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On integrability and cyclicity of cubic systems

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Abstract. In this paper we study the integrability of a few families of the complex cubic system. We have obtained necessary and sufficient conditions for existence of a local analytic first integral. Sufficiency of the obtained conditions was proven using different methods: time-reversibility, Darboux integrability and others. Using the obtained results on integrability of complex cubic system, we have obtained results for corresponding real cubic systems. Then the study of bifurcation of limit cycles from each component of the center variety of real system was performed.

Keywords: two dimensional systems, cubic systems, integrability, cyclicity, limit cycles.

2020 Mathematics Subject Classification: 34C05, 37C10.

1 Introduction

One of the main problems of qualitative theory is the problem of integrability. The integrability is not often seen phenomena, but never the least less important. A first integral determines the phase portrait of the plane system and for higher dimensional systems first integral can be used to reduce the dimension of the system, hence the importance. This problem can be linked to another problem of qualitative theory, the problem of distinguishing between a center or a focus. The so-called center problem goes back to Dulac [19], who published in year 1908 a paper on integrability of real quadratic ones. The integrability problem for quadratic system is resolved by Dulac, Kapteyn and others, see [19, 30–32, 39, 48, 50, 51]. Since the publication of Dulac's work, a lot of studies have been made on higher degrees systems, real and complex systems. The integrability conditions for some cubic systems were presented in [4, 14, 17, 18, 22, 36–38, 43, 47] and for results on higher degree systems see [5, 6, 8, 23, 24, 45].

When the systems that contain a center are known, there appears the question: "What is the bound of the number of limit cycles that can bifurcate from the center under small perturbation of parameters of the system?" This is a part of the 16th Hilbert's problem, one of the twenty-three problems introduced by David Hilbert in 1900. It is stated as: "What is the maximum number of limit cycles of system $\dot{x} = P_n, \quad \dot{y} = Q_n$, where P_n and Q_n are polynomial of degree n or less? What are possible relative positions of the limit cycles?"

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In attempt to solve this open Hilbert's problem, the cyclicity problem became one of the main problems in the qualitative theory of differential equations (survey by J. Li, [34]).

The beginning of the study of cyclicity problem goes back to Bautin, who introduced the concept of cyclicity [3]. In the seminar paper of Bautin it was proven that the minimal bound on the number of limit cycles of quadratic system is 3. Since then a lot of studies were made on this problem. For quadratic systems it was believed for some time that there are only 3 limit cycles that can bifurcate, but some examples of quadratic systems with 4 limit cycles were constructed [7,49]. Due to the faulty proof of Dulac on the fixed number of limit cycles of fixed polynomial system, see [19], was his statement a big uncertainty for some time. But one step closer to reviling the correctness of it were Chicone and Shafer [9] in year 1983, where it was proven that a fixed quadratic system has a finite number of limit cycles in any bounded region. The result was extended to the whole phase plane by Bamón [2] and Romanovski [42]. Dulac's Theorem for an arbitrary polynomial system was then proven by Ecalle [20] and Il'yaschenko [27]. Even though a lot of studies on this problem is done, the question on the uniform bound on the number of limit cycles in polynomial systems of fixed degree remains unknown. For more results on cyclicity see [25,26,28,33,44,46,52–55,57].

In this paper we present results of integrability of a complex family of cubic polynomial systems of the following form

$$\begin{aligned} \dot{x} &= x - a_{10}x^2 - a_{20}x^3 - a_{11}x^2y - a_{02}xy^2 - a_{-13}y^3, \\ \dot{y} &= -y + b_{01}y^2 + b_{3,-1}x^3 + b_{20}x^2y + b_{11}xy^2 + b_{02}y^3. \end{aligned} \tag{1.1}$$

The computations for the general family (1.1) were complicated, hence we studied four different subfamilies of it. We explore integrability of the systems (1.1) where

$$\begin{aligned} &1)\ a_{-13} = b_{3,-1} = 1, \quad 2)\ a_{-13} = b_{3,-1} = 0, \quad 3)\ a_{-13} = 1, \quad b_{3,-1} = 0, \\ &4)\ a_{-13} = 0, \quad b_{3,-1} = 1. \end{aligned} \tag{1.2}$$

By choosing these specific subfamilies we enable determination of general conditions for integrability of complex systems of the form (1.1). In our case it is only necessary to study three of four cases, since the involution $a_{ij} \leftrightarrow b_{ji}$ transforms case 3) into case 4). As it will be shown in Section 3, obtained conditions for these subsystems can be transformed to more general system, where a_{-13} and $b_{3,-1}$ are arbitrary. The approach is describe into details in the same section.

The main result of this paper is presented here.

Theorem 1.1. *The system* (1.1) *is integrable if and only if one of the following conditions holds:*

1. $a_{11} = a_{-13} = a_{02} = b_{11} = b_{02} = 0,$
2. $a_{11} = a_{-13} = a_{02} = b_{11} = b_{3,-1} = b_{20} = 0,$
3. $a_{11} = a_{20} = b_{11} = b_{3,-1} = b_{20} = 0,$
4. $a_{11} - b_{11} = a_{-13} = b_{3,-1} = a_{20} + b_{20} = a_{02} + b_{02} = 0,$
5. $a_{11} - b_{11} = a_{20}^2a_{-13} - b_{02}^2b_{3,-1} = a_{02}b_{02}b_{3,-1} - a_{20}b_{20}a_{-13} = a_{02}a_{20} - b_{20}b_{02} = a_{02}^2b_{3,-1} - b_{20}^2a_{-13} = a_{10}^2b_{02} - a_{20}b_{01}^2 = a_{10}^2a_{-13}b_{20} - a_{02}b_{01}^2b_{3,-1} = a_{10}^2a_{20}a_{-13} - b_{02}b_{01}^2b_{3,-1} = a_{10}^2a_{02} - b_{01}^2b_{20} = a_{10}^4a_{-13} - b_{01}^4b_{3,-1} = 0,$

6. $a_{11} = a_{10} = b_{01} = b_{11} = 3a_{-13}b_{3,-1} + 4b_{20}b_{02} = a_{20} + 3b_{20} = 3a_{02} + b_{02} = 0$,
7. $a_{11} - b_{11} = a_{10} = b_{01} = a_{02} - 3b_{02} = 3a_{20} - b_{20} = 0$.

Using obtained components of center variety of complex system (1.1), we have computed the center variety of the general real system which complexification is complex systems (1.1), Theorem 4.1. In Section 4 we have researched the cyclicity of each real component.

2 Preliminaries

Let us study the system

$$\begin{aligned} \dot{u} &= au + bv + f_1(u,v), \\ \dot{v} &= cu + dv + f_2(u,v). \end{aligned} \tag{2.1}$$

The behavior of the nondegenerate singular point at the origin of two-dimensional systems (2.1) is the same as for the linearized system of (2.1), that is the system

$$\dot{u} = au + bv, \quad \dot{v} = cu + dv,$$

except in the case of center. In the case of two purely imaginary eigenvalues of the linearized system the singularity can be either a focus or a center. In that case some additional study needs to be done.

The important theorem, which is the link between the center-focus problem and the integrability problem, studied in this paper, is the Poincaré–Lyapunov Theorem [35,40].

