there. Therefore, our proof is based on a different approach which is inspired by [1], namely, we make use of a modified Miranda's theorem (cf. [24]). In addition, we are also able to prove that the minimizer of the constrained problem is also a sign-changing solution via the quantitative deformation lemma and degree theory.

Now we can present our first main result.

Theorem 1.2. *Assume that* (f_1)–(f_3) *hold. Then, there exists* $\lambda^* > 0$ *such that for all* $\lambda \geq \lambda^*$, *problem* (1.1) *has a least energy sign-changing solution* u_b.

Another goal of this paper is to establish the so-called energy doubling property (cf. [37]), i.e., the energy of any sign-changing solution of problem (1.1) is strictly larger than twice the ground state energy. For the semilinear equation problem (1.13), the conclusion is trivial. Indeed, if we denote the Nehari manifold associated to problem (1.13) by

$$\mathcal{N} = \{u \in E \setminus \{0\} \mid \langle J'(u), u\rangle = 0\}$$

and define

$$c = \inf_{u\in\mathcal{N}} J(u) \tag{1.15}$$

then it is easy to verify that $u^{\pm} \in \mathcal{N}$ for any sign-changing solution $u \in E$ for problem (1.13). Moreover, if the nonlinearity $f(x,t)$ satisfies some conditions (see [3]) which is analogous to (f_1)–(f_3), we can deduce that

$$J(w) = J(w^+) + J(w^-) \geq 2c. \tag{1.16}$$

We point out that the minimizer of (1.14) is indeed a ground state solution of problem (1.11) and $c > 0$ is the least energy of all weak solutions of problem (1.11). Therefore, by (1.15), it follows that the energy of any sign-changing solution of problem (1.11) is larger than twice the least energy. When $b > 0$, a similar result was obtained by Shuai [29] in a bounded domain Ω. We are also interested in that whether property (1.15) is still true for problem (1.1). To answer this question, we have the following result:

Theorem 1.3. *Assume that (f_1)–(f_3) hold. Then, there exists $\lambda^{**} > 0$ such that for all $\lambda \geq \lambda^{**}$, the $c^* := \inf_{u\in\mathcal{N}_b^\lambda} I_b^\lambda(u) > 0$ is achieved and $I_b^\lambda(u) > 2c^*$, where $\mathcal{N}_b^\lambda = \{u \in E \setminus \{0\} \mid \langle (I_b^\lambda)'(u), u\rangle = 0\}$ and u is the least energy sign-changing solution obtained in Theorem 1.2. In particular, $c^* > 0$ is achieved either by a positive or a negative function.*

The plan of this paper is as follows: Section 2 covers the proof of the achievement of least energy for the constraint problem (1.1), Section 3 is devoted to the proof of our main theorems.

Throughout this paper, we use standard notations. For simplicity, we use "$\to$" and "$\rightharpoonup$" to denote the strong and weak convergence in the related function space respectively. Various positive constants are denoted by C and C_i. We use ":=" to denote definitions and $B_r(x) := \{y \in \mathbb{R}^N \mid |x-y| < r\}$. We denote a subsequence of a sequence $\{u_n\}$ as $\{u_n\}$ to simplify the notation unless specified.

2 Some technical lemmas

Now, fixed $u \in E$ with $u^{\pm} \neq 0$, we define function $\psi_u : [0,\infty) \times [0,\infty) \to \mathbb{R}$ and mapping $T_u : [0,\infty) \times [0,\infty) \to \mathbb{R}^2$ by

$$\psi_u(\alpha,\beta) = I_b^\lambda(\alpha u^+ + \beta u^-) \tag{2.1}$$

and

$$T_u(\alpha,\beta) = \left(\langle (I_b^\lambda)'(\alpha u^+ + \beta u^-), \alpha u^+\rangle, \langle (I_b^\lambda)'(\alpha u^+ + \beta u^-), \beta u^-\rangle\right). \tag{2.2}$$

Lemma 2.1. *Assume that (f_1)–(f_3) hold, if $u \in E$ with $u^{\pm} \neq 0$, then there is the unique maximum point pair (α_u, β_u) of the function ψ such that $\alpha_u u^+ + \beta_u u^- \in \mathcal{M}_b^\lambda$.*

Proof. Our proof will be divided into three steps.

Step 1. For any $u \in E$ with $u^{\pm} \neq 0$, in the following, we will prove the existence of α_u and β_u.

From (f_1) and (f_2), for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$|f(x,t)| \leq \varepsilon|t| + C_\varepsilon|t|^{\theta-1} \quad \text{for all } t \in \mathbb{R}. \tag{2.3}$$

Then, by the Sobolev embedding theorem, we have that

$$\begin{aligned}
&\langle (I_b^\lambda)'(\alpha u^+ + \beta u^-), \alpha u^+ \rangle \\
&\quad \geq \alpha^2\|u^+\|^2 + b\alpha^4\|u^+\|^4 + b\alpha^2\beta^2\|u^+\|^2\|u^-\|^2 \\
&\qquad - \lambda\alpha^2\varepsilon\int_\Omega |u^+|^2 dx - \lambda C_\varepsilon \alpha^\theta \int_\Omega |u^+|^\theta dx - \alpha^{2^{**}}\int_\Omega |u|^{2^{**}} dx \\
&\quad \geq \alpha^2\|u^+\|^2 + b\alpha^4\|u^+\|^4 - \lambda\alpha^2\varepsilon C_1\|u^+\|^2 - \lambda C_\varepsilon\alpha^\theta C_2\|u^+\|^\theta - C_3\alpha^{2^{**}}\|u^+\|^{2^{**}} \\
&\quad = (1-\lambda\varepsilon C_1)\,\alpha^2\|u^+\|^2 + b\alpha^4\|u^+\|^4 - \lambda C_\varepsilon\alpha^\theta C_2\|u^+\|^\theta - C_3\alpha^{2^{**}}\|u^+\|^{2^{**}}.
\end{aligned}$$

Choose $\varepsilon > 0$ such that $(1-\lambda\varepsilon C_1) > 0$. Since $2^{**}, \theta > 4$, we have that $\langle (I_b^\lambda)'(\alpha u^+ + \beta u^-), \alpha u^+ \rangle > 0$ for α small enough and all $\beta \geq 0$.

Similarly, we obtain that $\langle (I_b^\lambda)'(\alpha u^+ + \beta u^-), \beta u^- \rangle > 0$ for β small enough and all $\alpha \geq 0$.

Therefore, there exists $\delta_1 > 0$ such that

$$\langle (I_b^\lambda)'(\delta_1 u^+ + \beta u^-), \delta_1 u^+ \rangle > 0, \quad \langle (I_b^\lambda)'(\alpha u^+ + \delta_1 u^-), \delta_1 u^- \rangle > 0 \tag{2.4}$$

for all $\alpha, \beta \geq 0$.

On the other hand, by (f_2) and (f_3), we have that

$$f(x,t)t > 0, \quad t \neq 0; \quad F(x,t) \geq 0, \quad t \in \mathbb{R}. \tag{2.5}$$

In fact, by (f_2) and (f_3), we obtain that $f(x,t) > 0 (< 0)$ for $t > 0 (< 0)$ and almost every $x \in \Omega$. Moreover, by (f_2) and continuity of f, it follows that $f(x,0) = 0$ for almost every $x \in \Omega$. Therefore, $F(x,t) \geq 0$ for $t \geq 0$ and almost every $x \in \Omega$.

If $t < 0$, by (f_3), we have

$$F(x,t) = \int_0^t \frac{f(x,s)}{s^3} s^3 ds \geq \frac{f(x,t)}{t^3}\int_0^t s^3 ds = \frac{1}{4} f(x,t)t > 0, \quad \text{a.e. } x \in \Omega,$$

since $t \leq s < 0$ and $f(x,t) < 0$ for a.e. $x \in \Omega$.

From the above arguments, we conclude that (2.5) holds.

