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Structurally unstable quadratic vector fields of codimension two: families possessing a finite saddle-node and an infinite saddle-node

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Abstract. In 1998, Artés, Kooij and Llibre proved that there exist 44 structurally stable topologically distinct phase portraits modulo limit cycles, and in 2018 Artés, Llibre and Rezende showed the existence of at least 204 (at most 211) structurally unstable topologically distinct codimension-one phase portraits, modulo limit cycles. Artés, Oliveira and Rezende (2020) started the study of the codimension-two systems by the set (AA), of all quadratic systems possessing either a triple saddle, or a triple node, or a cusp point, or two saddle-nodes. They got 34 topologically distinct phase portraits modulo limit cycles. Here we consider the sets (AB) and (AC). The set (AB) contains all quadratic systems possessing a finite saddle-node and an infinite saddle-node obtained by the coalescence of an infinite saddle with an infinite node. The set (AC) describes all quadratic systems possessing a finite saddle-node and an infinite saddle-node, obtained by the coalescence of a finite saddle (respectively, finite node) with an infinite node (respectively, infinite saddle). We obtain all the potential topological phase portraits of these sets and we prove their realization. From the set (AB) we got 71 topologically distinct phase portraits modulo limit cycles and from the set (AC) we got 40 ones.

Keywords: quadratic differential system, structural stability, codimension two, phase portrait, saddle-node.

2020 Mathematics Subject Classification: Primary: 34A34, 34C23, 34C40. Secondary: 58-02.

1 Introduction and statement of the main results

Mathematicians are fascinated in closing problems. Having a question solved or even sign with a "q.e.d" a question asked in the past is a pleasure which is directly proportional to the time elapsed between the formulation of the question and the moment of the answer.

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The advent of the differential calculus opened the possibility of solving many questions that medieval mathematicians asked, but at the same time it opened the possibility of formulating many new other questions. The search for primitive functions that could not be expressed algebraically or with a finite number of analytic terms complicated the future research lines, and even new areas of Mathematics were created to give answers to these questions. And beside the problem of finding a primitive to a differential equation in a single dimension, if we add the possibility of more dimensions, the problem becomes much more difficult.

Therefore, it took almost 200 years between the appearance of the first system of linear differential equations and its complete resolution by Laplace in 1812. After the resolution of linear differential systems, for any dimension, it seemed natural to address the classification of quadratic differential systems. However, it was found that the problem would not have an easy and fast solution. Unlike the linear systems that can be solved analytically, quadratic systems (or higher degree systems) do not generically admit a solution of that kind, calculable in a finite number of terms.

Therefore, for the resolution of non-linear differential systems, another strategy was chosen and it allowed the creation of a new area of knowledge in Mathematics: the Qualitative Theory of Ordinary Differential Equations [27]. Since we are not able to give a concrete mathematical expression to the solution of a system of differential equations, this theory intends to express by means of a complete and precise drawing the behavior of any particle located in a vector field governed by such a differential equation, i.e. its phase portrait.

Even with all the reductions made to the problem until now, there are still difficulties. The most expressive difficulty is that the phase portraits of differential systems may have invariant sets as limit cycles and graphics. A linear system cannot generate limit cycles; at most they can present a completely circular phase portrait where all the orbits are periodic. But a differential system in the plane, polynomial or not, and starting with the quadratic ones, may present several limit cycles. It is natural to find an infinite number of these cycles in non-polynomial problems, but the intuition seems to indicate that a polynomial system should not have an infinite number of limit cycles in a similar way as it cannot have an infinite number of isolated singular points. And because the number of singular points is linked to the degree of the polynomial system, it also seems logical to think that the number of limit cycles could also have a similar link, either directly as the number of singular points, or even in an indirect way from the number the parameters of such systems.

In 1900, David Hilbert [21] proposed a set of 23 problems to be solved in the 20th century, and among them, the second part of his well-known 16th problem asks for the maximum number of limit cycles that a polynomial differential system in the plane with degree n may have. More than one hundred years after, we do not have an uniform upper bound for this generic problem, only for specific families of such a system.

During discussions, in 1966 Coppel [16] expressed the belief that we could obtain the classification of phase portraits of quadratic systems by purely algebraic means. That is, by means of algebraic equalities and inequalities, it should be possible to determine the phase portrait of a quadratic system. This claim was not easy to refute at that time, since the isolated finite singular points of a quadratic system can be found by means of the resultant that is of fourth degree, and its solutions can be calculated algebraically, like those of infinity. Moreover, at that time it was known how to generate limit cycles by a Hopf bifurcation, whose conditions are also determined algebraically.

On the other hand, in 1991, Dumortier and Fiddelears [17] showed that, starting with the quadratic systems (and following all the higher-degree systems), there exist geometric and topological phenomena in phase portraits of such a system whose determination cannot be fixed by means of algebraic expressions. More specifically, most part of the connections among separatrices and the occurrence of double or semi-stable limit cycles cannot be algebraically determined.

Therefore, the complete classification of quadratic systems is a very difficult task at the moment and it depends on the solution of the second part of Hilbert's 16th problem, even at least partially for the quadratic case.

Even so, a lot of problems have been appearing related to quadratic systems to which it has been possible to give an answer. In fact, there are more than one thousand articles published that are directly related to quadratic systems. John Reyn, from Delft University (Netherlands), prepared a bibliography that was published several times until his retirement (see [28,30–33]). It is worth mentioning that in the last two decades many other articles related to quadratic systems have appeared, so that the number of one thousand published papers on the subject may have been widely exceeded.

Many of the questions proposed and the problems solved have dealt with subclassifications of quadratic systems, that is, classifications of systems that shared some characteristic in common. For instance, we have systems with a center [26,35,36,38], with a weak focus of third order [3,24], with a nilpotent singularity [22], without real singular points [20], with two invariant lines [28] and so on, up to a thousand articles. In some of them complete answers could be given, including the problem of limit cycles (the existence and the number of limit cycles), but in other cases, the classification was done modulo limit cycles, that is, all the possible phase portraits without taking into account the presence and number of cycles. Since in quadratic systems a limit cycle can only surround a single finite singular point, which must necessarily be a focus [16], then it is enough to identify the outermost limit cycle of a nesting of cycles with a point, and interpret the stability of that point as the outer stability of this cycle, and study everything that can happen to the phase portrait in the rest of the space.

Within the families of quadratic systems that were studied in the 20th century, we would highlight the study of the structurally stable quadratic systems, modulo limit cycles. That is, the goal was to determine how many and which phase portraits of a quadratic system cannot be modified by small perturbations in their coefficients. To obtain a structurally stable system modulo limits cycles we need a few conditions: we do not allow the existence of multiple singular points and the existence of connections of separatrices. Centers, weak foci, semi-stable cycles, and all other unstable elements belong to the quotient modulo limit cycles. This systematic analysis [2] showed that the structurally stable quadratic systems have a total of 44 topologically distinct phase portraits.

From this scenario we observe that if we intent to work with classification of phase portraits of quadratic systems before the solution of the second part of Hilbert's 16th problem, this will have to be done modulo limit cycles.

Additionally, the entire family of quadratic systems by definition depends on twelve parameters, but due to the action of the group of the real affine transformations and time rescaling, this family ultimately depends on five parameters, but this is still a large number.

There are two ways to carry out a systematic study of all the phase portraits of the quadratic systems. One of them is the one initiated by Reyn in which he began by studying the phase portraits of all the quadratic systems in which all the finite singular points have coalesced with infinite singular points [29]. Later, he studied those in which exactly three finite singular points have coalesced with points of infinity, so there remains one real finite singularity. And then he completed the study of the cases in which two finite singular points have coalesced with points of infinity, originating two real points, or one double point, or two complex points. His work on finite multiplicity three was incomplete and the one on finite multiplicity four was inaccessible.

In another approach, instead of working from the highest degrees of degeneracy to the lower ones, is going to reverse direction. We already know that the structurally stable quadratic systems produce 44 topologically distinct phase portrait, as already mentioned before. In [6] the authors classified the structurally unstable quadratic systems of codimension one modulo limit cycles, which are systems having one and only one of the simplest structurally unstable objects: a saddle-node of multiplicity two (finite or infinite), a separatrix from one saddle point to another, or a separatrix forming a loop for a saddle point with its divergence nonzero. All the phase portraits of codimension one are split into four sets according to the possession of a structurally unstable element: (A) possessing a finite semi-elemental saddle-node, (B) possessing an infinite semi-elemental saddle-node $\overline{\binom{0}{2}}SN$, (C) possessing an infinite semi-elemental saddle-node $\overline{\binom{1}{1}}SN$, and (D) possessing a separatrix connection. This last set is split into five subsets according to the type of the connection: (a) finite-finite (heteroclinic orbit), (b) loop (homoclinic orbit), (c) finite-infinite, (d) infinite-infinite between symmetric points, and (e) infinite-infinite between adjacent points. The study of the codimension-one systems was done in approximately 20 years and finally it was obtained at least 204 (and at most 211) topologically distinct phase portraits of codimension one modulo limit cycles.

The next step is to study the structurally unstable quadratic systems of codimension two (see [12]), modulo limit cycles. Up to now, we have mentioned many times the word "codimension" and this is a clear concept in Geometry. However, in this classification we want to obtain topologically distinct phase portraits, and we want to group them according to their level of degeneracy. So, what was clear for structurally stable phase portraits and for codimension-one phase portraits (modulo limit cycles) may become a little weird if we continue in this same way, so we must give a definition of codimension adapted to this specific set that we want to classify.

Definition 1.1. We say that a phase portrait of a quadratic vector field is structurally stable if any sufficiently small perturbation in the parameter space leaves the phase portrait topologically equivalent to the previous one.

Definition 1.2. We say that a phase portrait of a quadratic vector field is structurally unstable of codimension $k \in \mathbb{N}$ if any sufficiently small perturbation in the parameter space either leaves the phase portrait topologically equivalent the previous one or it moves it to a lower codimension one, and there is at least one perturbation that moves it to the codimension $k-1$.

Remark 1.3.

1. When applying these definitions, modulo limit cycles, to phase portraits with centers, it would say that some phase portraits with centers would be of codimension as low as two, while geometrically they occupy a much smaller region in $\mathbb{R}^{12}$. So, the best way to avoid inconsistencies in the definitions is to tear apart the phase portraits with centers, that we know they are in number 31 (see [36]), and just work with systems without centers.

2. Starting in cubic systems, the definition of topologically equivalence, modulo limit cycles, becomes more complicated since we can have limit cycles having only one singu-

larity in its interior or more than one. So we cannot collapse the limit cycle because its interior is also relevant for the phase portrait.

3. Moreover, our definition of codimension needs also more precision starting with cubic systems due to new phenomena that may happen there.

Let $P_n(\mathbb{R}^2)$ be the set of all vector fields in $\mathbb{R}^2$ of the form $X(x,y) = (P(x,y), Q(x,y))$, with P and Q polynomials in the variables x and y of degree at most n (with $n \in \mathbb{N}$). In this set we consider the *coefficient topology* by identifying each vector field $X \in P_n(\mathbb{R}^2)$ with a point of $\mathbb{R}^{(n+1)(n+2)}$ (see more details in [6]). According to the previous definition concerning codimension two, and also according to the previously known results of codimension one, we have the result.

Theorem 1.4. *A polynomial vector field in $P_2(\mathbb{R}^2)$ is structurally unstable of codimension two modulo limit cycles if and only if all its objects are stable except for the break of exactly two stable objects. In other words, we allow the presence of two unstable objects of codimension one or one of codimension two.*

In what follows, instead of talking about codimension one modulo limit cycles, we will simply say *codimension one**. Analogously we will simply say *codimension two** instead of talking about codimension two modulo limit cycles.

Combining the classes of *codimension one** quadratic vector fields one to each other, we obtain 10 new classes, where one of them is split into 15 subsets, according to Tables 1.1 and 1.2.

	(A)	(B)	(C)	(D)
(A)	(AA)	-	-	-
(B)	(AB)	(BB)	-	-
(C)	(AC)	(BC)	(CC)	-
(D)	(AD) (5 cases)	(BD) (5 cases)	(CD) (5 cases)	see Table 1.2

Table 1.1: Sets of structurally unstable quadratic vector fields of codimension two considered from combinations of the classes of *codimension one**: (A), (B), (C), and (D) (which in turn is split into (a), (b), (c), (d), and (e)).

	(a)	(b)	(c)	(d)	(e)
(a)	(aa)				
(b)	(ab)	(bb)			
(c)	(ac)	(bc)	(cc)		
(d)	(ad)	(bd)	(cd)	(dd)	
(e)	(ae)	(be)	(ce)	(de)	(ee)

Table 1.2: Sets of structurally unstable quadratic vector fields of *codimension two** in the class (DD) (see Table 1.1).

Geometrically, the *codimension two** classes can be described as follows. Let X be a *codimension one** quadratic vector field. We have the following classes:

(AA) When X already has a finite saddle-node and either a finite saddle (respectively a finite node) of X coalesces with the finite saddle-node, giving birth to a semi-elemental triple saddle: $\overline{s}_{(3)}$ (respectively a triple node: $\overline{n}_{(3)}$), or when both separatrices of the saddle-node limiting its parabolic sector coalesce, giving birth to a cusp of multiplicity two: $\widehat{cp}_{(2)}$, or when another finite saddle-node is formed, having then two finite saddle-nodes: $\overline{sn}_{(2)}+\overline{sn}_{(2)}$. Since the phase portraits with $\overline{s}_{(3)}$ and with $\overline{n}_{(3)}$ would be topologically equivalent to structurally stable phase portraits and we are mainly interested in new phase portraits, we will skip them in this classification. Anyway, we may find them in the papers [11] and [13].

(AB) When X already has a finite saddle-node and an infinite saddle, and an infinite node of X coalesce with a finite saddle-node: $\overline{sn}_{(2)}+\overline{\binom{0}{2}}SN$.

(AC) When X already has a finite saddle-node and a finite saddle (respectively node), and an infinite node (respectively saddle) of X coalesce: $\overline{sn}_{(2)}+\overline{\binom{1}{1}}SN$.

(AD) When X has already a finite saddle-node and a separatrix connection is formed, considering all five types of class (D).

(BB) When an infinite saddle (respectively an infinite node) of X coalesces with an existing infinite saddle-node $\overline{\binom{0}{2}}SN$ of X, leading to a triple saddle: $\overline{\binom{0}{3}}S$ (respectively a triple node: $\overline{\binom{0}{3}}N$). This case is irrelevant to the production of new phase portraits since all the possible phase portraits that may produce are topologically equivalent to an structurally stable one.

(BC) When a finite antisaddle (respectively finite saddle) of X coalesces with an existing infinite saddle-node $\overline{\binom{0}{2}}SN$ of X, leading to a nilpotent elliptic saddle $\widehat{\binom{1}{2}}E-H$ (respectively nilpotent saddle $\widehat{\binom{1}{2}}HHH-H$). Or it may also happen that a finite saddle (respectively node) coalesces with an elemental node (respectively saddle) in a phase portrait having already an $\overline{\binom{0}{2}}SN$, having then in total $\overline{\binom{1}{1}}SN+\overline{\binom{0}{2}}SN$.

(BD) When we have an infinite saddle-node $\overline{\binom{0}{2}}SN$ plus a separatrix connection, considering all five types of class (D).

(CC) This case has two possibilities:

i) a finite saddle (respectively finite node) of X coalesces with an existing infinite saddle-node $\overline{\binom{1}{1}}SN$, leading to an semi-elemental triple saddle $\overline{\binom{2}{1}}S$ (respectively an semi-elemental triple node $\overline{\binom{2}{1}}N$),

ii) a finite saddle (respectively node) and an infinite node (respectively saddle) of X coalesce plus an another existing infinite saddle-node $\overline{\binom{1}{1}}SN$, leading to two infinite saddle-nodes $\overline{\binom{1}{1}}SN+\overline{\binom{1}{1}}SN$.

The first case is irrelevant to the production of new phase portraits since all the possible phase portraits that may produce are topologically equivalent to an structurally stable one.

(CD) When we have an infinite saddle-node $\overline{\binom{1}{1}}SN$ plus a saddle to saddle connection, considering all five types of class (D).

(DD) When we have two saddle to saddle connections, which are grouped as follows:

(aa) two finite-finite heteroclinic connections;

(ab) a finite-finite heteroclinic connection and a loop;

(ac) a finite-finite heteroclinic connection and a finite-infinite connection;

(ad) a finite-finite heteroclinic connection and an infinite-infinite connection between symmetric points;

(ae) a finite-finite heteroclinic connection and an infinite-infinite connection between adjacent points;

(bb) two loops;

(bc) a loop and a finite-infinite connection;

(bd) a loop and an infinite-infinite connection between symmetric points;

(be) a loop and an infinite-infinite connection between adjacent points;

(cc) two finite-infinite connections;

(cd) a finite-infinite connection and an infinite-infinite connection between symmetric points;

(ce) a finite-infinite connection and an infinite-infinite connection between adjacent points;

(dd) two infinite-infinite connections between symmetric points;

(de) an infinite-infinite connection between symmetric points and an infinite-infinite connection between adjacent points;

(ee) two infinite-infinite connections between adjacent points.

Some of these cases have also been proved to be empty in an on course paper [8].

In [12] the authors begin the study of codimension-two quadratic systems. The approach is the same used in the previous two works [2,6]. One must start by looking for all the potential topological phase portraits (i.e. phase portraits that can be drawn on paper) of codimension two modulo limit cycles, and then try to realize all of them (i.e. to find examples of quadratic differential systems whose phase portraits are exactly those phase portraits obtained previously) or to show that some of them are non-realizable or impossible (i.e. in case of absence of examples for the realization of a phase portrait, say Ψ, it is necessary to prove that there is no quadratic differential system whose phase portrait is topologically equivalent to Ψ).

In [12] the authors have considered the set (AA) obtained by the coalescence of two finite singular points, yielding either a triple saddle, or a triple node, or a cusp point, or two saddle-nodes. They obtained all the potential topological phase portraits modulo limit cycles of the set (AA) and proved their realization. In their study they got 34 new topologically distinct phase portraits (of codimension two) in the Poincaré disc modulo limit cycles. Moreover, they also proved the impossibility of one phase portrait among the 204 phase portraits from [6]. Therefore, in [6] they actually have at least 203 (and at most 210) topologically distinct phase portraits of codimension one modulo limit cycles. Additionally, more recent studies (in a preprint level) have shown the impossibility of another phase portrait among the 203 cited above. In that study it was also verified that, in fact, there exist at least 202 (and at most 209) topologically distinct phase portraits of codimension one modulo limit cycles.

In this paper we intend to contribute to the classification of the phase portraits of planar quadratic differential systems of codimension two, modulo limit cycles. According to what was explained before, since there are more than 10 cases of codimension two to be analyzed, it is impracticable to write a single paper with all the results. So, in [12] the authors have decided to split this study in several papers and this present article is the second one of this series. We indicate [2, 6, 12] for more details of the context of this study as well for all related definitions.

Here we present all the global phase portraits of the vector fields $X \in P_2(\mathbb{R}^2)$ belonging to sets (AB) and (AC) and we study their realization. The set (AB) contains all quadratic systems possessing a finite saddle-node $\overline{sn}_{(2)}$ and an infinite saddle-node of type $\overline{\binom{0}{2}}SN$ obtained by the coalescence of an infinite saddle with an infinite node. The set (AC) describes all quadratic systems possessing a finite saddle-node $\overline{sn}_{(2)}$ and an infinite saddle-node of type $\overline{\binom{1}{1}}SN$, obtained by the coalescence of a finite saddle (respectively, a finite node) with an infinite node (respectively, an infinite saddle). Notice that the finite singularity that coalesces with an infinite singularity cannot be the finite saddle-node since then what we would obtain at infinity would not be a saddle-node of type $\overline{\binom{1}{1}}SN$ but a multiplicity three singularity. Even this is also a *codimension two** case and somehow can be considered inside the set (AC), we have preferred to put it into the set (CC), which will be studied in a future paper.

We point out that in each picture representing a phase portrait we only draw the *skeleton of separatrices*, according to the next definition.

Definition 1.5. Let $p(X) \in P_n(\mathbb{S}^2)$ (respectively $X \in P_n(\mathbb{R}^2)$). A *separatrix* of $p(X)$ (respectively of X) is an orbit which is either a singular point (respectively a finite singular point), or a limit cycle, or a trajectory which lies in the boundary of a hyperbolic sector at a singular point (respectively a finite singular point). In [25] the author proved that the set formed by all separatrices of $p(X)$, denoted by $S(p(X))$, is closed. The open connected components of $\mathbb{S}^2 \setminus S(p(X))$ are called *canonical regions* of $p(X)$. We define a *separatrix configuration* as the union of $S(p(X))$ plus one representative solution chosen from each canonical region. Two separatrix configurations S_1 and S_2 of vector fields of $P_n(\mathbb{S}^2)$ (respectively $P_n(\mathbb{R}^2)$) are said to be *topologically equivalent* if there exists an orientation-preserving homeomorphism of $\mathbb{S}^2$ (respectively $\mathbb{R}^2$) which maps the trajectories of S_1 onto the trajectories of S_2. The *skeleton of separatrices* is defined as the union of $S(p(X))$ without the representative solution of each canonical region. Thus, a skeleton of separatrices can still produce different separatrix configurations.

Let $\sum_0^2$ denote the set of all planar structurally stable vector fields and $\sum_i^2(S)$ denote the set of all structurally unstable vector fields $X \in P_2(\mathbb{R}^2)$ of codimension i, modulo limit cycles belonging to the set S, where S is a set of vector fields with the same type of instability modulo orientation. For instance, in this paper we consider the sets $\sum_2^2(AB)$ and $\sum_2^2(AC)$, which denote, respectively, the set of all structurally unstable vector fields $X \in P_2(\mathbb{R}^2)$ of *codimension two** belonging to the sets (AB) and (AC).

The main goal of this paper is to prove the following two theorems.

Theorem 1.6. *If $X \in \sum_2^2(AB)$, then its phase portrait on the Poincaré disc is topologically equivalent modulo orientation and modulo limit cycles to one of the* 71 *phase portraits of Figures 1.1 to 1.3.*

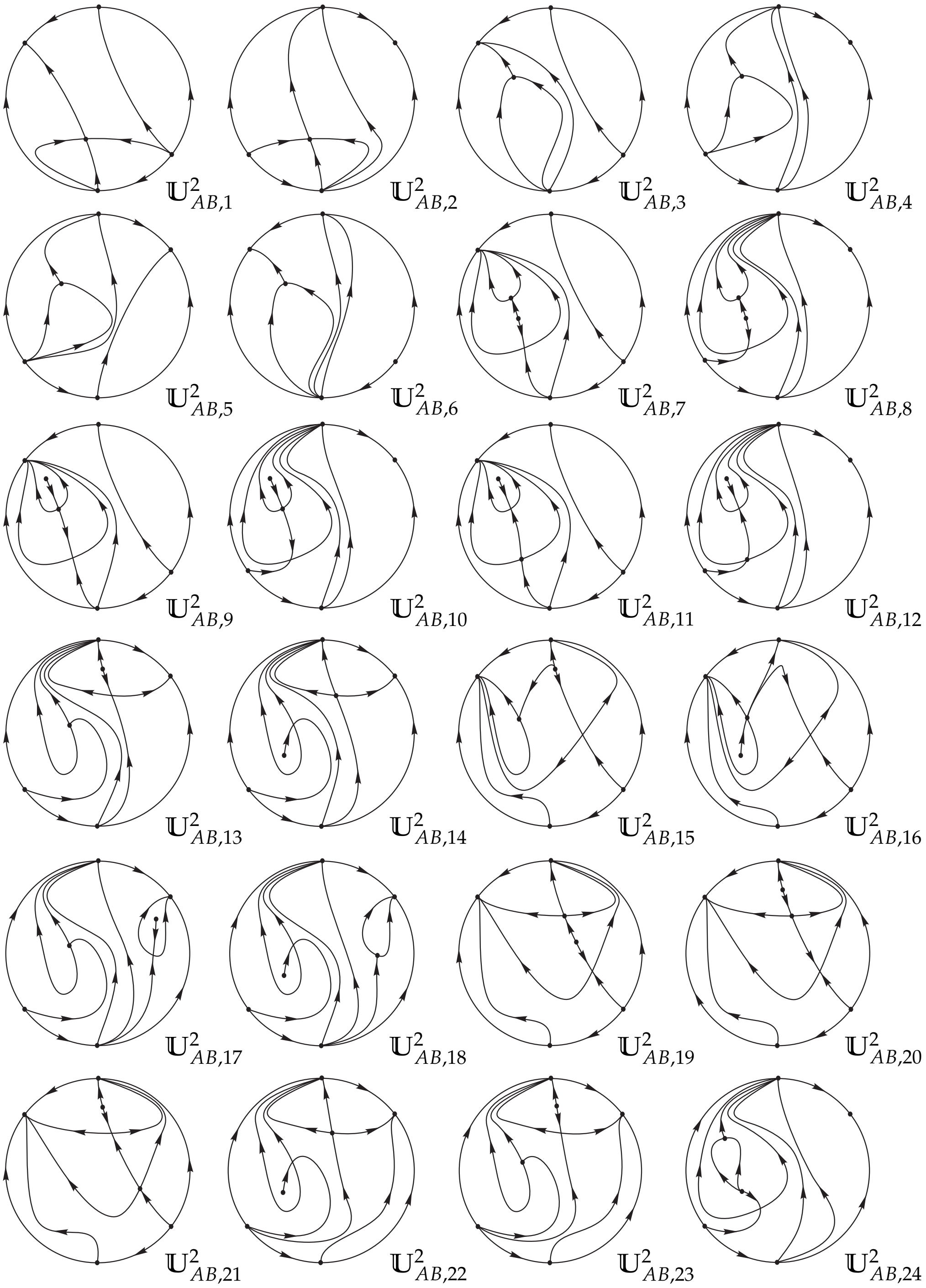


Figure 1.1: Structurally unstable quadratic phase portraits of *codimension two** of the set (AB).

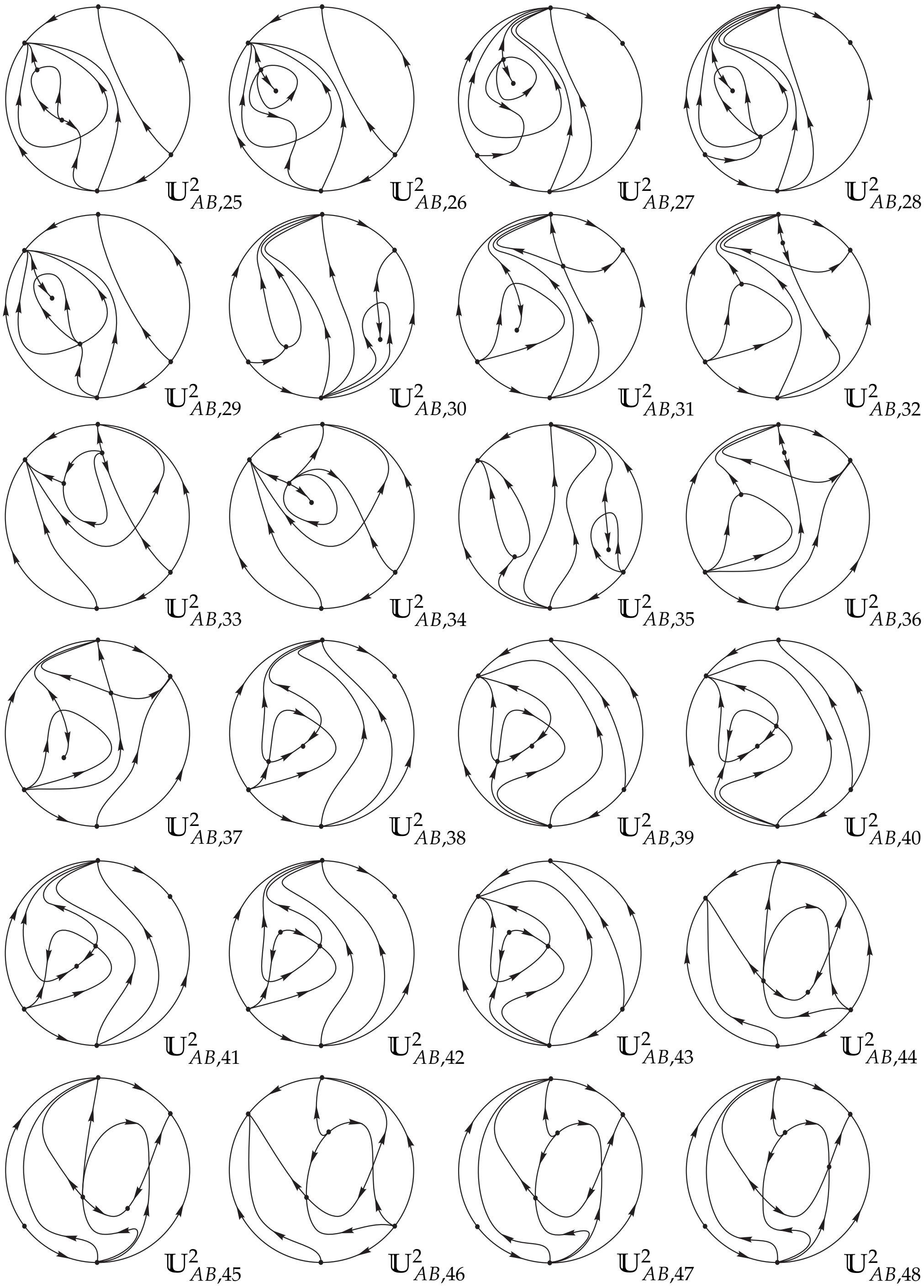


Figure 1.2: *(Cont.)* Structurally unstable quadratic phase portraits of *codimension two** of the set (AB).

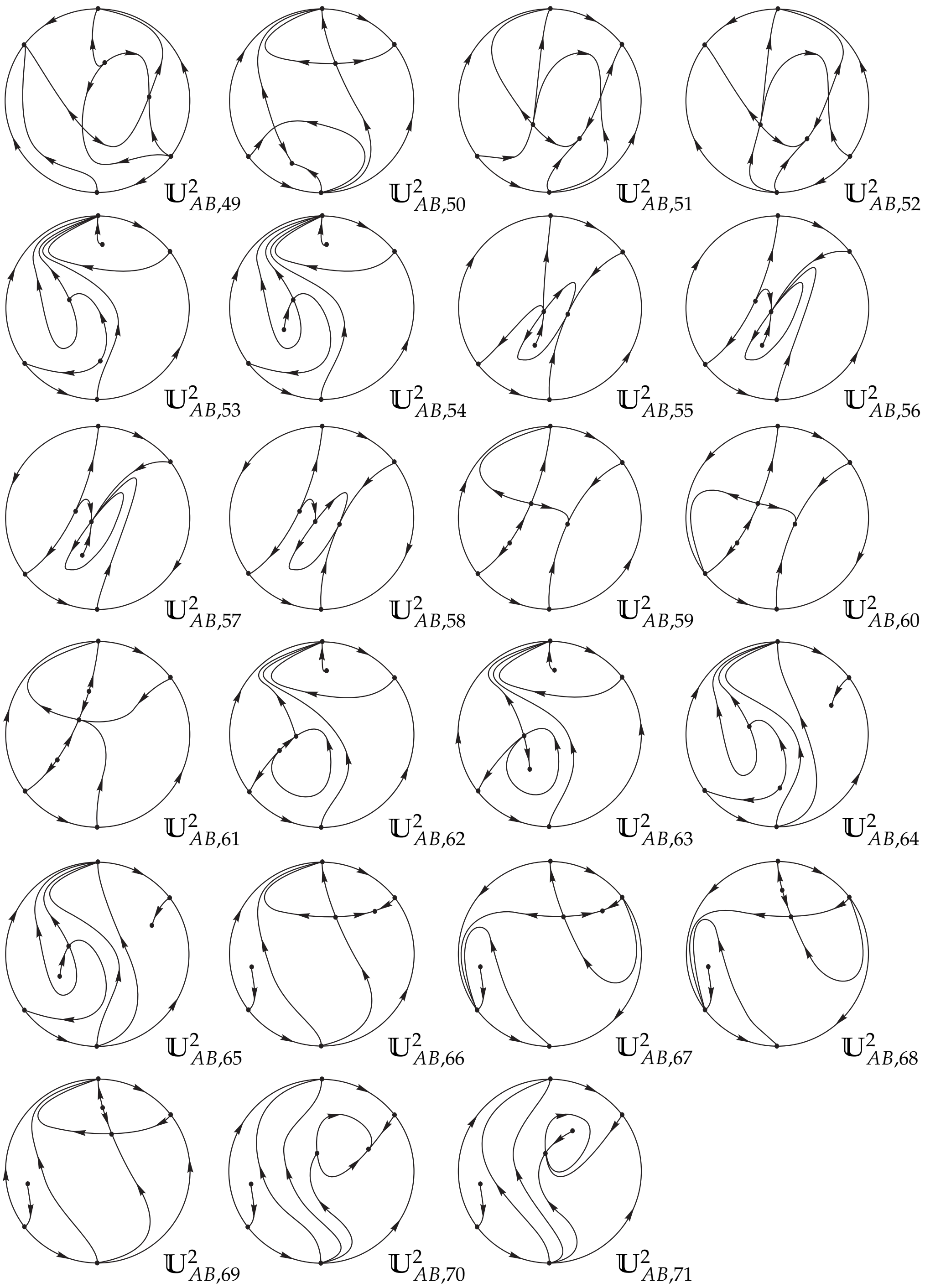


Figure 1.3: *(Cont.)* Structurally unstable quadratic phase portraits of *codimension two** of the set (AB).

Theorem 1.7. *If $X \in \sum_2^2(AC)$, then its phase portrait on the Poincaré disc is topologically equivalent modulo orientation and modulo limit cycles to one of the* 40 *phase portraits of Figures 1.4 and 1.5.*

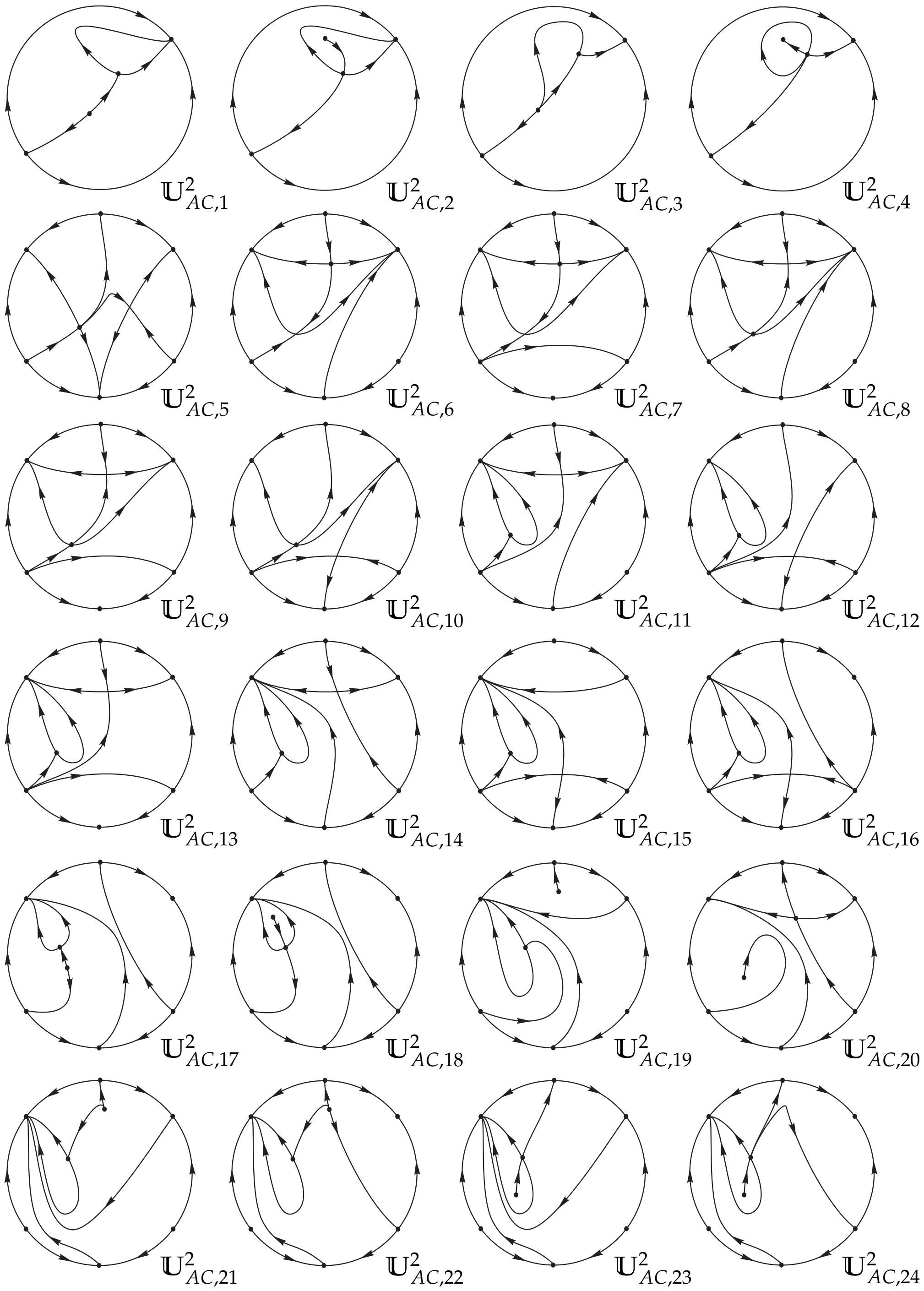


Figure 1.4: Structurally unstable quadratic phase portraits of *codimension two** of the set (AC).

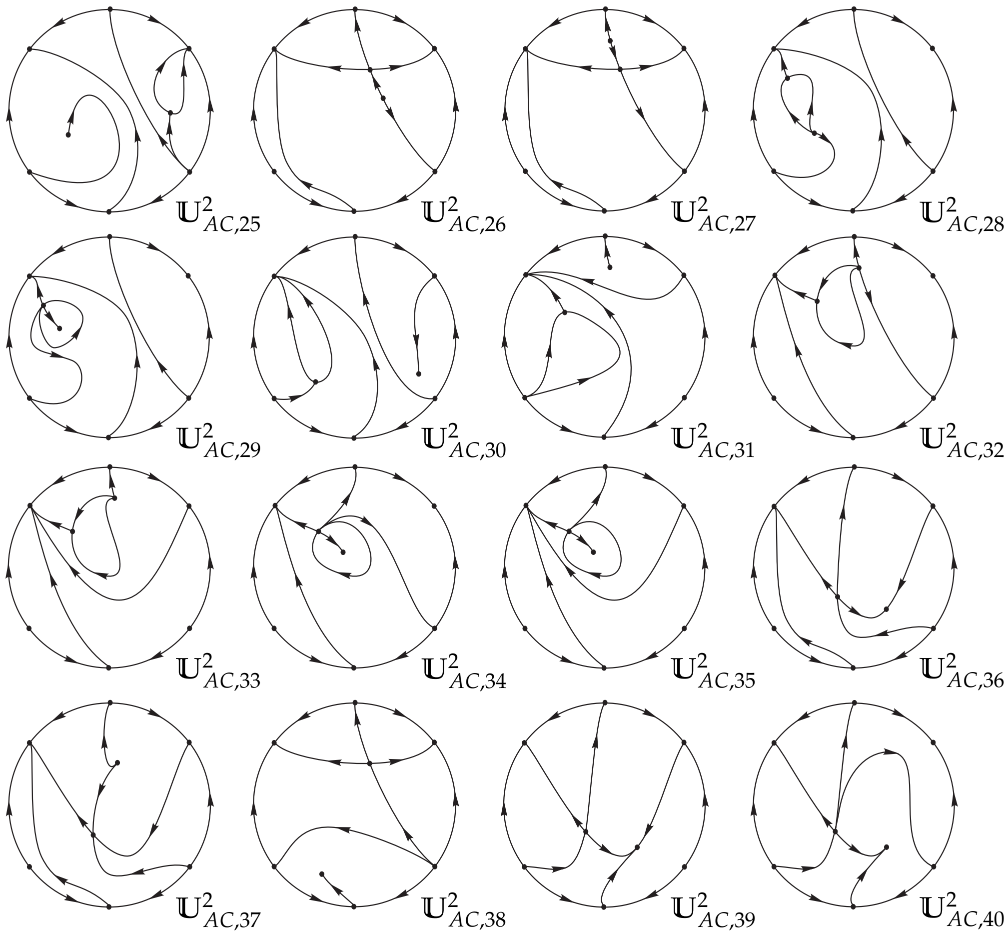


Figure 1.5: *(Cont.)* Structurally unstable quadratic phase portraits of *codimension two*** of the set (AC).

