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Existence of a homoclinic orbit in a generalized Liénard type system

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Abstract. The object of this paper is to study the existence and nonexistence of an important orbit in a generalized Liénard type system. This trajectory is doubly asymptotic to an equilibrium solution, i.e., an orbit which lies in the intersection of the stable and unstable manifolds of a critical point. Such an orbit is called a homoclinic orbit.

Keywords: Liénard system, homoclinic orbit, planar system, dynamical systems

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1 Introduction

Consider the planar system

$$\begin{aligned} \dot{x} &= P(Q(y) - F(x)) \\ \dot{y} &= -g(x), \end{aligned} \tag{1.1}$$

which is a generalized Liénard type system, where P, Q, F and g are continuous functions satisfying suitable assumptions in order to ensure the existence of a unique solution to the initial value problems. Moreover, suppose that the following assumptions hold under which the origin is the unique critical point of system (1.1).

$$\begin{gathered} P(u) \text{ and } Q(y) \text{ are strictly increasing and } F(0) = P(0) = Q(0) = 0, \\ uP(u) > 0 \text{ for } u \neq 0,\ yQ(y) > 0 \text{ for } y \neq 0 \text{ and } xg(x) > 0 \text{ for } x \neq 0. \end{gathered}$$

System (1.1) includes the classical Liénard system as a special case, which is of great importance in various applications (see [1] to [23] and the references cited therein).

Definition 1.1. In system (1.1), a trajectory is said to be a homoclinic orbit if its α- and ω-limit sets are the origin (see Fig. 1.1).

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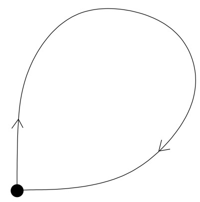

Figure 1.1: Homoclinic orbit

The main purpose of this paper is to give an implicit necessary and sufficient condition and some explicit sufficient conditions on $F(x)$, $g(x)$, $P(u)$ and $Q(y)$ under which system (1.1) has homoclinic orbits. These results extend and improve the results presented for special cases of system (1.1) in [3, 11, 19].

The existence of homoclinic orbit is an important problem in nonlinear dynamical systems and the theory of ordinary differential equations. The results about the existence of homoclinic orbits for the other systems, such as the Lorenz system, Schrödinger systems, predator–prey systems and Hamiltonian systems can be found in [13, 18, 22, 23], respectively. Moreover, various systems and equations such as generalized Euler equation [4] and predator–prey systems [22] can be transformed to the Liénard type systems.

The existence of homoclinic orbits in the Liénard-type systems is closely connected with the stability of the zero solution and the center problem (see [6, 11, 19, 21]). If system (1.1) has a homoclinic orbit, then the zero solution is no longer stable. A homoclinic orbit and a center cannot exist together in system (1.1). Our subject also has a near relation with the global attractivity of the origin and oscillation of solutions and so on (see [9, 12, 20]).

The curve $\Gamma = \{(x,y)|y = Q^{-1}(F(x))\}$ is called the characteristic curve of (1.1). Let

$$\Gamma_1 = \{(x,y) \mid y = Q^{-1}(F(x)) \text{ and } x > 0\},$$

and

$$\Gamma_2 = \{(x,y) \mid y = Q^{-1}(F(x)) \text{ and } x < 0\}.$$

Then, $\Gamma = \Gamma_1 \bigcup \Gamma_2 \bigcup (0,0)$. Positive and negative orbits of (1.1) passing through $p \in \mathbb{R}^2$ are shown by $O^+(p)$ and $O^-(p)$, respectively.

The following definitions are presented to state our main results.

Definition 1.2. System (1.1) has property (Z_1^+) (resp., (Z_3^+)) if there exists a point $p(x_0, y_0) \in \Gamma_1$ (resp., $p(x_0, y_0) \in \Gamma_2$), such that the $O^+(p)$ of (1.1) starting at p approaches the origin through only the first (resp., third) quadrant (see Fig. 1.2).

Definition 1.3. System (1.1) has property (Z_2^-) (resp., (Z_4^-)) if there exists a point $p(x_0, y_0) \in \Gamma_2$ (resp., $p(x_0, y_0) \in \Gamma_1$), such that the $O^-(p)$ of (1.1) starting at p approaches the origin through only the second (resp., fourth) quadrant.

If system (1.1) has both properties (Z_1^+) and (Z_2^-), then a homoclinic orbit exists in the upper half-plane. Similarly, if system (1.1) has both properties (Z_3^+) and (Z_4^-), then a homoclinic orbit exists in the lower half-plane. In this paper we will find conditions for deciding whether system (1.1) has homoclinic orbit.

