The asymptotic behavior of solutions
to a class of inhomogeneous problems:
an Orlicz–Sobolev space approach

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Abstract. The asymptotic behavior of the sequence $\{v_n\}$ of nonnegative solutions for a class of inhomogeneous problems settled in Orlicz–Sobolev spaces with prescribed Dirichlet data on the boundary of domain Ω is analysed. We show that $\{v_n\}$ converges uniformly in Ω as $n \to \infty$, to the distance function to the boundary of the domain.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the family of problems

$$\begin{cases} -\operatorname{div}\left(\frac{\varphi_n(|\nabla v|)}{|\nabla v|}\nabla v\right) = \lambda e^v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where for each positive integer n, the mappings $\varphi_n : \mathbb{R} \to \mathbb{R}$ are odd, increasing homeomorphisms of class C^1 satisfying *Lieberman-type condition*

$$N - 1 < \varphi_n^- - 1 \leq \frac{t\varphi_n'(t)}{\varphi_n(t)} \leq \varphi_n^+ - 1 < \infty, \qquad \forall\, t \geq 0 \tag{1.2}$$

for some constants φ_n^- and φ_n^+ with $1 < \varphi_n^- \leq \varphi_n^+ < \infty$,

$$\varphi_n^- \to \infty \quad \text{as } n \to \infty, \tag{1.3}$$

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and such that

> there exists a real constant $\beta > 1$ with the property that $\varphi_n^+ \leq \beta \varphi_n^-$, $\forall\, n \geq 1$ (1.4)

and

$$\lim_{n \to \infty} \varphi_n(1)^{1/\varphi_n^-} = 1. \qquad (1.5)$$

For some examples of functions satisfying conditions (1.2)–(1.5) the reader is referred to [5, p. 4398]. Here we just point out the fact that in the particular case when $\varphi_n(t) = |t|^{n-2}t$, $n \geq 2$, the differential operator involved in problem (1.1) is the n-Laplacian, which for sufficiently smooth functions v is defined as $\Delta_n v := \operatorname{div}(|\nabla v|^{n-2}\nabla v)$. In this particular case problem (1.1) becomes

$$\begin{cases} -\Delta_n v = \lambda e^v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \qquad (1.6)$$

which has been extensively studied in the literature (see, e.g. [3, 7, 12, 14, 15, 18, 19, 32]). An existence result concerning problem (1.6) for each given $n > N$ and $\lambda > 0$ sufficiently small was proved by Aguilar Crespo & Peral Alonso in [3] by using a fixed-point argument while Mihăilescu *et al.* [32] showed a similar result by using variational techniques. Moreover, in [32] was studied the asymptotic behavoir of solutions as $n \to \infty$. More precisely, it was proved that there exists $\lambda^\star > 0$ (which does not depend on n) such that for each $n > N$ and each $\lambda \in (0, \lambda^\star)$ problem (1.6) possesses a nonnegative solution $u_n \in W_0^{1,n}(\Omega)$ and the sequence of solutions $\{u_n\}$ converges uniformly in $\overline{\Omega}$, as $n \to \infty$, to the unique viscosity solution of the problem

$$\begin{cases} \min\{|\nabla u| - 1, -\Delta_\infty u\} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (1.7)$$

which is precisely the distance function to the boundary of the domain $\operatorname{dist}(\cdot, \partial\Omega)$ (see [26, Lemma 6.10]). The result from [32] was extended to the case of equations involving variable exponent growth conditions by Mihăilescu & Fărcășeanu in [14]. Motivated by these results the goal of this paper is to investigate the asymptotic behaviour of the solutions of the family of problems (1.1), as $n \to \infty$, for $\lambda > 0$ sufficiently small. We will show that the results from [32] and [14] continue to hold true in the case of the family of problems (1.1). In particular, our results generalise the results from [32] and complement the results from [14].

The paper is organized as follows. In Section 2 we give the definitions of the Orlicz and Orlicz–Sobolev spaces which represent the natural functional framework where the problems of type (1.1) should be investigated. Section 3 is devoted to the proof of the existence of weak solutions for problem (1.1) when λ is sufficiently small. Finally, in Section 4 we analyse the asymptotic behavior of the sequence of solutions found in the previous section, as $n \to \infty$, and we prove its uniform convergence to the distance function to the boundary of the domain.

2   Orlicz and Orlicz–Sobolev spaces

In this section we provide a brief overview on the Orlicz and Orlicz–Sobolev spaces and we recall the definitions and some of their main properties. For more details about these spaces the reader can consult the books [2, 22, 33, 34] and papers [4, 9, 10, 20, 21].

First, we will introduce the Orlicz spaces. We assume that the function φ is an odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ of class C^1. We define $\Phi : [0, \infty) \to \mathbb{R}$ by

$$\Phi(t) = \int_0^t \varphi(s) \, ds.$$

Note that Φ is a *Young function*, that is Φ vanishes when $t = 0$, Φ is continuous, Φ is convex and $\lim_{t \to \infty} \Phi(t) = \infty$. Moreover, since $\Phi(0) = 0$ if and only if $t = 0$, $\lim_{t \to 0} \frac{\Phi(t)}{t} = 0$ and $\lim_{t \to \infty} \frac{\Phi(t)}{t} = \infty$, then Φ is called a *N-function* (see [1, 2]). Next, we define the function $\Phi^\star : [0, \infty) \to \mathbb{R}$ given by

$$\Phi^\star(t) = \int_0^t \varphi^{-1}(s) \, ds \, .$$

$\Phi^\star$ is called the *complementary function of* Φ. The functions Φ and $\Phi^\star$ satisfy

$$\Phi^\star(t) = \sup_{s \geq 0} \left(st - \Phi(s) \right) \quad \text{for any } t \geq 0.$$

We note that $\Phi^\star$ is also a *N*-function, too.

Throughout this paper, we will assume that

$$0 < \varphi^- - 1 \leq \frac{t \varphi'(t)}{\varphi(t)} \leq \varphi^+ - 1 < \infty, \quad \text{for all } t > 0 \tag{2.1}$$

for some positive constants φ^- and φ^+. By [28, Lemma 1.1] (see also [31, Lemma 2.1]) we deduce that

$$1 < \varphi^- \leq \frac{t \varphi(t)}{\Phi(t)} \leq \varphi^+ < \infty, \quad \text{for all } t > 0. \tag{2.2}$$

By relation (2.2) it follows that for each $t > 0$ and $s \in (0, 1]$ we have

$$-\ln s^{\varphi^-} = \int_{st}^t \frac{\varphi^-}{\tau} \, d\tau \leq \int_{st}^t \frac{\varphi(\tau)}{\Phi(\tau)} \, d\tau = \ln \Phi(t) - \ln \Phi(st) \leq \int_{st}^t \frac{\varphi^+}{\tau} \, d\tau = -\ln s^{\varphi^+}$$

or

$$s^{\varphi^+} \Phi(t) \leq \Phi(st) \leq s^{\varphi^-} \Phi(t), \quad \forall \, t > 0, \, s \in (0, 1] \, . \tag{2.3}$$

Similarly, for each $t > 0$ and $s > 1$ we have

$$\ln s^{\varphi^-} = \int_t^{st} \frac{\varphi^-}{\tau} \, d\tau \leq \int_t^{st} \frac{\varphi(\tau)}{\Phi(\tau)} \, d\tau = \ln \Phi(st) - \ln \Phi(t) \leq \int_t^{st} \frac{\varphi^+}{\tau} \, d\tau = \ln s^{\varphi^+}$$

or

$$s^{\varphi^-} \Phi(t) \leq \Phi(st) \leq s^{\varphi^+} \Phi(t), \quad \forall \, t > 0, \, s > 1 \, . \tag{2.4}$$

Inequalities (2.3) and (2.4) can be reformulated as follows

$$\min\{s^{\varphi^-}, s^{\varphi^+}\} \Phi(t) \leq \Phi(st) \leq \max\{s^{\varphi^-}, s^{\varphi^+}\} \Phi(t) \quad \text{for any } s, t > 0 \, . \tag{2.5}$$

Similarly, by [31, Lemma 2.1] we deduce that

$$\min\{s^{\varphi^- - 1}, s^{\varphi^+ - 1}\} \varphi(t) \leq \varphi(st) \leq \max\{s^{\varphi^- - 1}, s^{\varphi^+ - 1}\} \varphi(t), \quad \forall \, s, t > 0 \, . \tag{2.6}$$

Next, if we let $s = \varphi^{-1}(t)$ then we have

$$\frac{t (\varphi^{-1})'(t)}{\varphi^{-1}(t)} = \frac{\varphi(s)}{\varphi'(s) s} \, .$$

By (2.1) we deduce that

$$\frac{1}{\varphi^+ - 1} \le \frac{t(\varphi^{-1})'(t)}{\varphi^{-1}(t)} \le \frac{1}{\varphi^- - 1}, \qquad \forall\, t > 0.$$

The above relation implies that

$$1 < \frac{\varphi^+}{\varphi^+ - 1} \le \frac{t\varphi^{-1}(t)}{\Phi^\star(t)} \le \frac{\varphi^-}{\varphi^- - 1} < \infty \quad \text{for all } t > 0. \tag{2.7}$$

Examples. We point out some example of functions φ which are odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$, and φ and the corresponding primitive Φ satisfy condition (2.2) (see [10, Examples 1–3, p. 243]):

1. $\varphi(t) = |t|^{p-2}t$, $\Phi(t) = \frac{|t|^p}{p}$ with $p > 1$ and $\varphi^- = \varphi^+ = p$.

