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Analysis of singular one-dimensional linear boundary value problems using two-point Taylor expansions

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Abstract. We consider the second-order linear differential equation $(x^2-1)y''+f(x)y'+g(x)y=h(x)$ in the interval $(-1,1)$ with initial conditions or boundary conditions (Dirichlet, Neumann or mixed Dirichlet–Neumann). The functions f, g and h are analytic in a Cassini disk $\mathcal{D}_r$ with foci at $x=\pm 1$ containing the interval $[-1,1]$. Then, the two end points of the interval may be regular singular points of the differential equation. The two-point Taylor expansion of the solution $y(x)$ at the end points ± 1 is used to study the space of analytic solutions in $\mathcal{D}_r$ of the differential equation, and to give a criterion for the existence and uniqueness of analytic solutions of the boundary value problem. This method is constructive and provides the two-point Taylor approximation of the analytic solutions when they exist.

Keywords: second-order linear differential equations, regular singular point, boundary value problem, Frobenius method, two-point Taylor expansions.

2020 Mathematics Subject Classification: 34A25, 34B05, 41A58.

1 Introduction

In [6] we considered the second-order linear equation $y''+f(x)y'+g(x)y=h(x)$ in the interval $(-1,1)$ with initial conditions or boundary conditions of the type Dirichlet, Neumann or mixed Dirichlet–Neumann. The functions f, g and h are analytic in a Cassini disk with foci at $x=\pm 1$ containing the interval $[-1,1]$. Then, the end points of the interval, where the boundary data are given, are regular points of the differential equation. The two-point Taylor expansion of the solution $y(x)$ at the end points ± 1 was used to give a criterion for the existence and uniqueness of analytic solutions of the initial or boundary value problem and approximate the solutions when they exist. In [1] we have considered problems that have an extra difficulty: one of the end points of the interval is a regular singular point of the differential equation, that is, we have considered the equation $(x+1)y''+f(x)y'+g(x)y=h(x)$.

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In this paper we continue our investigation considering problems where both end points of the interval are regular singular points of the differential equation. We consider initial or boundary value problems of the form

$$\begin{cases} (x^2-1)y''+f(x)y'+g(x)y=h(x) \quad \text{in} \ (-1,1), \\ B\begin{pmatrix} y(-1) \\ y(1) \\ y'(-1) \\ y'(1) \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \end{cases} \tag{1.1}$$

where f, g and h are analytic in a Cassini disk with foci at $x=\pm 1$ containing the interval $[-1,1]$ (we give more details in the next section), $\alpha,\beta\in\mathbb{C}$ and B is a 2×4 matrix of rank two which defines the initial conditions or the boundary conditions (Dirichlet, Neumann or mixed).

The consideration of the interval $(-1,1)$ is not a restriction, as any real interval (a,b) can be transformed into the interval $(-1,1)$ by means of an affine change of the independent variable. The form of the differential equation in (1.1) is not a restriction either: consider the differential equation $(x^2-1)^2u''(x)+(x^2-1)F(x)u'(x)+G(x)u(x)=0$, with F and G analytic at $x=\pm 1$. After the change of the dependent variable $u=(x-1)^\lambda(x+1)^\mu y$, with λ a solution of the equation $4\lambda(\lambda-1)+2F(1)\lambda+G(1)=0$ and μ a solution of the equation $4\mu(\mu-1)-2F(-1)\mu+G(-1)=0$, the equation may be written in the form $(x^2-1)y''+f(x)y'+g(x)y=0$, with f and g analytic at $x=\pm 1$. On the other hand, the points $x=\pm 1$ are both indeed regular singular points of the differential equation $(x^2-1)y''+f(x)y'+g(x)y=h(x)$ when $|f(\pm 1)|+|g(\pm 1)|+|h(\pm 1)|\neq 0$; if $f(\pm 1)=g(\pm 1)=h(\pm 1)=0$, then both, $x=\pm 1$, are regular points, and problem (1.1) is the regular problem analyzed in [6]. If $f(1)=g(1)=h(1)=0$ and $|f(-1)|+|g(-1)|+|h(-1)|\neq 0$, then only one end point is a regular singular point of the equation, and problem (1.1) has been analyzed in [1]. We omit these restrictions here and then, the regular case studied in [6] or the cases studied in [1] may be considered particular cases of the more general one analyzed in this paper.

A standard theorem for the existence and uniqueness of solution of (1.1) is based on the knowledge of the two-dimensional linear space of solutions of the homogeneous equation $(x^2-1)y''+f(x)y'+g(x)y=0$ [2, Chapter 4, Section 1]. When f are g are constants or in some other particular situation, it is possible to find the general solution of the equation (sometimes via the Green function [2, Chapter 4], [7, Chapters 1 and 3])). But this is not possible in general situations and that standard criterion for the existence and uniqueness of solution of (1.1) is not practical. Other well-known criterion for the existence and uniqueness of solution of (1.1) is based on the Lax–Milgram theorem when (1.1) is an elliptic problem [3]. In any case, the determination of the existence and uniqueness of solution of (1.1) requires a non-systematic detailed study of the problem, like for example the study of the eigenvalue problem associated to (1.1) [2, Chapter 4, Section 2], [7, Chapter 7].

When f, g and h are analytic in a disk with center at $x=0$ and containing the interval $[-1,1]$, we may consider the initial value problem

$$\begin{cases} (x^2-1)y''+f(x)y'+g(x)y=h(x), \quad x\in(-1,1), \\ y(0)=y_0, \quad y'(0)=y'_0, \end{cases} \tag{1.2}$$

with $y_0,y'_0\in\mathbb{C}$. Using the Frobenius method we can approximate the solution of this problem by its Taylor polynomial of degree $N\in\mathbb{N}$ at $x=0$, $y_N(x)=\sum_{n=0}^{N}c_kx^k$, where the coefficients

c_k are affine functions of $c_0 = y_0$ and $c_1 = y_0'$. By imposing the boundary conditions given in (1.1) over $y_N(x)$, we obtain an algebraic linear system for y_0 and y_0'. The existence and uniqueness of solution to this algebraic linear system gives us information about the existence and uniqueness of solution of (1.1). This procedure, although theoretically possible, has a difficult practical implementation since the data of the problem are given at $x = \pm 1$, not at $x = 0$ (see [6] for further details).

In [6] we improved the ideas of the previous paragraph for the regular case (when $f(-1) = g(-1) = h(-1) = 0$) using, not the standard Taylor expansion in the associated initial value problem (1.2), but a two-point Taylor expansion [4] at the end points $x = \pm 1$ directly in the differential equation and in the boundary conditions. The convergence region for a two-point Taylor expansion is a Cassini disk (see Figure 2.1), and this Cassini disk avoids the possible singularities of the coefficient functions located near the interval $[-1, 1]$ more efficiently than the standard Taylor disk [5].

In this paper we investigate if a two-point Taylor expansion at the end points $x = \pm 1$ also works for the more general problem (1.1), in particular when both, $x = -1$ and $x = 1$ are regular singular points of the equation. Thus, we use the two-point Taylor expansion of the solution $y(x)$ to give a criterion for the existence and uniqueness of analytic solutions based on the data of the problem, not based on the knowledge of the general solution of the differential equation.

The paper is organized as follows. In the next section we introduce some elements of the theory of two-point Taylor expansions and study the space S of analytic solutions of the differential equation $(x^2 - 1)y'' + f(x)y' + g(x)y = h(x)$. In Section 3 we derive the two-point Taylor expansion at the end points $x = \pm 1$ of the functions of S (when S is nonempty). In Section 4 we give an algebraic characterization of S that we use, in Section 5, to formulate a criterion of existence and uniqueness of analytic solutions of problem (1.1). Section 6 includes some illustrative examples and Section 7 a few final remarks. The analysis of this paper paper follows the same pattern as the analysis of [5].

2 Global analytic solutions of the differential equation

Assume that the coefficient functions f, g and h in (1.1) are analytic in the Cassini disk $\mathcal{D}_r = \{z \in \mathbb{C} \mid |z^2 - 1| < r\}$ with foci at $z = \pm 1$ and Cassini's radius r, with $r > 1$ (see [4]). The requirement $r > 1$ assures that the interval $[-1, 1]$ is contained into the Cassini disk $\mathcal{D}_r$ (see Figure 2.1). Then, the three functions f, g and h, admit a two-point Taylor series in $\mathcal{D}_r$ of the form [4],

$$f(z) = \sum_{n=0}^{\infty} [f_n^0 + f_n^1 z](z^2 - 1)^n, \quad g(z) = \sum_{n=0}^{\infty} [g_n^0 + g_n^1 z](z^2 - 1)^n, \quad h(z) = \sum_{n=0}^{\infty} [h_n^0 + h_n^1 z](z^2 - 1)^n, \tag{2.1}$$

where the coefficients of the expansions of f are [4]

$$\begin{aligned} f_0^0 &:= \frac{f(1) + f(-1)}{2}, \quad f_0^1 := \frac{f(1) - f(-1)}{2}, \\ f_n^0 &:= \sum_{k=0}^{n} \frac{(n+k-1)!}{(n-k-1)!} \frac{(-1)^k f^{(n-k)}(1) + (-1)^n f^{(n-k)}(-1)}{n!\,k!\,2^{n+k+1}}, \quad n = 1, 2, 3, \ldots, \\ f_n^1 &:= \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!} \frac{(-1)^k f^{(n-k)}(1) - (-1)^n f^{(n-k)}(-1)}{n!\,k!\,2^{n+k+1}}, \quad n = 1, 2, 3, \ldots \end{aligned} \tag{2.2}$$

The coefficients g_n^0 and g_n^1 of the expansion of g and the coefficients h_n^0 and h_n^1 of the expansion of h are defined by means of similar formulas. The three expansions in (2.1) converge absolute and uniformly to the respective functions f, g and h in $\mathcal{D}_r$ (see [4]). The regular case analyzed in [6] corresponds to the particular situation $f_0^0 = f_0^1 = g_0^0 = g_0^1 = h_0^0 = h_0^1 = 0$ (that is equivalent to $f(\pm 1) = g(\pm 1) = h(\pm 1) = 0$).

