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Positive solutions for second-order differential equations with singularities and separated integral boundary conditions

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. We study the existence of positive solutions for second-order differential equations with separated integral boundary conditions. The nonlinear part of the equation involves the derivative and may be singular for the second and third space variables. The result ensures existence of a positive solution when the parameters are in certain ranges. The proof depends on general properties of the associated Green's function and the Krasnosel'skii–Guo fixed point theorem applied to a perturbed Hammerstein integral operator. Both numerical and analytical examples are constructed to illustrate applications of the theorem to a group of equations. The result generalizes previous work.

Keywords: fixed point, Green's function, Hammerstein integral operator, positive solution, singular boundary value problem.

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1 Introduction

We are interested in the following singular Boundary Value Problem (BVP) for second-order differential equations with non-local boundary conditions involving integrals:

$$\begin{cases} u''(t) + f(t, u(t), u'(t)) = 0, \quad t \in [0,1], \\ \theta u(0) - \alpha u'(0) = \int_0^1 g_1(s) u(s) ds, \\ \gamma u(1) + \beta u'(1) = \int_0^1 g_2(s) u(s) ds, \end{cases} \tag{1.1}$$

where the parameters $\theta, \alpha, \beta > 0, \gamma \geq 0$. The nonlinear function f is continuous, non-negative on $[0,1] \times (0,\infty) \times (0,\infty)$ and may be singular at zero on its space variables.

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When $\gamma = 0$, BVP (1.1) reduces to the problem studied in [17]. It also includes the anti-symmetric boundary conditions $u(0) = u'(0), u(1) = -u'(1)$ [13]. The local singular BVP studied in [16] is a special case of the boundary conditions of (1.1) when $\gamma = 0$, and $g_1 = g_2 = 0$ as well. Similar boundary conditions have been studied for fractional differential equations in [2] which assumed that $\theta = \gamma$ and $\alpha = -\beta$.

In the study of BVPs and their applications, nonlocal boundary conditions usually involve discrete multi-point boundary conditions. Previously, the following three-point BVPs have been extensively studied [3–5, 10, 12]:

$$u(0) = 0, \quad u(1) = \alpha u(\eta),$$

or

$$u'(0) = 0, \quad u(1) = \alpha u(\eta),$$

where $0 < \eta < 1$, α is a parameter. Later, the boundary conditions were further extended to involve integrals and functionals [7–9, 13–15]. In particular, in [14], existence of multiple positive solutions for nonlocal BVPs involving various integral conditions were obtained for the case that the nonlinear function f does not involve the first-order derivative. On the other side, results on non-existence of positive solutions for different types of nonlocal BVPs were discussed in [11].

Our main result on the existence of positive solutions of BVP (1.1) is proved by using the similar techniques that were applied in [17] and originally developed by Webb and Infante [13]. The idea is to restrict the singular function f to a subset $[0, 1] \times [\rho_1, \infty) \times [\rho_2, \infty)$ of $[0, 1] \times (0, \infty) \times (0, \infty)$, where $\rho_1, \rho_2 > 0$ are properly selected such that problem (1.1) can be converted to the following perturbed Hammerstein integral operator of the form

$$Fu(t) = \int_0^1 G(t, s) f(s, u(s), u'(s)) ds + r(t)\eta[u] + w(t)\xi[u], \tag{1.2}$$

where $\eta[u]$ and $\xi[u]$ are positive linear functionals on $C[0, 1]$, r and w satisfy certain upper bound conditions. Then existence of a positive solution for problem (1.1) is equivalent to a fixed point problem for the operator F.

For convenience, we give the following definition of an order cone P in a Banach space and the well-known Krasnosel'skii–Guo fixed point theorem on a cone P that will be applied to prove the existence result in Section 3.

Definition 1.1. A cone P in a Banach space X is a closed convex set such that $\lambda x \in P$ for every $x \in P$ and for all $\lambda \geq 0$, and satisfying $P \cap (-P) = \{0\}$.

For any $r > 0$, we denote $\Omega_r = \{x \in X : \|x\| < r\}$ and $\partial\Omega_r = \{x \in X : \|x\| = r\}$.

Theorem 1.1 (Krasnosel'skii–Guo [6]). Let $T : P \to P$ be a compact map. Assume that there exist two positive constants r, R with $r \neq R$ such that

$$\|Tu\| \leq \|u\| \text{ for every } u \in P \text{ with } \|u\| = r,$$

and

$$\|Tu\| \geq \|u\| \text{ for every } u \in P \text{ with } \|u\| = R.$$

Then there exists $u_0 \in P$ such that $Tu_0 = u_0$ and $\min\{r, R\} \leq \|u_0\| \leq \max\{r, R\}$.

In Section 2, we first prove some properties of the Green's function G in (1.2) that are essential in the construction of the cone for our proof.