Therefore, choose $\alpha = \delta_2^* > \delta_1$, if $\beta \in [\delta_1, \delta_2^*]$ and δ_2^* is large enough, it follows that

$$\begin{aligned}
&\langle (I_b^\lambda)'(\delta_2^* u^+ + \beta u^-), \delta_2^* u^+ \rangle \\
&\quad \leq \tau(\delta_2^*)^2\|u^+\|^2 + b(\delta_2^*)^4\|u^+\|^4 + b(\delta_2^*)^4\|u^+\|^2\|u^-\|^2 - (\delta_2^*)^{2^{**}}\int_\Omega |u^+|^{2^{**}} dx \leq 0.
\end{aligned}$$

Similarly, we have that

$$\begin{aligned}
&\langle (I_b^\lambda)'(\alpha u^+ + \delta_2^* u^-), \delta_2^* u^- \rangle \\
&\quad \leq \tau(\delta_2^*)^2\|u^-\|^2 + b(\delta_2^*)^4\|u^+\|^4 + b(\delta_2^*)^4\|u^+\|^2\|u^-\|^2 - (\delta_2^*)^{2^{**}}\int_\Omega |u^-|^{2^{**}} dx \leq 0.
\end{aligned}$$

Let $\delta_2 > \delta_2^*$ be large enough, we obtain that

$$\langle (I_b^\lambda)'(\delta_2^* u^+ + \beta u^-), \delta_2^* u^+ \rangle < 0 \quad \text{and} \quad \langle (I_b^\lambda)'(\alpha u^+ + \delta_2^* u^-), \delta_2^* u^- \rangle < 0 \tag{2.6}$$

for all $\alpha, \beta \in [\delta_1, \delta_2]$.

Combining (2.4) and (2.6) with Miranda's theorem [24], there exists $(\alpha_u, \beta_u) \in (0, +\infty) \times (0, +\infty)$ such that $T_u(\alpha, \beta) = (0, 0)$, i.e., $\alpha u^+ + \beta u^- \in \mathcal{M}_b^\lambda$.

Step 2. In this step, we prove the uniqueness of the pair (α_u, β_u).

• Case $u \in \mathcal{M}_b^\lambda$.

If $u \in \mathcal{M}_b^\lambda$, we have that

$$\|u^+\|_E^2 + b\|u^+\|^4 + b\|u^+\|^2\|u^-\|^2 = \lambda \int_\Omega f(x, u^+)u^+ dx + \int_\Omega |u^+|^{2^{**}} dx \tag{2.7}$$

and

$$\|u^-\|_E^2 + b\|u^-\|^4 + b\|u^+\|^2\|u^-\|^2 = \lambda \int_\Omega f(x, u^-)u^- dx + \int_\Omega |u^-|^{2^{**}} dx. \tag{2.8}$$

We show that $(\alpha_u, \beta_u) = (1, 1)$ is the unique pair of numbers such that $\alpha_u u^+ + \beta_u u^- \in \mathcal{M}_b^\lambda$.

Let (α_0, β_0) be a pair of numbers such that $\alpha_0 u^+ + \beta_0 u^- \in \mathcal{M}_b^\lambda$ with $0 < \alpha_0 \le \beta_0$. Hence, one has that

$$\alpha_0^2\|u^+\|_E^2 + b\alpha_0^4\|u^+\|^4 + b\alpha_0^2\beta_0^2\|u^+\|^2\|u^-\|^2 = \lambda \int_\Omega f(x, \alpha_0 u^+)\alpha_0 u^+ dx + \alpha_0^{2^{**}} \int_\Omega |u^+|^{2^{**}} dx \tag{2.9}$$

and

$$\begin{aligned} &\beta_0^2\|u^-\|_E^2 + b\beta_0^4\|u^-\|^4 + b\alpha_0^2\beta_0^2\|u^+\|^2\|u^-\|^2 \\ &\quad = \lambda \int_\Omega f(x, \beta_0 u^-)\beta_0 u^- dx + \beta_0^{2^{**}} \int_\Omega |u^-|^{2^{**}} dx. \end{aligned} \tag{2.10}$$

According to $0 < \alpha_0 \le \beta_0$ and (2.10), we have that

$$\frac{\|u^-\|_E^2}{\beta_0^2} + b\|u^-\|^4 + b\|u^+\|^2\|u^-\|^2 \ge \lambda \int_\Omega \frac{f(x, \beta_0 u^-)}{(\beta_0 u^-)^3}(u^-)^4 dx + \beta_0^{2^{**}-4} \int_\Omega |u^-|^{2^{**}} dx. \tag{2.11}$$

If $\beta_0 > 1$, by (2.8) and (2.11), one has that

$$\left(\frac{1}{\beta_0^2} - 1\right)\|u^-\|_E^2 \ge \lambda \int_\Omega \left[\frac{f(x, \beta_0 u^-)}{(\beta_0 u^-)^3} - \frac{f(x, u^-)}{(u^-)^3}\right](u^-)^4 dx + (\beta_0^{2^{**}-4} - 1)\int_\Omega |u^-|^{2^{**}} dx.$$

Thus, for any $\beta_0 > 1$, the left side of the above inequality is negative, the right-hand side above is greater than zero by condition (f_3), which is a contradiction. Therefore, we conclude that $0 < \alpha_0 \le \beta_0 \le 1$.

Similarly, by (2.9) and $0 < \alpha_0 \le \beta_0$, we have that

$$\left(\frac{1}{\alpha_0^2} - 1\right)\|u^+\|_E^2 \le \lambda \int_\Omega \left[\frac{f(x, \alpha_0 u^+)}{(\alpha_0 u^+)^3} - \frac{f(x, u^-)}{(u^+)^3}\right](u^+)^4 dx + (\alpha_0^{2^{**}-4} - 1)\int_\Omega |u^+|^{2^{**}} dx.$$

According to condition (f_3), we obtain that $\alpha_0 \ge 1$.

Consequently, $\alpha_0 = \beta_0 = 1$.

• Case $u \notin \mathcal{M}_b^\lambda$.

Suppose that there exist (α_1, β_1), (α_2, β_2) such that

$$\omega_1 = \alpha_1 u^+ + \beta_1 u^- \in \mathcal{M}_b^\lambda \quad \text{and} \quad \omega_2 = \alpha_2 u^+ + \beta_2 u^- \in \mathcal{M}_b^\lambda.$$

Hence

$$\omega_2 = \left(\frac{\alpha_2}{\alpha_1}\right)\alpha_1 u^+ + \left(\frac{\beta_2}{\beta_1}\right)\beta_1 u^- = \left(\frac{\alpha_2}{\alpha_1}\right)\omega^+ + \left(\frac{\beta_2}{\beta_1}\right)\omega^- \in \mathcal{M}_b^\lambda.$$

By $\omega_1 \in \mathcal{M}_b^\lambda$, one has that

$$\frac{\alpha_2}{\alpha_1} = \frac{\beta_2}{\beta_1} = 1.$$

Hence, $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$.

Step 3. In this step, we will prove that (α_u, β_u) is the unique maximum point of ψ_u on $[0,\infty) \times [0,\infty)$.

In fact, by (2.3), we have that

$$\begin{aligned}
\psi_u(\alpha,\beta) &= I_b^\lambda(\alpha u^+ + \beta u^-) \\
&= \frac{1}{2}\|\alpha u^+ + \beta u^-\|_E^2 + \frac{b}{4}\|\alpha u^+ + \beta u^-\|^4 \\
&\quad -\lambda \int_\Omega F(x, \alpha u^+ + \beta u^-)dx - \frac{1}{2^{**}}\int_\Omega |\alpha u^+ + \beta u^-|^{2^{**}}dx \\
&= \frac{\alpha^2}{2}\|u^+\|_E^2 + \frac{\beta^2}{2}\|u^-\|_E^2 + \frac{b\alpha^4}{4}\|u^+\|^4 + \frac{b\beta^4}{4}\|u^-\|^4 + \frac{b\alpha^2\beta^2}{2}\|u^+\|^2\|u^-\|^2 \\
&\quad -\lambda\int_\Omega F(x,\alpha u^+)dx - \lambda\int_\Omega F(x,\beta u^-)dx - \frac{\alpha^{2^{**}}}{2^{**}}\int_\Omega |u^+|^{2^{**}}dx - \frac{\beta^{2^{**}}}{2^{**}}\int_\Omega |u^-|^{2^{**}}dx \\
&\le \frac{\tau\alpha^2}{2}\|u^+\|^2 + \frac{\tau\beta^2}{2}\|u^-\|^2 + \frac{b\alpha^4}{4}\|u^+\|^4 + \frac{b\beta^4}{4}\|u^-\|^4 + \frac{b\alpha^2\beta^2}{2}\|u^+\|^2\|u^-\|^2 \\
&\quad -\frac{\alpha^{2^{**}}}{2^{**}}\int_\Omega |u^+|^{2^{**}}dx - \frac{\beta^{2^{**}}}{2^{**}}\int_\Omega |u^-|^{2^{**}}dx,
\end{aligned}$$

which implies that $\lim_{|(\alpha,\beta)|\to\infty}\psi(\alpha,\beta) = -\infty$ thanks to $2^{**} > 4$.