It states the following:

Theorem 2.1. *The system*

$$\begin{aligned} \dot{u} &= \lambda u - v + \tilde{P}(u,v) = \lambda u - v + \sum_{j+k=2}^{n} A_{jk} u^j v^k, \\ \dot{v} &= u + \lambda v + \tilde{Q}(u,v) = u + \lambda v + \sum_{j+k=2}^{n} B_{jk} u^j v^k. \end{aligned} \tag{2.2}$$

on $\mathbb{R}^2$ has a center in the origin if and only if it there exists the a formal first integral of the form $\psi(u,v) = u^2 + v^2 + \cdots$

By transformation $x = u + iv$ the real system can be transformed to

$$\dot{x} = ix + P\left(\frac{(x+\bar{x})}{2}, \frac{(x-\bar{x})}{2i}\right) + iQ\left(\frac{(x+\bar{x})}{2}, \frac{(x-\bar{x})}{2i}\right) = i(x + X_1(x,\bar{x})).$$

The complex system obtained after (complex) time transformation $idt = d\tau$ is

$$\dot{x} = \lambda x + i\left(x - \sum_{p+q=2}^{n} a_{pq} x^{p+1} \bar{x}^q\right). \tag{2.3}$$

The system (2.3) for $\lambda = 0$, with $\bar{x} \to y$, $\bar{a}_{pq} \to b_{qp}$ and after time rescaling is written as

$$\begin{aligned} \dot{x} &= x - \sum_{p+q=2}^{n} a_{pq} x^{p+1} y^q = P_1(x,y), \\ \dot{y} &= -y + \sum_{p+q=2}^{n} b_{qp} x^q y^q = Q_1(x,y), \end{aligned} \tag{2.4}$$

where $P_1(x,y)$ and $Q_1(x,y)$ are polynomials of degree at most n.

The system (2.4) is *locally analytically integrable* if and only if it admits a formal first integral in the form

$$\psi(x,y) = xy + \sum_{l+m\geq 3} v_{l-1,m-1} x^l y^m. \tag{2.5}$$

Since the first integral is constant on any solution, it is obvious that it needs to satisfy $\mathcal{X}\psi(x,y) = \frac{\partial\psi}{\partial x}P_1 + \frac{\partial\psi}{\partial y}Q_1 \equiv 0$.

The construction of the first integral in the form (2.5) yields a series for which $\mathcal{X}\psi(x,y)$ reduces to

$$\mathcal{X}\psi(x,y) = \frac{\partial\psi}{\partial x}P_1 + \frac{\partial\psi}{\partial y}Q_1 := \sum_{k_1+k_2\geq 2} g_{k_1,k_2} x^{k_1} y^{k_2} \dots \tag{2.6}$$

The coefficients g_{k_1,k_2} of series (2.6) can be obtain with some computations from

$$\begin{aligned}\mathcal{X}\psi(x,y) = &\left(y + \sum_{l+m\geq 3} l v_{l-1,m-1} x^{l-1} y^m\right)\left(x - \sum_{p+q=2}^{n} a_{pq} x^{p+1} y^q\right)\\ &+\left(x + \sum_{l+m\geq 3} m v_{l-1,m-1} x^l y^{m-1}\right)\left(-y + \sum_{p+q=2}^{n} b_{qp} x^q y^q\right)\end{aligned} \tag{2.7}$$

and are of the form

$$g_{k_1,k_2} = (k_1-k_2)v_{k_1,k_2} - \sum_{\substack{s_1+s_2=0,\\ s_1,s_2\geq -1}}^{k_1+k_2-1} ((s_1+1)a_{k_1-s_1,k_2-s_2} - (s_2+1)b_{k_1-s_1,k_2-s_2})v_{s_1,s_2}. \tag{2.8}$$

In order for the series $\psi(x,y)$ to be a first integral each coefficient g_{k_1,k_2} must be equal to zero. By step-by-step construction of series (2.5), we see that for $k_1 \neq k_2$ the coefficients $v_{l,m}$ can be chosen so that $g_{k_1,k_2} = 0$. But when $k_1 = k_2 = i$ this is not the case and g_{k_1,k_2} depends on previous $v_{l,m}$. The polynomial of coefficients of the system (2.4) appearing in (2.6),

$$g_{i,i} = \sum_{\substack{s_1+s_2=0,\\ s_1,s_2\geq -1}}^{2k-1} ((s_1+1)a_{k-s_1,k-s_2} - (s_2+1)b_{k-s_1,k-s_2})v_{s_1,s_2},$$

is called *i-th focus quantity* and the ideal $\mathcal{B} = \langle g_{1,1}, g_{2,2}, \dots\rangle$ is called the *Bautin ideal*. The ideal generated by the first k focus quantities is denoted by $\mathcal{B}_k$. The variety of the ideal $\mathcal{B}$, $\mathbf{V}(\mathcal{B})$, is called *the center variety*.

The ideals $\mathcal{B}_1, \mathcal{B}_2, \dots$ form the ascending chain of ideals,
$\mathcal{B}_1 \subseteq \dots \subseteq \mathcal{B}_{k-1} \subseteq \mathcal{B}_k \subseteq \dots$, and by the Hilbert Basis Theorem, this chain stabilizes at some k.

Hence in order to obtain subfamilies of the system (1.1) which are locally integrable it is necessary to compute irreducible decomposition of $\mathbf{V}(\mathcal{B}_k)$, where k is the number for which the ascending chain of $\mathcal{B}_k$ stabilizes. For obtained conditions it remains to be shown that these conditions are sufficient, i.e. find the first integral of the form (2.5). For more detailed on this see [1,44].

From obtained center variety of any polynomial family one can produce, using different approaches, a bound for the cyclicity of the system. An efficient computational technique which we used in this paper and which allows estimation of the generic cyclicity of a family of centers was described in the paper by Christopher [10].

Before the formulation of the theorem presented in [10], let us explain some notations and give some additional definitions.

Denote with $(\lambda, (A, B))$ the coefficient string $(\lambda, A_{20}, \dots, B_{0n})$ and with $E((\lambda, (A, B)))$ the space of parameters of the family (2.2). For the family (2.3) the coefficient string is $(\lambda, a) = (\lambda, a_{p_1 q_1}, \dots, a_{p_l q_l})$, where l is the number of coefficients of the system (2.3) and $E((\lambda, a))$ is the space of parameters. By $g_{kk}^{\mathbb{R}}$ the polynomial obtained by substitution of coefficients b_{ji} with $\overline{a}_{ij}$ in the polynomial g_{kk} is denoted and let $\mathcal{B}_k^{\mathbb{R}}$ be the ideal $\mathcal{B}_k^{\mathbb{R}} = \langle g_{11}^{\mathbb{R}}, g_{22}^{\mathbb{R}}, \dots, g_{kk}^{\mathbb{R}} \rangle$.

Since the parameters of the system (2.2) and of the system (2.3) are connected, the definition is given for the complex system (2.3).