Hara and Yoneyama in [9] considered system (1.1) with $Q(y) = y$ and $P(u) = u$ and presented some sufficient conditions under which the system has or fails to have property (Z_1^+). Also, Sugie presented an implicit necessary and sufficient condition for system (1.1) with $P(u) = u$ to have property (Z_1^+) [19]. Next, Aghajani and Moradifam in [3] considered

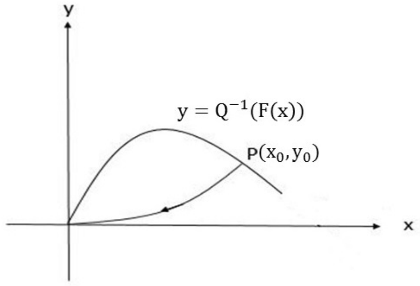

Figure 1.2: Property (Z_1^+)

system (1.1) with $P(u) = u$ and gave an implicit necessary and sufficient condition for the system to have property (Z_1^+) which improved some results in [19].

In the next section an implicit necessary and sufficient condition and some explicit sufficient conditions are provided for system (1.1) to have property (Z_1^+). Since some nonlinear functions are added to the classical Liénard system in this article, our results are proper extensions of the known ones in [3], [9], [11] and [19].

2 Necessary and sufficient conditions for property of (Z_1^+)

In this section we will give necessary and sufficient conditions for system (1.1) to have properties (Z_1^+) and (Z_2^-). First, consider the following lemma about asymptotic behavior of solutions of (1.1).

Lemma 2.1. *For each point $H(c, Q^{-1}(F(c)))$ with $c > 0$ or $c < 0$, the positive or negative semi-orbit of (1.1) starting at H crosses the negative y-axis if the following condition holds.*

(A_1) *There exists a $\delta > 0$ such that $F(x) < 0$ for $-\delta < x < \delta$ or $F(x)$ has an infinite number of positive zeroes clustering at $x = 0$.*

Remark 2.2. Lemma 2.1 implies that system (1.1) fails to have properties (Z_1^+) and (Z_2^-) if **(A_1)** holds. Hence, hereafter we assume that there exists a $\delta > 0$ such that $F(x) > 0$ for $-\delta < x < \delta$.

Theorem 2.3. *System (1.1) has property (Z_1^+) if and only if there exist a constant $\delta > 0$ and a continuous function $\phi(x)$ such that*

$$0 \le \phi(x) < F(x) \quad \text{and} \quad \int_0^x \frac{-g(\eta)}{P(\phi(\eta) - F(\eta))} d\eta \le Q^{-1}(\phi(x)) \tag{2.1}$$

for $0 < x < \delta$.

Proof. First, note that the positive semi-orbit of (1.1) starting at $H(x_0, Q^{-1}(F(x_0)))$ is considered as a solution $y(x)$ of

$$\frac{dy}{dx} = \frac{-g(x)}{P(Q(y) - F(x))}, \tag{2.2}$$

with $y(x_0) = Q^{-1}(F(x_0))$.

Sufficiency: Suppose that system (1.1) fails to have property (Z_1^+). Thus, there exist a point $H(x_0, Q^{-1}(F(x_0)))$ and $x_0 > 0$ such that the positive semi-orbit of (1.1) starting at H does not approach the origin through the first quadrant. Taking the vector field of (1.1) into account, it is obvious that the positive semi-orbit rotates in clockwise direction about the origin. For this reason, it crosses the curve $y = Q^{-1}(\phi(x))$ and meets the y-axis at a point $(0, y_1)$ with $y_1 < 0$. Let

$$x_1 = \inf\{x : 0 < x < \delta \text{ and } y(x) > Q^{-1}(\phi(x))\}.$$

Then, $(x_1, y(x_1))$ is the intersection point of $O^+(H)$ and the curve $y = Q^{-1}(\phi(x))$ nearest to the origin, that is $y(x_1) = Q^{-1}(\phi(x_1))$ and $y < Q^{-1}(\phi(x))$ for $0 < x < x_1$. Hence, from (2.1), it can be concluded that

$$\begin{aligned} Q^{-1}(\phi(x_1)) < y(x_1) - y_1 &= \int_0^x \frac{-g(\eta)}{P(Q(y(\eta)) - F(\eta))} d\eta \\ &< \int_0^{x_1} \frac{-g(\eta)}{P(\phi(\eta) - F(\eta))} d\eta \le Q^{-1}(\phi(x_1)), \end{aligned}$$

which is a contradiction.

