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Optimal harvesting for a stochastic competition system with stage structure and distributed delay

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Abstract. A stochastic competition system with harvesting and distributed delay is investigated, which is described by stochastic differential equations with distributed delay. The existence and uniqueness of a global positive solution are proved via Lyapunov functions, and an ergodic method is used to obtain that the system is asymptotically stable in distribution. By using the comparison theorem of stochastic differential equations and limit superior theory, sufficient conditions for persistence in mean and extinction of the stochastic competition system are established. We thereby obtain the optimal harvest strategy and maximum net economic revenue by the optimal harvesting theory of differential equations.

Keywords: stochastic differential equation, distributed delay, competition system, stability in distribution, optimal harvesting strategy.

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1 Introduction

In nature, relationships between species can be classified as either competition, predator-prey, or mutualism. Because of limited natural resources, competition among populations is widespread. Many scholars have researched competition models. Early studies mainly considered deterministic models [5, 16]. Individual organisms experience a growth process, from infancy to adulthood, immaturity to maturity, and adulthood to old age, with viability varying by age. Young individuals have a weaker ability to cope with environmental disturbances, predators, and competitors' survival pressure, while the survival ability of adult individuals is strong, and they are able to conceive the next generation. The stage-structured model is popular among scholars, and the study of the stage-structured deterministic model, as a single-species model [7] or two-species competitive model [14], is comprehensive. Predator-prey models with stage structures have been discussed in the literature [4, 17, 18]. X. Y. Huang et al. presented the sufficient conditions of extinction for a two-species competitive stage-structured system with harvesting [6].

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The effects of population competition are not immediate, hence, it is necessary to consider time delays in the governing equations [9, 15, 20]. We propose a competitive model with distributed delay and harvesting,

$$\begin{cases} dx_1 = (a_{11}x_2 - a_{12}x_1^2 - sx_1)dt, \\ dx_2 = \left(a_{21}x_1 - a_{22}x_2^2 - d_1x_2 \int_{-\infty}^{t} f_1(t-v)x_3(v)dv - \beta x_2\right) dt, \\ dx_3 = \left(x_3 \left(r\left(1 - \frac{x_3}{k_3}\right) - d_2 \int_{-\infty}^{t} f_2(t-v)x_2(v)dv - qE\right)\right) dt, \end{cases} \tag{1.1}$$

where x_i is the density of the ith species, $i = 1, 2, 3$, where x_1, x_2, respectively represent the juveniles and adults of one of two species. a_{11} is the birth rate of juveniles and a_{21} is the transformation rate from juveniles to adults. a_{12}, a_{22} denote inter-specific competitive coefficients of x_1 and x_2. Considering x_1 is young and not competitive, we assume that only x_2 and x_3 are competitive. d_1 and d_2 are the loss rates of populations x_2 and x_3 in competition. r and k_3 are respectively the intrinsic growth rate and environmental capacity of species x_3. The sum of the death and conversion rates of juveniles x_1 and the sum of the death rates of adults x_2 are expressed by s and β, respectively. q is the catchability coefficient of species x_3. E denotes the effort used to harvest the population x_3. All of the parameters are assumed to be positive constants. The kernel $f_i : [0, \infty) \to [0, \infty)$ is normalized as

$$\int_0^{\infty} f_i(v)dv = 1, \qquad i = 1, 2.$$

For the distributed delay, MacDonald [10] initially proposed that it is reasonable to use a Gamma distribution,

$$f_i(t) = \frac{t^n \alpha_i^{n+1} e^{-\alpha_i t}}{n!}, \qquad i = 1, 2,$$

as a kernel, where $\alpha_i > 0, i = 1, 2$ denote the rate of decay of effects of past memories, and n is called the order of the delay kernel $f_i(t)$. They are nonnegative integers.

This article mainly considers the weak kernel case, i.e., $f_i = \alpha_i e^{-\alpha_i t}$ for $n = 0$. The strong kernel case can be considered similarly. Let

$$u_1 = \int_{-\infty}^{t} f_1(t-v)x_3(v)dv, \qquad u_2 = \int_{-\infty}^{t} f_2(t-v)x_2(v)dv.$$

Then, by the linear chain technique [13], the system (1.1) is transformed to the following equivalent system:

$$\begin{cases} dx_1 = (a_{11}x_2 - a_{12}x_1^2 - sx_1)dt, \\ dx_2 = (a_{21}x_1 - a_{22}x_2^2 - d_1x_2u_1 - \beta x_2)dt, \\ dx_3 = \left(x_3 \left(r\left(1 - \frac{x_3}{k_3}\right) - d_2u_2 - qE\right)\right) dt, \\ du_1 = \alpha_1(x_3 - u_1)dt, \\ du_2 = \alpha_2(x_2 - u_2)dt, \end{cases} \tag{1.2}$$

In addition, the population must be disturbed by realistic environmental noise, which is important in the study of bio-mathematical models [12, 15, 19], such as rainfall, wind, and drought. White noise is introduced to indicate the effects on the system disturbance. It is assumed that environmental disturbances will manifest themselves mainly as disturbances in population density x_i $(i = 1, 2, 3)$ of a system (1.2). Further, the following system of stochastic differential equations is obtained:

$$\begin{cases} dx_1 = (a_{11}x_2 - a_{12}x_1^2 - sx_1)dt + \sigma_1 x_1 dB_1(t), \\ dx_2 = (a_{21}x_1 - a_{22}x_2^2 - d_1 x_2 u_1 - \beta x_2)dt + \sigma_1 x_2 dB_1(t), \\ dx_3 = \left(x_3\left(r\left(1 - \dfrac{x_3}{k_3}\right) - d_2 u_2 - qE\right)\right)dt + \sigma_2 x_3 dB_2(t), \\ du_1 = \alpha_1(x_3 - u_1)dt, \\ du_2 = \alpha_2(x_2 - u_2)dt, \end{cases} \tag{1.3}$$

where $B_i(t)$, $i = 1, 2$, are independent standard Brownian motions and σ_i^2, $i = 1, 2$, represent the intensity of the white noise. Because x_1 and x_2 live together, they are affected by the same noise.

The following assumption applies throughout this paper.

Assumption 1.1. Because of limited environmental supply and interspecific and intra-specific constraints, species x_i must have environmental capacity k_i.

2 Existence and uniqueness of the global positive solution

Theorem 2.1. *For any initial value $x(0) = \big(x_1(0), x_2(0), x_3(0), u_1(0), u_2(0)\big) \in R_+^5$, there is a unique solution $x(t) = \big(x_1(t), x_2(t), x_3(t), u_1(t), u_2(t)\big)$ of system (1.3) on $t \geqslant 0$. Furthermore, the solution will remain in R_+^5 with probability 1.*

Proof. System (1.3) is locally Lipschitz continuous, so for any initial value $x(0) = \big(x_1(0), x_2(0), x_3(0), u_1(0), u_2(0)\big) \in R_+^5$, there is a unique maximal local solution $x(t) = \big(x_1(t), x_2(t), x_3(t), u_1(t), u_2(t)\big)$ for $t \in [0, \tau_e)$ a.s., where τ_e is the explosion time [1].

We must show that $\tau_e = \infty$ a.s. Let $m_0 > 0$ be sufficiently large that the initial value $x_i(0)$ is in the interval $\left[\frac{1}{m_0}, m_0\right]$. For each $m > m_0$, define a stopping time,

$$\tau_m = \inf\left\{t \in [0, \tau_e) : x_i(t) \notin \left(\frac{1}{m}, m\right),\ i = 1, 2, 3\right\}.$$

Obviously, τ_m increases as $m \to \infty$. Let $\tau_\infty = \lim_{m\to\infty} \tau_m$. Hence $\tau_\infty \leq \tau_e$ a.s., which is enough to certify $\tau_\infty = \infty$ a.s.

In contrast, there is a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$, such that

$$P\{\tau_\infty \leq T\} > \varepsilon.$$

Hence an integer $m_1 > m_0$ exists, and for arbitrary $m > m_1$,

$$P\{\tau_m \leq T\} \leq \varepsilon.$$

A Lyapunov function $V : R_+^5 \to R_+$ is defined as

$$V(x) = x_1 - 1 - \ln x_1 + x_2 - a - a\ln\frac{x_2}{a} + x_3 - b - b\ln\frac{x_3}{b} + \frac{1}{\alpha_1}(u_1 - 1 - \ln u_1) + \frac{1}{\alpha_2}(u_2 - 1 - \ln u_2),$$

where a, b are positive constants to be determined later. The nonnegativity of this function can be seen because

$$\omega - 1 - \ln \omega \geq 0 \quad \text{for any } \omega > 0.$$

Let $T > 0$ be a random positive constant. For any $0 \leq t \leq \tau_m \wedge T$, using Itô's formula, one obtains

$$dV(x) = LV(x)dt + \sigma_1(x_1 - 1)dB_1(t) + \sigma_1(x_2 - 1)dB_1(t) + \sigma_2(x_3 - 1)dB_2(t), \tag{2.1}$$

