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Existence of steady-state solutions for a class of competing systems with cross-diffusion and self-diffusion

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Abstract. We focus on a system of two competing species with cross-diffusion and self-diffusion. By constructing an appropriate auxiliary function, we derive the sufficient conditions such that there are no coexisting steady-state solutions to the model. It is worth noting that the auxiliary function constructed above is applicable to Dirichlet, Neumann and Robin boundary conditions.

Keywords: cross-diffusion, self-diffusion, steady-state solution, maximum principle.

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1 Introduction

In this work, we study the steady-state solutions of the following competing systems with cross-diffusion and self-diffusion:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta[(d_1 + a_{11}u + a_{12}v)u] + u(a_1 - b_1 u - c_1 v), & x \in \Omega, t > 0, \\ \dfrac{\partial v}{\partial t} = \Delta[(d_2 + a_{21}u + a_{22}v)v] + v(a_2 - b_2 u - c_2 v), & x \in \Omega, t > 0, \\ \alpha_1 u + \beta_1 \dfrac{\partial u}{\partial \nu} = \alpha_2 v + \beta_2 \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) \geq 0,\ v(x,0) = v_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with smooth boundary, u and v are the densities of two competing species, α_i, β_i and a_{ij} $(i, j = 1, 2)$ are nonnegative constants, a_i, b_i, c_i and d_i $(i = 1, 2)$ are all positive constants, a_{11} and a_{22} stand for the self-diffusion pressures, while a_{12} and a_{21} are the cross-diffusion pressures, a_1, a_2 represent the intrinsic growth rates of the two species, b_1, c_2 describe the intra-specific competitions, while b_2, c_1 describe the inter-specific competitions, and d_1, d_2 are their diffusion rates.

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System (1.1) was first proposed by Shigesada, Kawasaki and Teramoto [10] in 1979 to investigate the spatial segregation of interacting species. In the last several decades, a great deal of mathematical effort has been devoted to the study of the model. For the smooth solutions of the system (1.1) with homogeneous Neumann boundary conditions, [4] and [11] obtained the global existence and boundedness in a bounded convex domain. We refer to [2,3,5–7] for the study of the positive steady-state solutions. For instance, Lou and Ni [2] established the sufficient conditions for the existence and nonexistence of nonconstant steady-state solutions in the strong and weak competition case, respectively. When $a_{21} = a_{22} = 0$, Lou et al. [5] provided the parameters ranges such that the system has no nonconstant positive solutions for $a_{11} = 0$ and $a_{11} \neq 0$, respectively.

For literatures about the system (1.1) under homogeneous Dirichlet boundary conditions, see [1,8,12,14] and references therein. In [9], by the decomposing operators and the theory of fixed point, Ryn and Ahn discussed the existence of the positive coexisting steady-state of system (1.1) for two competing species or predator-prey species.

Motivated by [5], we introduce the effect of self-diffusion and consider the model under three boundary conditions. Our purpose is to establish the sufficient conditions such that the following system has no coexisting solutions:

$$\begin{cases} \Delta[(d_1+\alpha v)u] + u(a_1 - b_1 u - c_1 v) = 0, & x \in \Omega, \\ \Delta[(d_2+\beta v)v] + v(a_2 - b_2 u - c_2 v) = 0, & x \in \Omega, \\ \alpha_1 u + \beta_1 \dfrac{\partial u}{\partial \nu} = \alpha_2 v + \beta_2 \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $\alpha_i \geq 0, \beta_i \geq 0$ and $\alpha_i + \beta_i > 0$ for $i = 1,2$. In what follows, we always assume that $\alpha \geq 0, \beta \geq 0, a_i > 0, b_i > 0, c_i > 0$ and $d_i > 0$ for $i = 1,2$. To achieve that, the main tools we use are the strong maximum principle, Hopf's boundary lemma and the divergence theorem. Since u and v represent species densities, we are interested in the nonnegative classical solution (u,v) of (1.2), which means that $(u,v) \in (C^1(\overline{\Omega}) \cap C^2(\Omega))^2$, $u,v \geq 0$ in $\overline{\Omega}$, and satisfies (1.2) in the pointwise sense.

