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Periodic solutions of second order Hamiltonian systems with nonlinearity of general linear growth

Guanggang Liu ✉

School of Mathematical Sciences, Liaocheng University, Liaocheng 252000, P. R. China

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Abstract. In this paper we consider a class of second order Hamiltonian system with the nonlinearity of linear growth. Compared with the existing results, we do not assume an asymptotic of the nonlinearity at infinity to exist. Moreover, we allow the system to be resonant at zero. Under some general conditions, we will establish the existence and multiplicity of nontrivial periodic solutions by using the Morse theory and two critical point theorems.

Keywords: second order Hamiltonian systems, periodic solutions, Morse theory, critical groups.

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1 Introduction

Consider the following second order Hamiltonian systems

$$-\ddot{x} = V_x(t, x), \tag{1.1}$$

where $V \in C^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ with $V(t + T, x) = V(t, x)$ for some $T > 0$. During the past forty years, the existence and multiplicity of periodic solutions for second order Hamiltonian systems have been extensively studied by variational methods. There has been a lot of results under various suitable solvability conditions, such as the sublinear conditions (see [14, 18, 22, 23, 27, 28] and references therein), the superlinear conditions (see [3, 8, 9, 16, 17, 21, 24, 29] and references therein), and the asymptotically linear conditions (see [2, 6, 10, 15, 19, 20, 30] and references therein).

In this paper, we shall study the existence and multiplicity of nontrivial periodic solutions for (1.1) when the nonlinearity $V_x(t, x)$ has linear growth. Compared with the existing results, we do not make any assumptions at infinity on the asymptotic behaviors of the nonlinearity $V_x(t, x)$. Specifically, we do not require the system to be asymptotically linear at infinity. Instead, we assume that there exists a T-periodic symmetric matrix function $A_\infty(t)$ such that for some $K > 0$,

$$V_{xx}(t, x) \geq A_\infty(t) \quad (\text{or } V_{xx}(t, x) \leq A_\infty(t)), \qquad \forall t \in [0, T],\ |x| \geq K,$$

✉ Email: lgg112@163.com

where for two symmetric matrices A and B, $A \leq B$ means that $B - A$ is semi-positively definite. Under this general linear growth condition, we will construct a sequence of approximate systems and use the Morse theory and two critical point theorems to establish the existence and multiplicity of nontrivial periodic solutions for the system. The idea of our proof is closely related to the work of Liu, Su and Wang [13], where they dealt with the existence of nontrivial solutions of elliptic problems. Note that in [13] the authors assumed that the elliptic problem was nonresonant at zero. By contrast, here we allow system (1.1) to be resonant at zero. On the other hand, system (1.1) with periodic boundary condition is rather different from the elliptic problems with Dirichlet boundary condition. These lead us to need some new technique.

Now let us say some words about the idea of the proof. We first construct a sequence of approximate systems which are asymptotically linear and non-resonant at infinity. Then in a crucial step we establish the L^∞ bound to the solutions of the approximate systems whose Morse index is controlled by the Morse index at infinity. Finally, we use the Morse theory and two critical point theorems to obtain the nontrivial periodic solutions with the controlled Morse index for the approximate systems, therefore using the previous L^∞ estimate they are also the nontrivial periodic solutions of the original system.

We make the following assumptions:

(H_1) $V(t,x) \in C^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ with $V(t,0) = 0$ and $V(t+T,x) = V(t,x)$;

(H_2) There exist $C_1 > 0$ and $C_2 > 0$ such that

$$|V_x(t,x)| \leq C_1(1+|x|), \qquad |V_{xx}(t,x)| \leq C_2, \qquad t \in [0,T],\ x \in \mathbb{R}^N;$$

(H_3) $V_x(t,x) = A_0(t)x + (G_0)_x(t,x)$, where $A_0(t)$ is a T-periodic continuous symmetric matrix function and $(G_0)_x(t,x) = o(|x|)$ as $|x| \to 0$;

$(H_4^\pm)$ There exists $\delta > 0$ such that

$$\pm G_0(t,x) > 0, \qquad \forall t \in [0,T],\ 0 < |x| < \delta;$$

$(H_5^\pm)$ There exists a T-periodic continuous symmetric matrix function $A_\infty(t)$ such that for some $K > 0$,

$$\pm V_{xx}(t,x) \geq \pm A_\infty(t), \qquad \forall t \in [0,T],\ |x| \geq K.$$

Let $E := H_T^1(\mathbb{R}, \mathbb{R}^N)$, the Hilbert space of T-periodic functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the inner product

$$\langle x, y \rangle = \int_0^T (\dot{x} \cdot \dot{y} + x \cdot y)dt, \qquad \forall x, y \in E,$$

and norm $\|x\| = \langle x, x \rangle^{\frac{1}{2}}$. We define the functional I on E by

$$I(x) = \frac{1}{2}\int_0^T |\dot{x}(t)|^2 dt - \int_0^T V(t,x)dt. \tag{1.2}$$

By (H_1) and (H_2), $I \in C^2(E, \mathbb{R})$ and the critical points of I in E are T-periodic solutions of (1.1).

Clearly, the set $\sigma = \{(\frac{2k\pi}{T})^2 \mid k \in \mathbb{Z}^+\}$ is the set of the eigenvalues of

$$-\ddot{x} = \lambda x \tag{1.3}$$

with T-periodic boundary condition. Consider the eigenvalue problem of the following system

$$-\ddot{x} - A_\infty x = \lambda x \tag{1.4}$$

with T-periodic boundary condition. Without loss of generality, in $(H_5^\pm)$ by considering $A_\infty(t) \mp \epsilon I_N$ instead of $A_\infty(t)$ for ϵ small if necessary we may assume that 0 is not the eigenvalue of (1.4). Let $\lambda_1 < \lambda_2 < \cdots < \lambda_l < 0 < \lambda_{l+1} < \lambda_{l+2} < \cdots$ be distinct eigenvalues of (1.4). Clearly, $\lambda_i \to \infty$ as $i \to \infty$. Let $E(\lambda_i)$ be the eigenspace of (1.4) corresponding to λ_i, $i \in \mathbb{Z}^+$.

We define the linear operator $\tilde{L}$ on E by

$$\langle \tilde{L}x, y\rangle := \int_0^T \dot{x} \cdot \dot{y} dt, \qquad \forall x, y \in E.$$

Then $\tilde{L}$ is a bounded self-adjoint operator. Define the linear operators B_0 and B_∞ on E by

$$\langle B_0 x, y\rangle := \int_0^T A_0(t)x \cdot y dt, \qquad \forall x, y \in E$$

and

$$\langle B_\infty x, y\rangle := \int_0^T A_\infty(t)x \cdot y dt, \qquad \forall x, y \in E.$$

Then B_0 and B_∞ are bounded self-adjoint compact operators on E. Let $L_0 := \tilde{L} - B_0$ and $L_\infty := \tilde{L} - B_\infty$. Since 0 is not an eigenvalue of (1.4), we have that L_∞ is a non-degenerate operator on E. Denote by E_0^+, E_0^-, E_∞^+ and E_∞^- the positive and negative spectral subspaces of L_0 and L_∞ respectively, and let $E_0^0 = \ker L_0$. Then there exists a constant $c_0 > 0$ such that for any $x \in E_0^+$ and $y \in E_0^-$,

$$\langle L_0 x, x\rangle \geq c_0\|x\|^2, \qquad \langle L_0 y, y\rangle \leq -c_0\|y\|^2. \tag{1.5}$$

Clearly,

$$E_\infty^- = \bigoplus_{i=1}^{l} E(\lambda_i), \qquad E_\infty^+ = \bigoplus_{i=l+1}^{\infty} E(\lambda_i),$$

$$E = E_0^+ \bigoplus E_0^0 \bigoplus E_0^- = E_\infty^+ \bigoplus E_\infty^-.$$

Set

$$i_0^0 = \dim E_0^0, \qquad i_0^- = \dim E_0^-, \qquad i_\infty^- = \dim E_\infty^-.$$

By (H_3), we see that $x = 0$ is a periodic solutions of (1.1) which is called trivial periodic solution. Our aim is to find nontrivial periodic solutions of (1.1). Now we give our main results as follows.