where

$$\begin{aligned}
LV(x) = & \left(1 - \frac{1}{x_1}\right)(a_{11}x_2 - a_{12}x_1^2 - sx_1) + \left(1 - \frac{a}{x_2}\right)(a_{21}x_1 - a_{22}x_2^2 - d_1x_2u_1 - \beta x_2) \\
& + (x_3 - b)\left(r\left(1 - \frac{x_3}{k_3}\right) - d_2u_2 - qE\right) + \sigma_1^2 + \frac{1}{2}\sigma_2^2 + \left(1 - \frac{1}{u_1}\right)(x_3 - u_1) \\
& + \left(1 - \frac{1}{u_2}\right)(x_2 - u_2) \\
\leq & \ (a_{11} - \beta + a_{22}a + 1)x_2 - a_{22}x_2^2 - a_{12}x_1^2 + (a_{21} - s + a_{12})x_1 \\
& + \left(r - qE + \frac{r}{k_3}b + 1\right)x_3 - \frac{r}{k_3}x_3^2 + (ad_1 - 1)u_1 + (bd_2 - 1)u_2 \\
& + s + a\beta - br + bqE + 2 + \sigma_1^2 + \frac{1}{2}\sigma_2^2 \\
\leq & \ M + (ad_1 - 1)u_1 + (bd_2 - 1)u_2 + s + \frac{\beta}{d_1} - \frac{r}{d_2} + \frac{qE}{d_2} + 2 + \sigma_1^2 + \frac{1}{2}\sigma_2^2,
\end{aligned} \tag{2.2}$$

where $M = \sup\{-a_{22}x_2^2 + (a_{11} - \beta + \frac{a_{22}}{d_1} + 1)x_2 - a_{12}x_1^2 + (a_{21} - s + a_{12})x_1\} - \frac{r}{k_3}x_3^2 + (r - qE + \frac{r}{k_3d_2} + 1)x_3\}$.
Choose $a = \frac{1}{d_1}$, $b = \frac{1}{d_2}$ such that $ad_1 - 1 = 0$, $bd_2 - 1 = 0$. Then one obtains

$$LV(x) \leq M + s + \frac{\beta}{d_1} - \frac{r}{d_2} + \frac{qE}{d_2} + 2 + \sigma_1^2 + \frac{1}{2}\sigma_2^2 = K_1. \tag{2.3}$$

The following proof is similar to that of Bao and Yuan [2].

Apply inequality (2.3) to equation (2.1), and integrate from 0 to $\tau_m \wedge T$ to obtain

$$\begin{aligned}
\int_0^{\tau_m \wedge T} d\big(V(x(v))\big)dv \leq \int_0^{\tau_m \wedge T} Kdv + \int_0^{\tau_m \wedge T} \sigma_1(x_1 - 1)dB_1(v) + \int_0^{\tau_m \wedge T} \sigma_1(x_2 - 1)dB_1(v) \\
+ \int_0^{\tau_m \wedge T} \sigma_2(x_3 - 1)dB_2(v).
\end{aligned}$$

Taking the expectations, the above inequality becomes

$$E\big(V\big(x(\tau_m \wedge T)\big)\big) \leq V\big(x(0)\big) + E\big(K_1(\tau_m \wedge T)\big),$$

i.e.,

$$E\big(V\big(x(\tau_m \wedge T)\big)\big) \leq V\big(x(0)\big) + K_1T.$$

For each $u \geq 0$, define $\mu(u) = \inf\{V(x), |x_i| \geq u, i = 1, 2, 3\}$. Clearly, if $u \to \infty$, then $\mu(u) \to \infty$. One can see that

$$\mu(m)P(\tau_m \leq T) \leq E\big(V\big(x(\tau_m)\big)I_{\{\tau_m \leq T\}}\big) \leq V\big(x(0)\big) + K_1T.$$

When $m \to \infty$, it is easy to see that $P(\tau_\infty \leq T) = 0$. Owing to the arbitrariness of T, $P(\tau_\infty = \infty) = 1$. The proof is completed. □

3 Stability in distribution

Lemma 3.1. *Suppose* $x(t) = \big(x_1(t), x_2(t), x_3(t), u_1(t), u_2(t)\big)$ *is a solution of system* (1.3) *with any given initial value. Then there exists a constant* $K_2 > 0$*, such that* $\limsup_{t\to+\infty} E|x(t)| \le K_2$.

Proof. The proof is similar to Theorem 3.1 in paper [2], and hence is omitted here. □

Then one can further prove the following theorem.

Theorem 3.2. *If* $a_{12} > 2(a_{11} \vee a_{21})$, $a_{22} > \alpha_2 + (a_{11} \vee a_{21})$, $r > \alpha_1 k_3$, $\alpha_1 > d_1$, $\alpha_2 > d_2$, *then system* (1.3) *will be asymptotically stable in distribution, i.e., when* $t \to +\infty$*, there is a unique probability measure* $\mu(\cdot)$ *such that the transition probability density* $p(t, \phi, \cdot)$ *of* $x(t)$ *converges weakly to* $\mu(\cdot)$ *with any given initial value* $\phi(t) \in R^5_+$.

Proof. Let $x^\phi(t)$ and $x^\varphi(t)$ be two solutions of system (1.3), with initial values $\phi(\theta) \in R^5_+$ and $\varphi(t) \in R^5_+$, respectively. Applying Itô's formula to

$$V(t) = \sum_{i=1}^{3} |\ln x_i^\phi(t) - \ln x_i^\varphi(t)| + \sum_{j=1}^{2} |\ln u_i^\phi(t) - \ln u_i^\varphi(t)|$$

yields

$$\begin{aligned}
d^+V(t) = &\sum_{i=1}^{3} \operatorname{sgn}\big(x_i^\phi(t) - x_i^\varphi(t)\big) d\big(\ln x_i^\phi(t) - \ln x_i^\varphi(t)\big) \\
&+ \sum_{j=1}^{2} \operatorname{sgn}\big(u_i^\phi(t) - u_i^\varphi(t)\big) d\big(\ln u_i^\phi(t) - \ln u_i^\varphi(t)\big) \\
\le\, & a_{11} \left| \frac{x_2^\phi(t)}{x_1^\phi(t)} - \frac{x_2^\varphi(t)}{x_1^\varphi(t)} \right| dt - a_{12}|x_1^\phi(t) - x_1^\varphi(t)|dt - (a_{22} - \alpha_2)|x_2^\phi(t) - x_2^\varphi(t)|dt \\
&+ a_{21} \left| \frac{x_1^\phi(t)}{x_2^\phi(t)} - \frac{x_1^\varphi(t)}{x_2^\varphi(t)} \right| dt - (\alpha_1 - d_1)|u_1^\phi(t) - u_1^\varphi(t)|dt - \left(\frac{r}{k_3} - \alpha_1\right)|x_3^\phi(t) - x_3^\varphi(t)|dt \\
&- (\alpha_2 - d_2)|u_2^\phi(t) - u_2^\varphi(t)|dt \\
\le & -\big(a_{12} - 2(a_{11} \vee a_{21})\big)|x_1^\phi(t) - x_1^\varphi(t)|dt - \big(a_{22} - \alpha_2 - 2(a_{11} \vee a_{21})\big)|x_2^\phi(t) - x_2^\varphi(t)|dt \\
&- \left(\frac{r}{k_3} - \alpha_1\right)|x_3^\phi(t) - x_3^\varphi(t)|dt - (\alpha_1 - d_1)|u_1^\phi(t) - u_1^\varphi(t)|dt \\
&- (\alpha_2 - d_2)|u_2^\phi(t) - u_2^\varphi(t)|dt.
\end{aligned}$$

Therefore,

$$\begin{aligned}
E(V(t)) \le V(0) &- \big(a_{12} - 2(a_{11} \vee a_{21})\big) \int_0^t E|x_1^\phi(v) - x_1^\varphi(v)|dv \\
&- \big(a_{22} - \alpha_2 - 2(a_{11} \vee a_{21})\big) \int_0^t E|x_2^\phi(v) - x_2^\varphi(v)|dv - \left(\frac{r}{k_3} - \alpha_1\right) \int_0^t E|x_3^\phi(v) - x_3^\varphi(v)|dv \\
&- (\alpha_1 - d_1) \int_0^t E|u_1^\phi(v) - u_1^\varphi(v)|dv - (\alpha_2 - d_2) \int_0^t E|u_2^\phi(v) - u_2^\varphi(v)|dv.
\end{aligned}$$

Because $V(t) \geq 0$, according to the inequality above,

$$\begin{aligned}&\big(a_{12}-2(a_{11}\vee a_{21})\big)\int_0^t E|x_1^{\phi}(v)-x_1^{\varphi}(v)|dv+\big(a_{22}-\alpha_2-2(a_{11}\vee a_{21})\big)\int_0^t E|x_2^{\phi}(v)-x_2^{\varphi}(v)|dv\\&\quad+\left(\frac{r}{k_3}-\alpha_1\right)\int_0^t E|x_3^{\phi}(v)-x_3^{\varphi}(v)|dv+(\alpha_1-d_1)\int_0^t E|u_1^{\phi}(v)-u_1^{\varphi}(v)|dv\\&\quad+(\alpha_2-d_2)\int_0^t E|u_2^{\phi}(v)-u_2^{\varphi}(v)|dv\leq V(0)<\infty.\end{aligned}$$

That is,

$$E|x_i^{\phi}(v)-x_i^{\varphi}(v)|\in L^1[0,+\infty),\quad i=1,2,3\quad\text{and}\quad E|u_j^{\phi}(v)-u_j^{\varphi}(v)|\in L^1[0,+\infty),\quad j=1,2.$$

Moreover, it can be seen from the first equation of system (1.3) that

$$E\big(x_1(t)\big)=x_1(0)+\int_0^t\big[a_{11}E\big(x_2(v)\big)-a_{12}E\big(x_1^2(v)\big)-sE\big(x_1(v)\big)\big]dv.$$

Thus $E\big(x_1(t)\big)$ is a continuously differentiable function. By Lemma 3.1,

$$\frac{dE\big(x_1(t)\big)}{dt}\leq a_{11}E\big(x_2(t)\big)\leq K_2.$$

Hence $E\big(x_1(t)\big)$ is uniformly continuous. Using the same method on the other equations of system (1.3), one can obtain that $E\big(x_2(t)\big)$, $E\big(x_3(t)\big)$, $E\big(u_1(t)\big)$, and $E\big(u_2(t)\big)$ are uniformly continuous. According to [3],

$$\lim_{t\to\infty}E|x_i^{\phi}(t)-x_i^{\varphi}(t)|=0\quad a.s.,\qquad\lim_{t\to\infty}E|u_j^{\phi}(t)-u_j^{\varphi}(t)|=0\quad a.s.\tag{3.1}$$

Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_t$ contains all $\mathbb{P}$-null sets). Suppose $p(t,\phi,dy)$ is the transition probability density of the process $x(t)$, and $p(t,\phi,A)$ is the probability of event $x^{\phi}(t)\in A$ with initial value $\phi(\theta)\in R_+^5$. By Lemma 3.1 and Chebyshev's inequality, the family of transition probability $p(t,\phi,A)$ is tight. So, a compact subset $\mathcal{K}\in R_+^5$ can be obtained such that $p(t,\phi,\mathcal{K})\geq 1-\epsilon^*$ for any $\epsilon^*>0$.

