The irreducible characters of Sylow p-subgroups of split finite groups of Lie type

by

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A thesis submitted to
The University of Birmingham
for the degree of
DOCTOR OF PHILOSOPHY

School of Mathematics
College of Engineering and Physical Sciences
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March 24th, 2016

ABSTRACT

Let G be a split finite group of Lie type defined over $\mathbb{F}_q$, where $q = p^e$ is a prime power and p is not a very bad prime for G. Let U be a Sylow p-subgroup of G. In this thesis, we provide a full parametrization of the set $\mathrm{Irr}(U)$ of irreducible characters of U when G is of rank 5 or less. In particular, for every character $\chi \in \mathrm{Irr}(U)$ we determine an abelian subquotient of U such that χ is obtained by an inflation, followed by an induction of a linear character of this subquotient.

The characters are given in most cases as the output of an algorithm that has been implemented in the computer system GAP, whose validity is proved in this thesis using classical results in representation theory and properties of the root system associated to G. We also develop a method to determine a parametrization of the remaining irreducible characters, which applies for every split finite group of Lie type of rank at most 5, and lays the groundwork to provide such a parametrization in rank 6 and higher.

ACKNOWLEDGEMENTS

First and foremost, I would like to express my deepest gratitude to Dr. Simon Goodwin, my PhD supervisor, for his patient guidance. His attentive listening and detailed feedback on my work have been extremely helpful throughout my PhD. I deeply acknowledge his efforts to look after me through his highly valuable comments, and at the same time to make me an independent researcher. The development and completion of this work would not have been possible without his guidance.

I am also very much indebted to Dr. Tung Le and Dr. Kay Magaard. Their enthusiasm towards research and determination to pursue a mathematical goal has had a great impact on my research and myself. The collaboration with them has determined a major contribution to this work.

I am particularly grateful to the School of Mathematics of the University of Birmingham for the financial support during my PhD. I would also like to thank several members of the staff, in particular Mrs. Janette Lowe for her consistent support in my PhD progression, and all the members of the algebra research group, for their help, their mathematical enthusiasm and many useful discussions. Sincere thanks go to present and past postgraduate students, who have been pleasant company, and in some cases solid friendship.

I wish to thank Prof. Radha Kessar and Prof. Sergey Shpectorov for attentively reading this work, and for their precious feedback.

Part of my PhD has been spent by the Technische Universität Kaiserslautern. This place has been an amazing research environment; here many useful ideas for this work have been developed. I wish to sincerely thank Prof. Gunter Malle for his financial and mathematical support during my research visits, and all postdoctoral students for their help and company.

I would like to thank Tania, for passing me her positive energy and attitude towards research and life, and especially for covering me in happiness and love.

Many thanks are also due to my friends Arvin, Chema, George, Giulia, Mar, Onur and Seyit for being close to me in this experience, and to my current and former flatmates for their patience.

Ringrazio Wilma e Roberto per il loro affetto e la loro vicinanza. Un sentito grazie anche ai miei amici in Italia, per non smettere di esserci.

Per finire, un ringraziamento speciale. La mia passione per la matematica e per la ricerca si sviluppa da uno stimolo alla curiosità, dalla pazienza e dall'interesse delle persone che per prime mi hanno mostrato le figure geometriche ed insegnato a far di conto. La mia vera fortuna è essere stato cresciuto da queste persone, e da loro accompagnato fino a questo traguardo. A mia mamma Patrizia, a mio papà Giuseppe e a mio fratello Andrea dedico questa tesi.

CONTENTS

INTRODUCTION

A problem of major interest in representation theory of finite groups is to determine the irreducible characters of finite groups of Lie type. These groups form, in a sense, most of the finite nonabelian simple groups, as we know from the classification of finite simple groups [GLS1].

Let p be a prime, and let G be a finite group of Lie type defined over the field $\mathbb{F}_q$ with q elements, where $q = p^e$ for some $e \in \mathbb{Z}_{\geq 1}$. The groundbreaking methods introduced by Deligne and Lusztig in [DL76], involving geometric methods and ℓ-adic cohomology, provided a general procedure for constructing ordinary irreducible characters of G. A system for computing and processing the generic character table of G is developed in [CHEVIE], using the computer algebra system GAP3 [GAP3]. In particular, character tables of finite groups of Lie type of low rank are completely determined in this way.

The problem of studying modular irreducible characters, that is, characters over a field of positive characteristic, is in general wide open. Motivation for the research in this thesis is the representation theory of G in non-defining characteristic $\ell \neq p$. The standard approach for its study is by determining the decomposition numbers of irreducible ordinary characters into irreducible Brauer characters. The case of defining characteristic generally comes down to the representation theory of the underlying algebraic group, and involves different methods; we do not consider this case in this thesis.

A successful approach to the representation theory of G in non-defining characteristics is by inducing ordinary characters of certain classes of subgroups; this gives projective characters of G. In [GH97], decomposition numbers are determined for all classical types assuming that ℓ is a *linear prime* for G, that is, both ℓ and the order of q modulo ℓ are odd. The main idea is to use Harish-Chandra induction of characters of proper Levi subgroups of G. The induction of another type of characters, namely generalized Gelfand-Graev characters arising from unipotent subgroups of G, is of major importance in the more recent works [DM15] and [DM16]. In these works decomposition numbers are obtained, respectively, when $\ell \mid q+1$ and ℓ is a good prime for G, and when $\ell \mid q^2+1$ up to few unknowns in types ${}^2\mathrm{E}_6(q^2)$ and $\mathrm{F}_4(q)$ (one also has to assume that p is a good prime for G of types C_r, $r \leq 4$ or F_4).

The methods in [DM15] and [DM16] provide us with motivation to study the irreducible characters of a maximal unipotent subgroup of G, namely a Sylow p-subgroup U of G. The character theory of U is also used in [HN14], where decomposition numbers of groups of Lie types B_3 and C_3 are obtained via ordinary irreducible characters of parabolic subgroups; for example, see [Him11] and [HH13] for similar applications of the character theory of parabolic subgroups to the modular representation theory of G in certain low rank cases.

Independently of the above, the problem of studying the irreducible characters and conjugacy classes of U has attracted a lot of interest for many years, with motivation going back to the work of Higman in the 1960s, and significant progress by several authors, especially in the last decade.

Before we go into more detail on this, we fix some more notation. We often write $U(G)$ for the fixed Sylow p-subgroup U of the finite group of Lie type G. If G is of type Y_r, with r the rank of G, then we also write $\mathrm{UY}_r(q)$, or more simply U_{Y_r} for $U(G)$. We denote by $\mathrm{Irr}(U)$ the set of irreducible characters of U. We denote by $k(U)$ (respectively $k(U,D)$) the number of elements of $\mathrm{Irr}(U)$ (respectively the number of elements of $\mathrm{Irr}(U)$ of degree D). By Φ^+ we denote the subset of positive roots of the root system Φ associated to G.

There are several open problems about the representation theory of U. In fact, for G

of a fixed type, a generic expression for $k(U)$ as a function of q is not known in general; it appears to be a very difficult problem to obtain such an expression. A conjecture attributed to Higman, see for example [Hig60], states that $k(U_{\mathrm{A}_r})$ can be expressed as a polynomial in q with integer coefficients for every $r \geq 1$. Lehrer then conjectured in [Leh74] that every character degree in U_{A_r} is a power of q, and that $k(U_{\mathrm{A}_r}, q^d)$ can be expressed as a polynomial in q with integer coefficients for every $d \geq 0$. Finally, Isaacs conjectured in [Is07] that the expressions of such polynomials $k(U_{\mathrm{A}_r}, q^d)$ in $v := q-1$ should have non-negative coefficients. Although the last statement was proven to hold for $r \leq 12$ [AVL03], and for every $d \leq 8$ [Mar11], the above conjectures are still open. The recent works [HP11] and [PS15] suggest that they might not hold. These conjectures naturally generalize to all classical types.