Theorem 1.1. *Assume that (H_1), (H_2), (H_3) hold. Then (1.1) has at least one nontrivial periodic solution in each of the following cases:*

(1) (H_4^+), (H_5^+) and $i_0^- + i_0^0 < i_\infty^- - 1$;

(2) (H_4^-), (H_5^+) and $i_0^- < i_\infty^- - 1$;

(3) (H_4^+), (H_5^-) and $i_0^- + i_0^0 > i_\infty^- + 1$;

(4) (H_4^-), (H_5^-) and $i_0^- > i_\infty^- + 1$.

Theorem 1.2. *Assume that* (H_1), (H_2), (H_3) *hold, and* $V(t,-x) = V(t,x)$ *for any* $(t,x) \in \mathbb{R} \times \mathbb{R}^N$.

(1) *If* (H_4^+), (H_5^+) *hold and* $i_0^- + i_0^0 < i_\infty^- - 1$, *then* (1.1) *has at least* $i_\infty^- - i_0^- - i_0^0 - 1$ *pairs of nontrivial periodic solutions;*

(2) *If* (H_4^-), (H_5^+) *hold and* $i_0^- < i_\infty^- - 1$, *then* (1.1) *has at least* $i_\infty^- - i_0^- - 1$ *pairs of nontrivial periodic solutions;*

(3) *If* (H_4^+), (H_5^-) *hold and* $i_0^- + i_0^0 > i_\infty^- + 1$, *then* (1.1) *has at least* $i_0^- + i_0^0 - i_\infty^- - 1$ *pairs of nontrivial periodic solutions;*

(4) *If* (H_4^-), (H_5^-) *hold and* $i_0^- > i_\infty^- + 1$, *then* (1.1) *has at least* $i_0^- - i_\infty^- - 1$ *pairs of nontrivial periodic solutions.*

Remark 1.3. In what follows, we assume that $x = 0$ is an isolated critical point of I in E. In fact, if $x = 0$ is not an isolated critical point of I, then I has infinitely many critical points and therefore (1.1) has infinitely many periodic solutions. Therefore Theorem 1.1 and 1.2 hold naturally.

The paper is organized as follows. In Section 2, we construct a sequence of approximate systems and establish the L^∞ bound to the solutions of these approximate systems with appropriate Morse indexes. In Section 3, we will give the proof of Theorem 1.1 by using Morse theory and previous estimate. In Section 4, we will prove Theorem 1.2 by using two critical point theorems for even functional and previous estimate.

2 Preliminaries

In this section we give some important preliminary lemmas. Let H be a real Hilbert space and $J \in C^2(H, \mathbb{R})$. Denote $K(J) = \{u \in H \mid J'(u) = 0\}$. For $u \in K(J)$, we denote the Morse index of u by $m^-(J''(u))$ which is the dimension of the negative spectral subspace of $J''(u)$. The augmented Morse index of u is defined by

$$m^*(J''(u)) = m^-(J''(u)) + \dim \ker(J''(u)),$$

where $\ker(J''(u))$ is the kernel of $J''(u)$.

To construct a sequence of approximate systems of (1.1), we first construct a sequence of approximate functions $V_k(t,x)$. The following result is from [13].

Lemma 2.1. *Assume that* (H_1), (H_2) *and* (H_5^+) *(resp.* (H_5^-)*) hold. Then there exists a sequence functions* $V_k(t,x) \in C^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ *satisfying the following properties:*

(a) $V_k(t+T,x) = V_k(t,x)$ *and there exists an increasing sequence of real numbers* $R_k \to \infty$ $(k \to \infty)$ *such that*
$$V_k(t,x) = V(t,x), \qquad \forall |x| \le R_k, t \in [0,T];$$

(b) *there exist* $C_1' > 0$ *and* $C_2' > 0$ *independent of k such that*
$$|(V_k)_x(t,x)| \le C_1'(1+|x|), \qquad |(V_k)_{xx}(t,x)| \le C_2';$$

(c) *for each* $k \in \mathbb{Z}^+$, $(V_k)_{xx}(t,x) \ge A_\infty(t)$ *(resp.* $(V_k)_{xx}(t,x) \le A_\infty(t)$*) for all* $t \in [0,T]$, $|x| \ge K$;

(d) there is $\gamma > 0$ independent of k such that $(\frac{2p\pi}{T})^2 < \gamma < (\frac{2(p+1)\pi}{T})^2$ for some $p \in \mathbb{Z}^+$, and for each $k \in \mathbb{Z}^+$ fixed,

$$V_k(t,x) = \frac{\gamma}{2}|x|^2 + o(|x|^2), \qquad (V_k)_x(t,x) = \gamma x + o(|x|), \qquad (V_k)_{xx}(t,x) = \gamma I_N + o(1)$$

as $|x| \to \infty$;

(e) if $V(t,-x) = V(t,x)$, $\forall t \in [0,T], x \in \mathbb{R}^N$, then for every $k \in \mathbb{Z}^+$, $V_k(t,-x) = V_k(t,x)$, $\forall t \in [0,T], x \in \mathbb{R}^N$.

Let

$$I_k(x) := \frac{1}{2}\int_0^T |\dot{x}|^2 dt - \psi_k(x), \qquad x \in E, \tag{2.1}$$

where

$$\psi_k(x) := \int_0^T V_k(t,x)dt.$$

Clearly, $I_k(x) \in C^2(E,\mathbb{R})$ and the critical points of I_k correspond to the periodic solutions of the following system

$$-\ddot{x} = (V_k)_x(t,x). \tag{2.2}$$

By Lemma 2.1 (a) and Remark 1.3, $x = 0$ is also an isolated critical point of I_k for every $k \in \mathbb{Z}^+$. Define the linear operator $B_\gamma : E \to E$ by

$$\langle B_\gamma x, y\rangle := \int_0^T \gamma x \cdot y dt, \qquad \forall x, y \in E.$$

Let $L_\gamma := \widetilde{L} - B_\gamma$, then by Lemma 2.1, L_γ is a non-degenerate bounded linear self-adjoint operator on E. We have the decomposition $E = E_\gamma^- \oplus E_\gamma^+$, where E_γ^- and E_γ^+ are the negative and positive spectral subspaces of L_γ. Then there exists a constant $c_\gamma > 0$ such that for any $x \in E_\gamma^+$ and $y \in E_\gamma^-$,

$$\langle L_\gamma x, x\rangle \geq c_\gamma \|x^2\|, \qquad \langle L_\gamma y, y\rangle \leq -c_\gamma \|y^2\|. \tag{2.3}$$