Hence, (α_u, β_u) is the unique critical point of ψ_u in $[0,\infty)\times[0,\infty)$. So it is sufficient to check that a maximum point cannot be achieved on the boundary of $[0,\infty)\times[0,\infty)$. By contradiction, we suppose that $(0,\beta_0)$ is a maximum point of ψ_u with $\beta_0 \ge 0$. Then, we have that

$$\begin{aligned}
\psi_u(\alpha,\beta_0) = &\frac{\alpha^2}{2}\|u^+\|_E^2 + \frac{b\alpha^4}{4}\|u^+\|^4 - \lambda\int_\Omega F(x,\alpha u^+)dx - \frac{\alpha^{2^{**}}}{2^{**}}\int_\Omega |u^+|^{2^{**}}dx \\
&+\frac{\beta_0^2}{2}\|u^-\|_E^2 + \frac{b\beta_0^4}{4}\|u^-\|^4 - \lambda\int_\Omega F(x,\beta_0 u^-)dx - \frac{\beta^{2^{**}}}{2^{**}}\int_\Omega |u^-|^{2^{**}}dx \\
&+\frac{b\alpha^2\beta_0^2}{2}\|u^+\|^2\|u^-\|^2.
\end{aligned}$$

Therefore, it is obvious that

$$\begin{aligned}
(\psi_u)'_\alpha(\alpha,\beta_0) &= \alpha\|u^+\|_E^2 + b\alpha^3\|u^+\|^4 + b\alpha\beta_0^2\|u^+\|^2\|u^-\|^2 \\
&\quad -\lambda\int_\Omega f(x,\alpha u^+)u^+dx - \alpha^{2^{**}-1}\int_\Omega |u^+|^{2^{**}}dx \\
&\ge \alpha\|u^+\|^2 + b\alpha^3\|u^+\|^4 + b\alpha\beta_0^2\|u^+\|^2\|u^-\|^2 \\
&\quad -\lambda\int_\Omega f(x,\alpha u^+)u^+dx - \alpha^{2^{**}-1}\int_\Omega |u^+|^{2^{**}}dx \\
&> 0,
\end{aligned}$$

if α is small enough. That is, ψ_u is an increasing function with respect to α if α is small enough. This yields the contradiction. Similarly, ψ_u can not achieve its global maximum on $(\alpha, 0)$ with $\alpha \ge 0$. $\square$

Lemma 2.2. *Assume that* (f_1)–(f_3) *hold, if* $u \in E$ *with* $u^{\pm} \neq 0$ *such that* $\langle (I_b^{\lambda})'(u), u^{\pm} \rangle \leq 0$. *Then, the unique maximum point of* ψ_u *on* $[0,\infty) \times [0,\infty)$ *satisfies* $0 < \alpha_u, \beta_u \leq 1$.

Proof. In fact, if $\alpha_u \geq \beta_u > 0$. On the one hand, by $\alpha_u u^+ + \beta_u u^- \in \mathcal{M}_b^{\lambda}$, we have

$$\begin{aligned} &\alpha_u^2 \|u^+\|_E^2 + b\alpha_u^4 \|u^+\|^4 + b\alpha_u^4 \|u^+\|^2 \|u^-\|^2 \\ &\geq \alpha_u^2 \|u^+\|_E^2 + b\alpha_u^4 \|u^+\|^4 + b\alpha_u^2 \beta_u^2 \|u^+\|^2 \|u^-\|^2 \\ &= \lambda \int_{\Omega} f(x, \alpha_u u^+) \alpha_u u^+ dx + \alpha^{2^{**}} \int_{\Omega} |u^+|^{2^{**}} dx. \end{aligned} \tag{2.12}$$

On the other hand, by $\langle (I_b^{\lambda})'(u), u^+ \rangle \leq 0$, we have

$$\|u^+\|_E^2 + b\|u^+\|^4 + b\|u^+\|^2 \|u^-\|^2 \leq \lambda \int_{\Omega} f(x, u^+) u^+ dx + \int_{\Omega} |u^+|^{2^{**}} dx. \tag{2.13}$$

So, according to (2.12) and (2.13), we have that

$$\left(\frac{1}{\alpha_u^2} - 1 \right) \|u^+\|_E^2 \geq \lambda \int_{\Omega} \left[\frac{f(x, \alpha_u u^+)}{(\alpha_u u^+)^3} - \frac{f(x, u^+)}{(u^+)^3} \right] (u^+)^4 dx + (\alpha_u^{2^{**}-2} - 1) \int_{\Omega} |u^+|^{2^{**}} dx.$$

Thanks to condition (f_3), we conclude that $\alpha_u \leq 1$. Thus, we have that $0 < \alpha_u, \beta_u \leq 1$. □

Lemma 2.3. *Let* $c_b^{\lambda} = \inf_{u \in \mathcal{M}_b^{\lambda}} I_b^{\lambda}(u)$, *then we have that* $\lim_{\lambda \to \infty} c_b^{\lambda} = 0$.

Proof. For any $u \in \mathcal{M}_b^{\lambda}$, we have

$$\|u^{\pm}\|_E^2 + b\|u^{\pm}\|^4 + b\|u^+\|^2 \|u^-\|^2 = \lambda \int_{\Omega} f(x, u^{\pm}) u^{\pm} dx + \int_{\Omega} |u^{\pm}|^{2^{**}} dx.$$

Then, by (2.3) and Sobolev inequalities, we have that

$$\|u^{\pm}\|^2 \leq \lambda \int_{\Omega} f(x, u^{\pm}) u^{\pm} dx + \int_{\Omega} |u^{\pm}|^{2^{**}} dx \leq \lambda \varepsilon C_1 \|u^{\pm}\|^2 + \lambda C_{\varepsilon} C_2 \|u^{\pm}\|^{\theta} + C_3 \|u^{\pm}\|^{2^{**}}.$$

Thus, we get

$$(1 - \lambda \varepsilon C_1) \|u^{\pm}\|^2 \leq \lambda C_{\varepsilon} C_2 \|u^{\pm}\|^{\theta} + C_3 \|u^{\pm}\|^{2^{**}}.$$

Choosing ε small enough such that $1 - \lambda \varepsilon C_1 > 0$, since $2^{**} > 4$, there exists $\rho > 0$ such that

$$\|u^{\pm}\| \geq \rho \quad \text{for all} \;\; u \in \mathcal{M}_b^{\lambda}. \tag{2.14}$$

On the other hand, for any $u \in \mathcal{M}_b^{\lambda}$, it is obvious that $\langle (I_b^{\lambda})'(u), u \rangle = 0$. Thanks to (f_2) and (f_3), we obtain that

$$\Theta(x, t) := f(x, t)t - 4F(x, t) \geq 0 \tag{2.15}$$

and is increasing when $t > 0$ and decreasing when $t < 0$ for almost every $x \in \Omega$. Then, we have

$$I_b^{\lambda}(u) = I_b^{\lambda}(u) - \frac{1}{4} \langle (I_b^{\lambda})'(u), u \rangle \geq \frac{1}{4} \|u\|^2.$$

From above discussions, we have that $I_b^{\lambda}(u) > 0$ for all $u \in \mathcal{M}_b^{\lambda}$. Therefore, I_b^{λ} is bounded below on $\mathcal{M}_b^{\lambda}$, that is $c_b^{\lambda} = \inf_{u \in \mathcal{M}_b^{\lambda}} I_b^{\lambda}(u)$ is well defined.