This paper is organized as follows. In Section 2 we make a brief description of phase portraits of codimensions zero and one that are needed in this paper.

In Section 3 we prove Theorem 1.6 and in Section 4 we prove Theorem 1.7. We point out that in order to verify the realization of the corresponding phase portraits we compute each one of them with the numerical program P4 [1,18].

Once again, remember that by modulo limit cycles we mean all eyes with limit cycles are assimilated with the unique singular point (a focus) within such an eye, i.e. we may say that the phase portraits are *blind* to limit cycles. Additionally, the phase portraits are also blind with respect to distinguishing if a singular point is a focus or a node, because these are not topological properties. But as the phase portraits are not blind to detecting other important features like various types of graphics, in Section 5 we discuss about the existence of graphics and also limit cycles in this study.

2 Quadratic vector fields of codimension zero and one

In this section we summarize all the needed results from the book of Artés, Llibre and Rezende [6]. The following three results are the restriction of Theorem 1.1 from book [6] to the sets (A), (B), and (C), respectively (see page 4). We denote by $\sum_1^2(A)$ (respectively, $\sum_1^2(B)$, and $\sum_1^2(C)$) the set of all structurally unstable vector fields $X \in P_2(\mathbb{R}^2)$ of *codimension one** belonging to the set (A) (respectively, (B), and (C)).

Theorem 2.1. *If $X \in \sum_1^2(A)$, then its phase portrait on the Poincaré disc is topologically equivalent modulo orientation and modulo limit cycles to one of the* 69 *phase portraits of Figures* 2.1 *to* 2.3, *and all of them are realizable.*

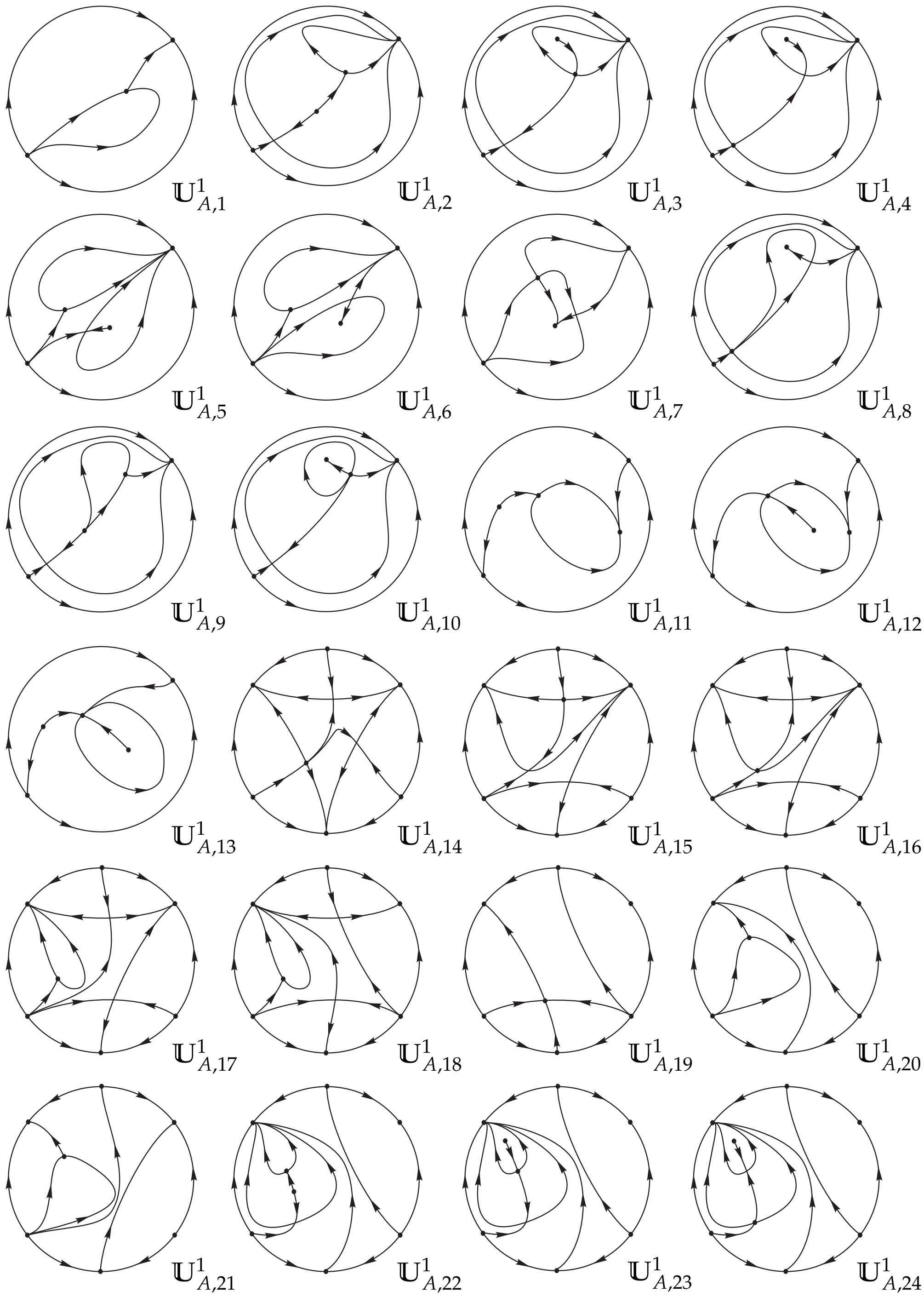


Figure 2.1: Unstable quadratic systems of *codimension one** of the set (A) (cases with a finite saddle-node $\overline{sn}_{(2)}$).

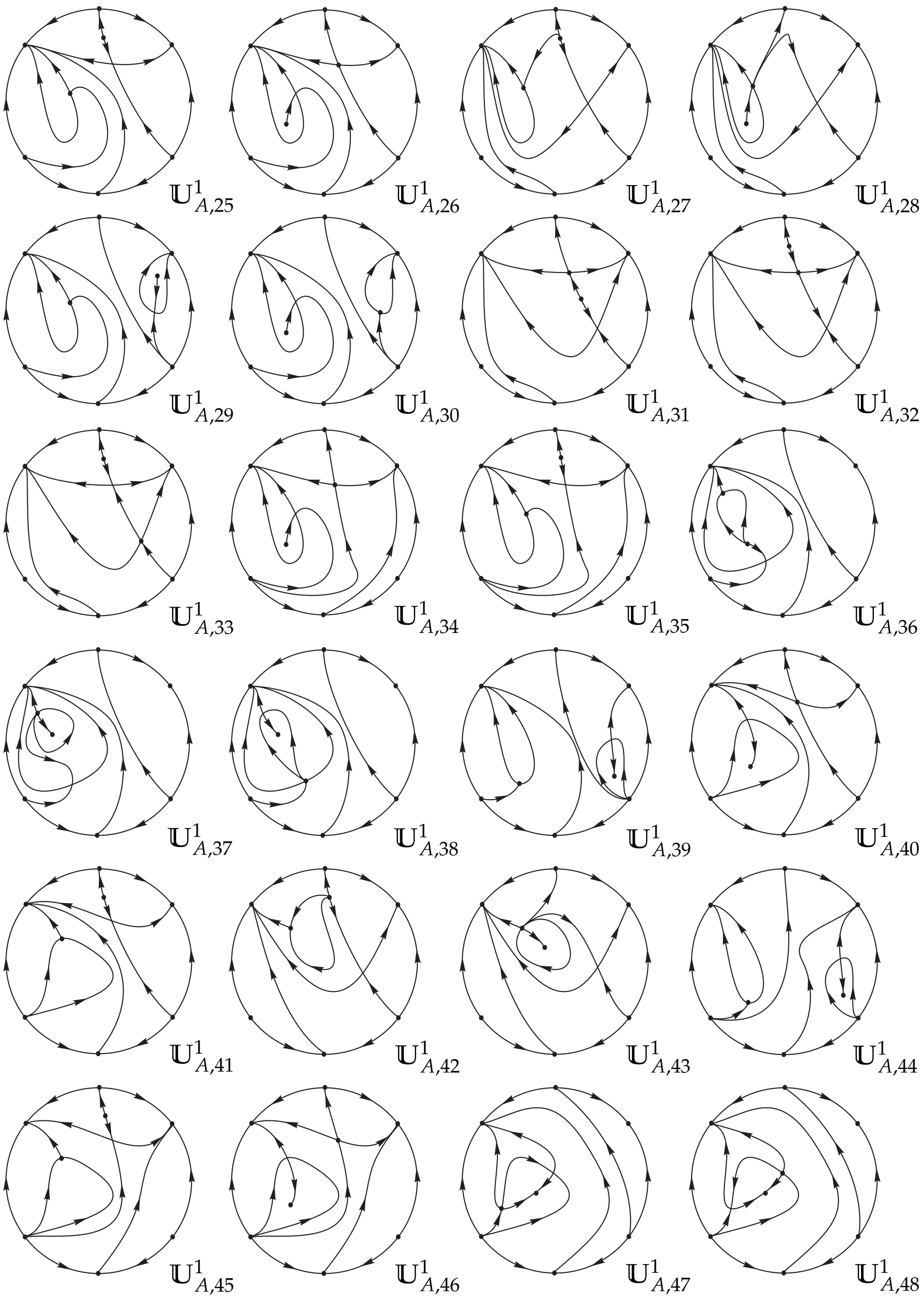


Figure 2.2: *(Cont.)* Unstable quadratic systems of *codimension one** of the set (A) (cases with a finite saddle-node $\overline{sn}_{(2)}$).

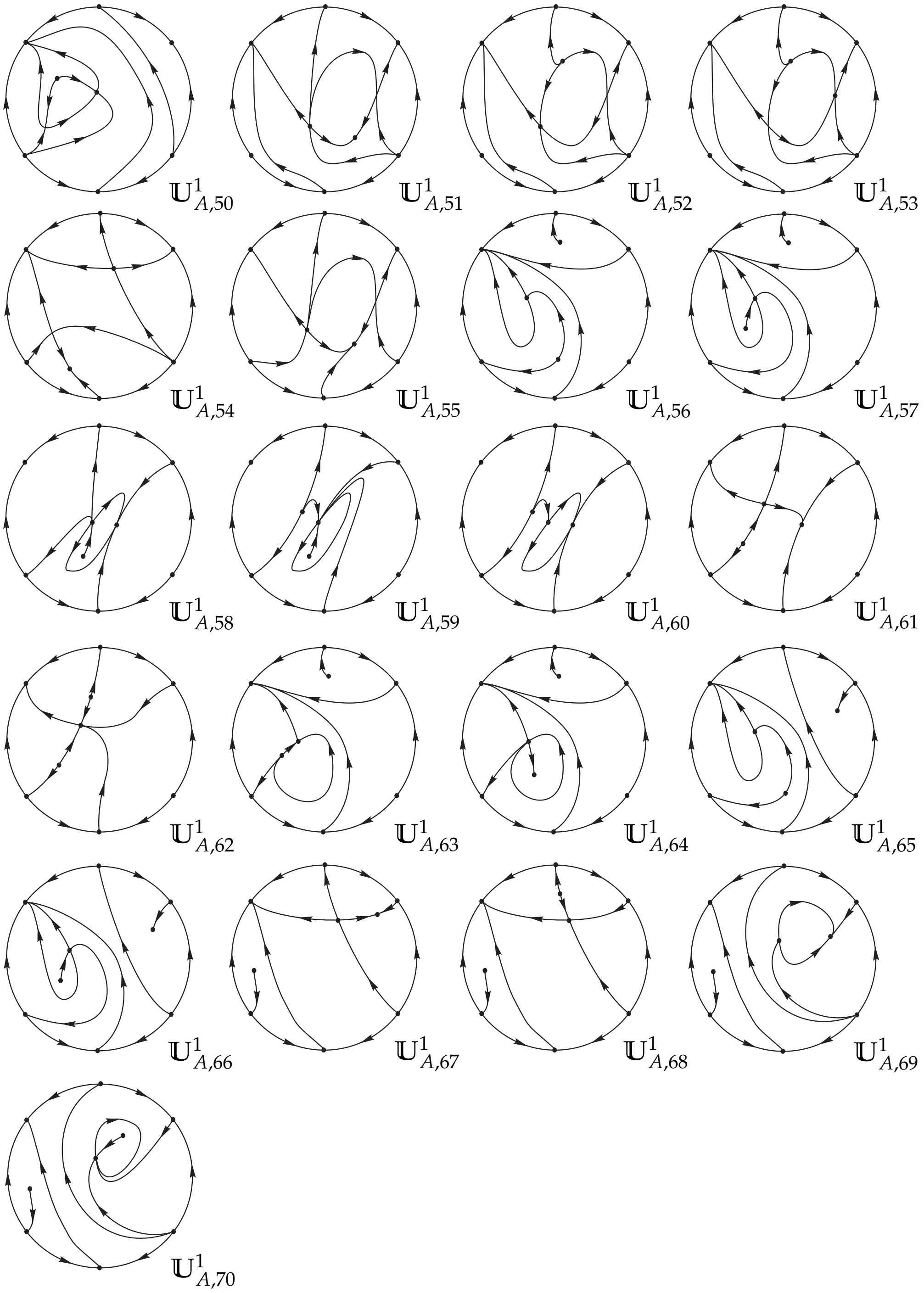


Figure 2.3: *(Cont.)* Unstable quadratic systems of *codimension one** of the set (A) (cases with a finite saddle-node $\overline{sn}_{(2)}$).

Remark 2.2. In [12] the authors proved that the phase portrait $\mathbb{U}^1_{A,49}$ from Figure 1.4 of [6] is actually impossible. Therefore, in our Figures 2.1 to 2.3 we have simply "skipped" this phase portrait, since all of the remaining ones are proved to be realizable in [6]. We present this impossible phase portrait in Figure 2.8 and there we denote it by $\mathbb{U}^{1,I}_{A,49}$.

Theorem 2.3. *If $X \in \sum_1^2(B)$, then its phase portrait on the Poincaré disc is topologically equivalent modulo orientation and modulo limit cycles to one of the* 40 *phase portraits of Figures* 2.4 *and* 2.5*, and all of them are realizable.*

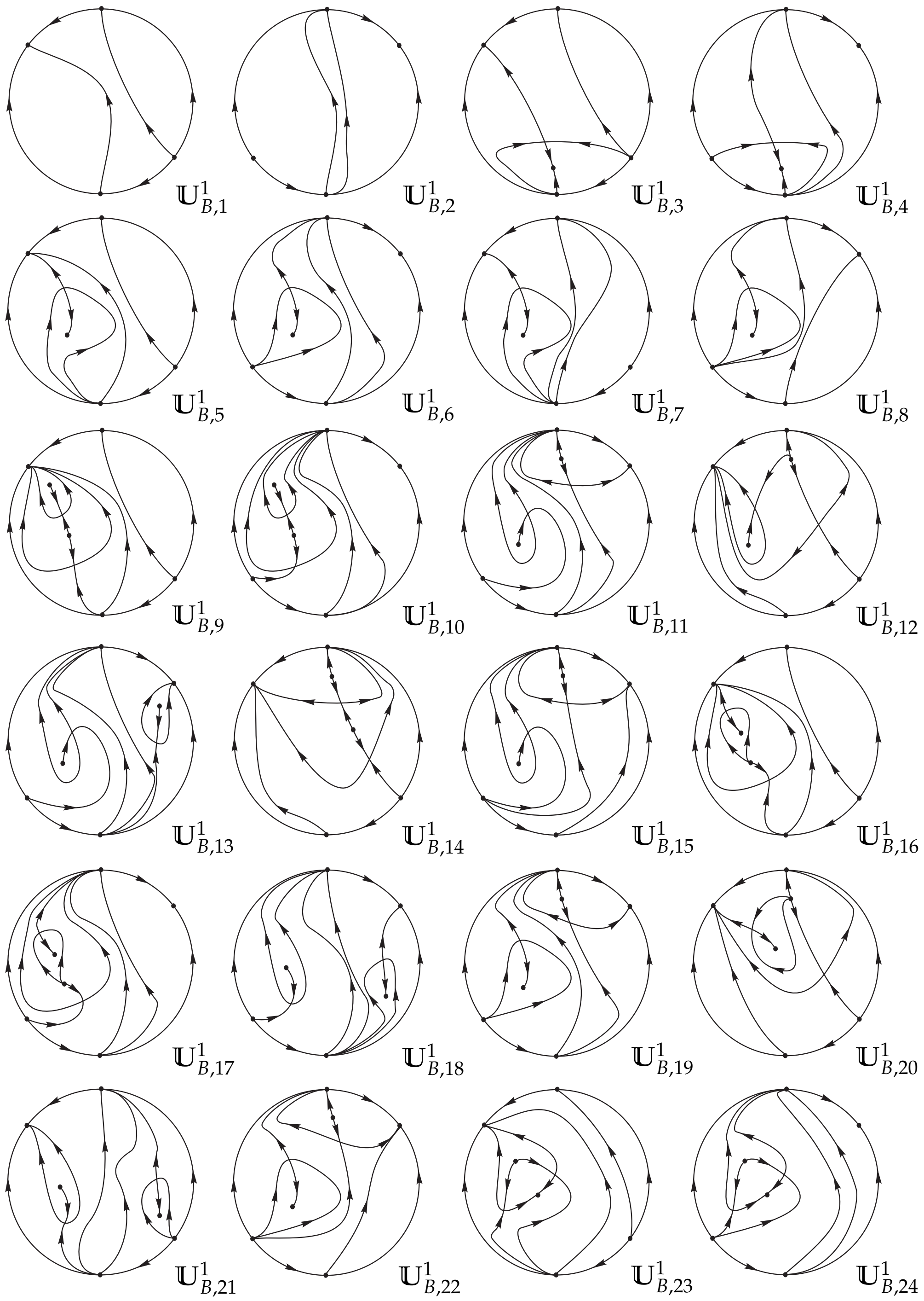


Figure 2.4: Unstable quadratic systems of *codimension one** of the set (B) (cases with an infinite saddle-node of type $\overline{\binom{0}{2}}SN$).

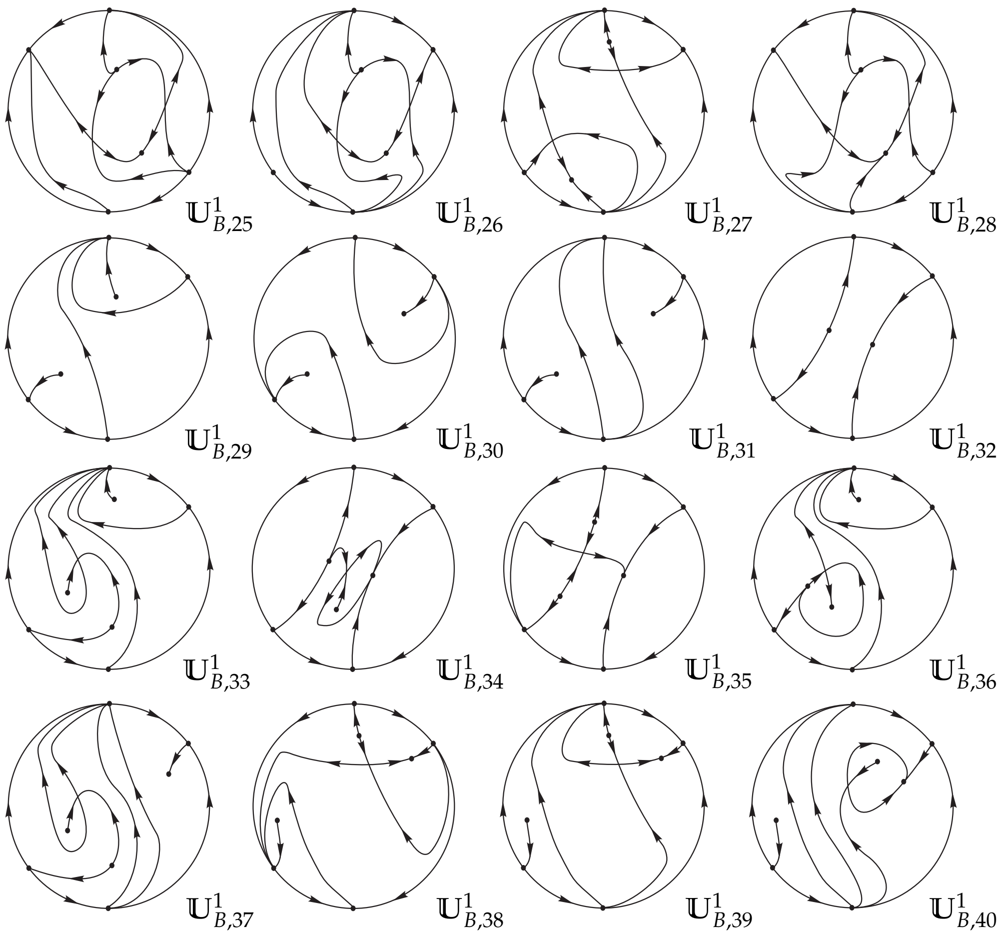


Figure 2.5: *(Cont.)* Unstable quadratic systems of *codimension one** of the set (B) (cases with an infinite saddle-node of type $\overline{\binom{0}{2}}SN$).

Theorem 2.4. *If $X \in \sum_1^2(C)$, then its phase portrait on the Poincaré disc is topologically equivalent modulo orientation and modulo limit cycles to one of the* 32 *phase portraits of Figures 2.6 and 2.7, and all of them are realizable.*

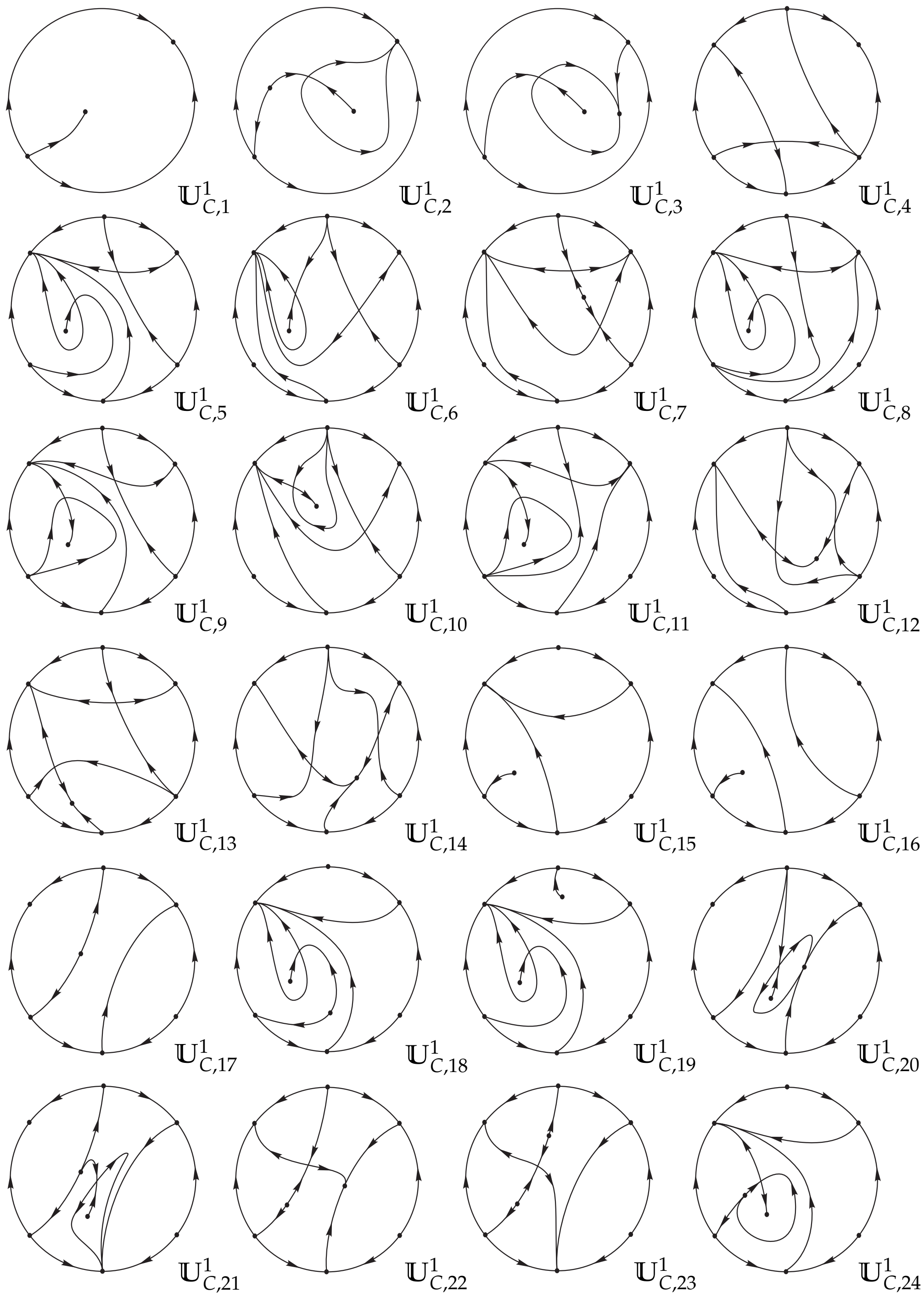


Figure 2.6: Unstable quadratic systems of *codimension one** of the set (C) (cases with an infinite saddle-node of type $\overline{\binom{1}{1}}SN$).

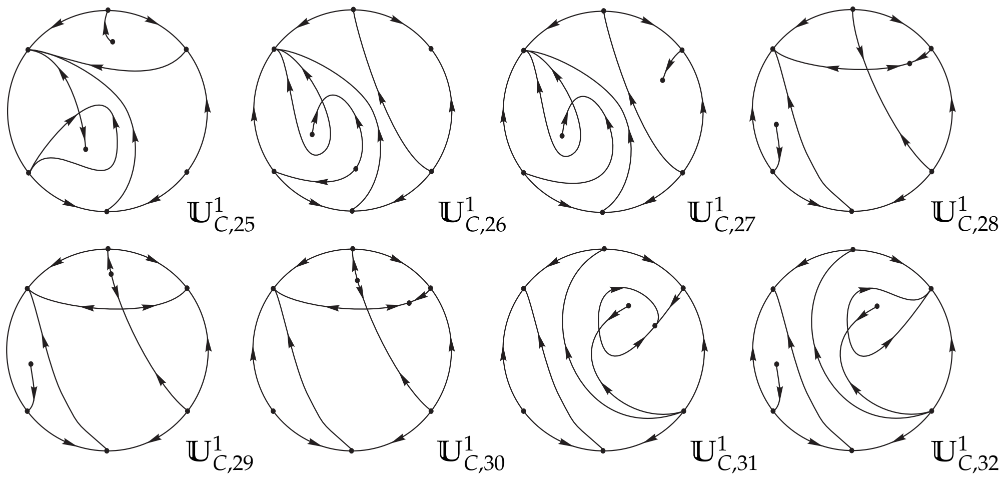


Figure 2.7: *(Cont.)* Unstable quadratic systems of *codimension one** of the set (C) (cases with an infinite saddle-node of type $\overline{\binom{1}{1}}SN$).

Before we state our next theorem, consider the following remark.

Remark 2.5. Consider all the impossible phase portraits from the book [6]. In that book these phase portraits are described with a specific notation. However, in this paper we changed a little bit their notation in order to associate each impossible phase portrait with the set in which such a phase portrait is proved to be impossible, but we keep the respective indexes. For instance, in that book we have the presence of the impossible phase portrait $\mathbb{U}^1_{I,105}$, which is a non-realizable case from the set (A). Such a phase portrait is denoted in this paper by $\mathbb{U}^{1,I}_{A,105}$. We also use this new notation for phase portraits which are proved to be impossible in the sets (B) and (C).

The next result describes which phase portraits were discarded in the set (A) in [6] because they were not realizable, but their role now is important in the process of discarding impossible phase portraits of *codimension two**.

Theorem 2.6. *In order to obtain a phase portrait of a structurally unstable quadratic vector field of codimension one* from the set (A) it is necessary and sufficient to coalesce a finite saddle and a finite node from a structurally stable quadratic vector field, which leads to a finite saddle-node, and after some small perturbation it disappears. For the vector fields in the set (A), the following statements hold.*

(a) In Table 2.1 we see in the first and fifth columns the structurally stable quadratic vector fields (following the notation present in [2, 6]) which, after the coalescence of singularities cited above, lead to at least one phase portrait of codimension one from the set (A).*

(b) Inside this set (A), we have a total of 77 topologically distinct phase portraits according to the different α-limit or ω-limit of the separatrices of their saddles, 7 of which are proved non-realizable in [6] and another one is proved non-realizable in [12] (all of these eight non-realizable phase portraits are given in Table 2.2). These numbers are given in the second and sixth columns of Table 2.1.

(c) From these potential phase portraits, most of them are realizable. That is, even though there is the topological possibility of their existence, some of them break some analytical property which makes them not realizable inside quadratic vector fields. We have a total of 69 realizable phase portraits. In the third and seventh columns of Table 2.1 we present the number of realizable

cases coming from the bifurcation of each structurally stable phase portrait, and in the fourth and eighth columns we present the bifurcated phase portraits of codimension one associated to each one.*

(d) *There are then 8 non-realizable cases from the set (A) which we now collect in a single picture (see Figure 2.8) and denote by $\mathbb{U}^{1,I}_{A,k}$, where $\mathbb{U}^{1,I}_{A}$ stands for Impossible of codimension one* from the set (A) and $k \in \{1, 2, 3, 49, 103, 104, 105, 106\}$, see Remark 2.5. These phase portraits are all drawn in [6]. Anyway, we provide Table 2.2 in order to relate easily (giving also the page where they appear first and the page they are proved to be impossible).*

SSQVF [2]	$\#_\mathbf{p}$	$\#_\mathbf{r}$	**SU1 [6]**	**SSQVF [2]**	$\#_\mathbf{p}$	$\#_\mathbf{r}$	**SU1 [6]**
$S^2_{2,1}$	1	1	$\mathbb{U}^1_{A,1}$	$S^2_{10,6}$	2	2	$\mathbb{U}^1_{A,34}, \mathbb{U}^1_{A,35}$
$S^2_{3,1}$	3	3	$\mathbb{U}^1_{A,2}, \mathbb{U}^1_{A,3}, \mathbb{U}^1_{A,4}$	$S^2_{10,7}$	4	3	$\mathbb{U}^1_{A,36}, \mathbb{U}^1_{A,37}, \mathbb{U}^1_{A,38}$
$S^2_{3,2}$	1	1	$\mathbb{U}^1_{A,5}$	$S^2_{10,8}$	1	1	$\mathbb{U}^1_{A,39}$
$S^2_{3,3}$	1	1	$\mathbb{U}^1_{A,6}$	$S^2_{10,9}$	2	2	$\mathbb{U}^1_{A,40}, \mathbb{U}^1_{A,41}$
$S^2_{3,4}$	1	1	$\mathbb{U}^1_{A,7}$	$S^2_{10,10}$	4	2	$\mathbb{U}^1_{A,42}, \mathbb{U}^1_{A,43}$
$S^2_{3,5}$	3	3	$\mathbb{U}^1_{A,8}, \mathbb{U}^1_{A,9}, \mathbb{U}^1_{A,10}$	$S^2_{10,11}$	1	1	$\mathbb{U}^1_{A,44}$
$S^2_{5,1}$	3	3	$\mathbb{U}^1_{A,11}, \mathbb{U}^1_{A,12}, \mathbb{U}^1_{A,13}$	$S^2_{10,12}$	2	2	$\mathbb{U}^1_{A,45}, \mathbb{U}^1_{A,46}$
$S^2_{7,1}$	1	1	$\mathbb{U}^1_{A,14}$	$S^2_{10,13}$	4	4	$\mathbb{U}^1_{A,47}, \mathbb{U}^1_{A,48}, \mathbb{U}^1_{A,50}$
$S^2_{7,2}$	2	2	$\mathbb{U}^1_{A,15}, \mathbb{U}^1_{A,16}$	$S^2_{10,14}$	4	3	$\mathbb{U}^1_{A,51}, \mathbb{U}^1_{A,52}, \mathbb{U}^1_{A,53}$
$S^2_{7,3}$	1	1	$\mathbb{U}^1_{A,17}$	$S^2_{10,15}$	1	1	$\mathbb{U}^1_{A,54}$
$S^2_{7,4}$	1	1	$\mathbb{U}^1_{A,18}$	$S^2_{10,16}$	1	1	$\mathbb{U}^1_{A,55}$
$S^2_{9,1}$	1	1	$\mathbb{U}^1_{A,19}$	$S^2_{12,1}$	2	2	$\mathbb{U}^1_{A,56}, \mathbb{U}^1_{A,57}$
$S^2_{9,2}$	1	1	$\mathbb{U}^1_{A,20}$	$S^2_{12,2}$	3	3	$\mathbb{U}^1_{A,58}, \mathbb{U}^1_{A,59}, \mathbb{U}^1_{A,60}$
$S^2_{9,3}$	1	1	$\mathbb{U}^1_{A,21}$	$S^2_{12,3}$	2	2	$\mathbb{U}^1_{A,61}, \mathbb{U}^1_{A,62}$
$S^2_{10,1}$	3	3	$\mathbb{U}^1_{A,22}, \mathbb{U}^1_{A,23}, \mathbb{U}^1_{A,24}$	$S^2_{12,4}$	3	2	$\mathbb{U}^1_{A,63}, \mathbb{U}^1_{A,64}$
$S^2_{10,2}$	2	2	$\mathbb{U}^1_{A,25}, \mathbb{U}^1_{A,26}$	$S^2_{12,5}$	2	2	$\mathbb{U}^1_{A,65}, \mathbb{U}^1_{A,66}$
$S^2_{10,3}$	3	2	$\mathbb{U}^1_{A,27}, \mathbb{U}^1_{A,28}$	$S^2_{12,6}$	2	2	$\mathbb{U}^1_{A,67}, \mathbb{U}^1_{A,68}$
$S^2_{10,4}$	2	2	$\mathbb{U}^1_{A,29}, \mathbb{U}^1_{A,30}$	$S^2_{12,7}$	3	2	$\mathbb{U}^1_{A,69}, \mathbb{U}^1_{A,70}$
$S^2_{10,5}$	3	3	$\mathbb{U}^1_{A,31}, \mathbb{U}^1_{A,32}, \mathbb{U}^1_{A,33}$				

Table 2.1: Potential and realizable bifurcated phase portraits for a given structurally stable quadratic vector field. In this table, **SSQVF** stands for structurally stable quadratic vector fields, $\#_\mathbf{p}$ (respectively $\#_\mathbf{r}$) for the number of topologically potential (respectively realizable) phase portraits of *codimension one** bifurcated from the respective **SSQVF**, and **SU1** for the respective phase portraits of *codimension one**.

SSQVF [2]	**Page** [6]	**Impossible** [6]	**SSQVF** [2]	**Page** [6]	**Impossible** [6]
$\mathbb{S}^2_{10,3}$	70	$\mathbb{U}^{1,I}_{A,1}$	$\mathbb{S}^2_{10,14}$	77	$\mathbb{U}^{1,I}_{A,3}$
$\mathbb{S}^2_{10,7}$	(73) 190	$\mathbb{U}^{1,I}_{A,103}$	$\mathbb{S}^2_{12,4}$	(80) 191	$\mathbb{U}^{1,I}_{A,105}$
$\mathbb{S}^2_{10,10}$	75; 191	$\mathbb{U}^{1,I}_{A,2}$; $\mathbb{U}^{1,I}_{A,104}$	$\mathbb{S}^2_{12,7}$	(82) 188	$\mathbb{U}^{1,I}_{A,106}$
$\mathbb{S}^2_{10,13}$	76	$\mathbb{U}^{1,I}_{A,49}$ (see [12])			

Table 2.2: Non-realizable phase portraits from the set (A) which could bifurcate (if existed) from structurally stable quadratic vector fields. The first and fourth columns indicate the structurally stable quadratic vector field (**SSQVF**) which suffers a bifurcation, the second and fifth columns indicate the pages where they appear in [6] and the third and sixth columns present the corresponding impossible phase portraits (remember that phase portrait $\mathbb{U}^1_{A,49}$ from Figure 1.4 of [6] is proved to be impossible in [12]).

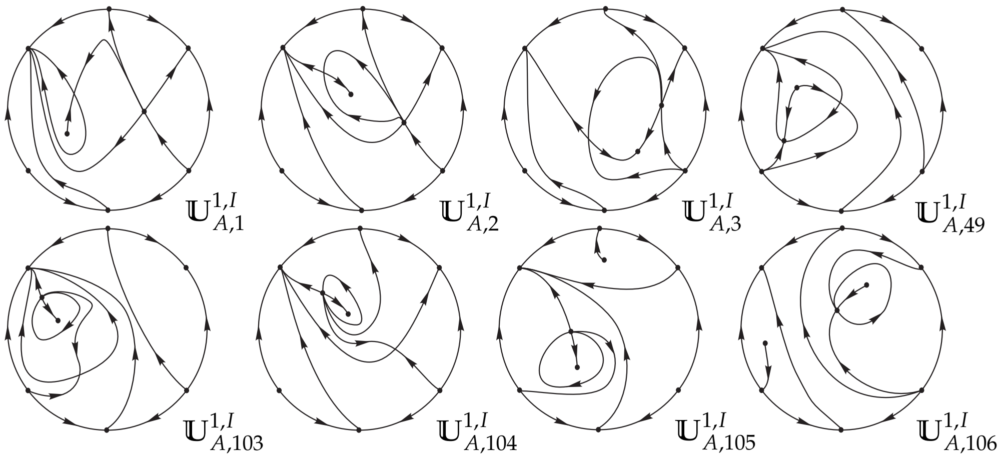


Figure 2.8: Phase portraits of the non-realizable structurally unstable quadratic vector fields of *codimension one** from the set (A).

In what follows we present an analogous theorem regarding discarded phase portraits from the set (B) in [6].

Theorem 2.7. *In order to obtain a phase portrait of a structurally unstable quadratic vector field of codimension one** *from the set* (B) *it is necessary and sufficient to coalesce an infinite saddle with an infinite node from a structurally stable quadratic vector field, which leads to an infinite saddle-node of type* $\overline{\binom{0}{2}}SN$*, and after some small perturbation it disappears. For the vector fields in set* (B)*, the following statements hold.*

(a) In Table 2.3 we see in the first and fifth columns the structurally stable quadratic vector fields (following the notation present in [2, 6]) which, after the coalescence of singularities cited above, lead to at least one phase portrait of codimension one* *from the set* (B)*.*

(b) Inside this set (B)*, we have a total of* 55 *topologically distinct phase portraits according to the different* α*-limit or* ω*-limit of the separatrices of their saddles,* 15 *of which are non-realizable (they are given in Table 2.4). These numbers are given in the second and sixth columns of Table 2.3.*

(c) From these potential phase portraits, most of them are realizable. That is, even though there is the topological possibility of their existence, some of them break some analytical property which

makes them not realizable inside quadratic vector fields. We have a total of 40 *realizable phase portraits. In the third and seventh columns of Table 2.3 we present the number of realizable cases coming from the bifurcation of each structurally stable phase portrait, and in the fourth and eighth columns we present the bifurcated phase portraits of codimension one** *associated to each one.*

(d) *There are then* 15 *non-realizable cases from the set (B) which we now collect in a single picture (see Figure 2.9) and denote by* $\mathbb{U}^{1,I}_{B,k}$, *where* $\mathbb{U}^{1,I}_{B}$ *stands for Impossible of codimension one** *from the set (B) and* $k \in \{4, 5, 6, 7, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117\}$, *see Remark 2.5. These phase portraits are all drawn in [6]. Anyway, we provide Table 2.4 in order to relate easily (giving also the page where they appear first and the page they are proved to be impossible).*

SSQVF [2]	$\#_\mathbf{p}$	$\#_\mathbf{r}$	**SU1 [6]**	**SSQVF [2]**	$\#_\mathbf{p}$	$\#_\mathbf{r}$	**SU1 [6]**
$\mathbb{S}^2_{8,1}$	2	2	$\mathbb{U}^1_{B,1}, \mathbb{U}^1_{B,2}$	$\mathbb{S}^2_{10,12}$	2	1	$\mathbb{U}^1_{B,22}$
$\mathbb{S}^2_{9,1}$	2	2	$\mathbb{U}^1_{B,3}, \mathbb{U}^1_{B,4}$	$\mathbb{S}^2_{10,13}$	2	2	$\mathbb{U}^1_{B,23}, \mathbb{U}^1_{B,24}$
$\mathbb{S}^2_{9,2}$	2	2	$\mathbb{U}^1_{B,5}, \mathbb{U}^1_{B,6}$	$\mathbb{S}^2_{10,14}$	2	2	$\mathbb{U}^1_{B,25}, \mathbb{U}^1_{B,26}$
$\mathbb{S}^2_{9,3}$	2	2	$\mathbb{U}^1_{B,7}, \mathbb{U}^1_{B,8}$	$\mathbb{S}^2_{10,15}$	2	1	$\mathbb{U}^1_{B,27}$
$\mathbb{S}^2_{10,1}$	2	2	$\mathbb{U}^1_{B,9}, \mathbb{U}^1_{B,10}$	$\mathbb{S}^2_{10,16}$	1	1	$\mathbb{U}^1_{B,28}$
$\mathbb{S}^2_{10,2}$	2	1	$\mathbb{U}^1_{B,11}$	$\mathbb{S}^2_{11,1}$	1	1	$\mathbb{U}^1_{B,29}$
$\mathbb{S}^2_{10,3}$	2	1	$\mathbb{U}^1_{B,12}$	$\mathbb{S}^2_{11,2}$	2	2	$\mathbb{U}^1_{B,30}, \mathbb{U}^1_{B,31}$
$\mathbb{S}^2_{10,4}$	2	1	$\mathbb{U}^1_{B,13}$	$\mathbb{S}^2_{11,3}$	1	1	$\mathbb{U}^1_{B,32}$
$\mathbb{S}^2_{10,5}$	2	1	$\mathbb{U}^1_{B,14}$	$\mathbb{S}^2_{12,1}$	2	1	$\mathbb{U}^1_{B,33}$
$\mathbb{S}^2_{10,6}$	2	1	$\mathbb{U}^1_{B,15}$	$\mathbb{S}^2_{12,2}$	1	1	$\mathbb{U}^1_{B,34}$
$\mathbb{S}^2_{10,7}$	2	2	$\mathbb{U}^1_{B,16}, \mathbb{U}^1_{B,17}$	$\mathbb{S}^2_{12,3}$	1	1	$\mathbb{U}^1_{B,35}$
$\mathbb{S}^2_{10,8}$	2	1	$\mathbb{U}^1_{B,18}$	$\mathbb{S}^2_{12,4}$	2	1	$\mathbb{U}^1_{B,36}$
$\mathbb{S}^2_{10,9}$	2	1	$\mathbb{U}^1_{B,19}$	$\mathbb{S}^2_{12,5}$	2	1	$\mathbb{U}^1_{B,37}$
$\mathbb{S}^2_{10,10}$	2	1	$\mathbb{U}^1_{B,20}$	$\mathbb{S}^2_{12,6}$	2	2	$\mathbb{U}^1_{B,38}, \mathbb{U}^1_{B,39}$
$\mathbb{S}^2_{10,11}$	2	1	$\mathbb{U}^1_{B,21}$	$\mathbb{S}^2_{12,7}$	2	1	$\mathbb{U}^1_{B,40}$

Table 2.3: Potential and realizable bifurcated phase portraits for a given structurally stable quadratic vector field. In this table, **SSQVF** stands for structurally stable quadratic vector fields, $\#_\mathbf{p}$ (respectively $\#_\mathbf{r}$) for the number of topologically potential (respectively realizable) phase portraits of *codimension one** bifurcated from the respective **SSQVF**, and **SU1** for the respective phase portraits of *codimension one**.