The remainder of this work is organized as follows. In Section 2, we show that the nonnegative classical solutions are strictly positive if they are not identically equal to zero, which plays a key role in the proof of main theorems. Section 3 constructs an auxiliary function, which can be used to produce contradictions, and thus parameter ranges for nonexistence of coexisting steady-state solutions will be obtained under three boundary conditions.

2 Preliminaries

Let us first give the following proposition by applying the strong maximum principle, which indicates that nonnegative classical solutions are strictly positive if they are nontrivial.

Proposition 2.1. *Suppose that (u,v) is a nonnegative classical solution of (1.2). Then if $u \not\equiv 0$, we have $u > 0$ in Ω, and if $v \not\equiv 0$, we have $v > 0$ in Ω.*

Proof. We only prove $u > 0$ in Ω whenever $u \not\equiv 0$, since the positivity of v in Ω can be proved in a similar way. Let $w = (d_1 + \alpha v)u$. Due to $d_1 > 0, \alpha \geq 0$ and $v \geq 0$ in $\overline{\Omega}$, it suffices to prove $w > 0$ in Ω. Otherwise, there is $x_0 \in \Omega$ such that $w(x_0) = \min_{x\in\overline{\Omega}} w(x) = 0$.

It follows from the first equation of (1.2) that

$$\Delta w + u(a_1 - b_1 u - c_1 v) = \Delta w + \frac{a_1 - b_1 u - c_1 v}{d_1 + \alpha v} \cdot w = 0.$$

Let

$$Lw = -\Delta w + cw \quad \text{with} \quad c = \frac{b_1 u + c_1 v}{d_1 + \alpha v}.$$

Then

$$c \geq 0 \quad \text{and} \quad Lw = \frac{a_1 w}{d_1 + \alpha v} \geq 0 \quad \text{in } \Omega.$$

So, an application of the strong maximum principle shows that w is constant in Ω, and thus $w = 0$, a contradiction to $u \not\equiv 0$. This completes the proof. □

Remark 2.2. When $\alpha_i = 0$ and $\beta_i > 0$ for $i = 1, 2$, that is, in the case of Neumann boundary conditions, we can get further that $u, v > 0$ in $\overline{\Omega}$ by Hopf's boundary lemma.

Next, we list two lemmas about the existence of positive solutions for single equation under Dirichlet or Robin boundary conditions, which can reveal the existence of semi steady-state solution of system (1.2). The following lemma comes from Theorem 2.1 in [8].

Lemma 2.3. *Consider the following problem:*

$$\begin{cases} -\Delta[(d + \gamma w)w] = w(a - bw), & x \in \Omega, \\ w = 0, & x \in \partial\Omega, \end{cases} \tag{2.1}$$

where a, b, d are positive constants and γ is nonnegative constant. Let $\lambda_1^d > 0$ denote the first eigenvalue of $-\Delta$ with the homogeneous Dirichlet boundary condition on $\partial\Omega$. If

$$\lambda_1^d < \frac{a}{d},$$

then problem (2.1) has a unique positive solution.

From now on, if $\lambda_1^d < \frac{a_1}{d_1}$ and $\lambda_1^d < \frac{a_2}{d_2}$, we denote u^* and v^* as the unique positive solution of systems

$$\begin{cases} -d_1 \Delta u + (b_1 u - a_1)u = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta[(d_2 + \beta v)v] + (c_2 v - a_2)v = 0, & x \in \Omega, \\ v = 0, & x \in \partial\Omega, \end{cases}$$

respectively.

For Robin boundary conditions, the corresponding result can be found in Theorem 2.10 of [9].