Let $u \in E$ with $u^{\pm} \neq 0$ be fixed. By Lemma 2.1, for each $\lambda > 0$, there exist $\alpha_\lambda, \beta_\lambda > 0$ such that $\alpha_\lambda u^+ + \beta_\lambda u^- \in \mathcal{M}_b^\lambda$. By using Lemma 2.1 again, we have that

$$\begin{aligned}
0 \leq c_b^\lambda = \inf_{u \in \mathcal{M}_b^\lambda} I_b^\lambda(u) &\leq I_b^\lambda(\alpha_\lambda u^+ + \beta_\lambda u^-) \\
&\leq \frac{1}{2}\|\alpha_\lambda u^+ + \beta_\lambda u^-\|_E^2 + \frac{b}{4}\|\alpha_\lambda u^+ + \beta_\lambda u^-\|^4 \\
&\leq \alpha_\lambda^2 \|u^+\|_E^2 + \beta_\lambda^2 \|u^-\|_E^2 + 2b\alpha_\lambda^4 \|u^+\|^4 + 2b\beta_\lambda^4 \|u^-\|^4.
\end{aligned}$$

To the end, we just prove that $\alpha_\lambda \to 0$ and $\beta_\lambda \to 0$ as $\lambda \to \infty$.

Let

$$\mathcal{T}_u = \{(\alpha_\lambda, \beta_\lambda) \in [0,\infty) \times [0,\infty) : T_u(\alpha_\lambda, \beta_\lambda) = (0,0), \lambda > 0\},$$

where T_u is defined as (2.2). By (2.3), we have that

$$\begin{aligned}
&\alpha_\lambda^{2^{**}} \int_\Omega |u^+|^{2^{**}} dx + \beta_\lambda^{2^{**}} \int_\Omega |u^-|^{2^{**}} dx \\
&\quad \leq \alpha_\lambda^{2^{**}} \int_\Omega |u^+|^{2^{**}} dx + \beta_\lambda^{2^{**}} \int_\Omega |u^-|^{2^{**}} dx \\
&\qquad + \lambda \int_\Omega f(x, \alpha_\lambda u^+)\alpha_\lambda u^+ dx + \lambda \int_\Omega f(x, \beta_\lambda u^-)\beta_\lambda u^- dx \\
&\quad = \|\alpha_\lambda u^+ + \beta_\lambda u^-\|_E^2 + b\|\alpha_\lambda u^+ + \beta_\lambda u^-\|^4 \\
&\quad \leq 2\tau^2\alpha_\lambda^2 \|u^+\|^2 + 2\tau^2\beta_\lambda^2 \|u^-\|^2 + 4b\alpha_\lambda^4 \|u^+\|^4 + 4b\beta_\lambda^4 \|u^-\|^4.
\end{aligned}$$

Hence, $\mathcal{T}_u$ is bounded. Let $\{\lambda_n\} \subset (0,\infty)$ be such that $\lambda_n \to \infty$ as $n \to \infty$. Then, there exist α_0 and β_0 such that $(\alpha_{\lambda_n}, \beta_{\lambda_n}) \to (\alpha_0, \beta_0)$ as $n \to \infty$.

Now, we claim $\alpha_0 = \beta_0 = 0$. Suppose, by contradiction, that $\alpha_0 > 0$ or $\beta_0 > 0$. By $\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^- \in \mathcal{M}_b^{\lambda_n}$, for any $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\|\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-\|_E^2 + b\|\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-\|^4 \\
&\quad = \lambda_n \int_\Omega f(x, \alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-)(\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-) dx + \int_\Omega |\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-|^{2^{**}} dx.
\end{aligned} \tag{2.16}$$

Thanks to $\alpha_{\lambda_n} u^+ \to \alpha_0 u^+$ and $\beta_{\lambda_n} u^- \to \beta_0 u^+$ in E, (2.3) and (2.4), we have that

$$\int_\Omega f(x, \alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-)(\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-) dx \to \int_\Omega f(x, \alpha_0 u^+ + \beta_0 u^-)(\alpha_0 u^+ + \beta_0 u^-) dx > 0$$

as $n \to \infty$.

It follows from $\lambda_n \to \infty$ as $n \to \infty$ and $\{\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-\}$ is bounded in E, which contradicts equality (2.16). Hence, $\alpha_0 = \beta_0 = 0$.

Hence, we conclude that $\lim_{\lambda \to \infty} c_b^\lambda = 0$. □

Lemma 2.4. *There exists $\lambda^* > 0$ such that for all $\lambda \geq \lambda^*$, the infimum c_b^λ is achieved.*

Proof. By the definition of c_b^λ, there exists a sequence $\{u_n\} \subset \mathcal{M}_b^\lambda$ such that

$$\lim_{n \to \infty} I_b^\lambda(u_n) = c_b^\lambda.$$

Obviously, $\{u_n\}$ is bounded in E. Then, up to a subsequence, still denoted by $\{u_n\}$, there exists $u \in E$ such that $u_n \rightharpoonup u$. Since the embedding $E \hookrightarrow L^t(\Omega)$ is compact for all $t \in (2, 2^{**})$ (see [27]), we have

$$u_n \to u \quad \text{in } L^t(\Omega), \quad u_n \to u \quad \text{a.e. } x \in \Omega.$$

Hence

$$\begin{aligned} u_n^{\pm} &\rightharpoonup u^{\pm} \quad \text{in } E, \\ u_n^{\pm} &\to u^{\pm} \quad \text{in } L^t(\Omega), \\ u_n^{\pm} &\to u^{\pm} \quad \text{a.e. } x \in \Omega. \end{aligned}$$

By Lemma 2.1, we have

$$I_b^{\lambda}(\alpha u_n^+ + \beta u_n^-) \le I_b^{\lambda}(u_n)$$

for all $\alpha, \beta \ge 0$.

Then, by the Brézis–Lieb lemma and Fatou's lemma, we have that

$$\begin{aligned} \liminf_{n\to\infty} I_b^{\lambda}(\alpha u_n^+ + \beta u_n^-) \ge{}& \frac{\alpha^2}{2} \lim_{n\to\infty} (\|u_n^+ - u^+\|_E^2 + \|u^+\|_E^2) \\ &+ \frac{\beta^2}{2} \lim_{n\to\infty} (\|u_n^- - u^-\|_E^2 + \|u^-\|_E^2) \\ &+ \frac{b\alpha^4}{4} \left[\lim_{n\to\infty} (\|u_n^+ - u^+\|^2 + \|u^+\|^2)\right]^2 \\ &+ \frac{b\beta^4}{4} \left[\lim_{n\to\infty} (\|u_n^- - u^-\|^2 + \|u^-\|^2)\right]^2 \\ &- \frac{\alpha^{2^{**}}}{2^{**}} \left[\lim_{n\to\infty} \int_\Omega |u_n^+ - u^+|^{2^{**}} dx + \lim_{n\to\infty} \int_\Omega |u^+|^{2^{**}} dx\right] \\ &- \frac{\beta^{2^{**}}}{2^{**}} \lim_{n\to\infty} \left[\int_\Omega |u_n^- - u^-|^{2^{**}} dx + \int_\Omega |u^-|^{2^{**}} dx\right] \\ &- \lambda \int_\Omega F(x, \alpha u^+) dx - \lambda \int_\Omega F(x, \beta u^-) dx + \frac{b\alpha^2\beta^2}{2} \liminf_{n\to\infty} (\|u_n^+\|^2 \|u_n^-\|^2) \\ \ge{}& I_b^{\lambda}(\alpha u^+ + \beta u^-) + \frac{\alpha^2}{2} \lim_{n\to\infty} \|u_n^+ - u^+\|_E^2 + \frac{\beta^2}{2} \lim_{n\to\infty} \|u_n^- - u^-\|_E^2 \\ &+ \frac{b\alpha^4}{2} \lim_{n\to\infty} \|u_n^+ - u^+\|^2 \|u^+\|^2 + \frac{b\beta^4}{2} \lim_{n\to\infty} \|u_n^- - u^-\|^2 \|u^-\|^2 \\ &+ \frac{b\alpha^4}{4} (\lim_{n\to\infty} \|u_n^+ - u^+\|^2)^2 + \frac{bt^4}{4} (\lim_{n\to\infty} \|u_n^- - u^-\|^2)^2 \\ &- \frac{\alpha^{2^{**}}}{2^{**}} \int_\Omega |u_n^+ - u^+|^{2^{**}} dx - \frac{\beta^{2^{**}}}{2^{**}} \int_\Omega |u_n^- - u^-|^{2^{**}} dx \\ \ge{}& I_b^{\lambda}(\alpha u^+ + \beta u^-) + \frac{\alpha^2}{2} A_1 + \frac{b\alpha^4}{2} A_1 \|u^+\|^2 + \frac{b\alpha^4}{4} A_1^2 - \frac{\alpha^{2^{**}}}{2^{**}} B_1 \\ &+ \frac{\beta^2}{2} A_2 + \frac{b\beta^4}{2} A_2 \|u^-\|^2 + \frac{b\beta^4}{4} A_2^2 - \frac{\beta^{2^{**}}}{2^{**}} B_2, \end{aligned}$$

where

$$\begin{aligned} A_1 &= \lim_{n\to\infty} \|u_n^+ - u^+\|^2, \quad A_2 = \lim_{n\to\infty} \|u_n^- - u^-\|^2, \\ B_1 &= \lim_{n\to\infty} |u_n^+ - u^+|_{2^{**}}^{2^{**}}, \quad B_2 = \lim_{n\to\infty} |u_n^- - u^-|_{2^{**}}^{2^{**}}. \end{aligned}$$