Denote

$$j_\infty^- = \dim E_\gamma^-.$$

By Lemma 2.1 (c), (d), if (H_5^+) holds, then $\gamma I_N \geq A_\infty(t)$, which implies that

$$E_\infty^- \subset E_\gamma^- \quad \text{and} \quad j_\infty^- \geq i_\infty^-. \tag{2.4}$$

If (H_5^-) holds, then $\gamma I_N \leq A_\infty(t)$, which implies that

$$E_\gamma^- \subset E_\infty^- \quad \text{and} \quad j_\infty^- \leq i_\infty^-. \tag{2.5}$$

Let

$$G_k(t,x) = V_k(t,x) - \frac{\gamma}{2}|x|^2, \qquad G_{0k}(t,x) = V_k(t,x) - \frac{1}{2}A_0(t)x\cdot x$$

and

$$\varphi_k(x) = \int_0^T G_k(t,x)dt, \qquad \varphi_{0k}(x) = \int_0^T G_{0k}(t,x)dt.$$

By (H_3), Lemma 2.1 (a), (d), we see that $(G_k)_x(t,x) = o(|x|)$ as $|x| \to \infty$ and $(G_{0k})_x(t,x) = o(|x|)$ as $|x| \to 0$. Then we have

$$\varphi_k'(x) = o(\|x\|) \quad \text{as } \|x\| \to \infty \quad \text{and} \quad \varphi_{0k}'(x) = o(\|x\|) \quad \text{as } \|x\| \to 0. \tag{2.6}$$

And we can rewrite the functional I_k by

$$I_k(x) = \frac{1}{2}\langle L_\gamma x, x\rangle - \varphi_k(x) = \frac{1}{2}\langle L_0 x, x\rangle - \varphi_{0k}(x), \qquad x \in E. \tag{2.7}$$

Lemma 2.2. *Assume that* (H_1), (H_2), (H_3) *and* (H_5^+) *(resp.* (H_5^-)*) hold. For every* $k \in \mathbb{Z}^+$*, if* x_k *is a critical point of* I_k *with* $m^-(I_k''(x_k)) \le i_\infty^- - 1$ *(resp.* $m^*(I_k''(x_k)) \ge i_\infty^- + 1$*), then there exists a constant* $\beta > 0$ *independent of* k *such that* $\|x_k\|_{L^\infty} \le \beta$.

Proof. We use an indirect argument. Assume that $\|x_k\|_{L^\infty} \to \infty$ as $k \to \infty$. By the Sobolev inequality $\|x\|_{L^\infty([0,T])} \le C\|x\|$, we have that $\|x_k\| \to \infty$ as $k \to \infty$.

Let

$$\bar{x}_k = \frac{x_k}{\|x_k\|}.$$

Then $\bar{x}_k$ satisfies

$$-\ddot{\bar{x}}_k = \frac{(V_k)_x(t, x_k)}{\|x_k\|}. \tag{2.8}$$

Up to a subsequence, we have that for some $\bar{x} \in E$, $\bar{x}_k \rightharpoonup \bar{x}$ in E, $\bar{x}_k \to \bar{x}$ in $L^2([0,T])$. And it follows from Proposition 1.2 in [20] that $\bar{x}_k$ converges uniformly to $\bar{x}$ on $[0,T]$. By (H_2), (H_3) and Lemma 2.1, there exists $C_1' > 0$ such that $|(V_k)_x(t, x_k)| \le C_1'|x_k|$. Thus for every k,

$$\left| \frac{(V_k)_x(t, x_k)}{\|x_k\|} \right| \le C_1'|\bar{x}_k|. \tag{2.9}$$

Multiplying (2.8) by $\bar{x}_k$, one has

$$1 = \|\bar{x}_k\|^2 \le (C_1' + 1)\|\bar{x}_k\|^2_{L^2([0,T])}.$$

Letting $k \to \infty$, we get

$$\|\bar{x}\|^2_{L^2([0,T])} \ge \frac{1}{C_1' + 1} > 0. \tag{2.10}$$

Now we show that up to a subsequence $\dot{\bar{x}}_k$ converges uniformly to $\dot{\bar{x}}$ on $[0,T]$. For any $t \in [0,T]$, by (2.8), (2.9) and Hölder inequality, we have

$$\begin{aligned}
|\dot{\bar{x}}_k(0)| &= \left| \dot{\bar{x}}_k(t) + \int_0^t \frac{(V_k)_x(s, x_k)}{\|x_k\|} ds \right| \\
&\le \left| \dot{\bar{x}}_k(t) \right| + \left| \int_0^t C_1' \left| \bar{x}_k(s) \right| ds \right| \\
&\le |\dot{\bar{x}}_k(t)| + C_1'\sqrt{T}\|\bar{x}_k\|_{L^2} \\
&\le |\dot{\bar{x}}_k(t)| + C_1'\sqrt{T},
\end{aligned}$$

thus

$$\begin{aligned}
\int_0^T |\dot{\bar{x}}_k(0)| dt &\le \int_0^T |\dot{\bar{x}}_k(t)| dt + \int_0^T C_1'\sqrt{T} dt \\
&\le \sqrt{T}\|\dot{\bar{x}}_k\|_{L^2} + C_1'\sqrt{T}T \\
&\le \sqrt{T} + C_1'\sqrt{T}T.
\end{aligned}$$

Hence

$$|\dot{\bar{x}}_k(0)| \le C_2,$$

where $C_2 = \frac{\sqrt{T}}{T} + C_1'\sqrt{T}$. Then for any $t \in [0, T]$,

$$\begin{aligned}
|\dot{\bar{x}}_k(t)| &= \left| \dot{\bar{x}}_k(0) + \int_0^t -\frac{(V_k)_x(s, x_k)}{\|x_k\|} ds \right| \\
&\leq |\dot{\bar{x}}_k(0)| + \left| \int_0^t C_1' |\bar{x}_k(s)| ds \right| \\
&\leq C_2 + C_1'\sqrt{T} \|\bar{x}_k\|_{L^2} \\
&\leq C_2 + C_1'\sqrt{T},
\end{aligned}$$

which implies that for every $k \in \mathbb{Z}^+$,

$$\|\dot{\bar{x}}_k(t)\|_{C^0} \leq C_2 + C_1'\sqrt{T}. \tag{2.11}$$

For any $\Delta t \in \mathbb{R}$, by (2.8) and (2.9) we have

$$\begin{aligned}
|\dot{\bar{x}}_k(t + \Delta t) - \dot{\bar{x}}_k(t)| &= \left| \int_t^{t+\Delta t} \ddot{\bar{x}}_k(s) ds \right| \\
&= \left| \int_t^{t+\Delta t} -\frac{(V_k)_x(t, x_k)}{\|x_k\|} ds \right| \\
&\leq \left| \int_t^{t+\Delta t} C_1' |\bar{x}_k| ds \right| \\
&\leq C_1' |\Delta t|^{\frac{1}{2}} \|\bar{x}_k\|_{L^2} \leq C_1' |\Delta t|^{\frac{1}{2}}.
\end{aligned} \tag{2.12}$$

Thus by (2.11) and (2.12), we have

$$\|\dot{\bar{x}}_k(t)\|_{C^{\frac{1}{2}}} \leq C.$$

Then by the Arzelà–Ascoli theorem, $\dot{\bar{x}}_k$ converges uniformly to $\dot{\bar{x}}$ on $[0, T]$.