Definition 2.2. For parameters (λ, a), let $n((\lambda, a), \epsilon)$ denote the number of limit cycles of the corresponding system (2.3) that lie wholly within an ϵ-neighborhood of the origin. The singularity at the origin for the system (2.3) with fixed coefficients $(\lambda^*, a^*) \in E((\lambda, a))$ has *cyclicity c with respect to the space $E((\lambda, a))$*, if there exist positive constant δ_0 and ϵ_0 such that for every pair ϵ and δ satisfying $0 < \epsilon < \epsilon_0$ and $0 < \delta < \delta_0$,

$$\max \{ n((\lambda, a), \epsilon) : |(\lambda, a) - (\lambda^*, a^*)| < \delta \} = c.$$

The approach for the estimation of the number of limit cycles of our system was based on the following theorem by C. Christopher [10]:

Theorem 2.3. *Suppose that s is a point on the center variety and that* $\operatorname{rank} J_p(\mathcal{B}_k^{\mathbb{R}}) = k$. *Then s lies on a component of the center variety of codimension at least k and there are bifurcations of* (2.3) *which produce k limit cycles locally from the center corresponding to the parameter value s.*

If furthermore, we know that s lies on a component of the center variety of codimension k, then s is smooth point of the variety, and the cyclicity of the center for the parameter value s is exactly $k-1$.

In the latter case, $k-1$ is also the cyclicity of generic point on this component of the center variety.

3 Results on integrability

Before presenting the main results on integrability we recall some important methods used approaching the problem of integrability.

The so-called Darboux method is based on Darboux factors and using them we can sometimes construct the Darboux integrals, more on this can be found in [11, 12, 44].

Definition 3.1. A nonconstant polynomial $f(x, y) \in \mathbb{C}[x, y]$ is called a *Darboux factor* of system (2.4) if there exists a polynomial $K(x, y) \in \mathbb{C}[x, y]$ such that

$$\mathcal{X} f = \frac{\partial f}{\partial x} P_1 + \frac{\partial f}{\partial y} Q_1 = K f. \tag{3.1}$$

The polynomial $K(x, y)$ is called a *cofactor* of $f(x, y)$ and it has degree at most n.

If sufficient number of Darboux factors are found, then so-called *Darboux first integral* can be constructed.

Let $f_1, \dots, f_s$ be Darboux factors such that $\alpha_j \in \mathbb{C}$ for $1 \leq j \leq s$. A first integral of system (2.4) of the form

$$H = f_1^{\alpha_1} \dots f_s^{\alpha_s}$$

is called a *Darboux first integral* of system (2.4).

For two specific systems of the form (2.4), *Hamiltonian system* and *time-reversible system*, it is known that the singularity of the origin is a center, see [44].

We recall that: System (2.4) is a *Hamiltonian system* if there is a function $H : \mathbb{C}^2 \to \mathbb{C}$ called Hamiltonian, such that $P_1 = -H_y$ and $Q_1 = H_x$.

Clearly, the Hamiltonian is a first integral of the system.

The definition of time-reversibility of the system is the following.

Definition 3.2. The system $\frac{dz}{dt} = F(z)$, where $z = (x, y) \in \mathbb{C}^2$, is time-reversible if there exists a transformation $T(x, y) = (\gamma x, \gamma^{-1} y)$, for $\gamma \in \mathbb{C} \setminus \{0\}$, such that

$$\frac{d(Tz)}{dt} = F(Tz).$$

In the proofs of the following theorems the results of [29] on time-reversibility of the cubic systems will be important.

Next we present the results on integrability of system (1.1).

Theorem 3.3. *System (1.1) with $a_{-13} = b_{3,-1} = 1$ is integrable if and only if one of the following conditions holds:*

1. $a_{11} - b_{11} = b_{01} = a_{10} = a_{02} - 3b_{02} = 3a_{20} - b_{20} = 0,$
2. $a_{11} - b_{11} = a_{20} + b_{02} = a_{02} + b_{20} = a_{10}^2 + b_{01}^2 = 0,$
3. $a_{11} - b_{11} = a_{20} - b_{02} = a_{02} - b_{20} = a_{10} - b_{01} = 0,$
4. $a_{11} - b_{11} = a_{20} - b_{02} = a_{02} - b_{20} = a_{10} + b_{01} = 0,$
5. $a_{11} = b_{11} = a_{10} = b_{01} = a_{20} + 3b_{20} = 3a_{02} + b_{02} = 4b_{20}b_{02} + 3 = 0.$

Proof. The computation of necessary conditions

With the computer algebra system Mathematica we were able to compute first nine non-zero focus quantities using algorithm presented in [44]. Due to the large size of the focus quantities, we present here only two

$$\begin{aligned}
g_{11} &= a_{01}a_{10} + a_{11} - b_{01}b_{10} - b_{11};\\
g_{22} &= (24a_{01}^2a_{10}^2 + 24a_{01}a_{10}a_{11} + 6a_{01}^2a_{20} + 3a_{02}a_{20} + 2a_{10}a_{-12}a_{20}\\
&\quad - 18a_{01}a_{10}^2b_{01} - 18a_{10}a_{11}b_{01} - 3a_{01}a_{20}b_{01} - 27a_{01}^2a_{10}b_{10}\\
&\quad + 3a_{02}a_{10}b_{10} - 27a_{01}a_{11}b_{10} + 2a_{10}^2a_{-12}b_{10} + 5a_{-12}a_{20}b_{10}\\
&\quad + 21a_{11}b_{01}b_{10} + 18a_{10}b_{01}^2b_{10} + 3a_{10}b_{02}b_{10} + 3a_{10}a_{-12}b_{10}^2\\
&\quad + 27a_{01}b_{01}b_{10}^2 - 24b_{01}^2b_{10}^2 - 6b_{02}b_{10}^2 - 2a_{-12}b_{10}^3 - 21a_{01}a_{10}b_{11}\\
&\quad + 18a_{10}b_{01}b_{11} + 27a_{01}b_{10}b_{11} - 24b_{01}b_{10}b_{11} + 2a_{10}a_{-12}b_{20}\\
&\quad - 3a_{01}b_{01}b_{20} - 3b_{02}b_{20} - 3a_{-12}b_{10}b_{20} + 2a_{01}^3b_{2,-1} + 3a_{01}a_{02}b_{2,-1}\\
&\quad - 4a_{11}a_{-12}b_{2,-1} + 2a_{10}a_{-13}b_{2,-1} - 3a_{01}^2b_{01}b_{2,-1} - 2a_{02}b_{01}b_{2,-1}\\
&\quad - 2a_{01}b_{01}^2b_{2,-1} - 5a_{01}b_{02}b_{2,-1} - 2b_{01}b_{02}b_{2,-1} - a_{-13}b_{10}b_{2,-1}\\
&\quad + 4a_{-12}b_{11}b_{2,-1} + a_{01}a_{-12}b_{3,-1} - 2a_{-12}b_{01}b_{3,-1})/3.
\end{aligned}$$

To obtain the necessary conditions for system to be integrable, the irreducible decomposition of integrability variety, $\mathbf{V}(B_9)$ needs to computed. The irreducible decomposition was computed using Singular [15] routine `minAssGTZ` [16].

Since the computation of irreducible decomposition is difficult, in many cases it is necessary to work in modular arithmetics instead of over the field of rational numbers. Since the obtained ideals have rational coefficients, the rational reconstruction needs to be done. For more informations on rational reconstruction algorithm see [53]. Working with modular arithmetics sometimes produces wrong conditions or do not produces all conditions, some can be lost. For this reason additional few steps need to be done.

The approach which can be used to check the conditions was suggested in [41].

The irreducible decomposition was computed over four different characteristics; 7919, 32003, 100109 and 104729. The approach described in [41] was not done completely, but in many cases computations are difficult even for more capable computers. But with high probability the list of conditions of Theorem 3.3 is complete.