That is, one has that

$$\begin{aligned} I_b^{\lambda}(\alpha u^+ + \beta u^-) &+ \frac{\alpha^2}{2} A_1 + \frac{b\alpha^4}{2} A_1 \|u^+\|^2 + \frac{b\alpha^4}{4} A_1^2 - \frac{\alpha^{2^{**}}}{2^{**}} B_1 \\ &+ \frac{\beta^2}{2} A_2 + \frac{b\beta^4}{2} A_2 \|u^-\|^2 + \frac{b\beta^4}{4} A_2^2 - \frac{\beta^{2^{**}}}{2^{**}} B_2 \le c_b^{\lambda} \end{aligned} \tag{2.17}$$

for all $\alpha \geq 0$ and all $\beta \geq 0$.

Now, we claim that $u^{\pm} \neq 0$.

In fact, since the situation $u^- \neq 0$ is analogous, we just prove $u^+ \neq 0$. By contradiction, we suppose $u^+ = 0$. Hence, let $\beta = 0$ in (2.17) and we have that

$$\frac{\alpha^2}{2}A_1 + \frac{b\alpha^4}{4}A_1^2 - \frac{\alpha^{2^{**}}}{2^{**}}B_1 \leq c_b^\lambda \tag{2.18}$$

for all $\alpha \geq 0$.

Case 1: $B_1 = 0$.

If $A_1 = 0$, that is, $u_n^+ \to u^+$ in E. In view of Lemma (2.14), we obtain $\|u^+\| > 0$, which contradicts our supposition. If $A_1 > 0$, by (2.18), we have that

$$\frac{\alpha^2}{2}A_1 + \frac{b\alpha^4}{4}A_1^2 \leq c_b^\lambda$$

for all $\alpha \geq 0$, which is absurd by Lemma 2.3. Anyway, we have a contradiction.

Case 2: $B_1 > 0$.

One one hand, by Lemma 2.3, there exists $\lambda^* > 0$ such that

$$c_b^\lambda < \frac{2}{N}S^{-2/N} \quad \text{for all } \lambda \geq \lambda^*, \tag{2.19}$$

where $S := \inf\left\{\int_\Omega |\Delta u|^2 dx : \int_\Omega |u|^{2^{**}} dx = 1\right\}$.

On the other hand, since $B_1 > 0$, we obtain $A_1 > 0$. Hence, in view of (2.18), we have that

$$\frac{2}{N}S^{-2/N} \leq \frac{2}{N}\left[\frac{A_1^{\frac{2^{**}}{2}}}{B_1}\right]^{\frac{2}{2^{**}-2}} \leq \max_{\alpha\geq 0}\left\{\frac{\alpha^2}{2}A_1 - \frac{\alpha^{2^{**}}}{2^{**}}B_1\right\} \leq \max_{\alpha\geq 0}\left\{\frac{\alpha^2}{2}A_1 + \frac{b\alpha^4}{4}A_1^2 - \frac{\alpha^{2^{**}}}{2^{**}}B_1\right\} \leq c_b^\lambda,$$

which is a contradiction. That is, we deduce that $u^{\pm} \neq 0$.

Next we prove $B_1 = B_2 = 0$.

Since the situation $B_2 = 0$ is analogous, we only prove $B_1 = 0$. By contradiction, we suppose that $B_1 > 0$.

Case 1: $B_2 > 0$.

According to $B_1, B_2 > 0$ and Sobolev embedding, we obtain that $A_1, A_2 > 0$. Let

$$\varphi(\alpha) = \frac{\alpha^2}{2}A_1 + \frac{b\alpha^4}{4}A_1^2 - \frac{\alpha^{2^{**}}}{2^{**}}B_1 \quad \text{for all } \ \alpha \geq 0.$$

It is easy to see that $\varphi(\alpha) > 0$ for $\alpha > 0$ small enough and $\varphi(\alpha) < 0$ for $\alpha < 0$ large enough. Hence, by continuous of $\varphi(\alpha)$, there exists $\hat{\alpha} > 0$ such that

$$\frac{\hat{\alpha}^2}{2}A_1 + \frac{b\hat{\alpha}^4}{4}A_1^2 - \frac{\hat{\alpha}^{2^{**}}}{2^{**}}B_1 = \max_{\alpha\geq 0}\left\{\frac{\alpha^2}{2}A_1 + \frac{b\alpha^4}{4}A_1^2 - \frac{\alpha^{2^{**}}}{2^{**}}B_1\right\}.$$

Similarly, there exists $\hat{\beta} > 0$ such that

$$\frac{\hat{\beta}^2}{2}A_2 + \frac{b\hat{\beta}^4}{4}A_2^2 - \frac{\hat{\beta}^{2^{**}}}{2^{**}}B_2 = \max_{\alpha\geq 0}\left\{\frac{\beta^2}{2}A_2 + \frac{b\beta^4}{4}A_2^2 - \frac{\beta^{2^{**}}}{2^{**}}B_2\right\}.$$

Since $[0,\hat{\alpha}]\times[0,\hat{\beta}]$ is compact and ψ is continuous, there exists $(\alpha_u,\beta_u)\in[0,\hat{\alpha}]\times[0,\hat{\beta}]$ such that

$$\psi(\alpha_u,\beta_u)=\max_{(\alpha,\beta)\in[0,\hat{\alpha}]\times[0,\hat{\beta}]}\psi(\alpha,\beta).$$

Now, we prove that $(\alpha_u,\beta_u)\in(0,\hat{\alpha})\times(0,\hat{\beta})$.

Note that, if β is small enough, we have that

$$\psi(\alpha,0)=I_b^\lambda(\alpha u^+)<I_b^\lambda(\alpha u^+)+I_b^\lambda(\beta u^-)\le I_b^\lambda(\alpha u^++\beta u^-)=\psi(\alpha,\beta)$$

for all $\alpha\in[0,\hat{\alpha}]$.

Hence, there exists $\beta_0\in[0,\hat{\beta}]$ such that

$$\psi(\alpha,0)\le\psi(\alpha,\beta_0)\quad\text{for all }\alpha\in[0,\hat{\alpha}].$$

That is, any point of $(\alpha,0)$ with $0\le\alpha\le\hat{\alpha}$ is not the maximizer of ψ. Hence, $(\alpha_u,\beta_u)\notin[0,\hat{\alpha}]\times\{0\}$. Similarly, we obtain $(\alpha_u,\beta_u)\notin\{0\}\times[0,\hat{\alpha}]$.

On the other hand, it is easy to see that

$$\frac{\alpha^2}{2}A_1+\frac{b\alpha^4}{2}A_1\|u^+\|^2+\frac{b\alpha^4}{4}A_1^2-\frac{\alpha^{2^{**}}}{2^{**}}B_1>0 \tag{2.20}$$

and

$$\frac{\beta^2}{2}A_2+\frac{b\beta^4}{2}A_2\|u^-\|^2+\frac{b\beta^4}{4}A_2^2-\frac{\beta^{2^{**}}}{2^{**}}B_2>0 \tag{2.21}$$

for $\alpha\in(0,\hat{\alpha}]$, $\beta\in(0,\hat{\beta}]$.