2 Preliminaries

Let $h_1(t) = \gamma(1-t) + \beta$, $h_2(t) = \alpha + \theta t$ and $m = \theta\gamma + \theta\beta + \alpha\gamma$. The following assumption ensures that problem (1.1) is non-resonant [5]:

(H1) $$\left(m - \int_0^1 g_1(s)h_1(s)ds\right)\left(m - \int_0^1 g_2(s)h_2(s)ds\right) - \int_0^1 g_1(s)h_2(s)ds \int_0^1 g_2(s)h_1(s)ds \neq 0.$$

This condition implies that BVP (2.1) has only the trivial solution:

$$\begin{cases} u''(t) = 0, \quad t \in [0,1], \\ \theta u(0) - \alpha u'(0) = \int_0^1 g_1(s)u(s)ds, \\ \gamma u(1) + \beta u'(1) = \int_0^1 g_2(s)u(s)ds. \end{cases} \tag{2.1}$$

Under condition (H1), BVP (1.1) can be converted to a fixed point problem for the nonlinear operator F in (1.2), where G is the Green's function of the problem

$$\begin{cases} u''(t) + y(t) = 0, \quad t \in [0,1], \\ \theta u(0) - \alpha u'(0) = 0, \\ \gamma u(1) + \beta u'(1) = 0, \end{cases} \tag{2.2}$$

r and w are the unique solutions of

$$\begin{cases} u''(t) = 0, \quad t \in [0,1], \\ \theta u(0) - \alpha u'(0) = 1, \\ \gamma u(1) + \beta u'(1) = 0, \end{cases} \tag{2.3}$$

and

$$\begin{cases} u''(t) = 0, \quad t \in [0,1], \\ \theta u(0) - \alpha u'(0) = 0, \\ \gamma u(1) + \beta u'(1) = 1, \end{cases} \tag{2.4}$$

respectively. By calculation, we can find that $r(t) = \frac{h_1(t)}{m}$, $w(t) = \frac{h_2(t)}{m}$, and

$$G(t,s) = \begin{cases} \dfrac{h_2(s)h_1(t)}{m}, & 0 \leq s \leq t \leq 1, \\ \dfrac{h_2(t)h_1(s)}{m}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.5}$$

Condition (H1) is equivalent to

$$\left(1 - \int_0^1 g_1(s)r(s)ds\right)\left(1 - \int_0^1 g_2(s)w(s)ds\right) - \int_0^1 g_1(s)w(s)ds \int_0^1 g_2(s)r(s)ds \neq 0. \tag{2.6}$$

Lemma 2.1. Let $\Phi(s) = G(s,s)$, then

$$c_0\Phi(s) \leq G(t,s) \leq \Phi(s), \quad \text{for } 0 < t, s < 1,$$

where

$$c_0 = \begin{cases} \frac{\alpha}{\alpha+\theta}, & \gamma = 0 \ \text{ or } \left(\gamma \neq 0, \text{ and } \frac{\beta}{\gamma} - \frac{\alpha}{\theta} \geq 1\right), \\ \frac{\beta}{\beta+\gamma}, & \gamma \neq 0, \ \frac{\beta}{\gamma} - \frac{\alpha}{\theta} \leq -1, \\ \frac{\alpha\beta}{(\alpha+\theta)(\gamma+\beta)}, & \gamma \neq 0, \ -1 < \frac{\beta}{\gamma} - \frac{\alpha}{\theta} < 1. \end{cases} \tag{2.7}$$

Proof: For both cases of $0 \leq s \leq t \leq 1$ and $0 \leq t \leq s \leq 1$, we can easily verify that $G(t,s) \leq G(s,s)$ from the inequalities: $h_1(t) \leq h_1(s)$ for $0 \leq s \leq t \leq 1$ and $h_2(t) \leq h_2(s)$ for $0 \leq t \leq s \leq 1$. Now consider

$$c_0 G(s,s) = \frac{c_0(-\theta\gamma s^2 + (\gamma\theta + \beta\theta - \alpha\gamma)s + \alpha(\gamma+\beta))}{m}.$$

(1) If $\gamma = 0$,

$$c_0 G(s,s) = \frac{c_0(\theta s + \alpha)}{\theta} < \frac{c_0(\theta+\alpha)}{\theta} = \frac{\alpha}{\theta} \leq \frac{(\alpha + \theta t \text{ (or } s))}{\theta} = G(t,s).$$

(2) If $\gamma \neq 0$, let $h(s) := -\theta\gamma s^2 + (\gamma\theta + \beta\theta - \alpha\gamma)s + \alpha(\gamma+\beta)$. Then h has the critical point:

$$s_0 = \frac{1}{2} + \frac{1}{2}\left(\frac{\beta}{\gamma} - \frac{\alpha}{\theta}\right).$$