2. $\varphi(t) = \log(1 + |t|^r)|t|^{p-2}t$, $\Phi(t) = \log(1 + |t|^r)\frac{|t|^p}{p} - \frac{r}{p}\int_0^{|t|} \frac{s^{p+r-1}}{1+s^r}\, ds$ with $p, r > 1$ and $\varphi^- = p$, $\varphi^+ = p + r$.

3. $\varphi(t) = \frac{|t|^{p-2}t}{\log(1+|t|)}$ for $t \ne 0$, $\varphi(0) = 0$, $\Phi(t) = \frac{|t|^p}{p\log(1+|t|)} + \frac{1}{p}\int_0^{|t|} \frac{s^p}{(1+s)(\log(1+s))^2}\, ds$ with $p > 2$ and $\varphi^- = p - 1$, $\varphi^+ = p = \liminf_{t\to\infty} \frac{\log \Phi(t)}{\log t}$.

For each bounded domain $\Omega \subset \mathbb{R}^N$, the *Orlicz space* $L^\Phi(\Omega)$ defined by the N-function Φ (see [1,2,9]) is the set of real-valued measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L^\Phi(\Omega)} := \sup\left\{ \int_\Omega u(x)v(x)\, dx;\ \int_\Omega \Phi^\star(|v(x)|)\, dx \le 1 \right\} < \infty.$$

Then, the Orlicz space $L^\Phi(\Omega)$ endowed with the *Orlicz norm* $\|\cdot\|_{L^\Phi(\Omega)}$ is a Banach space and its Orlicz norm $\|\cdot\|_{L^\Phi(\Omega)}$ is equivalent to the so-called *Luxemburg norm* defined by

$$\|u\|_\Phi := \inf\left\{ \mu > 0\,;\ \int_\Omega \Phi\left(\frac{u(x)}{\mu}\right)\, dx \le 1 \right\}. \tag{2.8}$$

In the case of Orlicz spaces, the following relations hold true (see, e.g. [17, Lemma 2.1]):

$$\|u\|_\Phi^{\varphi^+} \le \int_\Omega \Phi(|u(x)|)\, dx \le \|u\|_\Phi^{\varphi^-} \qquad \forall\, u \in L^\Phi(\Omega) \text{ with } \|u\|_\Phi < 1, \tag{2.9}$$

$$\|u\|_\Phi^{\varphi^-} \le \int_\Omega \Phi(|u(x)|)\, dx \le \|u\|_\Phi^{\varphi^+} \qquad \forall\, u \in L^\Phi(\Omega) \text{ with } \|u\|_\Phi > 1 \tag{2.10}$$

and

$$\int_\Omega \Phi(|u(x)|)\, dx = 1 \iff \|u\|_\Phi = 1, \qquad \forall\, u \in L^\Phi(\Omega). \tag{2.11}$$

Next, we recall that for each bounded domain $\Omega \subset \mathbb{R}^N$, the *Orlicz–Sobolev space* $W^{1,\Phi}(\Omega)$ defined by the N-function Φ is the set of all functions u such that u and its distributional derivatives of order 1 lie in Orlicz space $L^\Phi(\Omega)$. More exactly, $W^{1,\Phi}(\Omega)$ is the space given by

$$W^{1,\Phi}(\Omega) = \left\{ u \in L^\Phi(\Omega);\ \frac{\partial u}{\partial x_j} \in L^\Phi(\Omega),\ j \in \{1, \dots, N\} \right\}.$$

It is a Banach space with respect to the following norm

$$\|u\|_{1,\Phi} := \|u\|_{\Phi} + \| \, |\nabla u| \, \|_{\Phi}.$$

By $W_0^{1,\Phi}(\Omega)$ we denoted the closure of all functions of class C^{∞} with compact support over Ω with respect to norm of $W^{1,\Phi}(\Omega)$, i.e.

$$W_0^{1,\Phi}(\Omega) := \overline{C_0^{\infty}(\Omega)}^{\; \|\cdot\|_{1,\Phi}}.$$

Note that the norms $\|\cdot\|_{1,\Phi}$ and $\|\cdot\|_{W_0^{1,\Phi}} := \| \, |\nabla \cdot| \, \|_{\Phi}$ are equivalent on the Orlicz–Sobolev space $W_0^{1,\Phi}(\Omega)$ (see [21, Lemma 5.7]).

Under conditions (2.2) and (2.7), Φ and $\Phi^{\star}$ satisfy the Δ_2-*condition*, i.e.

$$\Phi(2t) \le C\Phi(t), \qquad \forall \, t \ge 0, \tag{2.12}$$

for some constant $C > 0$ (see [2, p. 232]). Therefore, $L^{\Phi}(\Omega)$, $W^{1,\Phi}(\Omega)$ and $W_0^{1,\Phi}(\Omega)$ are reflexive Banach spaces (see [2, Theorem 8.19] and [2, p. 232]).

Remark 2.1. For each real number $p > 1$ let $\varphi(t) = |t|^{p-2}t$, $t \in \mathbb{R}$. It can be shown that $\varphi^- = \varphi^+ = p$ as mentioned above in Example 1 and the corresponding Orlicz space $L^{\Phi}(\Omega)$ reduces to the *classical Lebesgue space* $L^p(\Omega)$ while the Orlicz–Sobolev spaces $W^{1,\Phi}(\Omega)$ and $W_0^{1,\Phi}(\Omega)$ become the *classical Sobolev spaces* $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$, respectively. Note also that by [2, Theorem 8.12] the Orlicz space $L^{\Phi}(\Omega)$ is continuously embedded in the Lebesgue spaces $L^q(\Omega)$ for each $q \in (1, \varphi^-]$.

3   Variational solutions for problem (1.1)

In this section we will show that there exists a certain constant $\lambda^{\star} > 0$ (independent of n) such that for each $\lambda \in (0, \lambda^{\star})$ problem (1.1) possesses a nonnegative weak solution for each integer $n \ge 1$.

We start by introducing the following notations: for each positive integer n we denote by Φ_n a primitive of the function φ_n. More precisely, we define $\Phi_n : [0, \infty) \to \mathbb{R}$ by

$$\Phi_n(t) := \int_0^t \varphi_n(s) \, ds \, .$$

Definition 3.1. We say that v_n is a *weak solution* of problem (1.1) if $v_n \in W_0^{1,\Phi_n}(\Omega)$ and the following relation holds true

$$\int_{\Omega} \frac{\varphi_n(|\nabla v_n|)}{|\nabla v_n|} \nabla v_n \nabla w \, dx = \lambda \int_{\Omega} e^{v_n} w \, dx, \qquad \forall \, w \in W_0^{1,\Phi_n}(\Omega). \tag{3.1}$$

Note that the integral from the right-hand side of relation (3.1) is well-defined since the Orlicz–Sobolev space $W_0^{1,\Phi_n}(\Omega)$ is continuously embedded in the classical Sobolev space $W_0^{1,\varphi_n^-}(\Omega)$ (see, e.g. [2, Theorem 8.12]) and for $\varphi_n^- > N$ we have $W_0^{1,\varphi_n^-}(\Omega) \subset L^{\infty}(\Omega)$. Moreover, we recall that Morrey's inequality holds true, i.e. there exists a positive constant C_n such that

$$\|v\|_{L^{\infty}(\Omega)} \le C_n \| \, |\nabla v| \, \|_{L^{\varphi_n^-}(\Omega)}, \qquad \forall \, v \in W_0^{1,\varphi_n^-}(\Omega) \, . \tag{3.2}$$