Let $\mathcal{P}(R_+^5)$ be probability measures on R_+^5. For any two measures $P_1,P_2\in\mathcal{P}$, we define the metric

$$d_{\mathbb{L}}(P_1,P_2)=\sup_{g\in\mathbb{L}}\left|\int_{R_+^5}g(x)P_1(dx)-\int_{R_+^5}g(x)P_2(dx)\right|,$$

where

$$\mathbb{L}=\{g:R_+^5\to R:\|g(x)-g(y)\|\leq\|x-y\|,|g(\cdot)|\leq 1\}.$$

For any $g\in\mathbb{L}$ and $t,\iota>0$, one obtains

$$\begin{aligned}|Eg\big(x^{\phi}(t+\iota)\big)-Eg\big(x^{\phi}(t)\big)|&=|E[E\big(g\big(x^{\phi}(t+\iota)\big)|\mathcal{F}_{\vartheta}\big)]-Eg\big(x^{\phi}(t)\big)|\\&=\left|\int_{R_+^5}E\big(g\big(x^{\xi}(t)\big)p(\vartheta,\phi,d\xi)\big)-Eg\big(x^{\phi}(t)\big)\right|\\&\leq 2p(\vartheta,\phi,U_K^c)+\int_{U_K}|E\big(g\big(x^{\xi}(t)\big)\big)-E\big(g\big(x^{\phi}(t)\big)\big)|p(\vartheta,\phi,d\xi),\end{aligned}$$

where $U_K = \{x \in R^5_+ : |x| \leq K\}$, and U^c_K is a complementary set of U_K. Since the family of $p(t, \phi, dy)$ is tight, for any given $\iota \geq 0$, there exists sufficiently large K such that $p(\iota, \phi, U^c_K) < \frac{\epsilon^*}{4}$. From (3.1), there exists $T > 0$ such that for $t \geq T$,

$$\sup_{g \in \mathbb{L}} |E(g(x^{\xi}(t))) - E(g(x^{\phi}(t)))| \leq \frac{\epsilon^*}{2}.$$

Consequently, it is easy to find that $|Eg(x^{\phi}(t+\iota)) - Eg(x^{\phi}(t))| \leq \epsilon^*$. By the arbitrariness of g, we have

$$\sup_{g \in \mathbb{L}} |Eg(x^{\phi}(t+\iota)) - Eg(x^{\phi}(t))| \leq \epsilon^*.$$

That is,

$$d_{\mathbb{L}}(p(t+\iota, \phi, \cdot), p(t, \phi, \cdot)) \leq \epsilon^*, \quad \forall t \geq T,\ \iota > 0.$$

Therefore, $\{p(t, 0, \cdot) : t \geq 0\}$ is Cauchy in $\mathcal{P}$ with metric $d_{\mathbb{L}}$. There is a unique $\mu(\cdot) \in \mathcal{P}(R^5_+)$ such that $\lim_{t\to\infty} d_{\mathbb{L}}(p(t, 0, \cdot), \mu(\cdot)) = 0$. In addition, it follows from (3.1) that

$$\lim_{t\to\infty} d_{\mathbb{L}}(p(t, \phi, \cdot), p(t, 0, \cdot)) = 0.$$

Hence

$$\lim_{t\to\infty} d_{\mathbb{L}}(p(t, \phi, \cdot), \mu(\cdot)) \leq \lim_{t\to\infty} d_{\mathbb{L}}(p(t, \phi, \cdot), p(t, 0, \cdot)) + \lim_{t\to\infty} d_{\mathbb{L}}(p(t, 0, \cdot), \mu(\cdot)) = 0.$$

The proof is completed. □

4 Optimal harvesting

For convenience, we introduce the following notation:

$$b_1 = s + \frac{1}{2}\sigma_1^2, \quad b_2 = \beta + \frac{1}{2}\sigma_1^2, \quad b_3 = r - \frac{1}{2}\sigma_2^2,$$
$$\Gamma_1 = a_{12}a_{22}b_3 - d_2(a_{21}(a_{11}k_2 - b_1) - a_{12}b_2),$$
$$f^* = \limsup_{t\to\infty} f(t), \quad f_* = \liminf_{t\to\infty} f(t), \quad \langle f \rangle = t^{-1}\int_0^t f(s)ds.$$

Lemma 4.1 ([8]). *For $x(t) \in R_+$, the following holds:*

(i) If there are positive constants T and δ_0 such that

$$\ln x(t) \leq \delta t - \delta_0 \int_0^t x(v)dv + \alpha B(t), \quad a.s.$$

for any $t \geq T$, where $\alpha, \delta_1, \delta_2$ are constants, then

$$\begin{cases} \langle x \rangle^* \leq \frac{\delta}{\delta_0}, \ a.s. & \text{if } \delta \geq 0, \\ \lim_{t\to\infty} x(t) = 0, \ a.s. & \text{if } \delta \leq 0. \end{cases}$$

(ii) If there are positive constants T, δ, and δ_0 such that

$$\ln x(t) \geq \delta t - \delta_0 \int_0^t x(v)dv + \alpha B(t), \quad a.s.$$

for any $t \geq T$, then $\langle x \rangle_* \geq \frac{\delta}{\delta_0}$ a.s.

Lemma 4.2 (Strong law of large numbers [11])**.** *Let $M = \{M_t\}_{t\geq 0}$ be a real-valued continuous local martingale vanishing at $t = 0$. Then*

$$\lim_{t\to\infty}\langle M, M\rangle_t = \infty \quad a.s. \quad \Rightarrow \quad \lim_{t\to\infty}\frac{M_t}{\langle M, M\rangle_t} = 0 \quad a.s.$$

and

$$\limsup_{t\to\infty}\frac{\langle M, M\rangle_t}{t} < \infty \quad a.s. \quad \Rightarrow \quad \lim_{t\to\infty}\frac{M_t}{t} = 0 \quad a.s.$$

Lemma 4.3 (Strong law of large numbers for local martingales [11])**.** *Let $M(t), t \geq 0$, be a local martingale vanishing at time $t = 0$ and define*

$$\rho_M(t) = \int_0^t \frac{d\langle M\rangle(s)}{(1+s)^2}, \qquad t \geq 0,$$

where $M(t) = \langle M, M\rangle(t)$ is a Meyers angle bracket process. Then

$$\lim_{t\to\infty}\frac{M(t)}{t} = 0 \quad a.s.,$$

provided

$$\lim_{t\to\infty}\rho_M(t) < \infty \quad a.s.$$

Lemma 4.4. *Let $(x_1(t), x_2(t), x_3(t), u_1(t), u_2(t))$ be the solution of system (1.3) with any initial value $(x_1(0), x_2(0), x_3(0), u_1(0), u_2(0)) \in R^5_+$. Then, if $\alpha_1 > \alpha_2$, then*

$$\lim_{t\to\infty}\frac{u_1(t)}{t} = 0, \qquad \lim_{t\to\infty}\frac{u_2(t)}{t} = 0, \quad a.s.$$

and

$$\langle u_1(t)\rangle = \langle x_3(t)\rangle - \frac{u_1(t) - u_1(0)}{\alpha_1 t}, \qquad \langle u_2(t)\rangle = \langle x_2(t)\rangle - \frac{u_2(t) - u_2(0)}{\alpha_2 t}.$$

Proof. Define $V^*(w) = (1 + w)^\theta$, where θ is a positive constant to be determined later, and

$$w(t) = x_1(t) + x_2(t) + x_3(t) + \frac{r}{2k_3\alpha_1}u_1^2(t) + \frac{a_{12}}{2\alpha_2}u_2^2(t).$$