The focus of this thesis is to describe in more detail the set $\mathrm{Irr}(U)$. A parametrization of $\mathrm{Irr}(U)$ is already known in literature for $U = U_{\mathrm{A}_r}$ and $r \leq 12$ [Ev11], and $U = U_{\mathrm{D}_4}$ [HLM11]. Moreover, the minimal degree almost faithful irreducible characters are parametrized for every type and rank when G is split and p is not a very bad prime for G in [HLM15].

The main goal of this thesis is to develop a method towards a complete parametrization of $\mathrm{Irr}(U)$, when G is a split finite group of Lie type and p is not a very bad prime for G. This is achieved in this work when the rank of G is 5 or less. The main result of this thesis is the following theorem.

Theorem A. *Let G be one of the groups $\mathrm{B}_4(q), \mathrm{B}_5(q), \mathrm{C}_4(q), \mathrm{C}_5(q)$ and $\mathrm{F}_4(q)$ for $p \neq 2$, and $\mathrm{D}_4(q), \mathrm{D}_5(q)$ for every p. The irreducible characters of $U(G)$ are completely parametrized in Tables D.1 to D.7. Moreover, each character $\chi \in \mathrm{Irr}(U(G))$ can be obtained as an inflation, followed by an induction of a linear character of a certain subquotient of $U(G)$ that can be determined from the information in Appendix D.*

Part of this result is contained in [GLMP15], where the parametrization as in Theorem A is obtained when G is of rank at most 4. The methods in this work develop those used in [HLM11] and [HLM15], and make significant further progress. From now on, we assume that G is split and p is not a very bad prime for G. We develop an algorithm, namely Algorithm 2.6, which works by a successive reduction of characters to smaller subquotients of U. Such subquotients are associated to certain pairs of subsets of Φ^+, called *cores*, which we define below. The irreducible characters of U are then obtained by a process of inflation and induction from these subquotients. The reduction is similar to the one used in [Ev11] for the parametrization in type A_r.

The algorithm yields a parametrization of nearly the entire set $\mathrm{Irr}(U)$ for G of rank less than or equal to 6, namely of all characters that arise from *abelian cores*, as explained later. In principle, this algorithm also works for rank 7 and higher, but the output would contain a very large number of *nonabelian cores*, and the situation is much more complicated to analyse.

We have implemented Algorithm 2.6 in the computer algebra system GAP3 by using CHEVIE [CHEVIE]. This immediately determines a parametrization of irreducible characters for types considered in Theorem A arising from abelian cores, which are collected in plain font in Appendix D. On the other hand, Table 2.2 gives a measure of the elements of $\mathrm{Irr}(U)$ which are not immediately parametrized by the algorithm. These are dealt with via an ad-hoc study, which we explain later. Labels for irreducible characters obtained in this way are collected in bold font in Appendix D.

The approach used in [HLM11], [HLM15] and this work is built on partitioning the irreducible characters of U in terms of the root subgroups that lie in their centre, but not in their kernel. Consequently, there are similarities to the theory of supercharacters, which were first studied for G of type A_r by André, see for example [An01]. This theory was fully developed by Diaconis and Isaacs in [DI08]. Subsequently it was applied to the characters of U for G of types B_r, C_r and D_r by André and Neto in [AN09].

We now go in more detail about the methods applied in our reduction. For $\alpha \in \Phi^+$, we denote by X_α the corresponding root subgroup of U. In the algorithm we consider certain subquotients of U, which we refer to as quattern groups. A *pattern subgroup* of U is a subgroup that is a product of root subgroups, and a *quattern group* is a quotient of a pattern subgroup by a normal pattern subgroup. We refer to Section 1.3 for a precise definition. A quattern group is determined by a subset $\mathcal{S}$ of Φ^+ and denoted by $X_\mathcal{S}$. Let $\mathcal{Z}$ be a subset of $\{\alpha \in \mathcal{S} \mid X_\alpha \subseteq Z(X_\mathcal{S})\}$, where $Z(X_\mathcal{S})$ is the centre of $X_\mathcal{S}$. We define

$$\mathrm{Irr}(X_\mathcal{S})_\mathcal{Z} = \{\chi \in \mathrm{Irr}(X_\mathcal{S}) \mid X_\alpha \not\subseteq \ker\chi \text{ for all } \alpha \in \mathcal{Z}\}.$$

At each stage of the algorithm, we are considering a pair $(\mathcal{S}, \mathcal{Z})$ as above. We attempt to apply one of two possible types of reductions to reduce $(\mathcal{S}, \mathcal{Z})$ to one or two pairs such that the irreducible characters in $\mathrm{Irr}(X_\mathcal{S})_\mathcal{Z}$ are in bijection with those irreducible characters corresponding to the pairs we have obtained in the reduction.

The first reduction is based on the elementary but powerful character theoretic result [HLM15, Lemma 2.1], which we refer to as the *reduction lemma*. In Lemma 2.1, we state and prove a specific version of this lemma, which is the basis of the reduction. This lemma shows that under certain conditions (which are straightforward to check) we can replace $(\mathcal{S}, \mathcal{Z})$ with $(\mathcal{S}', \mathcal{Z})$, where $\mathcal{S}'$ contains two fewer roots than $\mathcal{S}$, and we have a bijection between $\mathrm{Irr}(X_\mathcal{S})_\mathcal{Z}$ and $\mathrm{Irr}(X_{\mathcal{S}'})_\mathcal{Z}$.

The second reduction is more elementary and used when it is not possible to apply the first reduction. For this we choose a root α such that $\alpha \notin \mathcal{Z}$, but $X_\alpha \subseteq Z(X_\mathcal{S})$. Then $(\mathcal{S}, \mathcal{Z})$ is replaced with the two pairs $(\mathcal{S} \setminus \{\alpha\}, \mathcal{Z})$ and $(\mathcal{S}, \mathcal{Z} \cup \{\alpha\})$. The justification of this reduction is that $\mathrm{Irr}(X_\mathcal{S})_\mathcal{Z}$ can be partitioned into the characters for which X_α is contained in the kernel, namely $\mathrm{Irr}(X_{\mathcal{S}\setminus\{\alpha\}})_\mathcal{Z}$, and the characters for which X_α is not contained in the kernel, namely $\mathrm{Irr}(X_\mathcal{S})_{\mathcal{Z}\cup\{\alpha\}}$.

We first partition the characters in terms of the root subgroups that lie in their kernel, and then apply the reductions to each part of this partition. After we have successively applied these reductions as many times as possible, we are left with a set $\{(\mathcal{S}_1, \mathcal{Z}_1), \dots, (\mathcal{S}_m, \mathcal{Z}_m)\}$ for some $m \in \mathbb{Z}_{\geq 1}$ such that $\mathrm{Irr}(U)$ is in bijection with the disjoint union

$$\bigsqcup_{i=1}^{m} \mathrm{Irr}(X_{\mathcal{S}_i})_{\mathcal{Z}_i}.$$

We refer to the pairs $(\mathcal{S}_i, \mathcal{Z}_i)$ as *cores*. In many cases we have that $X_{\mathcal{S}_i}$ is abelian, in which case it is trivial to determine $\mathrm{Irr}(X_{\mathcal{S}_i})_{\mathcal{Z}_i}$. Correspondingly, the character labels as in Appendix D are given by the roots contained in $\mathcal{Z}_i$ and $\mathcal{S}_i \setminus \mathcal{Z}_i$. These labels are described in more detail in Section 2.4.

The more interesting cases are where the $X_{\mathcal{S}_i}$ are not abelian. We refer to these as nonabelian cores. In these cases, there is still some work required to determine $\mathrm{Irr}(X_{\mathcal{S}_i})_{\mathcal{Z}_i}$. Our approach to study nonabelian cores builds on work in [LM15], where certain nonabelian cores in types D_4, E_6 and E_8 were already studied. The complication in these cases is that although some version of the reduction lemma can be applied, as explained in Section 3.2, we reduce to a subquotient which is *not* a quattern group. Therefore we cannot apply Algorithm 2.6.