Necessity: Suppose that $O^+(H)$ approaches the origin through the first quadrant. Then, its corresponding solution $y(x)$ satisfies

$$y(x) \to 0^+ \quad \text{as } x \to 0. \tag{2.3}$$

Let $\delta = x_0$ and $\phi(x) = Q(y(x))$ for $0 < x < \delta$. It is obvious that $\phi(x) \ge 0$. Thus,

$$Q^{-1}(\phi(x)) = y(x) < Q^{-1}(F(x)),$$

and therefore, $\phi(x) < F(x)$ for $0 < x < \delta$. Also, from (2.3) it can be easily seen that

$$\begin{aligned} \int_0^x \frac{-g(\eta)}{P(\phi(\eta) - F(\eta))} d\eta = \int_0^x \frac{-g(\eta)}{P(Q(y(\eta)) - F(\eta))} d\eta &= y(x) - \lim_{\epsilon \to 0} y(\epsilon) \\ &= Q^{-1}(\phi(x)). \end{aligned}$$

Thus, (2.1) holds and the proof is complete. □

Remark 2.4. For $P(u) = u$, Theorem 2.3 gives the corresponding result of Sugie in [19].

Corollary 2.5. *Suppose that there exists $k \in (0, 1)$ and $\delta > 0$ such that*

$$\frac{1}{Q^{-1}(kF(x))} \int_0^x \frac{-g(\eta)}{P((k-1)F(\eta))} d\eta \le 1 \quad \text{for } 0 < x < \delta. \tag{2.4}$$

Then, system (1.1) has property (Z_1^+).

Proof. Let $\phi(x) = kF(x)$. The following inequality is obtained from (2.4).

$$\int_0^x \frac{-g(\eta)}{P(\phi(\eta) - F(\eta))} d\eta = \int_0^x \frac{-g(\eta)}{P((k-1)F(\eta))} d\eta \le Q^{-1}(kF(x)),$$

for $0 < x < \delta$. Thus, by Theorem 2.3 system (1.1) has property (Z_1^+). □

Corollary 2.6. *Suppose that $P(au) \leq aP(u)$ for $a \in (-1,0)$ and $u > 0$. If there exist $k \in (0,1)$ and $\delta > 0$ such that*

$$\frac{1}{(1-k)Q^{-1}(kF(x))}\int_0^x \frac{g(\eta)}{P(F(\eta))}d\eta \leq 1 \quad \text{for } 0 < x < \delta,$$

then system (1.1) has property (Z_1^+).

Remark 2.7. For $P(u) = u$ and $Q(y) = y$ and taking $k = \frac{1}{2}$, Corollary 2.6 gives the result of Hara and Yoneyama in [9].

Corollary 2.8. *If for every $k \in [0,1]$ there exists a constant $\gamma_k > 0$ such that*

$$\liminf_{x\to 0^+}\left(\frac{1}{Q^{-1}((k+\gamma_k)F(x))}\int_0^x \frac{-g(\eta)}{P((k-\gamma_k-1)F(\eta))}d\eta\right) > 1, \tag{2.5}$$

then system (1.1) fails to have property (Z_1^+).

Proof. Suppose that there exist a constant $\delta > 0$ and a continuous function ϕ such that condition (2.1) holds. Define $k' = \liminf_{x\to 0^+}\frac{\phi(x)}{F(x)}$. Then $0 \leq k' \leq 1$, and from the definition of k' it follows that for every $\epsilon > 0$, there exist a b and a sequence $\{x_n\}$ with $0 < b < \delta$, $0 < x_n \leq b$, and $x_n \to 0$ as $n \to +\infty$ such that

$$\frac{\phi(x)}{F(x)} > k' - \epsilon \quad \text{for } 0 < x \leq b \quad \text{and} \quad \frac{\phi(x_n)}{F(x_n)} < k' + \epsilon.$$

Hence,

$$\phi(x) > (k'-\epsilon)F(x) \quad \text{for } 0 < x \leq b \quad \text{and} \quad \phi(x_n) < (k'+\epsilon)F(x_n).$$

Thus, from (2.1) it can be concluded that

$$\begin{aligned} 0 &\geq \int_0^{x_n}\frac{-g(\eta)}{P(\phi(\eta)-F(\eta))}d\eta - Q^{-1}(\phi(x_n)) \\ &> \int_0^{x_n}\frac{-g(\eta)}{P((k'-\epsilon)F(\eta)-F(\eta))}d\eta - Q^{-1}((k'+\epsilon)F(x_n)). \end{aligned}$$

Consequently, for $n \geq 1$ the following inequality holds.

$$\frac{1}{Q^{-1}((k'+\epsilon)F(x_n))}\int_0^{x_n}\frac{-g(\eta)}{P((k'-\epsilon-1)F(\eta))}d\eta < 1. \tag{2.6}$$

Thus, (2.6) contradicts (2.5) and the proof is complete. □

Corollary 2.9. *Suppose that $P(au) \geq aP(u)$ for $a \in [-2,-1)$ and $u > 0$. If there exists $\beta \in (1,2]$ such that*

$$\liminf_{x\to 0^+}\left(\frac{1}{2Q^{-1}((\beta+1)F(x))}\int_0^x \frac{g(\eta)}{P(F(\eta))}d\eta\right) > 1, \tag{2.7}$$

then system (1.1) fails to have property (Z_1^+).