We claim that $\bar{x}(t) \neq 0$ a.e. in $[0, T]$. In fact, conversely, if $\bar{x}(t) = 0$ in a positive measure subset of $[0, T]$, then there exists a point $t_0 \in [0, T]$ such that $\bar{x}(t_0) = 0$ and $\dot{\bar{x}}(t_0) = 0$. Recall that $\bar{x}_k$ and $\dot{\bar{x}}_k$ converge uniformly to $\bar{x}$ and $\dot{\bar{x}}$ respectively on $[0, T]$, we have

$$\bar{x}_k(t_0) \to 0 \quad \text{and} \quad \dot{\bar{x}}_k(t_0) \to 0 \tag{2.13}$$

as $k \to \infty$. Let $\bar{y}_k := \dot{\bar{x}}_k$, then $(\bar{x}_k, \bar{y}_k)$ satisfies the following system

$$\begin{cases} \dot{\bar{x}}_k = \bar{y}_k, \\ \dot{\bar{y}}_k = -\frac{(V_k)_x(t, x_k)}{\|x_k\|}. \end{cases} \tag{2.14}$$

For any $t \in [0, T]$,

$$\begin{aligned}
|(\bar{x}_k(t), \bar{y}_k(t))| &= \left| (\bar{x}_k(t_0), \bar{y}_k(t_0)) + \int_{t_0}^t \left(\bar{y}_k(s), -\frac{(V_k)_x(s, x_k)}{\|x_k\|} \right) ds \right| \\
&\leq |(\bar{x}_k(t_0), \bar{y}_k(t_0))| + \left| \int_{t_0}^t \left| \left(\bar{y}_k(s), -\frac{(V_k)_x(s, x_k)}{\|x_k\|} \right) \right| ds \right| \\
&\leq |(\bar{x}_k(t_0), \bar{y}_k(t_0))| + \left| \int_{t_0}^t \sqrt{1 + C_1'^2}\, |(\bar{x}_k(s), \bar{y}_k(s))|\, ds \right|.
\end{aligned}$$

Thus by Gronwall's inequality, we have

$$|(\bar{x}_k(t), \bar{y}_k(t))| \leq |(\bar{x}_k(t_0), \bar{y}_k(t_0))| e^{\left| \int_{t_0}^t \sqrt{1 + C_1'^2} ds \right|} \leq C |(\bar{x}_k(t_0), \bar{y}_k(t_0))|, \tag{2.15}$$

where $C = e^{\sqrt{1+C_1'^2}T}$. Then letting $k \to \infty$ in (2.15), we get $\bar{x}(t) = 0$ and $\bar{y}(t) = 0$ for any $t \in [0, T]$, which is contrary to (2.10). Hence the claim is proved. Note that $\|x_k\| \to \infty$, then by this claim one has

$$|x_k| \to \infty \quad \text{a.e. in } [0, T] \tag{2.16}$$

as $k \to \infty$.

If (H_5^+) holds, then by (2.16), Lemma 2.1 (b), (c) and Fatou's Lemma, for any fixed $x \in E_\infty^- \setminus \{0\}$,

$$\begin{aligned}
\limsup_{k\to\infty} \langle I_k''(x_k)x, x\rangle &= \langle \tilde{L}x, x\rangle - \liminf_{k\to\infty} \int_0^T (V_k)_{xx}(t, x_k)x \cdot x dt \\
&\le \langle \tilde{L}x, x\rangle - \int_0^T \liminf_{k\to\infty} (V_k)_{xx}(t, x_k)x \cdot x dt \\
&\le \langle \tilde{L}x, x\rangle - \int_0^T A_\infty(t)x \cdot x dt \\
&= \langle L_\infty x, x\rangle < 0,
\end{aligned}$$

which implies that there exists $k(x) \in \mathbb{Z}^+$ such that $\langle I_k''(x_k)x, x\rangle < 0$ when $k \ge k(x)$. Note that E_∞^- is finite dimensional, there must exist $k_0 \in \mathbb{Z}^+$ independent of $x \in E_\infty^- \setminus \{0\}$ such that

$$\langle I_k''(x_k)x, x\rangle < 0$$

for all $x \in E_\infty^- \setminus \{0\}$ and $k \ge k_0$. This means that $m^-(I_k''(x_k)) \ge i_\infty^-$ for $k \ge k_0$, which leads to a contradiction.

If (H_5^-) holds, since E_∞^+ is infinite dimensional, the above argument cannot be used directly. To overcome this difficulty, we will split E_∞^+ into two parts. Let

$$M = \max_{t\in[0,T]} |A_\infty(t)|.$$

Since $\lambda_i \to \infty$ as $i \to \infty$, then there exists $i_0 \in \mathbb{Z}^+$ such that $\lambda_{i_0} \ge 2(M + C_2')$ where C_2' is the constant as in Lemma 2.1 (b). Let

$$E_1 = \bigoplus_{i=l+1}^{i_0-1} E(\lambda_i), \qquad E_2 = \bigoplus_{i=i_0}^{\infty} E(\lambda_i).$$

Then $E_\infty^+ = E_1 \oplus E_2$ and E_1 is finite dimensional. For any $y_1 \in E_2 \setminus \{0\}$, note that

$$\int_0^T (|\dot{y}_1|^2 - A_\infty y_1 \cdot y_1)dt \ge \lambda_{i_0} \int_0^T |y_1|^2 dt,$$

then

$$\begin{aligned}
\langle I_k''(x_k)y_1, y_1\rangle &= \int_0^T |\dot{y}_1|^2 dt - \int_0^T (V_k)_{xx}(t, x_k)y_1 \cdot y_1 dt \\
&\ge \lambda_{i_0} \int_0^T |y_1|^2 dt + \int_0^T A_\infty y_1 \cdot y_1 dt - \int_0^T (V_k)_{xx}(t, x_k)y_1 \cdot y_1 dt \\
&\ge \lambda_{i_0} \int_0^T |y_1|^2 dt - \int_0^T M|y_1|^2 dt - \int_0^T C_2'|y_1|^2 dt \\
&\ge \frac{\lambda_{i_0}}{2} \int_0^T |y_1|^2 dt > 0.
\end{aligned} \tag{2.17}$$

For any $y_2 \in E_1 \setminus \{0\}$, by (2.16), Lemma 2.1 (b), (c) and Fatou's Lemma,

$$\begin{aligned}\liminf_{k\to\infty}\langle I_k''(x_k)y_2, y_2\rangle &= \int_0^T |\dot{y}_2|^2 dt - \limsup_{k\to\infty}\int_0^T (V_k)_{xx}(t,x_k)y_2\cdot y_2 dt \\ &\geq \int_0^T |\dot{y}_2|^2 dt - \int_0^T \limsup_{k\to\infty}(V_k)_{xx}(t,x_k)y_2\cdot y_2 dt \\ &\geq \int_0^T |\dot{y}_2|^2 dt - \int_0^T A_\infty(t)y_2\cdot y_2 dt \\ &= \langle L_\infty y_2, y_2\rangle > 0,\end{aligned}$$

which implies that there exists $k(y_2) \in \mathbb{Z}^+$ such that $\langle I_k''(x_k)y_2, y_2\rangle > 0$ for $k \geq k(y_2)$. Note that E_1 is finite dimensional, there must exist $k_1 \in \mathbb{Z}^+$ independent of $y_2 \in E_1 \setminus \{0\}$ such that

$$\langle I_k''(x_k)y_2, y_2\rangle > 0 \tag{2.18}$$

for all $y_2 \in E_1 \setminus \{0\}$ and $k \geq k_1$. Hence by (2.17) and (2.18), for any $y \in E_\infty^+ \setminus \{0\}$ and every $k \geq k_1$,

$$\langle I_k''(x_k)y, y\rangle > 0.$$

This implies that $m^*(I_k''(x_k)) \leq i_\infty^-$ for $k \geq k_1$, which leads to a contradiction.