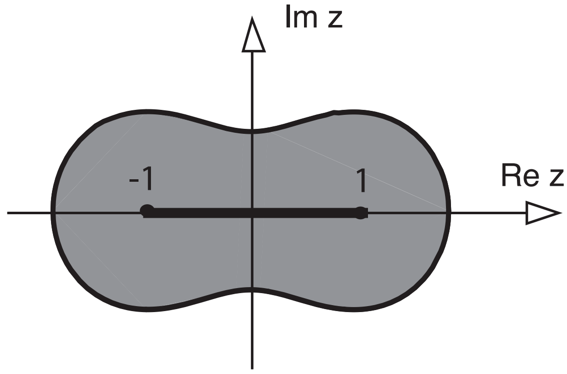

Figure 2.1: The Cassini disk $\mathcal{D}_r = \{z \in \mathbb{C} \mid |z^2 - 1| < r\}$ with foci at $z = \pm 1$ and radius $r > 1$ contains the real interval $[-1, 1]$.

As it is argued in [6], when $f(\pm 1) = g(\pm 1) = h(\pm 1) = 0$, any solution of the differential equation is analytic in $\mathcal{D}_r$. But the situation is different when $|f(1)| + |g(1)| + |h(1)| \neq 0$ and/or $|f(-1)| + |g(-1)| + |h(-1)| \neq 0$ (see [1]) and we need to introduce the following definition.

Definition 2.1. Denote by S_h the linear space of solutions of the homogeneous equation $(z^2 - 1)y'' + f(z)y' + g(z)y = 0$ that are analytic in $\mathcal{D}_r$. Denote by S the affine space of solutions of the complete equation $(z^2 - 1)y'' + f(z)y' + g(z)y = h(z)$ that are analytic in $\mathcal{D}_r$.

From Frobenius theory we know that the critical exponents of the homogeneous differential equation $(z^2 - 1)y'' + f(z)y' + g(z)y = 0$ at $z = -1$ are $\mu_1(-1) = 0$ and $\mu_2(-1) = 1 + f(-1)/2$. When $\mu_2(-1) \notin \mathbb{Z}$ ($f(-1) \notin 2\mathbb{Z}$), one independent solution of the homogeneous equation is analytic in $\mathcal{D}_r \setminus \{1\}$ and the other one is not, as it is of the form $(z+1)^{\mu_2(-1)}a(z)$ with $a(z)$ analytic in $\mathcal{D}_r$. When $\mu_2(-1) = 0, -1, -2, \ldots$, ($f(-1) \in -2\mathbb{N}$), one independent solution of $(z^2 - 1)y'' + f(z)y' + g(z)y = 0$ is analytic in $\mathcal{D}_r \setminus \{1\}$ and the other one is not, as it is of the form $a_1(z)\log(z+1) + (z+1)^{\mu_2(-1)}a_2(z)$ with $a_1(z)$ and $a_2(z)$ analytic in $\mathcal{D}_r \setminus \{1\}$. When $\mu_2(-1) \in \mathbb{N}$ ($f(-1) \in 2\mathbb{N} \cup \{0\}$), one independent solution of the homogeneous equation is analytic in $\mathcal{D}_r \setminus \{1\}$ (and it is canceled $\mu_2(-1)$ times at $z = -1$) and the other one is of the form $(z+1)^{\mu_2(-1)}a_1(z)\log(z+1) + a_2(z)$ with $a_1(z)$ and $a_2(z)$ analytic in $\mathcal{D}_r \setminus \{1\}$. Therefore, when $\mu_2(-1) \in \mathbb{N}$, may be only one or may be two independent solutions of $(z^2 - 1)y'' + f(z)y' + g(z)y = 0$ analytic at $z = -1$.

The discussion is similar at the point $z = 1$ replacing $f(-1)$ by $-f(1)$, that is, $\mu_1(1) = 0$ and $\mu_2(1) = 1 - f(1)/2$: when $f(1) \notin 2\mathbb{Z}$, one independent solution of the homogeneous equation is analytic in $\mathcal{D}_r \setminus \{-1\}$ and the other one is not. When $f(1) \in 2\mathbb{N}$, one independent solution of $(z^2 - 1)y'' + f(z)y' + g(z)y = 0$ is analytic in $\mathcal{D}_r \setminus \{-1\}$ and the other one is not. When $f(1) \in -2\mathbb{N} \cup \{0\}$, one independent solution of the homogeneous equation is analytic in $\mathcal{D}_r \setminus \{-1\}$ (and it is canceled $\mu_2(1)$ times at $z = 1$) and the other one may be or may be not analytic in $\mathcal{D}_r \setminus \{-1\}$. Therefore, when $\mu_2(1) \in \mathbb{N}$, may be only one or may be two independent solutions of $(z^2 - 1)y'' + f(z)y' + g(z)y = 0$ analytic at $z = 1$.

Then, all the possibilities may be summarized as follows: When $f(-1) \neq 0, 2, 4, \ldots$ or $f(1) \neq 0, -2, -4, \ldots$, then the homogeneous equation has only the null solution or a one-dimensional space of analytic solutions in $\mathcal{D}_r$. When $f(-1) = 0, 2, 4, \ldots$ and $f(1) = 0, -2, -4, \ldots$ then everything is possible: the homogeneous equation has only the null solution, it has a one-dimensional space or it has a two-dimensional space of analytic solutions in $\mathcal{D}_r$.

From the above discussion we conclude that

$$\dim(S_h) = \begin{cases} 0 \text{ or } 1 & \text{when} \quad f(-1) \neq 0, 2, 4, \ldots \text{ or } f(1) \neq 0, -2, -4, \ldots \\ 0, 1 \text{ or } 2 & \text{when} \quad f(-1) = 0, 2, 4, \ldots \text{ and } f(1) = 0, -2, -4, \ldots \end{cases}$$

On the other hand, it is clear that $S = y_p + S_h$, where $y_p(z)$ is a particular solution of $(z^2-1)y'' + f(z)y' + g(z)y = h(z)$ that is analytic in $\mathcal{D}_r$. The existence of that particular solution $y_p(z)$ is not guaranteed *a priori*; then, either $\dim(S) = \dim(S_h)$ or S is empty. (For example, the general solution of the equation $(z^2-1)y'' = 1$ is $y(z) = c_1 + c_2 z + z\log(\sqrt{(1-z)/(z+1)}) - \log(\sqrt{z^2-1})$, c_1, $c_2 \in \mathbb{C}$, then $\dim(S_h) = 2$ and S is empty. The general solution of the equation $(z^2-1)y'' - y' = 1$ is $y(z) = c_1 + c_2(\arcsin z + \sqrt{1-z^2}) - z$, c_1, $c_2 \in \mathbb{C}$, then $\dim(S_h) = \dim(S) = 1$.)

Once we have a picture of the spaces S and S_h in relation to the values of $f(\pm 1)$, we introduce the key point in the discussion of the paper. Any function $y(z) \in S$ or $y(z) \in S_h$ can be written in the form of a two-point Taylor expansion at the base points $z = \pm 1$ (see [4]),

$$y(z) = \sum_{n=0}^{\infty} [a_n + b_n z](z^2-1)^n, \quad z \in \mathcal{D}_r, \tag{2.3}$$

where the coefficients a_n and b_n are related to the values of the derivatives of $y(z)$ at $z = \pm 1$ in the same form as the coefficients f_n^0 and f_n^1 of f are related to the derivatives of f at $z = \pm 1$ in (2.2). If we can derive the coefficients a_n and b_n from $(z^2-1)y'' + f(z)y' + g(z)y = h(z)$, we will obtain the functions $y \in S$ in the form of a two-point Taylor series (2.3), when the space S is nonempty. This fact is not guaranteed *a priori* from the data of the problem. In the regular case $f(\pm 1) = g(\pm 1) = 0$, it is guaranteed that the dimension of S_h is two [6]. When only one of the end points is a regular singular point, then it is guaranteed that the dimension of S_h is, at least, one (see [1]).

In the more general case analyzed in this paper it is not guaranteed *a priori* that S_h or S are nonempty. Then, the existence of one analytic solution in $\mathcal{D}_r$ of the initial or boundary value problem (1.1) is not guaranteed *a priori* either; nor even when $h = 0$ (homogeneous case) or in the regular case $f(\pm 1) = g(\pm 1) = h(\pm 1) = 0$. In this paper we analyze the size of the spaces S_h and S and then, the existence and uniqueness of analytic solutions in $\mathcal{D}_r$ of the problem (1.1). We accomplish this task using that any function in S may be written in the form (2.3): in the remaining of the paper we replace the formal two-point Taylor series (2.3) in (1.1) and study if it is possible to obtain the coefficients a_n and b_n from the differential equation and the boundary conditions given in (1.1).

For any function $y(z)$ analytic in $\mathcal{D}_r$, the series (2.3) is absolute and uniformly convergent in the interval $[-1, 1]$, and we also have [6]

$$\begin{aligned} y'(z) &= \sum_{k=0}^{\infty} \{[(2k+1)b_k + 2(k+1)b_{k+1}] + 2(k+1)a_{k+1}z\}\,(z^2-1)^k, \\ y''(z) &= \sum_{k=0}^{\infty} 2(k+1)\,\{[(2k+1)a_{k+1} + 2(k+2)a_{k+2}] + [(2k+3)b_{k+1} + 2(k+2)b_{k+2}]z\}\,(z^2-1)^k, \end{aligned} \tag{2.4}$$

where the convergence of the series is absolute and uniform in the interval $[-1,1]$.