Assume that $\frac{\beta}{\gamma} - \frac{\alpha}{\theta} \geq 1$, $\max\{h(s), s \in [0,1]\} = h(1)$,

$$\begin{aligned} c_0 G(s,s) &\leq \frac{c_0(\alpha+\theta)\beta}{m} \leq \frac{c_0(\alpha+\theta)(\beta + \gamma(1 - t \text{ (or } s)))}{m} \\ &\leq \frac{(\alpha + \theta s \text{ (or } t))(\gamma + \beta - \gamma t \text{ (or } s))}{m} = G(t,s). \end{aligned}$$

On the other hand, if $\frac{\beta}{\gamma} - \frac{\alpha}{\theta} \leq -1$, $\max\{h(s), s \in [0,1]\} = h(0)$,

$$\begin{aligned} c_0 G(s,s) &\leq \frac{c_0\alpha(\gamma+\beta)}{m} \leq \frac{c_0(\alpha + \theta s \text{ (or } t))(\gamma+\beta)}{m} \\ &\leq \frac{(\alpha + \theta s \text{ (or } t))(\gamma + \beta - \gamma t \text{ (or } s))}{m} = G(t,s). \end{aligned}$$

In the case of $-1 < \frac{\beta}{\gamma} - \frac{\alpha}{\theta} < 1$, we have $-\alpha\gamma < \gamma\theta - \beta\theta$,

$$\max\{h(s), s \in [0,1]\} = \alpha(\gamma+\beta) + \frac{(\gamma\theta + \beta\theta - \alpha\gamma)^2}{4\theta\gamma} < \alpha(\gamma+\beta) + \theta\gamma.$$

Therefore,

$$\begin{aligned} c_0 h(s) < c_0(\alpha(\gamma+\beta) + \theta\gamma) &= \frac{\alpha}{\alpha+\theta}\left(\alpha\beta + \frac{\beta\theta\gamma}{\gamma+\beta}\right) \\ &< \alpha\beta < (\alpha + \theta s \text{ (or } t))(\gamma + \beta - \gamma t \text{ (or } s)), \end{aligned}$$

and

$$c_0 G(s,s) < \frac{(\alpha + \theta s \text{ (or } t))(\gamma + \beta - \gamma t \text{ (or } s))}{m} = G(t,s).$$

□

The following simple property of the constant c_0 will be useful in the sequel.

Property 2.2. Let c_0 be defined as (2.7). Then $c_0 \leq \min\left\{\frac{\alpha}{\alpha+\theta}, \frac{\beta}{\gamma+\beta}\right\}$.

Proof: If $\gamma = 0$, it is true since $c_0 = \frac{\alpha}{\alpha+\theta} < 1$. It is also clear for the case of $\gamma \neq 0$ and $-1 < \frac{\beta}{\gamma} - \frac{\alpha}{\theta} < 1$. Assume that $\gamma \neq 0$ and $\frac{\beta}{\gamma} - \frac{\alpha}{\theta} \geq 1$. Then $\frac{\beta}{\gamma} > \frac{\alpha}{\theta}$ implies $\frac{\gamma}{\beta+\gamma} < \frac{\theta}{\alpha+\theta}$. Hence $c_0 = \frac{\alpha}{\alpha+\theta} < \frac{\beta}{\beta+\gamma}$. Similarly, it can be shown that $c_0 = \frac{\beta}{\beta+\gamma} < \frac{\alpha}{\alpha+\theta}$ with the assumption of $\frac{\beta}{\gamma} - \frac{\alpha}{\theta} \leq -1$. □

3 Main result

Let $C^1[0,1]$ be the Banach space of continuously differentiable functions with the norm $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}$ and $\|u\|_\infty = \max\{|u(t)| : t \in [0,1]\}$. Following similar approaches of [13, 17], we consider the BVP for $\widetilde{f}$, the restriction of f on $[0,1] \times [\rho_1, \infty] \times [\rho_2, \infty]$, where $\rho_1 > 0$, $\rho_2 > 0$:

$$\begin{cases} u''(t) + \widetilde{f}(t, u(t), u'(t)) = 0, \quad t \in [0,1], \\ \theta u(0) - \alpha u'(0) = \int_0^1 g_1(s) u(s) ds, \\ \gamma u(1) + \beta u'(1) = \int_0^1 g_2(s) u(s) ds. \end{cases} \tag{3.1}$$

If u_0 is a positive solution of the regular BVP (3.1), then $u_0(t) \geq \rho_1 > 0$ and $u_0'(t) \geq \rho_2$, so u_0 is a positive solution of (1.1). In addition to (H1), we introduce more assumptions on function $\widetilde{f}$ and the coefficients θ, α, γ and β that appear in (3.1). Let

$$l_1 = \int_0^1 g_1(s) ds, \qquad l_2 = \int_0^1 g_2(s) ds,$$

and $m = \theta(\gamma + \beta) + \alpha\gamma$ as defined in Section 2. Assume there exist $0 < r < R$ and $K, k > 0$ such that:

(H2) $c_0 \min\left\{1, \frac{\alpha}{\theta}\right\} r \geq \rho_1$ and $c \min\left\{1, \frac{\alpha}{\theta}\right\} r \geq \rho_2$;

(H3) $\frac{\widetilde{f}(t,u,v)}{R} \leq K \leq \frac{2(m - \beta l_1 - \alpha l_2 - \theta l_2)}{\theta\gamma + 2(\theta + \alpha)\beta}$ for $(t, u, v) \in [0,1] \times \left[Rc_0 \min\left\{1, \frac{\alpha}{\theta}\right\}, R\right] \times \left[Rc \min\left\{1, \frac{\alpha}{\theta}\right\}, R\right]$;

(H4) $\frac{\widetilde{f}(t,u,v)}{r} \geq k \geq \frac{2(m - c_0 \beta \min\left\{1, \frac{\alpha}{\theta}\right\} l_1 - c_0(\alpha + \theta) \min\left\{1, \frac{\alpha}{\theta}\right\} l_2)}{(2\alpha + \theta)\beta}$ for $(t, u, v) \in [0,1] \times \left[rc_0 \min\left\{1, \frac{\alpha}{\theta}\right\}, r\right] \times \left[rc \min\left\{1, \frac{\alpha}{\theta}\right\}, r\right]$;

(H5) $(\theta l_2 - \gamma l_1) \min\left\{1, \frac{\alpha}{\theta}\right\} r - RK\gamma(\theta + \alpha) > 0$.

Of conditions (H2)–(H4), the constant c is defined as

$$c := \frac{-RK\gamma(\theta + \alpha) + (\theta l_2 - \gamma l_1) \min\left\{1, \frac{\alpha}{\theta}\right\} r}{(\alpha + \theta)(\gamma + \beta) RK + [(\gamma + \beta) l_1 + (\alpha + \theta) l_2] R}.$$

Since

$$\theta l_2 - \gamma l_1 < (\gamma + \beta) l_1 + (\alpha + \theta) l_2, \ \min\left\{1, \frac{\alpha}{\theta}\right\} r < R,$$

We have

$$(\theta l_2 - \gamma l_1) \min\left\{1, \frac{\alpha}{\theta}\right\} r - RK\gamma(\theta + \alpha) < (\alpha + \theta)(\gamma + \beta) RK + [(\gamma + \beta) l_1 + (\alpha + \theta) l_2] R.$$

Condition (H5) implies that $0 \leq c < 1$.

Theorem 3.1. Under the assumptions (H1)–(H5), the regular BVP (3.1) has a positive solution u satisfying

$$c_0 \min\left\{1, \frac{\alpha}{\theta}\right\} r \leq u(t) \leq R,$$

and

$$c \min\left\{1, \frac{\alpha}{\theta}\right\} r \leq u'(t) \leq R.$$

Proof: Similar as (1.2), we consider

$$(\widetilde{F}u)(t) = \int_0^1 G(t,s)\widetilde{f}(s,u(s),u'(s))ds + r(t)\int_0^1 g_1(s)u(s)ds + w(t)\int_0^1 g_2(s)u(s)ds \tag{3.2}$$

and its derivative

$$\begin{aligned}(\widetilde{F}u)'(t) = &\int_0^t \frac{-\gamma(\alpha+\theta s)}{m}\widetilde{f}(s,u(s),u'(s))ds + \int_t^1 \frac{\theta(\gamma+\beta-\gamma s)}{m}\widetilde{f}(s,u(s),u'(s))ds \\ &- \frac{\gamma}{m}\int_0^1 g_1(s)u(s)ds + \frac{\theta}{m}\int_0^1 g_2(s)u(s)ds.\end{aligned}$$

Define the cone P of $C^1[0,1]$ as

$$P = \left\{u \in C^1[0,1] : u(0) \geq \frac{\alpha}{\theta}\|u'\|_\infty,\ u'(t) \geq c\|u\|_\infty,\ u(t) \geq c_0\|u\|_\infty,\ t \in [0,1]\right\}. \tag{3.3}$$

Notice that the constant c in P involves the upper bound RK and the lower bound rk of $\widetilde{f}$ on the closed subsets. If $u \in P$, then

$$u(t) \geq c_0\|u\|_\infty \geq c_0 u(0) \geq c_0\frac{\alpha}{\theta}\|u'\|_\infty.$$

Hence

$$\begin{aligned}u(t) &\geq \max\left\{c_0\|u\|_\infty,\ c_0\frac{\alpha}{\theta}\|u'\|_\infty\right\} \\ &\geq c_0\min\left\{1,\frac{\alpha}{\theta}\right\}\max\left\{\|u\|_\infty,\|u'\|_\infty\right\} = c_0\min\left\{1,\frac{\alpha}{\theta}\right\}\|u\|.\end{aligned}$$