Therefore the lemma is proved. $\square$

3 Proof of Theorem 1.1

In this section, we will use Morse theory to prove the existence of nontrivial periodic solution for (1.1). Let H be a real Hilbert space and $J \in C^2(H,\mathbb{R})$ be a functional satisfying the (PS) condition, i.e., any sequence $\{u_n\} \subset H$ for which $J(u_n)$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence. Denote by $H_q(A,B)$ the q-th singular relative homology group of the topological pair (A,B) with coefficients in a field $\mathcal{F}$. Let u be an isolated critical point of J with $J(u) = c$. The groups

$$C_q(J,u) := H_q(J^c, J^c \setminus \{u\}), \qquad q \in \mathbb{Z}$$

are called the critical groups of J at u, where $J^c = \{u \in H \mid J(u) \leq c\}$. Denote $K = K(J) = \{u \in H \mid J'(u) = 0\}$. Suppose that $J(K)$ is bounded from below by $a \in \mathbb{R}$. The critical groups of J at infinity are defined by

$$C_q(J,\infty) := H_q(H, J^a), \qquad q \in \mathbb{Z}.$$

We say that J has a local linking structure at 0 with respect to the direct sum decomposition $H = H^- \oplus H^+$ if there exists $r > 0$ such that

$$J(u) > 0 \quad \text{for } u \in H^+ \text{ with } 0 < \|u\| \leq r, \qquad J(u) \leq 0 \quad \text{for } u \in H^- \text{ with } \|u\| \leq r.$$

The following results can be found in [1], [26] and [4].

Proposition 3.1 (See [1]). *Suppose J satisfies (PS) condition. If $K = \emptyset$, then $C_q(J,\infty) \cong 0, q \in \mathbb{Z}$. If $K = \{u_0\}$, then $C_q(J,\infty) \cong C_q(J,u_0), q \in \mathbb{Z}$.*

Proposition 3.2 (See [26]). *Let 0 be an isolated critical point of $J \in C^2(H, \mathbb{R})$ with Morse index μ_0 and nullity ν_0. Assume that J has a local linking structure at 0 with respect to the direct sum decomposition $H = H^- \oplus H^+$ and $k = \dim H^- < \infty$. If $k = \mu_0$ or $k = \mu_0 + \nu_0$, then*

$$C_q(J, u) = \delta_{q,k}\mathcal{F}, \qquad q \in \mathbb{Z}.$$

Let A be a nondegenerate bounded self-adjoint operator defined on H. According to its spectral decomposition, $H = H^+ \oplus H^-$, where H^+, H^- are invariant subspaces corresponding to the positive and negative spectrum of A respectively. Let

$$J(x) = \frac{1}{2}\langle Ax, x\rangle + g(x),$$

and the following assumptions are given:

(A_1) $A_\pm := A\mid_{H^\pm}$ has a bounded inverse on $H^\pm$;

(A_2) $\kappa := \dim H^- < \infty$;

(A_3) $g \in C^1(H, \mathbb{R}^1)$ has a compact derivative $g'(x)$ and $\|g'(x)\| = o(\|x\|)$ as $\|x\| \to \infty$.

Proposition 3.3 (See [4]). *Under the assumptions (A_1), (A_2) and (A_3), we have that J satisfies (PS) condition and $C_q(J, \infty) = \delta_{q,\kappa}\mathcal{F}$.*

Proposition 3.4 (See [4]). *Suppose that $J \in C^2(H, \mathbb{R})$ satisfies (PS) condition, and $K = \{u_1, \dots, u_k\}$, then*

$$\sum_{q=0}^{\infty} M_q t^q = \sum_{q=0}^{\infty} \beta_q t^q + (1+t)Q(t),$$

where $Q(t)$ is a formal series with nonnegative coefficients, $M_q = \sum_{i=0}^{k} \operatorname{rank} C_q(J, u_k)$ and $\beta_q = \operatorname{rank} C_q(J, \infty)$, $q = 0, 1, 2, \dots$

Now we compute the critical groups of I_k at zero and at infinity.

Lemma 3.5. *Assume that (H_1)–(H_3) hold. Then for every $k \in \mathbb{Z}^+$,*

(1) if (H_4^+) holds,

$$C_q(I_k, 0) = \delta_{q, i_0^- + i_0^0}\mathcal{F}, \qquad q \in \mathbb{Z}.$$

(2) if (H_4^-) holds,

$$C_q(I_k, 0) = \delta_{q, i_0^-}\mathcal{F}, \qquad q \in \mathbb{Z}.$$

Proof. (1) We first show that I_k has a local linking structure at 0 with respect to $E = E^- \oplus E^+$, where $E^- = E_0^- \oplus E_0^0$ and $E^+ = E_0^+$. For $x \in E_0^+$, by (1.5) and (2.6) we have

$$\begin{aligned} I_k(x) &= \frac{1}{2}\langle L_0 x, x\rangle - \varphi_{0k}(x) \\ &\geq \frac{c_0}{2}\|x\|^2 - o(\|x\|^2) \end{aligned} \tag{3.1}$$

as $\|x\| \to 0$. This means that there exists small $r > 0$ such that

$$I_k(x) > 0, \quad \text{for } x \in E_0^+ \text{ with } 0 < \|x\| \leq r. \tag{3.2}$$

For $x \in E_0^- \oplus E_0^0$, we write $x = x^- + x^0$ with $x^- \in E_0^-$ and $x^0 \in E_0^0$. Then

$$\begin{aligned} I_k(x) &= \frac{1}{2}\langle L_0 x^-, x^- \rangle - \int_0^T G_{0k}(t,x)dt \\ &\leq -\frac{c_0}{2}\|x^-\|^2 - \int_0^T G_{0k}(t,x)dt. \end{aligned} \tag{3.3}$$

By (H_4^+) and Lemma 2.1 (a),

$$\int_{|x|\leq \delta} G_{0k}(t,x)dt \geq 0. \tag{3.4}$$

If $|x| > \delta$, since E_0^0 is finite dimensional, we have

$$|x^-| \geq |x| - |x^0| \geq |x| - \|x^0\|_{L^\infty} \geq |x| - C\|x^0\| \geq |x| - C\|x\|,$$

thus let $0 < r < \frac{\delta}{3C}$, for $\|x\| \leq r$, we have

$$|x^-| \geq |x| - \frac{\delta}{3} \geq |x| - \frac{1}{3}|x| = \frac{2}{3}|x|. \tag{3.5}$$