Then, we have that

$$\begin{aligned}\frac{2}{N}S^{-2/N}\le{}&\frac{\hat{\alpha}^2}{2}A_1+\frac{b\hat{\alpha}^4}{4}A_1^2-\frac{\hat{\alpha}^{2^{**}}}{2^{**}}B_1+\frac{b\hat{\alpha}^4}{2}A_1\|u^+\|^2\\&+\frac{\beta^2}{2}A_2+\frac{b\beta^4}{2}A_2\|u^-\|^2+\frac{b\beta^4}{4}A_2^2-\frac{\beta^{2^{**}}}{2^{**}}B_2\end{aligned}$$

and

$$\begin{aligned}\frac{2}{N}S^{-2/N}\le{}&\frac{\tilde{\beta}^2}{2}A_2+\frac{b\tilde{\beta}^4}{4}A_2^2-\frac{\tilde{\beta}^{2^{**}}}{2^{**}}B_2+\frac{b\tilde{\beta}^4}{2}A_2\|u^-\|^2\\&+\frac{\alpha^2}{2}A_1+\frac{b\alpha^4}{2}A_1\|u^+\|^2+\frac{b\alpha^4}{4}A_1^2-\frac{\alpha^{2^{**}}}{2^{**}}B_1\end{aligned}$$

for all $\alpha\in[0,\tilde{\alpha}]$ and all $\beta\in[0,\tilde{\beta}]$.

Therefore, according to (2.17), we conclude that

$$\psi(\alpha,\hat{\beta})\le0,\quad\psi(\hat{\alpha},\beta)\le0$$

for all $\alpha\in[0,\hat{\alpha}]$ and all $\beta\in[0,\hat{\beta}]$.

Hence,$(\alpha_u,\beta_u)\notin\{\tilde{\alpha}\}\times[0,\tilde{\beta}]$ and $(\alpha_u,\beta_u)\notin[0,\tilde{\alpha}]\times\{\tilde{\beta}\}$.

Finally, we get that $(\alpha_u,\beta_u)\in(0,\hat{\alpha})\times(0,\hat{\beta})$. Hence, it follows that (α_u,β_u) is a critical point of ψ.

Hence, $\alpha_u u^++\beta_u u^-\in\mathcal{M}_b^\lambda$. From (2.17), (2.20), and (2.21), we have that

$$\begin{aligned}c_b^\lambda\ge{}&I_b^\lambda(\alpha_u u^++\beta_u u^-)+\frac{\alpha_u^2}{2}A_1+\frac{b\alpha_u^4}{2}A_1\|u^+\|^2+\frac{b\alpha_u^4}{4}A_1^2-\frac{\alpha^{2^{**}}}{2^{**}}B_1\\&+\frac{\beta_u^2}{2}A_2+\frac{b\beta_u^4}{2}A_2\|u^-\|^2+\frac{b\beta_u^4}{4}A_2^2-\frac{\beta^{2^{**}}}{2^{**}}B_2\\>{}&I_b^\lambda(\alpha_u u^++\beta_u u^-)\ge c_b^\lambda,\end{aligned}$$

which is a contradiction.

Case 2: $B_2 = 0$.

In this case, we can maximize in $[0, \hat{\alpha}] \times [0, \infty)$. Indeed, it is possible to show that there exist $\beta_0 \in [0, \infty)$ such that

$$I_b^\lambda(\alpha_u u^+ + \beta_u u^-) \le 0 \quad \text{for all } (\alpha, \beta) \in [0, \hat{\alpha}] \times [\beta_0, \infty).$$

Hence, there is $(\alpha_u, \beta_u) \in [0, \hat{\alpha}] \times [0, \infty)$ such that

$$\psi(\alpha_u, \beta_u) = \max_{(\alpha,\beta)\in[0,\hat{\alpha}]\times[0,\infty)} \psi(\alpha, \beta).$$

In the following, we prove that $(\alpha_u, \beta_u) \in (0, \hat{\alpha}) \times (0, \infty)$.

It is noted that $\psi(\alpha, 0) < \psi(\alpha, \beta)$ for $\alpha \in [0, \hat{\alpha}]$ and β small enough, so we have $(\alpha_u, \beta_u) \notin [0, \hat{\alpha}] \times \{0\}$.

Meanwhile, $\psi(0, \beta) < \psi(\alpha, \beta)$ for $\beta \in [0, \infty)$ and α small enough, then we have $(\alpha_u, \beta_u) \notin \{0\} \times [0, \infty)$.

On the other hand, it is obvious that

$$\frac{2}{N} S^{-2/N} \le \frac{\hat{\alpha}^2}{2} A_1 + \frac{b\hat{\alpha}^4}{4} A_1^2 - \frac{\alpha^{2^{**}}}{2^{**}} B_1 + \frac{b\hat{\alpha}^4}{2} A_2 \|u^+\|^2 + \frac{\beta^2}{2} A_2 + \frac{b\beta^4}{2} A_2 \|u^-\|^2 + \frac{b\beta^4}{4} A_2^2$$

for all $\beta \in [0, \infty)$.

Hence, we have that $\psi(\hat{\alpha}, \beta) \le 0$ for all $\beta \in [0, \infty)$. Thus, $(\alpha_u, \beta_u) \notin \{\hat{\alpha}\} \times [0, \infty)$. Hence, $(\alpha_u, \beta_u) \in (0, \hat{\alpha}) \times (0, \infty)$. That is, (α_u, β_u) is an inner maximizer of ψ in $[0, \hat{\alpha}) \times [0, \infty)$. Hence, $\alpha_u u^+ + \beta_u u^- \in \mathcal{M}_b^\lambda$.

Hence, in view of (2.20), we have that

$$\begin{aligned} c_b^\lambda &\ge I_b^\lambda(\alpha_u u^+ + \beta_u u^-) + \frac{\alpha_u^2}{2} A_1 + \frac{b\alpha_u^4}{2} A_1 \|u^+\|^2 + \frac{b\alpha_u^4}{4} A_1^2 - \frac{\alpha^{2^{**}}}{2^{**}} B_1 \\ &\quad + \frac{\beta_u^2}{2} A_2 + \frac{b\beta_u^4}{2} A_2 \|u^-\|^2 + \frac{b\beta_u^4}{4} A_2^2 \\ &> I_b^\lambda(\alpha_u u^+ + \beta_u u^-) \ge c_b^\lambda, \end{aligned}$$

which is a contradiction.

Therefore, from the above arguments, we have that $B_1 = B_2 = 0$.

Finally, we prove c_b^λ is achieved.

Since $u^\pm \ne 0$, by Lemma 2.1, there exist $\alpha_u, \beta_u > 0$ such that

$$\bar{u} := \alpha_u u^+ + \beta_u u^- \in \mathcal{M}_b^\lambda.$$

Furthermore, it is easy to see that

$$\langle (I_b^\lambda)'(u), u^\pm \rangle \le 0.$$

By Lemma 2.2, we obtain $0 < \alpha_u, \beta_u < 1$.

Since $u_n \in \mathcal{M}_b^\lambda$, according to Lemma 2.3, we get

$$I_b^\lambda(\alpha_u u_n^+ + \beta_u u_n^-) \le I_b^\lambda(u_n^+ + u_n^-) = I_b^\lambda(u_n).$$

Thanks to (f_3), $B_1 = B_2 = 0$ and that the norm in E is lower semicontinuous, we have that

$$
\begin{aligned}
c_b^\lambda &\le I_b^\lambda(\bar{u}) - \frac{1}{4}\langle (I_b^\lambda)'(\bar{u}), \bar{u}\rangle \\
&\le \frac{1}{4}\|\bar{u}\|_E^2 + \left(\frac{1}{4} - \frac{1}{2^{**}}\right)\int_\Omega |\bar{u}|^{2^{**}} dx + \frac{\lambda}{4}\int_\Omega [f(x,\bar{u})\bar{u} - 4F(x,\bar{u})]dx \\
&= \frac{1}{4}(\|\alpha_u u^+\|_E^2 + \|\beta_u u^-\|_E^2) + \left(\frac{1}{4} - \frac{1}{2^{**}}\right)\left[\int_\Omega |\alpha_u u^+|^{2^{**}} dx + \int_\Omega |\beta_u u^-|^{2^{**}} dx\right] \\
&\quad + \frac{\lambda}{4}\int_\Omega [f(x,\alpha_u u^+)(\alpha_u u^+) - 4F(x,\alpha_u u^+)]dx + \frac{\lambda}{4}\int_\Omega [f(x,\beta_u u^-)(\beta_u u^-) - 4F(x,\beta_u u^-)]dx \\
&\le \frac{1}{4}\|u\|_E^2 + \left(\frac{1}{4} - \frac{1}{2^{**}}\right)\int_\Omega |u|^{2^{**}} dx + \frac{\lambda}{4}\int_\Omega [f(x,u)u - 4F(x,u)]dx \\
&\le \liminf_{n\to\infty}\left[I_b^\lambda(u_n) - \frac{1}{4}\langle (I_b^\lambda)'(u_n), u_n\rangle\right] \le c_b^\lambda.
\end{aligned}
$$

Therefore, $\alpha_u = \beta_u = 1$, and c_b^λ is achieved by $u_b := u^+ + u^- \in \mathcal{M}_b^\lambda$. □

3 Proof of Theorems 1.2–1.3

In this section, we prove our main results. First, we prove Theorem 1.2. In fact, by means of Lemma 2.4, we just prove that the minimizer u_b for c_b^λ is indeed a sign-changing solution of problem (1.1).