The analysis of nonabelian cores often gives irreducible characters of U whose degrees are not powers of q when p is a bad prime. We recall the results of major relevance about degrees of irreducible characters of U. By studying "strong subgroups" of algebra groups, it was proven in [Is95] that every character degree in U_{A_r} is a power of q, proving one of the conjectures stated by Lehrer in [Leh74]. This result was later proven to hold, with similar methods, in types B_r, C_r and D_r if and only if $p \neq 2$ [San03]. Examples of characters of degree $q^7/3$ in $\mathrm{UE}_6(3^e)$ and $q^{16}/5$ in $\mathrm{UE}_8(5^e)$ are obtained in [LM15], by inflating a family of characters from a quattern group of U. Finally, via the analysis outlined in the proof of [HH09, Theorem 5.2], we get an irreducible character of degree $q/2$ (respectively $q/3$) in $\mathrm{Irr}(\mathrm{UG}_2(2^e))$ (respectively $\mathrm{Irr}(\mathrm{UG}_2(3^e))$) by looking at the character table of a Borel subgroup in [EY86] (respectively [Eno76]).

The typical situation that we get for G up to rank 5 for nonabelian cores is dealt with by applying the argument in Section 3.1, that is, by studying the representation theory of certain 3-dimensional groups over $\mathbb{F}_q$ that naturally appear in our new reduction process. In some of these cases, the behavior of good and bad primes is completely different. On the one hand, in the case of good primes we always obtain characters of degree q^d for some $d \geq 0$. This is proved for almost all split finite groups of Lie type, as stated in Theorem B later. In fact, we know this is true for all split finite groups of Lie type when $p \geq h$, where h is the Coxeter number of G; this is proved in [GMR15] by using the Kirillov orbit method. On the other hand, in the case of bad primes the analysis of these 3-dimensional groups often yields character degrees of the form q^d/p for some $d \geq 1$.

For the study of $\mathrm{Irr}(X_{\mathcal{S}_i})_{\mathcal{Z}_i}$ when $X_{\mathcal{S}_i}$ is not abelian, one has to go in detail into several computations, for example when computing orbits of characters by conjugation. The last two chapters of this work are devoted to expanding the computations and parametrizing $\mathrm{Irr}(X_{\mathcal{S}_i})_{\mathcal{Z}_i}$ in these cases. We again obtain such characters as an inflation, followed by an induction from a certain abelian subquotient of U, but now this subquotient is often *not* a quattern group. This completes the parametrization of $\mathrm{Irr}(U(G))$ stated in Theorem A.

We now point out some consequences of such parametrization. Firstly, we can now complete the state of the art for character degrees in exceptional types for bad primes. By inflation of characters, looking at suitable subgraphs of Dynkin diagrams, we easily see that the results in types D_4 for $p = 2$, E_6 for $p = 3$ and E_8 for $p = 5$ imply that for G of types $\mathrm{E}_6, \mathrm{E}_7$ and E_8 and p a bad prime for G, there exists $\chi \in \mathrm{Irr}(U)$ and some $d \in \mathbb{Z}$ such that $\chi(1) = q^d/p$ for every power q of p. Moreover, we find in Section 4.3 some irreducible characters of $\mathrm{UF}_4(3^e)$ of degree $q^4/3$. These results, together with [Is95], [San03], [HLM11] and other results in this thesis, allow us to state the following theorem.

Theorem B. *Let G be a split finite group of Lie type over $\mathbb{F}_q$.*

(i) If p is a bad prime for G, then there exists $\chi \in \mathrm{Irr}(U(G))$ and some $d \geq 1$ such that $\chi(1) = q^d/p$ for every power q of p.

(ii) If p is at least the Coxeter number of G, or if p is a good prime and G is not $\mathrm{E}_6(q)$, $\mathrm{E}_7(q)$ or $\mathrm{E}_8(q)$, then the degree of every irreducible character of $U(G)$ is a power of q.

As a second consequence of the parametrization, we obtain more information about the number of characters of a fixed degree. The work in [GMR15] determines expressions of $k(U(G), D)$ as polynomials in $v = q - 1$ with nonnegative coefficients for every G of rank at most 8, except E_8. These are proved to be valid when p is at least the Coxeter number of G, as the Kirillov method for adjoint orbits is applied. The work in this thesis extends the validity of such expressions to every good prime p when the rank of G is at most 5. Moreover, in contrast with the previous result, we find that the expression of $k(\mathrm{UF}_4(q), q^4)$ as a polynomial in v when $q = 3^e$ does not have integer coefficients, as we note from Table 4.3.

This work provides multiple directions for future work. The first one is about an application to the modular representation theory of G previously mentioned, namely to obtain decomposition numbers for G of type F_4. In [DM16], decomposition numbers for $\mathrm{F}_4(q)$ are obtained when p is a good prime, and even then not all decomposition numbers are determined. By inducing characters of $\mathrm{Irr}(U_{\mathrm{F}_4})$ obtained in this work, one hopes to get new decomposition numbers for $\mathrm{F}_4(q)$ when $p \geq 3$, on the one hand providing brand new results for $p = 3$, and on the other hand filling the gaps in the case where $p \geq 5$.

The parametrization of $\mathrm{Irr}(U)$ is one of the key steps for the construction of the generic character table of U. Thanks to the results in [GR09] and [GMR14], the generic conjugacy classes of U have been parametrized when p is a good prime for G of rank at most 7, except E_7. These have also been determined in [BG14] for bad primes when G is a group of rank at most 4, except F_4. We already have examples of computations of generic character tables, namely the recent work in [GLM15], which builds on such a parametrization of conjugacy classes, and on the parametrization of $\mathrm{Irr}(U_{\mathrm{D}_4})$ obtained in [HLM11]. The size of U_{D_4} allows to complete a case-by-case check with a limited use of computer algebra. A future approach for constructing other generic character tables, for example for $\mathrm{UF}_4(q)$ when $p \geq 5$, aims on the one hand to generalize the methods used in [GLM15], and on the other hand to implement an algorithm in CHEVIE or MAGMA that allows us to deal with several easy cases at once, similarly to what is done in Algorithm 2.6 for the parametrization of $\mathrm{Irr}(U)$. We also aim to develop methods to get a parametrization of conjugacy classes of $\mathrm{UF}_4(3^e)$ and possibly also in higher rank groups. This would allow us to construct generic character tables in the case of bad primes, taking advantage of the parametrization of irreducible characters provided in this work.

The reason why we stick with the assumption of p not very bad, is that we are allowed to use the "dictionary" provided in Section 1.3 between subgroups (respectively subquotients) of U and patterns (respectively quatterns) in Φ^+. Removing this assumption on p, a weaker version of Proposition 2.1 and Algorithm 2.6 could be formulated. This has partially been developed in [Fal16], and is a work in progress. Such achievement would lead to a parametrization of $\mathrm{Irr}(U)$ for types $\mathrm{B}_3, \mathrm{C}_3, \mathrm{B}_4, \mathrm{C}_4$ and F_4 when $p = 2$, towards the determination of missing decomposition numbers for G in these cases.

The advantage of working with split finite groups of Lie type is that we can describe the structure of certain subquotients of U in terms of the root subgroups they contain. In the case of a twisted group, the construction of root subgroups is more complicated; the algorithm presented in this work does not yet cover these cases. Nevertheless, a complete parametrization of the irreducible characters of a Sylow p-subgroup of ${}^3\mathrm{D}_4(q^3)$ is obtained in [Le13] by taking advantage of such a construction of root subgroups. Another direction for future research is to take advantage of the computational methods presented in this work and of the construction in [Le13] to obtain irreducible characters of Sylow p-subgroups of other twisted finite groups of Lie type, as for example ${}^2\mathrm{E}_6(q^2)$, for which some decomposition numbers have not yet been determined.