By Lemma 2.1 (b), (d), there exists $C_\delta > 0$ such that for $|x| > \delta$,

$$|G_{0k}(t,x)| \leq C_\delta |x|^3. \tag{3.6}$$

Hence, by (3.3)–(3.6), for $x \in E_0^- \oplus E_0^0$ with $\|x\| \leq r$, we have

$$\begin{aligned} I_k(x) &\leq -\frac{c_0}{2}\|x^-\|^2 - \int_0^T G_{0k}(t,x)dt \\ &\leq -\frac{c_0}{2}\|x^-\|^2 - \int_{|x|\leq\delta} G_{0k}(t,x)dt - \int_{|x|>\delta} G_{0k}(t,x)dt \\ &\leq -\frac{c_0}{2}\|x^-\|^2 + \int_{|x|>\delta} C_\delta |x|^3 dt \\ &\leq -\frac{c_0}{2}\|x^-\|^2 + C_\delta \int_{|x|>\delta} \left(\frac{3}{2}\right)^3 |x^-|^3 dt \\ &\leq -\frac{c_0}{2}\|x^-\|^2 + C'_\delta \|x^-\|^3. \end{aligned} \tag{3.7}$$

This implies that there exists $r > 0$ small enough such that

$$I_k(x) < 0, \quad \text{for } x \in E_0^- \oplus E_0^0 \text{ with } \|x\| \leq r \text{ and } \|x^-\| > 0. \tag{3.8}$$

On the other hand, for $x^0 \in E_0^0$, we can choose $r > 0$ small enough such that

$$0 < \|x^0\|_{L^\infty} < \delta, \quad \text{when } 0 < \|x^0\| \leq r.$$

Then for $x^0 \in E_0^0$ with $0 < \|x^0\| \leq r$, since $x^0 \in C^2([0,T], \mathbb{R}^N)$, there must exist $0 < t_1 < t_2 < T$ such that

$$0 < |x^0(t)| < \delta, \qquad \forall t \in [t_1, t_2].$$

Then by (H_4^+) and Lemma 2.1 (a), for $x^0 \in E_0^0$ with $0 < \|x^0\| \leq r$,

$$I_k(x^0) = -\int_0^T G_{0k}(t,x^0)dt = -\int_0^T G_0(t,x^0)dt \leq -\int_{t_1}^{t_2} G_0(t,x^0)dt < 0. \tag{3.9}$$

Hence, by (3.8) and (3.9), there exists $r > 0$ such that

$$I_k(x) < 0, \quad \text{for } x \in E_0^- \oplus E_0^0 \text{ with } 0 < \|x\| \le r. \tag{3.10}$$

Therefore, it follows from (3.2) and (3.10) that I_k has a local linking structure at 0 with respect to $E = E^- \oplus E^+$, where $E^- = E_0^- \oplus E_0^0$ and $E^+ = E_0^+$. Then by Proposition 3.2, we have

$$C_q(I_k, 0) = \delta_{q, i_0^- + i_0^0}\mathcal{F}, \qquad q \in \mathbb{Z}.$$

(2) By a similar argument as (1), we can prove that I_k has a local linking structure at 0 with respect to $E = E^- \oplus E^+$, where $E^- = E_0^-$ and $E^+ = E_0^+ \oplus E_0^0$. Then by Proposition 3.2, we have

$$C_q(I_k, 0) = \delta_{q, i_0^-}\mathcal{F}, \qquad q \in \mathbb{Z}. \qquad \square$$

Lemma 3.6. *Assume that (H_1)-(H_3), (H_5^+)(or (H_5^-)) hold. Then for every $k \in \mathbb{Z}^+$, I_k satisfies (PS) condition and the critical groups of I_k at infinity are*

$$C_q(I_k, \infty) = \delta_{q, j_\infty^-}\mathcal{F}, \qquad q \in \mathbb{Z}.$$

Proof. Note that

$$I_k(x) = \frac{1}{2}\langle L_\gamma x, x\rangle - \varphi_k(x)$$

Since L_γ is a nondegenerate operator on E, then $L_\gamma\,|_{E_\gamma^\pm}$ has a bounded inverse on $E_\gamma^\pm$. Recall that $\dim E_\gamma^- = j_\infty^- < \infty$. Thus the assumptions ($A_1$) and ($A_2$) in Proposition 3.3 are satisfied. On the other hand, note that $\varphi_k(x) \in C^2(E, \mathbb{R})$ has compact derivative $\varphi_k'(x)$ and $\varphi_k'(x) = o(\|x\|)$ as $\|x\| \to \infty$, then the assumption (A_3) in Proposition 3.3 is also satisfied. Hence, by Proposition 3.3, we have

$$C_q(I_k, \infty) = \delta_{q, j_\infty^-}\mathcal{F}, \qquad q \in \mathbb{Z}. \qquad \square$$

Remark 3.7. Since $I_k'(x) = L_\gamma x + \varphi_k'(x) = L_\gamma x + o(\|x\|)$ as $\|x\| \to \infty$ and L_γ is invertible, it is easy to see that the critical point set $K(I_k)$ is bounded for every $k \in \mathbb{Z}^+$. Then since I_k satisfies (PS) condition by Lemma 3.6, we conclude that $K(I_k)$ is a compact set for every $k \in \mathbb{Z}^+$.

Proof of Theorem 1.1. We only prove the result for the case (1), the proofs for the cases (2), (3) and (4) are similar.

For every $k \in \mathbb{Z}^+$, since $x = 0$ is an isolated critical point of I_k, there exists $\sigma > 0$ such that $I_k(x)$ has no nontrivial critical points in $B_\sigma(0) := \{x \mid \|x\| \le \sigma\}$. Since $i_0^- + i_0^0 < i_\infty^- - 1$, then by (2.4), Lemma 3.5 (1) and Lemma 3.6 we have

$$C_q(I_k, \infty) \ne C_q(I_k, 0)$$

for some $q \in \mathbb{Z}$. So by Proposition 3.1 and Remark 3.7, the set $K(I_k) \setminus \{0\}$ is not empty and compact. Denote $\mathcal{K}_k = K(I_k) \setminus \{0\}$.

Now we show that for every $k \in \mathbb{Z}^+$ there exists a nontrivial critical point $x_k \in \mathcal{K}_k$ such that

$$m^-(I_k''(x_k)) \le i_\infty^- - 1. \tag{3.11}$$

We use an indirect argument. Suppose that for any $x_k \in \mathcal{K}_k$,

$$m^-(I_k''(x_k)) > i_\infty^- - 1. \tag{3.12}$$

For $A \subset E$ and $a > 0$, set

$$N_a(A) := \{x \in E \mid \operatorname{dist}(x, A) < a\}.$$

Using the Marino–Prodi perturbation technique from [25], for any $\epsilon > 0$ and $0 < \tau < \min\{\frac{\sigma}{3}, 1\}$, we can obtain a C^2 functional J_k such that:

(i) $\|I_k - J_k\|_{C^2} < \epsilon$;

(ii) $I_k(x) = J_k(x)$, $x \in E \setminus N_{2\tau}(\mathcal{K}_k)$;

(iii) $I_k''(x) = J_k''(x)$ for any $x \in N_\tau(K(I_k))$, $K(J_k) \setminus \{0\} \subset N_\tau(\mathcal{K}_k)$, and the nontrivial critical points of J_k are all non-degenerate.