SSQVF [2]	Page [6]	Impossible [6]	SSQVF [2]	Page [6]	Impossible [6]
$\mathbb{S}^2_{10,2}$	86; 200	$\mathbb{U}^{1,I}_{B,107}$	$\mathbb{S}^2_{10,11}$	90; 200	$\mathbb{U}^{1,I}_{B,115}$
$\mathbb{S}^2_{10,3}$	86; 203	$\mathbb{U}^{1,I}_{B,108}$	$\mathbb{S}^2_{10,12}$	91; 200	$\mathbb{U}^{1,I}_{B,116}$
$\mathbb{S}^2_{10,4}$	87; 200	$\mathbb{U}^{1,I}_{B,109}$	$\mathbb{S}^2_{10,15}$	92; 200	$\mathbb{U}^{1,I}_{B,117}$
$\mathbb{S}^2_{10,5}$	87; 207	$\mathbb{U}^{1,I}_{B,110}$	$\mathbb{S}^2_{12,1}$	94	$\mathbb{U}^{1,I}_{B,4}$
$\mathbb{S}^2_{10,6}$	88; 200	$\mathbb{U}^{1,I}_{B,111}$	$\mathbb{S}^2_{12,4}$	96	$\mathbb{U}^{1,I}_{B,5}$
$\mathbb{S}^2_{10,8}$	89; 200	$\mathbb{U}^{1,I}_{B,112}$	$\mathbb{S}^2_{12,5}$	96	$\mathbb{U}^{1,I}_{B,6}$
$\mathbb{S}^2_{10,9}$	89; 200	$\mathbb{U}^{1,I}_{B,113}$	$\mathbb{S}^2_{12,7}$	97	$\mathbb{U}^{1,I}_{B,7}$
$\mathbb{S}^2_{10,10}$	90; 203	$\mathbb{U}^{1,I}_{B,114}$			

Table 2.4: Non-realizable phase portraits from the set (B) which could bifurcate (if existed) from structurally stable quadratic vector fields. The first and fourth columns indicate the structurally stable quadratic vector field (**SSQVF**) which suffers a bifurcation, the second and fifth columns indicate the pages where they appear in [6] and the third and sixth columns present the corresponding impossible phase portraits.

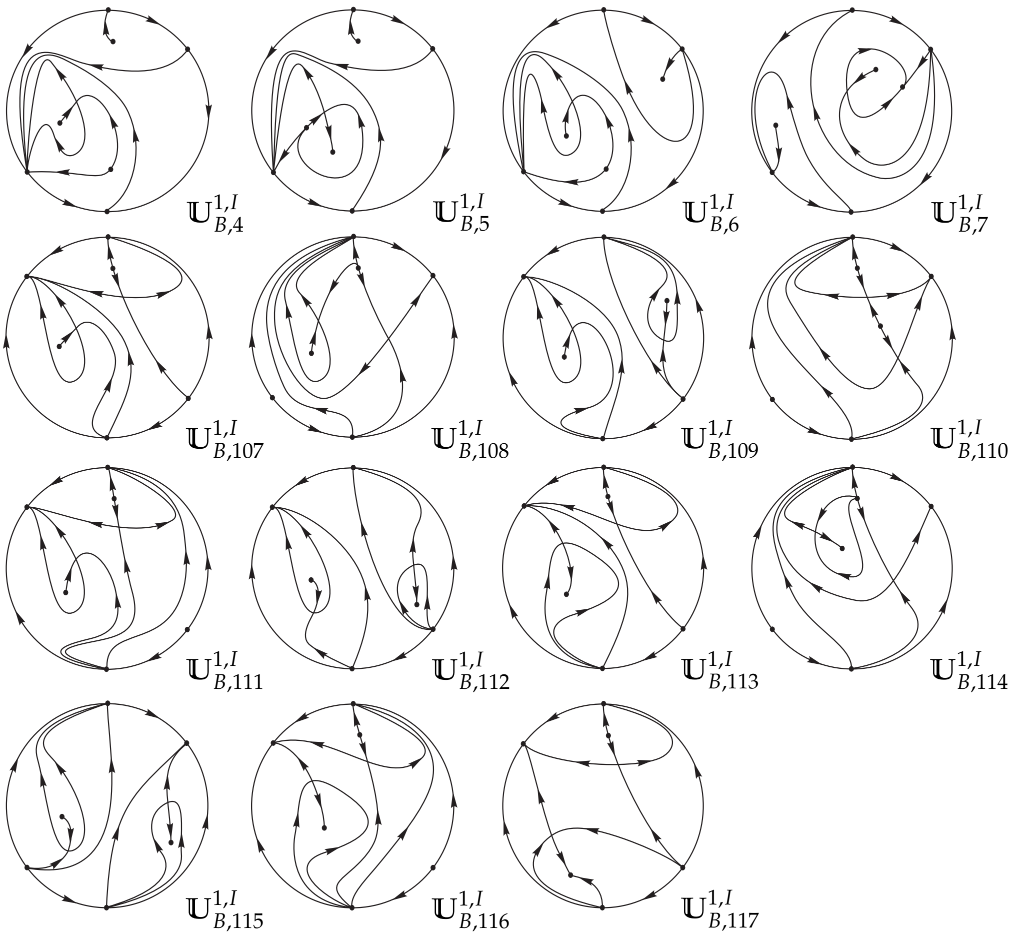


Figure 2.9: Phase portraits of the non-realizable structurally unstable quadratic vector fields of *codimension one** from the set (B).

Remark 2.8. Regarding the phase portraits of the non-realizable structurally unstable quadratic vector fields of *codimension one** from the set (B), we point out that in page 91 from [6], phase portrait $\mathbb{U}^{1,I}_{B,116}$ (which corresponds to B_{31} from such a book) is wrongly drawn. In fact, it possesses an extra infinite node and such a phase portrait should be drawn exactly as we present in our Figure 2.9.

Finally we present an analogous theorem regarding discarded phase portraits from the set (C) in [6].

Theorem 2.9. *In order to obtain a phase portrait of a structurally unstable quadratic vector field of codimension one* from the set (C) it is necessary and sufficient to coalesce a finite node (respectively, a finite saddle) with an infinite saddle (respectively, an infinite node) from a structurally stable quadratic vector field, which leads to an infinite saddle-node of type* $\overline{\binom{1}{1}}SN$, *and after some small perturbation, this saddle-node is split into a finite saddle (respectively, a finite node) and an infinite node (respectively, an infinite saddle). For the vector fields in the set* (C), *the following statements hold.*

(a) In Table 2.5 we see in the first and fifth columns the structurally stable quadratic vector fields (following the notation present in [2, 6]) which, after the coalescence of singularities cited above, lead to at least one phase portrait of codimension one from the set (C).*

(b) Inside this set (C), we have a total of 34 *topologically distinct phase portraits according to the different* α*-limit or* ω*-limit of the separatrices of their saddles, two of which are non-realizable (they are given in Table 2.6). These numbers are given in the second and sixth columns of Table 2.5.*

(c) From these potential phase portraits, only two of them are not realizable. That is, even though there is the topological possibility of their existence, two of them break some analytical property which makes them not realizable inside quadratic vector fields. We have a total of 32 *realizable phase portraits. In the third and seventh columns of Table 2.5 we present the number of realizable cases coming from the bifurcation of each structurally stable phase portrait, and in the fourth and eighth columns we present the bifurcated phase portraits of codimension one* associated to each one.*

(d) There are then two non-realizable cases from the set (C) which we present in Figure 2.10 and denote by $\mathbb{U}^{1,I}_{C,k}$, *where* $\mathbb{U}^{1,I}_{C}$ *stands for Impossible of codimension one* from the set (C) and* $k \in \{8, 9\}$, *see Remark 2.5. These phase portraits are drawn in [6]. Anyway, we provide Table 2.6 in order to relate easily (giving also the page where they appear first and the page they are proved to be impossible).*

SSQVF [2]	**#$_\mathbf{p}$**	**#$_\mathbf{r}$**	**SU1 [6]**	**SSQVF [2]**	**#$_\mathbf{p}$**	**#$_\mathbf{r}$**	**SU1 [6]**
$\mathbb{S}^2_{4,1}$	1	1	$\mathbb{U}^1_{C,1}$	$\mathbb{S}^2_{10,16}$	1	1	$\mathbb{U}^1_{C,14}$
$\mathbb{S}^2_{5,1}$	2	2	$\mathbb{U}^1_{C,2}, \mathbb{U}^1_{C,3}$	$\mathbb{S}^2_{11,1}$	1	1	$\mathbb{U}^1_{C,15}$
$\mathbb{S}^2_{9,1}$	1	1	$\mathbb{U}^1_{C,4}$	$\mathbb{S}^2_{11,2}$	1	1	$\mathbb{U}^1_{C,16}$
$\mathbb{S}^2_{10,2}$	2	1	$\mathbb{U}^1_{C,5}$	$\mathbb{S}^2_{11,3}$	1	1	$\mathbb{U}^1_{C,17}$
$\mathbb{S}^2_{10,3}$	1	1	$\mathbb{U}^1_{C,6}$	$\mathbb{S}^2_{12,1}$	2	2	$\mathbb{U}^1_{C,18}, \mathbb{U}^1_{C,19}$
$\mathbb{S}^2_{10,5}$	1	1	$\mathbb{U}^1_{C,7}$	$\mathbb{S}^2_{12,2}$	2	2	$\mathbb{U}^1_{C,20}, \mathbb{U}^1_{C,21}$
$\mathbb{S}^2_{10,6}$	1	1	$\mathbb{U}^1_{C,8}$	$\mathbb{S}^2_{12,3}$	2	2	$\mathbb{U}^1_{C,22}, \mathbb{U}^1_{C,23}$
$\mathbb{S}^2_{10,9}$	2	1	$\mathbb{U}^1_{C,9}$	$\mathbb{S}^2_{12,4}$	2	2	$\mathbb{U}^1_{C,24}, \mathbb{U}^1_{C,25}$
$\mathbb{S}^2_{10,10}$	1	1	$\mathbb{U}^1_{C,10}$	$\mathbb{S}^2_{12,5}$	2	2	$\mathbb{U}^1_{C,26}, \mathbb{U}^1_{C,27}$
$\mathbb{S}^2_{10,12}$	1	1	$\mathbb{U}^1_{C,11}$	$\mathbb{S}^2_{12,6}$	3	3	$\mathbb{U}^1_{C,28}, \mathbb{U}^1_{C,29}, \mathbb{U}^1_{30}$
$\mathbb{S}^2_{10,14}$	1	1	$\mathbb{U}^1_{C,12}$	$\mathbb{S}^2_{12,7}$	2	2	$\mathbb{U}^1_{C,31}, \mathbb{U}^1_{C,32}$
$\mathbb{S}^2_{10,15}$	1	1	$\mathbb{U}^1_{C,13}$				

Table 2.5: Potential and realizable bifurcated phase portraits for a given structurally stable quadratic vector field. In this table, **SSQVF** stands for structurally stable quadratic vector fields, #$_\mathbf{p}$ (respectively #$_\mathbf{r}$) for the number of topologically potential (respectively realizable) phase portraits of *codimension one** bifurcated from the respective **SSQVF**, and **SU1** for the respective phase portraits of *codimension one**.

SSQVF [2]	**Page [6]**	**Impossible [6]**
$\mathbb{S}^2_{10,2}$	101	$\mathbb{U}^{1,I}_{C,8}$
$\mathbb{S}^2_{10,9}$	103	$\mathbb{U}^{1,I}_{C,9}$

Table 2.6: Non-realizable phase portraits from the set (C) which could bifurcate (if existed) from structurally stable quadratic vector fields. The first column indicates the structurally stable quadratic vector field (**SSQVF**) which suffers a bifurcation, the second column indicates the pages where they appear in [6] and the third column present the corresponding impossible phase portrait.

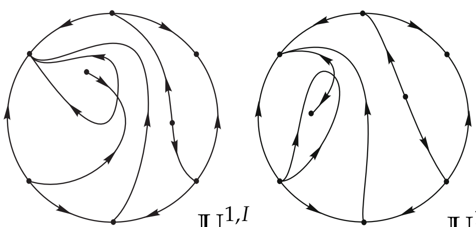


Figure 2.10: Phase portraits of the non-realizable structurally unstable quadratic vector fields of *codimension one** from the set (C).

An important result to study the impossibility of some phase portraits is Corollary 3.29 of [6].

Corollary 2.10. *If one of the structurally stable vector fields that bifurcates from a potential structurally unstable vector field of codimension one** *is not realizable, then this unstable system is also not realizable.*

Our aim is to prove the following result, which is the analogous of the previous corollary for the sets (AB) and (AC).

Theorem 2.11. *If one of the phase portraits of codimension one* that bifurcates from a potential codimension two* phase portrait from the sets (AB) and (AC) is not realizable, then this latter phase portrait is also not realizable.*

Proof. In what follows we prove the equivalent statement: If a potential *codimension two** phase portrait X from the sets (AB) and (AC) is realizable, then the phase portraits of *codimension one** that bifurcates from X are also realizable.
We start from the set (AB). We already know that a realizable phase portrait belongs to the set (AB) if and only if it has a finite saddle-node $\overline{sn}_{(2)}$ and an infinite saddle-node of type $\overline{\binom{0}{2}}SN$ obtained by the coalescence of an infinite saddle with an infinite node. In [14] the authors classified the set of all real quadratic polynomial differential systems with a finite semi-elemental saddle-node $\overline{sn}_{(2)}$ located at the origin of the plane and an infinite saddle-node of type $\overline{\binom{0}{2}}SN$ located in the bisector of first and third quadrants. Such a classification was done with respect to the normal form

$$\begin{aligned} \dot{x} &= gx^2 + 2hxy + (n - g - 2h)y^2, \\ \dot{y} &= y + lx^2 + (2g + 2h - 2l - n)xy + (l - 2g - 2h + 2n)y^2, \end{aligned} \tag{2.1}$$

where g, h, l, and n are real parameters. The parameter space of this normal form is a four-dimensional space, which can be projectivized, as it was done in [14] and the authors proved that all generic phenomena occur for $g = 1$. In the paper under discussion the authors used the *Invariant Theory* (developed in Sibirsky School – Moldova, see a very nice summary of this theory in Sec. 7 of [7]) in order to construct and study their bifurcation diagram. In Lemma 5.5 from the book [9] the authors proved that a necessary and sufficient condition for a generic quadratic system to possess an infinite saddle-node of type $\overline{\binom{0}{2}}SN$ and another simple infinite singularity is that the comitants η and $\widetilde{M}$ verify the conditions

$$\eta = 0, \quad \widetilde{M} \neq 0,$$

for all the possible values of the parameters of the system. Additionally, in Table 5.1 from that book the authors present the invariant polynomials which are responsible for the number, kinds (real or/and complex), and multiplicities of finite singularities of a generic quadratic system. In particular, they show that if the invariant polynomial $\mathbb{D}$ verifies the condition

$$\mathbb{D} = 0,$$

then we have a finite singularity of multiplicity at least two. In fact, for systems (2.1) calculations show that these systems verify such conditions, since for that normal form (with $g = 1$) we obtain

$$\eta = 0, \quad \widetilde{M} = -8(1 + 2h + l - n)^2(x - y)^2 \neq 0, \quad \mathbb{D} = 0.$$

Now, for $g = 1$, consider the perturbation of systems (2.1)

$$\begin{aligned} \dot{x} &= (1 - \varepsilon)x^2 + 2hxy + (n - 1 - 2h)y^2, \\ \dot{y} &= y + l(1 - \varepsilon)x^2 + ((2 + 2h - n)(1 - \varepsilon) - 2l)xy + (l - 2 - 2h + 2n)y^2, \end{aligned} \tag{2.2}$$

where $|\varepsilon|$ is small enough. For these systems, calculations show that

$$\eta = 4\varepsilon((1 + 2h + l - n)^2 - (-1 - 2h + n)^2\varepsilon)^2 \neq 0, \quad \mathbb{D} = 0.$$

So, according to Lemma 5.5 from the mentioned book, we have three distinct infinite singularities (all of them are real if $\varepsilon > 0$ and, if $\varepsilon < 0$, we have one real infinite singularity and two complex ones). Additionally, as $\mathbb{D} = 0$, perturbation (2.2) leaves unperturbed the finite saddle-node.
On the other hand, for $g = 1$ consider the perturbation of systems (2.1)

$$\begin{aligned} \dot{x} &= -\varepsilon + x^2 + 2hxy + (n - 1 - 2h)y^2, \\ \dot{y} &= -\varepsilon l + y + lx^2 + (2 + 2h - 2l - n)xy + (l - 2 - 2h + 2n)y^2, \end{aligned} \tag{2.3}$$

where $|\varepsilon|$ is small enough. For systems (2.3) we have

$$\eta = 0, \quad \widetilde{M} = -8(1 + 2h + l - n)^2(x - y)^2 \neq 0,$$

and

$$\mathbb{D} = -768\varepsilon(-1 + (2(1 + h)(-1 + l) + n)^2\varepsilon)^2(1 + 2h + h^2 - n + n^2((-1 + l)(1 + 2h + l) + n)\varepsilon).$$

According to Lemma 5.5 mentioned before, the perturbation (2.3) has not affected the infinite singular points and, according to Table 5.1 from the mentioned book, we no longer have finite multiple singularities, i.e. the perturbation splits the origin into two points (which are real or complex, depending on the sign of ε).
Therefore the result holds for the set (AB).
Now, consider the set (AC). A realizable phase portrait belongs to the set (AC) if and only if it has a finite saddle-node $\overline{sn}_{(2)}$ and an infinite saddle-node of type $\overline{\binom{1}{1}}SN$, obtained by the coalescence of a finite saddle (respectively, finite node) with an infinite node (respectively, infinite saddle). Remember that, as we discussed in page 8, the case in which the finite saddle-node is the finite singularity that coalesces with an infinite singularity will be considered in the future during the study of the set (CC). With the *Invariant Theory* as the main tool, in [10] we classified the set of all real quadratic polynomial differential systems with a finite semi-elemental saddle-node $\overline{sn}_{(2)}$ located at the origin of the plane and an infinite saddle-node of type $\overline{\binom{1}{1}}SN$. Such a classification was done with respect to the normal form

$$\begin{aligned} \dot{x} &= cx + cy - cx^2 + 2hxy, \\ \dot{y} &= ex + ey - ex^2 + 2mxy, \end{aligned} \tag{2.4}$$

where c, h, e, and m are real parameters, with the (nondegeneracy) condition $eh \neq cm$. The parameter space of this normal form is a four-dimensional space, which can be projectivized, as it was done in that paper where we proved that all generic phenomena occur for $h = 1$. In Lemma 5.2 from the book [9] the authors proved that a necessary and sufficient condition for a generic quadratic system to possess an infinite saddle-node of type $\overline{\binom{1}{1}}SN$ is that the comitants μ_0 and μ_1 verify the conditions

$$\mu_0 = 0, \quad \mu_1 \neq 0,$$

for all the possible values of the parameters of the system. Additionally, as in the previous case, from Table 5.1 it is possible to conclude that if the invariant polynomial $\mathbb{D}$ verifies the condition

$$\mathbb{D} = 0,$$

then we have a finite singularity of multiplicity at least two. Indeed, for systems (2.4) with $h = 1$ calculations show that such conditions are fulfilled, since

$$\mu_0 = 0, \quad \mu_1 = -8(e - cm)^2 x \neq 0, \quad \mathbb{D} = 0.$$

Now, for $h = 1$, consider the perturbation of systems (2.4)

$$\begin{aligned} \dot{x} &= cx + cy - cx^2 + 2xy + \varepsilon y^2, \\ \dot{y} &= ex + ey - ex^2 + 2mxy + \varepsilon y^2, \end{aligned} \tag{2.5}$$

where $|\varepsilon|$ is small enough, calculations show that for systems (2.5) the comitant μ_0 is given by

$$\mu_0 = \varepsilon(-4(1 - m)(e - cm) + (c - e)^2 \varepsilon).$$

So the perturbation under consideration splits the infinite saddle-node $\overline{\binom{1}{1}}SN$. Additionally, we conclude that the perturbation maintains the finite saddle-node, since for systems (2.5) calculations show that the invariant polynomial $\mathbb{D}$ vanishes.
Finally, for $h = 1$ (as we did for the set (AB)), consider the perturbation of systems (2.4)

$$\begin{aligned} \dot{x} &= -\varepsilon + cx + cy - cx^2 + 2xy, \\ \dot{y} &= -\varepsilon e + ex + ey - ex^2 + 2mxy, \end{aligned} \tag{2.6}$$

where $|\varepsilon|$ is small enough. For systems (2.6) we have

$$\mu_0 = 0, \quad \mu_1 = -4(e - cm)^2 x \neq 0,$$

and

$$\begin{aligned} \mathbb{D} =& 768\varepsilon(e - cm)^3 \left(16\varepsilon^2(e - m)^3 - 8(c - 1)e(e - cm)^2\right) \\ &+ 768\varepsilon^2(e - cm)^4 \left((9c(3c - 2) - 13)e^2 + 4(11 - 9c)em - 4m^2\right). \end{aligned}$$

According to the results (from the book [9]) presented before, we conclude that systems (2.6) have the infinite saddle-node $\overline{\binom{1}{1}}SN$ and do not have the finite saddle-node $\overline{sn}_{(2)}$, i.e. the perturbation (2.6) of systems (2.4) keeps the infinite saddle-node and splits the finite saddle-node.
Then the theorem also holds for the set (AC), as we wanted to prove. $\square$

As at the moment we are not interested in giving a proof for a general case of the previous theorem, in what follows we present a conjecture.

Conjecture 2.12. *If one of the phase portraits of codimension k that bifurcates from a potential codimension $k + 1$ phase portrait is not realizable, then this latter phase portrait is also not realizable.*

Remark 2.13. In Qualitative Theory of Ordinary Differential Equations is quite common to use the term "perturbation" to denote an infinitesimal modification of the parameters of a system such that a different phase portrait bifurcates from it. In this paper we use the term "evolution" in order to say that we "move a *codimension one** phase portrait to its border and detect which phase portraits are in the other side of this border", so with an evolution of a *codimension one** phase portrait we produce a *codimension two** phase portrait. In this sense we mean that we modify (in a continuous way) the first system inside the region of parameters in which it is defined up to the other side of the border of this region where we obtain a system having one codimension more. In a certain way, with this modification we are provoking an "evolution" of the first system. Note that we *contrast* "perturbation" with "evolution".

3 Proof of Theorem 1.6

In this section we present the proof of Theorem 1.6. More precisely, in Subsection 3.1 we obtain all the topologically potential phase portraits belonging to the set (AB) (we have 110 topologically distinct phase portraits) and we prove that 39 of them are impossible. In Subsection 3.2 we show the realization of each one of the remaining 71 phase portraits.

3.1 The topologically potential phase portraits

The main goal of this subsection is to obtain all the topologically potential phase portraits from the set (AB).

We already know that in the set (AB), the unstable objects of *codimension two** belong to the set of saddle-nodes $\{\overline{sn}_{(2)}+\overline{\binom{0}{2}}SN\}$. Considering all the different ways of obtaining phase portraits belonging to the set (AB) of *codimension two**, we have to consider all the possible ways of coalescing specific singular points in both sets (A) and (B). However, as the sets (AB) and (BA) are the same (i.e. their elements are obtained independently of the order of the evolution in the elements of the sets (A) or (B)), it is necessary to consider only all the possible ways of obtaining an infinite saddle-node of type $\overline{\binom{0}{2}}SN$ in each element from the set (A) (phase portraits possessing a finite saddle-node $\overline{sn}_{(2)}$). Anyway, in order to make things clear, in page 54 we discuss briefly how should we perform if we start by considering the set (B).

In order to obtain phase portraits from the set (AB) by starting our study from the set (A), we have to consider Theorem 2.7 and also Lemma 3.25 from [6] (regarding phase portraits from the set (B)) which we state as follows.

Lemma 3.1. *Suppose that a polynomial vector field X of codimension one* has an infinite saddle-node p of multiplicity two with $\rho_0 = (\partial P/\partial x + \partial Q/\partial y)_p \neq 0$ and first eigenvalue equal to zero.*

(a) Any perturbation of X in a sufficiently small neighborhood of this point will produce a structurally stable system (with one infinite saddle and one infinite node, or with no singular points in the neighborhood) or a system topologically equivalent to X.

(b) Both possibilities of structurally stable system (with one saddle and one node at infinity, or with no singular points in the neighborhood) are realizable.

Here we consider all the 69 realizable structurally unstable quadratic vector fields of *codimension one** from the set (A). In order to obtain a phase portrait of *codimension two** belonging to the set (AB) starting from a phase portrait of *codimension one** of the set (A), we keep the existing finite saddle-node and using Lemma 3.1 we build an infinite saddle-node of type $\overline{\binom{0}{2}}SN$ by the coalescence of an infinite saddle with an infinite node. On the other hand, from the phase portraits of *codimension two** from the set (AB), one can obtain phase portraits of *codimension one** belonging to the set (A) after perturbation of the infinite saddle-node $\overline{\binom{0}{2}}SN$ into an infinite saddle and an infinite node, or into complex singularities.

In what follows we denote by $\mathbb{U}^2_{AB,k}$, where $\mathbb{U}^2_{AB}$ stands for structurally unstable quadratic vector field of *codimension two** from the set (AB) and $k \in \{1,\ldots,71\}$ (note that the notation $\mathbb{U}^2_{AB}$ is simpler than $\mathbb{U}^2_{(AB)}$). The impossible phase portraits will be denoted by $\mathbb{U}^{2,I}_{AB,j}$, where $\mathbb{U}^{2,I}_{AB}$ stands for *Impossible of codimension two* from the set (AB)* and $j \in \mathbb{N}$. We need to enumerate also the impossible phase portraits, not for the completeness of this paper, but for the future papers in which someone will study *codimension three** families. Just in the same way as impossible *codimension one** phase portraits are a crucial tool for the study of our families.

Note that phase portraits $\mathbb{U}^1_{A,1}$ to $\mathbb{U}^1_{A,13}$ cannot have a phase portrait possessing an infinite saddle-node of type $\overline{\binom{0}{2}}SN$ as an evolution, since each one of them has only one infinite singularity. Analogously, phase portraits $\mathbb{U}^1_{A,14}$ to $\mathbb{U}^1_{A,18}$ cannot have a phase portrait possessing an infinite saddle-node of type $\overline{\binom{0}{2}}SN$ as an evolution, since each one of them has three infinite singularities (which are nodes).

Phase portrait $\mathbb{U}^1_{A,19}$ has phase portraits $\mathbb{U}^2_{AB,1}$ and $\mathbb{U}^2_{AB,2}$ as evolution (see Figure 3.1, where the arrows starting from the phase portrait $\mathbb{U}^1_{A,19}$ and pointing towards the phase portraits $\mathbb{U}^2_{AB,1}$ and $\mathbb{U}^2_{AB,2}$ indicate that these last two phase portraits are evolution of the phase portrait $\mathbb{U}^1_{A,19}$). After bifurcation we get phase portrait $\mathbb{U}^1_{A,1}$, in both cases, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear (split into two complex singularities). In Figure 3.1 we present the corresponding unfoldings on the right-hand side of the *codimension two** phase portraits.

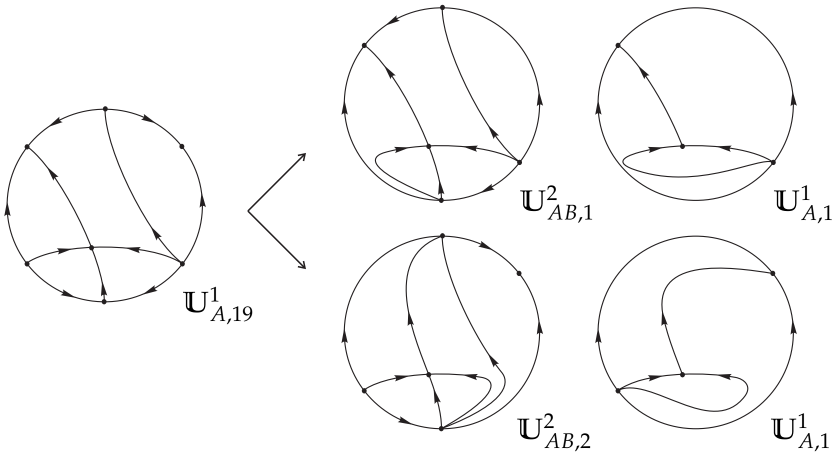


Figure 3.1: Unstable systems $\mathbb{U}^2_{AB,1}$ and $\mathbb{U}^2_{AB,2}$.

Note that $\mathbb{U}^1_{A,19}$ possesses two pairs of infinite nodes and only one pair of infinite saddles, so from $\mathbb{U}^1_{A,19}$ there are only two ways of obtaining a phase portrait possessing an infinite saddle-node of type $\overline{\binom{0}{2}}SN$, and these cases are represented exactly by the phase portraits $\mathbb{U}^2_{AB,1}$ and $\mathbb{U}^2_{AB,2}$ from Figure 3.1. From now on, we will always omit the proof of the nonexistence of other cases apart from those ones that we discuss by words or by presenting in figures, since the argument of nonexistence is in general quite simple.

Before we continue with the study of the remaining *codimension one** phase portraits, we highlight that it is very important to have the "structure" of all the figures very well understood, since the proofs of Theorems 1.6 and 1.7 require and are done based on several figures. So, in this paragraph we discuss about it. In the next cases, when from a *codimension one** phase portrait we have more than one *codimension two** phase portraits which are evolution of the *codimension one** phase portrait, we will present figures with the same "structure" of Figure 3.1. More precisely, all the arrows that appear starting from an unstable phase portrait of *codimension one** will have the same meaning as explained for Figure 3.1, i.e., they will point towards the phase portraits of *codimension two** which are evolution of the respective *codimension one** phase portrait. Moreover, we will present the corresponding unfoldings on the right-hand side of the *codimension two** phase portraits. On the other hand, when from a *codimension one** phase portrait we have only one *codimension two** phase portrait which is an evolution of the *codimension one** phase portrait, we will present figures like Figure 3.7, for instance, where on the left-hand side we have a *codimension one** phase portrait, on the center we have the corresponding *codimension two** phase portrait and on the right-hand side we have the respective unfolding of the *codimension two** phase portrait.

Phase portrait $\mathbb{U}^1_{A,20}$ has phase portraits $\mathbb{U}^2_{AB,3}$ and $\mathbb{U}^2_{AB,4}$ as evolution (see Figure 3.2). After bifurcation we get phase portrait $\mathbb{U}^1_{A,1}$, in both cases, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

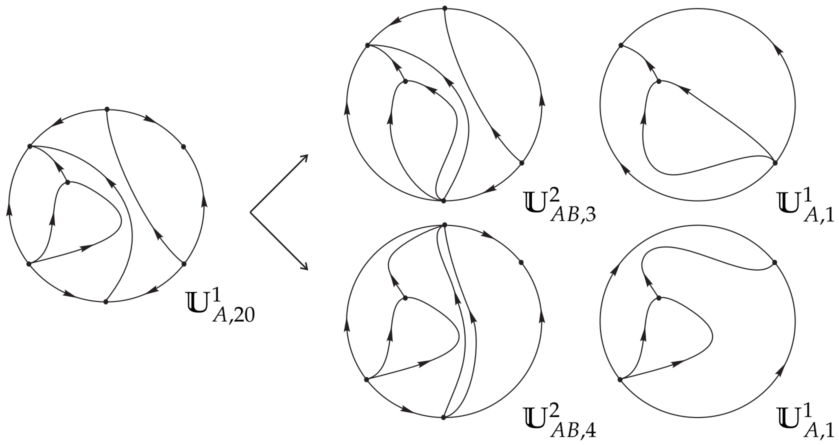


Figure 3.2: Unstable systems $\mathbb{U}^2_{AB,3}$ and $\mathbb{U}^2_{AB,4}$.

Phase portrait $\mathbb{U}^1_{A,21}$ has phase portraits $\mathbb{U}^2_{AB,5}$ and $\mathbb{U}^2_{AB,6}$ as evolution (see Figure 3.3). After bifurcation we get phase portrait $\mathbb{U}^1_{A,1}$, in both cases, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

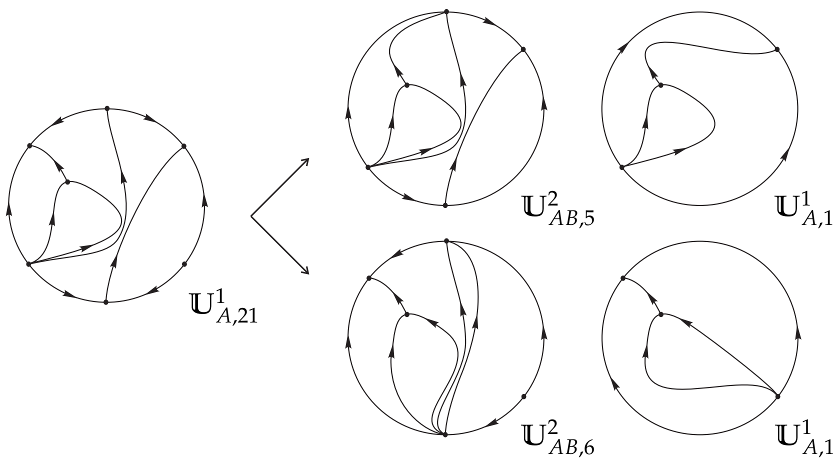


Figure 3.3: Unstable systems $\mathbb{U}^2_{AB,5}$ and $\mathbb{U}^2_{AB,6}$.

Phase portrait $\mathbb{U}^1_{A,22}$ has phase portraits $\mathbb{U}^2_{AB,7}$ and $\mathbb{U}^2_{AB,8}$ as evolution (see Figure 3.4). After bifurcation we get phase portrait $\mathbb{U}^1_{A,2}$, in both cases, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

Phase portrait $\mathbb{U}^1_{A,23}$ has phase portraits $\mathbb{U}^2_{AB,9}$ and $\mathbb{U}^2_{AB,10}$ as evolution (see Figure 3.5). After bifurcation we get phase portrait $\mathbb{U}^1_{A,3}$, in both cases, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

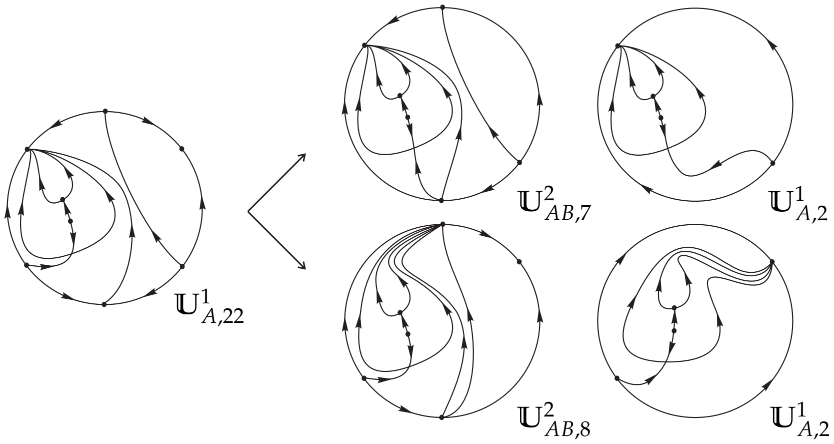


Figure 3.4: Unstable systems $\mathbb{U}^2_{AB,7}$ and $\mathbb{U}^2_{AB,8}$.

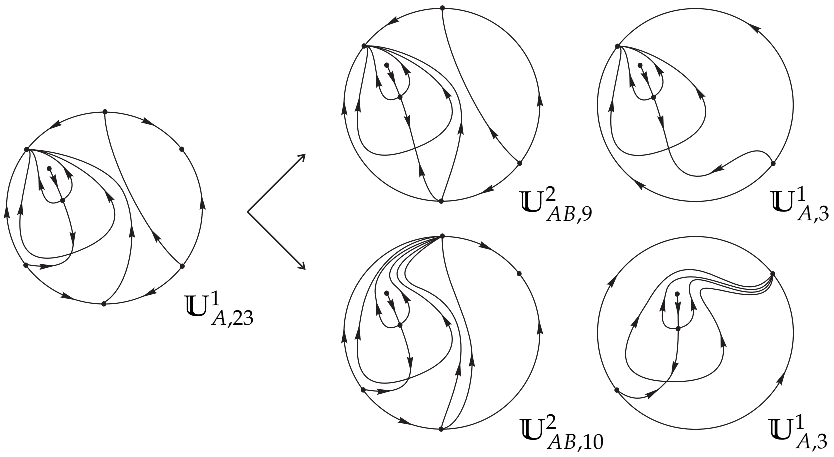


Figure 3.5: Unstable systems $\mathbb{U}^2_{AB,9}$ and $\mathbb{U}^2_{AB,10}$.

Phase portrait $\mathbb{U}^1_{A,24}$ has phase portraits $\mathbb{U}^2_{AB,11}$ and $\mathbb{U}^2_{AB,12}$ as evolution (see Figure 3.6). After bifurcation we get phase portrait $\mathbb{U}^1_{A,4}$, in both cases, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

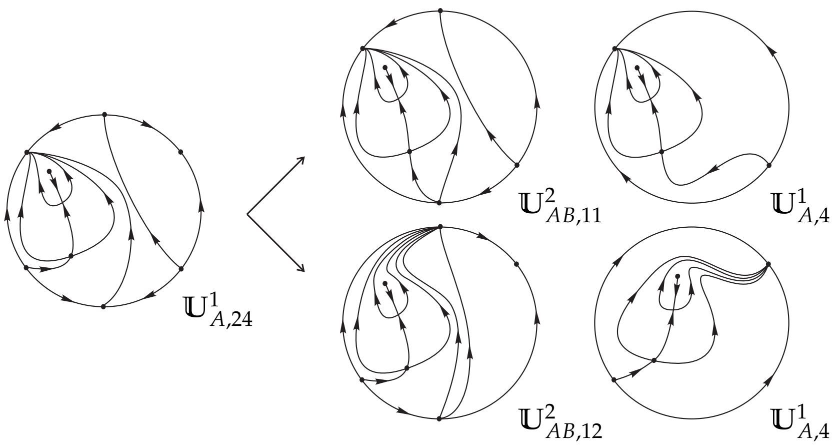


Figure 3.6: Unstable systems $\mathbb{U}^2_{AB,11}$ and $\mathbb{U}^2_{AB,12}$.