3 Two-point Taylor expansion representation of the functions of S

As it happens in the standard Frobenius method for initial value problems, when we replace f, g and h by their two-point Taylor expansions (2.1) in the differential equation $(z^2-1)y''+f(z)y'+g(z)y=h(z)$, and the solution $y(z)$ and its derivatives by their two-point Taylor expansions (2.3) and (2.4), we find that the coefficients a_n and b_n satisfy, for $n=0,1,2,\ldots$, a linear system of two recurrences

$$\begin{aligned}
&2(n+1)[(2n+f_0^1)a_{n+1}+f_0^0b_{n+1}]+2n(2n-1)a_n+2\sum_{k=0}^{n-1}(k+1)(f_{n-k}^0b_{k+1}+f_{n-k}^1a_{k+1})\\
&\qquad+\sum_{k=0}^{n}\left[(2k+1)f_{n-k}^0b_k+2(k+1)f_{n-k-1}^1a_{k+1}+g_{n-k}^0a_k+(g_{n-k}^1+g_{n-k-1}^1)b_k\right]=h_n^0,\\
&2(n+1)[(2n+f_0^1)b_{n+1}+f_0^0a_{n+1}]+2n(2n+1)b_n+2\sum_{k=0}^{n-1}(k+1)(f_{n-k}^0a_{k+1}+f_{n-k}^1b_{k+1})\\
&\qquad+\sum_{k=0}^{n}\left[(2k+1)f_{n-k}^1b_k+g_{n-k}^0b_k+g_{n-k}^1a_k\right]=h_n^1,
\end{aligned}\tag{3.1}$$

with $f_{-1}^0=g_{-1}^0=f_{-1}^1=g_{-1}^1:=0$. Then, in general, as it happens in the standard Frobenius method or in the particular regular boundary problem analyzed in [6], the computation of the coefficients a_n and b_n involve the previous coefficients a_0, $b_0,\ldots,a_{n-1}$ and b_{n-1}. But we find here a particularity that we do not find in the standard Frobenius method nor in the regular problem solved in [6]: in general, for a given $n=0,1,2,\ldots$, we can solve the linear system (3.1) for a_{n+1} and b_{n+1} if and only if

$$\begin{vmatrix}2n+f_0^1 & f_0^0\\ f_0^0 & 2n+f_0^1\end{vmatrix}\neq 0\Leftrightarrow\begin{cases}f(-1)\equiv f_0^0-f_0^1\neq 2n,\\ f(1)\equiv f_0^0+f_0^1\neq -2n.\end{cases}$$

Then, if $f(-1)/2$ and $-f(1)/2\notin\mathbb{N}\cup\{0\}$, we can solve the linear system (3.1) for a_{n+1} and b_{n+1} for any $n=0,1,2,\ldots$ But if $f(-1)/2$ or $-f(1)/2\equiv n_0\in\mathbb{N}\cup\{0\}$, then we can solve the system (3.1) for a_{n+1} and b_{n+1} for any $n=0,1,2,\ldots$, except for $n=n_0$. If $f(-1)/2$ and $-f(1)/2\in\mathbb{N}\cup\{0\}$, then we define $n_0\equiv\max\{f(-1)/2,-f(1)/2\}$. For convenience, when $f(-1)/2\notin\mathbb{N}\cup\{0\}$ and $-f(1)/2\notin\mathbb{N}\cup\{0\}$ we define $n_0=-1$.

Therefore, in any case, we can solve the linear system (3.1) for a_{n+1} and b_{n+1} for $n=n_0+1,n_0+2,n_0+3,\ldots$ This means that we obtain all the coefficients a_n and b_n needed in (2.3) for $n=n_0+2,n_0+3,n_0+4,\ldots$, as a function of the first $2(n_0+2)$ coefficients $a_0,b_0,a_1,b_1,\ldots,a_{n_0+1},b_{n_0+1}$. But these $2(n_0+2)$ first coefficients are not totally free, as they must satisfy the equations (3.1) for $n=0,1,2,\ldots,n_0$. These facts impose $2(n_0+1)$ linear restrictions (not all of them necessarily independent) to the $2(n_0+2)$ first coefficients $a_0,b_0,a_1,b_1,\ldots,a_{n_0+1},b_{n_0+1}$. Let us denote these equations by $L_k[a_0,b_0,a_1,b_1,\ldots,a_{n_0+1},b_{n_0+1}]=0$, $k=1,2,3,\ldots,2n_0+2$. In general, these equations are non homogeneous; they are homogeneous when $h(z)=0$.

In the particular case of the regular problem analyzed in [6] we have that $n_0=0$, since $f(\pm1)=0$. Then, we can obtain from system (3.1) all the coefficients a_n and b_n for $n\geq 2$ as a function of the first four coefficients a_0, b_0, a_1 and b_1. In this case, the above mentioned set of restrictions consists of the equations (3.1) for $n = 0$. But using that $f(\pm 1) = g(\pm 1) = h(\pm 1) = 0$ we see that these equations are the tautology $0 = 0$ and then, they do not introduce any linear dependence between the coefficients a_0, b_0, a_1 and b_1.

As a difference with the Frobenius method where we only have one recurrence relation for the sequence of standard Taylor coefficients, here we have a system of two recurrences (3.1). But moreover, the computation of the coefficients a_n, b_n for $n \geq n_0 + 2$ requires the initial seed $a_0, b_0, a_1, b_1, \ldots, a_{n_0+1}, b_{n_0+1}$. These $2n_0 + 4$ coefficients satisfy the above mentioned $2n_0 + 2$ equations $L_k = 0$. This does not mean that the linear space S_h or the affine space S may have dimension two or more, these spaces have, of course, dimension at most two. It is happening that, apart from the affine space S of (true) solutions of $(z^2 - 1)y'' + f(z)y' + g(z)y = h(z)$, there is a bigger space of formal solutions W defined by

$$W := \left\{ y(z) = \sum_{n=0}^{\infty} [a_n + b_n z](z^2 - 1)^n \;\middle|\; a_n, b_n \text{ given in (3.1) for } n \geq n_0 + 2; \right.$$
$$(a_0, b_0, a_1, b_1, \ldots, a_{n_0+1}, b_{n_0+1}) \in \mathbb{C}^{2n_0+4}$$
$$\left. \text{with } L_k[a_0, b_0, \ldots, a_{n_0+1}, b_{n_0+1}] = 0,\ k = 1, 2, 3, \ldots, 2n_0 + 2 \right\}.$$

Formally, all the two-point Taylor series in W are solutions of $(z^2 - 1)y'' + f(z)y' + g(z)y = h(z)$. But not all of them are convergent, only a subset: the affine space S of (true) solutions of $(z^2 - 1)y'' + f(z)y' + g(z)y = h(z)$, that may be written in the form

$$S = \left\{ y \in W \mid \sum_{n=0}^{\infty} [a_n + b_n z](z^2 - 1)^n \text{ is uniformly convergent in } [-1, 1] \right\}.$$

In the following section we derive a more practical characterization of the space S in the form of two extra linear equations for the coefficients $a_0, b_0, a_1, b_1, \ldots, a_{n_0+1}, b_{n_0+1}$. This characterization allows us to give some more precise information about the size of the space S.

4 Algebraic characterization of the space S

From (3.1) and the discussion below that formula, we see that we may solve (3.1) for (a_n, b_n) for $n \geq n_0 + 2$ in the schematic form

$$\begin{aligned} a_n &= \sum_{k=0}^{n-1} [A_{n,k} a_k + B_{n,k} b_k] + E_n, \\ b_n &= \sum_{k=0}^{n-1} [C_{n,k} a_k + D_{n,k} b_k] + F_n, \end{aligned} \tag{4.1}$$

where the coefficients $A_{n,k}$, $B_{n,k}$, $C_{n,k}$ and $D_{n,k}$ are functions of f_k^0, f_k^1, g_k^0 and g_k^1. The coefficients $E_{n,k}$ and $F_{n,k}$ are functions of h_k^0 and h_k^1, $k = 0, 1, 2, \ldots, n - 1$. For simplicity, we do not detail here these functions, as the precise value of these coefficients is not needed in the theoretical discussion. It is not needed either in computation in the particular examples, as it is more convenient the use of an algebraic manipulator to compute (a_n, b_n), $n \geq n_0 + 2$, directly from (3.1).