By Itô's formula,

$$dV^*(w) = LV^*(w)dt + \sigma_1(1+w)^{\theta-1}x_1 dB_1(t) + \sigma_1(1+w)^{\theta-1}x_2 dB_1(t) + \sigma_2(1+w)^{\theta-1}x_3 dB_2(t),$$

where

$$\begin{aligned}
LV^*(w) &= \theta(1+w)^{\theta-1}(a_{11}x_2 - a_{12}x_1^2 - sx_1 + a_{21}x_1 - a_{22}x_2^2 - d_1x_2u_1 - \beta x_2 + rx_3 \\
&\quad - \frac{r}{k_3}x_3^2 - d_2x_3u_2 - qEx_3 + \frac{r}{k_3}x_3u_1 - \frac{r}{k_3}u_1^2 + a_{12}x_2u_2 - a_{12}u_2^2) \\
&\quad + \frac{\sigma_1^2\theta(\theta-1)}{2}(1+w)^{\theta-2}(x_1^2+x_2^2) + \frac{\sigma_2^2\theta(\theta-1)}{2}(1+w)^{\theta-2}x_3^2 \\
&\le \theta(1+w)^{\theta-1}\Big(- a_{12}x_1^2 + (a_{21}-s)x_1 - a_{22}x_2^2 + (a_{11}-\beta)x_2 - \frac{r}{k_3}x_3^2 \\
&\quad + (r-qE)x_3 + \frac{r}{2k_3}x_3^2 + \frac{r}{2k_3}u_1^2 - \frac{r}{k_3}u_1^2 + \frac{a_{12}}{2}x_2^2 + \frac{a_{12}}{2}u_2^2 - a_{12}u_2^2\Big) \\
&\quad + \frac{\sigma_1^2\theta(\theta-1)}{2}(1+w)^{\theta-2}(x_1^2+x_2^2) + \frac{\sigma_2^2\theta(\theta-1)}{2}(1+w)^{\theta-2}x_3^2 \\
&= \theta(1+w)^{\theta-2}\Big((1+w)(-a_{12}x_1^2 + (a_{21}-s)x_1 - \Big(a_{22}+\frac{a_{12}}{2}\Big)x_2^2 + (a_{11}-\beta)x_2 \\
&\quad - \frac{r}{2k_3}x_3^2 + (r-qE)x_3 - \frac{r}{2k_3}u_1^2 - \frac{a_{12}}{2}u_2^2) + \frac{\sigma_1^2\theta(\theta-1)}{2}(x_1^2+x_2^2) \\
&\quad + \frac{\sigma_2^2\theta(\theta-1)}{2}x_3^2\Big) \\
&\le \theta(1+w)^{\theta-1}\Big(- a_{12}x_1^2 + (a_{21}-s+\alpha_1)x_1 - \Big(a_{22}+\frac{a_{12}}{2}\Big)x_2^2 \\
&\quad + (a_{11}-\beta+\alpha_1)x_2 - \frac{r}{2k_3}x_3^2 + (r-qE+\alpha_1)x_3 - \alpha_1 w + \frac{a_{12}}{2}\Big(\frac{\alpha_1}{\alpha_2}-1\Big)u_2^2\Big) \\
&\quad + \sigma_1^2\theta(\theta-1)(1+w)^{\theta-2}w^2 + \frac{\sigma_2^2\theta(\theta-1)}{2}(1+w)^{\theta-2}w^2 \\
&\le \theta(1+w)^{\theta-2}\Big((1+w)(-a_{12}x_1^2 + (a_{21}-s+\alpha_2)x_1 - a_{22}x_2^2 + (a_{11}-\beta+\alpha_2)x_2 \\
&\quad - \frac{r}{2k_3}x_3^2 + (r-qE+\alpha_2)x_3 - \alpha_2 w) + \frac{(2\sigma_1^2+\sigma_2^2)}{2}(\theta-1)w^2\Big) \\
&\le \theta(1+w)^{\theta-2}\left(-(\alpha_2 - \frac{(2\sigma_1^2+\sigma_2^2)}{2}(\theta-1))w^2 + (M_1-\alpha_2)w + M_1\right),
\end{aligned}$$

where

$$M_1 = \sup_{x_1,x_2,x_3\in(0,+\infty)}\Big\{ - a_{12}x_1^2 + (a_{21}-s-\alpha_1)x_1 - a_{22}x_2^2 + (a_{11}-\beta+\alpha_1)x_2 - \frac{r}{2k_3}x_3^2 + (r-qE+\alpha_1)x_3\Big\}.$$

Choose $\theta \in \big(1, \frac{2\alpha_2}{2\sigma_1^2+\sigma_2^2}+1\big)$ such that $\lambda^* = \alpha_2 - \frac{2\sigma_1^2+\sigma_2^2}{2}(\theta-1) > 0$. Then

$$\begin{aligned}
dV^* &\le \theta(1+w)^{\theta-2}\big(- \lambda^* w^2 + (M_1-\alpha_2)w + M_1\big)dt + \sigma_1(1+w)^{\theta-1}x_1dB_1(t) \\
&\quad + \sigma_1(1+w)^{\theta-1}x_2dB_1(t) + \sigma_2(1+w)^{\theta-1}x_3dB_2(t).
\end{aligned} \tag{4.1}$$

Hence, for $0 < \mu < \theta\lambda^*$, we have

$$\begin{aligned}
d\big(e^{\mu t}V^*(w)\big) &\le L\big(e^{\mu t}V^*(w)\big)dt + \sigma_1\theta e^{\mu t}(1+w)^{\theta-1}x_1dB_1(t) + \sigma_1\theta e^{\mu t}(1+w)^{\theta-1}x_2dB_1(t) \\
&\quad + \sigma_2\theta e^{\mu t}(1+w)^{\theta-1}x_3dB_2(t),
\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}L\big(e^{\mu t}V^*(w)\big) &\le \mu e^{\mu t}(1+w)^{\theta} + e^{\mu t}\theta(1+w)^{\theta-2}\big(-\lambda^* w^2 + (M_1-\alpha_2)w + M_1\big)\\ &= e^{\mu t}(1+w)^{\theta-2}\big(-(\theta\lambda^*-\mu)w^2 + (2\mu + M_1\theta - \alpha_2\theta)w + M_1\theta + \mu\big)\\ &\le e^{\mu t}M_2,\end{aligned}$$

where

$$M_2 = \sup_{w\in(0,+\infty)} (1+w)^{\theta-2}\big(-(\theta\lambda^*-\mu)w^2 + (2\mu + M_1\theta - \alpha_2\theta)w + M_1\theta + \mu\big).$$

Integrating from 0 to t and taking the expectation of two sides of (4.2) yields

$$\begin{aligned}E\big(e^{\mu t}V^*(w(t))\big) &= V^*\big(w(0)\big) + \int_0^t E\Big(L\big(e^{\mu\vartheta}V^*\big(w(\vartheta)\big)\big)\Big)d\vartheta\\ &\le (1+w(0))^{\theta} + \frac{M_2}{\mu}e^{\mu t}, \quad a.s.\end{aligned}$$

On account of the continuity of $V^*(w(t))$, there exists a constant $H>0$ such that

$$E\big((1+w(t))^{\theta}\big) \le H, \qquad t\ge 0, \ \ a.s. \tag{4.3}$$

From (4.1) and (4.3), for sufficiently small $\delta>0$, $n=1,2,\dots$,

$$E\Big(\sup_{n\delta\le t\le(n+1)\delta}(1+w(t))^{\theta}\Big) \le E\big((1+w(n\delta))^{\theta}\big) + I_1 + I_2, \tag{4.4}$$

where

$$\begin{aligned}I_1 &= \theta E\Big(\sup_{n\delta\le t\le(n+1)\delta}\Big|\int_{n\delta}^t (1+w)^{\theta-2}\Big(-\lambda^* w^2 + (M_1-\alpha_2)w + M_1\Big)dt\Big|\Big)\\ I_2 &= \sigma_1\theta E\Big(\sup_{n\delta\le t\le(n+1)\delta}\Big|\int_{n\delta}^t \Big(1+w(\vartheta)\Big)^{\theta-1} x_1(\vartheta)dB_1(\vartheta)\Big|\Big)\\ &\quad + \sigma_1\theta E\Big(\sup_{n\delta\le t\le(n+1)\delta}\Big|\int_{n\delta}^t \Big(1+w(\vartheta)\Big)^{\theta-1} x_2(\vartheta)dB_1(\vartheta)\Big|\Big)\\ &\quad + \sigma_2\theta E\Big(\sup_{n\delta\le t\le(n+1)\delta}\Big|\int_{n\delta}^t \Big(1+w(\vartheta)\Big)^{\theta-1} x_3(\vartheta)dB_2(\vartheta)\Big|\Big).\end{aligned}$$