Also

$$u'(t) \geq c\|u\|_\infty \geq c\frac{\alpha}{\theta}\|u'\|_\infty.$$

Therefore

$$u'(t) \geq c\max\left\{\|u\|_\infty,\ \frac{\alpha}{\theta}\|u'\|_\infty\right\} \geq c\min\left\{1,\frac{\alpha}{\theta}\right\}\|u\|.$$

Let

$$\Omega_1 = \{u \in C^1[0,1] : \|u\| < r\} \quad \textit{and} \quad \Omega_2 = \{u \in C^1[0,1] : \|u\| < R\}.$$

We show that $\widetilde{F} : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$. If $u \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$, then $\widetilde{F}u \in C^1[0,1]$, and

$$\begin{aligned}c_0\|\widetilde{F}u\|_\infty &\leq c_0\int_0^1 G(s,s)\widetilde{f}(s,u(s),u'(s))ds + c_0\frac{\beta+\gamma}{m}\int_0^1 g_1(s)u(s)ds \\ &\quad + c_0\frac{\alpha+\theta}{m}\int_0^1 g_2(s)u(s)ds \\ &\leq \int_0^1 G(t,s)\widetilde{f}(s,u(s),u'(s))ds + \frac{\beta+\gamma(1-t)}{m}\int_0^1 g_1(s)u(s)ds \\ &\quad + \frac{\alpha+\theta t}{m}\int_0^1 g_2(s)u(s)ds \\ &= \widetilde{F}u(t).\end{aligned} \tag{3.4}$$

Next, conditions (H3) and (H4) imply that $f(t,u,v) \leq RK$ for $(t,u,v) \in [0,1]\times[Rc_0\min\{1,\frac{\alpha}{\theta}\},R]\times\left[Rc\min\{1,\frac{\alpha}{\theta}\},R\right]$ and $f(t,u,v) \geq rk$ for $(t,u,v) \in [0,1]\times\left[rc_0\min\left\{1,\frac{\alpha}{\theta}\right\},r\right]\times\left[rc\min\left\{1,\frac{\alpha}{\theta}\right\},r\right]$.

For $u \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$, we obtain that

$$
\begin{aligned}
(\widetilde{F}u)'(t) = & \int_0^t \frac{-\gamma(\alpha+\theta s)}{m} \widetilde{f}(s,u(s),u'(s))ds + \int_t^1 \frac{\theta(\gamma+\beta-\gamma s)}{m} \widetilde{f}(s,u(s),u'(s))ds \\
& - \frac{\gamma}{m}\int_0^1 g_1(s)u(s)ds + \frac{\theta}{m}\int_0^1 g_2(s)u(s)ds \\
\geq & RK\frac{-\gamma(\alpha+\theta)}{m} + rk\int_t^1 \frac{\theta(\gamma+\beta-\gamma s)}{m}ds + \int_0^1 \frac{-\gamma g_1(s)+\theta g_2(s)}{m}u(s)ds \\
\geq & RK\frac{-\gamma(\alpha+\theta)}{m} + rk\frac{\theta(\gamma+\beta)}{m}(1-t) - rk\frac{\theta\gamma}{2m}(1-t^2) + \frac{\theta l_2 - \gamma l_1}{m} r \min\left\{1, \frac{\alpha}{\theta}\right\} \\
= & \frac{rk\theta\gamma}{2m}t^2 - \frac{rk\theta(\gamma+\beta)}{m}t - RK\frac{-\gamma(\alpha+\theta)}{m} + \frac{rk\theta(\gamma+\beta)}{m} - rk\frac{\theta\gamma}{2m} \\
& + \frac{\theta l_2 - \gamma l_1}{m} r \min\left\{1, \frac{\alpha}{\theta}\right\} = H(t).
\end{aligned}
\tag{3.5}
$$

If $\gamma = 0$, then

$$
H(t) = -\frac{rk\theta\beta}{m}t + \frac{rk\theta\beta}{m} + \frac{\theta l_2}{m} r \min\left\{1, \frac{\alpha}{\theta}\right\}.
$$

H is decreasing for $t \in [0,1]$. The minimum occurs at $t = 1$. When $\gamma \neq 0$, H is a quadratic function with the critical point $\frac{\gamma+\beta}{\gamma} > 1$. For $t \in [0,1]$, the minimum also occurs at $t = 1$. Hence

$$
(\widetilde{F}u)'(t) \geq H(1) = \frac{-RK\gamma(\alpha+\theta) + (\theta l_2 - \gamma l_1) r \; min\{1, \frac{\alpha}{\theta}\}}{m}.
\tag{3.6}
$$