The existence of the analytic first integral

Now we prove that under each of the conditions of Theorem 3.3 the system has a first integral.

Case 1. The system under conditions $a_{11} - b_{11} = b_{01} = a_{10} = a_{02} - 3b_{02} = 3a_{20} - b_{20} = 0$ is

$$\dot{x} = x - a_{20}x^3 - b_{11}x^2y - a_{02}xy^2 - y^3,$$
$$\dot{y} = -y + x^3 + b_{11}xy^2 + 3a_{20}x^2y + \frac{a_{02}}{3}y^3.$$

It is a Hamiltonian system. The first integral is $\psi(x,y) = xy - \frac{x^4}{4} - \frac{y^4}{4} - a_{20}x^3y - \frac{b_{11}}{2}x^2y^2 - \frac{a_{02}}{3}xy^3$.

Case 2. Conditions $a_{11} - b_{11} = a_{20} + b_{02} = a_{02} + b_{20} = a_{10}^2 + b_{01}^2 = 0$ satisfy the conditions for time-reversible cubic system written in [44], hence the system is time-reversible.

Case 3 and *Case 4.* systems are of form

$$\dot{x} = x - a_{10}x^2 - a_{20}x^3 - a_{11}x^2y - a_{02}xy^2 - y^3,$$
$$\dot{y} = -y \pm a_{10} + x^3 + a_{11}xy^2 + a_{02}x^2y + a_{20}y^3.$$

The system, the same as in Case 2, is time-reversible, since it satisfies the conditions for time-reversible cubic system.

Case 5. The conditions $a_{11} = b_{11} = a_{10} = b_{01} = a_{20} + 3b_{20} = 3a_{02} + b_{02} = 4b_{20}b_{02} + 3 = 0$ yield the system

$$\dot{x} = x - \frac{9}{4b_{02}}x^3 + \frac{b_{02}}{3}xy^2 - y^3,$$
$$\dot{y} = x^3 - y - \frac{3}{4b_{02}}x^2y + b_{02}y^3.$$

We obtain three Darboux factors of this system, one of degree four,

$$l_1(x,y) = 1 - \frac{3}{2b_{02}}x^2 + \frac{b_{02}^2}{9}x^4 - \frac{9}{4b_{02}^2}xy - \frac{4b_{02}^2}{9}xy + \frac{2b_{02}}{3}x^3y - \frac{2b_{02}}{3}y^2 + \frac{3}{2}x^2y^2 + \frac{3}{2b_{02}}xy^3 + \frac{9}{16b_{02}^2}y^4,$$

and two of degree six,

$$l_2(x,y) = 1 - \frac{9}{2b_{02}}x^2 + \frac{81}{16b_{02}^2}x^4 - \frac{b_{02}}{3}x^6 + 2b_{02}x^3y - \frac{3}{2}x^5y - 2b_{02}y^2 + \frac{3}{2}x^2y^2 - \frac{9}{4b_{02}}x^4y^2 + \frac{9}{2b_{02}}xy^3 - \frac{9}{8b_{02}^2}x^3y^3 - \frac{2b_{02}^2}{9}x^3y^3 + b_{02}^2y^4 - b_{02}x^2y^4 - \frac{3}{2}xy^5 + -\frac{3}{4b_{02}}y^6$$

and

$$l_3(x,y) = 1 - \frac{9x^2}{4b_{02}} - \frac{216b_{02}^2xy}{81+16b_{02}^4} - b_{02}y^2 + \frac{54b_{02}^2x^4}{81+16b_{02}^4} + \frac{9}{2}x^2y^2 + \frac{54b_{02}^2y^4}{81+16b_{02}^4} + \frac{b_{02}(243+16b_{02}^4)x^3y}{81+16b_{02}^4} + \frac{27(27+16b_{02}^4)xy^3}{4b_{02}(81+16b_{02}^4)} - \frac{8b_{02}^5x^6}{3(81+16b_{02}^4)} - \frac{24b_{02}^4x^5y}{81+16b_{02}^4} - \frac{90b_{02}^3x^4y^2}{81+16b_{02}^4} - \frac{180b_{02}^2x^3y^3}{81+16b_{02}^4} - \frac{405b_{02}x^2y^4}{2(81+16b_{02}^4)} - \frac{243xy^5}{2(81+16b_{02}^4)} - \frac{243y^6}{8b_{02}(81+16b_{02}^4)}.$$

Two of these three Darboux factors construct the first integral

$$\psi(x,y) = C(l_1^3l_2 - l_1^3l_2^{-1}) = xy + \dots,$$

where $C = \frac{6b_{02}^2}{81+16b_{02}^4}$ and $81 + 16b_{02}^4 \neq 0$.

In case $81 + 16b_{02}^4 = 0$, the first integral is of form

$$\psi(x,y) = \frac{1}{4}(4 - 4(-1)^{\frac{3}{4}}x^2 + ix^4 - 4(-1)^{\frac{1}{4}}x^3y + 4(-1)^{\frac{1}{4}}y^2 + 6x^2y^2 + 4(-1)^{\frac{3}{4}}xy^3 - iy^4). \quad \square$$

Theorem 3.4. *The system* (1.1) *with* $a_{-13} = b_{3,-1} = 0$ *is integrable if and only if one of the following conditions holds:*

1. $a_{11} = b_{11} = b_{20} = a_{20} = 0,$
2. $a_{11} = b_{11} = b_{20} = a_{02} = 0,$
3. $a_{11} = b_{11} = b_{02} = a_{02} = 0,$
4. $a_{11} - b_{11} = a_{02}a_{20} - b_{20}b_{02} = a_{20}b_{01}^2 - a_{10}^2b_{02} = a_{10}^2a_{02} - b_{01}^2b_{20} = 0,$
5. $a_{11} - b_{11} = a_{20} + b_{20} = a_{02} + b_{02} = 0.$

Proof. The computation of necessary conditions The computation of irreducible decomposition of variety of ideal $\mathcal{B}_9$ with additional conditions $a_{-13} = b_{3,-1} = 0$, was not too extensive and difficult, hence it was done over the field of rational numbers. This way conditions of Theorem 3.4 were obtained.

The existence of the analytic first integral

The system (1.1) with $a_{-13} = b_{3,-1} = 0$ is Lotka–Volterra system, which was studied in [18].

Case 1. The system under conditions $a_{11} = b_{11} = b_{20} = a_{20} = 0$ is equivalent to the system of Case 4 of Theorem 1.4 in [18].

Case 2. Conditions $a_{11} = b_{11} = b_{20} = a_{02} = 0$ yield the Case 3 of Theorem 1.4 in [18].

Case 3. Conditions $a_{11} = b_{11} = b_{02} = a_{02} = 0$ yield the system that is equivalent to the system of Case 5 of Theorem 1.4 in [18].

Case 4. The Case 4 is Case 2 of Theorem 1.4 in [18].

Case 5. Conditions $a_{11} - b_{11} = a_{20} + b_{20} = a_{02} + b_{02} = 0$ are conditions of Case 1 of Theorem 1.4 in [18]. □

Theorem 3.5. *The system* (1.1) *with $a_{-13} = 1$ and $b_{3,-1} = 0$ is integrable if and only if one of the following conditions holds:*

1. $a_{11} - b_{11} = b_{20} = a_{20} = a_{10} = 0,$
2. $a_{11} = b_{11} = b_{20} = a_{20} = 0,$
3. $a_{11} - b_{11} = a_{10} = b_{01} = 3a_{20} - b_{20} = a_{02} - 3b_{02} = 0,$
4. $a_{11} = b_{11} = a_{20} + 3b_{20} = b_{01} = b_{02} = a_{02} = a_{10} = 0.$

Proof. The computation of necessary conditions

The conditions were obtained similar as in case of Theorem 3.3.