Lemma 2.4. *Consider the following system:*

$$\begin{cases} -\Delta[(d(x) + \gamma w)w] = w(a(x) - bw), & x \in \Omega, \\ \delta w + \eta \dfrac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{2.2}$$

where γ is a nonnegative constant, b, δ, η are positive constants and $d(x), a(x) \in C^2(\overline{\Omega})$ with $d(x) > 0$ for all $x \in \overline{\Omega}$. If $\frac{\partial}{\partial \nu}(d(x)) \leq 0$ on $\partial\Omega$ and $\lambda_1(d(x), a(x), \delta, \eta) > 0$, then (2.2) has a unique positive solution, where

$$\lambda_1(d(x), a(x), \delta, \eta) = \frac{\int_\Omega \left(-|\nabla[d(x)\phi_1]|^2 + d(x)a(x)\phi_1^2\right) - \int_{\partial\Omega} d(x)\left[\frac{\delta}{\eta}d(x) - \frac{\partial d(x)}{\partial \nu}\right]\phi_1^2}{\|\sqrt{d(x)}\phi_1\|^2_{L^2(\Omega)}}$$

denotes the principal eigenvalue with eigenfunction ϕ_1 of the following eigenvalue problem:

$$\begin{cases} \Delta[(d(x)\phi] + a(x)\phi = \lambda\phi, & x \in \Omega, \\ \delta\phi + \eta\dfrac{\partial\phi}{\partial\nu} = 0, & x \in \partial\Omega. \end{cases}$$

Similarly, if $\lambda_1(d_1, a_1, \alpha_1, \beta_1) > 0$ and $\lambda_1(d_2, a_2, \alpha_2, \beta_2) > 0$, we write u^{**} and v^{**} as the unique positive solution of systems

$$\begin{cases} -d_1\Delta u + (b_1 u - a_1)u = 0, & x \in \Omega, \\ \alpha_1 u + \beta_1\dfrac{\partial u}{\partial\nu} = 0, & x \in \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta[(d_2 + \beta v)v] + (c_2 v - a_2)v = 0, & x \in \Omega, \\ \alpha_2 v + \beta_2\dfrac{\partial v}{\partial\nu} = 0, & x \in \partial\Omega, \end{cases}$$

respectively.

3 Steady-state solutions

Now we give the main theorems of this work. When the intra-competition and inter-competition parameters of one species are greater than inter-competition and intra-competition of the other, respectively, whereas the intrinsic growth rate is less than that of the other, we explore two different sufficient criteria for nonexistence of coexisting solutions of system (1.2).

3.1 Dirichlet boundary conditions

Theorem 3.1. *Let $\alpha_i > 0$, $\beta_i = 0$ for $i = 1, 2$, $\alpha \geq 0$ and $\beta \geq 0$. Assume that (u, v) is a nonnegative classical solution of (1.2). If*

$$\text{(i)} \quad b_1 > b_2,\ c_1 > c_2,\ a_1 < a_2,\ d_1 \geq d_2 \quad \text{and} \quad \alpha \geq \beta$$

or

$$\text{(ii)} \quad b_1 < b_2,\ c_1 < c_2,\ a_1 > a_2,\ d_1 \leq d_2 \quad \text{and} \quad \alpha \leq \beta,$$

then we have either

$$(u, v) \equiv (0, 0),$$

or

$$(u, v) = (u^*, 0) \quad \text{if} \quad \lambda_1^d < \frac{a_1}{d_1},$$

or

$$(u, v) = (0, v^*) \quad \text{if} \quad \lambda_1^d < \frac{a_2}{d_2}.$$

Proof. (i) By way of contradiction, suppose that $u \not\equiv 0$ and $v \not\equiv 0$. Thus, u and v are positive in Ω by Proposition 2.1. So, it is apparent from system (1.2) that:

$$\begin{cases} \dfrac{\Delta[(d_1+\alpha v)u]}{u} = -a_1 + b_1 u + c_1 v, & x \in \Omega, \\ \dfrac{\Delta[(d_2+\beta v)v]}{v} = -a_2 + b_2 u + c_2 v, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