Lastly, we mention the problem of determining a parametrization for the elements of $\mathrm{Irr}(U)$ in rank 6 and higher. On the one hand, we would like to classify simultaneously nonabelian cores with a similar structure, with methods similar to that outlined in Section 5.1. This would be useful especially in type E_6, where quattern groups arising from nonabelian cores are not as big as in other rank 6 types, but occur frequently; this is a work in progress [GLMP16]. On the other hand, we want to implement in CHEVIE or MAGMA a program to analyse a fixed nonabelian core, that is, to get the ad-hoc examination in Chapter 4 and Chapter 5 to be performed by a machine. For example, in each of types B_6 and C_6 a nonabelian core gives rise to a quattern group of order q^{43}. We do not yet know a parametrization of the irreducible characters arising from these cores.

This thesis is structured as follows. In Chapter 1 we present the main background results about finite groups of Lie type and the character theory of finite groups, and we introduce the notions of quatterns and quattern groups, stating their properties. In Chapter 2 we develop and formally state the algorithm that allows us to parametrize $\mathrm{Irr}(U)$ up to the study of nonabelian cores. We outline a method to analyse nonabelian cores in Chapter 3. This is applied in the next two chapters to parametrize $\mathrm{Irr}(U)$ when the rank of G is at most 5, namely in Chapter 4 for G of rank 4, and in Chapter 5 for G of rank 5. In Chapter 5 we also discuss how one might be able to proceed in higher rank cases. In the appendices we collect basic notation and we give the explicit parametrization of the characters.

CHAPTER 1

PRELIMINARIES AND BACKGROUND RESULTS

We present in this chapter the basic results and notation on which the thesis relies. We recall in Section 1.1 some properties of linear algebraic groups, which allow us to define split finite groups of Lie type and root data. In Section 1.2, we state the main results of character theory that are used in the subsequent chapters. We end with Section 1.3, recalling the notions of patterns and antichains for a root system, and introducing *quatterns*.

1.1 Finite groups of Lie type and root data

We recall in this section the definition of finite groups of Lie type, root systems and their properties, and we mention the basic results that lead to the classification of root data and split finite groups of Lie type. The main references we use are [DM] and [MT].

Let p be a prime, and let $q = p^e$ be a prime power for some $e \in \mathbb{Z}_{\geq 1}$. We denote by $k = \overline{\mathbb{F}}_p$ the algebraic closure of the finite field $\mathbb{F}_p$ with p elements.

We denote by $\mathbb{G}$ a *linear algebraic group* over k, that is, a group which is also an affine algebraic variety, such that multiplication and inversion are morphisms of varieties. A *homomorphism* of linear algebraic groups is a group homomorphism which is also a morphism of algebraic varieties. For $\mathbb{G}, \mathbb{G}'$ linear algebraic groups, let $\mathrm{Hom}(\mathbb{G}, \mathbb{G}')$ be the set of homomorphisms from $\mathbb{G}$ to $\mathbb{G}'$. If $\phi \in \mathrm{Hom}(\mathbb{G}, \mathbb{G}')$, we have that the kernel of ϕ is a closed subgroup of $\mathbb{G}$. In fact, $\ker \phi$ is in turn an algebraic group, as is every closed subgroup of $\mathbb{G}$. As in [MT, Theorem 1.7], there exists an $n \geq 1$ such that $\mathbb{G}$ is isomorphic to a closed subgroup of the group $\mathrm{GL}_n(k)$ of invertible $n \times n$ matrices over k.

We say that $\mathbb{G}$ is *connected* if it cannot be decomposed as a disjoint union of proper closed subsets. We call by $\mathbb{G}^\circ$ the connected component of $\mathbb{G}$ containing the identity element. We have that $\mathbb{G}^\circ$ is in turn an algebraic group, and a normal subgroup in $\mathbb{G}$.

We recall the *right* (respectively *left*) *conjugation* action of $\mathbb{G}$ on itself by $g^h := h^{-1}gh$ (respectively ${}^hg := hgh^{-1}$) for $g, h \in \mathbb{G}$. Let $\mathbb{H}$ be a closed subgroup of $\mathbb{G}$. We denote by $C_{\mathbb{G}}(\mathbb{H})$ the *centralizer* of $\mathbb{H}$ in $\mathbb{G}$, and by $N_{\mathbb{G}}(\mathbb{H})$ the *normalizer* of $\mathbb{H}$ in $\mathbb{G}$.

Let us denote by $\mathbb{G}_m$ the algebraic group $(k^\times, *)$, where $k^\times := k \setminus \{0\}$. A *torus* is an algebraic group isomorphic to $\mathbb{G}_m^s$ for some $s \in \mathbb{Z}_{\geq 1}$. Let $X(\mathbb{T}) = \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ be the *character group* of $\mathbb{T}$. We have that $X(\mathbb{T})$ is isomorphic to $\mathbb{Z}^s$, since the elements of $X(\mathbb{T})$ are of the form $(t_1, \ldots, t_s) \mapsto t_1^{a_1} \cdots t_s^{a_s}$ for $a_1, \ldots, a_s \in \mathbb{Z}$. The *cocharacter group* $Y(\mathbb{T}) = \mathrm{Hom}(\mathbb{G}_m, \mathbb{T})$ is also isomorphic to $\mathbb{Z}^s$; in particular, the element $t \mapsto (t^{a_1}, \ldots, t^{a_s})$ of $Y(\mathbb{T})$, for $a_1, \ldots, a_s \in \mathbb{Z}$, is mapped to $(a_1, \ldots, a_s) \in \mathbb{Z}^s$. We note that for every $\gamma \in Y(\mathbb{T})$ and $\chi \in X(\mathbb{T})$, there exists $c_{\chi,\gamma} \in \mathbb{Z}$ such that $\chi \circ \gamma(x) = x^{c_{\chi,\gamma}}$ for every $x \in k^\times$. Let us put $\langle \chi, \gamma \rangle := c_{\chi,\gamma}$. As in [MT, Proposition 3.6] the map defined by

$$\begin{aligned} \langle\ ,\ \rangle : X(\mathbb{T}) \times Y(\mathbb{T}) &\longrightarrow \mathbb{Z} \\ (\chi, \gamma) &\mapsto \langle \chi, \gamma \rangle, \end{aligned}$$

is a *perfect pairing*, that is, group homomorphisms from $X(\mathbb{T})$ to $\mathbb{Z}$ are of the form $\chi \mapsto \langle \chi, \gamma \rangle$ for some $\gamma \in Y(\mathbb{T})$ and vice versa, inducing a duality between $X(\mathbb{T})$ and $Y(\mathbb{T})$.

We denote by $R(\mathbb{G})$ the *solvable radical* of $\mathbb{G}$, that is, the maximal closed connected solvable normal subgroup of $\mathbb{G}$. Moreover, $R_u(\mathbb{G})$ denotes the *unipotent radical* of $\mathbb{G}$, which is the maximal closed connected unipotent normal subgroup of $\mathbb{G}$. We call $\mathbb{G}$ *simple* if $\mathbb{G} \neq 1$, and $\mathbb{G}$ has no nontrivial proper closed connected normal subgroups. We say that $\mathbb{G}$ is *semisimple* if it is connected and $R(\mathbb{G}) = 1$, and that $\mathbb{G}$ is *reductive* if $R_u(\mathbb{G}) = 1$.

From now on, we assume that $\mathbb{G}$ is a connected reductive algebraic group. By [MT, Theorem 8.21, Corollary 8.22], we have that

$$\mathbb{G} = Z(\mathbb{G})\, \mathbb{G}_1 \cdots \mathbb{G}_m$$

for some $m \geq 1$, where $Z(\mathbb{G})$ is the centre of $\mathbb{G}$, and $\mathbb{G}_1, \ldots, \mathbb{G}_m$ are simple algebraic groups, such that $\mathbb{G}_i \cap \mathbb{G}_j \subseteq Z(\mathbb{G})$ for every $1 \leq i < j \leq m$.