On the other hand,

$$
\begin{aligned}
c\|\widetilde{F}u\|_\infty \leq & \; c\int_0^1 \frac{(\alpha+\theta)(\gamma+\beta)}{m} \widetilde{f}(s,u(s),u'(s))ds + c\frac{\gamma+\beta}{m}\int_0^1 g_1(s)u(s)ds \\
& + c\frac{\alpha+\theta}{m}\int_0^1 g_2(s)u(s)ds \\
\leq & \; c\left[RK\frac{(\alpha+\theta)(\gamma+\beta)}{m} + \frac{(\gamma+\beta)R}{m}l_1 + \frac{(\alpha+\theta)R}{m}l_2\right] \\
= & \; \frac{-RK\gamma(\theta+\alpha) + (\theta l_2 - \gamma l_1)\min\left\{1, \frac{\alpha}{\theta}\right\} r}{m}.
\end{aligned}
\tag{3.7}
$$

From (3.6) and (3.7), we have

$$
(\widetilde{F}u)'(t) \geq c\|\widetilde{F}u\|_\infty \geq 0, \quad t \in [0,1].
\tag{3.8}
$$

The non-negative property of $(\widetilde{F}u)'$ ensures that

$$
\begin{aligned}
\frac{\alpha}{\theta}\|(\widetilde{F}u)'\|_\infty = & \frac{\alpha}{\theta}\max_{t\in[0,1]}(\widetilde{F}u)'(t) \\
\leq & \int_0^1 \frac{\alpha(\gamma+\beta-\gamma s)}{m} \widetilde{f}(s,u(s),u'(s))ds - \frac{\alpha\gamma}{\theta m}\int_0^1 g_1(s)u(s)ds + \frac{\alpha}{m}\int_0^1 g_2(s)u(s)ds \\
\leq & \int_0^1 \frac{\alpha(\gamma+\beta-\gamma s)}{m} \widetilde{f}(s,u(s),u'(s))ds + \frac{\gamma+\beta}{m}\int_0^1 g_1(s)u(s)ds + \frac{\alpha}{m}\int_0^1 g_2(s)u(s)ds \\
= & \widetilde{F}u(0).
\end{aligned}
\tag{3.9}
$$

Combining (3.4), (3.8) and (3.9), we obtain that $\widetilde{F}$ maps $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ to P.

Next, for $u \in P \cap \partial\Omega_2$, $\|u\| = R$,

$$Rc_0 \min\left\{1, \frac{\alpha}{\theta}\right\} \le u(t) \le R \quad \text{and} \quad Rc \min\left\{1, \frac{\alpha}{\theta}\right\} \le u'(t) \le R.$$

$$\begin{aligned}
\|\widetilde{F}u(t)\|_\infty &= \widetilde{F}u(1) \\
&= \int_0^1 G(1,s)\widetilde{f}(s,u(s),u'(s))ds + \frac{\beta}{m}\int_0^1 g_1(s)u(s)ds \\
&\quad + \frac{\alpha+\theta}{m}\int_0^1 g_2(s)u(s)ds \\
&\le KR\int_0^1 G(1,s)ds + \frac{\beta R}{m}l_1 + \frac{(\alpha+\theta)R}{m}l_2 \\
&= KR\left(\frac{\alpha\beta}{m} + \frac{\theta\beta}{2m}\right) + \frac{\beta R l_1 + (\alpha+\theta)Rl_2}{m},
\end{aligned}$$

and

$$\begin{aligned}
\|(\widetilde{F}u)'(t)\|_\infty &\le \int_0^1 \frac{\theta(\gamma+\beta-\gamma s)}{m}\widetilde{f}(s,u(s),u'(s))ds - \frac{\gamma}{m}\int_0^1 g_1(s)u(s)ds \\
&\quad + \frac{\theta}{m}\int_0^1 g_2(s)u(s)ds \\
&\le KR\int_0^1 \frac{\alpha(\gamma+\beta-\gamma s)}{m}ds + \frac{\theta l_2}{m}R \\
&= \frac{KR(\frac{\theta\gamma}{2} + \theta\beta) + \theta l_2}{m}.
\end{aligned}$$

Thus, (H3) implies

$$\begin{aligned}
\|(\widetilde{F}u)(t)\| &= \max\left\{\|Fu\|_\infty, \|(\widetilde{F}u)'\|_\infty\right\} \\
&\le \frac{K(\frac{\theta\gamma}{2} + \theta\beta + \alpha\beta) + \beta l_1 + (\alpha+\theta)l_2}{m}R \\
&\le R = \|u\|.
\end{aligned}$$

For $u \in P \cap \partial\Omega_1$, $\|u\| = r$,

$$rc_0 \min\left\{1, \frac{\alpha}{\theta}\right\} \le u(t) \le r \quad \text{and} \quad rc \min\left\{1, \frac{\alpha}{\theta}\right\} \le u'(t) \le r.$$