For a fixed $m \in \mathbb{N}$, $m \geq 2n_0 + 2$, and $n = 0, 1, 2, \ldots, m - n_0 - 1$, we define the vectors

$$v_n := (a_{n+n_0+2-m}, b_{n+n_0+2-m}, a_{n+n_0+3-m}, b_{n+n_0+3-m}, \ldots, a_{n+n_0}, b_{n+n_0}, a_{n+n_0+1}, b_{n+n_0+1}) \in \mathbb{C}^{2m},$$

with $a_{-k}=b_{-k}=0$ for $k\in\mathbb{N}$. In particular, we have

$$v_{m-n_0-2}=(a_0,b_0,a_1,b_1,\dots,a_{m-1},b_{m-1})$$

and

$$v_0=(0,0,\dots,0,0,a_0,b_0,a_1,b_1,\dots,a_{n_0+1},b_{n_0+1}).$$

For $n=0,1,2,\dots,m-n_0-2$, define the $(2m)\times(2m)$ matrix

$$M_n:=\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1\\ 0 & \dots & 0 & A_{n+n_0+2,0} & B_{n+n_0+2,0} & \dots & \dots & A_{n+n_0+2,n+n_0+1} & B_{n+n_0+2,n+n_0+1}\\ 0 & \dots & 0 & C_{n+n_0+2,0} & D_{n+n_0+2,0} & \dots & \dots & C_{n+n_0+2,n+n_0+1} & D_{n+n_0+2,n+n_0+1} \end{pmatrix}. \tag{4.2}$$

The only non-null elements of this matrix are the corresponding to the entries $m_{i,i+2}=1$, $i=1,2,3,\dots,2m-2$, and to the entries $m_{2m-1,k}$, $m_{2m,k}$, $k=0,1,2,\dots,n+n_0+1$. In particular we have

$$M_0=\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1\\ 0 & \dots & 0 & A_{n_0+2,0} & B_{n_0+2,0} & \dots & \dots & A_{n_0+2,n_0+1} & B_{n_0+2,n_0+1}\\ 0 & \dots & 0 & C_{n_0+2,0} & D_{n_0+2,0} & \dots & \dots & C_{n_0+2,n_0+1} & D_{n_0+2,n_0+1} \end{pmatrix}$$

and

$$M_{m-n_0-2}=\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1\\ A_{m,0} & B_{m,0} & A_{m,1} & B_{m,1} & \dots & \dots & \dots & A_{m,m-1} & B_{m,m-1}\\ C_{m,0} & D_{m,0} & C_{m,1} & D_{m,1} & \dots & \dots & \dots & C_{m,m-1} & D_{m,m-1} \end{pmatrix}.$$

We also need, for $n=0,1,2,\dots,m-n_0-2$, the definition of the vector

$$c_n:=(0,0,\dots,0,0,E_{n+2},F_{n+2})\in\mathbb{C}^{2m}.$$

Then, the system of recurrences (4.1) (that indeed represents (3.1)) can be written in matrix form

$$v_n=M_{n-1}v_{n-1}+c_{n-1},\quad n=1,2,3,\dots,m-n_0-1.$$

To find the solution of this linear recurrence for the vector v_n, we define recurrently the following matrices

$$\begin{aligned} \mathcal{M}_0 &:= M_0, \quad \mathcal{M}_n := M_n\mathcal{M}_{n-1},\\ \mathcal{C}_0 &:= c_0, \qquad \mathcal{C}_n := M_n\mathcal{C}_{n-1}+c_n,\quad n=1,2,3,\dots,m-n_0-2, \end{aligned}$$

or equivalently,

$$\mathcal{M}_n = \prod_{k=0}^{n} M_{n-k}, \quad \mathcal{C}_n = c_n + \sum_{k=0}^{n-1} [M_n \cdot M_{n-1} \cdots M_{k+1}] c_k, \quad n = 0, 1, 2, 3, \dots, m - n_0 - 2.$$

Then, we find

$$v_{m-n_0-1} = \mathcal{M}_{m-n_0-2} v_0 + \mathcal{C}_{m-n_0-2},$$

or, in an extended form

$$\begin{pmatrix} \star \\ \star \\ \cdot \\ \cdot \\ \cdot \\ \star \\ \star \\ a_m \\ b_m \end{pmatrix} = \begin{pmatrix} \star \\ \star \\ \cdot \\ \cdot \\ \cdot \\ \star \\ \star \\ \mathcal{B}_{2m-1} \\ \mathcal{B}_{2m} \end{pmatrix} + \begin{pmatrix} \star & \star & \star & \star & \star & \cdots & \star & \star \\ \star & \star & \star & \star & \star & \cdots & \star & \star \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \star & \star & \star & \star & \star & \cdots & \star & \star \\ \star & \star & \star & \star & \star & \cdots & \star & \star \\ \star & \cdots & \star & \mathcal{M}_{2m-1,2m-2n_0-3} & \mathcal{M}_{2m-1,2m-2n_0-2} & \cdots & \mathcal{M}_{2m-1,2m-1} & \mathcal{M}_{2m-1,2m} \\ \star & \cdots & \star & \mathcal{M}_{2m,2m-2n_0-3} & \mathcal{M}_{2m,2m-2n_0-2} & \cdots & \mathcal{M}_{2m,2m-1} & \mathcal{M}_{2m,2m} \end{pmatrix} \begin{pmatrix} 0 \\ \cdot \\ 0 \\ a_0 \\ b_0 \\ \cdot \\ \cdot \\ a_{n_0+1} \\ b_{n_0+1} \end{pmatrix},$$

where $\mathcal{M}_{i,j}$ are the entrances of the last two rows and last $2n_0 + 4$ columns of the matrix $\mathcal{M}_{m-n_0-2}$, $\mathcal{B}_i$ are the last two components of the vector $\mathcal{C}_{m-n_0-2}$ and the $\star$ denote complex (unspecified) numbers. The two-point Taylor series of an analytic function in $\mathcal{D}_r$ converges in $[-1, 1]$ if it converges at $z = 0$ [4]. And it converges at $z = 0$ if and only if $\lim_{m\to\infty}(a_m, b_m) = (0, 0)$. Then, taking the limit $m \to \infty$ into the above equation we find

$$\begin{pmatrix} \star \\ \star \\ \cdot \\ \cdot \\ \cdot \\ \star \\ \star \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \star & \star & \star & \star & \star & \dots & \star & \star \\ \star & \star & \star & \star & \star & \dots & \star & \star \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \star & \star & \star & \star & \star & \dots & \star & \star \\ \star & \star & \star & \star & \star & \dots & \star & \star \\ \star & \dots & \star & H_{1,1} & H_{1,2} & \dots & H_{1,2n_0+3} & H_{1,2n_0+4} \\ \star & \dots & \star & H_{2,1} & H_{2,2} & \dots & H_{2,2n_0+3} & H_{2,2n_0+4} \end{pmatrix} \begin{pmatrix} 0 \\ \cdot \\ 0 \\ a_0 \\ b_0 \\ \cdot \\ \cdot \\ a_{n_0+1} \\ b_{n_0+1} \end{pmatrix} + \begin{pmatrix} \star \\ \star \\ \cdot \\ \cdot \\ \cdot \\ \star \\ \star \\ \gamma_1 \\ \gamma_2 \end{pmatrix},$$

where we have denoted

$$\begin{aligned} H_{i,j} &:= \lim_{m\to\infty} \mathcal{M}_{2m+i-2,2m-2n_0+j-4}, \quad i = 1, 2, \quad j = 1, 2, 3, \dots, 2n_0 + 4, \\ \gamma_1 &= \lim_{m\to\infty} \mathcal{B}_{2m-1}, \quad \gamma_2 = \lim_{m\to\infty} \mathcal{B}_{2m}. \end{aligned} \tag{4.3}$$

Then, the two equations that we were looking for are, for $k = 1, 2$

$$H_k[a_0, b_0, \dots, a_{n_0+1}, b_{n_0+1}] := H_{k,1}a_0 + H_{k,2}b_0 + \dots + H_{k,2n_0+3}a_{n_0+1} + H_{k,2n_0+4}b_{n_0+1} + \gamma_k = 0. \tag{4.4}$$

Therefore, at this moment, we have found the more practical characterization of the space S of true solutions of $(z^2 - 1)y'' + f(z)y' + g(z)y = h(z)$ that we were looking for

$$\begin{aligned} S := \Bigg\{ y(z) = \sum_{n=0}^{\infty} [a_n + b_n z](z^2 - 1)^n \;\Bigg|\; & a_n, b_n \text{ given in (3.1) for } n \geq n_0 + 2; \\ & (a_0, b_0, a_1, b_1, \dots, a_{n_0+1}, b_{n_0+1}) \in \mathbb{C}^{2n_0+4} \text{ with } L_k[a_0, b_0, a_1, b_1, \dots, a_{n_0+1}, b_{n_0+1}] = 0 \\ & \text{for } k = 1, 2, 3, \dots, 2n_0 + 2, \text{ and } H_k[a_0, b_0, a_1, b_1, \dots, a_{n_0+1}, b_{n_0+1}] = 0 \text{ for } k = 1, 2 \Bigg\}. \end{aligned} \tag{4.5}$$

In other words, the $2n_0+4$ coefficients $a_0, b_0, a_1, b_1, \dots, a_{n_0+1}, b_{n_0+1}$ of the two-point Taylor expansion of any function in S must be a solution of the following linear system of $2n_0+4$ equations

$$\begin{cases} L_k[a_0, b_0, a_1, b_1, \dots, a_{n_0+1}, b_{n_0+1}] = 0, & k = 1, 2, 3, \dots, 2n_0+2, \\ H_k[a_0, b_0, a_1, b_1, \dots, a_{n_0+1}, b_{n_0+1}] = 0, & k = 1, 2. \end{cases} \tag{4.6}$$

This system is homogeneous when $h = 0$ (when $h_n^0 = h_n^1 = 0$) and non-homogeneous when $h \neq 0$. Let's denote $(4.6)_h$ the system (4.6) when $h = 0$. We know that $\dim(S_h) = 0$, 1 or 2. This means that $\mathrm{rank}[(4.6)_h] = 2n_0+2$, $2n_0+3$ or $2n_0+4$ and then, the homogeneous system has a one or two-dimensional space of solutions or $S_h = \{0\}$. On the other hand, we know that $\dim(S) = 1$ or 2, or S is empty. This means that there are five possibilities:

(i) $\mathrm{rank}[(4.6)] = \mathrm{rank}[(4.6)_h] = 2n_0+2$; then $\dim(S) = \dim(S_h) = 2$,

(ii) $\mathrm{rank}[(4.6)] = \mathrm{rank}[(4.6)_h] = 2n_0+3$; then $\dim(S) = \dim(S_h) = 1$,

(iii) $\mathrm{rank}[(4.6)] = \mathrm{rank}[(4.6)_h] = 2n_0+4$; then $S_h = \{0\}$ and $S = \{y_p\}$,

(iv) $\mathrm{rank}[(4.6)] = 2n_0+3$ or $2n_0+4$ and $\mathrm{rank}[(4.6)_h] = 2n_0+2$; then $\dim(S_h) = 2$ and S is empty,

(v) $\mathrm{rank}[(4.6)] = 2n_0+4$ and $\mathrm{rank}[(4.6)_h] = 2n_0+3$; then $\dim(S_h) = 1$ and S is empty.