Let

$$w = (d_1 + \alpha v)u. \tag{3.2}$$

Then, by (3.1) and conditions $b_1 > b_2$, $c_1 > c_2$ and $a_1 < a_2$, we have

$$\frac{\Delta w}{w} > \frac{\Delta[(d_2+\beta v)v]}{(d_1+\alpha v)v} \quad \text{in } \Omega. \tag{3.3}$$

We now define a function

$$p(s) = s^{\frac{d_2-d_1}{d_1}} (d_1 + \alpha s)^{\frac{2\beta - \alpha - \frac{d_2}{d_1}\alpha}{\alpha}} \quad \text{for } s > 0. \tag{3.4}$$

It is easy to verify that $p(s) > 0$ for any $s > 0$. Moreover, a direct calculation implies that

$$p'(s) = p(s) \left(\frac{d_2 - d_1}{d_1 s} + \frac{2\beta - \alpha - \frac{d_2}{d_1}\alpha}{d_1 + \alpha s} \right).$$

This, together with (3.3), yields that

$$\begin{aligned} &\operatorname{div}\left[(d_1+\alpha v)vp(v)\nabla w - wp(v)\nabla[(d_2+\beta v)v]\right] \\ &\quad = (d_1+\alpha v)vp(v)\Delta w + \nabla[(d_1+\alpha v)vp(v)] \cdot \nabla w \\ &\qquad - wp(v)\Delta[(d_2+\beta v)v] - \nabla[wp(v)] \cdot \nabla[(d_2+\beta v)v] \\ &\quad > \nabla[(d_1+\alpha v)vp(v)] \cdot \nabla w - \nabla[wp(v)] \cdot \nabla[(d_2+\beta v)v] \quad \text{in } \Omega. \end{aligned}$$

Furthermore, we can see that

$$\begin{aligned} &\nabla[(d_1+\alpha v)vp(v)] \cdot \nabla w - \nabla[wp(v)] \cdot \nabla[(d_2+\beta v)v] \\ &\quad = \Big[\alpha vp(v)\nabla v + (d_1+\alpha v)p(v)\nabla v + (d_1+\alpha v)vp'(v)\nabla v\Big] \cdot \Big[\alpha u\nabla v + (d_1+\alpha v)\nabla u\Big] \\ &\qquad - \Big[\alpha up(v)\nabla v + (d_1+\alpha v)p(v)\nabla u + (d_1+\alpha v)up'(v)\nabla v\Big] \cdot \Big[\beta v\nabla v + (d_2+\beta v)\nabla v\Big] \\ &\quad = |\nabla v|^2\Big[2\alpha^2 uvp(v) + d_1\alpha up(v) + d_1\alpha uvp'(v) + \alpha^2 uv^2 p'(v) - 2\alpha\beta uvp(v) \\ &\qquad - 2d_1\beta uvp'(v) - 2\alpha\beta uv^2 p'(v) - d_2\alpha up(v) - d_1 d_2 up'(v) - d_2\alpha uvp'(v)\Big] \\ &\qquad + \nabla u \cdot \nabla v\Big[3d_1\alpha vp(v) + d_1^2 p(v) + d_1^2 vp'(v) + 2d_1\alpha v^2 p'(v) + 2\alpha^2 v^2 p(v) \\ &\qquad + \alpha^2 v^3 p'(v) - 2d_1\beta vp(v) - 2\alpha\beta v^2 p(v) - d_1 d_2 p(v) - d_2\alpha vp(v)\Big] \\ &\quad \triangleq |\nabla v|^2 M(u,v) + \nabla u \cdot \nabla v N(u,v). \end{aligned}$$