Furthermore,

$$\begin{aligned}I_1 &\le \max\Big\{\lambda^*, \frac{1}{2}|M_1-\alpha_2|, M_1\Big\}\theta E\Big(\sup_{n\delta\le t\le(n+1)\delta}\Big|\int_{n\delta}^t (1+w)^{\theta-2}(w^2+2w+1)dt\Big|\Big)\\ &\le C_1\delta E\Big(\sup_{n\delta\le t\le(n+1)\delta}\Big(1+w(t)\Big)^{\theta}\Big),\end{aligned} \tag{4.5}$$

where $C_1 = \theta\max\{\lambda^*, \frac{1}{2}|M_1-\alpha_2|, M_1\}$. According to the Burkholder–Davis–Gundy inequality [1],

$$
\begin{aligned}
I_2 &\le \sqrt{32}\sigma_1\theta E\left(\left|\int_{n\delta}^{(n+1)\delta}\left(1+w(\vartheta)\right)^{2\theta-2}x_1^2(\vartheta)d\vartheta\right|^{\frac{1}{2}}\right)\\
&\quad+\sqrt{32}\sigma_1\theta E\left(\left|\int_{n\delta}^{(n+1)\delta}\left(1+w(\vartheta)\right)^{2\theta-2}x_2^2(\vartheta)d\vartheta\right|^{\frac{1}{2}}\right)\\
&\quad+\sqrt{32}\sigma_2\theta E\left(\left|\int_{n\delta}^{(n+1)\delta}\left(1+w(\vartheta)\right)^{2\theta-2}x_3^2(\vartheta)d\vartheta\right|^{\frac{1}{2}}\right)\\
&\le 2\sqrt{32}\sigma_1\theta E\left(\left|\int_{n\delta}^{(n+1)\delta}\left(1+w(\vartheta)\right)^{2\theta}d\vartheta\right|^{\frac{1}{2}}\right)+\sqrt{32}\sigma_2\theta E\left(\left|\int_{n\delta}^{(n+1)\delta}\left(1+w(\vartheta)\right)^{2\theta}d\vartheta\right|^{\frac{1}{2}}\right)\\
&\le 2\sqrt{32}\sigma_1\theta\sqrt{\delta}E\left(\sup_{n\delta\le t\le(n+1)\delta}\left(1+w(t)\right)^{\theta}\right)+\sqrt{32}\sigma_2\theta\sqrt{\delta}E\left(\sup_{n\delta\le t\le(n+1)\delta}\left(1+w(t)\right)^{\theta}\right)\\
&=(2\sigma_1+\sigma_2)\sqrt{32}\theta\sqrt{\delta}E\left(\sup_{n\delta\le t\le(n+1)\delta}\left(1+w(t)\right)^{\theta}\right).
\end{aligned}
\tag{4.6}
$$

By (4.4)-(4.6), we obtain that

$$
(1-C_1\delta-(2\sigma_1+\sigma_2)\sqrt{32}\theta\sqrt{\delta})E\left(\sup_{n\delta\le t\le(n+1)\delta}\left(1+w(t)\right)^{\theta}\right)\le H
$$

for a sufficiently small constant $\delta>0$ such that $C_1\delta+(2\sigma_1+\sigma_2)\sqrt{32}\theta\sqrt{\delta}\le\frac{1}{2}$. Then

$$
E\left(\sup_{n\delta\le t\le(n+1)\delta}\left(1+w(t)\right)^{\theta}\right)\le 2H.
$$

For arbitrary ϵ, according to Chebyshev's inequality,

$$
P\left(\sup_{n\delta\le t\le(n+1)\delta}\left(1+w(t)\right)^{\theta}>(n\delta)^{1+\epsilon}\right)\le\frac{E\left(\sup_{n\delta\le t\le(n+1)\delta}\left(1+w(t)\right)^{\theta}\right)}{(n\delta)^{1+\epsilon}}\le\frac{2H}{(n\delta)^{1+\epsilon}}.
$$

From the Borel–Cantelli lemma [11], we have that $\sup_{n\delta\le t\le(n+1)\delta}\left(1+w(t)\right)^{\theta}\le(n\delta)^{1+\epsilon},$ *a.s.* holds for all but finitely many n.

For $\epsilon\to 0$, we have $\limsup_{t\to+\infty}\frac{\ln(1+w(t))^{\theta}}{\ln t}\le 1,$ *a.s.* Hence

$$
\limsup_{t\to+\infty}\frac{\ln w(t)}{\ln t}\le\limsup_{t\to+\infty}\frac{\ln\left(1+w(t)\right)}{\ln t}\le\frac{1}{\theta}.
$$

For $\epsilon_0<1$, there exists $T>0$ such that

$$
\ln w(t)\le\left(\frac{1}{\theta}+\epsilon_0\right)\ln t,\quad\text{when } t\ge T.
$$

Thus

$$
\limsup_{t\to+\infty}\frac{w(t)}{t^2}\le\limsup_{t\to+\infty}t^{\frac{1}{\theta}+\epsilon_0-2}=0,
$$

i.e.,

$$\limsup_{t\to+\infty}\frac{x_1(t)+x_2(t)+x_3(t)+\frac{a_{12}}{2\alpha_2}u_2^2(t)+\frac{r}{2k_3\alpha_1}u_1^2(t)}{t^2}=0,$$

which, together with the positivity of $x_1(t), x_2(t), x_3(t), u_1(t), u_2(t)$, gives

$$\lim_{t\to\infty}\frac{u_1(t)}{t}=0,\qquad \lim_{t\to\infty}\frac{u_2(t)}{t}=0,\quad a.s.$$

Indeed, integration of the system (1.3) from 0 to t yields

$$\frac{u_1(t)-u_1(0)}{t}=\alpha_1\langle x_3(t)\rangle-\alpha_1\langle u_1(t)\rangle$$
$$\frac{u_2(t)-u_2(0)}{t}=\alpha_1\langle x_2(t)\rangle-\alpha_2\langle u_2(t)\rangle.$$

Thus

$$\langle u_1(t)\rangle=\langle x_3(t)\rangle-\frac{u_1(t)-u_1(0)}{\alpha_1 t},\qquad \langle u_2(t)\rangle=\langle x_2(t)\rangle-\frac{u_2(t)-u_2(0)}{\alpha_2 t}.$$ □

Next, to obtain the optimal harvest strategy of system (1.3), we establish the following auxiliary systems:

$$\begin{cases} dy_1(t)=y_1\big(a_{11}k_2-a_{12}y_1(t)-s\big)dt+\sigma_1 y_1(t)dB_1(t),\\ dy_2(t)=y_2\big(a_{21}y_1-a_{22}y_2(t)-\beta\big)dt+\sigma_1 y_2(t)dB_1(t),\\ dy_3(t)=\left(y_3(t)\left(r\left(1-\frac{y_3(t)}{k_3}\right)-d_2v(t)-qE\right)\right)dt+\sigma_2 y_3(t)dB_2(t),\\ dv(t)=\alpha_2(y_2(t)-v(t)). \end{cases}\tag{4.7}$$

On the basis of Lemma 4.4, we similarly obtain $\langle v(t)\rangle=\langle y_2(t)\rangle-\frac{v(t)-v(0)}{\alpha_2 t}$ and $\lim_{t\to\infty}\frac{v(t)}{t}=0$ a.s.

Theorem 4.5. *Under Assumption 1.1, if $a_{11}k_2-b_1>0$, $a_{21}(a_{11}k_2-b_1)-a_{12}b_2>0$, then the solution $(y_1(t), y_2(t), y_3(t), v(t))$ of system (4.7) with initial value $(y_1(0), y_2(0), y_3(0), v(0))$ meets the conditions*

$$\lim_{t\to\infty}\langle y_1(t)\rangle=\frac{a_{11}k_2-b_1}{a_{12}}\quad a.s.,\qquad \lim_{t\to\infty}\langle y_2(t)\rangle=\frac{a_{21}(a_{11}k_2-b_1)-a_{12}b_2}{a_{12}a_{22}}\quad a.s.$$

$$\begin{cases}\lim_{t\to\infty}\langle y_3(t)\rangle=0\ a.s. & \text{if } \Gamma_1<a_{12}a_{22}qE,\\ \lim_{t\to\infty}\langle y_3(t)\rangle=\frac{(\Gamma_1-a_{12}a_{22}qE)k_3}{a_{12}a_{22}r}\ a.s. & \text{if } \Gamma_1>a_{12}a_{22}qE.\end{cases}$$

Proof. By Itô's formula, we have

$$d\ln y_1(t)=(a_{11}k_2-a_{12}y_1(t)-b_1)dt+\sigma_1 dB_1(t),$$
$$d\ln y_2(t)=(a_{21}y_1-a_{22}y_2(t)-b_2)dt+\sigma_1 dB_1(t),$$
$$d\ln y_3(t)=\left(-\frac{r}{k_3}y_3(t)-d_2v(t)-qE+b_3\right)dt+\sigma_2 dB_2(t).$$

We integrate both sides of the above equation from 0 to t and divide by t to obtain

$$t^{-1}\ln\frac{y_1(t)}{y_1(0)} = -a_{12}\langle y_1(t)\rangle + a_{11}k_2 - b_1 + t^{-1}\sigma_1 B_1(t), \tag{4.8}$$

$$t^{-1}\ln\frac{y_2(t)}{y_2(0)} = a_{21}\langle y_1(t)\rangle - a_{22}\langle y_2(t)\rangle - b_2 + t^{-1}\sigma_1 B_1(t), \tag{4.9}$$

$$t^{-1}\ln\frac{y_3(t)}{y_3(0)} = -d_2\langle v(t)\rangle - \frac{r}{k_3}\langle y_3(t)\rangle - qE + b_3 + t^{-1}\sigma_2 B_2(t) \tag{4.10}$$

$$= -d_2\langle y_2(t)\rangle + d_2\frac{v(t)-v(0)}{\alpha_2 t} - \frac{r}{k_3}\langle y_3(t)\rangle - qE + b_3 + t^{-1}\sigma_2 B_2(t). \tag{4.11}$$

It is apparent that

$$\lim_{t\to\infty} t^{-1}\ln y_i(0) = 0, \qquad i = 1,2,3, \tag{4.12}$$

i.e., for any $\epsilon_1 > 0$, t is sufficiently large that

$$t^{-1}\ln y_1(t) \le -a_{12}\langle y_1(t)\rangle + a_{11}k_2 - b_1 + \epsilon_1 + t^{-1}\sigma_1 B_1(t),$$
$$t^{-1}\ln y_1(t) \ge -a_{12}\langle y_1(t)\rangle + a_{11}k_2 - b_1 - \epsilon_1 + t^{-1}\sigma_1 B_1(t).$$