Therefore,

- When $\mathrm{rank}[(4.6)_h] = 2n_0+4$, the unique analytic solution in $\mathcal{D}_r$ of the homogeneous equation $(z^2-1)y'' + f(z)y' + g(z)y = 0$ is the null solution and the complete equation $(z^2-1)y'' + f(z)y' + g(z)y = h(z)$ has a unique solution analytic in $\mathcal{D}_r$.
- When $\mathrm{rank}[(4.6)_h] = 2n_0+2$ then either, $\dim(S) = 2$ or S is empty.
- When $\mathrm{rank}[(4.6)_h] = 2n_0+3$ then either, $\dim(S) = 1$ or S is empty.

In the regular case we know that $\dim(S) = 2$ (it is proved in [6] that the only two equations $H_k = 0$ that define S in this case are linearly independent). But, in general, we need to compute the above ranks in order to obtain some information about the sizes of S and S_h.

4.1 Polynomial coefficients

When the coefficient functions f and g are polynomials, we can simplify the formulation of the above existence and uniqueness criterion. In general, the computation of the coefficients (a_n, b_n) requires a matrix M_n of size $(2m) \times (2m)$ with $m \geq n + n_0 + 2$. This means that we need matrices of increasing size to compute the coefficients when n increases. In the case of polynomial coefficients, the situation is different. The recurrences (3.1) are of constant order s independent of n and the computation of the coefficients a_n and b_n involves only the previous $2s$ coefficients a_{n-s}, $b_{n-s}, \dots, a_{n-1}$ and b_{n-1}. Thus, in this case, we do not need matrices of increasing size, but matrices of constant size $(2s) \times (2s)$.

The recurrence system (3.1) for polynomial coefficients is of the form

$$a_n = \sum_{k=n-s}^{n-1} [A_{n,k} a_k + B_{n,k} b_k] + E_n,$$

$$b_n = \sum_{k=n-s}^{n-1} [C_{n,k} a_k + D_{n,k} b_k] + F_n,$$

for a certain $s \in \mathbb{N}$, $n = n_0, n_0+1, n_0+2, \ldots$, with $a_{-k} = b_{-k} = 0$, $k \in \mathbb{N}$. The discussion is identical to the one developed in the general case analyzed above, but now we can eliminate the restriction $n \le m - n_0 - 2$. Moreover, we can simplify the computations because now, the size of the matrices M_n does not depend on n. We can now define the matrices M_n of fixed size $(2s) \times (2s)$ in the form

$$M_n := \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 \\ A_{n+2,n+2-s} & B_{n+2,n+2-s} & \ldots & A_{n+2,0} & B_{n+2,0} & \ldots & \ldots & A_{n+2,n+1} & B_{n+2,n+1} \\ C_{n+2,n+2-s} & D_{n+2,n+2-s} & \ldots & C_{n+2,0} & D_{n+2,0} & \ldots & \ldots & C_{n+2,n+1} & D_{n+2,n+1} \end{pmatrix}$$

instead of the form (4.2), with $A_{n,-k} = B_{n,-k} = C_{n,-k} = D_{n,-k} = 0$ for $k \in \mathbb{N}$. The computation of the system (4.6) is identical. The only difference is that now, the matrices $\mathcal{M}_m$ are of size $(2s) \times (2s)$ $\forall m \in \mathbb{N}$ and the vectors $\mathcal{C}_m \in \mathbb{R}^{2s}$ $\forall m \in \mathbb{N}$.

5 Existence and uniqueness criterion for the boundary value problem (1.1)

Once we have the algebraic description (4.5) of the space S of solutions analytic in $\mathcal{D}_r$ of the equation $(z^2-1)y'' + f(z)y' + g(z)y = h(z)$, we focus our attention on the boundary value problem (1.1) stated in the introduction. We introduce now the two boundary conditions in order to find an algebraic description of the solutions of (1.1). From (2.3) and (2.4) we have

$$\begin{pmatrix} y(-1) \\ y(1) \\ y'(-1) \\ y'(1) \end{pmatrix} = T \begin{pmatrix} a_0 \\ b_0 \\ a_1 \\ b_1 \end{pmatrix},$$

where T is the regular matrix

$$T = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 1 & 2 & 2 \end{pmatrix}.$$

(The first four coefficients a_0, b_0, a_1, b_1 of the two-point Taylor expansion (2.3) are related to $y(-1)$, $y(1)$, $y'(-1)$, $y'(1)$ by the matrix T^{-1}). Write the matrix BT, where B is the 2×4 matrix defining the boundary condition in (1.1), in the form

$$BT = \begin{pmatrix} R_{1,1} & R_{1,2} & R_{1,3} & R_{1,4} \\ R_{2,1} & R_{2,2} & R_{2,3} & R_{2,4} \end{pmatrix}.$$

Then, the boundary value problem (1.1) may be written in the following equivalent form that stresses the role of the first four coefficients of the two-point Taylor expansion of $y(x)$ in the boundary value equations

$$
\begin{cases}
(x^2-1)y''+f(x)y'+g(x)y = h(x) \ \text{ in } \ (-1,1), \\
R_1[a_0,b_0,a_1,b_1] := R_{1,1}a_0+R_{1,2}b_0+R_{1,3}a_1+R_{1,4}b_1-\alpha = 0, \\
R_2[a_0,b_0,a_1,b_1] := R_{2,1}a_0+R_{2,2}b_0+R_{2,3}a_1+R_{2,4}b_1-\beta = 0.
\end{cases}
\tag{5.1}
$$

When we add the above two algebraic equations R_1 and R_2 to the set of equations (4.6) that describe the space S of solutions of $(x^2-1)y''+f(x)y'+g(x)y = h(x)$, we find that the coefficients $a_0, b_0, \ldots, a_{n_0+1}, b_{n_0+1}$ of the two-point Taylor solutions $y(x)$ of (5.1) (if any) are solutions of the algebraic linear system

$$
\begin{cases}
L_k[a_0,b_0,a_1,b_1,\ldots,a_{n_0+1},b_{n_0+1}] = 0, & k=1,2,3,\ldots,2n_0+2, \\
H_k[a_0,b_0,a_1,b_1,\ldots,a_{n_0+1},b_{n_0+1}] = 0, & k=1,2, \\
R_k[a_0,b_0,a_1,b_1] = 0, & k=1,2.
\end{cases}
\tag{5.2}
$$

The remaining coefficients a_n, b_n for $n \geq n_0+2$ are obtained recursively from (3.1). The system (5.2) is a linear system of $2n_0+6$ equations with $2n_0+4$ unknowns (in the regular case, the system reduces to the last 4 equations). The existence and uniqueness of solutions of the system (5.2) is equivalent to the existence and uniqueness of solution of the problem (5.1). Then, we can finally formulate the following existence and uniqueness criterion for the boundary value problem (1.1).

Existence and uniqueness criterion

(i) If the system (5.2) *has not a solution, then problem* (1.1) *has not an analytic solution in $\mathcal{D}_r$.*

(ii) If the system (5.2) *has a unique solution, then problem* (1.1) *has a unique analytic solution in $\mathcal{D}_r$.*

(iii) If the system (5.2) *has a one-dimensional space of solutions, then problem* (1.1) *has a one-dimensional family of analytic solutions in $\mathcal{D}_r$.*

(iv) If the system (5.2) *has a two-dimensional space of solutions, then problem* (1.1) *has a two-dimensional family of analytic solutions in $\mathcal{D}_r$.*

Remark 5.1. According to the ranks of (4.6) and $(4.6)_h$ we have that

1. If rank[(4.6)] $=$ rank[$(4.6)_h$] $= 2n_0+3$, then (iv) is not possible.
2. If rank[(4.6)] $=$ rank[$(4.6)_h$] $= 2n_0+4$, then (iii) and (iv) are not possible.
3. If rank[(4.6)] $\neq$ rank[$(4.6)_h$], then only (i) is possible.

Remark 5.2. In practice, the coefficients of the two equations H_k in (5.2) are computed approximately, as the limits involved in their computation can be computed only approximately (see (4.3) and (4.4)). Therefore, the above existence and uniqueness criterion for solution of (1.1) is useful when system (5.2) is well conditioned. In order to determine the rank of system (5.2) and then, the dimension of the space of solutions, it is convenient to compute the limits of the determinants of the principal minors. On the other hand, the criterion is constructive as it provides an approximation to the solution of the form (2.3) once the coefficients $(a_0, b_0, \ldots, a_{n_0+1}, b_{n_0+1})$ are computed from (5.2).

Remark 5.3. When $-f(1)$ or $f(-1) \neq 0, 2, 4, \ldots$, the rank of the first $2n_0 + 4$ equations in (5.2) is, at least, $2n_0 + 3$ and (iv) is not possible. When $f(\pm 1) = g(\pm 1) = h(\pm 1) = 0$ (regular case), the system (5.2) only consists of the four last equations and the rank of the two equations $H_k = 0$, $k = 1, 2$, is 2. In any other case, the rank of the first $2n_0 + 4$ equations in the system (5.2) is not known *a priori*; it is calculated once we have computed the first $2n_0 + 4$ equations of system (5.2).