Since

$$\begin{aligned}
N(u,v) &= p(v)\Big[3d_1\alpha v + d_1^2 + 2\alpha^2 v^2 - 2d_1\beta v - 2\alpha\beta v^2 - d_1 d_2 - d_2\alpha v\Big] + p'(v)v(d_1+\alpha v)^2 \\
&= p(v)\Big[(d_1^2 - d_1 d_2) + (3d_1\alpha - 2d_1\beta - d_2\alpha)v + (2\alpha^2 - 2\alpha\beta)v^2 \\
&\quad + \frac{d_2-d_1}{d_1}(d_1+\alpha v)^2 + v(d_1+\alpha v)(2\beta - \alpha - \frac{d_2}{d_1}\alpha)\Big] \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
&M(u,v) \\
&= \alpha u p(v)(d_1 - d_2 + 2\alpha v - 2\beta v) + u p'(v)\big[-d_1 d_2 + (d_1\alpha - 2d_1\beta - d_2\alpha)v + (\alpha^2 - 2\alpha\beta)v^2\big] \\
&= \alpha u p(v)(d_1 - d_2 + 2\alpha v - 2\beta v) + u p'(v)(d_1+\alpha v)(-d_2 + \alpha v - 2\beta v) \\
&= \alpha u p(v)(d_1 - d_2 + 2\alpha v - 2\beta v) + u p(v)\frac{(d_2 - d_1 - 2\alpha v + 2\beta v)}{v}(-d_2 + \alpha v - 2\beta v) \\
&= u p(v)(d_1 - d_2 + 2\alpha v - 2\beta v)\left(\frac{d_2}{v} + 2\beta\right) \\
&\geq 0,
\end{aligned}$$

we conclude that

$$\operatorname{div}\Big[(d_1+\alpha v)vp(v)\nabla w - wp(v)\nabla[(d_2+\beta v)v]\Big] > 0 \quad \text{in } \Omega. \tag{3.5}$$

Now, let

$$\Omega_\varepsilon = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \varepsilon\} \quad \text{for any small } \varepsilon > 0.$$

Since $(u,v) \in (C^1(\overline{\Omega}) \cap C^2(\Omega))^2$, we know that $\Big[(d_1+\alpha v)vp(v)\nabla w - wp(v)\nabla[(d_2+\beta v)v]\Big] \in C^1(\overline{\Omega_\varepsilon})$. Then it follows from divergence theorem that

$$\begin{aligned}
&\int_\Omega \operatorname{div}\Big[(d_1+\alpha v)vp(v)\nabla w - wp(v)\nabla[(d_2+\beta v)v]\Big]\mathrm{d}x \\
&= \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \operatorname{div}\Big[(d_1+\alpha v)vp(v)\nabla w - wp(v)\nabla[(d_2+\beta v)v]\Big]\mathrm{d}x \\
&= \lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon} \Big[(d_1+\alpha v)vp(v)\nabla w - wp(v)\nabla[(d_2+\beta v)v]\Big]\cdot \nu \mathrm{d}S \\
&= \lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon} \left[(d_1+\alpha v)vp(v)\frac{\partial w}{\partial \nu} - wp(v)\frac{\partial[(d_2+\beta v)v]}{\partial \nu}\right]\mathrm{d}S \\
&= \lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon} \Bigg[\frac{\partial w}{\partial \nu} v^{\frac{d_2}{d_1}}(d_1+\alpha v)^{\frac{2\beta - \frac{d_2}{d_1}\alpha}{\alpha}} - \beta u \frac{\partial v}{\partial \nu} v^{\frac{d_2}{d_1}}(d_1+\alpha v)^{\frac{2\beta - \frac{d_2}{d_1}\alpha}{\alpha}} \\
&\qquad - \frac{u}{v}(d_2+\beta v)\frac{\partial v}{\partial \nu} v^{\frac{d_2}{d_1}}(d_1+\alpha v)^{\frac{2\beta - \frac{d_2}{d_1}\alpha}{\alpha}}\Bigg]\mathrm{d}S \\
&\triangleq \lim_{\varepsilon\to 0}(I_1(\varepsilon) + I_2(\varepsilon) + I_3(\varepsilon)).
\end{aligned} \tag{3.6}$$

Obviously, $I_1(\varepsilon)$ and $I_2(\varepsilon)$ both tend to zero as $\varepsilon \to 0$. To deal with the term $I_3(\varepsilon)$, we take

$$V = \left\{\varphi(x) \in C^1(\overline{\Omega}) \;\Big|\; \varphi|_\Omega > 0,\ \varphi|_{\partial\Omega} = 0,\ \frac{\partial \varphi}{\partial \nu}\Big|_{\partial\Omega} < 0\right\}.$$