The existence of the analytic first integral

Case 1. The corresponding system for conditions $a_{11} - b_{11} = b_{20} = a_{20} = a_{10} = 0$ is

$$\begin{aligned} \dot{x} &= x - a_{11}x^2y - a_{02}xy^2 - y^3, \\ \dot{y} &= -y + b_{01}y^2 + a_{11}xy^2 + b_{02}y^3. \end{aligned}$$

This system is time-reversible, hence integrable.

Case 2. In this case system is of the form

$$\begin{aligned} \dot{x} &= x - a_{10}x^2 - a_{02}xy^2 - y^3, \\ \dot{y} &= -y + b_{01}y^2 + b_{02}y^3. \end{aligned}$$

Darboux factors found for this system are

$$l_1(x,y) = y,\, l_{2,3}(x,y) = \frac{1}{2}\left(2 - b_{01}y \pm \sqrt{b_{01}^2 + 4b_{02}y}\right),$$

but using them we were not able to construct Darboux first integral or Darboux integrating factor. For this reason we looked for a first integral of the form $\psi(x,y) = \sum_{k=1}^{\infty} f_k(x)y^k$. The function $f_k(x)$ is defined by recursive differential equation

$$(k-2)b_{02}f_{k-2}(x) + (k-1)b_{01}f_{k-1}(x) - kf_k(x) - f'_{k-3}(x) + \\ - a_{02}xf'_{k-2}(x) + x(1 - a_{10}x)f'_k(x) = 0. \quad (3.2)$$

Using induction we show that for every odd number, $k = 2n-1$, is $f_{2n-1}(x) = \frac{p_n(x)}{(-1+a_{10}x)^{2n-1}}$ and for every even number, $k = 2n$, is $f_{2n}(x) = \frac{p_n(x)}{(-1+a_{10}x)^{2n}}$.

Proving first the assumption for odd numbers.

For $k = 1$: $f_1(x) = \frac{-x}{(-1+a_{10}x)}$. Let us assume that the assumption holds for all $l < 2n-1$. We need to show that it holds for $2n-1$. Using assumptions in (3.2) for every $l < 2n-1$ we obtain differential equation

$$\frac{p_n(x)}{x(-1+a_{10}x)^{2n-1}} = \frac{(2n-1)}{x(-1+a_{10}x)} f_{2n-1}(x) + f'_{2n-1}(x),$$

which has solution

$$\begin{aligned} f_{2n-1}(x) &= \frac{x^{2n-1}}{(-1)^{2n-1}(-1+a_{10}x)^{2n-1}} \int \frac{(-1)^{2n-1}p_n(x)(-1+a_{10}x)^{2n-1}}{x^{2n}(-1+a_{10}x)^{2n-1}} dx \\ &= \frac{x^{2n-1}}{(-1+a_{10}x)^{2n-1}} \int \frac{p_n(x)}{x^{2n}} dx = \frac{x^{2n-1}}{(-1+a_{10}x)^{2n-1}} \frac{p_n(x)}{x^{2n-1}} = \frac{p_n(x)}{(-1+a_{10}x)^{2n-1}}. \end{aligned}$$

In the same way this can be proven for even numbers k.

For $k = 2$: $f_2(x) = \frac{b_{01}x}{(-1+a_{10}x)^2}$ and

$$\frac{p_n(x)}{x(-1+a_{10}x)^{2n}} = \frac{2n}{x(-1+a_{10}x)} f_{2n}(x) + f'_{2n}(x)$$

needs to hold. Solving this differential equation we obtain $f_{2n}(x) = \frac{p_n(x)}{(-1+a_{10}x)^{2n}}$, as needed.

Case 3. The system corresponding to conditions $a_{11} - b_{11} = a_{10} = b_{01} = 3a_{20} - b_{20} = a_{02} - 3b_{02} = 0$ is

$$\begin{aligned} \dot{x} &= x - b_{11}x^2y - \frac{b_{20}}{3}x^3 - 3b_{02}xy^2 - y^3, \\ \dot{y} &= -y + b_{11}xy^2 + b_{20}x^2y + b_{02}y^3. \end{aligned}$$

This is Hamiltonian system and the first integral is

$$\psi(x,y) = xy - \frac{b_{20}}{3}x^3y - \frac{b_{11}}{2}x^2y^2 - b_{02}xy^3 - \frac{y^4}{4}.$$

Case 4. The system in this case is

$$\begin{aligned} \dot{x} &= x - a_{20}x^3 - y^3, \\ \dot{y} &= y(-1 + \frac{a_{20}}{3}x^2). \end{aligned}$$

Darboux factors of this system are $l_1(x,y) = y$, $l_2(x,y) = x - \frac{y^3}{4}$ and two Darboux factors of degree six,

$$l_3(x,y) = \frac{1}{9}(9 - 18a_{20}x^2 + 9a_{20}x^4 + 18a_{20}xy^3 - 2a_{20}^2x^3y^3 - 3a_{20}y^6)$$

and

$$l_4(x,y) = \frac{1}{6}(6 - 6a_{20}x^2 + 6a_{20}xy^3 - a_{20}y^6).$$

Using three of four Darboux factors we obtain first integral

$$\psi(x,y) = l_1 l_2 l_3^{-\frac{1}{3}} = xy + \cdots$$

□

Studying integrability of the systems of higher degrees is difficult, mostly because of computation of irreducible decomposition. Due to these problem we splitted the research of the system (1.1) to four cases, as explained before in Section 1. The fact is that by the involution of parameters $a_{ij} \leftrightarrow b_{ji}$ we can transforms case 3) of (1.2), where additional conditions are $a_{-13} = 1$ and $b_{3,-1} = 0$, into case 4), where $a_{-13} = 0$ and $b_{3,-1} = 1$. Hence only three of four cases needed to be studied. In theorems 3.3, 3.4 and 3.5 the obtained results are presented and in the proofs all procedures of obtaining these conditions are explained into details.

By fixing some coefficients and splitting the study of the system (1.1), the general conditions of integrability of this system were not obtained. But as it will be explained here the general conditions of integrability of the system (1.1) can be computed using conditions of Theorems 3.3, 3.4 and 3.5.

The main theory behind obtaining the general results is the elimination theory. More on this theory can be read in [13, Chapter 3] or [44, Chapter 1.3]. Before explaining the whole procedure for obtaining the general conditions, some important facts on the elimination theory need to be given.

Definition 3.6. Let $I = \langle f_1, \dots, f_m \rangle$ be ideal in $k[x_1, \dots, x_n]$ (with the implicit ordering of the variables $x_1 > x_2 > \dots > x_n$) and fix $l \in \{0, 1, \dots, n-1\}$. The *l-th elimination ideal of I is the ideal* $I_l = I \cap k[x_{l+1}, x_{l+2}, \dots, x_n]$*. Any point* $(a_{l+1}, \dots, a_n) \in \mathbf{V}(I_l)$ *is called partial solution of the system* $\{f = 0; f \in I\}$.