By [8, Proposition 3.1] we know that we can choose C_n as follows

$$C_n := \varphi_n^- |B(0,1)|^{-\frac{1}{\varphi_n^-}} N^{-\frac{N(\varphi_n^-+1)}{(\varphi_n^-)^2}} (\varphi_n^- - 1)^{\frac{N(\varphi_n^--1)}{(\varphi_n^-)^2}} (\varphi_n^- - N)^{\frac{N-(\varphi_n^-)^2}{(\varphi_n^-)^2}} [\lambda_1(\varphi_n^-)]^{\frac{N-\varphi_n^-}{(\varphi_n^-)^2}}, \qquad (3.3)$$

where $|B(0,1)|$ is the volume of the unit ball in $\mathbb{R}^N$ and for each real number $p \in (1, \infty)$, $\lambda_1(p)$ denotes the first eigenvalue for the p-Laplace operator with homogeneous Dirichlet boundary conditions, i.e.

$$\lambda_1(p) := \inf_{u \in C_0^\infty(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx}, \qquad \forall \, p \in (1, \infty).$$

By [8, Proposition 3.1] (see also [13, Theorem 3.2] for a similar result) it is well known that

$$\lim_{n \to \infty} C_n = \| \operatorname{dist}(\cdot, \partial\Omega) \|_{L^\infty(\Omega)}, \qquad (3.4)$$

where $\operatorname{dist}(x, \partial\Omega) := \inf_{y \in \partial\Omega} |x - y|$, $\forall \, x \in \Omega$, stands for the distance function to the boundary of Ω.

For each positive integer n and each positive real number λ we introduce the Euler–Lagrange functional associated to problem (1.1) as $J_{n,\lambda} : W_0^{1,\Phi_n}(\Omega) \to \mathbb{R}$ defined by

$$J_{n,\lambda}(v) := \int_\Omega \Phi_n(|\nabla v|) \, dx - \lambda \int_\Omega e^v \, dx, \qquad \forall \, v \in W_0^{1,\Phi_n}(\Omega).$$

Standard arguments can be used in order to show that $J_{n,\lambda} \in C^1(W_0^{1,\Phi_n}(\Omega), \mathbb{R})$ and

$$\langle J'_{n,\lambda}(v), w \rangle = \int_\Omega \frac{\varphi_n(|\nabla v|)}{|\nabla v|} \nabla v \nabla w \, dx - \lambda \int_\Omega e^v w \, dx, \qquad \forall \, v, \, w \in W_0^{1,\Phi_n}(\Omega).$$

Thus, it is clear that v_n is a weak solution of (1.1) if and only if v_n is a critical point of functional $J_{n,\lambda}$.

We point out that we cannot find critical points of $J_{n,\lambda}$ by using the Direct Method in the Calculus of Variations since in the case of our problem $J_{n,\lambda}$ is not coercive. For that reason we propose an analysis of problem (1.1) based on Ekeland's Variational Principle in order to find critical points of $J_{n,\lambda}$.

For each positive integer n we denote

$$\lambda_n^\star := \frac{1}{2|\Omega|} e^{-C_n \left[|\Omega| + \frac{1}{\Phi_n(1)} \right]^{1/\varphi_n^-}}, \qquad (3.5)$$

where C_n is the constant given by relation (3.3) and $|\Omega|$ stands for the N-dimensional Lebesgue measure of Ω. The starting point of our approach is the following lemma.

Lemma 3.2. *For each positive integer n let $\lambda_n^\star$ be given by relation (3.5). Then for each $\lambda \in (0, \lambda_n^\star)$ we have*

$$J_{n,\lambda}(v) \geq \frac{1}{2}, \qquad \forall \, v \in W_0^{1,\Phi_n}(\Omega) \quad \text{with} \quad \|v\|_{W_0^{1,\Phi_n}} = 1.$$

Proof. Let n be a positive integer arbitrary fixed. By relation (2.5) we get that $\Phi_n(s) \geq \Phi_n(1) s^{\varphi_n^-}$, for all $s > 1$ and thus,

$$s^{\varphi_n^-} \leq 1 + \frac{\Phi_n(s)}{\Phi_n(1)}, \qquad \forall \, s \geq 0.$$

Using this fact we deduce that

$$\int_{\Omega} |\nabla v|^{\varphi_n^-} \, dx \leq |\Omega| + \frac{1}{\Phi_n(1)} \int_{\Omega} \Phi_n(|\nabla v|) \, dx, \qquad \forall \, v \in W_0^{1,\Phi_n}(\Omega). \tag{3.6}$$

By the above inequality, and since for each $v \in W_0^{1,\Phi_n}(\Omega)$ with $\|v\|_{W_0^{1,\Phi_n}} := \| \, |\nabla v| \, \|_{\Phi_n} = 1$ we have $\int_{\Omega} \Phi_n(|\nabla v|) \, dx = 1$ (via relation (2.11)), it results

$$\| \, |\nabla v| \, \|_{L^{\varphi_n^-}(\Omega)} \leq \left[|\Omega| + \frac{1}{\Phi_n(1)} \right]^{1/\varphi_n^-}, \qquad \forall \, v \in W_0^{1,\Phi_n}(\Omega) \quad \text{with } \|v\|_{W_0^{1,\Phi_n}} = 1. \tag{3.7}$$

Next, taking into account that $W_0^{1,\Phi_n}(\Omega)$ is continuously embedded in $W_0^{1,\varphi_n^-}(\Omega)$ and using the fact that $\varphi_n^- > N$ and Morrey's inequality (3.2) we obtain

$$\begin{aligned}
J_{n,\lambda}(v) &= \int_{\Omega} \Phi_n(|\nabla v|) \, dx - \lambda \int_{\Omega} e^v \, dx \\
&\geq 1 - \lambda |\Omega| e^{\|v\|_{L^{\infty}(\Omega)}} \\
&\geq 1 - \lambda |\Omega| e^{C_n \| \, |\nabla v| \, \|_{L^{\varphi_n^-}(\Omega)}}, \qquad \forall \, v \in W_0^{1,\Phi_n}(\Omega) \text{ with } \|v\|_{W_0^{1,\Phi_n}} = 1.
\end{aligned}$$

Then for each $\lambda \in (0, \lambda_n^{\star})$, combining the above estimates with relation (3.7) we get

$$J_{n,\lambda}(v) \geq 1 - \lambda |\Omega| e^{C_n \left[|\Omega| + \frac{1}{\Phi_n(1)} \right]^{1/\varphi_n^-}} \geq 1 - \lambda_n^{\star} |\Omega| e^{C_n \left[|\Omega| + \frac{1}{\Phi_n(1)} \right]^{1/\varphi_n^-}} = \frac{1}{2},$$

for all $v \in W_0^{1,\Phi_n}(\Omega)$ with $\|v\|_{W_0^{1,\Phi_n}} = 1$. The proof of the lemma is complete. □

Lemma 3.3. *For each positive integer n let $\lambda_n^{\star}$ be given by relation (3.5). Define*

$$\lambda^{\star} := \inf_{n \in \mathbb{N}^*} \lambda_n^{\star}. \tag{3.8}$$

Then $\lambda^{\star} > 0$.

Proof. First, we show that there exists a positive constant $K > 0$ such that

$$\left[|\Omega| + \frac{1}{\Phi_n(1)} \right]^{1/\varphi_n^-} < K, \qquad \forall \, n \geq 1. \tag{3.9}$$

Indeed, since by (1.5) we have

$$\lim_{n \to \infty} \varphi_n(1)^{1/\varphi_n^-} = 1,$$

it yields that for each positive integer n large enough we get

$$\frac{1}{2} \leq \varphi_n(1)^{1/\varphi_n^-},$$

which implies that

$$\frac{1}{\varphi_n(1)} \leq 2^{\varphi_n^-}.$$

By (1.2) (via (2.1) and (2.2)) we find that for each positive integer n large enough the following inequalities hold true

$$\frac{1}{\Phi_n(1)} \leq \frac{\varphi_n^+}{\varphi_n(1)} \leq \varphi_n^+ 2^{\varphi_n^-} \leq \beta \varphi_n^- 2^{\varphi_n^-}.$$

Using the above relations we deduce that for each positive integer n large enough we obtain

$$\left[|\Omega| + \frac{1}{\Phi_n(1)}\right]^{1/\varphi_n^-} \leq \left[|\Omega| + \beta\varphi_n^- 2^{\varphi_n^-}\right]^{1/\varphi_n^-} \leq \left(\beta\varphi_n^- 2^{\varphi_n^-+1}\right)^{1/\varphi_n^-}.$$

Now, taking into account the fact that $\lim_{n\to\infty} \left(\beta\varphi_n^- 2^{\varphi_n^-+1}\right)^{1/\varphi_n^-} = 2$, the above approximations imply that relation (3.9) holds true.