Note that $a_{11}k_2 - b_1 > 0$. Let ϵ_1 be sufficiently small that $a_{11}k_2 - b_1 - \epsilon_1 > 0$. Then, by Lemma 4.1, we have

$$\lim_{t\to\infty}\langle y_1(t)\rangle \le \frac{a_{11}k_2 - b_1 + \epsilon_1}{a_{12}} \quad a.s, \qquad \lim_{t\to\infty}\langle y_1(t)\rangle \ge \frac{a_{11}k_2 - b_1 - \epsilon_1}{a_{12}} \quad a.s,$$

and by the arbitrariness of ϵ_1,

$$\lim_{t\to\infty}\langle y_1(t)\rangle = \frac{a_{11}k_2 - b_1}{a_{12}} \quad a.s. \tag{4.13}$$

Substitute (4.13) in (4.8) and note that $\lim\limits_{t\to\infty} t^{-1}\sigma_1 B_1(t) = 0$. Then, by Lemma 4.2,

$$\lim_{t\to\infty}\frac{\ln y_1(t)}{t} = 0. \tag{4.14}$$

Compute $a_{21}\times$(4.8)+$a_{12}\times$(4.9) to obtain

$$a_{21}t^{-1}\ln\frac{y_1(t)}{y_1(0)} + a_{12}t^{-1}\ln\frac{y_2(t)}{y_2(0)}$$
$$= -a_{12}a_{22}\langle y_2(t)\rangle + a_{21}(a_{11}k_2 - b_1) - a_{12}b_2 + (a_{21}+a_{12})t^{-1}\sigma_1 B_1(t), \tag{4.15}$$

and compute $a_{12}a_{22}\times$(4.10)$-d_2\times$(4.15) to obtain

$$\begin{aligned} &a_{12}a_{22}t^{-1}\ln\frac{y_3(t)}{y_3(0)} - a_{21}d_2t^{-1}\ln\frac{y_2(t)}{y_2(0)} - a_{12}d_2t^{-1}\ln\frac{y_1(t)}{y_1(0)} \\ &= \Gamma_1 - a_{12}a_{22}qE - \frac{r}{k_3}a_{12}a_{22}\langle y_3(t)\rangle + a_{12}a_{22}d_2\frac{v(t)-v(0)}{\alpha_2 t} \\ &\quad - d_2(a_{21}+a_{12})t^{-1}\sigma_1 B_1(t) + a_{12}a_{22}t^{-1}\sigma_2 B_2(t). \end{aligned} \tag{4.16}$$

Combining (4.12) with (4.14) yields that for any $0 < \epsilon_2 < a_{21}(a_{11}k_2 - b_1) - a_{12}b_2$, there exists $T_1 > 0$ such that

$$-\epsilon_2 < a_{21}t^{-1}\ln\frac{y_1(t)}{y_1(0)} + a_{12}t^{-1}\ln y_2(0) < \epsilon_2, \qquad t \ge T_1. \tag{4.17}$$

By (4.15) and (4.17), we can obtain that

$$a_{12}t^{-1}\ln y_2(t) \le -a_{12}a_{22}\langle y_2(t)\rangle + a_{21}(a_{11}k_2 - b_1) - a_{12}b_2 + \epsilon_2 + (a_{21}+a_{12})t^{-1}\sigma_1 B_1(t),$$
$$a_{12}t^{-1}\ln y_2(t) \ge -a_{12}a_{22}\langle y_2(t)\rangle + a_{21}(a_{11}k_2 - b_1) - a_{12}b_2 - \epsilon_2 + (a_{21}+a_{12})t^{-1}\sigma_1 B_1(t).$$

It then follows from Lemma 4.1 that

$$\lim_{t\to\infty}\langle y_2(t)\rangle \le \frac{a_{21}(a_{11}k_2-b_1)-a_{12}b_2+\epsilon_2}{a_{12}a_{22}} \quad a.s.,$$
$$\lim_{t\to\infty}\langle y_2(t)\rangle \ge \frac{a_{21}(a_{11}k_2-b_1)-a_{12}b_2-\epsilon_2}{a_{12}a_{22}} \quad a.s.$$

From the arbitrariness of ϵ_2, we can get that

$$\lim_{t\to\infty}\langle y_2(t)\rangle = \frac{a_{21}(a_{11}k_2-b_1)-a_{12}b_2}{a_{12}a_{22}} \quad a.s. \tag{4.18}$$

From (4.14), (4.15), and (4.18), one can observe that

$$\lim_{t\to\infty}\frac{\ln y_2(t)}{t} = 0. \tag{4.19}$$

Analogously, applying Lemmas 4.1 and 4.4 and combining (4.12), (4.14), and (4.19) with (4.16), one can see that when $\Gamma_1 > a_{12}a_{22}qE$, we have

$$\lim_{t\to\infty}\langle y_3(t)\rangle = \frac{k_3}{r}\left(\frac{\Gamma_1}{a_{12}a_{22}} - qE\right) \quad a.s. \tag{4.20}$$

From (4.14), (4.19), (4.20), and (4.16), one can see that

$$\lim_{t\to\infty}\frac{\ln y_3(t)}{t} = 0, \tag{4.21}$$

and if $\Gamma_1 < a_{12}a_{22}qE$, then

$$\lim_{t\to\infty}\langle y_3(t)\rangle = 0. \tag{4.22}$$

The proof is completed. □

Then, for system (1.3), we have the following theorem.

Theorem 4.6. *Under Assumption 1.1 and when $\alpha_1 > \alpha_2$:*

(i) if $a_{11}k_2 < b_1$ and $b_3 < qE$, then all x_1, x_2, and x_3 go to extinction almost surely, i.e., $\lim_{t\to\infty} x_1(t) = 0$, $\lim_{t\to\infty} x_2(t) = 0$, $\lim_{t\to\infty} x_3(t) = 0$.

(ii) if $a_{11} > b_1k_1$, $a_{21} > b_2k_2$, and $\Gamma_1 < a_{12}a_{22}qE$, then x_1, x_2 are persistent in mean a.s., and x_3 goes to extinction a.s.

(iii) if $a_{11}k_2 < b_1$ and $b_3 > qE$, then both x_1 and x_2 go to extinction a.s., and x_3 is persistent in mean a.s.

(iv) if $a_{11} > b_1k_1$, $a_{12}a_{22}r(a_{21} - b_2k_2) > d_1k_3(\Gamma_1 - a_{12}a_{22}qE)$, and $\Gamma_1 > a_{12}a_{22}qE$, then x_1, x_2, x_3 are all persistent in mean a.s.

Proof. By the stochastic comparison theorem, we obtain

$$x_1(t) \le y_1(t), \qquad x_2(t) \le y_2(t), \qquad x_3(t) \le y_3(t). \tag{4.23}$$

So, it follows from (4.14), (4.19), and (4.21) that

$$\lim_{t\to\infty} \frac{\ln x_1(t)}{t} = 0, \qquad \lim_{t\to\infty} \frac{\ln x_2(t)}{t} = 0, \qquad \lim_{t\to\infty} \frac{\ln x_3(t)}{t} = 0. \tag{4.24}$$

Applying Itô's formula to system (1.3) yields

$$\begin{aligned} d\ln x_1(t) &= \left(a_{11}\frac{x_2(t)}{x_1(t)} - a_{12}x_1(t) - b_1\right)dt + \sigma_1 dB_1(t),\\ d\ln x_2(t) &= \left(a_{21}\frac{x_1(t)}{x_2(t)} - a_{22}x_2(t) - d_1u_1(t) - b_2\right)dt + \sigma_1 dB_1(t),\\ d\ln x_3(t) &= \left(-\frac{r}{k_3}x_3(t) - d_2u_2(t) - qE + b_3\right)dt + \sigma_2 dB_2(t). \end{aligned}$$

Integrate both sides of the above three equations from 0 to t, and divide by t to obtain

$$t^{-1}\ln\frac{x_1(t)}{x_1(0)} = a_{11}t^{-1}\int_0^t \frac{x_2(v)}{x_1(v)}dv - a_{12}\langle x_1(t)\rangle - b_1 + t^{-1}\sigma_1 B_1(t), \tag{4.25}$$

$$\begin{aligned} t^{-1}\ln\frac{x_2(t)}{x_2(0)} &= a_{21}t^{-1}\int_0^t \frac{x_1(v)}{x_2(v)}dv - a_{22}\langle x_2(t)\rangle - d_1\langle u_1(t)\rangle - b_2 + t^{-1}\sigma_1 B_1(t)\\ &= a_{21}t^{-1}\int_0^t \frac{x_1(v)}{x_2(v)}dv - a_{22}\langle x_2(t)\rangle - d_1\langle x_3(t)\rangle + d_1\frac{u_1(t)-u_1(0)}{\alpha_1 t}\\ &\quad - b_2 + t^{-1}\sigma_1 B_1(t), \end{aligned} \tag{4.26}$$

$$\begin{aligned} t^{-1}\ln\frac{x_3(t)}{x_3(0)} &= -\frac{r}{k_3}\langle x_3(t)\rangle - d_2\langle u_2(t)\rangle - qE + b_3 + t^{-1}\sigma_2 B_2(t)\\ &= -\frac{r}{k_3}\langle x_3(t)\rangle - d_2\langle x_2(t)\rangle + d_2\frac{u_2(t)-u_2(0)}{\alpha_2 t} - qE + b_3 + t^{-1}\sigma_2 B_2(t). \end{aligned} \tag{4.27}$$

Now, let us prove conclusion (i). We use Lemma 4.2 to obtain

$$\lim_{t\to\infty}\frac{\sigma_1 B_1(t)}{t} = 0, \qquad \lim_{t\to\infty}\frac{\sigma_2 B_2(t)}{t} = 0.$$

Then, for arbitrary $\epsilon_3 > 0$, there exists $T_2 > 0$ such that

$$\left|\frac{\sigma_1 B_1(t)}{t}\right| < \frac{\epsilon_3}{4}, \qquad \left|\frac{\sigma_2 B_2(t)}{t}\right| < \frac{\epsilon_3}{4}, \qquad \left|\frac{\ln x_i(0)}{t}\right| < \frac{\epsilon_3}{4}, \qquad i = 1,2,3.$$

Using the specific property of the limit superior in (4.25) gives

$$t^{-1}\ln x_1(t) \le a_{11}k_2 - b_1 - a_{12}\langle x_1(t)\rangle + \epsilon_3, \qquad t > T_2.$$

By the assumption $a_{11}k_2 < b_1$, we can let ϵ_3 be sufficiently small that $a_{11}k_2 < b_1 - \epsilon_3$, and by Lemma 4.1, $\lim_{t\to\infty} x_1(t) = 0$ and $\lim_{t\to\infty}\langle x_1(t)\rangle = 0$.