By (iii), $J_k''(0) = I_k''(0)$, thus by Proposition 3.2 and Lemma 3.5, we have

$$C_q(J_k, 0) = C_q(I_k, 0) = \delta_{q, i_0^- + i_0^0}\mathcal{F}. \tag{3.13}$$

By (ii), $I_k(x) = J_k(x)$ for $x \in E \setminus N_{2\tau}(\mathcal{K}_k)$, then by Lemma 3.6, J_k also satisfies (PS) condition and

$$C_q(J_k, \infty) = C_q(I_k, \infty) = \delta_{q, j_\infty^-}\mathcal{F}. \tag{3.14}$$

Since $K(J_k) \subset N_\tau(K(I_k))$ and $K(I_k)$ is compact, $K(J_k)$ is also a compact set. Moreover, note that the notrivial critical points of J_k are all non-degenerate, we have that $K(J_k)$ is a finite set. Suppose that

$$K(J_k) \setminus \{0\} = \{x_{k1}, x_{k2}, x_{k3}, \dots, x_{kn}\}.$$

By (iii) and (3.12), we can choose τ small enough such that for all $1 \le i \le n$,

$$m^-(J_k''(x_{ki})) > i_\infty^- - 1. \tag{3.15}$$

By (3.13), (3.14), and Proposition 3.4 we have

$$t^{i_0^- + i_0^0} + \sum_{i=1}^{n} t^{m^-(J_k''(x_{ki}))} = t^{j_\infty^-} + (1+t)Q(t). \tag{3.16}$$

Note that $i_0^- + i_0^0 < i_\infty^- - 1$ and $i_\infty^- \le j_\infty^-$, it follows from (3.16) that $(1+t)Q(t)$ has a nonzero term with exponent $i_0^- + i_0^0$. Then this means that the left hand side of (3.16) has a nonzero term with exponent $i_0^- + i_0^0 - 1$ or $i_0^- + i_0^0 + 1$. Thus there exists a $1 \le i \le n$ such that

$$m^-(J_k''(x_{ki})) = i_0^- + i_0^0 - 1 \quad \text{or} \quad m^-(J_k''(x_{ki})) = i_0^- + i_0^0 + 1.$$

Since $i_0^- + i_0^0 < i_\infty^- - 1$, we have that $m^-(J_k''(x_{ki})) \le i_\infty^- - 1$ for some $1 \le i \le n$. This is contrary to (3.15), thus (3.11) is proved.

By Lemma 2.2 and (3.11), for every $k \in \mathbb{Z}^+$ the functional I_k has a nontrivial critical point x_k such that $\|x_k\|_{L^\infty} \le \beta$. By Lemma 2.1, for k large enough such that $R_k > \beta$, x_k is also a nontrivial critical point of I, and thus x_k is a nontrivial periodic solution of (1.1). $\square$

4 Proof of Theorem 1.2

We introduce two critical point theorems which will be used in proving Theorem 1.2. Let H be a Hilbert space. Assume that $J \in C^2(H, \mathbb{R})$ is an even functional, satisfies the (PS) condition, $J(0) = 0$ and $K(J)$ is a compact set. Let $B_a = \{y \in H \mid \|y\| \le a\}$ and $S_a = \partial B_a = \{y \in H \mid \|y\| = a\}$. The following two critical point theorems follow from Ghoussoub [7] and Chang [4] (see also [13]).

Proposition 4.1 (See [7]). *Assume $H = Y \bigoplus Z$, and let X be a subspace of H, satisfying* $\dim X = j > k = \dim Y$. *If there exist $R > r > 0$ and $\alpha > 0$ such that*

$$\inf J(S_r \cap Z) \geq \alpha, \qquad \sup J(S_R \cap X) \leq 0,$$

then J has $j - k$ pairs of nontrivial critical points $\{\pm u_1, \pm u_2, \dots, \pm u_{j-k}\}$ so that $m^-(J''(u_i)) \leq k + i$ for $i = 1, 2, \dots, j - k$.

Proposition 4.2 (See [4]). *Assume $H = Y \bigoplus Z$, and let X be a subspace of H, satisfying* $\dim X = j > k = \dim Y$. *If there exist $r > 0$ and $\alpha > 0$ such that*

$$\inf J(Z) > -\infty, \qquad \sup J(S_r \cap X) \leq -\alpha,$$

then J has at least $j - k$ pairs of nontrivial critical points $\{\pm u_1, \pm u_2, \dots, \pm u_{j-k}\}$ so that $m^*(J''(u_i)) \geq k + i - 1$ for $i = 1, 2, \dots, j - k$.

For every $k \in \mathbb{Z}^+$, by Lemma 2.1 (e), we see that $I_k(x)$ is an even functional on E. From Lemma 3.6 and Remark 3.7, I_k satisfies (PS) condition and $K(I_k)$ is compact. Now we give the proof of Theorem 1.2.

Proof of Theorem 1.2. (1) We will use Proposition 4.1 to prove this case. Let $Y = E_0^- \oplus E_0^0$, $Z = E_0^+$ and $X = E_\infty^-$. Then $E = Y \oplus Z$ and $\dim X = i_\infty^- > i_0^- + i_0^0 = \dim Y$.

For $x \in E_0^+$, by (1.5) and (2.6) we have

$$I_k(x) = \frac{1}{2}\langle L_0 x, x\rangle - \varphi_{0k}(x) \geq \frac{c_0}{2}\|x\|^2 + o(\|x\|^2) \tag{4.1}$$

as $\|x\| \to 0$. Then there exists $\alpha > 0$ and sufficiently small $r > 0$ such that $I_k(x) \geq \alpha$ for any $x \in S_r \cap E_0^+$, that is

$$\inf I_k(S_r \cap E_0^+) \geq \alpha. \tag{4.2}$$

On the other hand, recall that $E_\infty^- \subset E_\gamma^-$ in this case, then by (2.3) for $x \in E_\infty^-$ we have

$$I_k(x) = \frac{1}{2}\langle L_\gamma x, x\rangle - \varphi_k(x) \leq -\frac{c_\gamma}{2}\|x\|^2 + o(\|x\|^2) \tag{4.3}$$

as $\|x\| \to \infty$. Thus there exists $R > r$ such that $I_k(x) \leq 0$ for any $x \in S_R \cap E_\infty^-$, that is

$$\sup I_k(S_R \cap E_\infty^-) \leq 0. \tag{4.4}$$

For every $k \in \mathbb{Z}^+$, by (4.2), (4.4) and using Proposition 4.1, we have that $I_k(x)$ has $i_\infty^- - i_0^- - i_0^0$ pairs of nontrivial critical points $\{\pm x_k^1, \pm x_k^2, \dots, \pm x_k^{i_\infty^- - i_0^- - i_0^0}\}$ with $m^-(I_k''(x_k^i)) \leq i_0^- + i_0^0 + i$ for $i = 1, 2, \dots, i_\infty^- - i_0^- - i_0^0$. By Lemma 2.2, $\|x_k^i\|_{L^\infty} \leq \beta$ for $i = 1, 2, \dots, i_\infty^- - i_0^- - i_0^0 - 1$. Then for k large enough such that $R_k > \beta$, $\{\pm x_k^1, \pm x_k^2, \dots, \pm x_k^{i_\infty^- - i_0^- - i_0^0 - 1}\}$ are also nontrivial critical points of I, and therefore are nontrivial periodic solutions of (1.1).

(2) We will also use Proposition 4.1 to prove this case. Let $Y = E_0^-$, $Z = E_0^+ \oplus E_0^0$ and $X = E_\infty^-$. Then $E = Y \oplus Z$ and $\dim X = i_\infty^- > i_0^- = \dim Y$.