Geometrically, the elimination is the projection of $\mathbf{V}(I) \subset k^n$ on the lower dimensional subspace k^{n-l}.

The method for computing the elimination ideal I_l is provided in the following theorem.

Theorem 3.7. *Fix the lexicographic term order on the ring* $k[x_1, \dots, x_n]$ *with* $x_1 > x_2 > \dots > x_n$ *and let G be a Gröbner basis for an ideal I of* $k[x_1, \dots, x_n]$ *with respect to this order. Then for every l,* $0 \le l \le n-1$*, the set* $G_l := G \cap k[x_{l+1}, \dots, x_n]$ *is a Gröbner basis for the l-th elimination ideal* I_l.

The procedure of obtaining the general results is based on the following observations.

Taking the variables

$$x_1 \to ax, \quad y_1 \to by$$

changes the system (1.1) into the system

$$\begin{aligned} \dot{x_1} &= x_1 - \alpha_{10}x_1^2 - \alpha_{20}x_1^3 - \alpha_{01}x_1y_1 - \alpha_{11}x_1^2y_1 - \alpha_{-12}y_1^2 - \alpha_{02}x_1y_1^2 - \alpha_{-13}y_1^3, \\ \dot{y_1} &= -y_1 + \beta 2, -1x_1^2 + \beta 3, -1x_1^3 + \beta_{10}x_1y_1 + \beta_{02}x_1^2y_1 + \beta_{01}y_1^2 + \beta_{11}x_1y_1^2 + \beta_{02}y_1^3, \end{aligned} \tag{3.3}$$

where

$$\alpha_{10} = \frac{a_{10}}{a}, \qquad \beta_{2,-1} = \frac{bb_{21}}{a^2},$$

$$\alpha_{20} = \frac{a_{20}}{a^2}, \qquad \beta_{3,-1} = \frac{bb_{3,-1}}{a^3},$$

$$\alpha_{01} = \frac{a_{01}}{b}, \qquad \beta_{10} = \frac{b_{10}}{a},$$

$$\alpha_{11} = \frac{a_{11}}{ab}, \qquad \beta_{20} = \frac{b_{20}}{a^2},$$

$$\alpha_{-12} = \frac{aa_{-12}}{b^2}, \qquad \beta_{01} = \frac{b_{01}}{b},$$

$$\alpha_{02} = \frac{a_{02}}{b^2}, \qquad \beta_{11} = \frac{b_{11}}{ab},$$
$$\alpha_{-13} = \frac{aa_{-13}}{b^3}, \qquad \beta_{02} = \frac{b_{02}}{b^2}.$$

The focus quantities of both systems, (1.1) and (3.3), are different only by the constant factor. This constant factor does not make a difference for the center variety, hence the irreducible decomposition of both varieties generates the same conditions.

As it is seen from the system (3.3), each nonzero coefficient can be rescaled so that obtained coefficient is equal to 1. Similar, coefficients can be set equal to zero.

Hence by splitting our studies as presented in Section 1, the general results were not lost. These can be obtained with the approach described below.

For the case 1), where $a_{-13} = b_{3,-1} = 1$, the coefficients α_{-13} and $\beta_{3,-1}$ need to fulfil $\alpha_{-13} = ab^{-3}$ and $\beta_{3,-1} = a^{-3}b$, with additional restrictions $a \neq 0$ and $b \neq 0$. These additional restrictions can be written in the term of polynomial as $1 - wa$, respectively $1 - vb$. The other conditions of Theorem 3.3 change regarding $a_{i,j} = \alpha_{i,j}a^{-i}b^j$ and $b_{i,j} = \beta_{i,j}a^{-i}b^j$, where $i, j \in \{-1, \dots, 3\}$. This way ideals $I_1, \dots, I_5 \in \mathbb{C}[w, v, a, b, A, B]$, where $A = \{a_{10}, a_{20}, a_{11}, a_{02}, a_{-13}\}$ and $B = \{b_{01}, b_{02}, b_{11}, b_{20}, b_{3,-1}\}$ are formed,

$$\begin{aligned}
I_1 = \langle &1 - wa, 1 - vb, ab(a_{11} - b_{11}), bb_{01}, aa_{10}, b^2(a_{02} - 3b_{02}), a^2(3a_{20} - b_{20}), \\
&- a + b^3a_{-13}, -b + a^3b_{3,-1}\rangle \\
I_2 = \langle &1 - wa, 1 - vb, ab(a_{11} - b_{11}), a^2a_{20} + b^2b_{02}, b^2a_{02} + a^2b_{20}, a^2a_{10}^2 + b^2b_{01}^2, \\
&- a + b^3a_{-13}, -b + a^3b_{3,-1}\rangle \\
I_3 = \langle &1 - wa, 1 - vb, ab(a_{11} - b_{11}), a^2a_{20} - b^2b_{02}, b^2a_{02} - a^2b_{20}, aa_{10} - bb_{01}, \\
&- a + b^3a_{-13}, -b + a^3b_{3,-1}\rangle \\
I_4 = \langle &1 - wa, 1 - vb, ab(a_{11} - b_{11}), a^2a_{20} - b^2b_{02}, b^2a_{02} - a^2b_{20}, aa_{10} + bb_{01}, \\
&- a + b^3a_{-13}, -b + a^3b_{3,-1}\rangle \\
I_5 = \langle &1 - wa, 1 - vb, aba_{11}, abb_{11}, aa_{10}, bb_{01}, a^2(a_{20} + 3b_{20}), b^2(3a_{02} + b_{02}, \\
&3 + 4a^2b^2b_{02}b_{20}, -a + b^3a_{-13}, -b + a^3b_{3,-1}\rangle.
\end{aligned}$$

Similar we obtain ideals $I_6, \dots, I_{10}$ from conditions of Theorem 3.4. Ideals $I_{11}, \dots, I_{14}$ were gained from Theorem 3.5 and $I_{15}, \dots, I_{18}$ by involution of coefficients in conditions of Theorem 3.5.

From the obtained ideals $I_1, \dots, I_{18}$ we eliminate, using Singular routine `eliminate`, variables w, v, a and b. The elimination ideals are

$$J_1' = I_1 \cap \mathbb{C}[a_{10}, a_{20}, a_{11}, a_{02}, a_{-13}, b_{01}, b_{02}, b_{11}, b_{20}, b_{3,-1}], \dots, J_{18}'.$$