Proof. Suppose that (2.7) holds. Then, in (2.5) for every $k \in [0,1]$ let $\gamma_k = (\beta-1)k+1$. By this argument, we have $k-1-\gamma_k = 2k-\beta k-2$ and $k+\gamma_k = \beta k+1$. Since $1 < \beta \leq 2$ and $0 \leq k \leq 1$, then

$$-2 \leq 2k-\beta k-2 < -1, \qquad \frac{1}{2} \leq \frac{1}{2+(\beta-2)k} < 1 \quad \text{and} \quad \beta k+1 \leq \beta+1.$$

Now, put the last relations in the left-hand side of (2.5) and get

$$\begin{aligned}
&\liminf_{x\to 0^+}\left(\frac{1}{Q^{-1}((k+\gamma_k)F(x))}\int_0^x \frac{-g(\eta)}{P((k-\gamma_k-1)F(\eta))}d\eta\right)\\
&\quad=\liminf_{x\to 0^+}\left(\frac{1}{Q^{-1}((\beta k+1)F(x))}\int_0^x \frac{-g(\eta)}{P((2k-\beta k-2)F(\eta))}d\eta\right)\\
&\quad\geq\liminf_{x\to 0^+}\left(\frac{1}{(2+(\beta-2)k)Q^{-1}((\beta+1)F(x))}\int_0^x \frac{g(\eta)}{P(F(\eta))}d\eta\right)\\
&\quad\geq\liminf_{x\to 0^+}\left(\frac{1}{2Q^{-1}((\beta+1)F(x))}\int_0^x \frac{g(\eta)}{P(F(\eta))}d\eta\right)>1.
\end{aligned}$$

This completes the proof. □

By choosing $k = 0$ in the proof of Corollary 2.9, the following corollary can be presented with weaker conditions.

Corollary 2.10. *Suppose that $P(au) \geq aP(u)$ for $a \in [-2,-1)$ and $u > 0$. If*

$$\liminf_{x\to 0^+}\left(\frac{1}{2Q^{-1}(F(x))}\int_0^x \frac{g(\eta)}{P(F(\eta))}d\eta\right)>1, \tag{2.8}$$

then system (1.1) fails to have property (Z_1^+).

The following corollaries can be obtained as results of Theorem 2.3 which are very useful in applications.

Corollary 2.11. *Suppose that system (1.1) with $P(u) = P_1(u)$ has (resp., fails to have) property (Z_1^+). If $P_2(u) \leq P_1(u)$ (resp., $P_2(u) \geq P_1(u)$) for $u < 0$, then system (1.1) with $P(u) = P_2(u)$ has (resp., fails to have) property (Z_1^+).*

Corollary 2.12. *Suppose that system (1.1) with $Q(y) = Q_1(y)$ has (resp., fails to have) property (Z_1^+). If $Q_2(y) \leq Q_1(y)$ (resp., $Q_2(y) \geq Q_1(y)$) for $y > 0$ sufficiently small, then system (1.1) with $Q(y) = Q_2(y)$ has (resp., fails to have) property (Z_1^+).*

By the same way, we can prove the following theorem about property (Z_2^-).

Theorem 2.13. *System (1.1) has property (Z_2^-) if and only if there exist a constant $\delta > 0$ and a continuous function $\phi(x)$ such that*

$$0 \leq \phi(x) < F(x) \quad \text{and} \quad \int_0^x \frac{-g(\eta)}{P(\phi(\eta)-F(\eta))}d\eta \leq Q^{-1}(\phi(x))$$

for $-\delta < x < 0$.

Similarly, other obtained results (Corollaries 2.5–2.10) can be formulated for property (Z_2^-).

3 Some explicit results

Condition (2.1) is implicit necessary and sufficient for system (1.1) to possess property (Z_1^+). However, in some cases, it is very difficult to find a suitable function $\phi(x)$ with a constant δ satisfying (2.1). Therefore, in the following, some explicit sufficient conditions are provided for system (1.1) to have property (Z_1^+). The results can also be formulated for the property (Z_3^+), (Z_2^-) or (Z_4^-). We leave the details to the reader. To state the results, define

$$H(y) = \int_0^y Q(\eta)d\eta \quad \text{and} \quad G(x) = \int_0^x g(\eta)d\eta.$$

Also, the inverse function of $\omega(y) = H(y)\text{sgn}(y)$ is denoted by $H^{-1}(\omega)$.