From Lemma (4.4), for the above ϵ_3, there exists $T_3 > 0$ such that

$$\left|\frac{u_1(t)-u_1(0)}{t}\right| < \frac{\alpha_1\epsilon_3}{2d_1}, \qquad t \ge T_3.$$

Using limit superior in (4.26) gives

$$t^{-1}\ln x_2(t) \leq -b_2 + \epsilon_3 - a_{22}\langle x_2(t)\rangle - d_1\langle x_3(t)\rangle_*, \qquad t \geq T_3.$$

Let ϵ_3 be sufficiently small that $-b_2 + \epsilon_3 < 0$. Then $\lim_{t\to\infty} x_2(t) = 0$ by Lemma 4.1.

Similarly, from Lemma 4.4, there exists $T_4 > 0$ such that

$$\left|\frac{u_2(t) - u_1(0)}{t}\right| < \frac{\alpha_2\epsilon_3}{2d_2}, \qquad t \geq T_4.$$

Then

$$t^{-1}\ln x_3(t) \leq b_3 - qE + \epsilon_3 - \frac{r}{k_3}\langle x_3(t)\rangle - d_2\langle x_2(t)\rangle_*, \qquad t \geq T_4.$$

Because $b_3 < 0$, ϵ_3 is sufficiently small that $b_3 + \epsilon_5 < 0$, and we have $\lim_{t\to\infty} x_3(t) = 0$ by Lemma 4.1.

Next, we prove (ii). Because $\lim_{t\to\infty} y_3(t) = 0$ $a.s.$ when $\Gamma_1 < a_{12}a_{22}qE$ from Theorem 4.5 and (4.23), we know $\lim_{t\to\infty} x_3(t) = 0$ $a.s.$ Hence system (1.3) can be simplified to a stage-structured single-population model,

$$\begin{cases} dx_1(t) = \big(a_{11}x_2(t) - a_{12}x_1^2(t) - sx_1(t)\big)dt + \sigma_1 x_1(t)dB_1(t), \\ dx_2(t) = \big(a_{21}x_1(t) - a_{22}x_2^2(t) - \beta x_2(t)\big)dt + \sigma_1 x_2(t)dB_1(t). \end{cases}$$

Integrate both sides of the above equations from 0 to t and divide by t to obtain:

$$t^{-1}\ln\frac{x_1(t)}{x_1(0)} \geq \frac{a_{11}}{k_1} - a_{12}\langle x_1(t)\rangle - b_1 + t^{-1}\sigma_1 B_1(t), \tag{4.28}$$

$$t^{-1}\ln\frac{x_2(t)}{x_2(0)} \geq \frac{a_{21}}{k_2} - a_{22}\langle x_2(t)\rangle - b_2 + t^{-1}\sigma_1 B_1(t). \tag{4.29}$$

According to Lemma 4.1 and the proof of Theorem 4.5, one can obtain that

$$\lim_{t\to\infty}\langle x_1(t)\rangle \geq \frac{a_{11} - b_1k_1}{a_{12}k_1} > 0,$$

$$\lim_{t\to\infty}\langle x_2(t)\rangle \geq \frac{a_{21} - b_2k_2}{a_{22}k_2} > 0,$$

and the proof of (ii) is completed.

Similar to (ii), we can see that $\lim_{t\to\infty} x_1(t) = 0$, $\lim_{t\to\infty} x_2(t) = 0$, from (i) under the condition $a_{11}k_2 < b_1$, and system (1.3) can be simplified to a single-species model,

$$dx_3(t) = x_3(t)\left(r\left(1 - \frac{x_3(t)}{k_3(t)}\right) - qE\right)dt + \sigma_2 x_3(t)dB_2(t).$$

Therefore,

$$t^{-1}\ln\frac{x_3(t)}{x_3(0)} = -\frac{r}{k_3}\langle x_3(t)\rangle + b_3 - qE + t^{-1}\sigma_2 B_2(t).$$

Applying Lemma 4.1 and similar proof with Theorem 4.5 to the above equation, we obtain:

$$\lim_{t\to\infty}\langle x_3(t)\rangle = \frac{(b_3 - qE)k_3}{r} > 0.$$

Finally, we prove (iv). From (4.25)–(4.27), we obtain

$$t^{-1}\ln\frac{x_1(t)}{x_1(0)} \geq \frac{a_{11}}{k_1} - a_{12}\langle x_1(t)\rangle - b_1 + t^{-1}\sigma_1 B_1(t),$$

$$\begin{aligned} t^{-1}\ln\frac{x_2(t)}{x_2(0)} &\geq \frac{a_{21}}{k_2} - a_{22}\langle x_2(t)\rangle - d_1\langle x_3(t)\rangle^* + \frac{d_1\big(u_1(t)-u_1(0)\big)}{\alpha_1 t} - b_2 + t^{-1}\sigma_1 B_1(t) \\ &\geq \frac{a_{21}}{k_2} - a_{22}\langle x_2(t)\rangle - d_1\frac{k_3}{r}\left(\frac{\Gamma_1}{a_{12}a_{22}} - qE\right) + \frac{d_1\big(u_1(t)-u_1(0)\big)}{\alpha_1 t} - b_2 \\ &\quad + t^{-1}\sigma_1 dB_1(t), \end{aligned}$$

$$\begin{aligned} t^{-1}\ln\frac{x_3(t)}{x_3(0)} &\geq -\frac{r}{k_3}\langle x_3(t)\rangle - d_2\langle x_2(t)\rangle^* + \frac{d_2\big(u_2(t)-u_2(0)\big)}{\alpha_2 t} + b_3 - qE + t^{-1}\sigma_2 B_2(t) \\ &\geq -\frac{r}{k_3}\langle x_3(t)\rangle + \frac{d_2\big(u_2(t)-u_2(0)\big)}{\alpha_2 t} + \frac{\Gamma_1}{a_{12}a_{22}} - qE + t^{-1}\sigma_2 B_2(t). \end{aligned}$$

Simply, one can obtain that:

$$\begin{aligned} &\lim_{t\to\infty}\langle x_1(t)\rangle \geq \frac{a_{11}-k_1 b_1}{a_{12}k_1} > 0, \\ &\lim_{t\to\infty}\langle x_2(t)\rangle \geq \frac{a_{21}-b_2 k_2}{a_{22}k_2} - \frac{d_1 k_3}{a_{22} r}\left(\frac{\Gamma_1}{a_{12}a_{22}} - qE\right) > 0, \\ &\lim_{t\to\infty}\langle x_3(t)\rangle \geq \frac{k_3}{r}\left(\frac{\Gamma_1}{a_{12}a_{22}} - qE\right) > 0. \end{aligned}$$

□

Theorem 4.7. *If the conditions of Theorem 4.6 (iv) hold, then the optimal harvested efforts of species x_3 are*

$$E^\star = \frac{1}{2pq}\left(\frac{p\Gamma_1}{a_{12}a_{22}} - \frac{r}{k_3}c\right),$$

and the maximum expectation of net economic revenue is

$$m(E^\star) = \frac{k_3}{4pqr}\left(\frac{p\Gamma_1}{a_{12}a_{22}} - \frac{r}{k_3}c\right),$$

where p and $\frac{c}{x_3(t)}$ are respectively the unit price and unit cost of a commercially harvested population.

Proof. According to the conclusions of Theorems 4.5 and 4.6, we can obtain that

$$\lim_{t\to\infty}\langle x_3(t)\rangle \leq \frac{k_3}{r}\left(\frac{\Gamma_1}{a_{12}a_{22}} - qE\right), \qquad \lim_{t\to\infty}\langle x_3(t)\rangle \geq \frac{k_3}{r}\left(\frac{\Gamma_1}{a_{12}a_{22}} - qE\right).$$

Hence

$$\lim_{t\to\infty}\langle x_3(t)\rangle = \frac{k_3}{r}\left(\frac{\Gamma_1}{a_{12}a_{22}} - qE\right).$$

Then the net economic revenue is

$$\begin{aligned} m(E) &= \lim_{t\to\infty}\left(pE\frac{\int_0^t x_3(v)dv}{t} - \frac{c}{\frac{\int_0^t x_3(v)dv}{t}}\frac{\int_0^t x_3(v)dv}{t}E\right) \\ &= \frac{k_3}{r}\left(\frac{\Gamma_1}{a_{12}a_{22}}pE - pqE^2\right) - cE. \end{aligned}$$

Letting $\frac{dm(E)}{dE} = 0$, the optimal harvested efforts are

$$E^\star = \frac{1}{2pq}\left(\frac{p\Gamma_1}{a_{12}a_{22}} - \frac{r}{k_3}c\right),$$

and the maximum expectation of net economic revenue is

$$m(E^\star) = \frac{k_3}{4pqr}\left(\frac{p\Gamma_1}{a_{12}a_{22}} - \frac{r}{k_3}c\right). \qquad \square$$

5 Numerical analysis

We use some hypothetical parameter values to verify Theorems 4.6 and 4.7. We choose $k_1 = 50, k_2 = 50, k_3 = 100$, and initial values $x_1(0) = 5, x_2(0) = 5, x_3(0) = 8$. Assign different values to other parameters in Table 5.1, which satisfies Theorem 4.6 , to prove theoretical results. Fig. 5.1–Fig. 5.4 show the different survival states of the species, as demonstrated in Theorem 4.6.