Then Hopf's boundary lemma tells us that $\frac{\partial u(x_0)}{\partial \nu} < 0$ and $\frac{\partial v(x_0)}{\partial \nu} < 0$ for any $x_0 \in \partial\Omega$, and thus $u \in V$ and $v \in V$. Define

$$g(x) := \begin{cases} \dfrac{u(x)}{v(x)}, & x \in \Omega, \\ \dfrac{\partial u(x)}{\partial \nu} \Big/ \dfrac{\partial v(x)}{\partial \nu}, & x \in \partial\Omega. \end{cases}$$

Then by applying Lemma 2.4 in [13], we get $g(x) \in C\big(\overline{\Omega}, (0, +\infty)\big)$. Therefore $I_3(\varepsilon)$ also approaches to zero as $\varepsilon \to 0$.

As a result, $\int_\Omega \operatorname{div}\Big[(d_1 + \alpha v)vp(v)\nabla w - wp(v)\nabla[(d_2 + \beta v)v]\Big]\mathrm{d}x = 0$ because of Lebesgue dominated convergence theorem and the boundary conditions in (1.2), which contradicts (3.5). So either $(u, v) = (0, 0)$, or only one of them is equal to zero. When $v \equiv 0$, we have

$$\begin{cases} -\Delta u + \dfrac{1}{d_1}(b_1 u - a_1)u = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

By Lemma 2.3, we can see $(u, v) = (u^*, 0)$ if $\lambda_1^d < \frac{a_1}{d_1}$. Similarly, if $u \equiv 0$ and $\lambda_1^d < \frac{a_2}{d_2}$, then $(u, v) = (0, v^*)$. This finishes the proof of the first part.

(ii) Now, we also assume that $u \not\equiv 0$ and $v \not\equiv 0$. Then an application of Proposition 2.1 provides that u and v are positive in Ω. Hence,

$$\frac{\Delta w}{w} < \frac{\Delta[(d_2 + \beta v)v]}{(d_1 + \alpha v)v} \quad \text{in } \Omega,$$

according to the hypotheses $b_1 < b_2$, $c_1 < c_2$ and $a_1 > a_2$, where w is defined by (3.2).

Given $d_1 \leq d_2$ and $\alpha \leq \beta$, we know that

$$\operatorname{div}\Big[(d_1 + \alpha v)vp(v)\nabla w - wp(v)\nabla[(d_2 + \beta v)v]\Big] < 0 \quad \text{in } \Omega,$$

where $p(v)$ is defined as in (i).

Furthermore, again by divergence theorem, we can prove that

$$\int_\Omega \operatorname{div}\Big[(d_1 + \alpha v)vp(v)\nabla w - wp(v)\nabla[(d_2 + \beta v)v]\Big]\mathrm{d}x = 0,$$

a contradiction. By repeating the argument in (i), we complete the proof of Theorem 3.1. □

3.2 Neumann boundary conditions

In (3.6), if we consider Neumann boundary conditions, we can directly check that (3.6) equals to zero. Consequently, the following theorem is stated without proof.

Theorem 3.2. *Suppose that $\alpha_i = 0, \beta_i > 0$ for $i = 1, 2$ and (u, v) is a nonnegative classical solution of (1.2). If*

$$\text{(i)} \quad b_1 > b_2,\ c_1 > c_2,\ a_1 < a_2,\ d_1 \geq d_2 \quad \textit{and} \quad \alpha \geq \beta$$

or

$$\text{(ii)} \quad b_1 < b_2,\ c_1 < c_2,\ a_1 > a_2,\ d_1 \leq d_2 \quad \textit{and} \quad \alpha \leq \beta,$$

then either $(u, v) \equiv (0, 0)$, or $(u, v) = (\frac{a_1}{b_1}, 0)$, or $(u, v) = (0, \frac{a_2}{c_2})$.