Next, using (3.9) and the expression of $\lambda_n^\star$ we infer that

$$\lambda_n^\star = \frac{1}{2|\Omega|}e^{-C_n\left[|\Omega|+\frac{1}{\Phi_n(1)}\right]^{1/\varphi_n^-}} > \frac{1}{2|\Omega|}e^{-KC_n}, \qquad \forall\, n \geq 1.$$

Recalling that $\lim_{n\to\infty} C_n = \|\operatorname{dist}(\cdot,\partial\Omega)\|_{L^\infty(\Omega)}$ (by (3.4)) and taking into account that function $(1,\infty) \ni p \longrightarrow \lambda_1(p)$ is continuous (see, Lindqvist [29] or Huang [23]) we conclude from the above estimates that $\lambda^\star = \inf_{n\in\mathbb{N}^*} \lambda_n^\star > 0$. The proof of Lemma 3.3 is complete. $\qquad\square$

The main goal of this section is to prove the existence of weak solutions of problem (1.1) for each positive integer n. This result is the core of the following theorem.

Theorem 3.4. *Let $\lambda^\star > 0$ be given by (3.8). Then for each $\lambda \in (0,\lambda^\star)$ and each $n \in \mathbb{N}^\star$, problem (1.1) has a nonnegative solution $v_n \in \overline{B_1(0)} \subset W_0^{1,\Phi_n}(\Omega)$ identified by $J_{n,\lambda}(v_n) = \inf_{\overline{B_1(0)}} J_{n,\lambda}$, where $B_1(0)$ is the unit ball centered at the origin in the Orlicz–Sobolev space $W_0^{1,\Phi_n}(\Omega)$.*

Proof. We consider $\lambda \in (0,\lambda^\star)$ and $n \in \mathbb{N}^\star$ arbitrary fixed. For each $v \in W_0^{1,\Phi_n}(\Omega)$ with $\|v\|_{W_0^{1,\Phi_n}} \leq 1$, in view of relations (2.9) and (2.11), we have

$$\|v\|_{W_0^{1,\Phi_n}}^{\varphi_n^-} \geq \int_\Omega \Phi_n(|\nabla v|)\,dx \geq \|v\|_{W_0^{1,\Phi_n}}^{\varphi_n^+}. \tag{3.10}$$

Thus, taking into account (3.10), Morrey's inequality (3.2) and relation (3.6), for each $v \in \overline{B_1(0)} \subset W_0^{1,\Phi_n}(\Omega)$ we obtain

$$
\begin{aligned}
J_{n,\lambda}(v) &= \int_\Omega \Phi_n(|\nabla v|)\,dx - \lambda \int_\Omega e^v\,dx \\
&\geq \|v\|_{W_0^{1,\Phi_n}}^{\varphi_n^+} - \lambda|\Omega|e^{\|v\|_{L^\infty(\Omega)}} \\
&\geq -\lambda|\Omega|e^{C_n\|\,|\nabla v|\,\|_{L^{\varphi_n^-}(\Omega)}} \\
&\geq -\lambda|\Omega|e^{C_n\left[|\Omega|+\frac{1}{\Phi_n(1)}\right]^{1/\varphi_n^-}}.
\end{aligned}
$$

Computing $J_{n,\lambda}(0) = -\lambda|\Omega|$ we deduce that

$$J_{n,\lambda}(0) < 0$$

while by Lemma 3.2 we get

$$\inf_{\partial B_1(0)} J_{n,\lambda} \geq \frac{1}{2} > 0,$$

which imply that

$$\gamma_n := \inf_{\overline{B_1(0)}} J_{n,\lambda} \in (-\infty, 0).$$

We consider $\epsilon > 0$ such that

$$\epsilon < \inf_{\partial B_1(0)} J_{n,\lambda} - \inf_{B_1(0)} J_{n,\lambda}. \tag{3.11}$$

Ekeland's variational principle applied to $J_{n,\lambda}$ restricted to $\overline{B_1(0)}$ provides the existence of $v_\epsilon \in \overline{B_1(0)}$ having the properties

$$i)\ J_{n,\lambda}(v_\epsilon) < \inf_{B_1(0)} J_{n,\lambda} + \epsilon,$$

$$ii)\ J_{n,\lambda}(v_\epsilon) < J_{n,\lambda}(v) + \epsilon \left\| v - v_\epsilon \right\|_{W_0^{1,\Phi_n}} \quad \text{for all} \ v \neq v_\epsilon.$$

Since $\inf_{\overline{B_1(0)}} J_{n,\lambda} \leq \inf_{B_1(0)} J_{n,\lambda}$ and ϵ is chosen small such that (3.11) holds true, using relation *i)* above we arrive at

$$J_{n,\lambda}(v_\epsilon) < \inf_{B_1(0)} J_{n,\lambda} + \epsilon \leq \inf_{B_1(0)} J_{n,\lambda} + \epsilon < \inf_{\partial B_1(0)} J_{n,\lambda},$$

from which we deduce that v_ϵ is not an element on the boundary of the unit ball of space $W_0^{1,\Phi_n}(\Omega)$, $v_\epsilon \notin \partial B_1(0)$, and consequently, v_ϵ is an element in the interior of this ball, that means $v_\epsilon \in B_1(0)$.

Next, we focus on the functional $F_{n,\lambda} : \overline{B_1(0)} \to \mathbb{R}$ defined by $F_{n,\lambda}(v) = J_{n,\lambda}(v) + \epsilon \left\| v - v_\epsilon \right\|_{W_0^{1,\Phi_n}}$. Obviously, v_ϵ is a minimum point of $F_{n,\lambda}$ (via *ii)*) that infers

$$\frac{F_{n,\lambda}(v_\epsilon + tw) - F_{n,\lambda}(v_\epsilon)}{t} \geq 0$$

for small $t > 0$ and any $w \in B_1(0)$. Computing the above relation we find

$$\frac{J_{n,\lambda}(v_\epsilon + tw) - J_{n,\lambda}(v_\epsilon)}{t} + \epsilon \left\| w \right\|_{W_0^{1,\Phi_n}} \geq 0$$

and then passing to the limit as $t \to 0^+$ it yields that $\langle J'_{n,\lambda}(v_\epsilon), w \rangle + \epsilon \left\| w \right\|_{W_0^{1,\Phi_n}} \geq 0$ that implies $\left\| J'_{n,\lambda}(v_\epsilon) \right\|_{(W_0^{1,\Phi_n}(\Omega))^\star} \leq \epsilon$, where $(W_0^{1,\Phi_n}(\Omega))^\star$ is the dual space of $W_0^{1,\Phi_n}(\Omega)$.