We denote by $\mathbb{B}$ a *Borel* subgroup of $\mathbb{G}$, namely a maximal closed connected solvable subgroup of $\mathbb{G}$. As in [MT, Theorem 6.4, Corollary 6.5], all Borel subgroups are conjugate in $\mathbb{G}$, and all maximal tori are conjugate in $\mathbb{G}$. From now on, we fix a maximal torus $\mathbb{T}$ and a Borel subgroup $\mathbb{B}$, such that $\mathbb{T} \subseteq \mathbb{B}$. Let $\mathbb{U} := R_u(\mathbb{B})$ denote the unipotent radical of $\mathbb{B}$. Then we have that $\mathbb{B} = \mathbb{T}\mathbb{U}$, and $N_{\mathbb{G}}(\mathbb{U}) = \mathbb{B}$.

The following definition is of major importance in this work.

Definition 1.1 ([DM], Definition 0.25; [MT], Definition 9.1)**.** Let V be a real vector space, and let V^* be its dual, with duality map $\langle\, ,\rangle$. A *root system* Φ in V is a subset of V such that

(i) Φ is finite and spans V, and $0 \notin \Phi$,

(ii) If $\alpha, c\alpha \in \Phi$ for $c \in \mathbb{R}$, then $c \in \{\pm 1\}$,

(iii) for every $\alpha \in V$, there exists $\check{\alpha} \in V^*$, with $\langle \alpha, \check{\alpha} \rangle = 2$, such that the reflection

$$s_\alpha : V \longrightarrow V$$
$$x \mapsto x - \langle x, \check{\alpha} \rangle \alpha$$

stabilizes Φ, and

(iv) for every $\alpha, \beta \in \Phi$, we have $\langle \beta, \check{\alpha} \rangle \in \mathbb{Z}$.

There exists a scalar product $(\ , \ )$ on V which is invariant by s_α for every $\alpha \in \Phi$. We can then identify V with its dual by $\check{\alpha} \mapsto 2\alpha/(\alpha, \alpha)$. We say that Φ is *irreducible* if we cannot decompose it as a union of two orthogonal subsets with respect to the scalar product defined on V.

The following result provides us with a very important example of a root system. We refer to [DM, Theorem 0.31] and [MT, Definition 8.1]. We have that

$$\Phi := \{\alpha \in X(\mathbb{T}) \mid C_{\mathbb{G}}((\ker \alpha)^\circ) \supsetneq \mathbb{T}\}$$

is a root system in the subspace of $X(\mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{R}$ it generates, with duality $\langle \, , \rangle$. Moreover, by [MT, Proposition 9.11], we have that $\check{\Phi} \subseteq Y(\mathbb{T})$ is a root system in the subspace it generates in $Y(\mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{R}$. We recall that $X(\mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}\Phi$ if $\mathbb{G}$ is semisimple. There is an obvious notion of isomorphism of root systems, that is, a map that preserves the properties stated in Definition 1.1.

We denote by $(X, \Psi, Y, \check{\Psi})$ a *root datum* as in [MT, Definition 9.10], that is, X and Y are free abelian groups of finite rank in perfect pairing, which induces a duality between the associated root systems Ψ and $\check{\Psi}$. We can associate a root datum to $\mathbb{G}$ via the previously chosen torus $\mathbb{T} \subseteq \mathbb{G}$, namely the quadruple $(X(\mathbb{T}), \Phi, Y(\mathbb{T}), \check{\Phi})$. There is a natural notion of isomorphism of root data.

By [MT, Proposition 9.4], we can always pick a *basis* $\Pi = \{\alpha_1, \dots, \alpha_r\} \subseteq \Phi$ for Φ, that is, a basis of the vector space generated by Φ, such that every $\alpha \in \Phi$ can be written as $\alpha = c_1\alpha_1 + \cdots + c_r\alpha_r$ for some $c_1, \dots, c_r \in \mathbb{Z}$, with either $c_i \geq 0$ for all $i = 1, \dots, r$ or $c_i \leq 0$ for all $i = 1, \dots, r$. In the former case, we say that α is a *positive root*, and a *negative root* in the latter. We denote by Φ^+ (respectively Φ^-) the set of positive (respectively negative) roots in Φ. Then we have that $\Phi = \Phi^+ \sqcup \Phi^-$. The number r is called the *rank* of Φ (or of $\mathbb{G}$). Fixed $\alpha \in \Phi^+$, with $\alpha = c_1\alpha_1 + \cdots + c_r\alpha_r$, the *height* of α is the positive number $c_1 + \cdots + c_r$. As in [MT, Proposition 13.10], there exists a *unique* positive root of maximal height, which we denote by α_0. We call α_0 the *highest* root of Φ.

We obtain a classification of irreducible root systems via graphs called *Dynkin diagrams*, which are collected in Figure 1.1. More precisely, the statement is as follows.

Theorem 1.2 ([Hum], Theorem 11.4)**.** *Let Φ be an irreducible root system of rank r. Then the Dynkin diagram associated to Φ is one of the following,*

$$\mathrm{A}_r\,(r \geq 1), \quad \mathrm{B}_r\,(r \geq 2), \quad \mathrm{C}_r\,(r \geq 2), \quad \mathrm{D}_4\,(r \geq 4), \quad \mathrm{E}_r\,(r = 6, 7, 8), \quad \mathrm{F}_4, \quad \mathrm{G}_2.$$

Correspondingly, we say that Φ (or $\mathbb{G}$) is of *type* $\mathrm{A}_r, \mathrm{B}_r, \ldots, \mathrm{G}_2$. We recall some properties of a root system that can be deduced by looking at its Dynkin diagram; we refer to [Hum, Chapter 11] for a more complete overview. The number r of nodes of a Dynkin diagram is the rank of the corresponding root system. The numbering of the nodes determines a choice for the simple roots. A root system is irreducible if and only if the corresponding Dynkin diagram is connected. Moreover, two root systems are isomorphic if and only if they have the same Dynkin diagram.

By studying root data, we also obtain a classification of semisimple algebraic groups, namely [MT, Theorem 9.13], as follows.

Theorem 1.3. *Let $\mathbb{G}$ and $\mathbb{G}'$ be semisimple algebraic groups. Then $\mathbb{G}$ is isomorphic to $\mathbb{G}'$ if and only if the associated root data are isomorphic. Moreover, for each root datum there exists a semisimple algebraic group $\mathbb{G}$ which realizes it. In particular, $\mathbb{G}$ is simple if and only if its root system is irreducible.*

For every $\alpha \in \Phi^+$, there exist a subgroup $\mathbb{U}_\alpha$ of $\mathbb{U}$ and an isomorphism $x_\alpha : k \to \mathbb{U}_\alpha$,