From (H4), we obtain

$$\begin{aligned}
\|\widetilde{F}u\| &\ge \|\widetilde{F}u\|_\infty \\
&\ge \int_0^1 G(1,s)\widetilde{f}(s,u(s),u'(s))ds + \frac{\beta}{m}\int_0^1 g_1(s)u(s)ds + \frac{\alpha+\theta}{m}\int_0^1 g_2(s)u(s)ds \\
&\ge kr\int_0^1 G(1,s)ds + \frac{\beta rc_0 \min\left\{1,\frac{\alpha}{\theta}\right\} l_1}{m} + \frac{(\alpha+\theta)rc_0\min\left\{1,\frac{\alpha}{\theta}\right\} l_2}{m} \\
&\ge kr\left(\frac{\alpha\beta}{m} + \frac{\theta\beta}{2m}\right) + \frac{\beta rc_0\min\left\{1,\frac{\alpha}{\theta}\right\} l_1 + (\alpha+\theta)rc_0\min\left\{1,\frac{\alpha}{\theta}\right\} l_2}{m} \\
&= \frac{k(\alpha\beta + \frac{\theta\beta}{2}) + (\beta\min\left\{1,\frac{\alpha}{\theta}\right\} l_1 + (\alpha+\theta)\min\left\{1,\frac{\alpha}{\theta}\right\} l_2)c_0}{m}r \\
&\ge r = \|u\|.
\end{aligned}$$

It can be shown that $\widetilde{F}$ is compact on $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ following the standard arguments. Theorem 1.1 ensures that $\widetilde{F}$ has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$. □

4 Examples

We construct two examples to illustrate applications of Theorem 3.1. Example 4.1 represents a group of BVPs satisfying the conditions of Theorem 3.1 but results of [17] can not be applied. Example 4.3 shows that it is possible for BVPs satisfying all conditions of Theorem 3.1 to have multiple solutions including negative solutions.

Example 4.1. Consider the boundary value problem

$$\begin{cases} u''(t) + (0.5t+1)\left(\frac{0.01}{u(t)} + \frac{0.0001}{u'(t)}\right) = 0, \quad t \in [0,1], \\ u(0) - u'(0) = \int_0^1 g_1(s)u(s)ds, \\ 0.01u(1) + 4u'(1) = \int_0^1 g_2(s)u(s)ds, \end{cases} \tag{4.1}$$

where the parameters $\alpha = \theta = 1,\ \beta = 4,\ \gamma = 0.01$. Let g_1, g_2 be selected such that $l_1 = \frac{1}{16}$ and $l_2 = 1$. We can find that $c_0 = 0.5$, $m = 4.02$. For example, for $g_1(s) = \frac{s}{8}$, $g_2(s) = 2s$, we can verify that (H1) is true. Let $R = 2$ and $r = 0.1$. Condition (H3) is satisfied if

$$\frac{f(t,u,v)}{2} \le K < 0.22 < \frac{m - \beta l_1 - (\alpha+\theta) l_2}{\frac{\theta\gamma}{2} + \theta\beta + \alpha\beta} < 0.23,$$

for $(t,u,v) \in [0,1] \times [1,2] \times [2c,2]$, where $c = \frac{-0.04K+0.09375}{16.04K+4.50125}$. Since c is decreasing with respect to K, by calculation, we have $c \ge 0.011$. From

$$\frac{1}{2}(0.5t+1)\left(\frac{0.01}{u} + \frac{0.0001}{v}\right) < 0.01, \quad \text{for } (t,u,v) \in [0,1] \times [1,2] \times [2c,2],$$

we know that (H3) and (H5) are valid for $K \in [0.01, 0.22]$.

To find k satisfying condition (H4), we calculate that

$$0.5 > \frac{m - c_0\left(\beta \min\left\{1, \frac{\alpha}{\theta}\right\} l_1 + (\alpha+\theta)\min\left\{1, \frac{\alpha}{\theta}\right\} l_2\right)}{\frac{\theta\beta}{2} + \alpha\beta} > 0.49.$$

As $\frac{f(t,u,v)}{r} \ge 1.01$ for $(t,u,v) \in [0,1] \times [0.05, 0.1] \times [0.1c, 0.1]$, (H4) is satisfied for $k \in [0.50, 1.01]$. By Theorem (3.1), BVP (3.8) has a positive solution $u \in C^1[0,1]$ such that $0.05 \le \rho_1 \le u(t) \le 2$ and $0.0011 \le \rho_2 \le u'(t) \le 2$.

Remark 4.2. More generally, for all $\alpha = \theta, \beta = 4, \gamma = 0.01, l_2 = 1$, the calculation of Example 4.1 works as long as l_1 is small enough. The extreme case is $g_1(s) = 0$. Then the first boundary condition is reduced to $u(0) - u'(0) = 0$. We can verify that $c_0 = 0.5, m = 4.02\alpha$. Select the same values of R and r as that of Example 4.1, we can find the intervals for $K \in [0.01, 0.25]$ and $k \in [0.51, 1.01]$. The solution and its derivative are still in the same range as obtained in Example 4.1.