In consideration of that, we draw to the conclusion that there exists a sequence $\{v_m\}_m \subset B_1(0)$ such that

$$\lim_{m \to \infty} J_{n,\lambda}(v_m) = \gamma_n \quad \text{and} \quad \lim_{m \to \infty} J'_{n,\lambda}(v_m) = 0. \tag{3.12}$$

The sequence $\{v_m\}_m$ is certainly bounded in $W_0^{1,\Phi_n}(\Omega)$ since $v_m \in B_1(0)$ for all $m \in \mathbb{N}^\star$ and this fact induces the existence of $v_n \in W_0^{1,\Phi_n}(\Omega)$ such that, up to a subsequence, $\{v_m\}_m$ converges weakly to v_n in $W_0^{1,\Phi_n}(\Omega)$ and uniformly in Ω, since $\varphi_n^- > N$, as $m \to \infty$. Furthermore, we infer that

$$\lim_{m \to \infty} \int_\Omega e^{v_m}(v_m - v_n)\, dx = 0$$

and

$$\lim_{m \to \infty} \langle J'_{n,\lambda}(v_m), v_m - v_n \rangle = 0,$$

which imply that

$$\lim_{m \to \infty} \int_\Omega \frac{\varphi_n(|\nabla v_m|)}{|\nabla v_m|} \nabla v_m \nabla(v_m - v_n)\, dx = 0. \tag{3.13}$$

Owing to the weak convergence of sequence $\{v_m\}_m$ to v_n in $W_0^{1,\Phi_n}(\Omega)$, as $m \to \infty$, we have that

$$\lim_{m \to \infty} \langle J'_{n,\lambda}(v_n), v_m - v_n \rangle = 0$$

and it follows that

$$\lim_{m \to \infty} \int_\Omega \frac{\varphi_n(|\nabla v_n|)}{|\nabla v_n|} \nabla v_n \nabla (v_m - v_n) \, dx = 0. \tag{3.14}$$

Assembling relations (3.13) and (3.14), we conclude that

$$\lim_{m \to \infty} \int_\Omega \left[\frac{\varphi_n(|\nabla v_m|)}{|\nabla v_m|} \nabla v_m - \frac{\varphi_n(|\nabla v_n|)}{|\nabla v_n|} \nabla v_n \right] \nabla (v_m - v_n) \, dx = 0. \tag{3.15}$$

By [16, Lemma 3.2] we know that there exists a positive constant k_n such that

$$\left[\frac{\varphi_n(|\xi|)}{|\xi|} \xi - \frac{\varphi_n(|\eta|)}{|\eta|} \eta \right] \cdot (\xi - \eta) \geq k_n \frac{[\Phi_n(|\xi - \eta|)]^{\frac{\varphi_n^- + 2}{\varphi_n^- + 1}}}{[\Phi_n(|\xi|) + \Phi_n(|\eta|)]^{1/(\varphi_n^- + 1)}}, \qquad \forall \, \xi, \eta \in \mathbb{R}^N, \, \xi \neq \eta.$$

In our case, we established that there exist constant $k_n > 0$ so that

$$\int_\Omega \left[\frac{\varphi_n(|\nabla v_m|)}{|\nabla v_m|} \nabla v_m - \frac{\varphi_n(|\nabla v_n|)}{|\nabla v_n|} \nabla v_n \right] (\nabla v_m - \nabla v_n) \, dx$$

$$\geq k_n \int_\Omega \frac{[\Phi_n(|\nabla v_m - \nabla v_n|)]^{\frac{\varphi_n^- + 2}{\varphi_n^- + 1}}}{[\Phi_n(|\nabla v_m|) + \Phi_n(|\nabla v_n|)]^{1/(\varphi_n^- + 1)}} \, dx.$$

Due to relation (3.15) we deduce that

$$\lim_{m \to \infty} \int_\Omega \Phi_n(|\nabla(v_m - v_n)|) \left[\frac{\Phi_n(|\nabla(v_m - v_n)|)}{\Phi_n(|\nabla v_m|) + \Phi_n(|\nabla v_n|)} \right]^{1/(\varphi_n^- + 1)} dx = 0.$$

Since Φ_n is a convex function we obtain by relation (2.5) that

$$\Phi_n(|\nabla(v_m - v_n)|) \leq \frac{\Phi_n(2|\nabla v_m|) + \Phi_n(2|\nabla v_n|)}{2} \leq 2^{\varphi_n^+ - 1} \left[\Phi_n(|\nabla v_m|) + \Phi_n(|\nabla v_n|) \right].$$

Using assumption (1.4), the last two relations require

$$\lim_{m \to \infty} \int_\Omega \Phi_n(|\nabla(v_m - v_n)|) \, dx = 0 \,,$$

and (2.9) generates

$$\lim_{m \to \infty} \|v_m - v_n\|_{W_0^{1,\Phi_n}} = 0 \,.$$

That being the case, $\{v_m\}_m$ converges strongly to v_n in $W_0^{1,\Phi_n}(\Omega)$ as $m \to \infty$. Hence, relation (3.12) contribute to

$$J_{n,\lambda}(v_n) = \gamma_n < 0 \quad \text{and} \quad J'_{n,\lambda}(v_n) = 0. \tag{3.16}$$

As a result, v_n is the minimizer of $J_{n,\lambda}$ on $B_1(0)$, and also v_n is a critical point of the functional $J_{n,\lambda}$. Of course, v_n is really a weak solution of (1.1). Finally, note that $J_{n,\lambda}(|v|) \leq J_{n,\lambda}(v)$ for any $v \in W_0^{1,\Phi_n}(\Omega)$ and for this reason v_n is a nonnegative function on Ω.

The proof of Theorem 3.4 is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4 The asymptotic behavior of the sequence of solutions $\{v_n\}_n$ of problem (1.1) given by Theorem 3.4 as $n \to \infty$

The goal of this section is to prove the following result.

Theorem 4.1. *Let $\lambda^\star > 0$ be given by (3.8). For each $\lambda \in (0, \lambda^\star)$ and each $n \in \mathbb{N}^\star$ we denote by v_n the nonnegative weak solution of problem (1.1) given by Theorem 3.4. The sequence $\{v_n\}$ converges uniformly in Ω to $\mathrm{dist}(\cdot, \partial\Omega)$, the distance function to the boundary of Ω.*

In order to prove Theorem 4.1 we first establish the uniform Hölder estimates for the weak solutions of (1.1).

Lemma 4.2. *Let $\lambda^\star > 0$ be given by (3.8). Fix $\lambda \in (0, \lambda^\star)$ and let v_n be the nonnegative solution of problem (1.1) given by Theorem 3.4. Then there is a subsequence $\{v_n\}$ which converges uniformly in Ω, as $n \to \infty$, to a continuous function $v_\infty \in C(\overline{\Omega})$ with $v_\infty \geq 0$ in Ω and $v_\infty = 0$ on $\partial\Omega$.*

Proof. Let $q \geq N$ be an arbitrary real number. By (1.3) we can choose $q < \varphi_n^-$ for sufficiently large positive integer n. Using Hölder's inequality, relation (3.6), recalling that $v_n \in B_1(0) \subset W_0^{1,\Phi_n}(\Omega)$ and taking into account (2.9) we have

$$
\begin{aligned}
\left(\int_\Omega |\nabla v_n|^q \, dx \right)^{1/q} &\leq \left(\int_\Omega |\nabla v_n|^{\varphi_n^-} \, dx \right)^{1/\varphi_n^-} |\Omega|^{1/q - 1/\varphi_n^-} \\
&\leq \left[|\Omega| + \frac{1}{\Phi_n(1)} \int_\Omega \Phi_n(|\nabla v_n|) \, dx \right]^{1/\varphi_n^-} |\Omega|^{1/q - 1/\varphi_n^-} \\
&\leq \left[|\Omega| + \frac{1}{\Phi_n(1)} \|v_n\|_{W_0^{1,\Phi_n}}^{\varphi_n^-} \right]^{1/\varphi_n^-} |\Omega|^{1/q - 1/\varphi_n^-} \\
&\leq \left[|\Omega| + \frac{1}{\Phi_n(1)} \right]^{1/\varphi_n^-} |\Omega|^{1/q - 1/\varphi_n^-}.
\end{aligned}
$$

Thereupon, using (3.9) we find that sequence $\{|\nabla v_n|\}$ is uniformly bounded in $L^q(\Omega)$. It is clear that $q > N$ ensures that the embedding of $W_0^{1,q}(\Omega)$ into $C(\overline{\Omega})$ is compact. Keeping in mind the reflexivity of the Sobolev space $W_0^{1,q}(\Omega)$ we deduce that there exists a subsequence (not relabelled) of $\{v_n\}$ and a function $v_\infty \in C(\overline{\Omega})$ such that $v_n \rightharpoonup v_\infty$ weakly in $W_0^{1,q}(\Omega)$ and $v_n \to v_\infty$ uniformly in Ω as $n \to \infty$. In addition, the facts that $v_n \geq 0$ in Ω and $v_n = 0$ on $\partial\Omega$ for each $\varphi_n^- > N$ hint that $v_\infty \geq 0$ in Ω and $v_\infty = 0$ on $\partial\Omega$. The proof of Lemma 4.2 is complete. $\qquad\square$

In Theorem 4.5 below we show that function v_∞ given by Lemma 4.2 is the solution in the viscosity sense (see, Crandall, Ishii & Lions [11]) of a certain limiting problem. Accordingly, we adopt the usual strategy of first proving that continuous weak solutions of problem (1.1) at level n are indeed solutions in the viscosity sense. Before recalling the definition of viscosity solutions for this type of problems, let us note that if we assume for a moment that the solutions v_n of problem (1.1) are sufficiently smooth so that we can perform the differentiation in the PDE

$$
-\mathrm{div}\left(\frac{\varphi_n(|\nabla v_n|)}{|\nabla v_n|} \nabla v_n \right) = \lambda e^{v_n}, \quad \text{in } \Omega,
$$

we get

$$
-\frac{\varphi_n(|\nabla v_n|)}{|\nabla v_n|} \Delta v_n - \frac{|\nabla v_n| \varphi_n'(|\nabla v_n|) - \varphi_n(|\nabla v_n|)}{|\nabla v_n|^3} \Delta_\infty v_n = \lambda e^{v_n}, \quad \text{in } \Omega, \tag{4.1}
$$

where Δ stands for the Laplace operator, $\Delta v := \text{Trace}(D^2 v) = \sum_{i=1}^{N} \frac{\partial^2 v}{\partial x_i^2}$ and Δ_∞ stands for the ∞-Laplace operator,

$$\Delta_\infty v := \langle D^2 v \nabla v, \nabla v \rangle = \sum_{i,j=1}^{N} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j},$$

while $D^2 v$ denotes the Hessian matrix of v.