3.3 Robin boundary conditions

In this subsection, we consider the case in which α_i and β_i $(i = 1, 2)$ are both positive.

Theorem 3.3. *Let $\alpha_i > 0, \beta_i > 0$ for $i = 1, 2$ and (u, v) be a nonnegative classical solution of (1.2). If*

$$\text{(i)} \quad b_1 > b_2,\ c_1 > c_2,\ a_1 < a_2,\ d_1 \geq d_2, \alpha \geq \beta \quad \textit{and} \quad \frac{\alpha_1}{\beta_1} \geq \frac{\alpha_2}{\beta_2}$$

or

$$\text{(ii)} \quad b_1 < b_2,\ c_1 < c_2,\ a_1 > a_2,\ d_1 \leq d_2, \alpha \leq \beta \quad \textit{and} \quad \frac{\alpha_1}{\beta_1} \leq \frac{\alpha_2}{\beta_2},$$

then we have either

$$(u, v) \equiv (0, 0),$$

or

$$(u, v) = (u^{**}, 0) \quad \textit{if} \quad \lambda_1(d_1, a_1, \alpha_1, \beta_1) > 0,$$

or

$$(u, v) = (0, v^{**}) \quad \textit{if} \quad \lambda_1(d_2, a_2, \alpha_2, \beta_2) > 0.$$

Proof. We only prove (i), as (ii) can be proved in a same manner. According to the arguments of the proof of Theorem 3.1, we can obtain from the hypothesis $b_1 > b_2, c_1 > c_2, a_1 < a_2, d_1 \geq d_2$ and $\alpha \geq \beta$ that

$$\operatorname{div}\Big[(d_1 + \alpha v)vp(v)\nabla w - wp(v)\nabla[(d_2 + \beta v)v]\Big] > 0,$$

where the function $p(v)$ is introduced in (3.4).

We mention that $\frac{\partial u}{\partial \nu} = -\frac{\alpha_1}{\beta_1}u$, $\frac{\partial v}{\partial \nu} = -\frac{\alpha_2}{\beta_2}v$ on $\partial\Omega$. The boundary integral becomes that

$$\begin{aligned}
&\int_\Omega \operatorname{div}\Big[(d_1 + \alpha v)vp(v)\nabla w - wp(v)\nabla[(d_2 + \beta v)v]\Big]\mathrm{d}x \\
&= \int_{\partial\Omega} v^{\frac{d_2}{d_1}}(d_1 + \alpha v)^{\frac{2\beta - \frac{d_2}{d_1}\alpha}{\alpha}}\left[\alpha u\frac{\partial v}{\partial \nu} + (d_1 + \alpha v)\frac{\partial u}{\partial \nu} - 2\beta u\frac{\partial v}{\partial \nu} - d_2\frac{u}{v}\frac{\partial v}{\partial \nu}\right]\mathrm{d}S \\
&= \int_{\partial\Omega} v^{\frac{d_2}{d_1}}(d_1 + \alpha v)^{\frac{2\beta - \frac{d_2}{d_1}\alpha}{\alpha}}\left[uv\left(2\beta\frac{\alpha_2}{\beta_2} - \alpha\frac{\alpha_1}{\beta_1} - \alpha\frac{\alpha_2}{\beta_2}\right) + u\left(d_2\frac{\alpha_2}{\beta_2} - d_1\frac{\alpha_1}{\beta_1}\right)\right]\mathrm{d}S \\
&\leq 0,
\end{aligned}$$

due to $d_1 \geq d_2, \alpha \geq \beta$ and $\frac{\alpha_1}{\beta_1} \geq \frac{\alpha_2}{\beta_2}$, a contradiction. Thus, if $v = 0$, we have $(u, v) = (u^{**}, 0)$ when $\lambda_1(d_1, a_1, \alpha_1, \beta_1) > 0$. Similarly, $(u, v) = (0, v^{**})$ if $u = 0$ and $\lambda_1(d_2, a_2, \alpha_2, \beta_2) > 0$. This completes the proof. □

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