Phase portrait $\mathbb{U}^1_{A,25}$ has phase portrait $\mathbb{U}^2_{AB,13}$ as an evolution (see Figure 3.7). After

bifurcation we get phase portrait $\mathbb{U}^1_{A,5}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,25}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,1}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,107}$ of *codimension one**, see Figure 3.8. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,1}$ unfolds in $\mathbb{U}^1_{A,5}$.

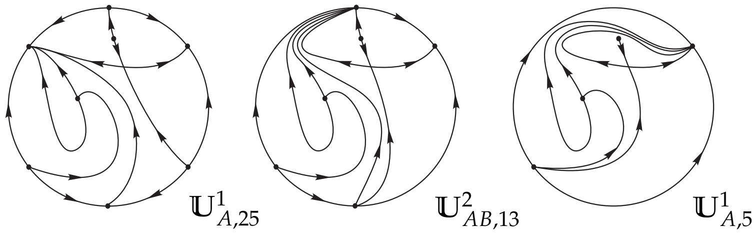


Figure 3.7: Unstable system $\mathbb{U}^2_{AB,13}$.

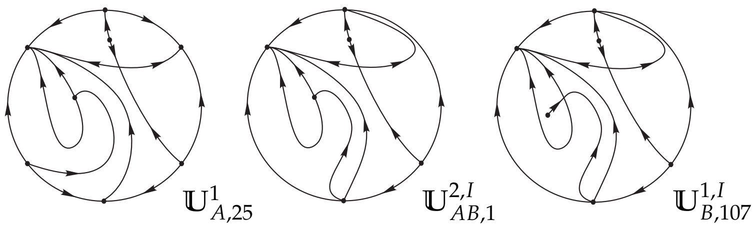


Figure 3.8: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,1}$.

Phase portrait $\mathbb{U}^1_{A,26}$ has phase portrait $\mathbb{U}^2_{AB,14}$ as an evolution (see Figure 3.9). After bifurcation we get phase portrait $\mathbb{U}^1_{A,5}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,26}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,2}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,107}$ of *codimension one**, see Figure 3.10. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,2}$ unfolds in $\mathbb{U}^1_{A,5}$.

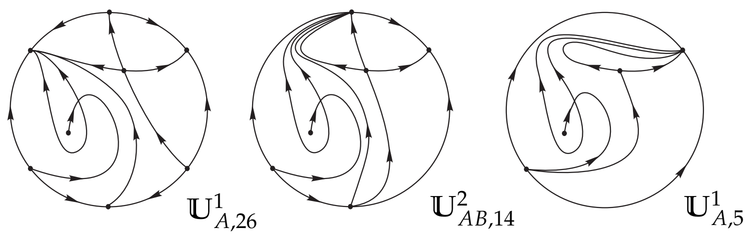


Figure 3.9: Unstable system $\mathbb{U}^2_{AB,14}$.

Phase portrait $\mathbb{U}^1_{A,27}$ has phase portrait $\mathbb{U}^2_{AB,15}$ as an evolution (see Figure 3.11). After bifurcation we get phase portrait $\mathbb{U}^1_{A,2}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,27}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,3}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,108}$ of *codimension one**, see Figure 3.12. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,3}$ unfolds in $\mathbb{U}^1_{A,2}$.

Phase portrait $\mathbb{U}^1_{A,28}$ has phase portrait $\mathbb{U}^2_{AB,16}$ as an evolution (see Figure 3.13). After

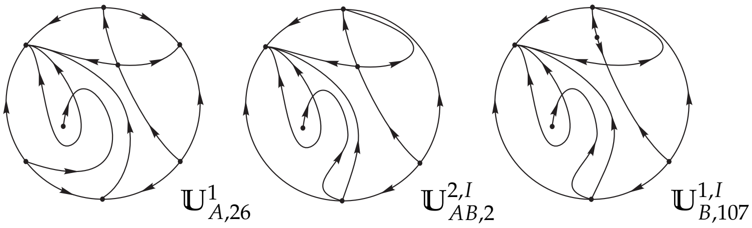


Figure 3.10: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,2}$.

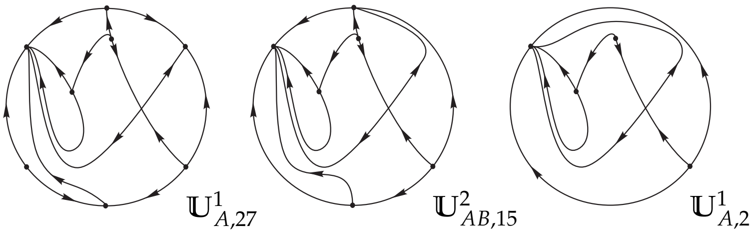


Figure 3.11: Unstable system $\mathbb{U}^2_{AB,15}$.

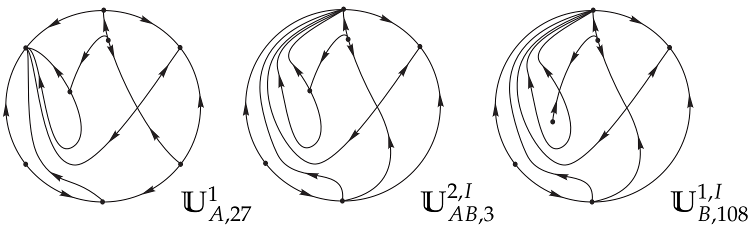


Figure 3.12: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,3}$.

bifurcation we get phase portrait $\mathbb{U}^1_{A,3}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,28}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,4}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,108}$ of *codimension one**, see Figure 3.14. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,4}$ unfolds in $\mathbb{U}^1_{A,3}$.

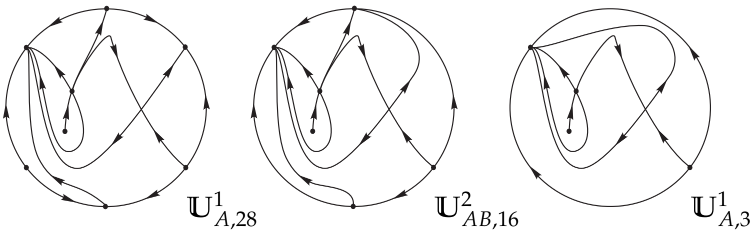


Figure 3.13: Unstable system $\mathbb{U}^2_{AB,16}$.

Phase portrait $\mathbb{U}^1_{A,29}$ has phase portrait $\mathbb{U}^2_{AB,17}$ as an evolution (see Figure 3.15). After bifurcation we get phase portrait $\mathbb{U}^1_{A,5}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,29}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,5}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,109}$ of *codimension one**, see Figure 3.16. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,5}$ unfolds in $\mathbb{U}^1_{A,5}$.

Phase portrait $\mathbb{U}^1_{A,30}$ has phase portrait $\mathbb{U}^2_{AB,18}$ as an evolution (see Figure 3.17). After

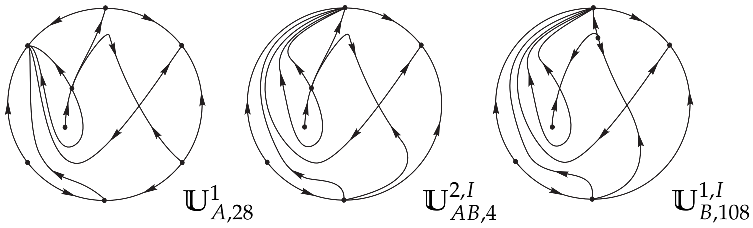


Figure 3.14: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,4}$.

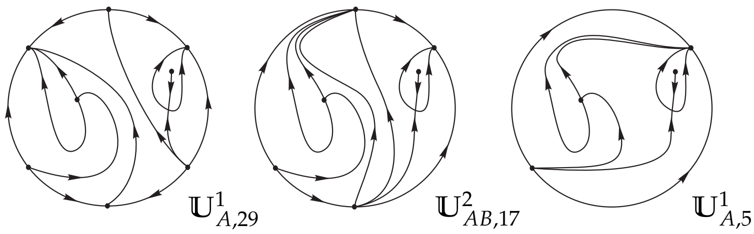


Figure 3.15: Unstable system $\mathbb{U}^2_{AB,17}$.

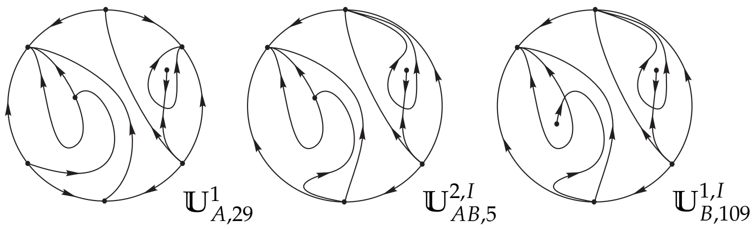


Figure 3.16: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,5}$.

bifurcation we get phase portrait $\mathbb{U}^1_{A,5}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,30}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,6}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,109}$ of *codimension one**, see Figure 3.18. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,6}$ unfolds in $\mathbb{U}^1_{A,5}$.

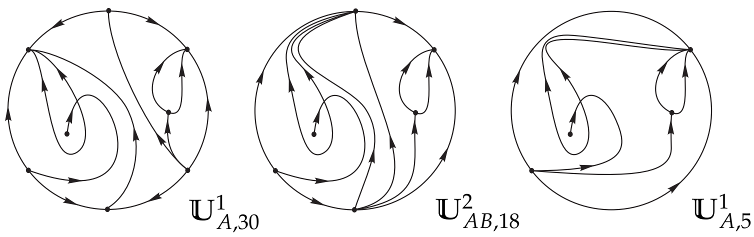


Figure 3.17: Unstable system $\mathbb{U}^2_{AB,18}$.

Phase portrait $\mathbb{U}^1_{A,31}$ has phase portrait $\mathbb{U}^2_{AB,19}$ (see Figure 3.19) as an evolution. After bifurcation we get phase portrait $\mathbb{U}^1_{A,2}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,31}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,7}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,110}$ of *codimension one**, see Figure 3.20. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,7}$ unfolds in $\mathbb{U}^1_{A,2}$.

Phase portrait $\mathbb{U}^1_{A,32}$ has phase portrait $\mathbb{U}^2_{AB,20}$ as an evolution (see Figure 3.21). After

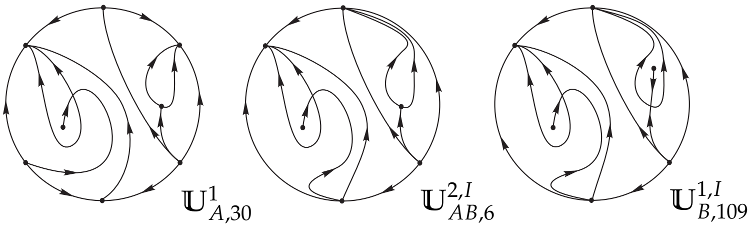


Figure 3.18: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,6}$.

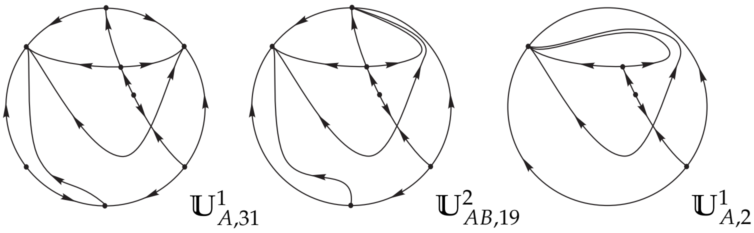


Figure 3.19: Unstable system $\mathbb{U}^2_{AB,19}$.

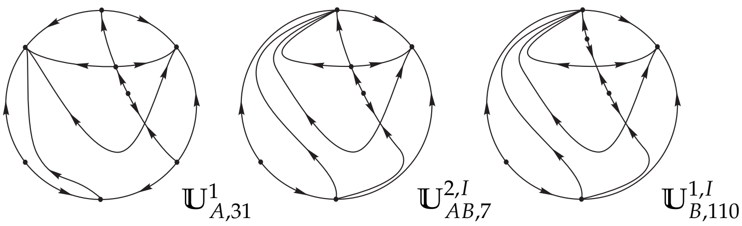


Figure 3.20: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,7}$.

bifurcation we get phase portrait $\mathbb{U}^1_{A,3}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,32}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,8}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,110}$ of *codimension one**, see Figure 3.22. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,8}$ unfolds in $\mathbb{U}^1_{A,3}$.

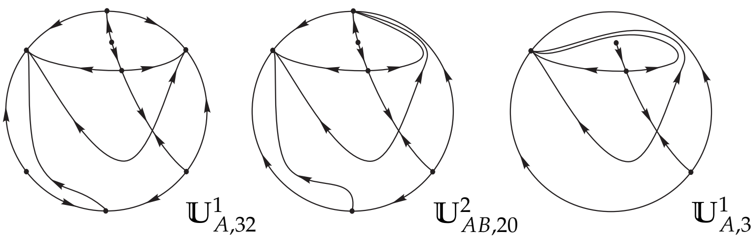


Figure 3.21: Unstable system $\mathbb{U}^2_{AB,20}$.

Phase portrait $\mathbb{U}^1_{A,33}$ has phase portrait $\mathbb{U}^2_{AB,21}$ as an evolution (see Figure 3.23). After bifurcation we get phase portrait $\mathbb{U}^1_{A,4}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,33}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,9}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,110}$ of *codimension one**, see Figure 3.24. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,9}$ unfolds in $\mathbb{U}^1_{A,4}$.

Phase portrait $\mathbb{U}^1_{A,34}$ has phase portrait $\mathbb{U}^2_{AB,22}$ as an evolution (see Figure 3.25). After

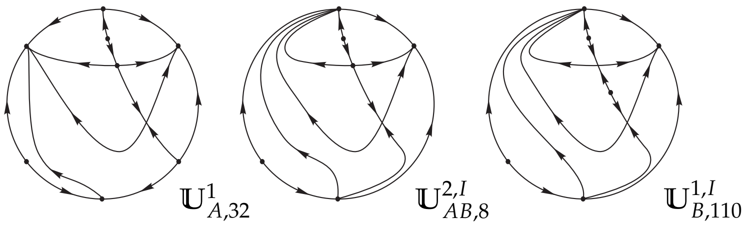


Figure 3.22: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,8}$.

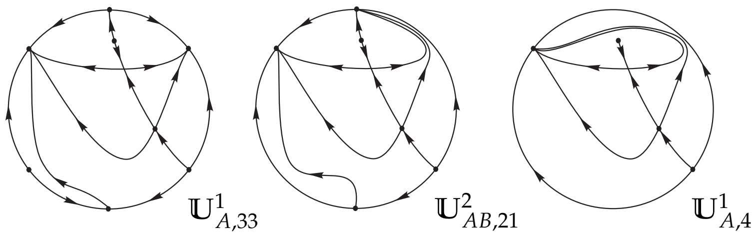


Figure 3.23: Unstable system $\mathbb{U}^2_{AB,21}$.

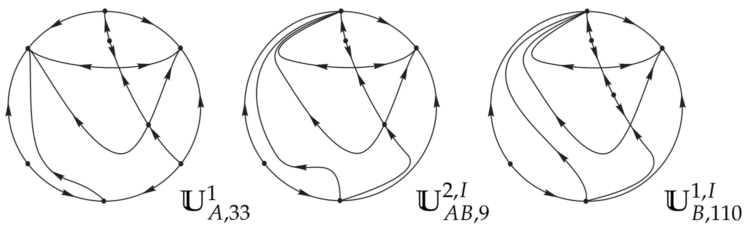


Figure 3.24: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,9}$.

bifurcation we get phase portrait $\mathbb{U}^1_{A,5}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,34}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,10}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,111}$ of *codimension one**, see Figure 3.26. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,10}$ unfolds in $\mathbb{U}^1_{A,5}$.

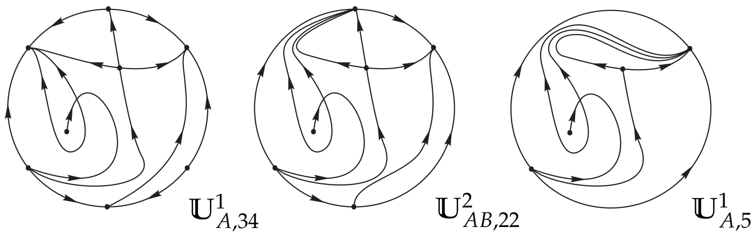


Figure 3.25: Unstable system $\mathbb{U}^2_{AB,22}$.

Phase portrait $\mathbb{U}^1_{A,35}$ has phase portrait $\mathbb{U}^2_{AB,23}$ as an evolution (see Figure 3.27). After bifurcation we get phase portrait $\mathbb{U}^1_{A,5}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,35}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,11}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,111}$ of *codimension one**, see Figure 3.28. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,11}$ unfolds in $\mathbb{U}^1_{A,5}$.

Phase portrait $\mathbb{U}^1_{A,36}$ has phase portraits $\mathbb{U}^2_{AB,24}$ and $\mathbb{U}^2_{AB,25}$ as evolution (see Figure 3.29).

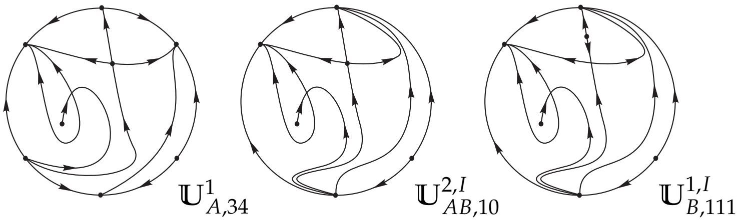


Figure 3.26: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,10}$.

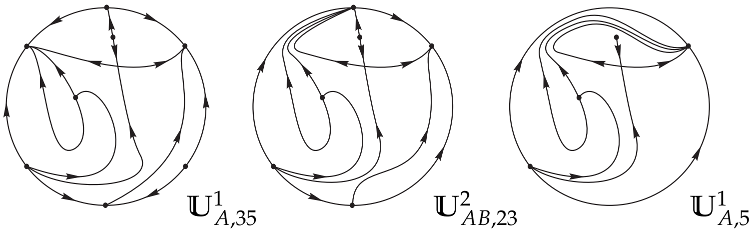


Figure 3.27: Unstable system $\mathbb{U}^2_{AB,23}$.

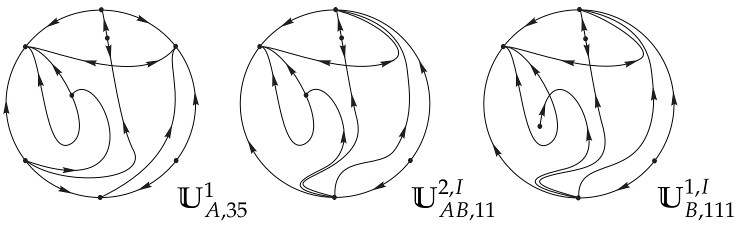


Figure 3.28: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,11}$.

After bifurcation we get phase portrait $\mathbb{U}^1_{A,9}$, in both cases, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

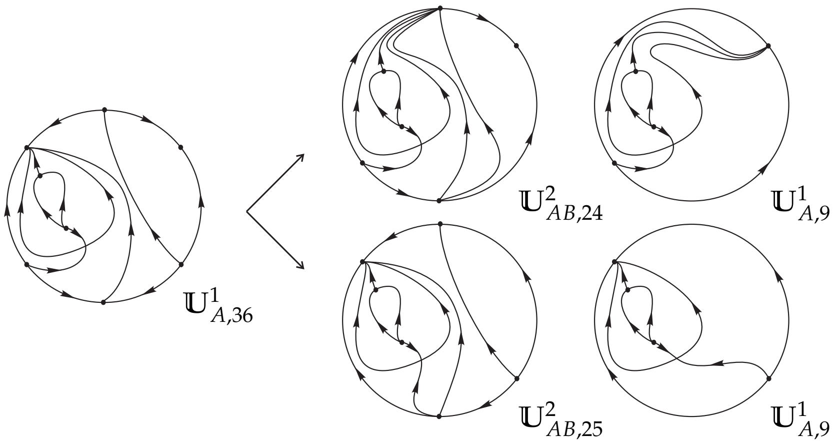


Figure 3.29: Unstable systems $\mathbb{U}^2_{AB,24}$ and $\mathbb{U}^2_{AB,25}$.

Phase portrait $\mathbb{U}^1_{A,37}$ has phase portraits $\mathbb{U}^2_{AB,26}$ and $\mathbb{U}^2_{AB,27}$ as evolution (see Figure 3.30). After bifurcation we get phase portrait $\mathbb{U}^1_{A,10}$, in both cases, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

Phase portrait $\mathbb{U}^1_{A,38}$ has phase portraits $\mathbb{U}^2_{AB,28}$ and $\mathbb{U}^2_{AB,29}$ as evolution (see Figure 3.31). After bifurcation we get phase portrait $\mathbb{U}^1_{A,8}$, in both cases, by making the infinite saddle-node

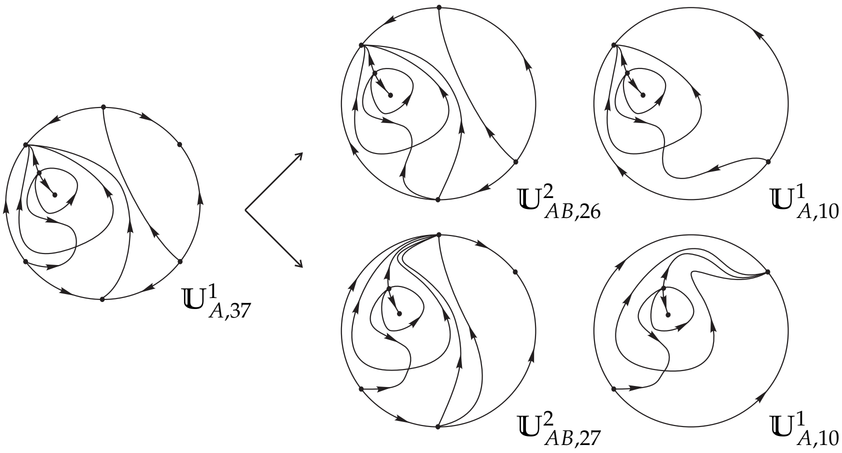


Figure 3.30: Unstable systems $\mathbb{U}^2_{AB,26}$ and $\mathbb{U}^2_{AB,27}$.

$\overline{\binom{0}{2}}SN$ disappear.

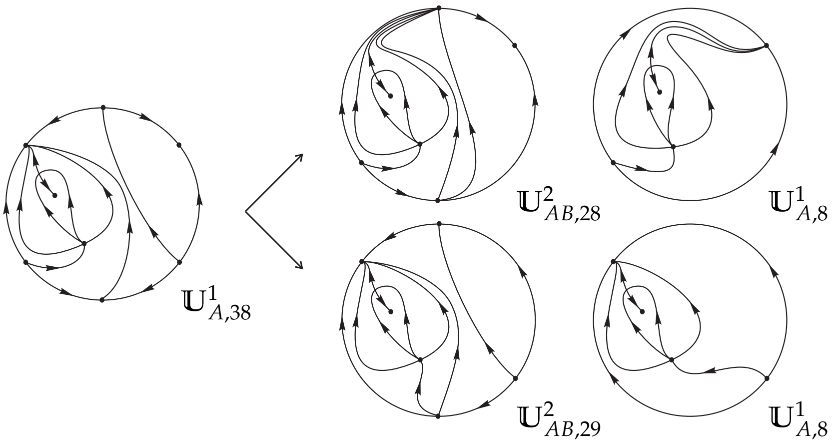


Figure 3.31: Unstable systems $\mathbb{U}^2_{AB,28}$ and $\mathbb{U}^2_{AB,29}$.

Phase portrait $\mathbb{U}^1_{A,39}$ has phase portrait $\mathbb{U}^2_{AB,30}$ as an evolution (see Figure 3.32). After bifurcation we get phase portrait $\mathbb{U}^1_{A,6}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,39}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,12}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,112}$ of *codimension one**, see Figure 3.33. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,12}$ unfolds in $\mathbb{U}^1_{A,6}$.

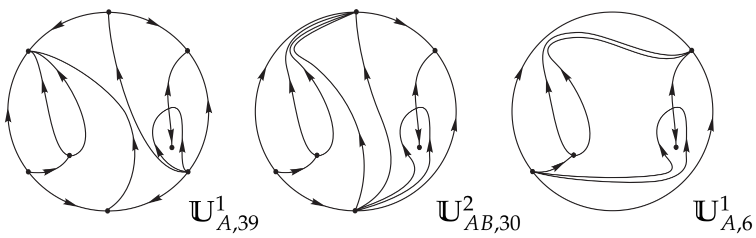


Figure 3.32: Unstable system $\mathbb{U}^2_{AB,30}$.

Phase portrait $\mathbb{U}^1_{A,40}$ has phase portrait $\mathbb{U}^2_{AB,31}$ as an evolution (see Figure 3.34). After

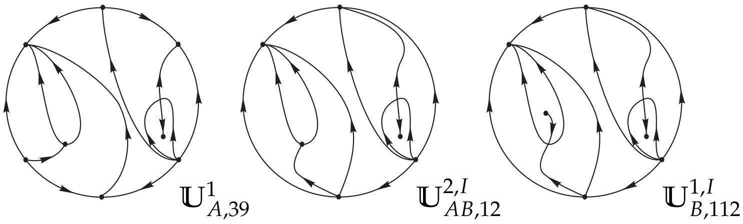


Figure 3.33: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,12}$.

bifurcation we get phase portrait $\mathbb{U}^1_{A,6}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,40}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,13}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,113}$ of *codimension one**, see Figure 3.35. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,13}$ unfolds in $\mathbb{U}^1_{A,6}$.

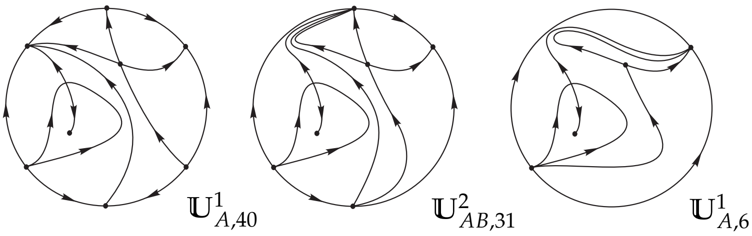


Figure 3.34: Unstable system $\mathbb{U}^2_{AB,31}$.

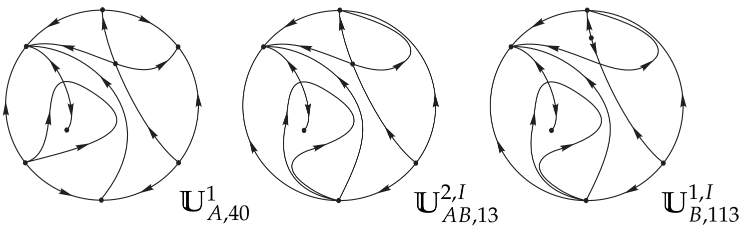


Figure 3.35: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,13}$.

Phase portrait $\mathbb{U}^1_{A,41}$ has phase portrait $\mathbb{U}^2_{AB,32}$ as an evolution (see Figure 3.36). After bifurcation we get phase portrait $\mathbb{U}^1_{A,6}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,41}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,14}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,113}$ of *codimension one**, see Figure 3.37. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,14}$ unfolds in $\mathbb{U}^1_{A,6}$.

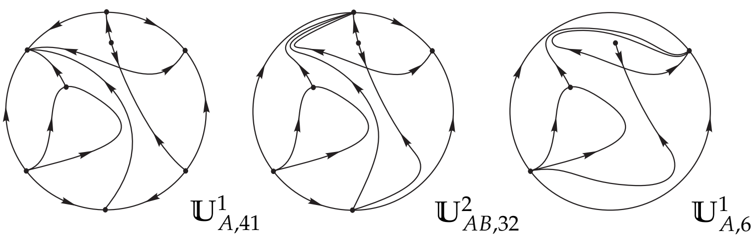


Figure 3.36: Unstable system $\mathbb{U}^2_{AB,32}$.

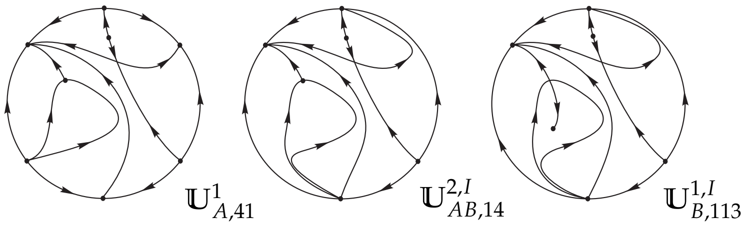


Figure 3.37: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,14}$.

Phase portrait $\mathbb{U}^1_{A,42}$ has phase portrait $\mathbb{U}^2_{AB,33}$ as an evolution (see Figure 3.38). After bifurcation we get phase portrait $\mathbb{U}^1_{A,9}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,42}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,15}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,114}$ of *codimension one**, see Figure 3.39. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,15}$ unfolds in $\mathbb{U}^1_{A,9}$.

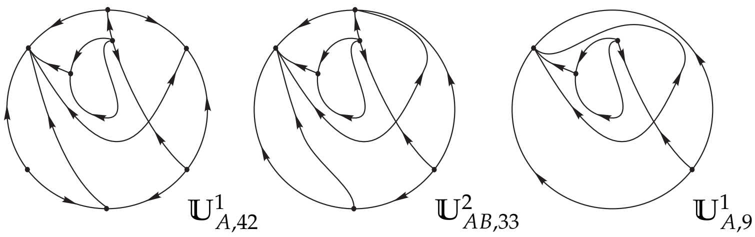


Figure 3.38: Unstable system $\mathbb{U}^2_{AB,33}$.

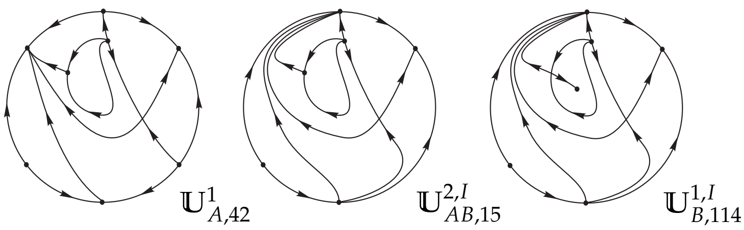


Figure 3.39: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,15}$.

Phase portrait $\mathbb{U}^1_{A,43}$ has phase portrait $\mathbb{U}^2_{AB,34}$ as an evolution (see Figure 3.40). After bifurcation we get phase portrait $\mathbb{U}^1_{A,10}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,43}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,16}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,114}$ of *codimension one**, see Figure 3.41. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,16}$ unfolds in $\mathbb{U}^1_{A,10}$.

Phase portrait $\mathbb{U}^1_{A,44}$ has phase portrait $\mathbb{U}^2_{AB,35}$ as an evolution (see Figure 3.42). After bifurcation we get phase portrait $\mathbb{U}^1_{A,6}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,44}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,17}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,115}$ of *codimension one**, see Figure 3.43. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,17}$ unfolds in $\mathbb{U}^1_{A,6}$.

Phase portrait $\mathbb{U}^1_{A,45}$ has phase portrait $\mathbb{U}^2_{AB,36}$ as an evolution (see Figure 3.44). After

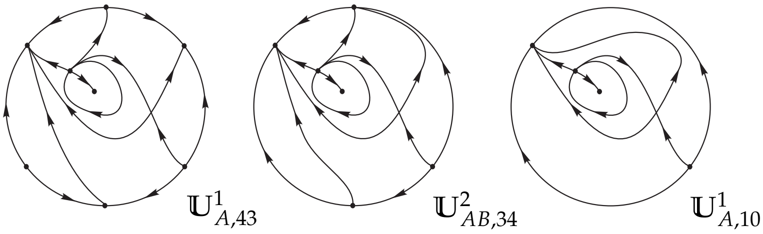


Figure 3.40: Unstable system $\mathbb{U}^2_{AB,34}$.

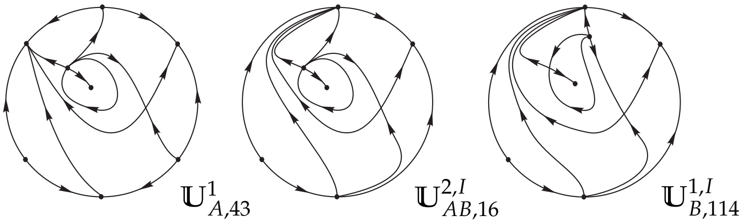


Figure 3.41: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,16}$.

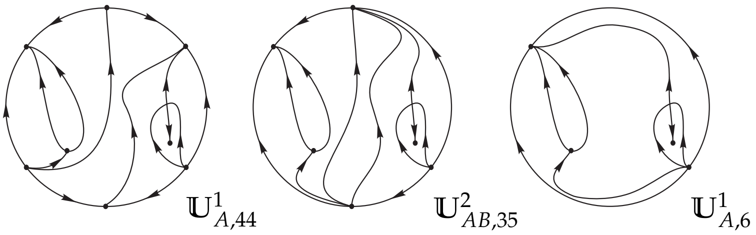


Figure 3.42: Unstable system $\mathbb{U}^2_{AB,35}$.

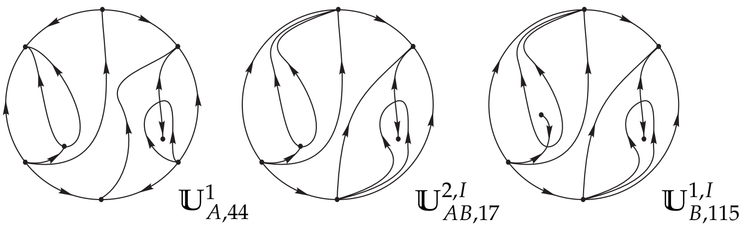


Figure 3.43: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,17}$.

bifurcation we get phase portrait $\mathbb{U}^1_{A,6}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,45}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,18}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,116}$ of *codimension one**, see Figure 3.45. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,18}$ unfolds in $\mathbb{U}^1_{A,6}$.

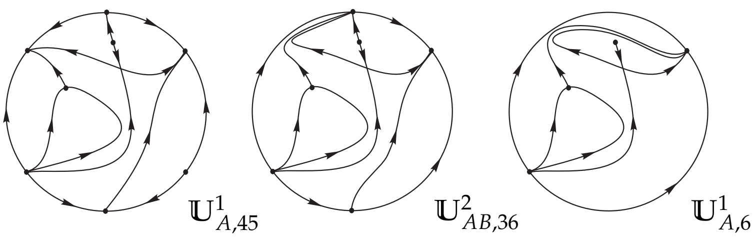


Figure 3.44: Unstable system $\mathbb{U}^2_{AB,36}$.

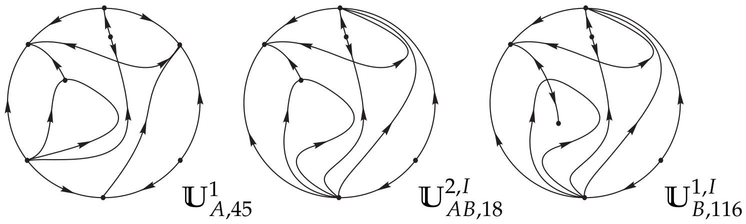


Figure 3.45: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,18}$.

Phase portrait $\mathbb{U}^1_{A,46}$ has phase portrait $\mathbb{U}^2_{AB,37}$ as an evolution (see Figure 3.46). After bifurcation we get phase portrait $\mathbb{U}^1_{A,6}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,46}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,19}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,116}$ of *codimension one**, see Figure 3.47. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,19}$ unfolds in $\mathbb{U}^1_{A,6}$.

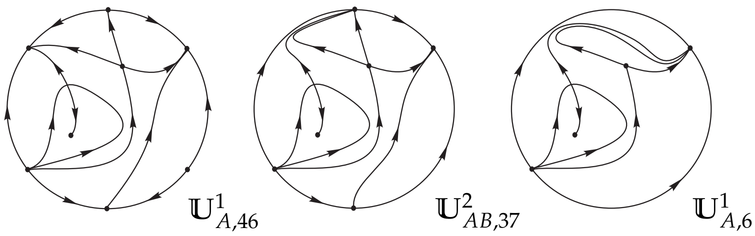


Figure 3.46: Unstable system $\mathbb{U}^2_{AB,37}$.

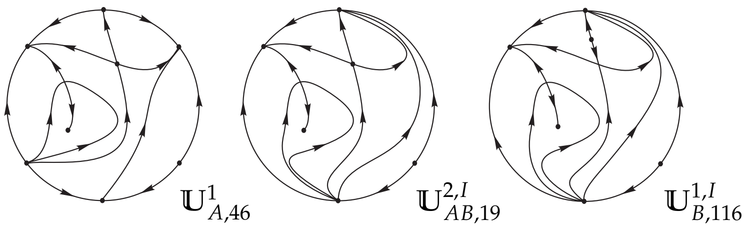


Figure 3.47: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,19}$.

Phase portrait $\mathbb{U}^1_{A,47}$ has phase portraits $\mathbb{U}^2_{AB,38}$ and $\mathbb{U}^2_{AB,39}$ as evolution (see Figure 3.48). After bifurcation we get phase portrait $\mathbb{U}^1_{A,7}$, in both cases, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

Phase portrait $\mathbb{U}^1_{A,48}$ has phase portraits $\mathbb{U}^2_{AB,40}$ and $\mathbb{U}^2_{AB,41}$ as evolution (see Figure 3.49). After bifurcation we get phase portrait $\mathbb{U}^1_{A,7}$, in both cases, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

Phase portrait $\mathbb{U}^1_{A,50}$ has phase portraits $\mathbb{U}^2_{AB,42}$ and $\mathbb{U}^2_{AB,43}$ as evolution (see Figure 3.50). After bifurcation we get phase portrait $\mathbb{U}^1_{A,7}$, in both cases, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

Phase portrait $\mathbb{U}^1_{A,51}$ has phase portraits $\mathbb{U}^2_{AB,44}$ and $\mathbb{U}^2_{AB,45}$ as evolution (see Figure 3.51). After bifurcation we get phase portrait $\mathbb{U}^1_{A,7}$, in both cases, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

Phase portrait $\mathbb{U}^1_{A,52}$ has phase portraits $\mathbb{U}^2_{AB,46}$ and $\mathbb{U}^2_{AB,47}$ as evolution (see Figure 3.52). After bifurcation we get phase portrait $\mathbb{U}^1_{A,7}$, in both cases, by making the infinite saddle-node

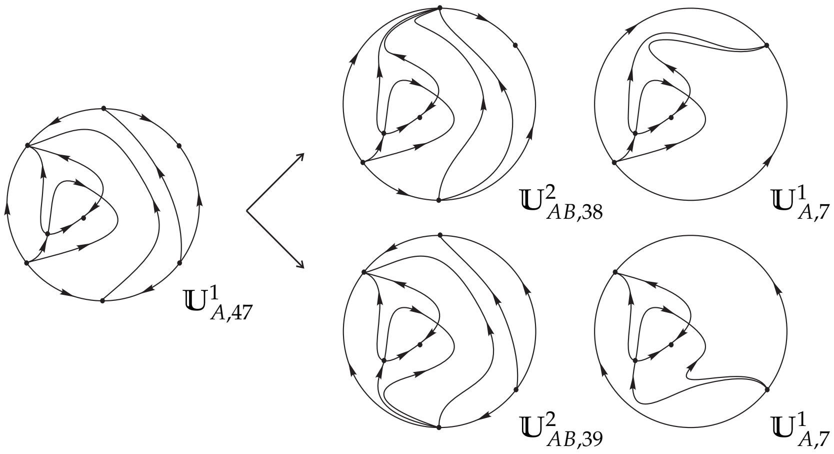


Figure 3.48: Unstable systems $\mathbb{U}^2_{AB,38}$ and $\mathbb{U}^2_{AB,39}$.