Theorem 3.1. *Suppose that $P(au) \le aP(u)$ for $a \in (-1,0)$ and $u > 0$ and there exist $\alpha > 0$ and $k \in [0,1)$ such that*

$$Q\left(\frac{x}{\alpha(1-k)}\right) \le kP^{-1}(\alpha Q(x)) \tag{3.1}$$

for $x > 0$ sufficiently small. Then, system (1.1) has property (Z_1^+) if

$$F(x) \ge P^{-1}(\alpha Q(H^{-1}(G(x)))), \tag{3.2}$$

for $x > 0$ sufficiently small.

Proof. From (3.1) it is obvious that

$$\frac{u}{\alpha(1-k)Q^{-1}(kP^{-1}(\alpha Q(u)))} \le 1, \tag{3.3}$$

for $u > 0$ sufficiently small. Since the function $u(x) = H^{-1}(G(x))$ is increasing and continuous on $[0,\infty)$ and $u(0) = 0$, by (3.2) we obtain

$$\frac{H^{-1}(G(x))}{\alpha(1-k)Q^{-1}(kP^{-1}(\alpha Q(H^{-1}(G(x)))))} \le 1, \tag{3.4}$$

for $x > 0$ sufficiently small. Since

$$\frac{d}{dx}H^{-1}(G(x)) = \frac{g(x)}{Q(H^{-1}(G(x)))},$$

from (3.4) we conclude that

$$\begin{aligned}
&\frac{1}{(1-k)Q^{-1}(kF(x))}\int_0^x \frac{g(\eta)}{P(F(\eta))}d\eta \\
&\quad \le \frac{1}{(1-k)Q^{-1}(kP^{-1}(\alpha Q(H^{-1}(G(x)))))}\int_0^x \frac{g(\eta)}{\alpha Q(H^{-1}(G(\eta)))}d\eta \\
&\quad = \frac{H^{-1}(G(x))}{\alpha(1-k)Q^{-1}(kP^{-1}(\alpha Q(H^{-1}(G(x)))))} \le 1,
\end{aligned}$$

for $x > 0$ sufficiently small. Hence, by Corollary 2.6 system (1.1) has property (Z_1^+). □

By choosing $\alpha = 2$, $k = \frac{1}{2}$ and $P(u) = u$, condition (3.1) holds for any function Q. In this case, the following corollary is obtained about property (Z_1^+) which is the corresponding result of Sugie in [19].

Corollary 3.2. *Suppose that*

$$F(x) \ge 2Q(H^{-1}(G(x))), \tag{3.5}$$

for $x > 0$ sufficiently small. Then, system (1.1) with $P(u) = u$ has property (Z_1^+).

Theorem 3.3. *Suppose that $\alpha > 0$ and $P(au) \geq aP(u)$ for $a \in [-2,-1)$ and $u > 0$. Also, assume that there exists $\beta \in (1,2]$ such that*

$$Q\left(\frac{x}{2\alpha}\right) \geq (\beta+1)P^{-1}(\alpha Q(x)) \tag{3.6}$$

for $x > 0$ sufficiently small. Then, system (1.1) fails to have property (Z_1^+) if

$$F(x) \leq P^{-1}(\lambda\alpha Q(H^{-1}(G(x)))), \tag{3.7}$$

for some $\lambda < 1$.

Proof. By (3.6) it is obvious that

$$\frac{u}{\alpha Q^{-1}((\beta+1)P^{-1}(\alpha Q(u)))} \geq 2.$$

By the similar argument to the proof of Theorem 3.1, it can be concluded that if (3.6) and (3.7) hold, then

$$\liminf_{x\to 0^+}\left(\frac{1}{2Q^{-1}((\beta+1)F(x))}\int_0^x \frac{g(\eta)}{P(F(\eta))}d\eta\right) > 1.$$

Hence, by Corollary 2.9 system (1.1) fails to have property (Z_1^+). □

4 Homoclinic orbit

In this section some results will be presented about the existence of homoclinic orbit in the upper half-plane for system (1.1). The following theorem is obtained by combining Theorem 2.3 and 2.13.

Theorem 4.1. *System (1.1) has homoclinic orbit in the upper half-plane if and only if there exist a constant $\delta > 0$ and a continuous function $\phi(x)$ such that*

$$0 \leq \phi(x) < F(x) \quad \textit{and} \quad \int_0^x \frac{-g(\eta)}{P(\phi(\eta)-F(\eta))}d\eta \leq Q^{-1}(\phi(x)) \tag{4.1}$$

for $0 < |x| < \delta$.