Then we compute irreducible decomposition (Singular routine `minAssGTZ`) of each obtained eliminated ideal, gaining ideals $J_1, \dots, J_{18}$:

$$\begin{aligned}
J_1 = \langle &b_{01}, a_{11} - b_{11}, 3a_{20} - b_{20}, a_{02} - 3b_{02}, a_{10}\rangle, \\
J_2 = \langle &a_{11} - b_{11}, a_{20}^2a_{-13} - b_{02}^2b_{3,-1}, a_{02}b_{02}b_{3,-1} - a_{20}a_{-13}b_{20}, a_{02}a_{20} - b_{20}b_{02}, \\
&a_{02}^2b_{3,-1} - a_{-13}b_{20}^2, a_{10}^2b_{02} - a_{20}b_{01}^2, a_{10}^2a_{-13}b_{20} - a_{02}b_{01}^2b_{3,-1}, \\
&a_{10}^2a_{20}a_{-13} - b_{01}^2b_{02}b_{3,-1}, a_{10}^2a_{02} - b_{01}^2b_{20}, a_{10}^4a_{-13} - b_{01}^4b_{3,-1}\rangle,
\end{aligned}$$

$$\begin{aligned}
J_3 = \langle & a_{11}-b_{11}, a_{20}^2a_{-13}-b_{02}^2b_{3,-1}, a_{02}b_{02}b_{3,-1}-a_{20}a_{-13}b_{20}, a_{02}a_{20}-b_{20}b_{02},\\
& a_{02}^2b_{3,-1}-a_{-13}b_{20}^2, a_{10}^2b_{02}-a_{20}b_{01}^2, a_{10}^2a_{-13}b_{20}-a_{02}b_{01}^2b_{3,-1},\\
& a_{10}^2a_{20}a_{-13}-b_{01}^2b_{02}b_{3,-1}, a_{10}^2a_{02}-b_{01}^2b_{20}, a_{10}^4a_{-13}-b_{01}^4b_{3,-1}\rangle,\\
J_4 = \langle & a_{11}-b_{11}, a_{20}^2a_{-13}-b_{02}^2b_{3,-1}, a_{02}b_{02}b_{3,-1}-a_{20}a_{-13}b_{20}, a_{02}a_{20}-b_{20}b_{02},\\
& a_{02}^2b_{3,-1}-a_{-13}b_{20}^2, a_{10}^2b_{02}-a_{20}b_{01}^2, a_{10}^2a_{-13}b_{20}-a_{02}b_{01}^2b_{3,-1},\\
& a_{10}^2a_{20}a_{-13}-b_{01}^2b_{02}b_{3,-1}, a_{10}^2a_{02}-b_{01}^2b_{20}, a_{10}^4a_{-13}-b_{01}^4b_{3,-1}\rangle,\\
J_5 = \langle & b_{11}, b_{01}, a_{11}, 3a_{-13}b_{3,-1}+4b_{20}b_{02}, a_{20}+3b_{20}, 3a_{02}+b_{02}, a_{10}\rangle, \ldots
\end{aligned}$$

By computing intersection of obtained ideals J_i, $J=\cap_{i=1}^{18} J_i$ (Singular routine `intersect`) and then using Singular routine `minAssGTZ` to compute irreducible decomposition of $\mathbf{V}(J)$, we obtain list of conditions from Theorem 1.1. For more details on this approach see [21].

4 Cyclicity of components of the center variety

In this section we will presented results connected to cyclicity of the specific family of real cubic system.

The researched real system was obtained from the complex system (1.1) by setting

$$\begin{aligned}
&a_{10}=A_{10}+iB_{10}, \quad b_{01}=A_{10}-iB_{10}, \quad a_{20}=A_{20}+iB_{20}, \quad b_{02}=A_{20}-iB_{20},\\
&a_{02}=A_{02}+iB_{02}, \quad b_{20}=A_{02}-iB_{02}, \quad a_{11}=A_{11}+iB_{11}, \quad b_{11}=A_{11}-iB_{11},\\
&\qquad a_{-13}=A_{-13}+iB_{-13}, \quad b_{3,-1}=A_{-13}-iB_{-13}.
\end{aligned} \tag{4.1}$$

In the same way, by setting (4.1), the real center variety was obtained from the center variety presented in Theorem 1.1. The studied real system is of the form

$$\begin{aligned}
\dot{x} = i(x &- (A_{10}x^2+A_{20}x^3+A_{11}x^2\bar{x}+A_{02}x\bar{x}^2+A_{-13}\bar{x}^3)\\
&- i(B_{10}x^2+B_{20}x^3+B_{11}x^2\bar{x}+B_{02}x\bar{x}^2+B_{-13}\bar{x}^3)).
\end{aligned} \tag{4.2}$$

Theorem 4.1. *The center variety in $\mathbb{R}^{10}$ of the real system (4.2) consists of the following irreducible components:*

1) $3B_{20}+B_{02}=B_{11}=B_{10}=3A_{20}-A_{02}=A_{10}=0,$

2) $B_{20}-3B_{02}=B_{11}=B_{10}=A_{20}+3A_{02}=A_{11}=A_{10}=A_{-13}^2+B_{-13}^2-4A_{02}^2-4B_{02}^2=0,$

3) $B_{11}=A_{02}B_{20}+A_{20}B_{02}=A_{02}^2B_{-13}-2A_{-13}A_{02}B_{02}-B_{-13}B_{02}^2=A_{20}A_{02}B_{-13}-2A_{-13}A_{20}B_{02}$
$+B_{-13}B_{20}B_{02}=A_{20}^2B_{-13}+2A_{-13}A_{20}B_{20}-B_{-13}B_{20}^2=2A_{10}A_{02}B_{10}+A_{10}^2B_{02}-B_{10}^2B_{02}$
$=2A_{10}A_{20}B_{10}-A_{10}^2B_{20}+B_{10}^2B_{20}=A_{10}^2A_{02}B_{-13}-A_{02}B_{10}^2B_{-13}-2A_{10}^2A_{-13}B_{02}+2A_{-13}B_{10}^2B_{02}$
$+2A_{10}B_{10}B_{-13}B_{02}=A_{10}^2A_{20}B_{-13}-A_{20}B_{10}^2B_{-13}+2A_{10}^2A_{-13}B_{20}-2A_{-13}B_{10}^2B_{20}$
$-2A_{10}B_{10}B_{-13}B_{20}=2A_{02}B_{10}^3B_{-13}+4A_{10}^2A_{-13}B_{10}B_{02}-4A_{-13}B_{10}^3B_{02}+A_{10}^3B_{-13}B_{02}$
$-5A_{10}B_{10}^2B_{-13}B_{02}=2A_{20}B_{10}^3B_{-13}-4A_{10}^2A_{-13}B_{10}B_{20}+4A_{-13}B_{10}^3B_{20}-A_{10}^3B_{-13}B_{20}$
$+5A_{10}B_{10}^2B_{-13}B_{20}=4A_{10}^3A_{-13}B_{10}-4A_{10}A_{-13}B_{10}^3+A_{10}^4B_{-13}-6A_{10}^2B_{10}^2B_{-13}+B_{10}^4B_{-13}=0,$

4) $B_{20}-B_{02}=B_{-13}=B_{11}=A_{02}+A_{20}=A_{-13}=0,$

5) $B_{02}=B_{-13}=B_{11}=A_{02}=A_{11}=A_{-13}=0.$

The dimension of these components is 5, 3, 5, 5, 4, respectively.

Proof. The center variety of the real system (4.2) was obtain from complex variety of Theorem 1.1. The change of coefficients in the way as written at (4.1) and then by elimination of complex unit i from obtained ideals produced the conditions of Theorem 4.1. The conditions 4), 5), 6), 7) of Theorem 1.1 yield conditions 4), 3), 2), 1) of this theorem and the condition 5) was obtain from 2). The other obtained conditions are subvarieties of 3), 4) and 5).

As we can see from 1), 4) and 5) the number of parameters in these components is equal to 5, 5 and 6. Hence the dimension is 5, 5 and 4, since the number of all parameters is 10 and 5 (respectively 5 and 4) parameters are free.