The key point in the discussion of the dimensions of S and S_h is system (4.6), and the key point in the discussion of the existence and uniqueness of problem (1.1) is system (5.2). In the examples of the following section we show how these systems are computed in practice and how the above criterium of existence and uniqueness may be implemented.

6 Examples

In the examples of this section we explore all the possible situations in relation to the values of $f(1)$ and $f(-1)$ and the sizes of the spaces S and S_h:

(i) $f(-1)/2$ and $-f(1)/2 \in \mathbb{N} \cup \{0\}$ and $\dim(S_h) = 2$, S is empty. Example 6.1.

(ii) $f(-1)/2$ and $-f(1)/2 \in \mathbb{N} \cup \{0\}$ and $\dim(S_h) = \dim(S) = 2$. Example 6.2.

(iii) $f(-1)/2$ or $-f(1)/2 \notin \mathbb{N} \cup \{0\}$ and $\dim(S_h) = \dim(S) = 1$. Example 6.3.

(iv) $f(-1)/2$ and $-f(1)/2 \in \mathbb{N} \cup \{0\}$ and $\dim(S_h) = \dim(S) = 1$. Example 6.4.

(v) $f(-1)/2$ or $-f(1)/2 \notin \mathbb{N} \cup \{0\}$ and $\dim(S_h) = 1$, S empty. Example 6.5.

(vi) $f(-1)/2$ and $-f(1)/2 \in \mathbb{N} \cup \{0\}$ and $\dim(S_h) = 1$, S empty. Example 6.6.

(vii) $f(-1)/2$ or $-f(1)/2 \notin \mathbb{N} \cup \{0\}$ and $S_h = \{0\}$, S non empty. Example 6.7.

(viii) $f(-1)/2$ and $-f(1)/2 \in \mathbb{N} \cup \{0\}$ and $S_h = \{0\}$, S non empty. Example 6.8.

In all the examples below, the parameters $a, b, c, d, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, C, \alpha$ and β are arbitrary complex numbers. The limits in the m index of the sequences (4.3) are approximated by the value of the sequences at $m = 10$. We have selected examples for which the general solution of the differential equation is known; in this way we may check the validity of the existence and uniqueness criterion of Section 5.

Example 6.1. Consider the boundary value problem

$$\begin{cases} (x^2 - 1)y'' - (x + 1)y' = 1 \quad \text{in } (-1, 1), \\ ay(-1) + by(1) + cy'(-1) + dy'(1) = \alpha, \\ \tilde{a}y(-1) + \tilde{b}y(1) + \tilde{c}y'(-1) + \tilde{d}y'(1) = \beta. \end{cases} \tag{6.1}$$

We have $f(x) = -(x + 1)$, $g(x) = 0$ and $h(x) = 1$. As $f(-1) = 0$ and $f(1) = -2$, the critical exponents at the points $x = \pm 1$ are $\mu_2(-1) = 1$ and $\mu_2(1) = 2$ respectively and $n_0 = 1$. For this example, the recurrence relations (3.1) may be written in the form $v_{n+1} = M_n v_n + c_n$ with $v_n = (a_n, b_n)$, $c_n = (0, 0)$, $n = 2, 3, \ldots$, and

$$M_n = \begin{pmatrix} \frac{1-2n}{2(n+1)} & 0 \\ -\frac{1}{2(n+1)} & -\frac{2n+1}{2(n+1)} \end{pmatrix}.$$

System (4.6) is given by

$$\begin{cases} b_0 + 2a_1 + 2b_1 = -1, \\ b_0 + 2a_1 + 2b_1 = 0, \\ -3b_1 + 4a_2 - 4b_2 = 0, \\ 3b_1 - 4a_2 + 4b_2 = 0, \\ 0.028031a_2 = 0, \\ 0.165074a_2 + 0.358179b_2 = 0, \end{cases}$$

which has no solution. The solution to the homogeneous system $(4.6)_h$ is $(a_1, b_1, a_2, b_2) = (-b_0/2, 0, 0, 0)$ with $a_0, b_0 \in \mathbb{C}$ free parameters. Then, $\dim(S_h) = 2$, but S is empty. This conclusion is the same one that we obtain from the knowledge of the general solution of the differential equation in (6.1)

$$y(x) = c_1 + c_2 x(x-2) + \frac{1}{8}[(x^2 - 2x - 3)\log(x+1) - (x-1)^2 \log(x-1) - 2x].$$

Example 6.2. Consider the boundary value problem

$$\begin{cases} (x^2-1)y'' - 2x^3 y' + 2(x^2+1)y = 0 \quad \text{in } (-1,1), \\ ay(-1) + by(1) + cy'(-1) + dy'(1) = \alpha, \\ \tilde{a}y(-1) + \tilde{b}y(1) + \tilde{c}y'(-1) + \tilde{d}y'(1) = \beta. \end{cases} \tag{6.2}$$

We have $f(x) = -2x^3$, $g(x) = 2(x^2+1)$ and $h(x) = 0$. As $f(-1) = 2$ and $f(1) = -2$, the critical exponents at the points $x = \pm 1$ are $\mu_2(-1) = \mu_2(1) = 2$ respectively and $n_0 = 1$. For this example, the recurrence relations (3.1) may be written in the form $v_{n+1} = M_n v_n + c_n$ with $v_n = (a_{n-1}, b_{n-1}, a_n, b_n)$, $c_n = (0,0,0,0)$, $n = 2, 3, \ldots$, and

$$M_n = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{2n-3}{2(n-1)(n+1)} & 0 & -\frac{(n-2)(2n-1)}{2(n-1)(n+1)} & 0 \\ 0 & \frac{1}{n+1} & 0 & -\frac{2n-1}{2(n+1)} \end{pmatrix}.$$

System (4.6)=$(4.6)_h$ is given by

$$\begin{cases} a_0 - a_1 = 0, \\ b_0 - 2b_1 = 0, \\ a_0 - a_1 = 0, \\ 0.015263a_1 - 0.030525a_2 = 0, \\ 0.150515b_1 - 0.35839b_2 = 0, \end{cases} \tag{6.3}$$

whose solution is $(a_1, b_1, a_2, b_2) = (a_0, b_0/2, a_0/2, 0.209988b_0)$ with $a_0, b_0 \in \mathbb{C}$ free parameters. As $\dim(S_h) = \dim(S) = 2$, the differential equation in (6.2) has a two-dimensional family of analytic solutions in $[-1,1]$, which agrees with the fact that the differential equation has two independent solutions e^{x^2} and $\sqrt{\pi}e^{x^2}\mathrm{erf}(x) + 2x$, both of them analytic in $[-1,1]$.

Now we apply the existence and uniqueness criterion of Section 5: the existence and uniqueness of solution of (6.2) is equivalent to the existence and uniqueness of solution of the linear system given by equations (6.3) and the boundary value conditions written in terms of the coefficients a_k and b_k

$$\begin{cases} (a+b)a_0 + (-a+b+c+d)b_0 + (-2c+2d)a_1 + (2c+2d)b_1 = \alpha, \\ (\widetilde{a}+\widetilde{b})a_0 + (-\widetilde{a}+\widetilde{b}+\widetilde{c}+\widetilde{d})b_0 + (-2\widetilde{c}+2\widetilde{d})a_1 + (2\widetilde{c}+2\widetilde{d})b_1 = \beta, \end{cases} \tag{6.4}$$

that, for this example, are given by

$$\begin{cases} (a+b-2c+2d)a_0 + (-a+b+2c+2d)b_0 = \alpha, \\ (\widetilde{a}+\widetilde{b}-2\widetilde{c}+2\widetilde{d})a_0 + (-\widetilde{a}+\widetilde{b}+2\widetilde{c}+2\widetilde{d})b_0 = \beta. \end{cases}$$

Then, problem (6.2) has a unique solution if and only if

$$\begin{pmatrix} a+b-2c+2d & -a+b+2c+2d \\ \widetilde{a}+\widetilde{b}-2\widetilde{c}+2\widetilde{d} & -\widetilde{a}+\widetilde{b}+2\widetilde{c}+2\widetilde{d} \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{6.5}$$

The existence and uniqueness condition obtained with this criterion coincides with the one provided by the knowledge of the family of analytic solutions of the differential equation given in (6.2)

$$y(x, C_1, C_2) = C_1 e^{x^2} + C_2(\sqrt{\pi} e^{x^2} \operatorname{erf}(x) + 2x).$$

The standard criterion of existence and uniqueness of solution of problem (6.2) depends on the existence of two complex numbers C_1 and C_2 that make $y(x, C_1, C_2)$ compatible with the boundary conditions in (6.2), that is,

$$\begin{pmatrix} (a+b-2c+2d)e & (-a+b+2(c+d))\left(2+e\sqrt{\pi}\operatorname{erf}(1)\right) \\ (\widetilde{a}+\widetilde{b}-2\widetilde{c}+2\widetilde{d})e & (-\widetilde{a}+\widetilde{b}+2(\widetilde{c}+\widetilde{d}))\left(2+e\sqrt{\pi}\operatorname{erf}(1)\right) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{6.6}$$

It can be checked that (6.5) and (6.6) are equivalent.

Example 6.3. Consider the boundary value problem

$$\begin{cases} (x^2-1)y'' + x(1-2x^2)y' + 2y = 0 \quad \text{in } (-1,1), \\ ay(-1) + by(1) + cy'(-1) + dy'(1) = \alpha, \\ \widetilde{a}y(-1) + \widetilde{b}y(1) + \widetilde{c}y'(-1) + \widetilde{d}y'(1) = \beta. \end{cases} \tag{6.7}$$

In this problem, $f(x) = x(1-2x^2)$, $g(x) = 2$ and $h(x) = 0$. We have $f(-1) = 1$ and $f(1) = -1$, so the critical exponents at the points $x = \pm 1$ are $\mu_2(-1) = \mu_2(1) = 3/2$ respectively and $n_0 = -1$.