The following two corollaries are obtained from Theorem 4.1, which provide explicit conditions for system (1.1) to have homoclinic orbit in upper half-plane. Note that, in Remark 2.2, it is assumed that there exists a $\delta > 0$ such that $F(x) > 0$ for $-\delta < x < \delta$.

Corollary 4.2. *Suppose that there exist $k \in (0,1)$ and $\delta > 0$ such that*

$$\frac{1}{Q^{-1}(kF(x))}\int_0^x \frac{-g(\eta)}{P((k-1)F(\eta))}d\eta \leq 1 \quad \textit{for } 0 < |x| < \delta. \tag{4.2}$$

Then, system (1.1) has homoclinic orbit in the upper half-plane.

Corollary 4.3. *Suppose that $P(au) \leq aP(u)$ for $a \in (-1,0)$ and $u > 0$. If there exist $k \in (0,1)$ and $\delta > 0$ such that*

$$\frac{1}{(1-k)Q^{-1}(kF(x))}\int_0^x \frac{g(\eta)}{P(F(\eta))}d\eta \leq 1 \quad \textit{for } 0 < |x| < \delta, \tag{4.3}$$

then system (1.1) has homoclinic orbit in the upper half-plane.

Remark 4.4. Suppose that F is an even and g is an odd function. It is easy to see that system (1.1) has property (Z_1^+) if and only if it has property (Z_2^-). Therefore, if system (1.1) has property (Z_1^+), then it has a homoclinic orbit in the upper half-plane.

Similarly, Theorem 3.1 and Corollary 3.2 and some other results can be formulated about property (Z_2^-) and the existence of homoclinic orbits in the upper half-plane. Turning our attention to the lower half-plane, all presented results can be formulated about properties (Z_3^+) and (Z_4^-) and finally about the existence of homoclinic orbit in the lower half-plane.

In the following, two examples will be presented to illustrate our results and show the applications of the results.

Example 4.5. Consider the following Gause-type Predator-Prey system

$$\begin{aligned} \dot{u} &= ur(u) - vsf(u) \\ \dot{v} &= v(q(u) - D), \end{aligned} \tag{4.4}$$

with $f(u) = u$, $r(u) = \beta - \gamma|u - \alpha|$, $q(u) = u^2$, $D = \alpha^2$ and $\beta > \alpha\gamma$. In system (4.4), $u(t)$ and $v(t)$ represent prey and predator densities, the function $f(u)$ is functional response, $q(u)$ is the growth rate of the predator, $r(u)$ is the growth rate of the prey in the absence of any predators, and $D > 0$ is the natural death rate of the predator in the absence of any prey. The constants α, β and γ are positive ecological parameters. System (4.4) has the positive equilibrium $E^* = (\alpha, \beta)$. By the change of variables

$$x = u - \alpha, \qquad y = \ln\beta - \ln v \quad \text{and} \quad dt = uds,$$

system (4.4) will be transformed into system (1.1) with

$$P(u) = u, \qquad Q(y) = \beta(1 - e^{-y}), \qquad F(x) = \gamma|x| \quad \text{and} \quad g(x) = x + \alpha - \frac{\alpha^2}{x + \alpha}. \tag{4.5}$$

Functions $F(x)$ and $g(x)$ are defined on $(-\alpha, +\infty)$ and satisfy $F(0) = 0$ and $xg(x) > 0$ for $x \neq 0$. Also, $Q(y)$ is defined on $\mathbb{R}$ satisfying $Q(0) = 0$ and $yQ(y) > 0$ for $y \neq 0$. The inverse function of $Q(y)$ is $Q^{-1}(y) = \ln\left(\frac{\beta}{\beta - y}\right)$ where defined on $(-\infty, \beta)$. For $0 < x < \frac{\beta}{k\gamma}$, by using Corollary 4.3, it can be written that

$$\begin{aligned} \frac{1}{(1-k)Q^{-1}(kF(x))} \int_0^x \frac{g(\eta)}{P(F(\eta))} d\eta &= \frac{1}{\gamma(1-k)\ln\left(\dfrac{\beta}{\beta - k\gamma x}\right)} \left(x + \alpha \ln\left(1 + \frac{x}{\alpha}\right)\right) \\ &< \frac{2\beta}{\gamma^2(1-k)k}. \end{aligned}$$

By choosing $k = \frac{1}{2}$, it can be concluded that

$$\frac{1}{(1-k)Q^{-1}(kF(x))} \int_0^x \frac{g(\eta)}{P(F(\eta))} d\eta < \frac{8\beta}{\gamma^2}.$$