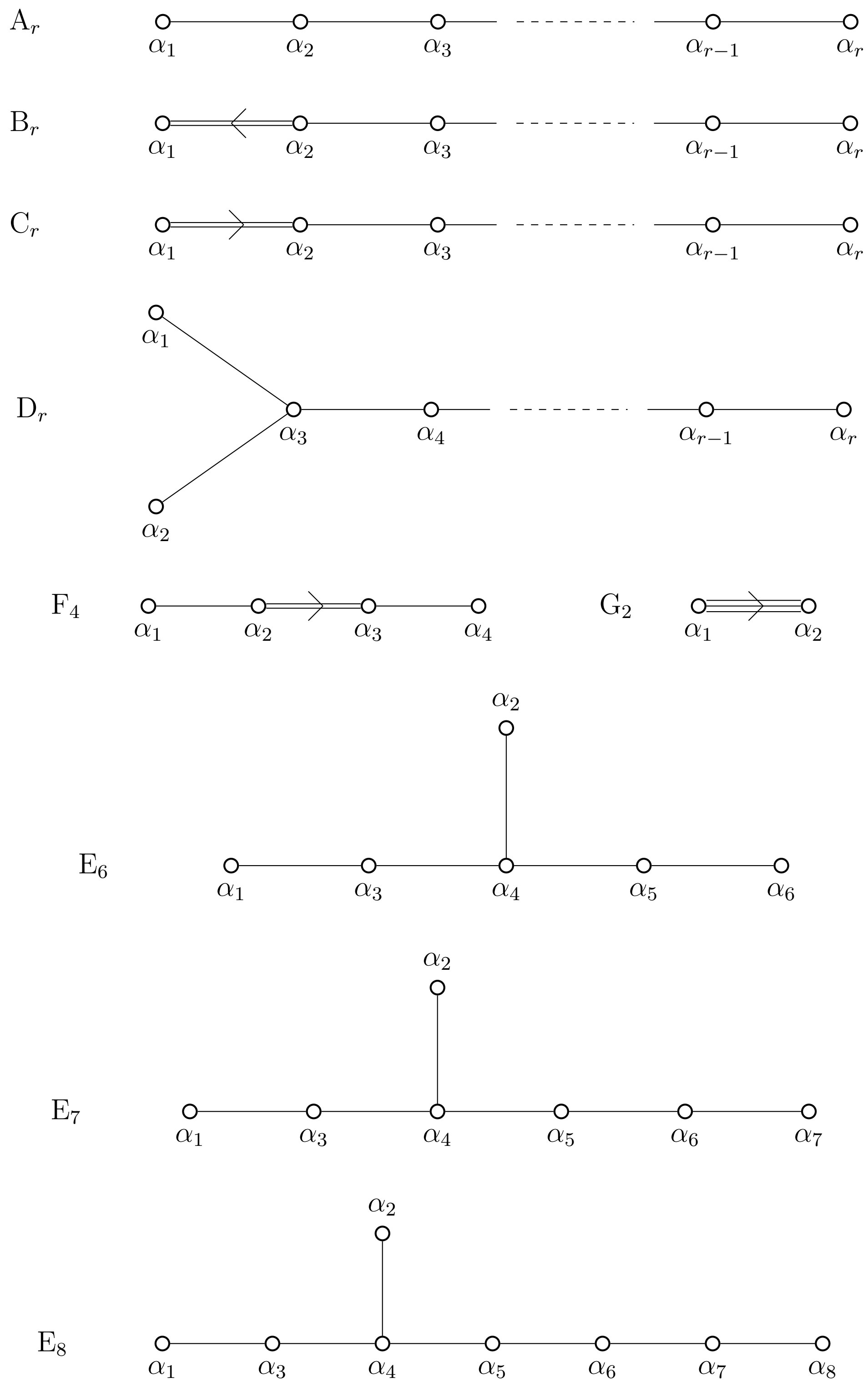

Figure 1.1: The Dynkin diagrams. Simple roots are labelled as in CHEVIE.

such that

$$\mathbb{U} = \prod_{\alpha \in \Phi^+} \mathbb{U}_\alpha,$$

and for every $t \in \mathbb{T}$ and $c \in k$, we have ${}^t x_\alpha(c) = x_\alpha(\alpha(t)c)$. The subgroups of $\mathbb{U}$ of the form $\mathbb{U}_\alpha$ for $\alpha \in \Phi^+$ are called *root subgroups*, and the elements of the form $x_\alpha(c)$ of $\mathbb{U}_\alpha$ for $c \in k$ are called *root elements*. For every $\alpha, \beta \in \Phi^+$, we have

$$[\mathbb{U}_\alpha, \mathbb{U}_\beta] \subseteq \prod_{i,j \in \mathbb{Z}_{\geq 1} | i\alpha + j\beta \in \Phi^+} \mathbb{U}_{i\alpha + j\beta}.$$

We have a Frobenius field automorphism $F_q : k \to k$, defined by $F_q(x) = x^q$. The field of fixed points of k under F_q is $\mathbb{F}_q$. In order to extend this notion to $\mathbb{G}$, we shall assume that $\mathbb{G}$ is *defined over* $\mathbb{F}_q$, that is, it is defined as an algebraic variety by a set I of polynomials with coefficients in $\mathbb{F}_q$. We refer to [MT, Section 21.1] for this construction. The automorphism F_q of $\mathbb{F}_q$ acts on the polynomials that define I by acting on their coefficients, thus it leaves I invariant. Thus F_q also acts on $\mathbb{G}$, set of common zeroes of I. The corresponding map $F : \mathbb{G} \to \mathbb{G}$ obtained via the action of F_q is called a *Frobenius morphism* of $\mathbb{G}$ with respect to I. In particular, for a linear embedding $\varphi : \mathbb{G} \to \mathrm{GL}_n(k)$, we have that $\varphi \circ F = F_q \circ \varphi$, where on the right hand side of this equality $F_q : \mathrm{GL}_n(k) \to \mathrm{GL}_n(k)$ is such that $(F_q(a))_{i,j} = F_q(a_{i,j}) = a_{i,j}^q$.

We say that $F : \mathbb{G} \to \mathbb{G}$ is a *Steinberg morphism* if F^m is a Frobenius morphism for some $m \geq 1$. We are now ready to introduce the class of groups of major importance in this work. The following definition is as stated in [MT, Definition 21.6]; there might be some ambiguity in literature.

Definition 1.4. We say that G is a *finite group of Lie type* if there exists a connected reductive linear algebraic group $\mathbb{G}$ defined over $\mathbb{F}_q$, and a Steinberg morphism F defined on $\mathbb{G}$, such that $G = \mathbb{G}^F$ is the set of fixed points of $\mathbb{G}$ under F.

As a consequence of the Lang-Steinberg theorem [MT, Theorem 21.7], we can always choose a maximal torus $\mathbb{T}$ and a Borel subgroup $\mathbb{B}$ of a connected reductive algebraic group $\mathbb{G}$ defined over $\mathbb{F}_q$ endowed with a Steinberg morphism F, such that $\mathbb{T} \subseteq \mathbb{B}$, and both $\mathbb{T}$ and $\mathbb{B}$ are F-stable. From now on, we assume that G is a *split* finite group of Lie type, that is, G is a finite group of Lie type, and there exists an F-stable maximal torus $\mathbb{T}$ of $\mathbb{G}$ such that $F(t) = t^q$ for every $t \in \mathbb{T}$. Moreover, we fix a choice of $\mathbb{B}$ Borel subgroup and $\mathbb{T} \subseteq \mathbb{B}$ maximal torus, such that $F(\mathbb{T}) = \mathbb{T}$ and $F(\mathbb{B}) = \mathbb{B}$.

We write $B = \mathbb{B}^F$, and we get $B = TU$, with $T = \mathbb{T}^F$ and $U = \mathbb{U}^F$. Then U is a Sylow p-subgroup of G. We also denote this choice of Sylow p-subgroup in G by $U(G)$. Since G is split, by [MT, Section 23.2] the isomorphisms of the form $x_\alpha : k \to \mathbb{U}_\alpha$ for every $\alpha \in \Phi$ can be chosen such that they restrict to isomorphisms $x_\alpha|_{\mathbb{F}_q} : \mathbb{F}_q \to X_\alpha$, where

$$X_\alpha := \mathbb{U}_\alpha^F = \{x_\alpha(t) \mid t \in \mathbb{F}_q\} \cong \mathbb{F}_q.$$

By abuse of notation, we also denote by x_α its restriction to $\mathbb{F}_q$. We fix such a choice of x_α, for every $\alpha \in \Phi$. The notion of root elements and root subgroups also makes sense in U. A presentation for U is given by the following commutator relations, also called Chevalley relations,

$$[x_\alpha(s), x_\beta(r)] = \prod_{i,j \in \mathbb{Z} \mid i\alpha + j\beta \in \Phi^+} x_{i\alpha + j\beta}(c_{i,j}^{\alpha,\beta} (-r)^i s^j) \qquad (1.1.1)$$

for every $r, s \in \mathbb{F}_q$, for $\alpha, \beta \in \Phi^+$ and some $c_{i,j}^{\alpha,\beta} \in \mathbb{Z} \setminus \{0\}$ called *Lie structure constants*. As proved in [Car, Section 5.2], the parametrizations of the root subgroups can be chosen so that the structure constants $c_{i,j}^{\alpha,\beta}$ are always ± 1, ± 2, ± 3, where ± 2 occurs only for G of type B_r, C_r, F_4 and G_2, and ± 3 only occurs for G of type G_2. Moreover, the structure constants are uniquely determined up to a fixed choice of signs for *extraspecial pairs*, as defined in [Car, Section 4.2]. Our fixed choice of signs for the groups of our interest is determined in Appendix B, according to the choice recorded in the computer algebra system MAGMA [MAGMA].