Remark that (4.1) can be reformulated as

$$H_n(v_n, \nabla v_n, D^2 v_n) = 0, \quad \text{in } \Omega$$

with function H_n defined as follows

$$H_n(y, z, S) := -\frac{\varphi_n(|z|)}{|z|} \text{Trace } S - \frac{|z|\varphi_n'(|z|) - \varphi_n(|z|)}{|z|^3} \langle Sz, z \rangle - \lambda e^y,$$

where $y \in \mathbb{R}$, z is a vector in $\mathbb{R}^N$ and S stands for a real symmetric matrix in $\mathbb{M}^{N \times N}$.

Since our main objective in this section is the asymptotic analysis of solutions $\{v_n\}$ as $n \to \infty$, we are now ready to give the definition of viscosity solutions for the homogeneous Dirichlet boundary value problem associated to degenerate elliptic PDE of the type

$$\begin{cases} H_n(v, \nabla v, D^2 v) = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Definition 4.3.

i) An upper semicontinuous function v is a viscosity subsolution of problem (4.2) if $v \le 0$ on $\partial\Omega$ and, whenever $x_0 \in \Omega$ and $\Psi \in C^2(\Omega)$ are such that $v(x_0) = \Psi(x_0)$ and $v(x) < \Psi(x)$ if $x \in B(x_0, r) \setminus \{x_0\}$ for some $r > 0$, we have $H_n(\Psi(x_0), \nabla\Psi(x_0), D^2\Psi(x_0)) \le 0$.

ii) A lower semicontinuous function v is a viscosity supersolution of problem (4.2) if $v \ge 0$ on $\partial\Omega$ and, whenever $x_0 \in \Omega$ and $Y \in C^2(\Omega)$ are such that $v(x_0) = Y(x_0)$ and $v(x) > Y(x)$ if $x \in B(x_0, r) \setminus \{x_0\}$ for some $r > 0$, we have $H_n(Y(x_0), \nabla Y(x_0), D^2 Y(x_0)) \ge 0$.

iii) A continuous function v is a viscosity solution of problem (4.2) if it is both viscosity supersolution and viscosity subsolution of problem (4.2).

In the sequel, functions Ψ and Y stand for test functions touching the graph of v from above and below, respectively.

Our goal now is to prove that any continuous weak solution of (1.1) is also viscosity solution of (1.1) and in order to establish this result we follow the approach by Juutinen, Lindqvist & Manfredi in [27, Lemma 1.8] (see also [35, Lemma 1] for a similar approach but in the framework of inhomogeneous differential operators).

Lemma 4.4. *A continuous weak solution of problem* (1.1) *is also a viscosity solution of* (1.1).

Proof. Firstly, we prove that if v_n is a continuous weak solution of problem (1.1) for a fixed positive integer n, then it is a viscosity subsolution of problem (1.1). We begin by considering $x_n^0 \in \overline{\Omega}$ and a test function $\Psi_n \in C^2(\overline{\Omega})$ such that $v_n(x_n^0) = \Psi_n(x_n^0)$ and $v_n - \Psi_n$ has a strict local maximum at x_n^0, that is $v_n(y) < \Psi_n(y)$ if $y \in B(x_n^0, \rho) \setminus \{x_n^0\}$ for some $\rho > 0$.

Next, we have to show that

$$- \operatorname{div}\left(\frac{\varphi_n(|\nabla \Psi_n(x_n^0)|)}{|\nabla \Psi_n(x_n^0)|} \nabla \Psi_n(x_n^0) \right) \leq \lambda e^{\Psi_n(x_n^0)}$$

or

$$-\frac{\varphi_n(|\nabla \Psi_n(x_n^0)|)}{|\nabla \Psi_n(x_n^0)|} \Delta \Psi_n(x_n^0) - \frac{|\nabla \Psi_n(x_n^0)|\varphi_n'(|\nabla \Psi_n(x_n^0)|) - \varphi_n(|\nabla \Psi_n(x_n^0)|)}{|\nabla \Psi_n(x_n^0)|^3} \Delta_\infty \Psi_n(x_n^0) \leq \lambda e^{\Psi_n(x_n^0)}.$$

Arguing *ad contrarium*, suppose that this is not the case of the above assertion. In other words, we admit that there exists a radius $\rho_n > 0$ such that $B(x_n^0, \rho_n) \subset \Omega$ from the Euclidean space $\mathbb{R}^N$ such that

$$-\frac{\varphi_n(|\nabla \Psi_n(y)|)}{|\nabla \Psi_n(y)|} \Delta \Psi_n(y) - \frac{|\nabla \Psi_n(y)|\varphi_n'(|\nabla \Psi_n(y)|) - \varphi_n(|\nabla \Psi_n(y)|)}{|\nabla \Psi_n(y)|^3} \Delta_\infty \Psi_n(y) > \lambda e^{\Psi_n(y)}$$

for all $y \in B(x_n^0, \rho_n)$. For ρ_n small enough, we may presume that $v_n - \Psi_n$ has a strict local maximum at x_n^0, that is $v_n(y) < \Psi_n(y)$ if $y \in B(x_n^0, \rho_n) \setminus \{x_n^0\}$. This fact implies that actually

$$\sup_{\partial B(x_n^0, \rho_n)} (v_n - \Psi_n) < 0.$$

Thus, we may consider a perturbation of the test function Ψ_n defined as

$$\overline{w}_n(y) := \Psi_n(y) + \frac{1}{2} \sup_{y \in \partial B(x_n^0, \rho_n)} [v_n - \Psi_n](y)$$

that has the properties

- $\overline{w}_n(x_n^0) < v_n(x_n^0)$;
- $\overline{w}_n > v_n$ on $\partial B(x_n^0, \rho_n)$;
- $-\operatorname{div}\left(\frac{\varphi_n(|\nabla \overline{w}_n|)}{|\nabla \overline{w}_n|} \nabla \overline{w}_n \right) > \lambda e^{\Psi_n}$ in $B(x_n^0, \rho_n)$.

Multiplying the above inequality by the positive part of the function $v_n - \overline{w}_n$, i.e. $(v_n - \overline{w}_n)^+$, that vanishes on the boundary of the ball $B(x_n^0, \rho_n)$, and integrating on $B(x_n^0, \rho_n)$, we get

$$\int_{\mathcal{M}_n} \frac{\varphi_n(|\nabla \overline{w}_n(x)|)}{|\nabla \overline{w}_n(x)|} \nabla \overline{w}_n(x) \ [\nabla v_n(x) - \nabla \overline{w}_n(x)] \ dx > \lambda \int_{\mathcal{M}_n} e^{\Psi_n(x)} [v_n(x) - \overline{w}_n(x)] \ dx, \quad (4.3)$$

where the set $\mathcal{M}_n := \{x \in B(x_n^0, \rho_n); \ \overline{w}_n(x) < v_n(x)\}$.