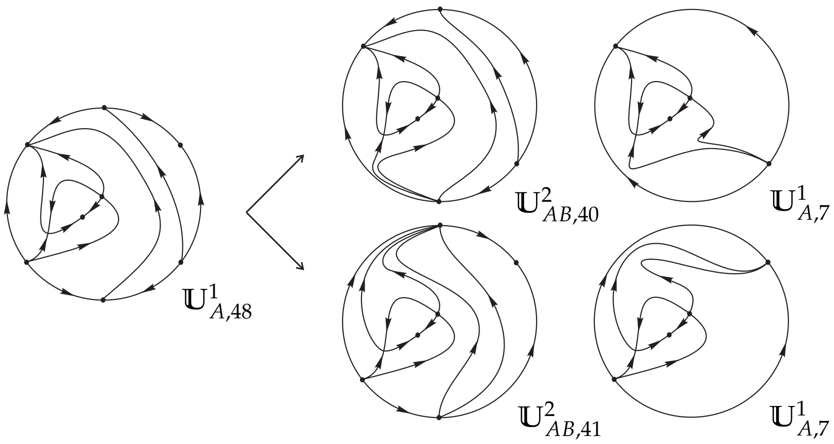


Figure 3.49: Unstable systems $\mathbb{U}^2_{AB,40}$ and $\mathbb{U}^2_{AB,41}$.

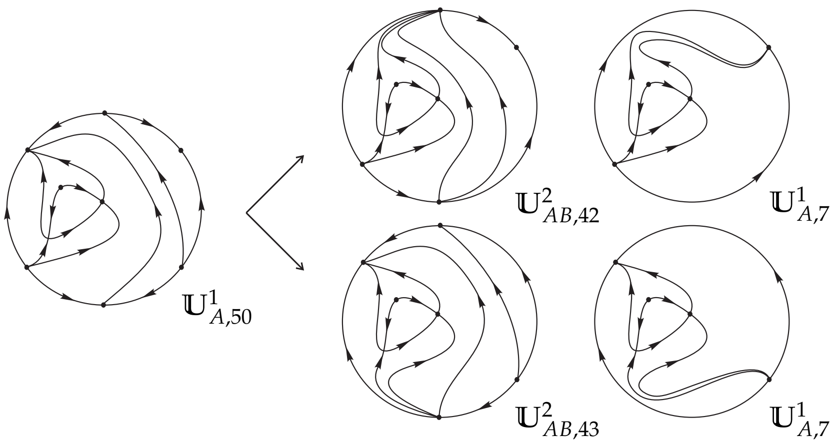


Figure 3.50: Unstable systems $\mathbb{U}^2_{AB,42}$ and $\mathbb{U}^2_{AB,43}$.

$\overline{\binom{0}{2}}SN$ disappear.

Phase portrait $\mathbb{U}^1_{A,53}$ has phase portraits $\mathbb{U}^2_{AB,48}$ and $\mathbb{U}^2_{AB,49}$ as evolution (see Figure 3.53). After bifurcation we get phase portrait $\mathbb{U}^1_{A,7}$, in both cases, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

Phase portrait $\mathbb{U}^1_{A,54}$ has phase portrait $\mathbb{U}^2_{AB,50}$ as an evolution (see Figure 3.54). After

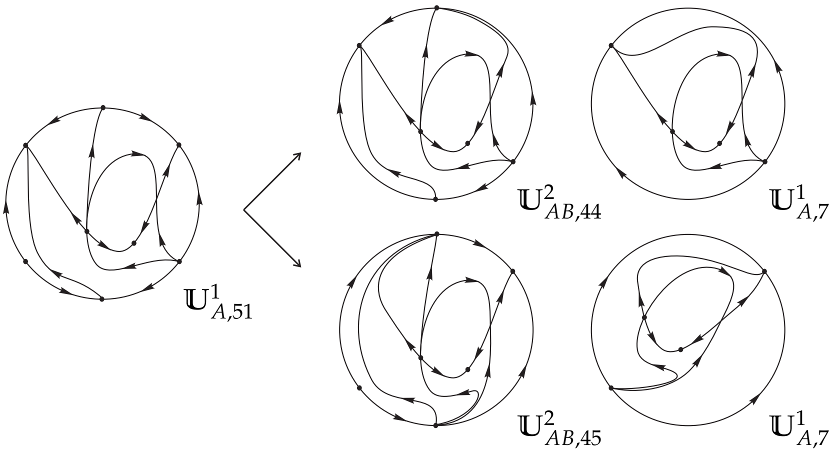


Figure 3.51: Unstable systems $\mathbb{U}^2_{AB,44}$ and $\mathbb{U}^2_{AB,45}$.

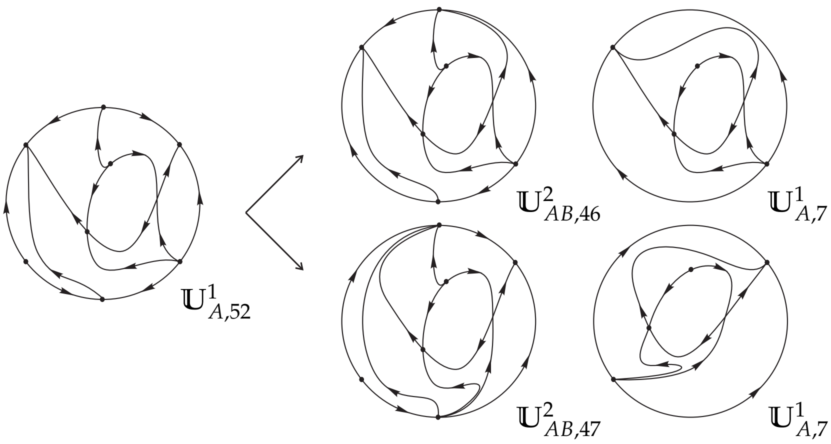


Figure 3.52: Unstable systems $\mathbb{U}^2_{AB,46}$ and $\mathbb{U}^2_{AB,47}$.

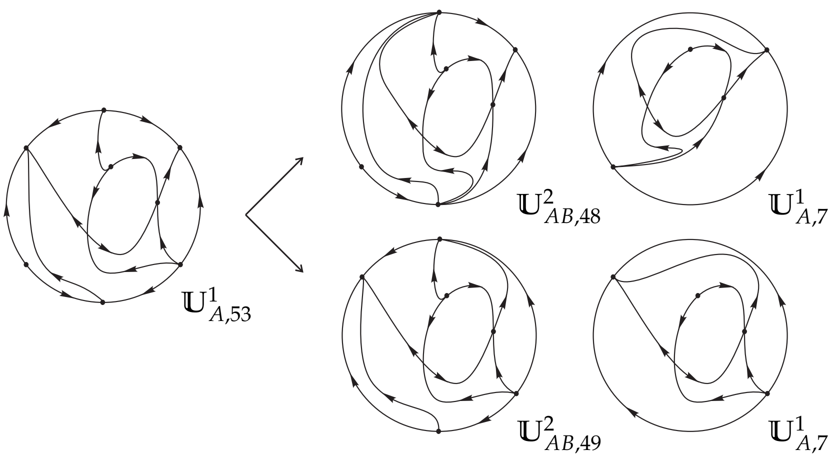


Figure 3.53: Unstable systems $\mathbb{U}^2_{AB,48}$ and $\mathbb{U}^2_{AB,49}$.

bifurcation we get phase portrait $\mathbb{U}^1_{A,6}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,54}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,20}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,117}$ of *codimension one**, see

Figure 3.55. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,20}$ unfolds in $\mathbb{U}^{1}_{A,6}$.

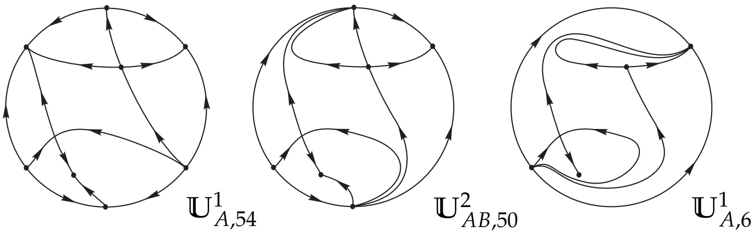


Figure 3.54: Unstable system $\mathbb{U}^{2}_{AB,50}$.

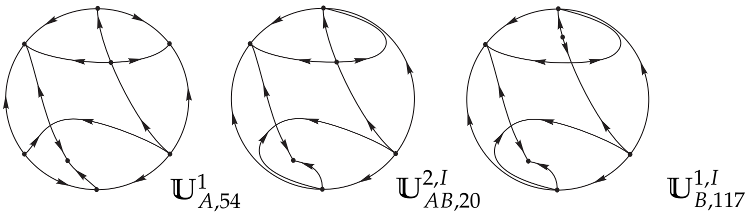


Figure 3.55: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,20}$.

Phase portrait $\mathbb{U}^{1}_{A,55}$ has phase portraits $\mathbb{U}^{2}_{AB,51}$ and $\mathbb{U}^{2}_{AB,52}$ as evolution (see Figure 3.56). After bifurcation we get phase portrait $\mathbb{U}^{1}_{A,7}$, in both cases, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

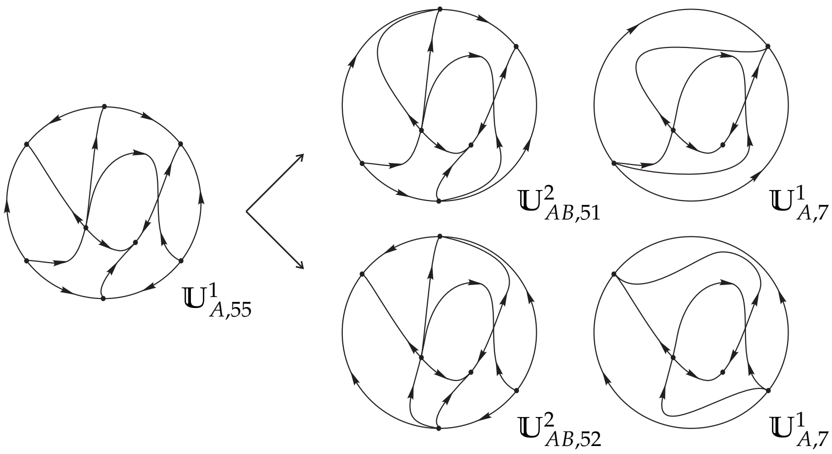


Figure 3.56: Unstable systems $\mathbb{U}^{2}_{AB,51}$ and $\mathbb{U}^{2}_{AB,52}$.

Phase portrait $\mathbb{U}^{1}_{A,56}$ has phase portrait $\mathbb{U}^{2}_{AB,53}$ as an evolution (see Figure 3.57). After bifurcation we get phase portrait $\mathbb{U}^{1}_{A,11}$, modulo limit cycle, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^{1}_{A,56}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,21}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,4}$ of *codimension one**, see Figure 3.58. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,21}$ also unfolds in an impossible phase portrait because after bifurcation we would get a limit cycle surrounding more than one finite singular points, and this is not possible in quadratic systems (see Lemma 3.14 from [6]).

Phase portrait $\mathbb{U}^{1}_{A,57}$ has phase portrait $\mathbb{U}^{2}_{AB,54}$ as an evolution (see Figure 3.59). After bifurcation we get phase portrait $\mathbb{U}^{1}_{A,12}$, modulo limit cycle, by making the infinite saddle-node

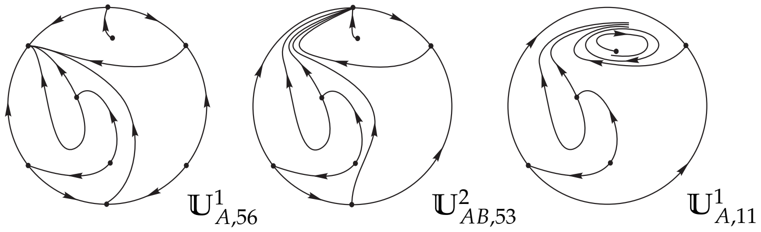


Figure 3.57: Unstable system $\mathbb{U}^2_{AB,53}$.

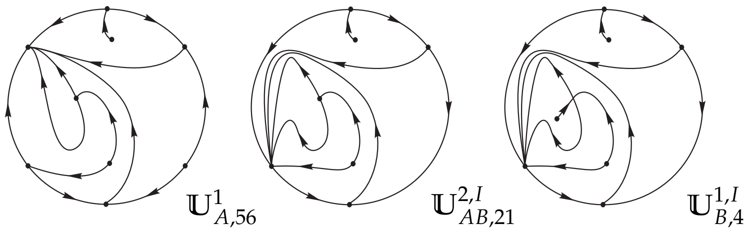


Figure 3.58: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,21}$.

$\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,57}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,22}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,4}$ of *codimension one**, see Figure 3.60. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,22}$ also unfolds in an impossible phase portrait, as in $\mathbb{U}^{2,I}_{AB,21}$.

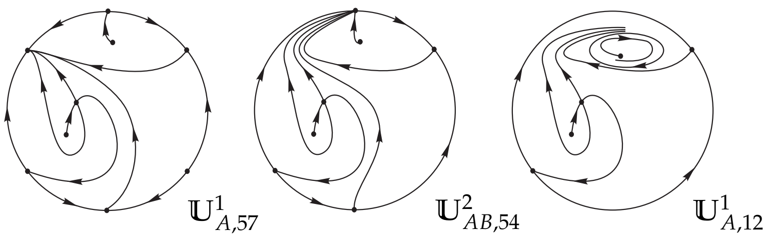


Figure 3.59: Unstable system $\mathbb{U}^2_{AB,54}$.

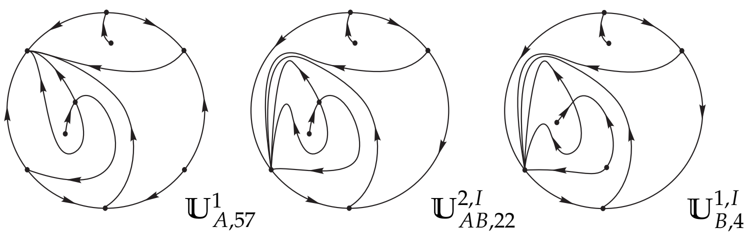


Figure 3.60: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,22}$.

Phase portrait $\mathbb{U}^1_{A,58}$ has phase portrait $\mathbb{U}^2_{AB,55}$ as an evolution (see Figure 3.61). After bifurcation we get phase portrait $\mathbb{U}^1_{A,12}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,58}$ has a second phase portrait which is not presented since it is topologically equivalent to $\mathbb{U}^2_{AB,55}$.

Phase portrait $\mathbb{U}^1_{A,59}$ has phase portraits $\mathbb{U}^2_{AB,56}$ and $\mathbb{U}^2_{AB,57}$ as evolution (see Figure 3.62). After bifurcation we get phase portrait $\mathbb{U}^1_{A,13}$, in both cases, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

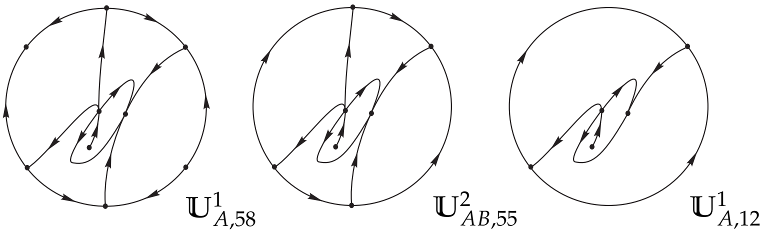


Figure 3.61: Unstable system $\mathbb{U}^2_{AB,55}$.

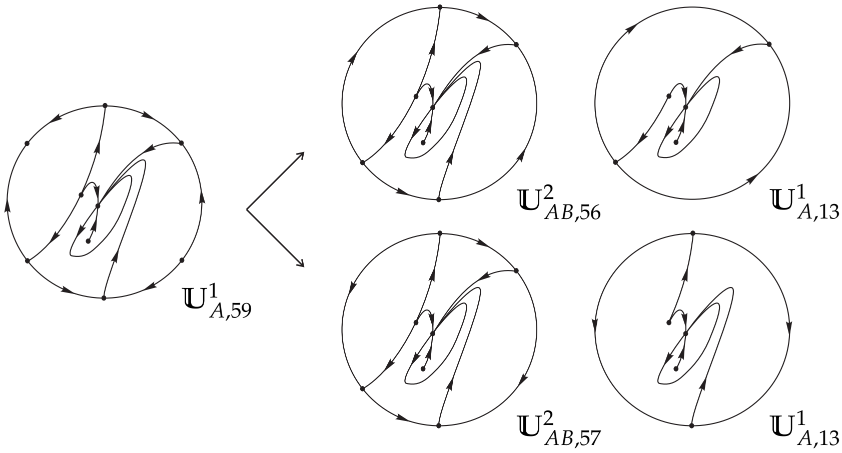


Figure 3.62: Unstable systems $\mathbb{U}^2_{AB,56}$ and $\mathbb{U}^2_{AB,57}$.

Phase portrait $\mathbb{U}^1_{A,60}$ has phase portrait $\mathbb{U}^2_{AB,58}$ as an evolution (see Figure 3.63). After bifurcation we get phase portrait $\mathbb{U}^1_{A,11}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,60}$ has a second phase portrait which is not presented since it is topologically equivalent to $\mathbb{U}^2_{AB,58}$.

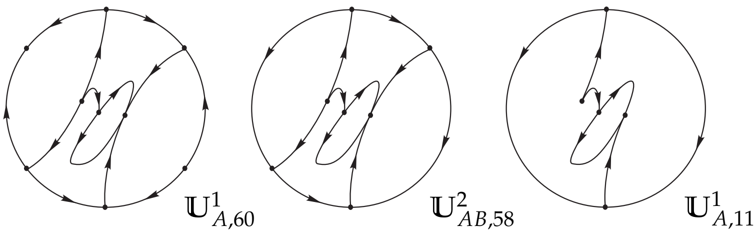


Figure 3.63: Unstable system $\mathbb{U}^2_{AB,58}$.

Phase portrait $\mathbb{U}^1_{A,61}$ has phase portraits $\mathbb{U}^2_{AB,59}$ and $\mathbb{U}^2_{AB,60}$ as evolution (see Figure 3.64). After bifurcation we get phase portraits $\mathbb{U}^1_{A,11}$ and $\mathbb{U}^1_{A,12}$, respectively, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

Phase portrait $\mathbb{U}^1_{A,62}$ has phase portrait $\mathbb{U}^2_{AB,61}$ as an evolution (see Figure 3.65). After bifurcation we get phase portrait $\mathbb{U}^1_{A,13}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,62}$ has a second phase portrait which is not presented since it is topologically equivalent to $\mathbb{U}^2_{AB,61}$.

Phase portrait $\mathbb{U}^1_{A,63}$ has phase portrait $\mathbb{U}^2_{AB,62}$ as an evolution (see Figure 3.66). After bifurcation we get phase portrait $\mathbb{U}^1_{A,11}$, modulo limit cycle, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,63}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,23}$ as an evolution. By

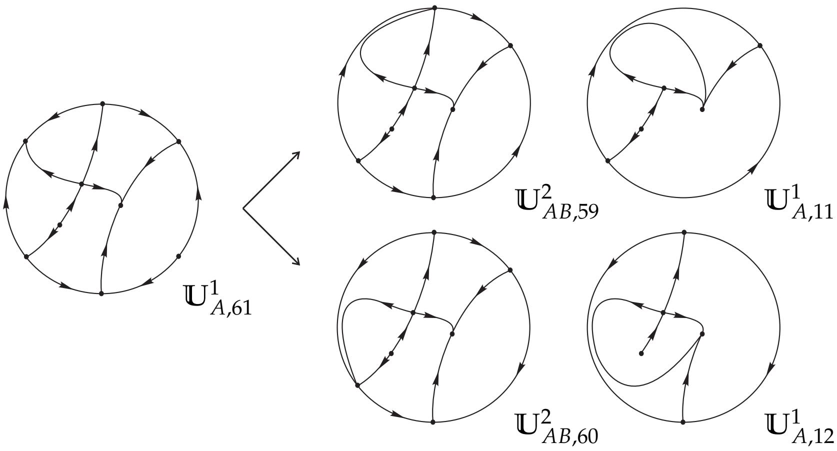


Figure 3.64: Unstable systems $\mathbb{U}^2_{AB,59}$ and $\mathbb{U}^2_{AB,60}$.

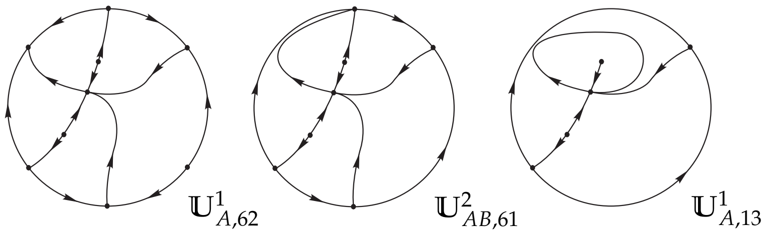


Figure 3.65: Unstable system $\mathbb{U}^2_{AB,61}$.

Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,5}$ of *codimension one**, see Figure 3.67. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,23}$ also unfolds in an impossible phase portrait, as in $\mathbb{U}^{2,I}_{AB,21}$.

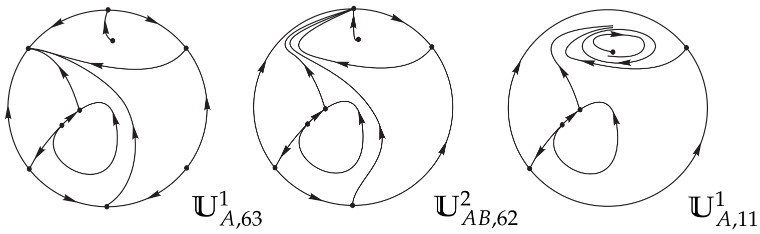


Figure 3.66: Unstable system $\mathbb{U}^2_{AB,62}$.

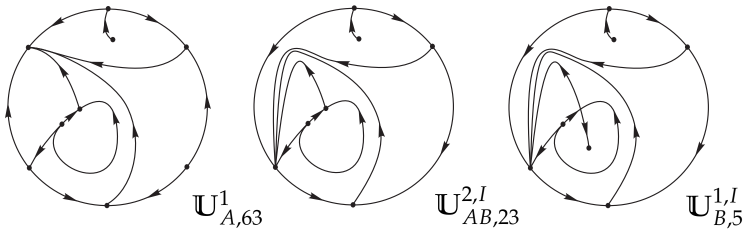


Figure 3.67: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,23}$.

Phase portrait $\mathbb{U}^1_{A,64}$ has phase portrait $\mathbb{U}^2_{AB,63}$ as an evolution (see Figure 3.68). After bifurcation we get phase portrait $\mathbb{U}^1_{A,13}$, modulo limit cycle, by making the infinite saddle-node

$\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,64}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,24}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,5}$ of *codimension one**, see Figure 3.69. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,24}$ also unfolds in an impossible phase portrait, as in $\mathbb{U}^{2,I}_{AB,21}$.

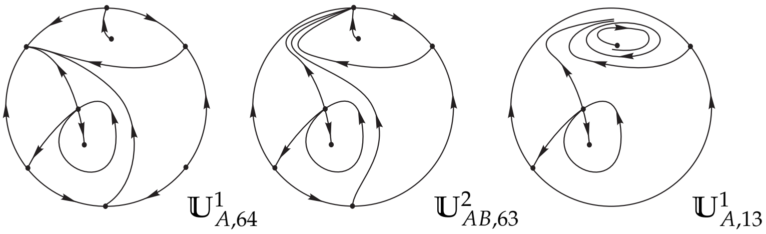


Figure 3.68: Unstable system $\mathbb{U}^2_{AB,63}$.

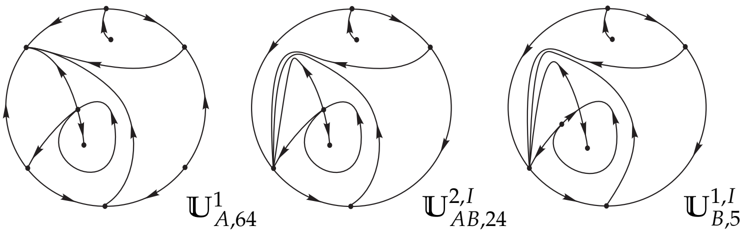


Figure 3.69: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,24}$.

Phase portrait $\mathbb{U}^1_{A,65}$ has phase portrait $\mathbb{U}^2_{AB,64}$ as an evolution (see Figure 3.70). After bifurcation we get phase portrait $\mathbb{U}^1_{A,11}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,65}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,25}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,6}$ of *codimension one**, see Figure 3.71. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,25}$ also unfolds in an impossible phase portrait, as in $\mathbb{U}^{2,I}_{AB,21}$.

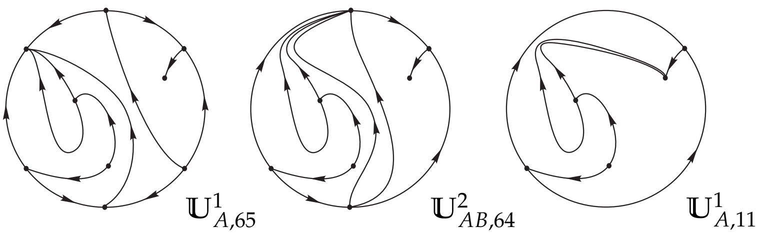


Figure 3.70: Unstable system $\mathbb{U}^2_{AB,64}$.

Phase portrait $\mathbb{U}^1_{A,66}$ has phase portrait $\mathbb{U}^2_{AB,65}$ as an evolution (see Figure 3.72). After bifurcation we get phase portrait $\mathbb{U}^1_{A,12}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,66}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,26}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,6}$ of *codimension one**, see Figure 3.73. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,26}$ also unfolds in an impossible phase portrait, as in $\mathbb{U}^{2,I}_{AB,21}$.

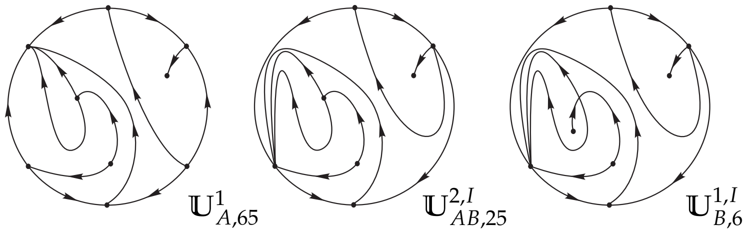


Figure 3.71: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,25}$.

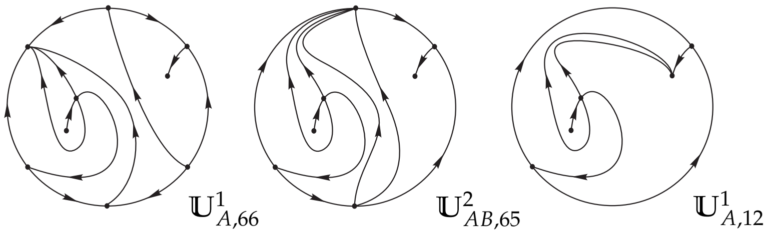


Figure 3.72: Unstable system $\mathbb{U}^2_{AB,65}$.

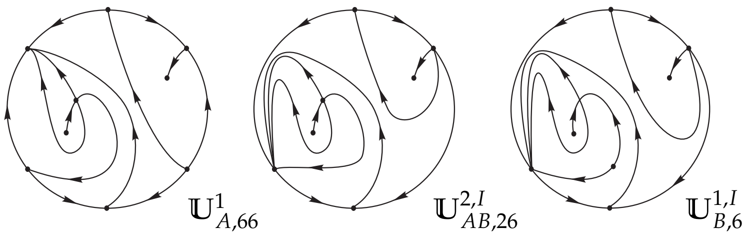


Figure 3.73: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,26}$.

Phase portrait $\mathbb{U}^1_{A,67}$ has phase portraits $\mathbb{U}^2_{AB,66}$ and $\mathbb{U}^2_{AB,67}$ as evolution (see Figure 3.74). After bifurcation we get phase portraits $\mathbb{U}^1_{A,11}$ and $\mathbb{U}^1_{A,13}$ (being this last one modulo limit cycles), respectively, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

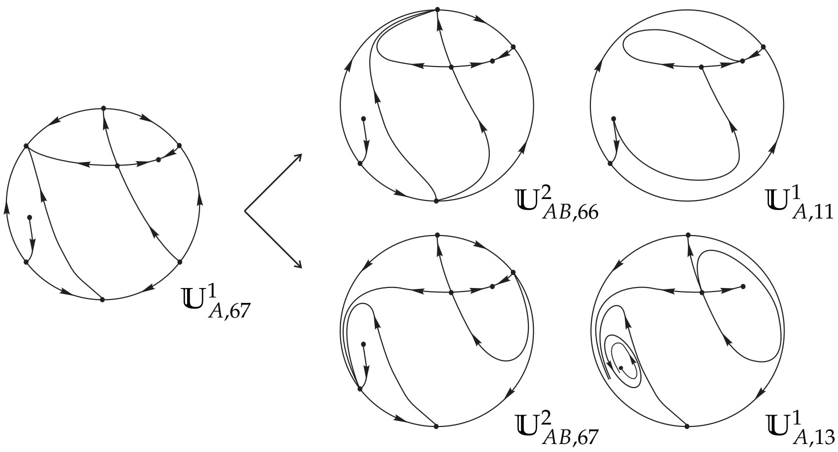


Figure 3.74: Unstable systems $\mathbb{U}^2_{AB,66}$ and $\mathbb{U}^2_{AB,67}$.

Phase portrait $\mathbb{U}^1_{A,68}$ has phase portraits $\mathbb{U}^2_{AB,68}$ and $\mathbb{U}^2_{AB,69}$ as evolution (see Figure 3.75). After bifurcation we get phase portraits $\mathbb{U}^1_{A,11}$ (modulo limit cycles) and $\mathbb{U}^1_{A,13}$, respectively, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear.

Phase portrait $\mathbb{U}^1_{A,69}$ has phase portrait $\mathbb{U}^2_{AB,70}$ as an evolution (see Figure 3.76). After

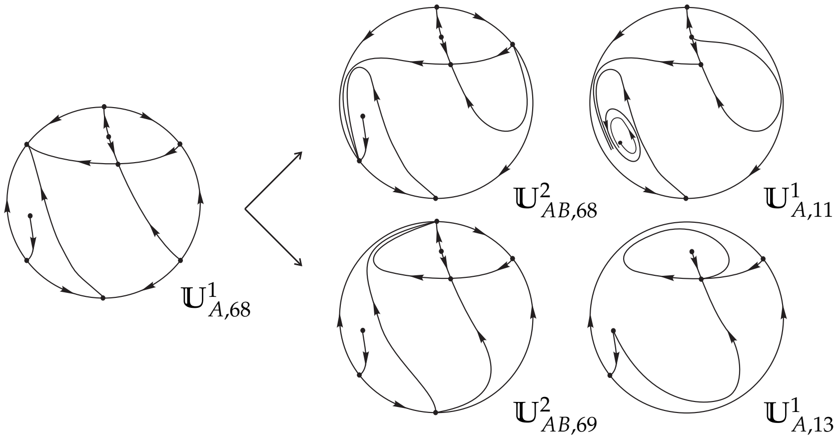


Figure 3.75: Unstable systems $\mathbb{U}^2_{AB,68}$ and $\mathbb{U}^2_{AB,69}$.

bifurcation we get phase portrait $\mathbb{U}^1_{A,11}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,69}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,27}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,7}$ of *codimension one**, see Figure 3.77. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,27}$ also unfolds in an impossible phase portrait, as in $\mathbb{U}^{2,I}_{AB,21}$.

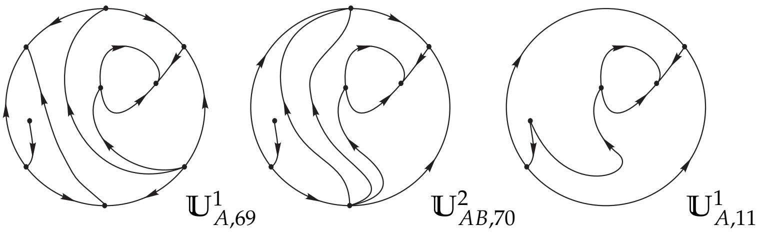


Figure 3.76: Unstable system $\mathbb{U}^2_{AB,70}$.

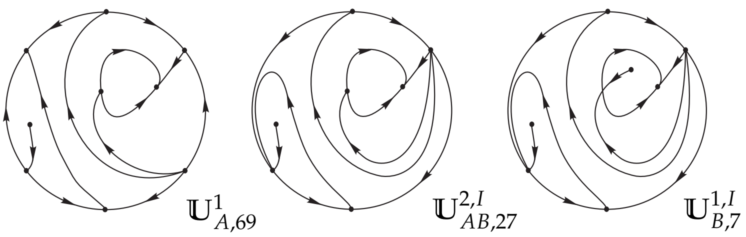


Figure 3.77: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,27}$.

Phase portrait $\mathbb{U}^1_{A,70}$ has phase portrait $\mathbb{U}^2_{AB,71}$ as an evolution (see Figure 3.78). After bifurcation we get phase portrait $\mathbb{U}^1_{A,13}$, by making the infinite saddle-node $\overline{\binom{0}{2}}SN$ disappear. Moreover, $\mathbb{U}^1_{A,70}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,28}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{B,7}$ of *codimension one**, see Figure 3.79. We observe that, in the set (A), $\mathbb{U}^{2,I}_{AB,28}$ also unfolds in an impossible phase portrait, as in $\mathbb{U}^{2,I}_{AB,21}$.

Therefore, we have just finished obtaining all the 71 topologically potential phase portraits

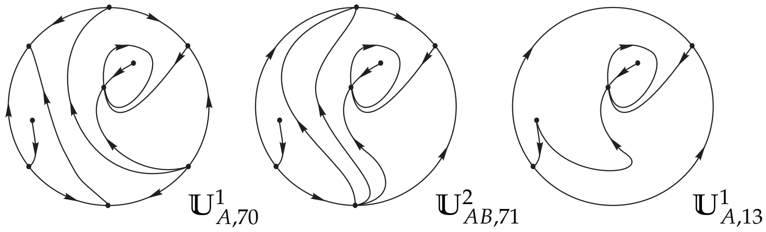


Figure 3.78: Unstable system $\mathbb{U}^2_{AB,71}$.

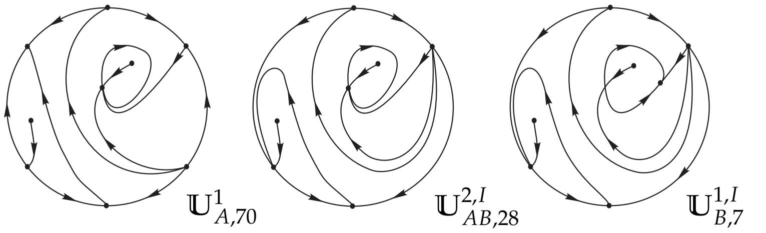


Figure 3.79: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,28}$.

of *codimension two** from the set (AB) presented in Figures 1.1 to 1.3.

Now we explain how one can obtain these 71 phase portraits by starting the study from the set (B). Let us consider all the 40 realizable structurally unstable quadratic vector fields of *codimension one** from the set (B). In order to obtain a phase portrait of *codimension two** belonging to the set (AB) starting from a phase portrait of *codimension one** of the set (B), we keep the existing infinite saddle-node $\overline{\binom{0}{2}}SN$ and by using Theorem 2.6 we build a finite saddle-node $\overline{sn}_{(2)}$ by the coalescence of a finite saddle with a finite node. On the other hand, from the phase portraits of *codimension two** from the set (AB), there exist two ways of obtaining phase portraits of *codimension one** also belonging to the set (B) after perturbation: splitting $\overline{sn}_{(2)}$ into a saddle and a node, or moving it to complex singularities (see Remark 3.2).

Remark 3.2. We recall that, in quadratic differential systems, the finite singular points are zeroes of a polynomial of degree four. Supposing that we have a singular point of multiplicity two, then the remaining singular points are zeroes of a quadratic polynomial. Therefore, these other two points can be two simple singular points, a double point (a saddle-node) or two complex conjugate singular points.

According to these facts, if a phase portrait does not possess finite singularities (for instance, $\mathbb{U}^1_{B,1}$ and $\mathbb{U}^1_{B,2}$) or if it possesses only two finite antisaddles (as for instance $\mathbb{U}^1_{B,29}$ to $\mathbb{U}^1_{B,32}$), it is not possible to obtain a phase portrait from it which belongs to the set (AB).

The main goal of this section is to obtain all the topologically potential phase portraits from the set (AB) and then prove their realization or show that they are not possible. So we have to be sure that no other phase portrait can be found if one does some evolution in all elements of the set (B) in order to obtain a phase portrait belonging to the set (AB). We point out that we have done this verification, i.e. we have also considered each element from the set (B) and produced a coalescence (when it was possible) of a finite saddle with a finite node and we also have obtained the 71 topologically potential phase portraits of *codimension two** from the set (AB) presented in Figures 1.1 to 1.3. In what follows we show the result (modulo limit cycles) of this study. We point out that we will not give all the details of this study. We will not even mention anything about why there are no more potential cases to be considered an evolution of a *codimension one** phase portrait, since we believe that this can be easily verified by the reader. Additionally, we will present pictures only of the impossible phase portraits obtained in order to explain their impossibility and we will not mention anything about phase portraits which are topologically equivalent to phase portraits already obtained.

It is important to remark that the realizable phase portraits that we will obtain from the set (B) to the set (AB) will coincide exactly with those ones previously found. However, the non-realizable ones that we will find from (B) will be different from those ones coming from (A). The reason is that the arguments used to prove the impossibility of those coming from (A) were precisely that they would bifurcate in some impossible from (B) and now, they will be those ones that bifurcate in some impossible from (A).

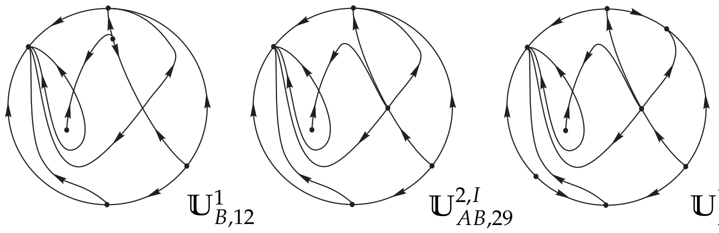
In Table 3.1 we present the study of phase portraits $\mathbb{U}^1_{B,3}$ to $\mathbb{U}^1_{B,11}$. In the first column we present the corresponding phase portrait from the set (B), in the second column we indicate its corresponding phase portrait belonging to the set (AB) i.e. after producing a finite saddle-node $\overline{sn}_{(2)}$, and in the third column we show the corresponding phase portrait after we make this finite saddle-node $\overline{sn}_{(2)}$ disappear.

phase portrait from the set (B)	**phase portrait from the set** (AB)	**phase portrait from the set** (B)
$\mathbb{U}^1_{B,3}$	$\mathbb{U}^2_{AB,1}$	$\mathbb{U}^1_{B,1}$
$\mathbb{U}^1_{B,4}$	$\mathbb{U}^2_{AB,2}$	$\mathbb{U}^1_{B,2}$
$\mathbb{U}^1_{B,5}$	$\mathbb{U}^2_{AB,3}$	$\mathbb{U}^1_{B,1}$
$\mathbb{U}^1_{B,6}$	$\mathbb{U}^2_{AB,4}$	$\mathbb{U}^1_{B,2}$
$\mathbb{U}^1_{B,7}$	$\mathbb{U}^2_{AB,6}$	$\mathbb{U}^1_{B,2}$
$\mathbb{U}^1_{B,8}$	$\mathbb{U}^2_{AB,5}$	$\mathbb{U}^1_{B,1}$
$\mathbb{U}^1_{B,9}$	$\mathbb{U}^2_{AB,7}$ $\mathbb{U}^2_{AB,9}$ $\mathbb{U}^2_{AB,11}$	$\mathbb{U}^1_{B,8}$
$\mathbb{U}^1_{B,10}$	$\mathbb{U}^2_{AB,8}$ $\mathbb{U}^2_{AB,10}$ $\mathbb{U}^2_{AB,12}$	$\mathbb{U}^1_{B,7}$
$\mathbb{U}^1_{B,11}$	$\mathbb{U}^2_{AB,13}$ $\mathbb{U}^2_{AB,14}$	$\mathbb{U}^1_{B,4}$ $\mathbb{U}^1_{B,7}$

Table 3.1: Phase portraits from the set (AB) obtained from evolution of elements of the set (B).

Phase portrait $\mathbb{U}^1_{B,12}$ has phase portraits $\mathbb{U}^2_{AB,15}$ and $\mathbb{U}^2_{AB,16}$ as evolution. After bifurcation we get phase portrait $\mathbb{U}^1_{B,3}$ (for both cases) by making the finite saddle-node $\overline{sn}_{(2)}$ disappear. Moreover, $\mathbb{U}^1_{B,12}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,29}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original infinite saddle-node $\overline{\binom{0}{2}}SN$ into an infinite saddle and an infinite node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{A,1}$ of *codimension one**, see Figure 3.80. We point out that, in the set (B), the corresponding unfolding of $\mathbb{U}^{2,I}_{AB,29}$ does not exist, since if such a phase portrait does exist, it would be an evolution of the impossible phase portrait $\mathbb{I}_{9,1}$ (see Figure 4.4 from [6]), which contradicts Theorem 2.11.