If $0 < 8\beta \leq \gamma^2$, then

$$\frac{1}{(1-k)Q^{-1}(kF(x))} \int_0^x \frac{g(\eta)}{P(F(\eta))} d\eta < 1.$$

By a similar argument, it can be shown that for $-\alpha < x < 0$

$$\frac{1}{(1-k)Q^{-1}(kF(x))} \int_0^x \frac{g(\eta)}{P(F(\eta))} d\eta < 1.$$

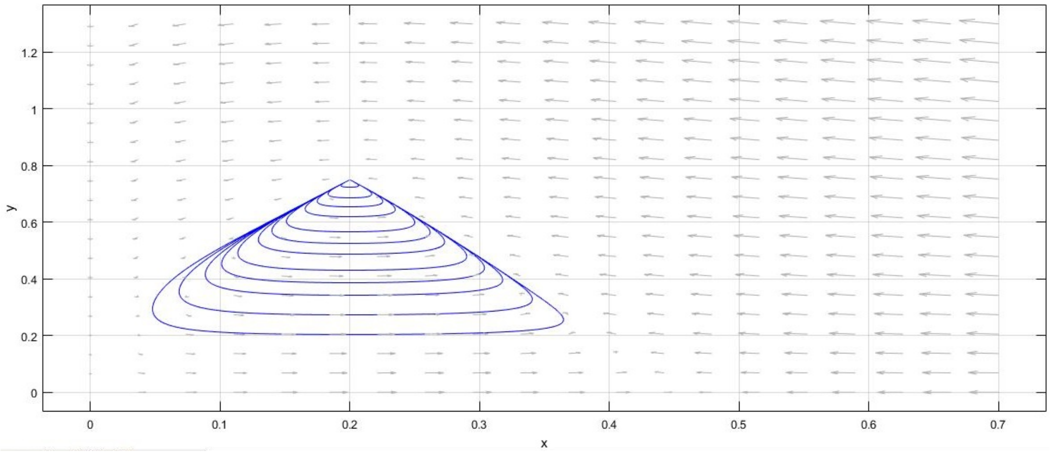

Figure 4.1: Phase portrait for system (4.4) with $\alpha = 0.2, \beta = 0.75$ and $\gamma = 3$.

Therefore, by Corollary 4.3 this system has a homoclinic orbit in the upper half-plane (see Fig. 4.1).

Remark 4.6. Sugie and Kimoto in [22], under the assumption $Q(y) \le my$ for $y > 0$, showed that system (1.1) with functions in (4.5) has homoclinic orbits in the upper half-plane if $0 < 8\beta \le \gamma^2$. In this work, the existence of homoclinic orbits has been presented without the assumption $Q(y) \le my$ for $y > 0$.

Example 4.7. Consider system (1.1) with functions

$$P(u) = u^3, \qquad Q(y) = \operatorname{sgn}(y)\sqrt{|y|}, \qquad F(x) = \sqrt[4]{|x|} \quad \text{and} \quad g(x) = x. \tag{4.6}$$

By Corollary 2.5, it can be written that

$$\frac{1}{Q^{-1}(kF(x))} \int_0^x \frac{-g(\eta)}{P((k-1)F(\eta))} d\eta = \frac{4\sqrt[4]{x^3}}{5k^2(1-k)^3} \le 1$$

for $0 < x < \left(\frac{3}{5}\right)^4 \frac{1}{5\sqrt[3]{5}}$. Therefore, this system has property (Z_1^+). Since F is even and g is odd, Remark 4.4 implies that this system has a homoclinic orbit in the upper half-plane (see Fig. 4.2).

The next example shows a new application which comes from articles treating the Liénard equation with the differential operator related to the relativistic acceleration, that is

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-(\dot{x})^2}}\right) + f(x)\dot{x} + g(x) = 0, \tag{4.7}$$

which, nowadays, is a quite interesting topic in works concerning the case of generalized Liénard equations. The existence of a stable limit cycle and periodic solutions of relativistic Liénard equation (4.7) has been investigated by Mawhin and Villari in [15]. Now, we apply our results to a special case of this equation.

Equation (4.7) can easily be transformed to system (1.1) with

$$P(u) = \frac{u}{\sqrt{1+u^2}}, \qquad Q(y) = y \quad \text{and} \quad F(x) = \int_0^x f(\eta) d\eta.$$