On the other hand, taking the test function in relation (3.1) to be

$$w : \Omega \to \mathbb{R}, \ w(x) = \begin{cases} (v_n - \overline{w}_n)^+(x), & \text{if } x \in B(x_n^0, \rho_n), \\ 0, & \text{if } x \in \Omega \setminus B(x_n^0, \rho_n), \end{cases}$$

we obtain

$$\int_{B(x_n^0, \rho_n)} \frac{\varphi_n(|\nabla v_n(x)|)}{|\nabla v_n(x)|} \nabla v_n(x) \ \nabla (v_n - \overline{w}_n)^+(x) \ dx = \lambda \int_{B(x_n^0, \rho_n)} e^{v_n(x)} (v_n - \overline{w}_n)^+(x) \ dx$$

or

$$\int_{\mathcal{M}_n} \frac{\varphi_n(|\nabla v_n(x)|)}{|\nabla v_n(x)|} \nabla v_n(x) \ \nabla (v_n - \overline{w}_n)(x) \ dx = \lambda \int_{\mathcal{M}_n} e^{v_n(x)} (v_n - \overline{w}_n)(x) \ dx$$

since $v_n \leq \overline{w}_n$ in the ball $B(x_n^0, \rho_n)$ outside $\mathcal{M}_n$.

Applying the subtraction of the above equality from inequality (4.3) it produces

$$\int_{\mathcal{M}_n} \left[\frac{\varphi_n(|\nabla \overline{w}_n|)}{|\nabla \overline{w}_n|} \nabla \overline{w}_n - \frac{\varphi_n(|\nabla v_n|)}{|\nabla v_n|} \nabla v_n \right] (\nabla v_n - \nabla \overline{w}_n) \, dx$$

$$> \lambda \int_{\mathcal{M}_n} \left(e^{\Psi_n} - e^{v_n} \right) (v_n - \overline{w}_n) \, dx \geq 0 \quad (4.4)$$

with the aid of the facts that $v_n < \Psi_n$ on $B(x_n^0, \rho_n) \setminus \{x_n^0\}$ and $\overline{w}_n < v_n$ on $\mathcal{M}_n \subset B(x_n^0, \rho_n)$.

Cauchy–Schwarz inequality implies

$$\int_{\mathcal{M}_n} \left[\varphi_n(|\nabla v_n|) - \varphi_n(|\nabla \overline{w}_n|) \right] (|\nabla v_n| - |\nabla \overline{w}_n|) dx$$

$$\leq \int_{\mathcal{M}_n} \left[\frac{\varphi_n(|\nabla v_n|)}{|\nabla v_n|} \nabla v_n - \frac{\varphi_n(|\nabla \overline{w}_n|)}{|\nabla \overline{w}_n|} \nabla \overline{w}_n \right] \nabla(v_n - \overline{w}_n) dx$$

and combined with relation (4.4) leads to

$$\int_{\mathcal{M}_n} \left[\varphi_n(|\nabla \overline{w}_n|) - \varphi_n(|\nabla v_n|) \right] (|\nabla \overline{w}_n| - |\nabla v_n|) dx < 0$$

which is a contradiction with the statement that φ_n is an increasing function on $\mathbb{R}$. Actually, it follows that v_n is a viscosity subsolution of problem (1.1).

On the other hand, v_n is a viscosity supersolution of problem (1.1) with similar arguments as above adapted for this case and therefore, these details will be omitted. The proof of Lemma 4.4 is complete. □

By Lemma 4.2 we may select a subsequence $\{v_n\}$ that converges uniformly to v_∞ in Ω as $n \to \infty$. Next, we will focus to identify the limit equation verified by v_∞. The following theorem encloses the main result regarding the asymptotic behavior of the solutions $\{v_n\}$ of problem (1.1).

Theorem 4.5. *Let v_∞ be the function achieved as the uniform limit of a subsequence of $\{v_n\}$ in Lemma 4.2. Then v_∞ is a solution in the viscosity sense of problem*

$$\begin{cases} \min\{-\Delta_\infty v, |\nabla v| - 1\} = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.5)$$

Proof. First, we investigate if v_∞ is a viscosity supersolution of (4.5). We consider $y_0 \in \Omega$ and a test function $Y \in C^2(\Omega)$ such that $v_\infty - Y$ has a strict local minimum point at y_0. We claim that the uniform convergence of $\{v_n\}$ shown in Lemma 4.2 allows us to extract, up to a subsequence, $\{y_n\} \subset \Omega$ such that y_n converges to y_0 and moreover $v_n - Y$ achieves a strict local minimum point at y_n. Indeed, since y_0 is a strict minimum point of $v_\infty - Y$ it follows that $v_\infty(y_0) = Y(y_0)$ and $v_\infty(y) > Y(y)$ for every y in a punctured neighborhood of y_0, let's say $B(y_0, r) \setminus \{y_0\}$ with $r > 0$ fixed in such a manner that $B(y_0, 2r) \subset \Omega$. For any positive ρ with $\rho < r$ we get

$$\inf_{B(y_0, r) \setminus B(y_0, \rho)} (v_\infty - Y) > 0.$$

By the uniform convergence of $\{v_n\}$ to v_∞ in Ω and in particular in $\overline{B(y_0, r)}$, for any positive integer n sufficiently large, the function $v_n - Y$ attains its zero minimum value in $B(y_0, \rho)$ and

thus, the minimum point of $v_n - Y$ will be represented by $y_n \in B(y_0, \rho)$. Considering a sequence $\rho_k \to 0^+$ as $k \to \infty$, we can construct a subsequence $\{n_k\}$ such that y_{n_k} converges to y_0 as $k \to \infty$. The claim now holds true after an appropriate relabelling of the indices. In other words, taking into account that $v_n, v_\infty \in C(\overline{\Omega})$ for any positive integer n sufficiently large, the uniform convergence of sequence $\{v_n\}$ to v_∞ in Ω implies that since Y touches v_∞ from below at y_0, then there are points $y_n \to y_0$ such that

$$v_n(y) - Y(y) > 0 = v_n(y_n) - Y(y_n) \quad \text{for all } y \in B(y_0, \rho) \setminus \{y_0\}$$

for some subsequence (see [6, Theorem 3.1] or [30, Lemma 11]).

Keeping in mind that in view of Lemma 4.4, v_n is a continuous viscosity solution of (1.1) we have

$$- \frac{\varphi_n(|\nabla Y(y_n)|)}{|\nabla Y(y_n)|} \Delta Y(y_n) - \frac{|\nabla Y(y_n)| \varphi'_n(|\nabla Y(y_n)|) - \varphi_n(|\nabla Y(y_n)|)}{|\nabla Y(y_n)|^3} \Delta_\infty Y(y_n) \geq \lambda e^{Y(y_n)} .$$
(4.6)

Since $\lambda e^{Y(y_n)} > 0$ for any $\lambda \in (0, \lambda^*)$, it follows that $|\nabla Y(y_n)| > 0$ for each positive integer n. Recalling inequality (2.6) states

$$\min\{s^{\varphi_n^- - 1}, s^{\varphi_n^+ - 1}\} \varphi_n(t) \leq \varphi_n(st) \leq \max\{s^{\varphi_n^- - 1}, s^{\varphi_n^+ - 1}\} \varphi_n(t), \qquad \forall \, s, t \geq 0 \qquad (4.7)$$

and keeping in mind (1.3), for each positive integer n sufficiently large, the functions $A_n, B_n :$ $[0, \infty) \to \mathbb{R}$,

$$A_n(t) := \begin{cases} \dfrac{t \varphi'_n(t) - \varphi_n(t)}{t^3}, & \text{if } t > 0, \\ 0, & \text{if } t = 0, \end{cases} \qquad B_n(t) := \begin{cases} \dfrac{\varphi_n(t)}{t}, & \text{if } t > 0, \\ 0, & \text{if } t = 0 \end{cases}$$

are continuous. Moreover, function B_n is of class C^1 since $A_n(t) = t^{-1} B'_n(t)$ for $t > 0$. According to (1.2) and (1.3), we deduce that

$$\frac{|\nabla Y(y_n)|^3}{|\nabla Y(y_n)| \varphi'_n(|\nabla Y(y_n)|) - \varphi_n(|\nabla Y(y_n)|)} > 0.$$

Inequality (4.6) multiplied with the above positive quantity in both sides becomes

$$- \frac{\varphi_n(|\nabla Y(y_n)|) |\nabla Y(y_n)|^2}{|\nabla Y(y_n)| \varphi'_n(|\nabla Y(y_n)|) - \varphi_n(|\nabla Y(y_n)|)} \Delta Y(y_n) - \Delta_\infty Y(y_n)$$
$$\geq \frac{\lambda e^{Y(y_n)} |\nabla Y(y_n)|^3}{|\nabla Y(y_n)| \varphi'_n(|\nabla Y(y_n)|) - \varphi_n(|\nabla Y(y_n)|)} . \qquad (4.8)$$

On the other hand, we obtain

$$\frac{\varphi_n(|\nabla Y(y_n)|) |\nabla Y(y_n)|^2}{|\nabla Y(y_n)| \varphi'_n(|\nabla Y(y_n)|) - \varphi_n(|\nabla Y(y_n)|)} = \frac{|\nabla Y(y_n)|^2}{\dfrac{|\nabla Y(y_n)| \varphi'_n(|\nabla Y(y_n)|)}{\varphi_n(|\nabla Y(y_n)|)} - 1} \leq \frac{|\nabla Y(y_n)|^2}{\varphi_n^- - 2}, \quad (4.9)$$

where in the latter inequality we use Lieberman-type condition (1.2).