In Table 3.2 we present the study of phase portraits $\mathbb{U}^1_{B,13}$ to $\mathbb{U}^1_{B,15}$. In the first column we present the corresponding phase portrait from the set (B), in the second column we indicate its corresponding phase portrait belonging to the set (AB) i.e. after producing a finite saddle-



$\mathbb{U}^1_{B,12}$ $\mathbb{U}^{2,I}_{AB,29}$ $\mathbb{U}^{1,I}_{A,1}$

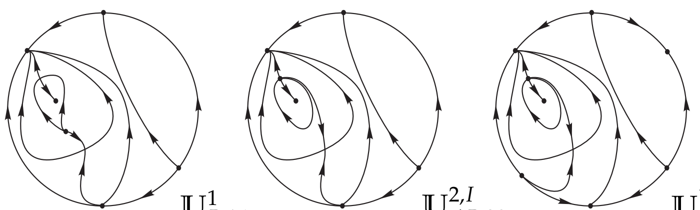
Figure 3.80: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,29}$.

node $\overline{sn}_{(2)}$, and in the third column we show the corresponding phase portrait after we make this finite saddle-node $\overline{sn}_{(2)}$ disappear.

phase portrait from the set (B)	**phase portrait from the set** (AB)	**phase portrait from the set** (B)
$\mathbb{U}^1_{B,13}$	$\mathbb{U}^2_{AB,17}$ $\mathbb{U}^2_{AB,18}$	$\mathbb{U}^1_{B,6}$ $\mathbb{U}^1_{B,7}$
$\mathbb{U}^1_{B,14}$	$\mathbb{U}^2_{AB,19}$ $\mathbb{U}^2_{AB,20}$ $\mathbb{U}^2_{AB,21}$	$\mathbb{U}^1_{B,3}$
$\mathbb{U}^1_{B,15}$	$\mathbb{U}^2_{AB,23}$ $\mathbb{U}^2_{AB,22}$	$\mathbb{U}^1_{B,3}$ $\mathbb{U}^1_{B,5}$

Table 3.2: Phase portraits from the set (AB) obtained from evolution of elements of the set (B).

Phase portrait $\mathbb{U}^1_{B,16}$ has phase portraits $\mathbb{U}^2_{AB,29}$, $\mathbb{U}^2_{AB,25}$, and $\mathbb{U}^2_{AB,26}$ as evolution. After bifurcation we get phase portraits $\mathbb{U}^1_{B,5}$, $\mathbb{U}^1_{B,8}$ and $\mathbb{U}^1_{B,8}$ (being this last one modulo limit cycle), respectively, by making the finite saddle-node $\overline{sn}_{(2)}$ disappear. Moreover, $\mathbb{U}^1_{B,16}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,30}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original infinite saddle-node $\overline{\binom{0}{2}}SN$ into an infinite saddle and an infinite node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{A,103}$ of *codimension one**, see Figure 3.81. We observe that, in the set (B), $\mathbb{U}^{2,I}_{AB,30}$ unfolds in $\mathbb{U}^1_{B,8}$ (modulo limit cycles).



$\mathbb{U}^1_{B,16}$ $\mathbb{U}^{2,I}_{AB,30}$ $\mathbb{U}^{1,I}_{A,103}$

Figure 3.81: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,30}$.

Phase portrait $\mathbb{U}^1_{B,17}$ has phase portraits $\mathbb{U}^2_{AB,28}$, $\mathbb{U}^2_{AB,24}$, and $\mathbb{U}^2_{AB,27}$ as evolution. After bifurcation we get phase portraits $\mathbb{U}^1_{B,6}$, $\mathbb{U}^1_{B,7}$ and $\mathbb{U}^1_{B,7}$ (being this last one modulo limit cycle), respectively, by making the finite saddle-node $\overline{sn}_{(2)}$ disappear. Moreover, $\mathbb{U}^1_{B,17}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,31}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original infinite saddle-node $\overline{\binom{0}{2}}SN$ into an infinite saddle and an infinite node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{A,103}$ of *codimension one**, see Figure 3.82. We observe that, in the set (B), $\mathbb{U}^{2,I}_{AB,31}$ unfolds in $\mathbb{U}^{1}_{B,7}$ (modulo limit cycles).

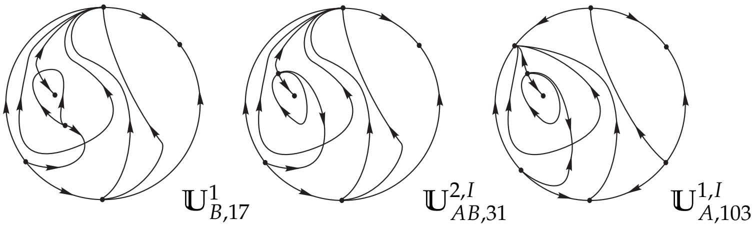


Figure 3.82: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,31}$.

Phase portrait $\mathbb{U}^{1}_{B,18}$ has phase portrait $\mathbb{U}^{2}_{AB,30}$ as an evolution and after bifurcation we get phase portrait $\mathbb{U}^{1}_{B,7}$, by making the finite saddle-node $\overline{sn}_{(2)}$ disappear. Moreover, $\mathbb{U}^{1}_{B,18}$ has a second phase portrait as an evolution which is topologically equivalent to $\mathbb{U}^{2}_{AB,30}$.

Phase portrait $\mathbb{U}^{1}_{B,19}$ has phase portraits $\mathbb{U}^{2}_{AB,32}$ and $\mathbb{U}^{2}_{AB,31}$ as evolution. After bifurcation we get phase portraits $\mathbb{U}^{1}_{B,4}$ and $\mathbb{U}^{1}_{B,6}$, respective, by making the finite saddle-node $\overline{sn}_{(2)}$ disappear.

Phase portrait $\mathbb{U}^{1}_{B,20}$ has phase portraits $\mathbb{U}^{2}_{AB,33}$ and $\mathbb{U}^{2}_{AB,34}$ as evolution. After bifurcation we get phase portrait $\mathbb{U}^{1}_{B,3}$, in both cases (being one of them modulo limit cycles), by making the finite saddle-node $\overline{sn}_{(2)}$ disappear. Moreover, $\mathbb{U}^{1}_{B,20}$ has the impossible phase portraits $\mathbb{U}^{2,I}_{AB,32}$ and $\mathbb{U}^{2,I}_{AB,33}$ as evolution. By Theorem 2.11 such phase portraits are impossible because by splitting the original infinite saddle-node $\overline{\binom{0}{2}}SN$ into an infinite saddle and an infinite node we obtain the impossible phase portraits $\mathbb{U}^{1,I}_{A,2}$ and $\mathbb{U}^{1,I}_{A,104}$, respectively, of *codimension one**, see Figure 3.83. We point out that, in the set (B), the corresponding unfolding of $\mathbb{U}^{2,I}_{AB,32}$ does not exist (by the exactly same reason that we have discussed in $\mathbb{U}^{2,I}_{AB,29}$) and the corresponding unfolding of $\mathbb{U}^{2,I}_{AB,33}$ is $\mathbb{U}^{1}_{B,3}$ (modulo limit cycles).

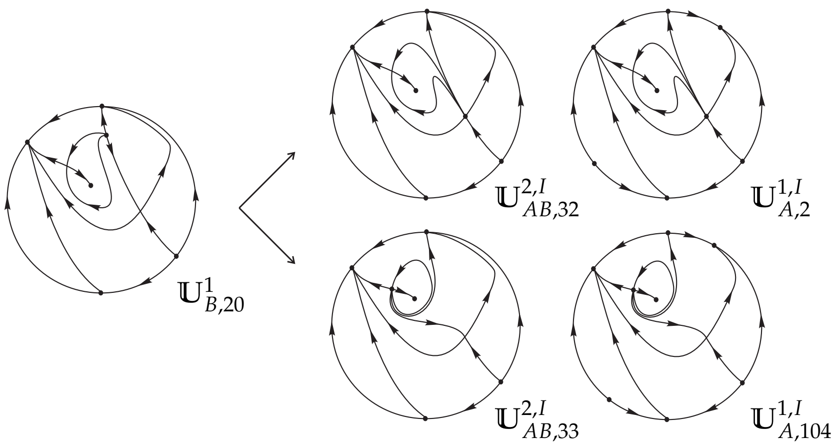
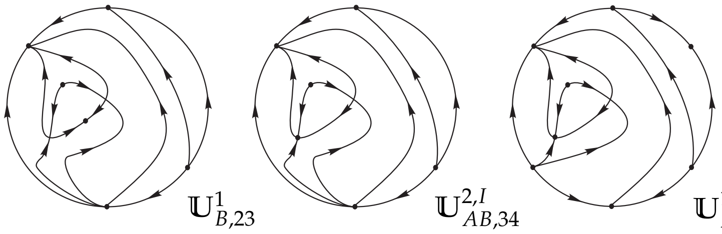


Figure 3.83: Impossible unstable phase portraits $\mathbb{U}^{2,I}_{AB,32}$ and $\mathbb{U}^{2,I}_{AB,33}$.

Phase portrait $\mathbb{U}^{1}_{B,21}$ has phase portrait $\mathbb{U}^{2}_{AB,35}$ as an evolution and after bifurcation we get phase portrait $\mathbb{U}^{1}_{B,6}$, by making the finite saddle-node $\overline{sn}_{(2)}$ disappear. Moreover, $\mathbb{U}^{1}_{B,21}$ has a second phase portrait as an evolution which is topologically equivalent to $\mathbb{U}^{2}_{AB,35}$.

Phase portrait $\mathbb{U}^{1}_{B,22}$ has phase portraits $\mathbb{U}^{2}_{AB,36}$ and $\mathbb{U}^{2}_{AB,37}$ as evolution. After bifurcation we get phase portraits $\mathbb{U}^{1}_{B,3}$ and $\mathbb{U}^{1}_{B,8}$, respective, by making the finite saddle-node $\overline{sn}_{(2)}$ disappear.

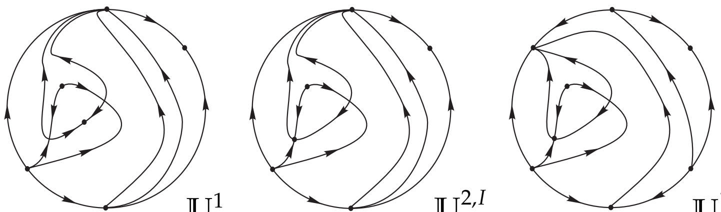
Phase portrait $\mathbb{U}^1_{B,23}$ has phase portraits $\mathbb{U}^2_{AB,39}$, $\mathbb{U}^2_{AB,40}$, and $\mathbb{U}^2_{AB,43}$ as evolution. After bifurcation we get phase portraits $\mathbb{U}^1_{B,5}$ (for the two first cases) and $\mathbb{U}^1_{B,8}$ (for the third case), by making the finite saddle-node $\overline{sn}_{(2)}$ disappear. Moreover, $\mathbb{U}^1_{B,23}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,34}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original infinite saddle-node $\overline{\binom{0}{2}}SN$ into an infinite saddle and an infinite node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{A,49}$ of *codimension one**, see Figure 3.84. We observe that, in the set (B), $\mathbb{U}^{2,I}_{AB,34}$ unfolds in $\mathbb{U}^1_{B,8}$.



$\mathbb{U}^1_{B,23}$ $\mathbb{U}^{2,I}_{AB,34}$ $\mathbb{U}^{1,I}_{A,49}$

Figure 3.84: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,34}$.

Phase portrait $\mathbb{U}^1_{B,24}$ has phase portraits $\mathbb{U}^2_{AB,38}$, $\mathbb{U}^2_{AB,41}$, and $\mathbb{U}^2_{AB,42}$ as evolution. After bifurcation we get phase portraits $\mathbb{U}^1_{B,6}$ (for the two first cases) and $\mathbb{U}^1_{B,7}$ (for the third case), by making the finite saddle-node $\overline{sn}_{(2)}$ disappear. Moreover, $\mathbb{U}^1_{B,24}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,35}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original infinite saddle-node $\overline{\binom{0}{2}}SN$ into an infinite saddle and an infinite node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{A,49}$ of *codimension one**, see Figure 3.85. We observe that, in the set (B), $\mathbb{U}^{2,I}_{AB,35}$ unfolds in $\mathbb{U}^1_{B,7}$.



$\mathbb{U}^1_{B,24}$ $\mathbb{U}^{2,I}_{AB,35}$ $\mathbb{U}^{1,I}_{A,49}$

Figure 3.85: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,35}$.

Phase portrait $\mathbb{U}^1_{B,25}$ has phase portraits $\mathbb{U}^2_{AB,46}$, $\mathbb{U}^2_{AB,49}$, and $\mathbb{U}^2_{AB,44}$ as evolution. After bifurcation we get phase portraits $\mathbb{U}^1_{B,3}$ (for the two first cases) and $\mathbb{U}^1_{B,8}$ (for the third case), by making the finite saddle-node $\overline{sn}_{(2)}$ disappear. Moreover, $\mathbb{U}^1_{B,25}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,36}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original infinite saddle-node $\overline{\binom{0}{2}}SN$ into an infinite saddle and an infinite node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{A,3}$ of *codimension one**, see Figure 3.86. We point out that, in the set (B), the corresponding unfolding of $\mathbb{U}^{2,I}_{AB,36}$ does not exist (by the exactly same reason that we have discussed in $\mathbb{U}^{2,I}_{AB,29}$).

Phase portrait $\mathbb{U}^1_{B,26}$ has phase portraits $\mathbb{U}^2_{AB,47}$, $\mathbb{U}^2_{AB,48}$, and $\mathbb{U}^2_{AB,45}$ as evolution. After bifurcation we get phase portraits $\mathbb{U}^1_{B,4}$ (for the two first cases) and $\mathbb{U}^1_{B,7}$ (for the third case), by making the finite saddle-node $\overline{sn}_{(2)}$ disappear. Moreover, $\mathbb{U}^1_{B,26}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,37}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original infinite saddle-node $\overline{\binom{0}{2}}SN$ into an infinite saddle and an infinite node

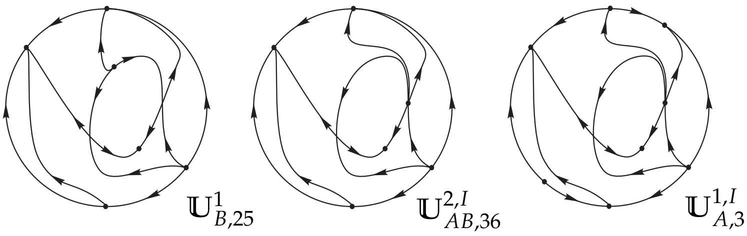


Figure 3.86: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,36}$.

we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{A,3}$ of *codimension one**, see Figure 3.87. We point out that, in the set (B), the corresponding unfolding of $\mathbb{U}^{2,I}_{AB,37}$ does not exist (by the exactly same reason that we have discussed in $\mathbb{U}^{2,I}_{AB,29}$).

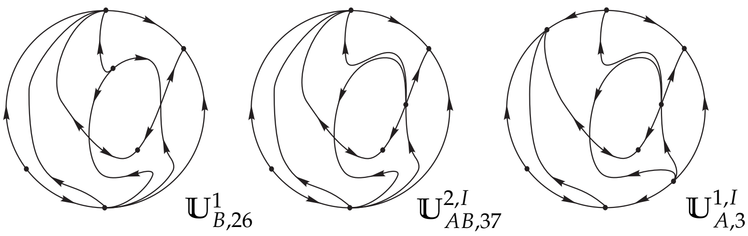


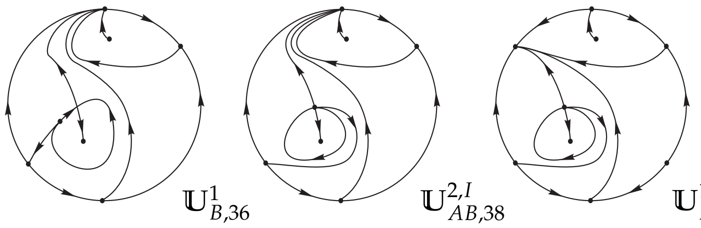
Figure 3.87: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,37}$.

In Table 3.3 we present the study (modulo limit cycles) of phase portraits $\mathbb{U}^{1}_{B,27}$ to $\mathbb{U}^{1}_{B,35}$. In the first column we present the corresponding phase portrait from the set (B), in the second column we indicate its corresponding phase portrait belonging to the set (AB) i.e. after producing a finite saddle-node $\overline{sn}_{(2)}$, and in the third column we show the corresponding phase portrait after we make this finite saddle-node $\overline{sn}_{(2)}$ disappear.

phase portrait from the set (B)	**phase portrait from the set** (AB)	**phase portrait from the set** (B)
$\mathbb{U}^{1}_{B,27}$	$\mathbb{U}^{2}_{AB,50}$	$\mathbb{U}^{1}_{B,4}$
$\mathbb{U}^{1}_{B,28}$	$\mathbb{U}^{2}_{AB,51}$ $\mathbb{U}^{2}_{AB,52}$	$\mathbb{U}^{1}_{B,3}$ $\mathbb{U}^{1}_{B,4}$
$\mathbb{U}^{1}_{B,33}$	$\mathbb{U}^{2}_{AB,53}$ $\mathbb{U}^{2}_{AB,54}$	$\mathbb{U}^{1}_{B,29}$
$\mathbb{U}^{1}_{B,34}$	$\mathbb{U}^{2}_{AB,55}$ $\mathbb{U}^{2}_{AB,56}$ $\mathbb{U}^{2}_{AB,57}$ $\mathbb{U}^{2}_{AB,58}$	$\mathbb{U}^{1}_{B,32}$
$\mathbb{U}^{1}_{B,35}$	$\mathbb{U}^{2}_{AB,61}$ $\mathbb{U}^{2}_{AB,59}$ $\mathbb{U}^{2}_{AB,60}$	$\mathbb{U}^{1}_{B,29}$ $\mathbb{U}^{1}_{B,32}$

Table 3.3: Phase portraits from the set (AB) obtained from evolution of elements of the set (B).

Phase portrait $\mathbb{U}^{1}_{B,36}$ has phase portraits $\mathbb{U}^{2}_{AB,62}$ and $\mathbb{U}^{2}_{AB,63}$ as evolution. After bifurcation we get phase portrait $\mathbb{U}^1_{B,29}$, for both cases (being one of them modulo limit cycles), by making the finite saddle-node $\overline{sn}_{(2)}$ disappear. Moreover, $\mathbb{U}^1_{B,36}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,38}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original infinite saddle-node $\overline{\binom{0}{2}}SN$ into an infinite saddle and an infinite node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{A,105}$ of *codimension one**, see Figure 3.88. We observe that, in the set (B), $\mathbb{U}^{2,I}_{AB,38}$ unfolds in $\mathbb{U}^1_{B,29}$ (modulo limit cycles).



$\mathbb{U}^I_{B,36}$ $\mathbb{U}^{2,I}_{AB,38}$ $\mathbb{U}^{1,I}_{A,105}$

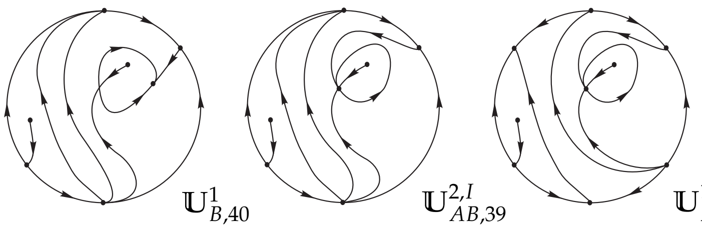
Figure 3.88: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,38}$.

Phase portrait $\mathbb{U}^1_{B,37}$ has phase portraits $\mathbb{U}^2_{AB,64}$ and $\mathbb{U}^2_{AB,65}$ as evolution. After bifurcation we get phase portrait $\mathbb{U}^1_{B,31}$, for both cases, by making the finite saddle-node $\overline{sn}_{(2)}$ disappear.

Phase portrait $\mathbb{U}^1_{B,38}$ has phase portraits $\mathbb{U}^2_{AB,68}$ and $\mathbb{U}^2_{AB,67}$ as evolution. After bifurcation we get phase portraits $\mathbb{U}^1_{B,29}$ and $\mathbb{U}^1_{B,30}$, respective, by making the finite saddle-node $\overline{sn}_{(2)}$ disappear.

Phase portrait $\mathbb{U}^1_{B,39}$ has phase portraits $\mathbb{U}^2_{AB,69}$ and $\mathbb{U}^2_{AB,66}$ as evolution. After bifurcation we get phase portraits $\mathbb{U}^1_{B,29}$ and $\mathbb{U}^1_{B,31}$, respective, by making the finite saddle-node $\overline{sn}_{(2)}$ disappear.

Phase portrait $\mathbb{U}^1_{B,40}$ has phase portraits $\mathbb{U}^2_{AB,70}$ and $\mathbb{U}^2_{AB,71}$ as evolution. After bifurcation we get phase portrait $\mathbb{U}^1_{B,31}$, for both cases (being one of them modulo limit cycles), by making the finite saddle-node $\overline{sn}_{(2)}$ disappear. Moreover, $\mathbb{U}^1_{B,40}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AB,39}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original infinite saddle-node $\overline{\binom{0}{2}}SN$ into an infinite saddle and an infinite node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{A,106}$ of *codimension one**, see Figure 3.89. We observe that, in the set (B), $\mathbb{U}^{2,I}_{AB,39}$ unfolds in $\mathbb{U}^1_{B,31}$ (modulo limit cycles).



$\mathbb{U}^1_{B,40}$ $\mathbb{U}^{2,I}_{AB,39}$ $\mathbb{U}^{1,I}_{A,106}$

Figure 3.89: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AB,39}$.

3.2 The realization of the potential phase portraits

In the previous subsection we have produced all the topologically potential phase portraits for structurally unstable quadratic systems of *codimension two** belonging to the set $\sum_2^2(AB)$. And from them, we have discarded 33 which are not realizable due to their respective unfoldings of *codimension one** being impossible.

In this subsection we aim to give specific examples for the remaining 71 different topological classes of structurally unstable quadratic systems of *codimension two** belonging to the set $\sum_2^2(AB)$ and presented in Figures 1.1 to 1.3.

In [2] the authors showed that for each structurally stable phase portrait with limit cycles there exists a realizable structurally stable phase portrait without limit cycles so that modulo limit cycles they are equivalent. On the contrary, due to the large number of cases, in [6] the authors did not follow the same procedure for the realizable structurally unstable phase portraits of *codimension one**. Since this present paper is directly derived from this second study, here we have found examples of *codimension two** phase portraits with no evidence of limit cycles, but we have not proved the absence of the infinitesimal ones (i.e. the ones born by Hopf-bifurcation).

In [14] the authors classified, with respect to a specific normal form, the set of all real quadratic polynomial differential systems with a finite semi-elemental saddle-node $\overline{sn}_{(2)}$ located at the origin of the plane and an infinite saddle-node of type $\overline{\binom{0}{2}}SN$ (obtained by the coalescence of an infinite saddle with an infinite node) located in the bisector of first and third quadrants. In [10] the authors show that phase portrait V_{171} from [14] is not topologically equivalent to V_{170} (i.e. the equivalence presented in Table 65 from the mentioned paper is not correct) and in [10] the authors present the correct picture of phase portrait V_{171}.

Remark 3.3. The study of a bifurcation diagram of a certain family of quadratic systems produces not only the class of phase portraits that we look for, but also all of those of their closure according to the normal form that we consider. Even though the study is mainly algebraic, analytic and numerical tools are also required. This implies that these studies may be not complete and subject to the existence of possible "islands" which could contain an undetected phase portrait. The border of that "island" could mean the connection of two separatrices, and its interior could contain a different phase portrait from the ones stated in the main theorem. In [14] the authors studied a bifurcation diagram in which the most generic phase portraits correspond to elements of the set (AB). In Section 7 of that paper the authors said that the bifurcation diagram they obtained is completely coherent, i.e. by taking any two points in the parameter space and joining them by a continuous curve, along this curve the changes in phase portraits that occur when crossing the different bifurcation surfaces could be completely explained. Nevertheless, at that moment, the authors could not be sure that the bifurcation diagram was the complete bifurcation diagram for the family they consider in their paper, due to the possibility of "islands" inside the bifurcation diagram. The topological study that we do in this paper solves partially this problem, since we prove that all the realizable phase portraits of class (AB) do really exist, and no other topological possibility does. However, the possible existence of "islands" in the bifurcation diagram still persists since they can be related to double limit cycles, as discussed in Section 7 of [14].

By using the phase portraits of generic regions of the bifurcation diagram from [14] plus the correct V_{171} presented in [10] we realize all the 71 unstable systems of *codimension two** of the set (AB), i.e. we can give concrete examples of all structurally unstable phase portraits from the set (AB).

Consider systems (2.1), which were studied in [14] and describe quadratic systems having a finite semi-elemental saddle-node $\overline{sn}_{(2)}$ and an infinite saddle-node of type $\overline{\binom{0}{2}}SN$ located in the endpoints of the bisector of the first and third quadrants.

In Tables 3.4 and 3.6 we present one representative from each generic region of the bifurcation diagram of [14] (as described before) corresponding to each phase portrait of *codimension* *two** from the set (AB) and, therefore, we conclude the proof of Theorem 1.6.

Cod 2*	**[14]**	**g**	**h**	**ł**	**n**
$\mathbb{U}^2_{AB,1}$	V_{23}	1	0	1/2	10
$\mathbb{U}^2_{AB,2}$	V_{84}	1	91/100	1	2304/625
$\mathbb{U}^2_{AB,3}$	V_{22}	1	0	9/10	10
$\mathbb{U}^2_{AB,4}$	V_{85}	1	22/25	1	2304/625
$\mathbb{U}^2_{AB,5}$	V_{20}	1	0	18	10
$\mathbb{U}^2_{AB,6}$	V_{21}	1	−2	1	10
$\mathbb{U}^2_{AB,7}$	V_{1}	1	−21/5	18	10
$\mathbb{U}^2_{AB,8}$	V_{2}	1	−5	10	10
$\mathbb{U}^2_{AB,9}$	V_{190}	1	3/5	−33/10	−1
$\mathbb{U}^2_{AB,10}$	V_{191}	1	3/5	−3	−1
$\mathbb{U}^2_{AB,11}$	V_{25}	1	173/80	6	10
$\mathbb{U}^2_{AB,12}$	V_{31}	1	112/25	6	30
$\mathbb{U}^2_{AB,13}$	V_{9}	1	−5	11/10	10
$\mathbb{U}^2_{AB,14}$	V_{121}	1	−9999/100000	4/25	81/100
$\mathbb{U}^2_{AB,15}$	V_{147}	1	−6/5	5	−1
$\mathbb{U}^2_{AB,16}$	V_{66}	1	5	−15	10
$\mathbb{U}^2_{AB,17}$	V_{7}	1	−9/2	13/5	10
$\mathbb{U}^2_{AB,18}$	V_{136}	1	−59999/100000	7/10	4/25
$\mathbb{U}^2_{AB,19}$	V_{64}	1	11/5	−4	10
$\mathbb{U}^2_{AB,20}$	V_{145}	1	−4/5	5	−1
$\mathbb{U}^2_{AB,21}$	V_{13}	1	−5	1/2	10
$\mathbb{U}^2_{AB,22}$	V_{83}	1	9201/10000	−15	2304/625
$\mathbb{U}^2_{AB,23}$	V_{10}	1	−5	7/10	10
$\mathbb{U}^2_{AB,24}$	V_{141}	1	−69/100	601/1000	9/100
$\mathbb{U}^2_{AB,25}$	V_{144}	1	−7999/10000	6397/10000	1/25
$\mathbb{U}^2_{AB,26}$	V_{172}	1	−1/10	−3	−1
$\mathbb{U}^2_{AB,27}$	V_{173}	1	−7/100	−31/20	−1
$\mathbb{U}^2_{AB,28}$	V_{41}	1	44773/10000	11/5	30
$\mathbb{U}^2_{AB,29}$	V_{69}	1	11/5	6	10

Table 3.4: Correspondence between *codimension two** phase portraits of the set (AB) and phase portraits from generic regions of the bifurcation diagram presented in [14]. In the first column we present the *codimension two** phase portraits from the set (AB) in the present paper, in the second column we show the corresponding phase portraits from [14] given by normal form (2.1), and in the other columns we present the values of the parameters g, h, ł, and n of (2.1) which realizes such phase portrait (remember that the correct phase portrait V_{171} is presented in [10]).

Cod 2*	**[14]**	**g**	**h**	**ł**	**n**
$\mathbb{U}^2_{AB,30}$	V_{15}	1	$-21/5$	3	10
$\mathbb{U}^2_{AB,31}$	V_{114}	1	$-211/2000$	$9549/50000$	$4/5$
$\mathbb{U}^2_{AB,32}$	V_{109}	1	$-41/400$	$99999/100000$	$4/5$
$\mathbb{U}^2_{AB,33}$	V_{154}	1	$-7/5$	$8/25$	-1
$\mathbb{U}^2_{AB,34}$	V_{102}	1	$481/2000$	-10	1
$\mathbb{U}^2_{AB,35}$	V_{129}	1	$-5499/10000$	$3/4$	$81/400$
$\mathbb{U}^2_{AB,36}$	V_{108}	1	$-41/400$	$11/10$	$4/5$
$\mathbb{U}^2_{AB,37}$	V_{78}	1	$9201/10000$	-50	$2304/625$
$\mathbb{U}^2_{AB,38}$	V_{42}	1	$44777/10000$	$203/100$	30
$\mathbb{U}^2_{AB,39}$	V_{71}	1	$223/100$	6	10
$\mathbb{U}^2_{AB,40}$	V_{170}	1	$-9/50$	-3	-1
$\mathbb{U}^2_{AB,41}$	V_{171}	1	$-3/40$	$-3/2$	-1
$\mathbb{U}^2_{AB,42}$	V_{142}	1	$-69/100$	$6007/10000$	$9/100$
$\mathbb{U}^2_{AB,43}$	V_{143}	1	$-7999/10000$	$27/50$	$1/25$
$\mathbb{U}^2_{AB,44}$	V_{104}	1	$573/1250$	-8	$19/10$
$\mathbb{U}^2_{AB,45}$	V_{123}	1	$-39/400$	$1/100$	$81/100$
$\mathbb{U}^2_{AB,46}$	V_{155}	1	$-7/5$	$3/10$	-1
$\mathbb{U}^2_{AB,47}$	V_{165}	1	$-1/5$	$-13/10$	-1
$\mathbb{U}^2_{AB,48}$	V_{37}	1	3	$11/10$	10
$\mathbb{U}^2_{AB,49}$	V_{44}	1	$22/5$	2	10
$\mathbb{U}^2_{AB,50}$	V_{110}	1	$-41/400$	$9/10$	$4/5$
$\mathbb{U}^2_{AB,51}$	V_{46}	1	$11/5$	$9/10$	10
$\mathbb{U}^2_{AB,52}$	V_{49}	1	$23/5$	$9/10$	10
$\mathbb{U}^2_{AB,53}$	V_{6}	1	-5	3	10
$\mathbb{U}^2_{AB,54}$	V_{189}	1	$37/50$	$-147/100$	-1
$\mathbb{U}^2_{AB,55}$	V_{61}	1	$4501/1000$	-1	10
$\mathbb{U}^2_{AB,56}$	V_{53}	1	6	$-1/10000$	10
$\mathbb{U}^2_{AB,57}$	V_{107}	1	$9/25$	$-1/2$	$41/25$
$\mathbb{U}^2_{AB,58}$	V_{149}	1	$-11/10$	$3/2$	-1
$\mathbb{U}^2_{AB,59}$	V_{62}	1	3	-1	10
$\mathbb{U}^2_{AB,60}$	V_{198}	1	$-2/5$	$11/10$	-1
$\mathbb{U}^2_{AB,61}$	V_{51}	1	6	$1/5$	10
$\mathbb{U}^2_{AB,62}$	V_{138}	1	$-3/5$	$7/10$	$9/100$
$\mathbb{U}^2_{AB,63}$	V_{177}	1	$3/100$	$-9/10$	-1
$\mathbb{U}^2_{AB,64}$	V_{3}	1	-5	6	10
$\mathbb{U}^2_{AB,65}$	V_{192}	1	$3/5$	$-123/50$	-1
$\mathbb{U}^2_{AB,66}$	V_{122}	1	$-39/400$	$31/1000$	$81/100$
$\mathbb{U}^2_{AB,67}$	V_{169}	1	$-1/5$	$-7/10$	-1
$\mathbb{U}^2_{AB,68}$	V_{113}	1	$-39/400$	$1/10$	$81/100$

Table 3.5: Continuation of Table 3.4.

Cod 2*	[14]	g	h	ł	n
$\mathbb{U}^2_{AB,69}$	V_{166}	1	$-1/5$	$-53/50$	-1
$\mathbb{U}^2_{AB,70}$	V_{140}	1	$-69/100$	$63/100$	$9/100$
$\mathbb{U}^2_{AB,71}$	V_{174}	1	$-41/1000$	$-133/100$	-1

Table 3.6: Continuation of Table 3.5.

4 Proof of Theorem 1.7

In this section we present the proof of Theorem 1.7. The procedure is the same as used in the previous section. In Subsection 4.1 we obtain all the topologically potential phase portraits possessing the saddle-nodes $\overline{sn}_{(2)}$ and $\overline{\binom{1}{1}}SN$ (we have 45 phase portraits) and we prove that five of them are impossible. In Subsection 4.2 we show the realization of each one of the remaining 40 phase portraits.

4.1 The topologically potential phase portraits

The main goal of this subsection is to obtain all the topologically potential phase portraits from the set (AC).

As we said before, inside the set (AC), the unstable objects of *codimension two** that we are considering in this paper belong to the set of saddle-nodes $\left\{\overline{sn}_{(2)}+\overline{\binom{1}{1}}SN\right\}$. Considering all the different ways of obtaining phase portraits belonging to the set (AC) of *codimension two**, we have to consider all the possible ways of coalescing specific singular points in both sets (A) and (C). However, as the sets (AC) and (CA) are the same (i.e. their elements are obtained independently of the order of evolution in elements of the sets (A) or (C)), it is necessary to consider only all the possible ways of obtaining an infinite saddle-node of type $\overline{\binom{1}{1}}SN$ in each element from the set (A) (phase portraits possessing a finite saddle-node $\overline{sn}_{(2)}$). Anyway, in order to make things clear, in page 77 we discuss briefly how should we perform if we start by considering the set (C).

In order to obtain phase portraits from the set (AC) by starting our study from the set (A), we have to consider Theorem 2.9 and also Lemma 3.26 from [6] (regarding phase portraits from the set (C)) which we state as follows.

Lemma 4.1. *Assume that a codimension one* polynomial vector field X has an infinite singular point p being a saddle-node of multiplicity two with $\rho_0 = (\partial P/\partial x + \partial Q/\partial y)_p \neq 0$ and second eigenvalue equal to zero.*

(a) Any perturbation of X in a sufficiently small neighborhood of this point will produce a structurally stable system (with one infinite saddle and one finite node, or vice versa) or a system topologically equivalent to X.

(b) Both possibilities of structurally stable systems are realizable.

(c) If the saddle-node is the only unstable object in the region of definition and we consider the perturbation which leaves a saddle and a node in a small neighborhood, then the node is ω-limit or α-limit (depending on its stability) of at least one of the separatrices of the saddle.

(d) In the case that after bifurcation the node remains at infinity and the saddle moves to the finite plane, then the separatrices of this new saddle have their α- and ω-limits fixed according to next rule:

(1) The separatrix γ that corresponds to the one of the saddle-node different from the infinity line must maintain the same α- or ω-limit set.

(2) The separatrix (belonging to the same eigenspace of γ) which appears after bifurcation must go to the node that remains at infinity, and this will be the only separatrix which can arrive to this node in this side of the infinity.

(3) The two separatrices which were the infinite line in the unstable phase portrait, and that now are two separatrices of the saddle drawn on the finite plane, must end at the same infinite node where they ended before the bifurcation (if a node was adjacent to the saddle-node) or in the same α- or ω-limit point of the finite separatrix of the adjacent infinite saddle. In case that the saddle-node is the only infinite singular point, then both separatrices go to the symmetric point which will remain as a node.

Here we consider all 69 realizable structurally unstable quadratic vector fields of *codimension one** from the set (A). In order to obtain a phase portrait of *codimension two** belonging to the set (AC) starting from a phase portrait of *codimension one** of the set (A), we keep the existing finite saddle-node and using Lemma 4.1 we build an infinite saddle-node of type $\overline{\binom{1}{1}}SN$ by the coalescence of a finite node (respectively, finite saddle) with an infinite saddle (respectively, infinite node). As we said before, we point out that the finite singularity that coalesces with an infinite singularity cannot be the finite saddle-node since then what we would obtain at infinity would not be a saddle-node of type $\overline{\binom{1}{1}}SN$ but a multiplicity three singularity. Even though this is also a *codimension two** case and somehow can be considered inside the set (AC), we have preferred to put it into the set (CC) where two possibilities will be needed to be studied: either two finite singularities coalescing with different infinite singularities, or two finite singularities coalescing with the same infinite singularity. On the other hand, from the phase portraits of *codimension two** from the set (AC), one can obtain phase portraits of *codimension one** also belonging to the set (A) after perturbation by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$ into a finite saddle (respectively, finite node) and an infinite node (respectively, infinite saddle). More precisely, after bifurcation the point that has arrived to infinity remains there with the same local behavior, and the one which was at infinity moves into the real plane at the other side of the infinity line.

As in the previous section, in what follows we denote by $\mathbb{U}^2_{AC,k}$, where $\mathbb{U}^2_{AC}$ stands for structurally unstable quadratic vector field of *codimension two** from the set (AC) and $k \in \{1, \ldots, 40\}$. The impossible phase portraits will be denoted by $\mathbb{U}^{2,I}_{AC,j}$, where $\mathbb{U}^{2,I}_{AC}$ stands for *Impossible of codimension two** *from the set (AC)* and $j \in \mathbb{N}$.

We point out that in this study we do not present phase portraits which are topologically equivalent to phase portraits already obtained. Additionally, as we explained clearly about how we obtain an infinite saddle-node of type $\overline{\binom{1}{1}}SN$ from a phase portrait from the set (A), we will not mention anything about why we do not have no more possibilities (of obtaining an infinite saddle-node of type $\overline{\binom{1}{1}}SN$) beyond those ones that we will present.

Phase portrait $\mathbb{U}^1_{A,1}$ cannot have a phase portrait possessing an infinite saddle-node of type $\overline{\binom{1}{1}}SN$ as an evolution, since $\mathbb{U}^1_{A,1}$ has only the finite saddle-node $\overline{sn}_{(2)}$ and only the infinite node.

Phase portrait $\mathbb{U}^1_{A,2}$ has phase portrait $\mathbb{U}^2_{AC,1}$ as an evolution (see Figure 4.1). After bifurcation we get phase portrait $\mathbb{U}^1_{A,11}$ by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

Phase portrait $\mathbb{U}^1_{A,3}$ has phase portrait $\mathbb{U}^2_{AC,2}$ as an evolution (see Figure 4.2). After bifur-

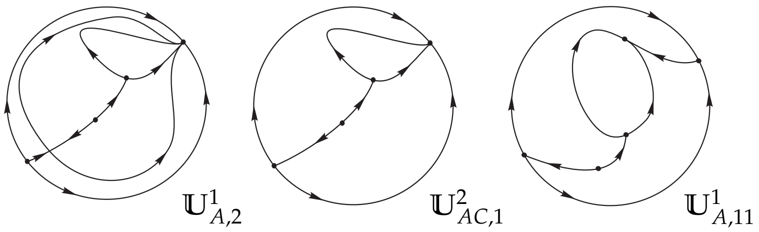


Figure 4.1: Unstable system $\mathbb{U}^2_{AC,1}$.

cation we get phase portrait $\mathbb{U}^1_{A,12}$ by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

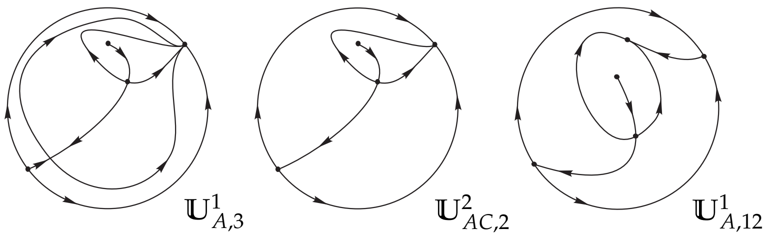


Figure 4.2: Unstable system $\mathbb{U}^2_{AC,2}$.