Parameter	Fig. 5.1 values	Fig. 5.2 values	Fig. 5.3 values	Fig. 5.4–5.5 values
a_{11}	0.04	13.8	0.03	12.8
a_{12}	0.1	0.2	0.05	0.85
a_{21}	0.2	0.24	0.5	0.24
a_{22}	0.1	0.2	0.1	0.5
s	0.7	0.25	0.5	0.25
d_1	0.1	0.2	0.1	0.01
d_2	0.35	0.2	0.35	0.01
r	1	1.25	1.1	2
q	0.45	0.5	0.5	0.55
E	3	3	2	3.5
β	0.1	0.004	0.1	0.004
σ_1	1.62	0.05	1.42	0.1
σ_2	0.2	0.2	0.2	0.2
α_1	0.5	0.5	0.5	0.5
α_2	0.4	0.4	0.4	0.2

Table 5.1: Parameter values.

Regarding the optimal harvesting effort, we still select the same parameters with the Fig. 5.4. By Theorem 4.7, we obtain $E^* = 1.788$. Therefore, the optimal harvesting policy exists, we show it in Fig. 5.5. The maximum expectation of net economic revenue exists when $E^* = 1.788$.

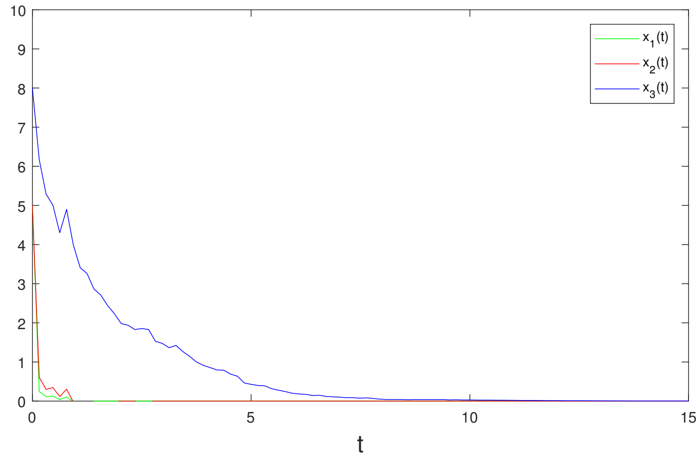

Figure 5.1: x_1, x_2 and x_3 all go to extinction.

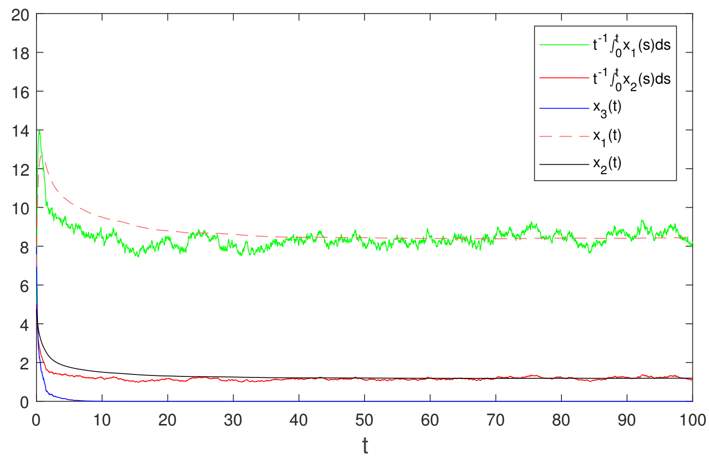

Figure 5.2: x_1 and x_2 are permanent, x_3 goes to extinction.

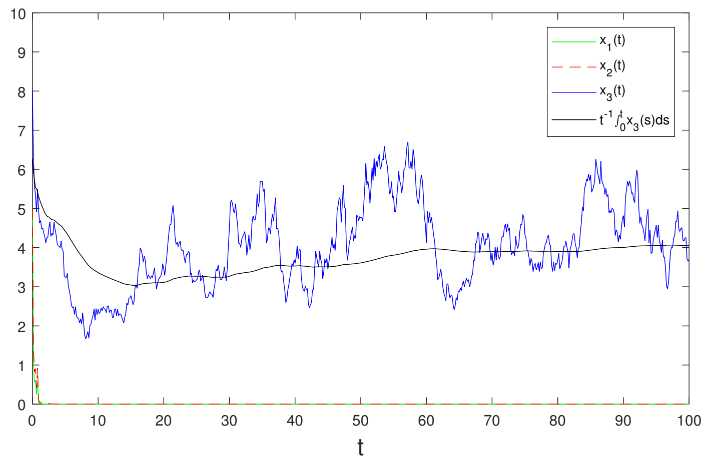

Figure 5.3: x_1,x_2 go to extinction, x_3 is permanent.

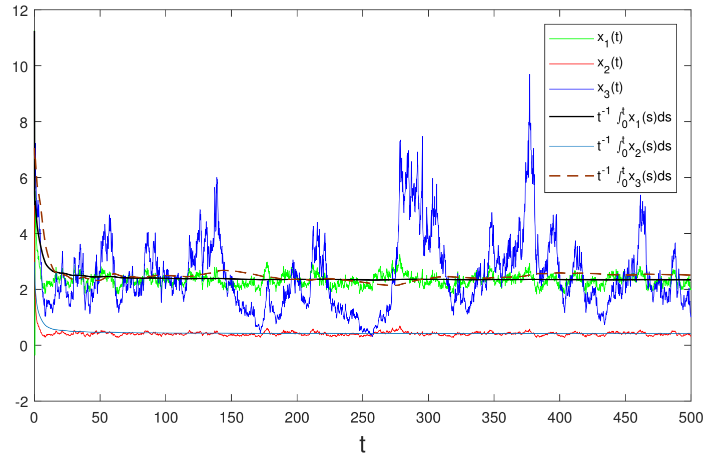

Figure 5.4: x_1, x_2 and x_3 are permanent.

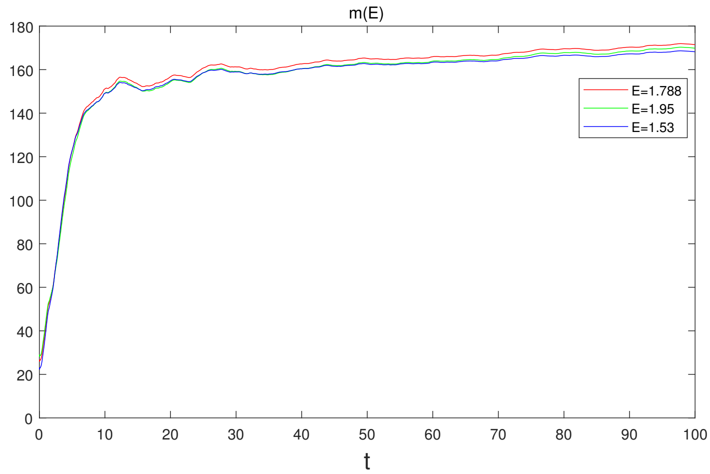

Figure 5.5: The optimal harvesting effort and the maximum of net economic revenue.

6 Conclusion

We investigated the dynamics of a stochastic stage-structured competitive system with distributed delay and harvesting. We took a weak kernel case as an example for convenience, and we similarly discuss the strong kernel case. Our objective was to study the optimal harvest strategy and the maximum net economic revenue. Some main results are as follows:

(i) The existence and uniqueness of the positive solution of system (1.3) was proved, using a Lyapunov function to ensure the rationality of the system and provide support for later results.

(ii) We showed that when $a_{12} > 2(a_{11} \vee a_{21})$, $a_{22} > \alpha_2 + (a_{11} \vee a_{21})$, $r > \alpha_1 k_3, \alpha_1 > d_1$, $\alpha_2 > d_2$, system (1.3) would be asymptotically stable in distribution.

(iii) The research of the optimal harvest and maximum expectation of net economic revenue of stochastic models has clear practical significance. Species extinction must be strictly prevented during fishing. First, sufficient conditions for persistence in mean and extinction were established. The optimal harvested efforts were

$$E^\star = \frac{1}{2pq}\left(\frac{p\Gamma_1}{a_{12}a_{22}} - \frac{r}{k_3}c\right),$$

and the maximum expectation of net economic revenue was

$$m(E^\star) = \frac{k_3}{4pqr}\left(\frac{p\Gamma_1}{a_{12}a_{22}} - \frac{r}{k_3}c\right).$$

We only considered the effect of white noise and delay on the dynamics of the stage-structured competitive system. It is also interesting to consider the effect of telephone noise, toxins, and Markovian switching, and these will be topics of our further research.

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