For this example, the recurrence relations (3.1) may be written in the form $v_{n+1} = M_n v_n + c_n$ with $v_n = (a_{n-1}, b_{n-1}, a_n, b_n)$, $c_n = (0,0,0,0)$, $n = 2, 3, \ldots$, and

$$M_n = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{2(n-1)}{(n+1)(2n-1)} & 0 & -\frac{2n^2-4n+1}{(n+1)(2n-1)} & 0 \\ 0 & \frac{1}{n+1} & 0 & -\frac{2n-1}{2(n+1)} \end{pmatrix}.$$

System (4.6) = $(4.6)_h$ is given by

$$\begin{cases} a_0 - a_1 = 0, \\ b_0 - 2b_1 = 0, \\ -a_1 + 2a_2 = 0, \\ -2b_0 + b_1 + 4b_2 = 0, \\ 0.009033a_1 - 0.018067a_2 = 0, \\ 0.111897b_1 - 0.266438b_2 = 0, \end{cases} \tag{6.8}$$

whose solution is $(b_0, a_1, b_1, a_2, b_2) = (0, a_0, 0, a_0/2, 0)$, with $a_0 \in \mathbb{C}$ a free parameter. As $\dim(S_h) = \dim(S) = 1$, the differential equation in (6.7) has a one-dimensional family of analytic solutions in $[-1,1]$, which agrees with the fact that the differential equation has two independent solutions, e^{x^2-1} and $e^{x^2-1}\int^x e^{-t^2}\sqrt{1-t^2}dt$, and just one of them is analytic in $[-1,1]$.

Now we apply the existence and uniqueness criterion of Section 5: the existence and uniqueness of solution of (6.7) is equivalent to the existence and uniqueness of solution of the linear system given by equations (6.8) and (6.4), that, for this example, are given by

$$\begin{cases} (a+b)a_0 + (-2c+2d)a_0 = \alpha, \\ (\tilde{a}+\tilde{b})a_0 + (-2\tilde{c}+2\tilde{d})a_0 = \beta. \end{cases}$$

Then, problem (6.7) has a unique solution if and only if

$$\frac{\alpha}{a+b-2c+2d} = \frac{\beta}{\tilde{a}+\tilde{b}-2\tilde{c}+2\tilde{d}} \tag{6.9}$$

with $a+b-2c+2d \neq 0$ and $\tilde{a}+\tilde{b}-2\tilde{c}+2\tilde{d} \neq 0$. The existence and uniqueness condition obtained with this criterion coincides with the one provided by the knowledge of the family of analytic solutions of the differential equation given in (6.7)

$$y(x, C) = Ce^{x^2-1}.$$

The standard criterion of existence and uniqueness of solution of problem (6.7) depends on the existence of a complex number C that makes $y(x, C)$ compatible with the boundary conditions in (6.7), that is,

$$\begin{cases} aC + bC - 2cC + 2dC = \alpha, \\ \tilde{a}C + \tilde{b}C - 2\tilde{c}C + 2\tilde{d}C = \beta. \end{cases} \tag{6.10}$$

It can be checked that conditions (6.9) and (6.10) are the same.

Example 6.4. Consider the boundary value problem

$$\begin{cases} (x^2-1)y'' - 2y = -2 \quad \text{in } (-1,1), \\ ay(-1) + by(1) + cy'(-1) + dy'(1) = \alpha, \\ \tilde{a}y(-1) + \tilde{b}y(1) + \tilde{c}y'(-1) + \tilde{d}y'(1) = \beta. \end{cases} \tag{6.11}$$

We have $f(x) = 0$, $g(x) = -2$ and $h(x) = -2$. As $f(-1) = f(1) = 0$, the critical exponents at the points $x = \pm 1$ are $\mu_2(-1) = \mu_2(1) = 1$ respectively and $n_0 = 0$. For this example, the recurrence relations (3.1) may be written in the form $v_{n+1} = M_n v_n + c_n$ with $v_n = (a_n, b_n)$, $c_n = (0,0)$, $n = 1, 2, \ldots$, and

$$M_n = \begin{pmatrix} -\frac{(n-1)(2n+1)}{2n(n+1)} & 0 \\ 0 & -\frac{2n-1}{2n} \end{pmatrix}.$$

System (4.6) and $(4.6)_h$ are given, respectively, by

$$\begin{cases} a_0 = 1, \\ b_0 = 0, \\ 0.176197 b_1 = 0, \end{cases} \qquad \begin{cases} a_0 = 0, \\ b_0 = 0, \\ 0.176197 b_1 = 0, \end{cases} \tag{6.12}$$

whose respective solutions are $(a_0, b_0, b_1) = (1, 0, 0)$ and $(a_0, b_0, b_1) = (0, 0, 0)$ with $a_1 \in \mathbb{C}$ a free parameter. As $\dim(S_h) = \dim(S) = 1$, the differential equation in (6.11) has a one-dimensional family of analytic solutions in $[-1, 1]$, which agrees with the fact that the homogeneous differential equation has two independent solutions, $x^2 - 1$ and $(x^2 - 1)\log((x+1)/(1-x)) - 2x$, and just one is analytic in $[-1, 1]$.

Now we apply the existence and uniqueness criterion of Section 5: the existence and uniqueness of solution of (6.11) is equivalent to the existence and uniqueness of solution of the linear system given by equations (6.12) and (6.4). Then, problem (6.11) has a unique solution if and only if

$$\frac{\alpha - a - b}{d - c} = \frac{\beta - \tilde{a} - \tilde{b}}{\tilde{d} - \tilde{c}} \tag{6.13}$$

with $c \neq d$ and $\tilde{c} \neq \tilde{d}$.

The existence and uniqueness condition obtained with this criterion coincides with the one provided by the knowledge of the family of analytic solutions of the differential equation given in (6.11)

$$y(x, C) = C(x^2 - 1) + 1.$$

The standard criterion of existence and uniqueness of solution of problem (6.11) depends on the existence of a complex number C that makes $y(x, C)$ compatible with the boundary conditions in (6.11), that is,

$$a + b - 2cC + 2dC = \alpha, \quad \tilde{a} + \tilde{b} - 2\tilde{c}C + 2\tilde{d}C = \beta \tag{6.14}$$

It can be checked that equations (6.14) and (6.13) are equivalent.

Example 6.5. Consider the boundary value problem

$$\begin{cases} (x^2 - 1)y'' + y' = x \quad \text{in } (-1, 1), \\ ay(-1) + by(1) + cy'(-1) + dy'(1) = \alpha, \\ \tilde{a}y(-1) + \tilde{b}y(1) + \tilde{c}y'(-1) + \tilde{d}y'(1) = \beta. \end{cases}$$

We have $f(x) = 1$, $g(x) = 0$ and $h(x) = x$. As $f(-1) = f(1) = 1$, the critical exponents at the points $x = \pm 1$ are $\mu_2(-1) = 3/2$ and $\mu_2(1) = 1/2$ respectively and $n_0 = -1$. For this example, the recurrence relations (3.1) may be written in the form $v_{n+1} = M_n v_n + c_n$ with $v_n = (a_n, b_n)$, $c_n = (0,0)$, $n = 1, 2, \ldots$, and

$$M_n = \begin{pmatrix} -\frac{2n^2}{2n^2+3n+1} & 0 \\ \frac{n}{2n^2+3n+1} & -\frac{2n+1}{2n+2} \end{pmatrix}.$$

Systems (4.6) and $(4.6)_h$ are given, respectively, by

$$\begin{cases} b_0 + 2b_1 = 0, \\ 2a_1 = 1, \\ 0.024569a_1 = 0, \\ -0.132232a_1 + 0.336376b_1 = 0, \end{cases} \qquad \begin{cases} b_0 + 2b_1 = 0, \\ 2a_1 = 0, \\ 0.024569a_1 = 0, \\ -0.132232a_1 + 0.336376b_1 = 0. \end{cases}$$

The first system has no solution; the solution of the second one is $(b_0, a_1, b_1) = (0, 0, 0)$ and $a_0 \in \mathbb{C}$ a free parameter. For this example, $\dim(S_h) = 1$ but S is empty, which agrees with the fact that the solution to the differential equation is

$$y(x) = c_1 \left[\sqrt{1-x^2} + 2 \arctan\left(\frac{x}{\sqrt{1-x^2}} \right) \right] + c_2 \\ - 2x + \sqrt{1-x^2} \arctan\left(\frac{x}{\sqrt{1-x^2}} \right) - \frac{1}{2} \arctan\left(\frac{x}{\sqrt{1-x^2}} \right)^2,$$

that is not analytic in $[-1, 1]$ for any value of (c_1, c_2).