For $x \in E_0^+ \oplus E_0^0$, we write $x = x^+ + x^0$ where $x^+ \in E_0^+$ and $x^0 \in E_0^0$. For $x \in (E_0^+ \cap S_r) \oplus (E_0^0 \cap B_r)$, by (1.5) we have

$$\begin{aligned} I_k(x) &= \frac{1}{2}\langle L_0 x^+, x^+\rangle - \varphi_{0k}(x^+ + x^0) \\ &\geq \frac{c_0}{2}\|x^+\|^2 - o(\|x^+ + x^0\|^2) \\ &\geq \frac{c_0}{4} r^2 \end{aligned} \tag{4.5}$$

provided r is small enough.

Now we consider I_k on $(E_0^+ \cap B_r) \oplus (E_0^0 \cap S_r)$. For $x \in (E_0^+ \cap B_r) \oplus (E_0^0 \cap S_r)$, by (1.5) we have that

$$I_k(x) = \frac{1}{2}\langle L_0 x^+, x^+\rangle - \varphi_{0k}(x^+ + x^0)$$
$$\geq -\varphi_{0k}(x^+ + x^0)$$
$$\geq -\frac{1}{4}r^2 \tag{4.6}$$

provided r is small enough. Inspired by [12], we define a function $g : E_0^0 \cap S_r \to \mathbb{R}$ by

$$g(x^0) = \inf\{I_k(x^+ + x^0) \mid x^+ \in E_0^+ \cap B_r\}.$$

Then by (4.6), g is well defined and continuous. For any fixed $x^0 \in E_0^0 \cap S_r$, by a standard minimization method, we see that $g(x^0)$ is attained at some $\bar{x}^+ \in E_0^+ \cap B_r$, i.e.,

$$g(x^0) = I_k(\bar{x}^+ + x^0).$$

By the Sobolev inequality $\|x\|_{L^\infty} \leq C\|x\|$, we can choose r small enough such that

$$\|\bar{x}^+ + x^0\|_{L^\infty} < \delta.$$

Thus by (H_4^-),

$$G_{0k}(t, (\bar{x}^+ + x^0)(t)) < 0$$

for any t satisfying $(\bar{x}^+ + x^0)(t) \neq 0$. Since $x^0 \in E_0^0 \cap S_r$, then $\bar{x}^+ + x^0$ is not identically equal to zero. This implies that

$$\int_0^T G_{0k}(t, (\bar{x}^+ + x^0)(t))dt < 0$$

and

$$g(x^0) = I_k(\bar{x}^+ + x^0) = \frac{1}{2}\langle L_0 \bar{x}^+, \bar{x}^+\rangle - \int_0^T G_{0k}(t, (\bar{x}^+ + x^0)(t))dt > 0.$$

Since E_0^0 is finite dimensional, $E_0^0 \cap S_r$ is a compact set. Then there exists $\alpha_0 > 0$ such that

$$g(x^0) \geq \alpha_0, \qquad \forall x^0 \in E_0^0 \cap S_r.$$

Hence, by the definition of g we have

$$I_k(x^+ + x^0) \geq g(x^0) \geq \alpha_0, \qquad \forall x^+ + x^0 \in (E_0^+ \cap B_r) \oplus (E_0^0 \cap S_r). \tag{4.7}$$

Let $\alpha = \min\{\alpha_0, \frac{c_0}{4}r^2\}$. Notice that

$$\partial[(E_0^+ \cap B_r) \oplus (E_0^0 \cap B_r)] = [(E_0^+ \cap S_r) \oplus (E_0^0 \cap B_r)] \cup [(E_0^+ \cap B_r) \oplus (E_0^0 \cap S_r)],$$

then by (4.5) and (4.7) we have

$$I_k(x^+ + x^0) \geq \alpha, \qquad \forall x^+ + x^0 \in \partial[(E_0^+ \cap B_r) \oplus (E_0^0 \cap B_r)]. \tag{4.8}$$

Taking $\alpha > 0$ smaller if necessary, we obtain

$$I_k(x^+ + x^0) \geq \alpha, \qquad \forall x^+ + x^0 \in (E_0^+ \oplus E_0^0) \cap S_r, \tag{4.9}$$

that is

$$\inf I_k((E_0^+ \oplus E_0^0) \cap S_r) \geq \alpha. \tag{4.10}$$

By (H_5^+), it is easy to see that (4.4) also holds in this case. Then by (4.4) and (4.10), using Proposition 4.1 and a similar argument as the case (1), we can prove that system (1.1) has at least $i_\infty^- - i_0^- - 1$ pairs of nontrivial periodic solutions;.

(3) We will use Proposition 4.2 to prove this case. Let $Y = E_\infty^-$, $Z = E_\infty^+$ and $X = E_0^- \oplus E_0^0$. Then $E = Y \oplus Z$ and $\dim X = i_0^- + i_0^0 > i_\infty^- = \dim Y$.

For $x \in E_\infty^+$, note that $E_\infty^+ \subset E_\gamma^+$ by (2.5) in this case, we have

$$I_k(x) = \frac{1}{2}\langle L_\gamma x, x\rangle - \varphi_k(x) \geq \frac{c_\gamma}{2}\|x\|^2 - o(\|x\|^2) \tag{4.11}$$

as $\|x\| \to \infty$. Then there exists $M_k > 0$ such that

$$I_k(x) \geq 0, \qquad \forall x \in E_\infty^+ \text{ with } \|x\| \geq M_k. \tag{4.12}$$

On the other hand, by Lemma 2.1, there exists a constant $C_1' > 0$ such that

$$|V_k(t,x)| \leq C_1'|x|^2.$$

Thus for $x \in E_\infty^+$ with $\|x\| \leq M_k$, we have

$$I_k(x) = \frac{1}{2}\int_0^T |\dot{x}|^2 dt - \int_0^T V_k(t,x)dt \geq -C_1'\int_0^T |x|^2 dt \geq -C_1' M_k^2. \tag{4.13}$$

By (4.12) and (4.13), we have

$$\inf I_k(E_\infty^+) > -\infty. \tag{4.14}$$

For $x \in E_0^- \oplus E_0^0$, by using a similar argument as in obtaining (4.9), we have that there exist $r > 0$ and $\alpha > 0$ such that

$$I_k(x) \leq -\alpha, \qquad \forall x \in (E_0^- \oplus E_0^0) \cap S_r,$$

that is

$$\sup I_k((E_0^- \oplus E_0^0) \cap S_r) \leq -\alpha. \tag{4.15}$$

Then by (4.14) and (4.15), using Proposition 4.2 and a similar argument as the case (1), we can prove that the system (1.1) has at least $i_0^- + i_0^0 - i_\infty^- - 1$ pairs of nontrivial periodic solutions.

(4) We will also use Proposition 4.2 to prove this case. Let $Y = E_\infty^-$, $Z = E_\infty^+$ and $X = E_0^-$. Then $E = Y \oplus Z$ and $\dim X = i_0^- > i_\infty^- = \dim Y$.

It is easy to see that (4.14) also holds in this case. For $x \in E_0^-$,

$$I_k(x) = \frac{1}{2}\langle L_0 x, x\rangle - \varphi_{0k}(x) \leq -\frac{c_0}{2}\|x\|^2 + o(\|x\|^2) \tag{4.16}$$

as $\|x\| \to 0$. By (4.16), there exist $r > 0$ and $\alpha > 0$ such that

$$\sup I_k(E_0^- \cap S_r) \leq -\alpha. \tag{4.17}$$

By (4.14) and (4.17), using Proposition 4.2 and a similar argument as the case (1), we can prove that system (1.1) has at least $i_0^- - i_\infty^- - 1$ pairs of nontrivial periodic solutions. □

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