Phase portrait $\mathbb{U}^1_{A,4}$ cannot have a phase portrait possessing an infinite saddle-node of type $\overline{\binom{1}{1}}SN$ as an evolution. In fact, such a phase portrait possesses only an infinite node which receives four separatrices from finite singularities. Then by item $(d)-(2)$ of Lemma 4.1 the finite saddle cannot reach the infinite node. We point out that this same situation happens in many other phase portraits, such as in $\mathbb{U}^1_{A,5}$ to $\mathbb{U}^1_{A,8}$. Because it is quite simple to detect this phenomena, when we deal again with this situation we will skip all the details.

Phase portrait $\mathbb{U}^1_{A,9}$ has phase portrait $\mathbb{U}^2_{AC,3}$ as an evolution (see Figure 4.3). After bifurcation we get phase portrait $\mathbb{U}^1_{A,11}$ by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

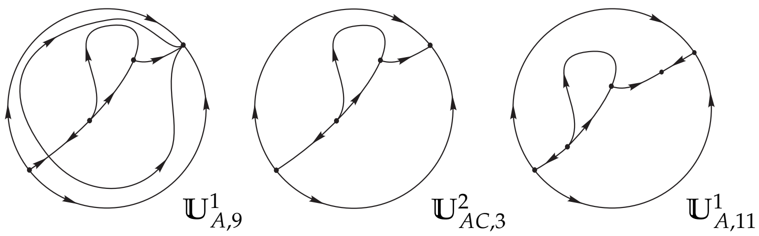


Figure 4.3: Unstable system $\mathbb{U}^2_{AC,3}$.

Phase portrait $\mathbb{U}^1_{A,10}$ has phase portrait $\mathbb{U}^2_{AC,4}$ as an evolution (see Figure 4.4). After bifurcation we get phase portrait $\mathbb{U}^1_{A,13}$ by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

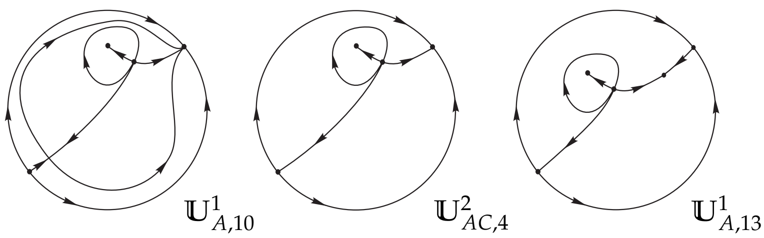


Figure 4.4: Unstable system $\mathbb{U}^2_{AC,4}$.

It is quite common that a given phase portrait of a certain codimension K be an unfolding of topologically distinct phase portraits of codimension $K+1$ (modulo limit cycles). This situation appears in this study. In the first column of Table 4.1 we present the phase portrait of the set (A), in the second column we indicate the corresponding phase portrait belonging to the set (AC), and in the third column we show the respective phase portrait after bifurcation. We point out that it is not necessary to present any explanation for the phase portraits present in the first column, since their corresponding elements from the third column already appeared and were explained before.

phase portrait from the set (A)	**phase portrait from the set** (AC)	**phase portrait from the set** (A)
$\mathbb{U}^1_{A,11}$	$\mathbb{U}^2_{AC,1}$ $\mathbb{U}^2_{AC,3}$	$\mathbb{U}^1_{A,2}$ $\mathbb{U}^1_{A,9}$
$\mathbb{U}^1_{A,12}$	$\mathbb{U}^2_{AC,2}$	$\mathbb{U}^1_{A,3}$
$\mathbb{U}^1_{A,13}$	$\mathbb{U}^2_{AC,4}$	$\mathbb{U}^1_{A,10}$

Table 4.1: Phase portraits from the set (AC) obtained from evolution of some elements of the set (A).

Phase portrait $\mathbb{U}^1_{A,14}$ has phase portrait $\mathbb{U}^2_{AC,5}$ as an evolution (see Figure 4.5). After bifurcation we get phase portrait $\mathbb{U}^1_{A,55}$ by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

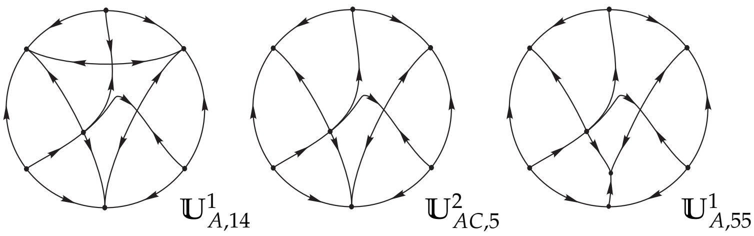


Figure 4.5: Unstable system $\mathbb{U}^2_{AC,5}$.

Phase portrait $\mathbb{U}^1_{A,15}$ has phase portraits $\mathbb{U}^2_{AC,6}$ and $\mathbb{U}^2_{AC,7}$ as evolution (see Figure 4.6). After bifurcation we get phase portraits $\mathbb{U}^1_{A,32}$ and $\mathbb{U}^1_{A,53}$, respectively, by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

Phase portrait $\mathbb{U}^1_{A,16}$ has phase portraits $\mathbb{U}^2_{AC,8}$, $\mathbb{U}^2_{AC,9}$, and $\mathbb{U}^2_{AC,10}$ as evolution (see Figure 4.7). After bifurcation we get phase portraits $\mathbb{U}^1_{A,33}$, $\mathbb{U}^1_{A,52}$, and $\mathbb{U}^1_{A,54}$, respectively, by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

Phase portrait $\mathbb{U}^1_{A,17}$ has phase portraits $\mathbb{U}^2_{AC,11}$, $\mathbb{U}^2_{AC,12}$, and $\mathbb{U}^2_{AC,13}$ as evolution (see Figure 4.8). After bifurcation we get phase portraits $\mathbb{U}^1_{A,35}$, $\mathbb{U}^1_{A,41}$, and $\mathbb{U}^1_{A,42}$, respectively, by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

Phase portrait $\mathbb{U}^1_{A,18}$ has phase portraits $\mathbb{U}^2_{AC,14}$, $\mathbb{U}^2_{AC,15}$, and $\mathbb{U}^2_{AC,16}$ as evolution (see Figure 4.9). After bifurcation we get phase portraits $\mathbb{U}^1_{A,25}$, $\mathbb{U}^1_{A,27}$, and $\mathbb{U}^1_{A,45}$, respectively, by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

Phase portraits $\mathbb{U}^1_{A,19}$, $\mathbb{U}^1_{A,20}$, and $\mathbb{U}^1_{A,21}$ cannot have a phase portrait possessing an infinite saddle-node of type $\overline{\binom{1}{1}}SN$ as an evolution since each one of them has only the finite saddle-node $\overline{sn}_{(2)}$.

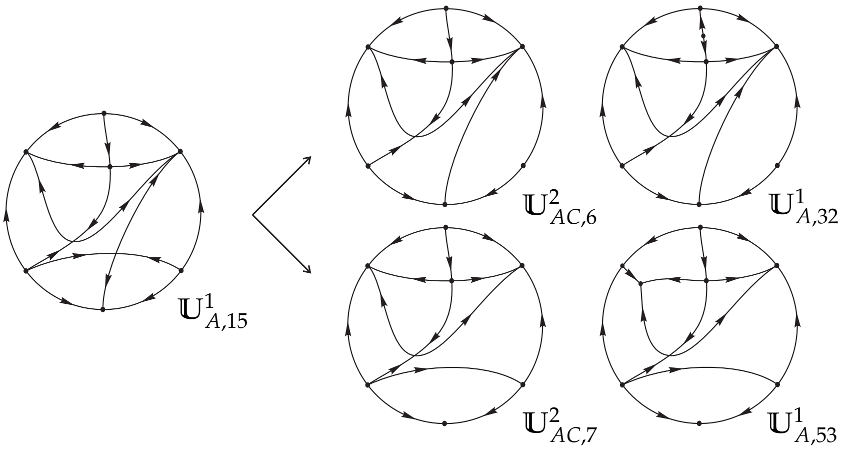


Figure 4.6: Unstable systems $\mathbb{U}^2_{AC,6}$ and $\mathbb{U}^2_{AC,7}$.

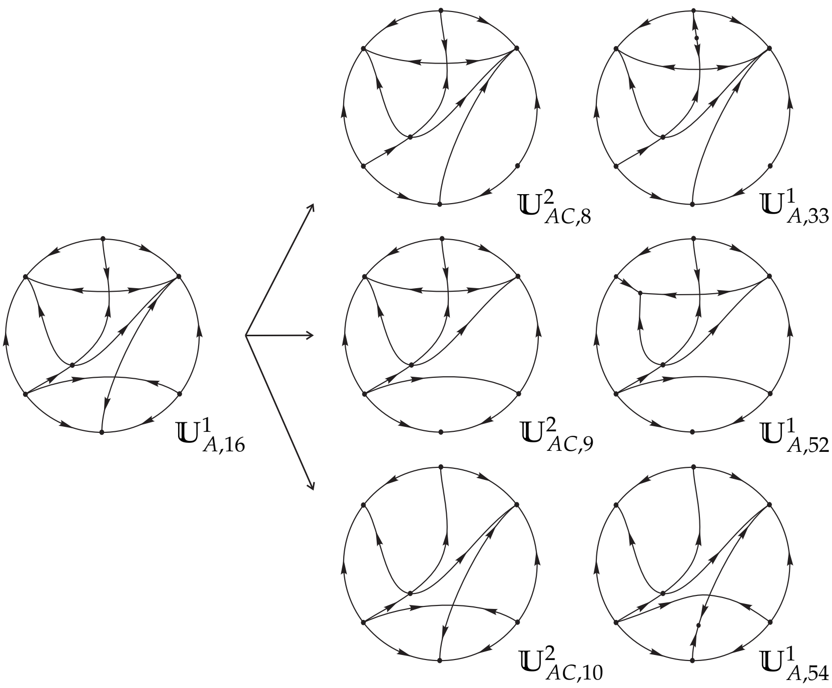


Figure 4.7: Unstable systems $\mathbb{U}^2_{AC,8}$, $\mathbb{U}^2_{AC,9}$, and $\mathbb{U}^2_{AC,10}$.

Phase portrait $\mathbb{U}^1_{A,22}$ has phase portrait $\mathbb{U}^2_{AC,17}$ as an evolution (see Figure 4.10). After bifurcation we get phase portrait $\mathbb{U}^1_{A,65}$ by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

Phase portrait $\mathbb{U}^1_{A,23}$ has phase portrait $\mathbb{U}^2_{AC,18}$ as an evolution (see Figure 4.11). After bifurcation we get phase portrait $\mathbb{U}^1_{A,66}$ by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

Phase portrait $\mathbb{U}^1_{A,24}$ cannot have a phase portrait possessing an infinite saddle-node of type $\overline{\binom{1}{1}}SN$ as an evolution since the finite saddle cannot reach the infinite node (by item $(d)-(2)$ of Lemma 4.1) and the finite node cannot reach the infinite saddle (because this elemental antisaddle is surrounded by the separatrices of the finite saddle).

Phase portrait $\mathbb{U}^1_{A,25}$ has three phase portraits as evolution.

1. $\mathbb{U}^2_{AC,19}$, see Figure 4.12, and after bifurcation we get phase portrait $\mathbb{U}^1_{A,56}$;

2. $\mathbb{U}^2_{AC,14}$, and its study was done when we spoke about $\mathbb{U}^1_{A,18}$;

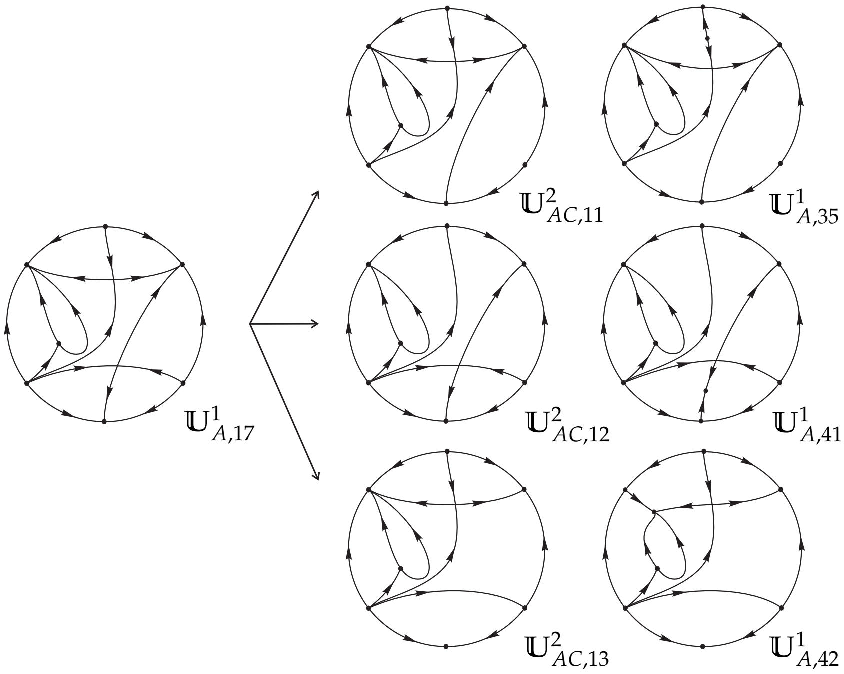


Figure 4.8: Unstable systems $\mathbb{U}^2_{AC,11}$, $\mathbb{U}^2_{AC,12}$, and $\mathbb{U}^2_{AC,13}$.

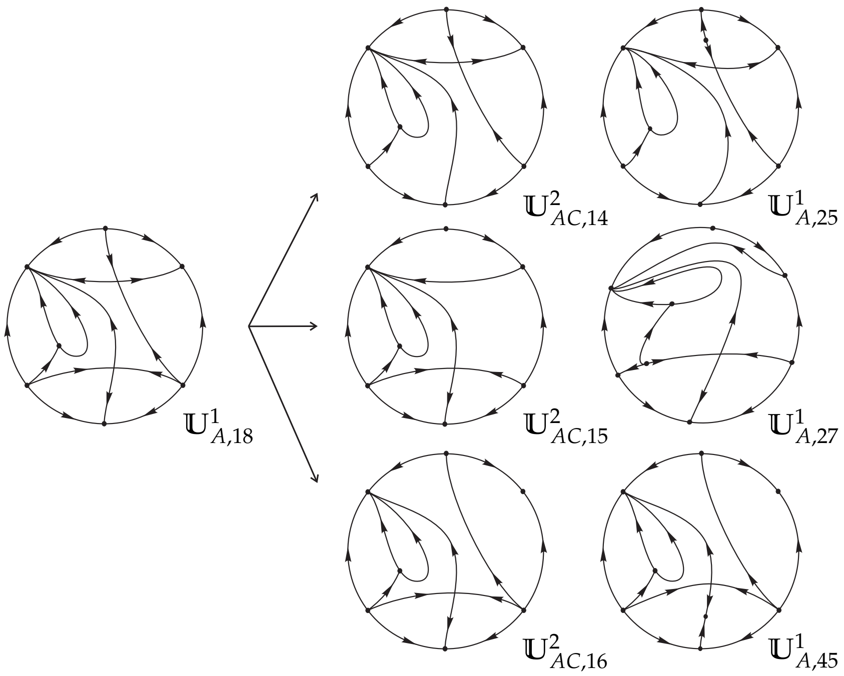


Figure 4.9: Unstable systems $\mathbb{U}^2_{AC,14}$, $\mathbb{U}^2_{AC,15}$, and $\mathbb{U}^2_{AC,16}$.

3. impossible phase portrait $\mathbb{U}^{2,I}_{AC,1}$. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{C,8}$ of *codimension one**, see Figure 4.13. We point out that, in the set (A), the corresponding unfolding of $\mathbb{U}^{2,I}_{AC,1}$ does not exist, since if such a phase portrait does exist, it would be an evolution of the impossible phase portrait $\mathbb{I}_{12,3}$ (see Figure 4.4 from [6]), which contradicts Theorem 2.11.

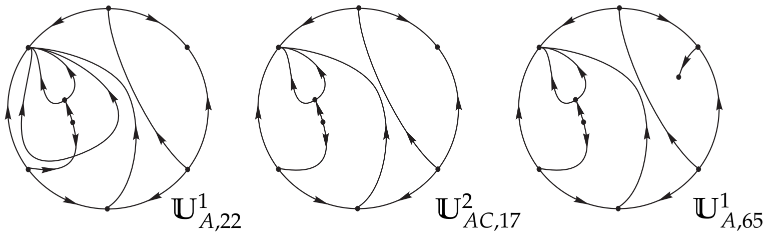


Figure 4.10: Unstable system $\mathbb{U}^2_{AC,17}$.

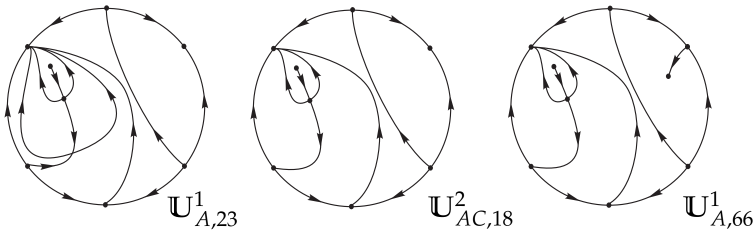


Figure 4.11: Unstable system $\mathbb{U}^2_{AC,18}$.

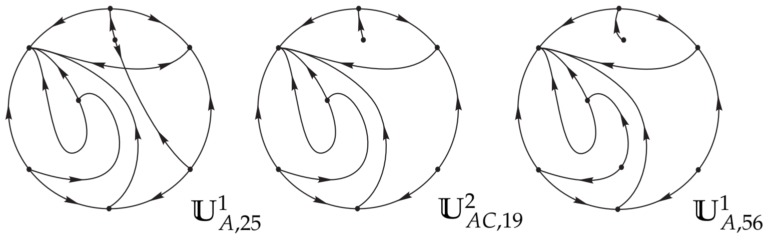


Figure 4.12: Unstable system $\mathbb{U}^2_{AC,19}$.

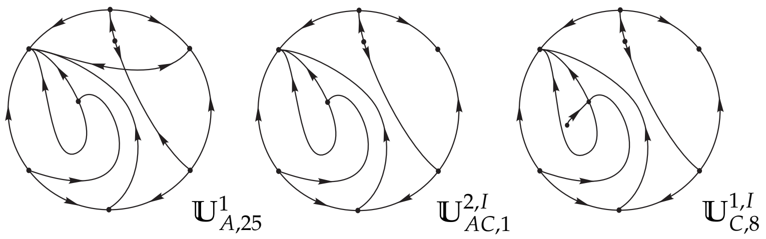


Figure 4.13: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AC,1}$.

Phase portrait $\mathbb{U}^1_{A,26}$ has phase portrait $\mathbb{U}^2_{AC,20}$ as an evolution (see Figure 4.14). After bifurcation we get phase portrait $\mathbb{U}^1_{A,67}$ by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

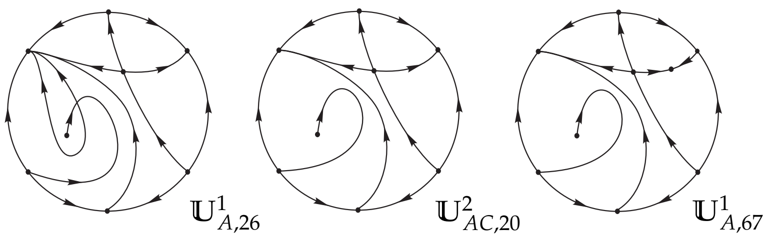


Figure 4.14: Unstable system $\mathbb{U}^2_{AC,20}$.

Phase portrait $\mathbb{U}^1_{A,27}$ has phase portraits $\mathbb{U}^2_{AC,21}$ and $\mathbb{U}^2_{AC,22}$ as evolution (see Figure 4.15). After bifurcation we get phase portraits $\mathbb{U}^1_{A,56}$ and $\mathbb{U}^1_{A,60}$, respectively, by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$. Moreover, $\mathbb{U}^1_{A,27}$ also has $\mathbb{U}^2_{AC,15}$ as an evolution, and this last one was mentioned before during the study of $\mathbb{U}^1_{A,18}$.

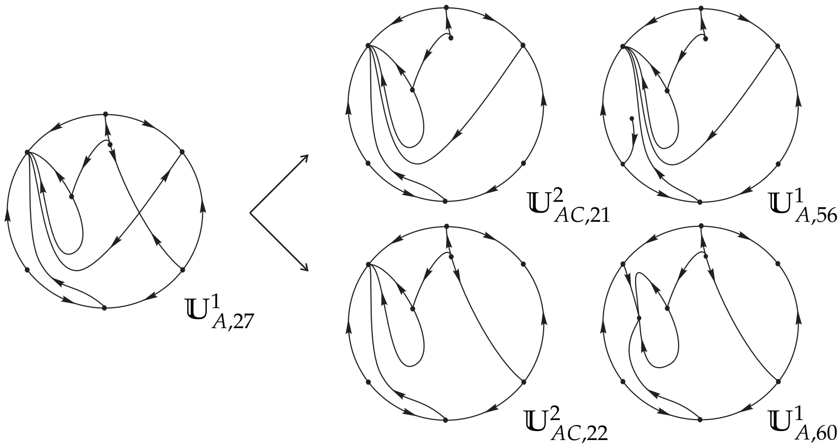


Figure 4.15: Unstable systems $\mathbb{U}^2_{AC,21}$ and $\mathbb{U}^2_{AC,22}$.

Phase portrait $\mathbb{U}^1_{A,28}$ has phase portraits $\mathbb{U}^2_{AC,23}$ and $\mathbb{U}^2_{AC,24}$ as evolution (see Figure 4.16). After bifurcation we get phase portraits $\mathbb{U}^1_{A,57}$ and $\mathbb{U}^1_{A,58}$, respectively, by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

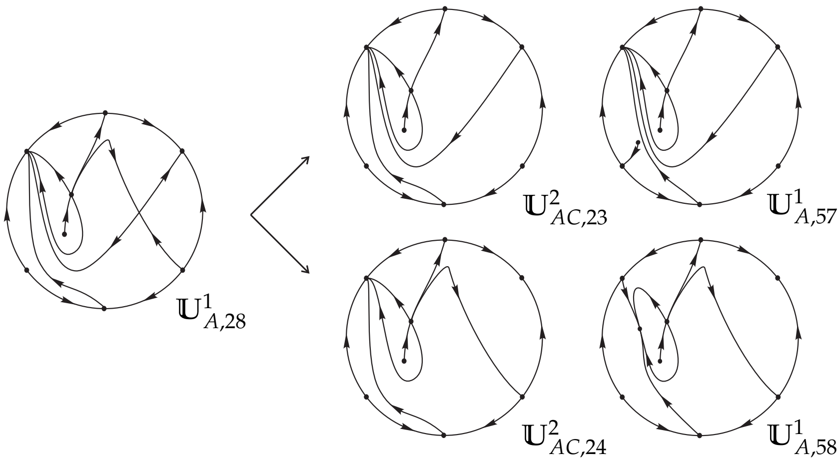


Figure 4.16: Unstable systems $\mathbb{U}^2_{AC,23}$ and $\mathbb{U}^2_{AC,24}$.

Phase portrait $\mathbb{U}^1_{A,29}$ cannot have a phase portrait possessing an infinite saddle-node of type $\overline{\binom{1}{1}}SN$ as an evolution since the finite saddle cannot reach the infinite node (by item $(d)-(2)$ of Lemma 4.1), the finite node cannot reach the infinite saddle (because this elemental antisaddle is surrounded by the separatrices of the finite saddle) and the finite saddle-node cannot go to infinity (as we have discussed during the analysis of $\mathbb{U}^1_{A,1}$).

Phase portrait $\mathbb{U}^1_{A,30}$ has phase portrait $\mathbb{U}^2_{AC,25}$ as an evolution (see Figure 4.17). After bifurcation we get phase portrait $\mathbb{U}^1_{A,69}$ by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

Phase portrait $\mathbb{U}^1_{A,31}$ has phase portrait $\mathbb{U}^2_{AC,26}$ as an evolution (see Figure 4.18). After bifurcation we get phase portrait $\mathbb{U}^1_{A,61}$ by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

Phase portrait $\mathbb{U}^1_{A,32}$ has phase portrait $\mathbb{U}^2_{AC,27}$ as an evolution (see Figure 4.19). After bifurcation we get phase portrait $\mathbb{U}^1_{A,61}$ by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$. Moreover,

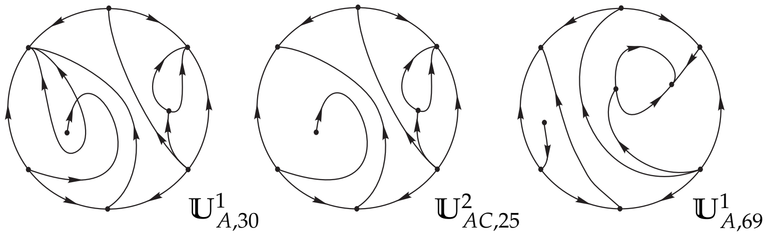


Figure 4.17: Unstable system $\mathbb{U}^2_{AC,25}$.

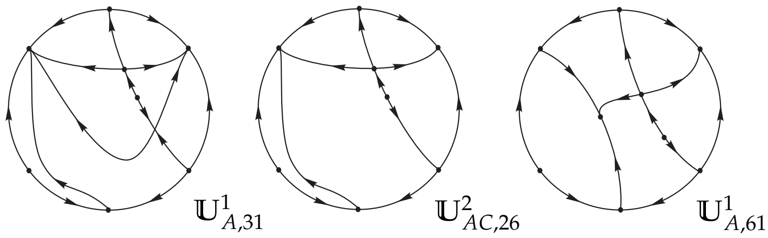


Figure 4.18: Unstable system $\mathbb{U}^2_{AC,26}$.

$\mathbb{U}^1_{A,32}$ also has $\mathbb{U}^2_{AC,6}$ as an evolution, and this last one was mentioned before during the study of $\mathbb{U}^1_{A,15}$.

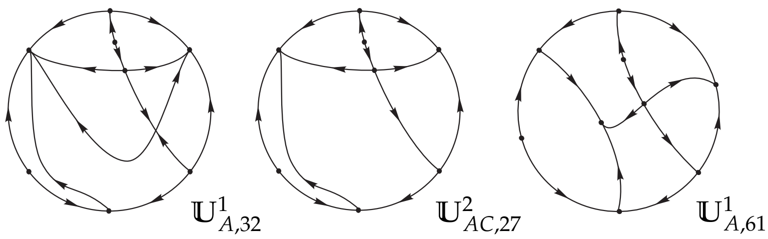


Figure 4.19: Unstable system $\mathbb{U}^2_{AC,27}$.

Phase portrait $\mathbb{U}^1_{A,33}$ has phase portrait $\mathbb{U}^2_{AC,8}$ as an evolution and this last one was mentioned before during the study of $\mathbb{U}^1_{A,16}$.

Phase portrait $\mathbb{U}^1_{A,34}$ cannot have a phase portrait possessing an infinite saddle-node of type $\overline{\binom{1}{1}}SN$ as an evolution, we can conclude this fact by using the same arguments as used for $\mathbb{U}^1_{A,29}$.

Phase portrait $\mathbb{U}^1_{A,35}$ has phase portrait $\mathbb{U}^2_{AC,11}$ as an evolution and this last one was mentioned before during the study of $\mathbb{U}^1_{A,17}$.

Phase portrait $\mathbb{U}^1_{A,36}$ has phase portrait $\mathbb{U}^2_{AC,28}$ as an evolution (see Figure 4.20). After bifurcation we get phase portrait $\mathbb{U}^1_{A,69}$ by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

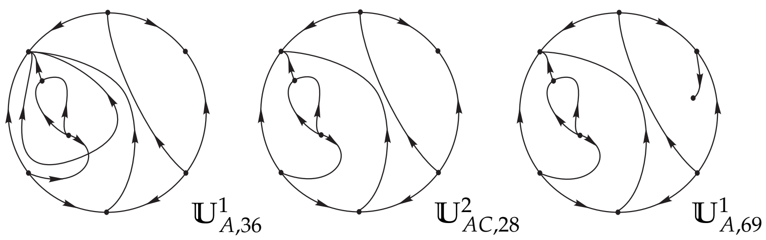


Figure 4.20: Unstable system $\mathbb{U}^2_{AC,28}$.

Phase portrait $\mathbb{U}^1_{A,37}$ has phase portrait $\mathbb{U}^2_{AC,29}$ as an evolution (see Figure 4.21). After bifurcation we get phase portrait $\mathbb{U}^1_{A,70}$ by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

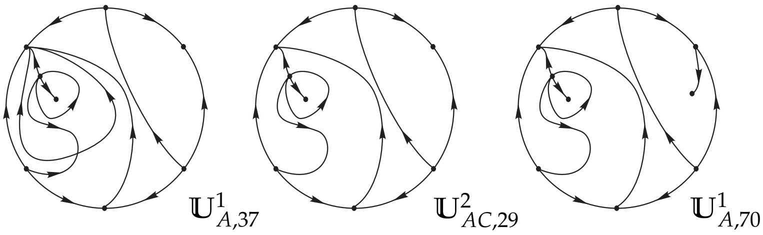


Figure 4.21: Unstable system $\mathbb{U}^2_{AC,29}$.

Phase portrait $\mathbb{U}^1_{A,38}$ cannot have a phase portrait possessing an infinite saddle-node of type $\overline{\binom{1}{1}}SN$ as an evolution.

Phase portrait $\mathbb{U}^1_{A,39}$ has phase portrait $\mathbb{U}^2_{AC,30}$ as an evolution (see Figure 4.22). After bifurcation we get phase portrait $\mathbb{U}^1_{A,65}$ by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

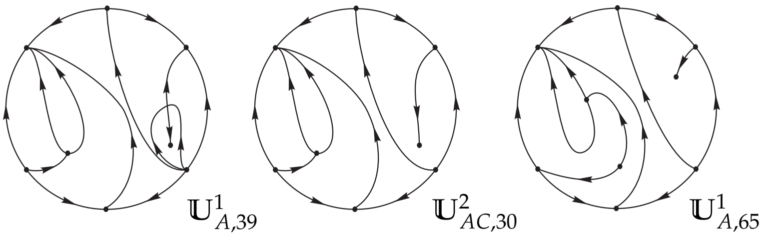


Figure 4.22: Unstable system $\mathbb{U}^2_{AC,30}$.

Phase portrait $\mathbb{U}^1_{A,40}$ cannot have a phase portrait possessing an infinite saddle-node of type $\overline{\binom{1}{1}}SN$ as an evolution.

Phase portrait $\mathbb{U}^1_{A,41}$ has three phase portraits as evolution.

1. $\mathbb{U}^2_{AC,31}$, see Figure 4.23, and after bifurcation we get phase portrait $\mathbb{U}^1_{A,63}$;

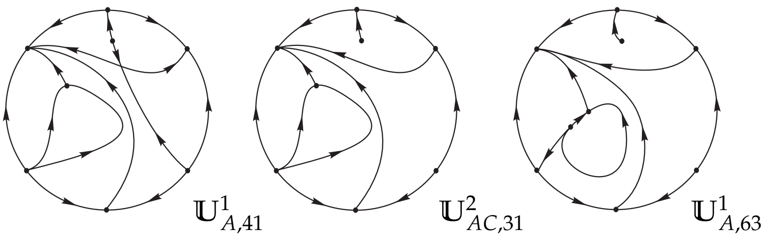


Figure 4.23: Unstable system $\mathbb{U}^2_{AC,31}$.

2. $\mathbb{U}^2_{AC,12}$, and its study was done when we spoke about $\mathbb{U}^1_{A,17}$;

3. impossible phase portrait $\mathbb{U}^{2,I}_{AC,2}$. By Theorem 2.11 such a phase portrait is impossible because by splitting the original finite saddle-node into a saddle and a node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{C,9}$ of *codimension one**, see Figure 4.24. We point out that, in the set (A), the corresponding unfolding of $\mathbb{U}^{2,I}_{AC,2}$ does not exist, since if such a phase portrait does exist, it would be an evolution of the impossible phase portrait $\mathbb{I}_{12,2}$ (see Figure 4.4 from [6]), which contradicts Theorem 2.11.

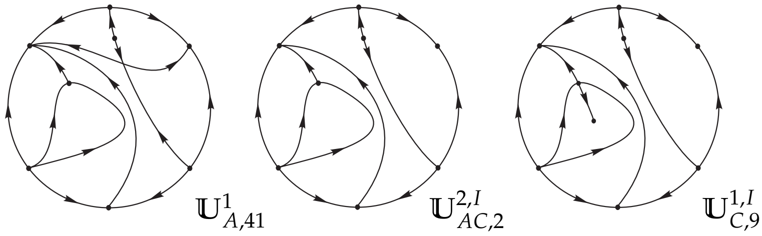


Figure 4.24: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AC,2}$.

Phase portrait $\mathbb{U}^1_{A,42}$ has phase portraits $\mathbb{U}^2_{AC,32}$ and $\mathbb{U}^2_{AC,33}$ as evolution (see Figure 4.25). After bifurcation we get phase portraits $\mathbb{U}^1_{A,60}$ and $\mathbb{U}^1_{A,63}$, respectively, by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$. Moreover, $\mathbb{U}^1_{A,42}$ also has $\mathbb{U}^2_{AC,13}$ as an evolution, and this last one was mentioned before during the study of $\mathbb{U}^1_{A,17}$.

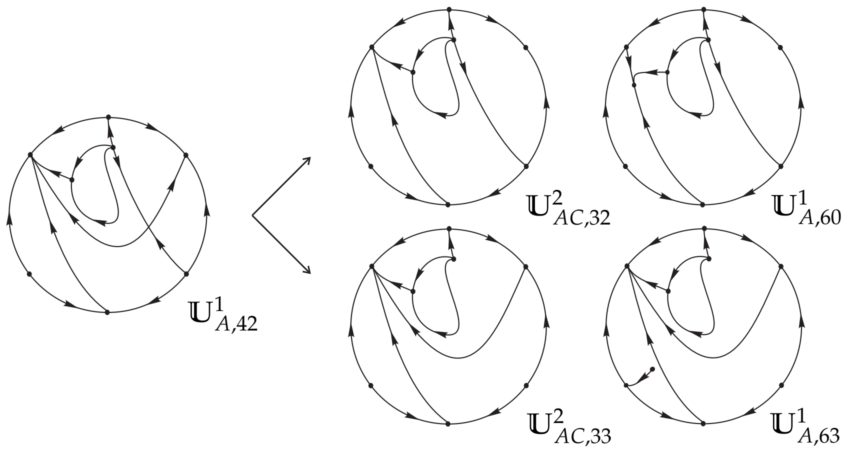


Figure 4.25: Unstable systems $\mathbb{U}^2_{AC,32}$ and $\mathbb{U}^2_{AC,33}$.

Phase portrait $\mathbb{U}^1_{A,43}$ has phase portraits $\mathbb{U}^2_{AC,34}$ and $\mathbb{U}^2_{AC,35}$ as evolution (see Figure 4.26). After bifurcation we get phase portraits $\mathbb{U}^1_{A,59}$ and $\mathbb{U}^1_{A,64}$, respectively, by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

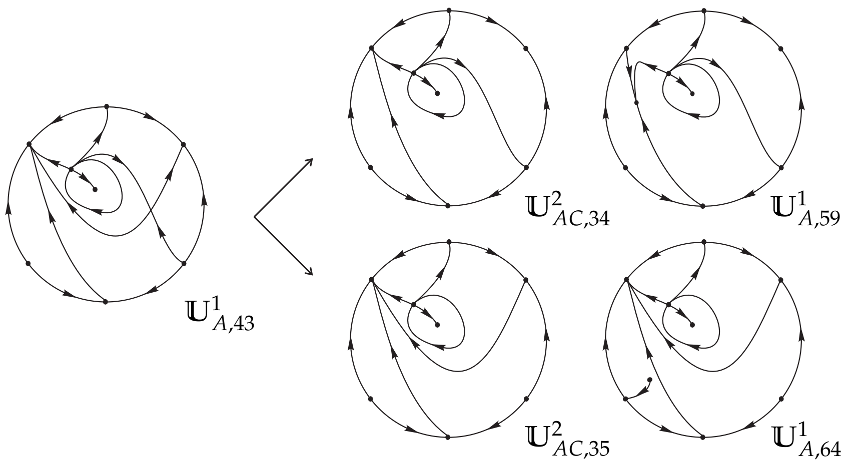


Figure 4.26: Unstable systems $\mathbb{U}^2_{AC,34}$ and $\mathbb{U}^2_{AC,35}$.

Phase portrait $\mathbb{U}^1_{A,44}$ cannot have a phase portrait possessing an infinite saddle-node of type $\overline{\binom{1}{1}}SN$ as an evolution.

Phase portrait $\mathbb{U}^1_{A,45}$ has phase portrait $\mathbb{U}^2_{AC,16}$ as an evolution, and this last one was mentioned before during the study of $\mathbb{U}^1_{A,18}$.

Phase portraits $\mathbb{U}^1_{A,46}$ to $\mathbb{U}^1_{A,48}$ and also $\mathbb{U}^1_{A,50}$ cannot have a phase portrait possessing an infinite saddle-node of type $\overline{\binom{1}{1}}SN$ as an evolution.

Phase portrait $\mathbb{U}^1_{A,51}$ has phase portrait $\mathbb{U}^2_{AC,36}$ as an evolution (see Figure 4.27). After bifurcation we get phase portrait $\mathbb{U}^1_{A,67}$ by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$.

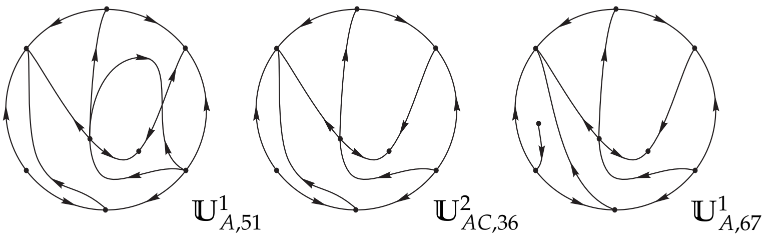


Figure 4.27: Unstable system $\mathbb{U}^2_{AC,36}$.

Phase portrait $\mathbb{U}^1_{A,52}$ has phase portrait $\mathbb{U}^2_{AC,37}$ as an evolution (see Figure 4.28). After bifurcation we get phase portrait $\mathbb{U}^1_{A,68}$, by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$. Moreover, $\mathbb{U}^1_{A,52}$ also has $\mathbb{U}^2_{AC,9}$ as an evolution, and this last one was mentioned before during the study of $\mathbb{U}^1_{A,16}$.

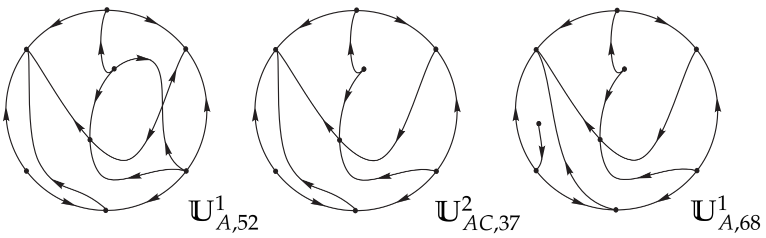


Figure 4.28: Unstable system $\mathbb{U}^2_{AC,37}$.

Phase portrait $\mathbb{U}^1_{A,53}$ has phase portrait $\mathbb{U}^2_{AC,7}$ as an evolution, and this last one was mentioned before during the study of $\mathbb{U}^1_{A,15}$.

Phase portrait $\mathbb{U}^1_{A,54}$ has phase portrait $\mathbb{U}^2_{AC,38}$ as an evolution (see Figure 4.29). After bifurcation we get phase portrait $\mathbb{U}^1_{A,68}$, by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$. Moreover, $\mathbb{U}^1_{A,54}$ also has $\mathbb{U}^2_{AC,10}$ as an evolution, and this last one was mentioned before during the study of $\mathbb{U}^1_{A,16}$.

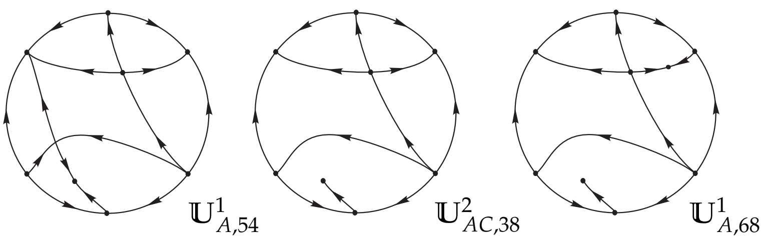


Figure 4.29: Unstable system $\mathbb{U}^2_{AC,38}$.