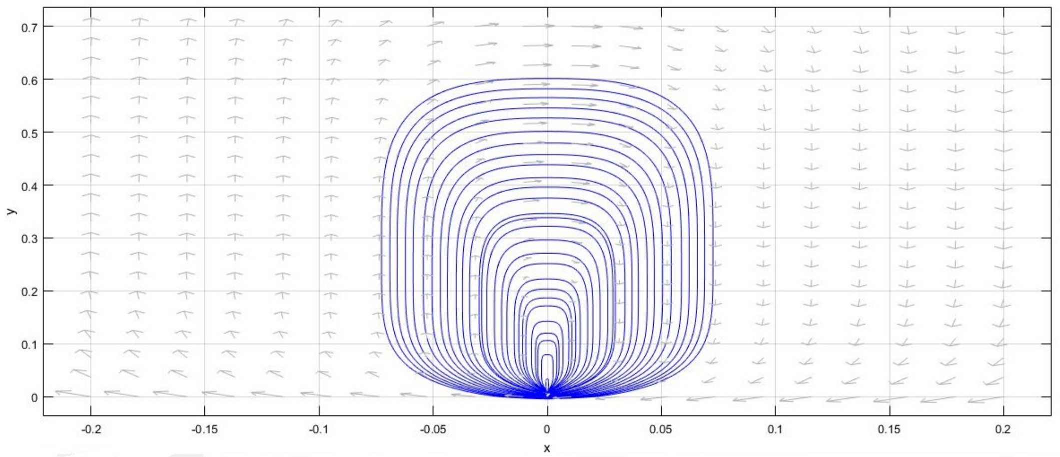

Figure 4.2: Phase portrait for system (4.6).

Example 4.8. Consider system (1.1) with

$$P(u) = \frac{u}{\sqrt{1+u^2}}, \qquad Q(y) = y, \qquad F(x) = x^2 \quad \text{and} \quad g(x) = \frac{x^3}{2\sqrt{1+x^4}}. \tag{4.8}$$

Since $P(au) \leq aP(u)$ for $-1 < a < 0$ and $u > 0$, from Corollary 2.6, by choosing $k = \frac{1}{2}$, we have

$$\frac{1}{(1-k)Q^{-1}(kF(x))} \int_0^x \frac{g(\eta)}{P(F(\eta))} d\eta = \frac{2}{x^2} \int_0^x \eta d\eta = 1.$$

Therefore, this system has property (Z_1^+). Since F is even and g is odd, Remark 4.4 implies that this system has a homoclinic orbit in the upper half-plane (see Fig. 4.3).

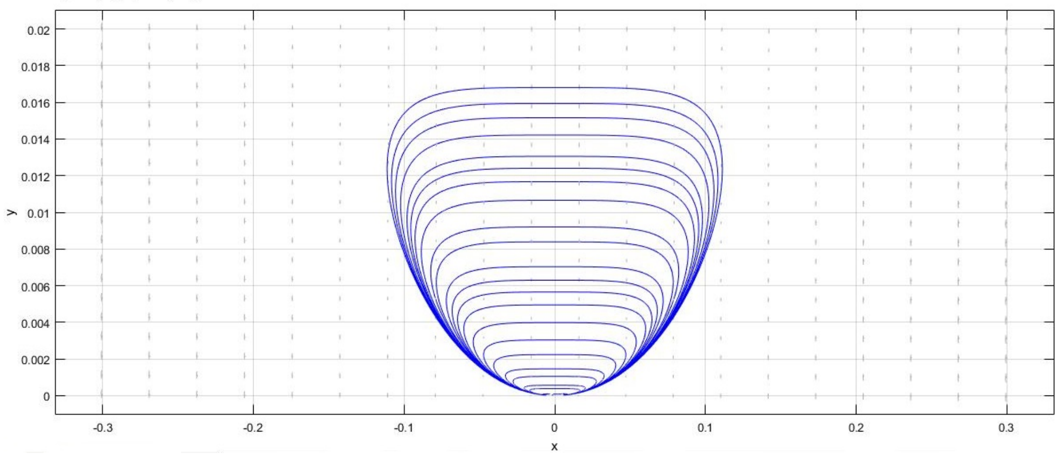

Figure 4.3: Phase portrait for system (4.8).

Example 4.9. Consider system (1.1) with

$$F(x) = x^m \quad (m > 0 \text{ and even number}), \qquad Q(y) = y^3$$
$$P(u) = u^3 \quad \text{and} \quad g(x) = |x^q| \operatorname{sgn}(x) \quad \text{with } q = \frac{10}{3}m + 1.$$

By choosing $k = \frac{1}{2}$, $\delta = \sqrt{\frac{q-3m+1}{8\sqrt[3]{2}}}$ and using Corollary 4.2 we have:

$$\frac{1}{Q^{-1}(kF(x))} \int_0^x \frac{-g(\eta)}{P((k-1)F(\eta))} d\eta = 8\sqrt[3]{2}\left(\frac{\int_0^x \eta^{q-3m} d\eta}{x^{\frac{m}{3}}}\right) = \frac{8\sqrt[3]{2}}{q-3m+1} x^{q-\frac{10}{3}m+1} < 1$$

for $0 < |x| < \sqrt{\frac{q-3m+1}{8\sqrt[3]{2}}}$.

Thus, system (1.1) has homoclinic orbit in the upper half-plane.

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