Let us decompose the highest root α_0 of Φ as

$$\alpha_0 = a_1\alpha_1 + \cdots + a_r\alpha_r,$$

where $\alpha_1, \ldots, \alpha_r$ are the simple roots of Φ, and $a_1, \ldots, a_r \in \mathbb{Z}_{\geq 1}$. We say that p is a *bad* prime for Φ if p divides some a_i for $i \in \{1, \ldots, r\}$. The prime p is said to be *very bad* for Φ if a constant $c_{i,j}^{\alpha,\beta}$ in the Chevalley relations of U is equal to p for some $\alpha, \beta \in \Phi^+$ and $i, j > 0$. For example, $p = 2$ is a bad prime for a root system of type D_4, since the highest root is $\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4$, but in this case $c_{i,j}^{\alpha,\beta} \in \{\pm 1\}$ for every $\alpha, \beta \in \Phi^+$ and $i, j \geq 0$, therefore 2 is not a very bad prime for D_4. The notation is consistent, in the sense that a very bad prime is a bad prime. When p is not a bad prime for Φ, we say that it is a *good* prime for Φ. The bad primes are recorded in Table 1.1; the bold font is used when the prime is very bad.

A_r	B_r	C_r	D_r	E_6	E_7	E_8	F_4	G_2
none	$\mathbf{2}$	$\mathbf{2}$	2	$2, 3$	$2, 3$	$2, 3, 5$	$\mathbf{2}, 3$	$\mathbf{2}, \mathbf{3}$

Table 1.1: The bad and very bad primes in every root system.

We have a standard (strict) partial order on Φ defined by $\alpha < \beta$ if $\beta - \alpha$ is a sum of positive roots. We call two roots α, β *comparable* if $\alpha \leq \beta$ or $\beta \leq \alpha$, and *incomparable* otherwise. In general, it is not true that if $\beta < \gamma$, then $\gamma - \beta$ is a positive root, as we see for example by choosing $\beta = \alpha_2$ and $\gamma = \alpha_1 + \alpha_2 + \alpha_3$ in type A_3. However, we have the following result.

Proposition 1.5. *Let $\alpha, \beta \in \Phi^+$, such that $\alpha < \beta$. Then there exist $\epsilon_1, \ldots, \epsilon_n \in \Pi$, such that*

$$\beta - \alpha = \epsilon_1 + \cdots + \epsilon_n,$$

and each partial sum $\alpha + \epsilon_1 + \cdots + \epsilon_i$ *for* $i = 1, \ldots, n$ *is an element of* Φ^+*.*

Proof. For every $\alpha, \beta \in \Phi^+$ with $\alpha < \beta$, we can decompose $\beta - \alpha = \nu_1 + \cdots + \nu_n$, for not necessarily distinct elements $\nu_1, \ldots, \nu_n \in \Pi$. We prove the claim by induction on n, the case $n = 1$ being trivial. Let then $\alpha, \beta \in \Phi^+$. Let us denote by (β_1, β_2) the scalar product of $\beta_1, \beta_2 \in \Phi^+$. We have that

$$0 < (\beta - \alpha, \beta - \alpha) = (\beta - \alpha, \sum_{i=1}^{n} \nu_i) = \sum_{i=1}^{n} (\beta, \nu_i) - (\alpha, \nu_i),$$

with $\nu_1, \ldots, \nu_n$ as above. Then there exists at least one $i \in \{1, \ldots, n\}$ such that $(\beta, \nu_i) > 0$ or $(\alpha, \nu_i) < 0$.

Let us suppose that $(\beta, \nu_i) > 0$. By [Hum, 9.4], we have that $\beta - \nu_i \in \Phi^+$. We then put $\epsilon_n := \nu_i$. By the inductive hypothesis, we have

$$(\beta - \epsilon_n) - \alpha = \epsilon_1 + \cdots + \epsilon_{n-1}$$

for $\epsilon_1, \ldots, \epsilon_{n-1}$, such that $\alpha + \epsilon_1 + \cdots + \epsilon_i \in \Phi^+$ for $i = 1, \ldots, n-1$, but of course we have that $\alpha + \epsilon_1 + \cdots + \epsilon_n = \beta$ is also a positive root. This proves the claim in this case.

Let us now assume $(\alpha, \nu_i) < 0$. Then in this case $\alpha + \nu_i \in \Phi^+$ by [Hum, 9.4]. We put $\epsilon_1 := \nu_i$. Then by induction we get

$$\beta - (\alpha + \epsilon_1) = \epsilon_2 + \cdots + \epsilon_n,$$

with $\alpha + \epsilon_1 + (\epsilon_2 + \cdots + \epsilon_i) \in \Phi^+$ for every $i = 2, \ldots, n$. The claim also follows in this case. □

We let $N = |\Phi^+|$. We fix an enumeration of $\Phi^+ = \{\alpha_1, \ldots, \alpha_N\}$ with $\Pi = \{\alpha_1, \ldots, \alpha_r\}$, such that $i < j$ whenever $\alpha_i < \alpha_j$. We abbreviate and write X_i for X_{α_i} and x_i for x_{α_i}. Each element of U can be written uniquely in the form $u = x_1(s_1)x_2(s_2)\cdots x_N(s_N)$, where $s_i \in \mathbb{F}_q$ for all $i = 1, \ldots, N$. In particular, the groups $X_1, \ldots, X_N$ generate U, and $|U| = q^N$.

To finish, we state a simplified version of [MT, Theorem 22.5] about the classification of split finite groups of Lie type, similar to Theorem 1.3 in the case of semisimple algebraic groups.

Theorem 1.6. *Two split finite groups of Lie type $G = \mathbb{G}^F$ and $G' = \mathbb{G}'^{F'}$ defined over $\mathbb{F}_q$ are isomorphic if and only if the root data corresponding to $\mathbb{G}$ and $\mathbb{G}'$ are isomorphic. Moreover, for a fixed prime power q and a root datum, there exists a unique split finite group of Lie type $G = \mathbb{G}^F$, such that $\mathbb{G}$ is associated to this root datum.*

1.2 Character theory of finite groups

We now recall some basic results about character theory of finite groups, from basic definitions about representations to Clifford theory and some of its deep consequences. Our main reference is [Is].

Let G be a finite group. We denote the right (respectively left) conjugation action by $g^h := h^{-1}gh$ (respectively ${}^hg := hgh^{-1}$) for $g, h \in G$. We denote by $Z(G)$ the *centre* of G, that is, the normal subgroup of G that consists of all elements $z \in G$ such that $z^g = z$ for every $g \in G$. We denote by $\mathbb{C}G$ the *group algebra* of G over $\mathbb{C}$, where the elements are formal sums of terms of the form $a_g g$, with $a_g \in \mathbb{C}$ and $g \in G$, and the operations are defined by

$$\left(\sum_{g\in G} a_g g\right) + \left(\sum_{g\in G} b_g g\right) := \sum_{g\in G}(a_g + b_g)g, \qquad \left(\sum_{g\in G} a_g g\right)\left(\sum_{g\in G} b_g g\right) := \sum_{g,g'\in G}(a_g b_{g'})(gg'),$$

where $a_g + b_g$ and $a_g b_{g'}$ are standard operations in $\mathbb{C}$, and gg' is the multiplication in G.