Example 6.6. Consider the boundary value problem

$$\begin{cases} (x^2-1)y'' + (1-x)y' + y = x \quad \text{in } (-1, 1), \\ ay(-1) + by(1) + cy'(-1) + dy'(1) = \alpha, \\ \tilde{a}y(-1) + \tilde{b}y(1) + \tilde{c}y'(-1) + \tilde{d}y'(1) = \beta. \end{cases}$$

We have $f(x) = 1 - x$, $g(x) = 1$ and $h(x) = x$. As $f(-1) = 2$ and $f(1) = 0$, the critical exponents at the points $x = \pm 1$ are $\mu_2(-1) = 2$ and $\mu_2(1) = 1$ respectively, and $n_0 = 1$. For this example, the recurrence relations (3.1) may be written in the form $v_{n+1} = M_n v_n + c_n$ with $v_n = (a_n, b_n)$, $c_n = (0, 0)$, $n = 2, 3, \ldots$, and

$$M_n = \begin{pmatrix} -\frac{(2n-1)^3}{8n(n^2-1)} & \frac{1}{8n(n^2-1)} \\ \frac{(1-2n)^2}{8n(n^2-1)} & \frac{1+2n+4n^2-8n^3}{8n(n^2-1)} \end{pmatrix}.$$

System (4.6) is given by

$$\begin{cases} a_0 - 2a_1 + b_0 + 2b_1 = 0, \\ 2a_1 - 2b_1 = 1, \\ a_2 + b_1 + b_2 = 0, \\ a_1 + 4a_2 + 3b_1 + 4b_2 = 0, \\ -0.080292a_2 + 0.005407b_2 = 0, \\ 0.236021a_2 - 0.527175b_2 = 0. \end{cases}$$

This system has no solution. For this example, the solution to the homogeneous system $(4.6)_h$ is $(b_0, a_1, b_1, a_2, b_2) = (-a_0, 0, 0, 0, 0)$ with $a_0 \in \mathbb{C}$ a free parameter. Then, $\dim(S_h) = 1$ but S is empty, which agrees with the fact that the solution of the differential equation is

$$y(x) = c_1(1-x) + c_2[2 + (1-x)\log(x-1)] - (x-1)\mathrm{Li}_2\left(\frac{1-x}{2} \right) \\ + (x-1)\log(1-x) + \log(2)(x-1)\log(x-1) - (x+1)\log(x+1) - 1,$$

that is not analytic in $[-1, 1]$ for any value of (c_1, c_2). (Here $\mathrm{Li}_2(z)$ is the polylogarithmic function.)

Example 6.7. Consider the boundary value problem

$$\begin{cases} (x^2-1)y'' + 4xy' + 2y = e^x \quad \text{in } (-1,1), \\ ay(-1) + by(1) + cy'(-1) + dy'(1) = \alpha, \\ \widetilde{a}y(-1) + \widetilde{b}y(1) + \widetilde{c}y'(-1) + \widetilde{d}y'(1) = \beta. \end{cases} \tag{6.15}$$

We have $f(x) = 4x$, $g(x) = 2$ and $h(x) = e^x$. As $f(-1) = -4$ and $f(1) = 4$, the critical exponents at the points $x = \pm 1$ are $\mu_2(-1) = \mu_2(1) = -1$ respectively and $n_0 = -1$. For this example, the recurrence relations (3.1) may be written in the form $v_{n+1} = M_n v_n + c_n$ with $v_n = (a_n, b_n)$, $c_n = (A_n, B_n)$, $n = 1, 2, \ldots,$

$$M_n = \begin{pmatrix} -\frac{2n+1}{2(n+2)} & 0 \\ 0 & -\frac{2n+3}{2(n+2)} \end{pmatrix},$$

$$A_n = \frac{1}{4(n+1)(n+2)} \sum_{k=0}^{n} \frac{(n+k-1)}{k!n!(n-k-1)!2^{n+k+1}} ((-1)^k e - (-1)^{n+1} e^{-1}),$$

and

$$B_n = \frac{1}{4(n+1)(n+2)} \sum_{k=0}^{n} \frac{(n+k-1)}{k!n!(n-k-1)!2^{n+k+1}} ((-1)^k e + (-1)^{n+1} e^{-1}).$$

System (4.6) is given by

$$\begin{cases} 2a_0 + 8a_1 = \cosh 1, \\ 6b_0 + 8b_1 = \sinh 1, \\ 0.056062a_1 = 0.002946, \\ 0.429814b_1 = 0.004012, \end{cases} \tag{6.16}$$

whose solution is $(a_0, b_0, a_1, b_1) = (0.561323, 0.183421, 0.0525542, 0.00933429)$. In this case, the solution to the system $(4.6)_h$ is $S_h = \{0\}$ and S is non empty.

This conclusion is the same that we obtain from the knowledge of the general solution of the differential equation in (6.15),

$$y(x) = -\frac{e^x}{1-x^2} + \frac{c_1}{1-x^2} + \frac{c_2 x}{1-x^2}.$$

There is only one analytic solution obtained for $(c_1, c_2) = (1, 1)$.

Now we apply the existence and uniqueness criterion of Section 5: the existence and uniqueness of solution of (6.15) is equivalent to the existence and uniqueness of solution of the linear system given by equations (6.16) and (6.4). Then, problem (6.15) has a unique solution if and only if

$$\begin{cases} 0.377902a + 0.744745b + 0.0969813c + 0.307198d = \alpha, \\ 0.377902\widetilde{a} + 0.744745\widetilde{b} + 0.0969813\widetilde{c} + 0.307198\widetilde{d} = \beta. \end{cases}$$

The same conditions may be obtained from the exact solution.

Example 6.8. Consider the boundary value problem

$$\begin{cases} (x^2-1)y'' + \frac{1}{4}y = 0 \quad \text{in } (-1,1), \\ ay(-1) + by(1) + cy'(-1) + dy'(1) = \alpha, \\ \widetilde{a}y(-1) + \widetilde{b}y(1) + \widetilde{c}y'(-1) + \widetilde{d}y'(1) = \beta. \end{cases} \tag{6.17}$$

We have $f(x) = 0$, $g(x) = 1/4$ and $h(x) = 0$. As $f(-1) = f(1) = 0$, the critical exponents at the points $x = \pm 1$ are $\mu_2(-1) = \mu_2(1) = 1$ respectively and $n_0 = 0$. For this example, the recurrence relations (3.1) may be written in the form $v_{n+1} = M_n v_n + c_n$ with $v_n = (a_n, b_n)$, $c_n = (0, 0)$, $n = 1, 2, \ldots$, and

$$M_n = \begin{pmatrix} -\frac{(1-4n)^2}{16n(1+n)} & 0 \\ 0 & -\frac{(1+4n)^2}{16n(1+n)} \end{pmatrix}.$$

System (4.6) = $(4.6)_h$ is given by

$$\begin{cases} a_0 = 0, \\ b_0 = 0, \\ 0.018792 a_1 = 0, \\ 0.360749 b_1 = 0, \end{cases} \tag{6.18}$$

whose solution is $(a_0, b_0, a_1, b_1) = (0, 0, 0, 0)$. Then, $S_h = S = \{0\}$ and the unique analytic solution in $(-1, 1)$ of the differential equation in problem (6.17) is the null solution. This conclusion is the same one that we obtain from the knowledge of the general solution of the differential equation in (6.17), since two independent solutions, none of them analytic in $[-1, 1]$ are

$${}_2F_1\left(-\frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, x^2\right), \quad x\,{}_2F_1\left(\frac{1}{4}, \frac{1}{4}, \frac{3}{2}, x^2\right).$$

Now we apply the existence and uniqueness criterion of Section 5: the existence and uniqueness of solution of (6.17) is equivalent to the existence and uniqueness of solution of the linear system given by equations (6.18) and (6.4). Then, problem (6.17) has a unique solution if and only if $\alpha = \beta = 0$.

7 Final remarks

In Section 2 we have detailed the dimensionality of the space S_h of analytic solutions in $\mathcal{D}_r$ of the homogeneous differential equation $(z^2 - 1)y'' + f(z)y' + g(z)y = 0$. The dimension of S_h is: (i) zero or one when $f(-1) \neq 0, 2, 4, \ldots$ or $f(1) \neq 0, -2, -4, \ldots$; (ii) zero, one or two when $f(-1) = 0, 2, 4, \ldots$ and $f(1) = 0, -2, -4, \ldots$; (iii) two when $f(\pm 1) = g(\pm 1) = 0$ (regular case). We have included the regular case analyzed in [6] as a particular case of the more general situation analyzed in this paper. The dimension of the space S of analytic solutions in $\mathcal{D}_r$ of the complete differential equation is either, the same as the dimension of S_h, or it is empty. A complete characterization of this space is given at the end of Section 4 from the study of the ranks of the algebraic linear systems (4.6) and $(4.6)_h$.

In Section 3 we have derived an algorithm to obtain the two-point Taylor expansion of the solutions of (1.1) (if any). In Section 5 we have given a straightforward and systematic criterion for the existence and uniqueness of analytic solutions of the boundary value problem (1.1). The criterion is very simple and establishes that the existence and uniqueness of solution of the boundary value problem (1.1) is equivalent to the existence and uniqueness of solution of the algebraic linear system (5.2). Two equations of this algebraic system are defined by the limits (4.3), whose exact computation is, in general, difficult. Then, in practice, the entrances of two of the equations of this algebraic system must be computed approximately and then, the solution is computed in an approximated form. Also, in practice, we must apply the above existence and uniqueness criterion for the solution of (1.1) using the approximate linear system. Then, the conclusions about the existence and uniqueness of solution are exact unless the system is ill-conditioned. In this case, the ranks of the coefficient matrix and/or of the augmented matrix of the system (5.2) sensibly depend on the precision in the computation of the approximate limits.

Formally, the criterion proposed in this paper is similar to the standard criterion based on the knowledge of the space of solutions: both criteria relate the existence and uniqueness of solution of the boundary value problem (1.1) to the existence and uniqueness of a solution of an algebraic linear system. As a difference with that standard criterion, our criterion does not require the knowledge of the general solution of the differential equation. This qualitative difference is essential when the general solution of the equation is not known. In this case, the standard criterion is not useful, whereas our criterion can always be applied (except in the case of ill-conditioning before discussed), as we have shown in the examples analyzed in Section 6.

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