Proof of Theorem 1.2. Since $u_b \in \mathcal{M}_b^\lambda$, we have $\langle (I_b^\lambda)'(u_b), u_b^+\rangle = \langle (I_b^\lambda)'(u_b), u_b^-\rangle = 0$. By Lemma 2.4, for $(\alpha,\beta) \in (\mathbb{R}_+ \times \mathbb{R}_+)\backslash(1,1)$, we have

$$
I_b^\lambda(\alpha u_b^+ + \beta u_b^-) < I_b^\lambda(u_b^+ + u_b^-) = c_b^\lambda. \tag{3.1}
$$

If $(I_b^\lambda)'(u_b) \neq 0$, then there exist $\delta > 0$ and $\theta > 0$ such that

$$
\|(I_b^\lambda)'(v)\| \ge \theta \quad \text{for all } \|v - u_b\| \ge 3\delta.
$$

Choose $\tau \in (0, \min\{1/2, \frac{\delta}{\sqrt{2}\|u_b\|}\})$. Let

$$
D := (1-\tau, 1+\tau) \times (1-\tau, 1+\tau)
$$

and

$$
g(\alpha,\beta) = \alpha u_b^+ + \beta u_b^- \quad \text{for all } (\alpha,\beta) \in D.
$$

In view of (3.1), it is easy to see that

$$
\bar{c}_\lambda := \max_{\partial\Omega} I_b^\lambda \circ g < c_{b,\lambda}. \tag{3.2}
$$

Let $\varepsilon := \min\{(c_b^\lambda - \bar{c}_\lambda)/2, \theta\delta/8\}$ and $S_\delta := B(u_b,\delta)$, according to Lemma 2.3 in [38], there exists a deformation $\eta \in C([0,1] \times D, D)$ such that

(a) $\eta(1,v) = v$ if $v \notin (I_b^\lambda)^{-1}([c_b^\lambda - 2\varepsilon, c_b^\lambda + 2\varepsilon] \cap S_{2\delta})$,

(b) $\eta(1, (I_b^\lambda)^{c_b^\lambda+\varepsilon} \cap S_\delta) \subset (I_b^\lambda)^{c_{b,\lambda}-\varepsilon}$,

(c) $I_b^\lambda(\eta(1,v)) \le I_b^\lambda)(v)$ for all $v \in E$.

First, we need to prove that

$$\max_{(\alpha,\beta)\in\bar{D}} I_b^\lambda(\eta(1,g(\alpha,\beta))) < c_b^\lambda. \tag{3.3}$$

In fact, it follows from Lemma 2.1 that $I_b^\lambda(g(\alpha,\beta)) \le c_b^\lambda < c_b^\lambda + \varepsilon$. That is,

$$g(\alpha,\beta) \in (I_b^\lambda)^{c_b^\lambda+\varepsilon}.$$

On the other hand, we have

$$\begin{aligned}\|g(\alpha,\beta)-u_b\|^2 &= \|(\alpha-1)u_b^+ + (\beta-1)u_b^-\| \\ &\le 2((\alpha-1)^2\|u_b^+\|^2 + (\beta-1)^2\|u_b^-\|^2) \\ &\le 2\tau\|u_b\|^2 < \delta^2,\end{aligned}$$

which shows that $g(\alpha,\beta) \in S_\delta$ for all $(\alpha,\beta) \in \bar{D}$.

Therefore, by (b), we have $I_b^\lambda(\eta(1,g(s,t))) < c_b^\lambda - \varepsilon$. Hence, (3.3) holds.

In the following, we prove that $\eta(1,g(D)) \cap \mathcal{M}_b^\lambda \neq \emptyset$, which contradicts the definition of c_b^λ.

Let $h(\alpha,\beta) := \eta(1,g(\alpha,\beta))$ and

$$\begin{aligned}\Psi_0(\alpha,\beta) :=&\ (\langle (I_b^\lambda)'(g(\alpha,\beta)), u_b^+\rangle, \langle (I_b^\lambda)'(g(\alpha,\beta)), u_b^-\rangle) \\ =&\ (\langle (I_b^\lambda)'(\alpha u_b^+ + \beta u_b^-), u_b^+\rangle, \langle (I_b^\lambda)'(\alpha u_b^+ + \beta u_b^-), u_b^-\rangle) \\ =:&\ (\varphi_u^1(\alpha,\beta), \varphi_u^2(\alpha,\beta))\end{aligned}$$

and

$$\Psi_1(\alpha,\beta) := \left(\frac{1}{\alpha}\langle (I_b^\lambda)'(h(\alpha,\beta)),\ (g(\alpha,\beta))^+\rangle, \frac{1}{\beta}\langle (I_b^\lambda)'(h(\alpha,\beta)), (h(\alpha,\beta))^-\rangle\right).$$

By the direct calculation, we have

$$\begin{aligned}\frac{\varphi_u^1(\alpha,\beta)}{\partial\alpha}\bigg|_{(1,1)} &= \|u_b^+\|_E^2 + 3b\|u_b^+\|^4 + b\|u_b^+\|^2\|u_b^-\|^2 \\ &\quad - (2^{**}-1)\int_\Omega |u_b^+|^{2^{**}}dx\end{aligned}$$

$$-\lambda\int_\Omega \partial_\alpha f(x,u_b^+)(u_b^+)^2dx,\ \frac{\varphi_u^1(\alpha,\beta)}{\partial\beta}\bigg|_{(1,1)} = 2b\|u_b^+\|^2\|u_b^-\|^2,\quad \frac{\varphi_u^2(\alpha,\beta)}{\partial\alpha}\ |_{(1,1)}= 2b\|u_b^+\|^2\|u_b^-\|^2,$$

$$\begin{aligned}\frac{\varphi_u^2(\alpha,\beta)}{\partial\beta}\bigg|_{(1,1)} &= \|u_b^-\|^2 + 3b\|u_b^-\|^4 + b\|u_b^+\|^2\|u_b^-\|^2 \\ &\quad - (2^{**}-1)\int_\Omega |u_b^-|^{2^{**}}dx - \lambda\int_\Omega \partial_\beta f(x,u_b^-)(u_b^-)^2dx.\end{aligned}$$

Let

$$M = \begin{bmatrix} \frac{\varphi_u^1(\alpha,\beta)}{\partial\alpha}\Big|_{(1,1)} & \frac{\varphi_u^2(\alpha,\beta)}{\partial\alpha}\Big|_{(1,1)} \\ \frac{\varphi_u^1(\alpha,\beta)}{\partial\beta}\Big|_{(1,1)} & \frac{\varphi_u^2(\alpha,\beta)}{\partial\beta}\Big|_{(1,1)} \end{bmatrix}.$$

By (f_3), for $t \neq 0$, we have

$$\partial_t f(x,t)t^2 - 3f(x,t)t > 0$$

for almost every $x \in \Omega$. Then, since $u_b \in \mathcal{M}_{b,\lambda}$, we have

$$
\begin{aligned}
&\det M \\
&= \frac{\varphi_u^1(\alpha,\beta)}{\partial\alpha}\bigg|_{(1,1)} \times \frac{\varphi_u^2(\alpha,\beta)}{\partial\beta}\bigg|_{(1,1)} - \frac{\varphi_u^1(\alpha,\beta)}{\partial\beta}\bigg|_{(1,1)} \times \frac{\varphi_u^2(\alpha,\beta)}{\partial\alpha}\bigg|_{(1,1)} \\
&= \left[2\|u_b^+\|^2+(2^{**}-4)\int_\Omega |u_b^+|^{2^{**}}dx+2b\|u_b^+\|^2\|u_b^-\|^2+\lambda\int_\Omega(\partial_\alpha f(x,u_b^+)(u_b^+)^2-3f(x,u_b^+)(u_b^+)dx\right] \\
&\quad \times\left[2\|u_b^-\|^2+(2^{**}-4)\int_\Omega |u_b^-|^{2^{**}}dx+2b\|u_b^+\|^2\|u_b^-\|^2+\lambda\int_\Omega(\partial_\beta f(x,u_b^-)(u_b^-)^2-3f(x,u_b^-)(u_b^-)dx\right] \\
&\quad -4b^2\|u_b^+\|^4\|u_b^-\|^4 \\
&> 0.
\end{aligned}
$$

Since $\Psi_0(\alpha,\beta)$ is a C^1 function and $(1,1)$ is the unique isolated zero point of Ψ_0, by using the degree theory, we deduce that $\deg(\Psi_0, D, 0) = 1$.