In relation (4.8) we pass to the limit as $n \to \infty$ and then using (1.3) we infer by relation (4.9) that

$$- \Delta_\infty Y(y_0) \geq \limsup_{n \to \infty} \frac{\lambda e^{Y(y_n)} |\nabla Y(y_n)|^3}{|\nabla Y(y_n)| \varphi'_n(|\nabla Y(y_n)|) - \varphi_n(|\nabla Y(y_n)|)} \qquad (4.10)$$

which hints that

$$-\Delta_\infty Y(y_0) \geq 0. \tag{4.11}$$

In the following we will show that

$$|\nabla Y(y_0)| - 1 \geq 0. \tag{4.12}$$

If we assume by contradiction that is not the case of the above claim, we get $|\nabla Y(y_0)| - 1 < 0$, that implies $|\nabla Y(y_n)| < 1$ for any positive integer n sufficiently large. Taking into consideration (1.2) and then inequality (4.7) we arrive at

$$
\begin{aligned}
\frac{\lambda e^{Y(y_n)}|\nabla Y(y_n)|^3}{|\nabla Y(y_n)|\varphi_n'(|\nabla Y(y_n)|) - \varphi_n(|\nabla Y(y_n)|)} &= \frac{|\nabla Y(y_n)|^3}{\frac{|\nabla Y(y_n)|\varphi_n'(|\nabla Y(y_n)|)}{\varphi_n(|\nabla Y(y_n)|)} - 1} \cdot \frac{\lambda e^{Y(y_n)}}{\varphi_n(|\nabla Y(y_n)|)} \\
&\geq \frac{|\nabla Y(y_n)|^3}{\varphi_n^+ - 2} \cdot \frac{\lambda e^{Y(y_n)}}{\varphi_n(|\nabla Y(y_n)|)} \\
&\geq \frac{|\nabla Y(y_n)|^3}{\varphi_n^+ - 2} \cdot \frac{\lambda e^{Y(y_n)}}{\varphi_n(1)|\nabla Y(y_n)|^{\varphi_n^- - 1}} \\
&= \left[\left(\frac{\lambda e^{Y(y_n)}}{(\varphi_n^+ - 2)\varphi_n(1)} \right)^{1/(\varphi_n^- - 4)} \frac{1}{|\nabla Y(y_n)|} \right]^{\varphi_n^- - 4}.
\end{aligned}
$$

Since by (1.5) we have $\lim_{n\to\infty} \varphi_n(1)^{1/\varphi_n^-} = 1$ we get using (1.4) that

$$\lim_{n\to\infty} \left(\frac{\lambda e^{Y(y_n)}}{(\varphi_n^+ - 2)\varphi_n(1)} \right)^{1/(\varphi_n^- - 4)} = 1.$$

Next, taking into account that $\lim_{n\to\infty} \dfrac{1}{|\nabla Y(y_n)|} = \dfrac{1}{|\nabla Y(y_0)|} > 1$ we obtain

$$\lim_{n\to\infty} \left(\frac{\lambda e^{Y(y_n)}}{(\varphi_n^+ - 2)\varphi_n(1)} \right)^{1/(\varphi_n^- - 4)} \frac{1}{|\nabla Y(y_n)|} = \frac{1}{|\nabla Y((y_0)|} > 1$$

and then, we deduce that there exists $\epsilon_0 > 0$ such that

$$\left(\frac{\lambda e^{Y(y_n)}}{(\varphi_n^+ - 2)\varphi_n(1)} \right)^{1/(\varphi_n^- - 4)} \frac{1}{|\nabla Y(y_n)|} \geq 1 + \epsilon_0 \quad \text{for all positive integer } n \text{ sufficiently large,}$$

which yields to

$$\limsup_{n\to\infty} \frac{\lambda e^{Y(y_n)}|\nabla Y(y_n)|^3}{|\nabla Y(y_n)|\varphi_n'(|\nabla Y(y_n)|) - \varphi_n(|\nabla Y(y_n)|)} \geq \lim_{n\to\infty} (1 + \epsilon_0)^{\varphi_n^- - 4} = +\infty,$$

a contradiction with (4.10). Thus, inequality (4.12) holds true.

   Assembling relations (4.11) and (4.12) we have $\min\{-\Delta_\infty Y(y_0), |\nabla Y(y_0)| - 1\} \geq 0$ which leads to the fact that v_∞ is a viscosity supersolution of (4.5).

   Now, it remains to see that in fact v_∞ is a viscosity subsolution of (4.5). We take a test function $\Psi \in C^2(\Omega)$ that touches the graph of v_∞ from above in a point $x_0 \in \Omega$, that means $v_\infty(x_0) = \Psi(x_0)$ and $v_\infty(x) < \Psi(x)$ for every x in a punctured neighborhood of x_0 and we have

to establish that $\min\{-\Delta_\infty\Psi(x_0), |\nabla\Psi(x_0)| - 1\} \leq 0$. We notice that if $|\nabla\Psi(x_0)| = 0$ then we have $\Delta_\infty\Psi(x_0) = 0$ and everything is clear. Then, it is sufficient to check that if $|\nabla\Psi(x_0)| > 0$ and also

$$|\nabla\Psi(x_0)| - 1 > 0, \tag{4.13}$$

we get $-\Delta_\infty\Psi(x_0) \leq 0$. Actually, the uniform convergence of subsequence of $\{v_n\}$ ensures again, as in the first part of this proof, the existence of a sequence $x_n \to x_0$ as $n \to \infty$ such that $v_n - \Psi$ has a strict local maximum point at x_n and

$$-\frac{\varphi_n(|\nabla\Psi(x_n)|)|\nabla\Psi(x_n)|^2}{|\nabla\Psi(x_n)|\varphi_n'(|\nabla\Psi(x_n)|) - \varphi_n(|\nabla\Psi(x_n)|)}\Delta\Psi(x_n) - \Delta_\infty\Psi(x_n)$$
$$\leq \frac{\lambda e^{\Psi(x_n)}|\nabla\Psi(x_n)|^3}{|\nabla\Psi(x_n)|\varphi_n'(|\nabla\Psi(x_n)|) - \varphi_n(|\nabla\Psi(x_n)|)}. \tag{4.14}$$

Passing to the limit as $n \to \infty$ in the above relation and using (4.13), inequality (4.7), and assumptions (1.3) and (1.5), we deduce that

$$-\Delta_\infty\Psi(x_0) \leq \liminf_{n\to\infty}\left[\left(\frac{\lambda e^{\Psi(x_n)}}{(\varphi_n^- - 2)\varphi_n(1)}\right)^{1/(\varphi_n^- - 4)}\frac{1}{|\nabla\Psi(x_n)|}\right]^{\varphi_n^- - 4} = 0$$

which implies that $-\Delta_\infty\Psi(x_0) \leq 0$. Thus, we conclude that v_∞ is a viscosity solution of problem (4.5). The proof of Theorem 4.5 is complete. $\qquad\square$

Next, we identify the limit of the entire sequence of weak solutions $\{v_n\}$ of problem (1.1).

Proof of Theorem 4.1 (concluded). It is well-known that problem (4.5) has as unique viscosity solution $\text{dist}(\cdot, \partial\Omega)$, namely the distance function to the boundary of Ω (see Jensen [25], or Juutinen [26, Lemma 6.10], or Ishibashi & Koike [24, p. 546]). As a consequence, Lemma 4.2 and Theorem 4.5 allow us to reach to the conclusion that the entire sequence $\{v_n\}$ converges uniformly to $\text{dist}(\cdot, \partial\Omega)$ in Ω, as $n \to \infty$. $\qquad\square$

Acknowledgements

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