The dimension of remaining components is not obvious as in three cases before. By the Theorem 2 of [13, Chapter 3.3] the upper bound of dimension can be determine from obtained rational parametrization. For the case 2) the parametrization is

$$\begin{aligned}
B_{10} = A_{10} = B_{11} = A_{11} = f_0 = 0, \quad & A_{02} = f_1(u_1, u_2, u_3)/g_2(u_3), \\
B_{02} = f_2(u_1, u_2, u_3)/g_2(u_3), \quad & A_{20} = f_3(u_1, u_2, u_3)/g_2(u_3), \\
B_{20} = f_4(u_1, u_2, u_3)/g_2(u_3), \quad & B_{-13} = f_5(u_1, u_2, u_3)/(g_1(u_2)g_2(u_3)), \\
A_{-13} = f_6(u_1, u_2, u_3)/(g_1(u_2)g_2(u_3)), &
\end{aligned}$$

where

$$\begin{gathered}
f_1(u_1, u_2, u_3) = u_1(1 - u_3^2), \quad f_2(u_1, u_2, u_3) = 2u_1u_3, \\
f_3(u_1, u_2, u_3) = -3u_1(1 - u_3^2), \quad f_4(u_1, u_2, u_3) = 6u_1u_3, \\
f_5(u_1, u_2, u_3) = -u_1(u_2 + u_3)(-1 + u_2u_3), \\
f_6(u_1, u_2, u_3) = \frac{1}{2}u_1(-1 - u_2 - u_3 + u_2u_3)(-1 + u_2 + u_3 + u_2u_3), \\
g_1(u_2) = 1 + u_2^2, \quad g_2(u_3) = 1 + u_3^2
\end{gathered}$$

and the components dimension is less or equal three, since these functions depends on three variables, u_1, u_2 and u_3. To know if the dimension is exactly three, Jacobian of the functions $f_0(u_1, u_2, u_3), \ldots, f_6(u_1, u_2, u_3)$ needs to be computed. The Jacobian in some arbitrary point, $u_1 = 1, u_2 = 4, u_3 = 2$, is three, hence the dimension is equal to three.

In the same way we obtain the dimension for component 3). The parametrization is

$$\begin{gathered}
B_{11} = f_0 = 0, \quad B_{10} = f_1(u_1, u_2, u_3, u_4, u_5) = u_1, \\
A_{10} = f_2(u_1, u_2, u_3, u_4, u_5) = u_2, \quad A_{20} = f_3(u_1, u_2, u_3, u_4, u_5) = u_3, \\
B_{20} = f_4(u_1, u_2, u_3, u_4, u_5)/(g_3(u_1, u_2)g_4(u_1, u_2)), \\
A_{02} = f_5(u_1, u_2, u_3, u_4, u_5) = u_4, \quad B_{02} = f_6(u_1, u_2, u_3, u_4, u_5)/(g_3(u_1, u_2)g_4(u_1, u_2)), \\
A_{-13} = f_7(u_1, u_2, u_3, u_4, u_5)/((f_1(u_1)f_2(u_2)g_3(u_1, u_2)g_4(u_1, u_2))),
\end{gathered}$$

where

$$\begin{gathered}
f_4(u_1, u_2, u_3, u_4, u_5) = -2u_1u_2u_3, \; f_6(u_1, u_2, u_3, u_4, u_5) = u_4, \\
f_7(u_1, u_2, u_3, u_4, u_5) = -2u_1u_2u_4, \; g_3(u_1, u_2) = u_1 - u_2, \\
g_4(u_1, u_2) = u_1 + u_2.
\end{gathered}$$

The dimension of this component is less or equal five and the Jacobian of $f_1(u_1, u_2, u_3, u_4, u_5)$, $\ldots, f_7(u_1, u_2, u_3, u_4, u_5)$ in random point $u_1 = 4, u_2 = 6, u_3 = 2, u_4 = 1, u_5 = 2$ is five, hence the dimension of this component of center variety is five. □

Theorem 4.2. *Let us define polynomials* $F_1 = A_{20}^2B_{-13} + 2A_{-13}A_{20}B_{20} - B_{-13}B_{20}^2$, $F_2 = (A_{02}B_{-13} - B_{02}A_{-13})(B_{02}B_{-13} + A_{02}A_{-13})$, $F_3 = 3A_{02}^2 + 2A_{02}A_{10}^2 - 8A_{02}A_{20} + 2A_{10}^2A_{20} - 3A_{20}^2 - 2A_{02}B_{10}^2 - 2A_{20}B_{10}^2$, $F_4 = (A_{10}^2 + B_{10}^2)(A_{02} + B_{02})(A_{02} - B_{02})$ *and* $F_5 = A_{10}^2B_{20} - B_{10}^2B_{20} - 2A_{10}A_{20}B_{10}$.

There are bifurcations of the system (4.2) which produce 3 limit cycles locally from the center corresponding to the parameter value p_1, where p_1 is a component 1) with $F_1(p_1) \neq 0$ of $\mathbb{R}^{10}$. The cyclicity of a generic point p_2 of component 2) with $F_2(p_2) \neq 0$ and of point p_4 with $F_4(p_4) \neq 0$ is 5. For the component 3) with $F_3(p_3) \neq 0$ the cyclicity is 4 and 6 for the component 5) with $F_5(p_5) \neq 0$.

Proof. Component 1) We choose an arbitrary point $p = (A_{10}, B_{10}, A_{20}, B_{20}, A_{02}, B_{02}, A_{11}, B_{11}, A_{-13}, B_{-13})$ of this component, $(0, 0, 1, 1, 3, -3, 2, 0, 1, 1)$, the rank of the Jacobian of the focus quantities, rank $J_p^{(k)} = 3$, is equal to three. By Theorem 2.3 the cyclicity of a generic point of this component is three.

Component 2) For the random point $p = (0, 0, -3, 3, 1, 1, 0, 0, -\sqrt{7}, 1)$ the rank of the Jacobian is five, rank $J_p^{(k)} = 5$, hence five limit cycles can bifurcate for these systems.

Component 3) The rank of Jacobian of the focus quantities at the point p, where $p = (2, 1, \frac{3}{4}, 1, -\frac{3}{4}, 1, 1, 0, \frac{7}{24}, 1)$ of the component 3) is equal to four, rank $J_p^{(4)} = 4$.

Component 4) For the point $p = (1, 1, -2, 3, 2, 3, 1, 0, 0, 0)$ of the component 4) there can bifurcate up to five limit cycles, since the rank of Jacobian at the point p is five, rank $J_p^{(5)} = 5$.

Component 5) The cyclicity of the component 5) is six, since the rank of Jacobian of the focus quantities at the point $p = (2, 3, 1, 1, 0, 0, 0, 0, 0, 0)$ of this component is six, rank $J_p^{(k)} = 6$. □

5 Conclusions

The main results in this paper are on integrability and cyclicity of cubic system. The computation of necessary conditions for system of the form (1.1) were difficult. It was impossible to compute over the field of rational numbers. To overcome the difficulties we have splitted our system into four subsystems, solved the integrability problem and from the integrability conditions for these subsystems we have reconstructed integrability variety of general system (1.1). From the results on integrability of complex cubic system, where seven conditions were obtained, see Theorem 1.1, we have obtained the conditions of associated real cubic systems. Results are presented in Thereom 4.1. For each of five obtained components of integrability variety of a real systems we studied the number of limit cycles that can bifurcate from it. It was shown that maximum limit cycles that can bifurcate from system (4.2) under some specific conditions is six. This number is, in comparison to result from Żołądek [56], where he proven that there are up to eleven limit cycles appearing, small.

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