Hence, combining (3.3) with (a), we obtain

$$g(\alpha,\beta) = h(\alpha,\beta) \quad \text{on } \partial D.$$

Consequently, we obtain $\deg(\Psi_1, D, 0) = 1$. Therefore, $\Psi_1(\alpha_0,\beta_0) = 0$ for some $(\alpha_0,\beta_0) \in D$ so that

$$\eta(1, g(\alpha_0,\beta_0)) = h(\alpha_0,\beta_0) \in \mathcal{M}_b^\lambda,$$

which is contradicted to (3.3).

From the above discussions, we deduce that u_b is a sign-changing solution for problem (1.1).

Finally, we prove that u has exactly two nodal domains. To this end, we assume by contradiction that

$$u_b = u_1 + u_2 + u_3$$

with

$$u_i \neq 0,\ u_1 \geq 0,\ u_2 \leq 0$$

and

$$\mathrm{suppt}(u_i) \cap \mathrm{suppt}(u_j) = \emptyset \quad \text{for } i \neq j,\ i,j = 1,2,3$$

and

$$\langle (I_b^\lambda)'(u), u_i\rangle = 0 \quad \text{for } i = 1,2,3.$$

Setting $v := u_1 + u_2$, we see that $v^+ = u_1$ and $v^- = u_2$, i.e., $v^\pm \neq 0$. Then, there exist a unique pair (α_v, β_v) of positive numbers such that

$$\alpha_v u_1 + \beta_v u_2 \in \mathcal{M}_b^\lambda.$$

Hence

$$I_b^\lambda(\alpha_v u_1 + \beta_v u_2) \geq c_b^\lambda.$$

Moreover, using the fact that $\langle (I_b^\lambda)'(u), u_i\rangle = 0$, we obtain $\langle (I_b^\lambda)'(v), v^\pm\rangle < 0$.

From Lemma 2.1 (ii), we have that

$$(\alpha_v, \beta_v) \in (0,1] \times (0,1].$$

On the other hand, we have that

$$
\begin{aligned}
0 = \frac{1}{4}\langle (I_b^\lambda)'(u), u_3\rangle &= \frac{1}{4}\|u_3\|^2 + \frac{b}{4}\|u_1\|^2\|u_3\|^2 + \frac{b}{4}\|u_2\|^2\|u_3\|^2 + \frac{b}{4}\|u_3\|^4 \\
&\quad - \frac{1}{2^{**}}\int_\Omega |u_3|^{2^{**}}dx - \frac{\lambda}{4}\int_\Omega f(x,u_3)u_3 dx \\
&< I_b^\lambda(u_3) + \frac{b}{4}\|u_1\|^2\|u_3\|^2 + \frac{b}{4}\|u_2\|^2\|u_3\|^2.
\end{aligned}
$$

Hence, by (2.15), we can obtain that

$$
\begin{aligned}
c_b^\lambda &\le I_b^\lambda(\alpha_v u_1 + \beta_v u_2) = I_b^\lambda(\alpha_v u_1 + \beta_v u_2) - \frac{1}{4}\langle (I_b^\lambda)'(\alpha_v u_1 + \beta_v u_2), (\alpha_v u_1 + \beta_v u_2)\rangle \\
&= \frac{1}{4}(\|\alpha_v u_1\|_E^2 + \|\beta_v u_2\|_E^2) + \frac{\lambda}{4}\int_\Omega [f(x,\alpha_v u_1)(\alpha_v u_1) - 4F(x,\alpha_v u_1)]dx \\
&\quad + \frac{\lambda}{4}\int_\Omega [f(x,\beta_v u_2)(\beta_v u_2) - 4F(x,\beta_v u_2)]dx \\
&\quad + \left(\frac{1}{4} - \frac{1}{2^{**}}\right)\int_\Omega \alpha_v^{2^{**}}|u_1|^{2^{**}}dx + \left(\frac{1}{4} - \frac{1}{2^{**}}\right)\int_\Omega \beta_v^{2^{**}}|u_2|^{2^{**}}dx \\
&\le \frac{1}{4}(\|u_1\|_E^2 + \|u_2\|_E^2) + \frac{\lambda}{4}\int_\Omega [f(x,u_1)u_1 - 4F(x,u_1)]dx \\
&\quad + \frac{\lambda}{4}\int_\Omega [f(x,u_2)u_2 - 4F(x,u_2)]dx + \left(\frac{1}{4} - \frac{1}{2^{**}}\right)\int_\Omega |u_1|^{2^{**}}dx + \left(\frac{1}{4} - \frac{1}{2^{**}}\right)\int_\Omega |u_2|^{2^{**}}dx \\
&= I_b^\lambda(u_1 + u_2) - \frac{1}{4}\langle (I_b^\lambda)'(u_1 + u_2), (u_1 + u_2)\rangle \\
&= I_b^\lambda(u_1 + u_2) + \frac{1}{4}\langle (I_b^\lambda)'(u), u_3\rangle + \frac{b}{4}\|u_1\|^2\|u_3\|^2 + \frac{b}{4}\|u_2\|^2\|u_3\|^2 \\
&< I_b^\lambda(u_1) + I_b^\lambda(u_2) + I_b^\lambda(u_3) + \frac{b}{4}(\|u_2\|^2 + \|u_3\|^2)\|u_1\|^2 \\
&\quad + \frac{b}{4}(\|u_1\|^2 + \|u_3\|^2)\|u_2\|^2 + \frac{b}{4}(\|u_1\|^2 + \|u_2\|^2)\|u_3\|^2 \\
&= I_b^\lambda(u) = c_b^\lambda,
\end{aligned}
$$

which is a contradiction, that is, $u_3 = 0$ and u_b has exactly two nodal domains. □

By Theorem 1.2, we obtain a least energy sign-changing solution u_b of problem (1.1). Next we prove that the energy of u_b is strictly more than twice the ground state energy.

Proof of Theorem 1.3. Similar to the proof of Lemma 2.3, there exists $\lambda_1^* > 0$ such that for all $\lambda \ge \lambda_1^*$, and for each $b > 0$, there exists $v_b \in \mathcal{N}_b^\lambda$ such that $I_b^\lambda(v_b) = c^* > 0$. By standard arguments (see Corollary 2.13 in [9]), the critical points of the functional I_b^λ on $\mathcal{N}_b^\lambda$ are critical points of I_b^λ in E, and we obtain $(I_b^\lambda)'(v_b) = 0$. That is, v_b is a ground state solution of (1.1).

According to Theorem 1.2, we know that the problem (1.1) has a least energy sign-changing solution u_b, which changes sign only once when $\lambda \ge \lambda^*$.

Let

$$\lambda^{**} = \max\{\lambda^*, \lambda_1^*\}.$$

Suppose that $u_b = u_b^+ + u_b^-$. As in the proof of Lemma 2.1, there exist $\alpha_{u_b^+} > 0$ and $\beta_{u_b^-} > 0$ such that

$$\alpha_{u_b^+} u_b^+ \in \mathcal{N}_b^\lambda, \quad \beta_{u_b^-} u_b^- \in \mathcal{N}_{b,\lambda}.$$

Furthermore, Lemma 2.1 implies that $\alpha_{u_b^+}, \beta_{u_b^-} \in (0,1)$.
Therefore, in view of Lemma 2.1, we have that

$$2c^* \le I_b^\lambda(\alpha_{u_b^+} u_b^+) + I_b^\lambda(\beta_{u_b^-} u_b^-) \le I_b^\lambda(\alpha_{u_b^+} u_b^+ + \beta_{u_b^-} u_b^-) < I_b^\lambda(u_b^+ + u_b^-) = c_b^\lambda.$$

Hence, it follows that $c^* > 0$ cannot be achieved by a sign-changing function. □

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