Phase portrait $\mathbb{U}^1_{A,55}$ has phase portraits $\mathbb{U}^2_{AC,39}$ and $\mathbb{U}^2_{AC,40}$ as evolution (see Figure 4.30). After bifurcation we get phase portraits $\mathbb{U}^1_{A,61}$ and $\mathbb{U}^1_{A,62}$, respectively, by splitting the infinite saddle-node $\overline{\binom{1}{1}}SN$. Moreover, $\mathbb{U}^1_{A,55}$ also has $\mathbb{U}^2_{AC,5}$ as an evolution, and this last one was mentioned before during the study of $\mathbb{U}^1_{A,14}$.

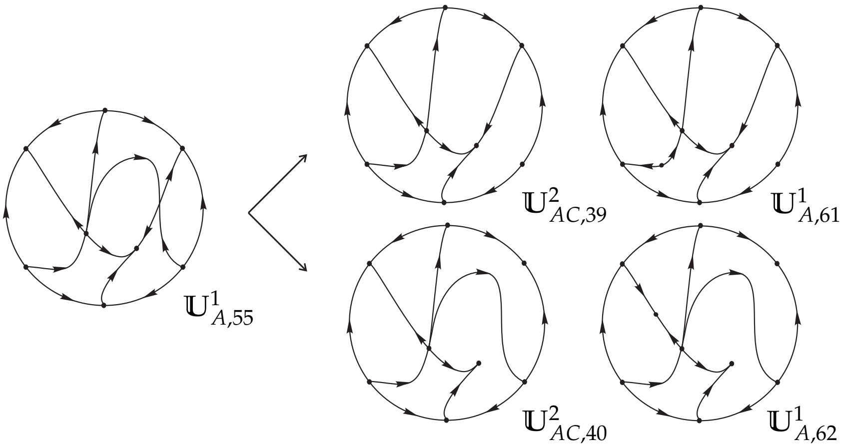


Figure 4.30: Unstable systems $\mathbb{U}^2_{AC,39}$ and $\mathbb{U}^2_{AC,40}$.

Phase portrait $\mathbb{U}^1_{A,56}$ has phase portraits $\mathbb{U}^2_{AC,19}$ and $\mathbb{U}^2_{AC,21}$ as evolution. These two phase portraits were obtained during the study of $\mathbb{U}^1_{A,25}$ and $\mathbb{U}^1_{A,27}$, respectively.

Phase portrait $\mathbb{U}^1_{A,57}$ has phase portrait $\mathbb{U}^2_{AC,23}$ as an evolution and this last one was obtained during the study of $\mathbb{U}^1_{A,28}$.

Phase portrait $\mathbb{U}^1_{A,58}$ has phase portrait $\mathbb{U}^2_{AC,24}$ as an evolution and this last one was obtained during the study of $\mathbb{U}^1_{A,28}$. Moreover, $\mathbb{U}^1_{A,58}$ has a second phase portrait which is topologically equivalent to $\mathbb{U}^2_{AC,24}$.

Phase portrait $\mathbb{U}^1_{A,59}$ has phase portrait $\mathbb{U}^2_{AC,34}$ as an evolution and this last one was obtained during the study of $\mathbb{U}^1_{A,43}$. Moreover, $\mathbb{U}^1_{A,59}$ has the impossible phase portrait $\mathbb{U}^{2,I}_{AC,3}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the obtained infinite saddle-node $\overline{\binom{1}{1}}SN$ into a finite saddle and an infinite node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{A,104}$ of *codimension one**, see Figure 4.31. We observe that, in the set (C), $\mathbb{U}^{2,I}_{AC,3}$ unfolds in $\mathbb{U}^1_{C,17}$ (modulo limit cycles).

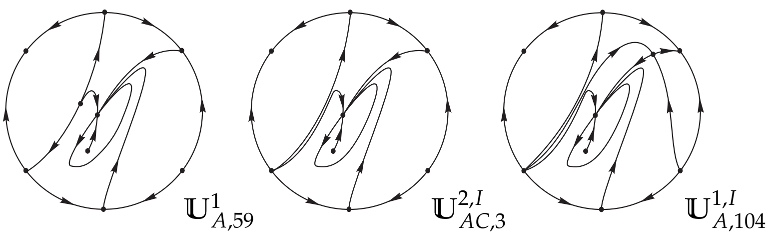


Figure 4.31: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AC,3}$.

In the first column of Table 4.2 we present the remaining phase portraits of the set (A), in the second column we indicate its corresponding phase portrait belonging to the set (AC), and in the third column we show the corresponding phase portrait after bifurcation. We point out that it is not necessary to present any explanation for the phase portraits present in the first column, since their corresponding elements from the third column already appeared and were explained before.

Therefore, we have just finished obtaining all the 40 topologically potential phase portraits of *codimension two** from the set (AC) presented in Figures 1.4 and 1.5.

phase portrait from the set (*A*)	**phase portrait from the set (*AC*)**	**phase portrait from the set (*A*)**
$\mathbb{U}^1_{A,60}$	$\mathbb{U}^2_{AC,22}$ $\mathbb{U}^2_{AC,32}$	$\mathbb{U}^1_{A,27}$ $\mathbb{U}^1_{A,42}$
$\mathbb{U}^1_{A,61}$	$\mathbb{U}^2_{AC,26}$ $\mathbb{U}^2_{AC,27}$ $\mathbb{U}^2_{AC,39}$	$\mathbb{U}^1_{A,31}$ $\mathbb{U}^1_{A,32}$ $\mathbb{U}^1_{A,55}$
$\mathbb{U}^1_{A,62}$	$\mathbb{U}^2_{AC,40}$	$\mathbb{U}^1_{A,55}$
$\mathbb{U}^1_{A,63}$	$\mathbb{U}^2_{AC,31}$ $\mathbb{U}^2_{AC,33}$	$\mathbb{U}^1_{A,41}$ $\mathbb{U}^1_{A,42}$
$\mathbb{U}^1_{A,64}$	$\mathbb{U}^2_{AC,35}$	$\mathbb{U}^1_{A,43}$
$\mathbb{U}^1_{A,65}$	$\mathbb{U}^2_{AC,17}$ $\mathbb{U}^2_{AC,30}$	$\mathbb{U}^1_{A,22}$ $\mathbb{U}^1_{A,39}$
$\mathbb{U}^1_{A,66}$	$\mathbb{U}^2_{AC,18}$	$\mathbb{U}^1_{A,23}$
$\mathbb{U}^1_{A,67}$	$\mathbb{U}^2_{AC,20}$ $\mathbb{U}^2_{AC,36}$	$\mathbb{U}^1_{A,26}$ $\mathbb{U}^1_{A,51}$
$\mathbb{U}^1_{A,68}$	$\mathbb{U}^2_{AC,37}$ $\mathbb{U}^2_{AC,38}$	$\mathbb{U}^1_{A,52}$ $\mathbb{U}^1_{A,54}$
$\mathbb{U}^1_{A,69}$	$\mathbb{U}^2_{AC,25}$ $\mathbb{U}^2_{AC,28}$	$\mathbb{U}^1_{A,30}$ $\mathbb{U}^1_{A,36}$
$\mathbb{U}^1_{A,70}$	$\mathbb{U}^2_{AC,29}$	$\mathbb{U}^1_{A,37}$

Table 4.2: Phase portraits from the set (AC) obtained from evolution of some elements of the set (A).

Now we explain how one can obtain these 40 phase portraits by starting the study from the set (C). Let us consider all the 32 realizable structurally unstable quadratic vector fields of *codimension one** from the set (C). In order to obtain a phase portrait of *codimension two** belonging to the set (AC) starting from a phase portrait of *codimension one** of the set (C), we keep the existing infinite saddle-node $\overline{\binom{1}{1}}SN$ and by using Theorem 2.6 we build a finite saddle-node $\overline{sn}_{(2)}$ by the coalescence of a finite node with a finite saddle. On the other hand, from the phase portraits of *codimension two** from the set (AC), there exist two ways of obtaining phase portraits of *codimension one** also belonging to the set (C) after perturbation: splitting $\overline{sn}_{(2)}$ into a saddle and a node, or moving it to complex singularities (remember Remark 3.2).

According to these facts, if a phase portrait possesses only a finite saddle-node, as $\mathbb{U}^1_{C,1}$ for instance, it is not possible to obtain a phase portrait from it which belongs to the set (AC). Moreover, in some cases when one makes the finite saddle-node disappear, it is possible to find a phase portrait possessing a limit cycle, as happens for instance with phase portrait $\mathbb{U}^1_{C,3}$ (see Figure 4.32). In such a figure we present the two potential phase portraits which can be obtained by forming a finite saddle-node and then by making it disappear. Indeed, phase portrait $\mathbb{U}^1_{C,3}$ has phase portraits $\mathbb{U}^2_{AC,3}$ and $\mathbb{U}^2_{AC,4}$ as evolution, respectively, by the coalescence of the finite saddle with each one of the two finite nodes. After bifurcation, by making the finite saddle-node disappear, from $\mathbb{U}^2_{AC,3}$ we get $\mathbb{U}^1_{C,1}$ and from $\mathbb{U}^2_{AC,4}$ we obtain $\mathbb{U}^1_{C,1}$, being this last one with a limit cycle. However, as our classification of phase portraits is always done modulo limit cycles we simply say that in this case from $\mathbb{U}^2_{AC,4}$ we have $\mathbb{U}^1_{C,1}$. This situation also happens when we perform analogous studies of phase portraits $\mathbb{U}^1_{C,20}$, $\mathbb{U}^1_{C,24}$, and $\mathbb{U}^1_{C,31}$, as we will see in the sequence.

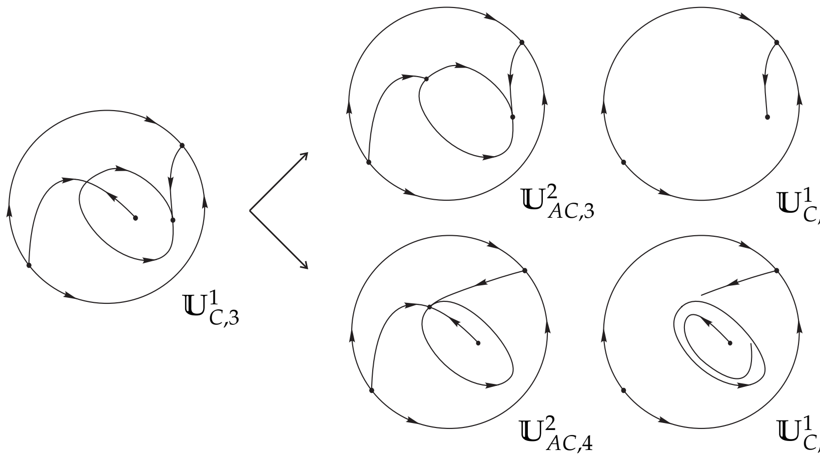


Figure 4.32: Unstable systems $\mathbb{U}^2_{AC,3}$ and $\mathbb{U}^2_{AC,4}$ from phase portrait $\mathbb{U}^1_{C,3}$.

The main goal of this section is to obtain all the topologically potential phase portraits from the set (AC) and then prove their realization or show that they are not possible. So we have to be sure that no other phase portrait can be found if one does some evolution in all elements of the set (C) in order to obtain a phase portrait belonging to the set (AC). We point out that we have done this verification, i.e. we have also considered each element from the set (C) and produced a coalescence (when it was possible) of a finite node with a finite saddle and we also have obtained the 40 topologically potential phase portraits of *codimension two** from the set (AC) presented in Figures 1.4 and 1.5. Moreover, doing this verification we have not found the impossible phase portraits $\mathbb{U}^{2,I}_{AC,1}$ and $\mathbb{U}^{2,I}_{AC,2}$ (this was expected since the corresponding unfoldings of *codimension one** are impossible in the set (C)). In Table 4.3 we present the study of phase portraits $\mathbb{U}^1_{C,2}$ to $\mathbb{U}^1_{C,19}$. In the first column of the mentioned table we present the phase portrait of the set (C), in the second column we indicate its corresponding phase portrait belonging to the set (AC) i.e. after producing a finite saddle-node $\overline{sn}_{(2)}$, and in the third column we show the corresponding phase portrait after we make this finite saddle-node $\overline{sn}_{(2)}$ disappear. Note that the sequence of indexes in the first column is not consecutive since in some phase portraits from the set (C) it is not possible to produce a finite saddle-node $\overline{sn}_{(2)}$ and then it is not possible to obtain a phase portrait belonging to the set (AC).

Phase portrait $\mathbb{U}^1_{C,20}$ has phase portraits $\mathbb{U}^2_{AC,32}$ and $\mathbb{U}^2_{AC,34}$ as evolution. After bifurcation we get phase portrait $\mathbb{U}^1_{C,17}$ for both cases (being one of them modulo limit cycles), by making the finite saddle-node $\overline{sn}_{(2)}$ disappear. Moreover, phase portrait $\mathbb{U}^1_{C,20}$ also has a phase portrait which is topologically equivalent to impossible phase portrait $\mathbb{U}^{2,I}_{AC,3}$, obtained before during the study of phase portrait $\mathbb{U}^{1,I}_{A,59}$. Again, by Theorem 2.11 such a phase portrait is impossible because by splitting the original infinite saddle-node $\overline{\binom{1}{1}}SN$ into a finite saddle and an infinite node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{A,104}$ of *codimension one**, see Figure 4.33. Also, in the set (C), $\mathbb{U}^{2,I}_{AC,3}$ unfolds in $\mathbb{U}^1_{C,17}$ (modulo limit cycles).

Phase portrait $\mathbb{U}^1_{C,21}$ has phase portraits $\mathbb{U}^2_{AC,22}$ and $\mathbb{U}^2_{AC,24}$ as evolution. After bifurcation we get phase portrait $\mathbb{U}^1_{C,17}$ for both cases, by making the finite saddle-node $\overline{sn}_{(2)}$ disappear.

Phase portrait $\mathbb{U}^1_{C,22}$ has phase portraits $\mathbb{U}^2_{AC,40}$ and $\mathbb{U}^2_{AC,39}$ as evolution. After bifurcation we get phase portraits $\mathbb{U}^1_{C,15}$ and $\mathbb{U}^1_{C,17}$, respectively, by making the finite saddle-node $\overline{sn}_{(2)}$

phase portrait from the set (C)	phase portrait from the set (AC)	phase portrait from the set (C)
$\mathbb{U}^1_{C,2}$	$\mathbb{U}^2_{AC,1}$ $\mathbb{U}^2_{AC,2}$	$\mathbb{U}^1_{C,1}$
$\mathbb{U}^1_{C,3}$	$\mathbb{U}^2_{AC,3}$ $\mathbb{U}^2_{AC,4}$	$\mathbb{U}^1_{C,1}$
$\mathbb{U}^1_{C,5}$	$\mathbb{U}^2_{AC,14}$	$\mathbb{U}^1_{C,4}$
$\mathbb{U}^1_{C,6}$	$\mathbb{U}^2_{AC,15}$	$\mathbb{U}^1_{C,4}$
$\mathbb{U}^1_{C,7}$	$\mathbb{U}^2_{AC,6}$ $\mathbb{U}^2_{AC,8}$	$\mathbb{U}^1_{C,4}$
$\mathbb{U}^1_{C,8}$	$\mathbb{U}^2_{AC,11}$	$\mathbb{U}^1_{C,4}$
$\mathbb{U}^1_{C,9}$	$\mathbb{U}^2_{AC,12}$	$\mathbb{U}^1_{C,4}$
$\mathbb{U}^1_{C,10}$	$\mathbb{U}^2_{AC,13}$	$\mathbb{U}^1_{C,4}$
$\mathbb{U}^1_{C,11}$	$\mathbb{U}^2_{AC,16}$	$\mathbb{U}^1_{C,4}$
$\mathbb{U}^1_{C,12}$	$\mathbb{U}^2_{AC,7}$ $\mathbb{U}^2_{AC,9}$	$\mathbb{U}^1_{C,4}$
$\mathbb{U}^1_{C,13}$	$\mathbb{U}^2_{AC,10}$	$\mathbb{U}^1_{C,4}$
$\mathbb{U}^1_{C,14}$	$\mathbb{U}^2_{AC,5}$	$\mathbb{U}^1_{C,4}$
$\mathbb{U}^1_{C,18}$	$\mathbb{U}^2_{AC,21}$ $\mathbb{U}^2_{AC,23}$	$\mathbb{U}^1_{C,15}$
$\mathbb{U}^1_{C,19}$	$\mathbb{U}^2_{AC,19}$	$\mathbb{U}^1_{C,15}$

Table 4.3: Phase portraits from the set (AC) obtained from evolution of elements of the set (C).

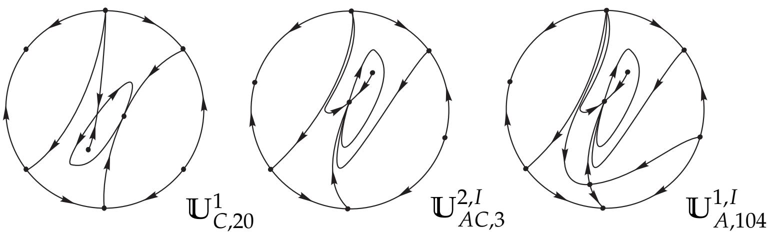


Figure 4.33: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AC,3}$ (see again Figure 4.31).

disappear.

Phase portrait $\mathbb{U}^1_{C,23}$ has phase portraits $\mathbb{U}^2_{AC,26}$ and $\mathbb{U}^2_{AC,27}$ as evolution. After bifurcation we get phase portrait $\mathbb{U}^1_{C,17}$ for both cases, by making the finite saddle-node $\overline{sn}_{(2)}$ disappear.

Phase portrait $\mathbb{U}^1_{C,24}$ has phase portraits $\mathbb{U}^2_{AC,33}$ and $\mathbb{U}^2_{AC,35}$ as evolution. After bifurcation we get phase portrait $\mathbb{U}^1_{C,15}$ for both cases (being one of them modulo limit cycles), by making the finite saddle-node $\overline{sn}_{(2)}$ disappear. Moreover, phase portrait $\mathbb{U}^1_{C,24}$ also has the impossible phase portrait $\mathbb{U}^{2,I}_{AC,4}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original infinite saddle-node $\overline{\binom{1}{1}}SN$ into a finite saddle and an infinite node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{A,104}$ of *codimension one**, see Figure 4.34. We observe that, in the set (C), $\mathbb{U}^{2,I}_{AC,4}$ unfolds in $\mathbb{U}^1_{C,15}$ (modulo limit cycles).

In Table 4.4 we present the study of phase portraits $\mathbb{U}^1_{C,25}$ to $\mathbb{U}^1_{C,30}$ and we follow the same

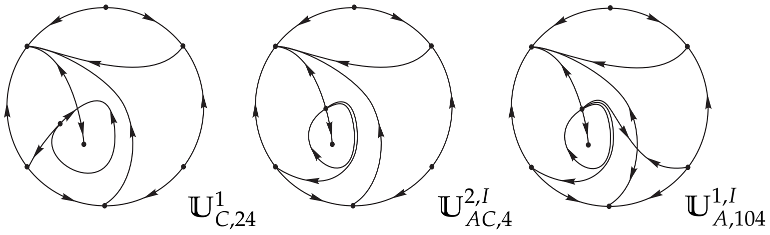


Figure 4.34: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AC,4}$.

pattern used in Table 4.3.

phase portrait from the set (C)	**phase portrait from the set (AC)**	**phase portrait from the set (C)**
$\mathbb{U}^{1}_{C,25}$	$\mathbb{U}^{2}_{AC,31}$	$\mathbb{U}^{1}_{C,15}$
$\mathbb{U}^{1}_{C,26}$	$\mathbb{U}^{2}_{AC,17}$ $\mathbb{U}^{2}_{AC,18}$	$\mathbb{U}^{1}_{C,16}$
$\mathbb{U}^{1}_{C,27}$	$\mathbb{U}^{2}_{AC,30}$	$\mathbb{U}^{1}_{C,16}$
$\mathbb{U}^{1}_{C,28}$	$\mathbb{U}^{2}_{AC,38}$	$\mathbb{U}^{1}_{C,15}$
$\mathbb{U}^{1}_{C,29}$	$\mathbb{U}^{2}_{AC,20}$	$\mathbb{U}^{1}_{C,16}$
$\mathbb{U}^{1}_{C,30}$	$\mathbb{U}^{2}_{AC,37}$ $\mathbb{U}^{2}_{AC,36}$	$\mathbb{U}^{1}_{C,15}$ $\mathbb{U}^{1}_{C,16}$

Table 4.4: Phase portraits from the set (AC) obtained from evolution of elements of the set (C).

Phase portrait $\mathbb{U}^{1}_{C,31}$ has phase portraits $\mathbb{U}^{2}_{AC,28}$ and $\mathbb{U}^{2}_{AC,29}$ as evolution. After bifurcation we get phase portrait $\mathbb{U}^{1}_{C,16}$ for both cases (being one of them modulo limit cycles), by making the finite saddle-node $\overline{sn}_{(2)}$ disappear. Moreover, phase portrait $\mathbb{U}^{1}_{C,31}$ also has the impossible phase portrait $\mathbb{U}^{2,I}_{AC,5}$ as an evolution. By Theorem 2.11 such a phase portrait is impossible because by splitting the original infinite saddle-node $\overline{\binom{1}{1}}SN$ into an infinite saddle and a finite node we obtain the impossible phase portrait $\mathbb{U}^{1,I}_{A,106}$ of *codimension one**, see Figure 4.35. We observe that, in the set (C), $\mathbb{U}^{2,I}_{AC,5}$ unfolds in $\mathbb{U}^{1}_{C,16}$ (modulo limit cycles).

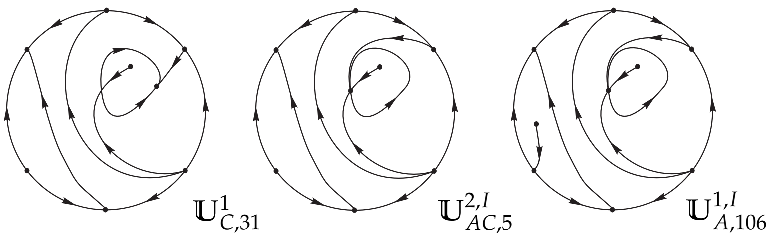


Figure 4.35: Impossible unstable phase portrait $\mathbb{U}^{2,I}_{AC,5}$.

4.2 The realization of the potential phase portraits

In the previous subsection we have produced all the 42 topologically potential phase portraits for structurally unstable quadratic systems of *codimension two** belonging to the set $\sum_2^2(AC)$.

And from them, we have already discarded two which are not realizable due to their respective unfoldings of *codimension one** being impossible.

In this subsection we aim to give specific examples for the 40 different topological classes of structurally unstable quadratic systems of *codimension two** belonging to the set $\sum_2^2(AC)$ and presented in Figures 1.4 and 1.5. As in the previous section (see page 61), we point out that we have found examples with no evidence of limit cycles, but we have not proved the absence of infinitesimal ones.

In [10] the authors classified, with respect to a specific normal form, the set of all real quadratic polynomial differential systems with a finite semi-elemental saddle-node $\overline{sn}_{(2)}$ located at the origin of the plane and an infinite saddle-node of type $\overline{\binom{1}{1}}SN$ obtained by the coalescence of a finite antisaddle (respectively, finite saddle) with an infinite saddle (respectively, infinite node).

As we have discussed in the previous section, the study of a bifurcation diagram of a certain family of quadratic systems, produces not only the class of phase portraits looked for, but also all those of their closure according to the normal form used. Even though the study is mainly algebraic, often, also analytic and numerical tools are required. This makes that these studies may be not complete and subject to the existence of possible "islands" which contain an undetected phase portrait. The border of that "island" could mean the connection of two separatrices, and the interior contain a different phase portrait from the ones stated in the theorem. The topological study that we do in this paper solves partially this problem, since we prove that all the realizable phase portraits of class (AC) do really exist, and no other topological possibility does. However, the possible existence of "islands" in the bifurcation diagram still persists since they can be related with double limit cycles, as discussed in Section 6 of [10].

By using the phase portraits of generic regions of the bifurcation diagram of the mentioned paper we realize all the 40 unstable systems of *codimension two** of the set (AC), i.e. we can give concrete examples of all structurally unstable phase portraits from the set (AC).

Consider systems (2.4). Such a normal form was studied in [10] and it describes quadratic polynomial differential systems which have a finite semi-elemental saddle-node $\overline{sn}_{(2)}$, a finite elemental singularity and an infinite saddle-node of type $\overline{\binom{1}{1}}SN$.

In Tables 4.5 and 4.6 we present one representative from each generic region of the bifurcation diagram of [10] corresponding to each phase portrait of *codimension two** from the set (AC) and, therefore, we conclude the proof of Theorem 1.7.

Cod 2*	**[10]**	**c**	**e**	**h**	**m**
$\mathbb{U}^2_{AC,1}$	V_{38}	-10	30	1	4
$\mathbb{U}^2_{AC,2}$	V_{1}	6	$81/2$	1	4
$\mathbb{U}^2_{AC,3}$	V_{33}	-7	$5/2$	1	4
$\mathbb{U}^2_{AC,4}$	V_{53}	2	$47/50$	1	$37/100$
$\mathbb{U}^2_{AC,5}$	V_{13}	-1	-10	1	4
$\mathbb{U}^2_{AC,6}$	V_{4}	7	15	1	4
$\mathbb{U}^2_{AC,7}$	V_{21}	$-9/4$	-10	1	4
$\mathbb{U}^2_{AC,8}$	V_{92}	-3	$7/2$	1	$-6/5$
$\mathbb{U}^2_{AC,9}$	V_{10}	$1/2$	$-11/2$	1	4
$\mathbb{U}^2_{AC,10}$	V_{63}	$-2/5$	$1/50$	1	$-1/4$
$\mathbb{U}^2_{AC,11}$	V_{95}	-3	$31/10$	1	$-6/5$
$\mathbb{U}^2_{AC,12}$	V_{73}	$-19/10$	$17/20$	1	$-3/4$
$\mathbb{U}^2_{AC,13}$	V_{8}	$3/2$	$-9/2$	1	4
$\mathbb{U}^2_{AC,14}$	V_{93}	-1	$11/10$	1	$-6/5$
$\mathbb{U}^2_{AC,15}$	V_{6}	$24/5$	$-4/5$	1	4
$\mathbb{U}^2_{AC,16}$	V_{68}	-3	$2/5$	1	$-1/4$
$\mathbb{U}^2_{AC,17}$	V_{39}	-25	30	1	4
$\mathbb{U}^2_{AC,18}$	V_{3}	$45/2$	98	1	4
$\mathbb{U}^2_{AC,19}$	V_{62}	$-1/40$	$1/50$	1	$-1/4$
$\mathbb{U}^2_{AC,20}$	V_{80}	$-6/5$	$1207/1000$	1	-1
$\mathbb{U}^2_{AC,21}$	V_{81}	$29/50$	$-3/5$	1	$-6/5$
$\mathbb{U}^2_{AC,22}$	V_{36}	-1	4	1	4
$\mathbb{U}^2_{AC,23}$	V_{23}	$-9/2$	-17	1	4
$\mathbb{U}^2_{AC,24}$	V_{112}	$1/2$	42	1	-10
$\mathbb{U}^2_{AC,25}$	V_{77}	$-5/4$	$629/500$	1	$-49/50$
$\mathbb{U}^2_{AC,26}$	V_{90}	$-9/5$	$881/400$	1	$-6/5$
$\mathbb{U}^2_{AC,27}$	V_{2}	1	7	1	4
$\mathbb{U}^2_{AC,28}$	V_{35}	$-1747/50$	30	1	4
$\mathbb{U}^2_{AC,29}$	V_{49}	10	$5156/625$	1	$51/100$
$\mathbb{U}^2_{AC,30}$	V_{65}	$-23/50$	$1151/10000$	1	$-1/4$
$\mathbb{U}^2_{AC,31}$	V_{59}	$-1/50$	$1/40$	1	$-1/4$
$\mathbb{U}^2_{AC,32}$	V_{29}	$-3/2$	$1/2$	1	4
$\mathbb{U}^2_{AC,33}$	V_{82}	$1341/2000$	$-3/5$	1	$-6/5$
$\mathbb{U}^2_{AC,34}$	V_{102}	$1/100$	$31/10$	1	$-5/2$

Table 4.5: Correspondence between *codimension two** phase portraits of the set (AC) and phase portraits from Figures 1 and 2 in [10]. In the first column we present the *codimension two** phase portraits from the set (AC) in the present paper, in the second column we show the corresponding phase portraits from Figures 1 and 2 in [10] given by normal form (2.4), and in the other columns we present the values of the parameters c, e, h, and m of (2.4) which realizes such phase portrait.

Cod 2*	**[10]**	**c**	**e**	**h**	**m**
$\mathbb{U}^2_{AC,35}$	V_{26}	$-687/50$	-17	1	4
$\mathbb{U}^2_{AC,36}$	V_{20}	$-21/10$	$-41/5$	1	4
$\mathbb{U}^2_{AC,37}$	V_{51}	10	$151/20$	1	$3/4$
$\mathbb{U}^2_{AC,38}$	V_{71}	$-1/10000$	$3/125$	1	$-1/4$
$\mathbb{U}^2_{AC,39}$	V_{14}	$-3/2$	-4	1	4
$\mathbb{U}^2_{AC,40}$	V_{55}	$1/100$	$1/100$	1	$-1/4$

Table 4.6: Continuation of Table 4.5.

5 Graphics and limit cycles

Even though the goal of this paper deals little with graphics and limit cycles, there is no doubt that these are two of the most important elements in qualitative theory of ordinary differential equations.

Limit cycles are the most elusive phenomena in phase portraits. They may appear either by a bifurcation of a weak focus (Hopf bifurcation), by a bifurcation of a graphic, or by a bifurcation of a multiple limit cycle, and only the first case can be fully algebraically controlled. The other cases are generically nonalgebraic. In fact, weak foci can be considered among graphics, since they can be seen as graphics reduced to a single point.

Our goal to find all the topologically different phase portraits modulo limit cycles bypasses this big problem, but it is not an irrelevant goal. Whenever the mathematical community finally gets the complete set of phase portraits of quadratic systems (or whatever other family), the subset of the phase portraits modulo limit cycles will be the base for such a classification. It is expected to obtain more than one thousand (maybe even up to 2000) different phase portraits of quadratic systems modulo limit cycles. For quite many of them it will be trivial to determine that they will not have limit cycles (in the case they do not have a finite antisaddle). But for all the others, it will be necessary to determine exactly how many different phase portraits can be obtained from that skeleton by adding limit cycles. Up to now and up to our knowledge, there are very few nontrivial skeletons of phase portraits which could theoretically have limit cycles, and for which the absence of limit cycles has been proved. To be more precise, we are only completely sure of one of them, namely the structurally stable phase portrait $S^2_{7,1}$. This phase portrait was obtained in [2] and was conjectured by statistical tools to be incompatible with limit cycles in [4] and this conjecture was proved in [5]. Also in [4] some other phase portraits are conjectured (by statistical data) to be incompatible with limit cycles, but no proof is available yet. Apart from these last ones, other candidates can be found in Class I of [37]. In that paper the authors produce three normal forms (denoted by I, II and III) and they prove that any system with limit cycle can be transformed in an element of them. The three classes have no intersection since they deal with the number of finite singularities that have gone to infinity ($\geq$ 2, 1 and 0, respectively). And in [37] it is also proved that systems from Class I have at most one limit cycle. There is still no conclusive study of phase portraits from Class I, but some phase portraits of this class have already been found having one limit cycle and some others with no limit cycle (see [15, 23, 34]). For the cases with limit cycle, it is closed the fact that such phase portraits can have at most one limit cycle, and if a conclusive study is done and results are confirmed, the cases with no limit cycle would add to the phase portrait $S^2_{7,1}$ as skeletons of phase portraits without limit cycles. For all other skeletons of phase portraits found up to now, there is not a single proof determining which is the maximum number of limit cycles that each one may have. There are many other papers related to the maximum number of limit cycles, but they are mostly linked to a certain normal form. Most of them simply prove that a specific normal form may have just one limit cycle. But this does not imply that the skeletons of phase portraits obtained in that normal form may have more limit cycles in the entire classification.

Up to now, it is known that there are examples of phase portraits of quadratic systems with four limit cycles distributed into two nests around two foci, more precisely, three limit cycles in one nest and the fourth limit cycle in the other nest. And even though it is conjectured that the effective maximum is four with the distribution just mentioned, there is still no conclusive proof. The phase portraits for which there are examples with four limit cycles belong to three skeletons of phase portraits, namely, the structurally stables $\mathbb{S}^2_{4,1}$ and $\mathbb{S}^2_{11,2}$ from [2], and the *codimension one** $\mathbb{U}^1_{B,31}$ from [6]. The proof that they may have at least four limit cycles appears in several papers since they appear in classifications with a weak focus of order three, already having a limit cycle around a strong focus.

But not even if the maximum bound was four, we would not be close to obtain all the phase portraits of quadratic systems. Any of the three skeletons mentioned before may have the topologically different configurations $(0,0)$, $(1,0)$, $(2,0)$, $(3,0)$, $(1,1)$, $(2,1)$, and $(3,1)$. That is, seven different configurations. But even that is not a criterion (that is, multiply the number of skeletons by 7) to obtain a simple upper bound for the total number of phase portraits. There are phase portraits like $\mathbb{S}^2_{5,1}$ from [2] which has three finite antisaddles. One of them receives (or emits) a single separatrix, the second one receives (or emits) exactly two separatrices, and the third one receives (or emits) exactly three separatrices. So, the fact that a limit cycle could be surrounding any of the three antisaddles would generate a topologically different phase portrait. And in case there were two nests of limit cycles, and assuming that they could have up to four limit cycles, the number of cases would increase up to 25 possibilities. But from these 25 possibilities, up to now only six have been confirmed to exist. We are collecting a large database and recording the maximum number of limit cycles found in each one of the skeletons classified up to now.

With all these facts we want to remark that the topological classification of phase portraits modulo limit cycles is important since it produces a complete set of skeletons from which all the complete set of phase portraits must be located. For each particular skeleton, it must be studied if it contains none, one, two or up to three antisaddles around which the limit cycles may be located. If there is a complete collection of phase portraits modulo limit cycles, and if an upper bound of limit cycles is found, it will give a quite rough upper bound for the number of different phase portraits. But the real number will need a deeper study case by case. Nowadays, the moment that we could have a complete topological classification is quite far away. However, the topological classification modulo limit cycles is within reach, although they are not easily reachable yet.

Let us now talk about graphics. Graphics are also very important because they can become the bifurcation edge which leads to the birth of limit cycles. There has been a lot of literature related to graphics, and one of the most relevant papers is [19] where the authors list a set of 121 different graphics whose finite cyclicity needs to be proved in order to prove the finiteness part of Hilbert 16th problem for quadratic systems. The graphics in this list can be of different types. Many of them imply the connection of one (or more) couple of separatrices, finite or infinite. Other graphics are formed simply because a separatrix arrives to the nodal part of a saddle-node (finite or infinite) or an even more degenerate singularity in coexistence with other properties of the phase portrait. Unfortunately, most of these graphics cannot be detected by means of algebraic tools. In many studies of families of systems where a complete bifurcation is given in the parameter space, after all the algebraic bifurcations are given, the use of continuity and coherence arguments allows the detection of some other nonalgebraic bifurcations where these graphics appear.

Our methodical study of phase portraits of quadratic systems modulo limit cycles started with codimension zero (structurally stable) [2] and of course these phase portraits cannot have any graphic at all. The second step was the classification of codimension-one phase portraits (modulo limit cycles), and in that study we could start finding some graphics, but not too many. Precisely, we found graphic (F_2^1) from [19] in $\mathbb{U}^1_{A,37}$, $\mathbb{U}^1_{A,43}$, $\mathbb{U}^1_{A,64}$, and $\mathbb{U}^1_{A,70}$. This graphic is formed simply by one finite saddle-node which sends its center manifold (separatrix of zero eigenvalue) to its own nodal part. We also have graphic (I^2_{19}) from [19] in $\mathbb{U}^1_{B,29}$, $\mathbb{U}^1_{B,30}$ (twice), $\mathbb{U}^1_{B,33}$, $\mathbb{U}^1_{B,36}$, and $\mathbb{U}^1_{B,38}$. This graphic is formed by one elemental infinite saddle which sends one of its separatrices to the nodal part of an infinite adjacent saddle-node formed by the coalescence of two infinite singularities. There are no graphics in the set (C) of codimension-one phase portraits (modulo limit cycles, see page 4). Finally, in the set (D) (see again page 4) we found the graphics (F_1^1), (H_1^1), and (I_1^2) from [19]. The first one is just a loop of a finite elemental saddle, the second one is a separatrix connection between opposite infinite elemental saddles, and the third one is a separatrix connection between adjacent infinite elemental saddles. The loop is present in $\mathbb{U}^1_{D,1}$, $\mathbb{U}^1_{D,6}$, $\mathbb{U}^1_{D,7}$, $\mathbb{U}^1_{D,8}$, $\mathbb{U}^1_{D,9}$, $\mathbb{U}^1_{D,12}$, $\mathbb{U}^1_{D,19}$, $\mathbb{U}^1_{D,20}$, $\mathbb{U}^1_{D,22}$, $\mathbb{U}^1_{D,23}$, $\mathbb{U}^1_{D,30}$, $\mathbb{U}^1_{D,31}$, $\mathbb{U}^1_{D,32}$, $\mathbb{U}^1_{D,46}$, $\mathbb{U}^1_{D,47}$, $\mathbb{U}^1_{D,48}$, $\mathbb{U}^1_{D,49}$, $\mathbb{U}^1_{D,50}$, $\mathbb{U}^1_{D,51}$, $\mathbb{U}^1_{D,52}$, $\mathbb{U}^1_{D,53}$, and $\mathbb{U}^1_{D,54}$. The second graphic appears in $\mathbb{U}^1_{D,10}$ and $\mathbb{U}^1_{D,11}$. And the third one can be seen in $\mathbb{U}^1_{D,28}$, $\mathbb{U}^1_{D,29}$, $\mathbb{U}^1_{D,37}$, $\mathbb{U}^1_{D,38}$, and $\mathbb{U}^1_{D,39}$. No other graphic from these last five may appear, since all the remaining 116 imply higher codimension.

Thus, in our current study of phase portraits of *codimension two** with a finite saddle-node and an infinite saddle-node, the only graphics that we can see will be those ones which are inherited from the respective phase portraits of *codimension one** already having a graphic. No new graphic may appear from the consolidation of the two different instabilities we mix here. In the studies of the sets (AD), (BD), and (CD) we will start incorporating more graphics from [19], since we will find, for example, saddle-nodes forming a loop instead of an elemental saddle. Also the set (DD) will provide graphics with two separatrix connections. Anyway, the graphics will appear in larger numbers when *codimension three** is studied.

There is another important fact, related to stability and graphics, to comment about the classification that we are working with. As mentioned in Section 1, in [6] it is claimed that there are at least 204 structurally unstable phase portraits of *codimension one** and at most 211. Two papers have found two mistakes in that book and the newly proved numbers are 202 and 209, respectively. The seven cases that have not been found correspond to cases which are conjectured as impossible and some arguments are given to support that conjecture. We point out that all the seven cases conjectured impossible contain a graphic, more precisely the polycycles (F_2^1) or (H_1^1). These phase portraits consist in an skeleton of separatrices which depending on the stability of the focus inside the polycycle (compared to other stabilities outside it) may lead or not to a realizable phase portrait. That is, they lead to a phase portrait which is already known to exist, or lead to a phase portrait which (up to our knowledge) never appeared before in any paper. The normal techniques which have allowed us to prove the impossibility of hundreds of phase portraits are useless in these seven cases. All we can say about these seven phase portraits is that in case they exist, some perturbations from them would produce phase portraits with a limit cycle that we have not found anywhere. Using the tools of perturbations related to stability that we use in this paper, we may claim that if one of those phase portraits with a limit cycle could be proven impossible, then the related unstable phase portrait with a polycycle would be also impossible. However, the opposite is not true. If the phase portrait with a limit cycle does exist, it is not sure that the related unstable phase portrait with a polycycle may exist. There is the possibility that by means of a rotated vector field one could pass from one to the other, but it is not guaranteed.

So, we see once more the importance of graphics and limit cycles in the classification of phase portraits. The fact that we talk so little about limit cycles is simply because we want to do the classification modulo limit cycles in order to have a good base upon which we or others may add the limit cycles. And the fact that we talk so little about graphics is because at the level of codimension that we are in this stage, there appear very few of the 121 graphics described in [19].

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