Example 4.3. The following problem is in the form of BVP (1.1):

$$\begin{cases} u''(t) + \frac{\ln(t+2)}{10^4(t+2)^2 u(t)} + \frac{1}{10^4(t+2)^3 u'(t)} + \frac{0.98u'(t)}{t+2} = 0, \\ u(0) - u'(0) = \xi_1 \int_0^1 su(s)ds, \\ 10^{-4}u(1) + u'(1) = \xi_2 \int_0^1 su(s)ds, \end{cases} \tag{4.2}$$

where $\theta = \alpha = 1$, $\gamma = 10^{-4}$, $\beta = 1$, $g_1(s) = \xi_1 s$ and $g_2(s) = \xi_2 s$. Select $\xi_1 = \frac{4\ln 2 - 2}{3 - \ln 729 + \ln 256}$, $\xi_2 = \frac{3\ln 3 + 10^4}{7500(3 - \ln 729 + \ln 256)}$, then $0.1979 < l_1 < 0.1980$, $0.3413 < l_2 < 0.3414$. It is easy to find that $c_0 = \frac{1}{2}$, and $m = 1.0002$. Let $r = 0.02$, $R = 1$, we can verify that all conditions (H1)-(H5) are satisfied. In fact, equation (4.2) is exact, we can find that $u_1(t) = 0.1\ln(t+2)$ and $u_2(t) = -0.1\ln(t+2)$ are two solutions of problem (4.2). This shows the existence of multiple and negative solutions.

Different from Example 4.3, Example 4.1 cannot be solved analytically. In order to validate this result of Example 4.1, we use the sinc-collocation numerical method based on the derivative interpolation to obtain a numerical solution of BVP (4.1). The sinc-collocation method is a highly efficient numerical technique with exponential rate of convergence. The details of the approach can be found in [1]. The numerical algorithm is coded in Python. The graphs of the obtained solutions u and u' for both cases of $g_1(s) = 0$ and $g_1(s) = \frac{s}{8}$ are depicted in Figures 4.1 and 4.2 respectively. Clearly they all satisfy the bounds obtained from Example 4.1.

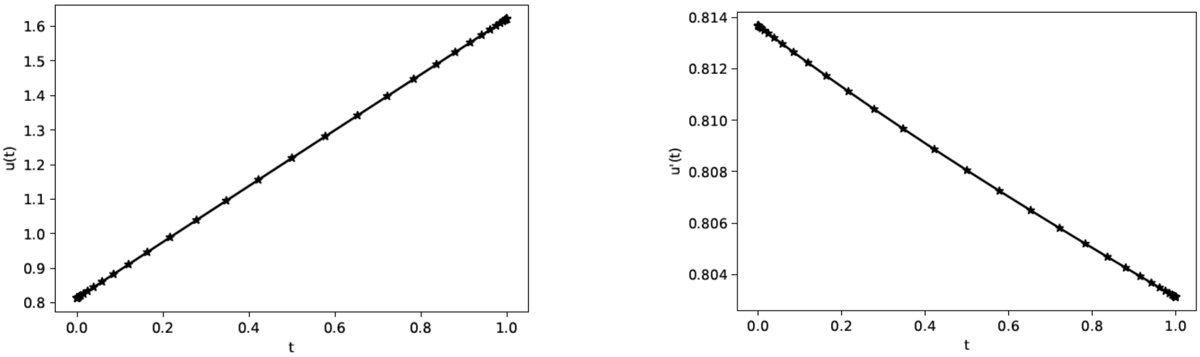

Figure 4.1: Numerical solution of BVP (4.1) ($g_1 = 0$, $g_2 = 2s$)

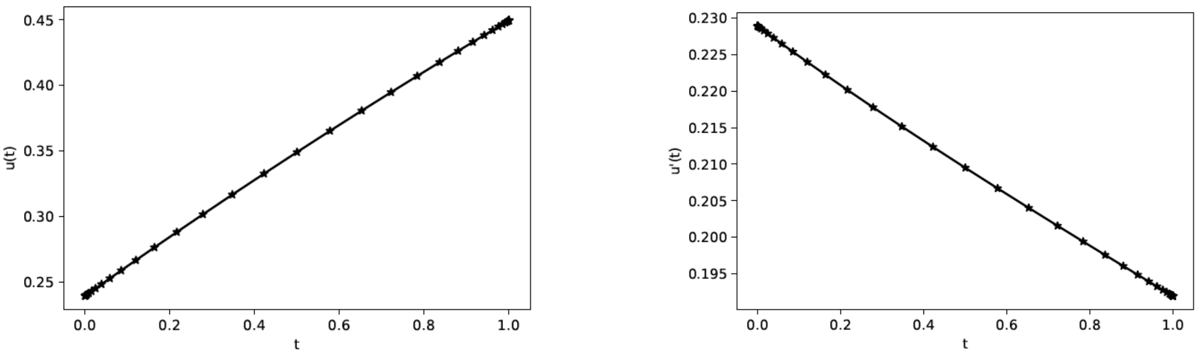

Figure 4.2: Numerical solution of BVP (4.1) ($g_1 = \frac{s}{8}$, $g_2 = 2s$)

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