An *ordinary representation* of G is a group homomorphism $\rho : G \to \mathrm{GL}(V)$, where V is a finite dimensional vector space over $\mathbb{C}$. We can make V into an $\mathbb{C}G$-module by setting

$g.v := \rho(g)(v)$ for every $g \in G$, and extending linearly to $\mathbb{C}G$. We say that ρ is an *irreducible representation* if the corresponding $\mathbb{C}G$-module V is irreducible, that is, if it contains no proper nontrivial $\mathbb{C}G$-submodules.

The *ordinary character* χ afforded by the representation ρ of G is the map from G to $\mathbb{C}$ defined by $\chi(g) = \mathrm{Tr}(\rho(g))$, where $\mathrm{Tr} : \mathrm{GL}(V) \to \mathbb{C}$ is the matrix trace map. The value $\chi(1) = \dim V$ is called the *degree* of χ. We say that ρ is a *linear representation* if $\dim(V) = 1$. The corresponding character is called a *linear character*.

Let $\bar{z}$ denote the complex conjugate of $z \in \mathbb{C}$. The following *inner product* is defined for characters χ, ψ of G,

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\psi(g)}.$$

We say that a character χ of G is *irreducible* if it is afforded by an irreducible representation. A character χ of G is irreducible if and only if $\langle \chi, \chi \rangle = 1$.

Let us call $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_s\}$ the set of all irreducible characters of G. Then $\mathrm{Irr}(G)$ is an orthogonal set with respect to $\langle\ ,\ \rangle$, that is, $\langle \chi_i, \chi_j \rangle$ is equal to 1 if $i = j$, and is 0 otherwise. Moreover, for every character χ of G, we have $\chi = a_1\chi_1 + \cdots + a_s\chi_s$, where $a_i = \langle \chi, \chi_i \rangle$. In other words, the set of characters of G is exactly

$$\mathbb{Z}_{\geq 0} \mathrm{Irr}(G) := \{a_1\chi_1 + \cdots + a_s\chi_s \mid \chi_1, \ldots, \chi_s \in \mathrm{Irr}(G),\ a_1, \ldots, a_s \in \mathbb{Z}_{\geq 0}\}.$$

The number $\langle \chi, \chi_i \rangle$ is called the *multiplicity* of χ_i in χ, and we say that χ_i is an *irreducible constituent* of χ if $\langle \chi, \chi_i \rangle \neq 0$. We write 1_G for the trivial character of G, such that $1_G(g) = 1$ for every $g \in G$. Of course, 1_G is an irreducible character, as is any linear character of G.

If $N \trianglelefteq G$ is a normal subgroup of a finite group G, then we can define the *inflation* map

$$\mathrm{Inf}_N^G : \mathbb{Z}_{\geq 0}\mathrm{Irr}(G/N) \longrightarrow \mathbb{Z}_{\geq 0}\mathrm{Irr}(G), \qquad \mathrm{Inf}_N^G(\chi)(g) := \chi(gN),$$

which is injective and preserves the degree. Moreover, the image of $\mathrm{Irr}(G/N)$ via inflation is contained in $\mathrm{Irr}(G)$. Omitting N, we sometimes write $\tilde{\chi}$, or also χ by abuse of notation, in place of $\mathrm{Inf}_N^G(\chi)$. Moreover, if H is a subgroup of G, we have an *induction* map

$$\mathrm{Ind}_H^G : \mathbb{Z}_{\geq 0}\mathrm{Irr}(H) \longrightarrow \mathbb{Z}_{\geq 0}\mathrm{Irr}(G), \qquad \mathrm{Ind}_H^G(\psi)(g) := \frac{1}{|H|} \sum_{\substack{x \in G: \\ g^x \in H}} \psi(g^x).$$

One easily verifies that $\mathrm{Ind}_H^G(\psi)(1) = \psi(1)|G|/|H|$. We sometimes write ψ^G in place of $\mathrm{Ind}_H^G(\psi)$. For a character $\chi \in \mathrm{Irr}(G)$, we have a *restriction* map

$$\mathrm{Res}_H^G : \mathbb{Z}_{\geq 0}\mathrm{Irr}(G) \longrightarrow \mathbb{Z}_{\geq 0}\mathrm{Irr}(H), \qquad \mathrm{Res}_H^G(\chi)(h) := \chi(h).$$

We often write $\chi|_H$ instead of Res_H^G. We recall that *Frobenius reciprocity* for a character χ of G and a character ψ of H states that

$$\langle \mathrm{Res}_H^G(\chi), \psi \rangle = \langle \chi, \mathrm{Ind}_H^G(\psi) \rangle.$$

Moreover, we recall that if $N \trianglelefteq G$, then we have

$$1_N^G = \sum_{\chi \in \mathrm{Irr}(G/N)} \chi(1)\tilde{\chi}. \tag{1.2.1}$$

For a character $\eta \in \mathrm{Irr}(H)$, we denote

$$\mathrm{Irr}(G \mid \eta) := \{\chi \in \mathrm{Irr}(G) \mid \langle \chi, \eta^G \rangle \neq 0\} = \{\chi \in \mathrm{Irr}(G) \mid \langle \chi|_H, \eta \rangle \neq 0\}.$$

For a character $\chi \in \mathrm{Irr}(G)$, we define

$$\ker(\chi) = \{g \in G \mid \chi(g) = \chi(1)\},$$

the *kernel* of χ, and we define

$$Z(\chi) = \{g \in G \mid |\chi(g)| = \chi(1)\},$$

the *centre* of χ. The reason for the names lies in the fact that if ρ is the representation corresponding to χ, then $\ker \chi = \ker \rho$, and $Z(\chi)$ is the subgroup of elements in G such that $\rho(g) = \zeta_g \cdot \mathrm{id}$ for some $\zeta_g \in \mathbb{C}$, that is, $\rho(g) \in Z(\rho(G))$.

Given $g \in G$, a normal subgroup N of G and an irreducible character ψ of N, we write ${}^g\psi$ for the character of N defined by ${}^g\psi(x) = \psi(x^g)$ for every $x \in N$. This is in turn an irreducible character of N. This naturally defines an action of G on $\mathrm{Irr}(N)$.

We define the *tensor product* of $\chi_1, \chi_2 \in \mathrm{Irr}(G)$ by $(\chi_1 \otimes \chi_2)(g) = \chi_1(g)\chi_2(g)$. The following property, namely [Is, Problem 5.3], follows by direct computations.

Lemma 1.7. *Let H be a subgroup of G, let χ be a character of $\mathrm{Irr}(G)$ and let ψ be a character of $\mathrm{Irr}(H)$. Then*

$$(\chi|_H \otimes \psi)^G = \chi \otimes \psi^G.$$

The following commutativity property of induction and inflation will be used several times later.

Lemma 1.8. *Let $N \leq H \leq G$, with $N \trianglelefteq G$, and let $\psi \in \mathrm{Irr}(H/N)$. Then we have*

$$\mathrm{Inf}_{G/N}^{G} \mathrm{Ind}_{H/N}^{G/N} \psi = \mathrm{Ind}_{H}^{G} \mathrm{Inf}_{H/N}^{H} \psi. \tag{1.2.2}$$

Proof. Let $g \in G$. We have that

$$(\mathrm{Ind}_H^G \mathrm{Inf}_{H/N}^H \psi)(g) = \frac{1}{|H|} \sum_{\substack{x \in G: \\ g^x \in H}} \mathrm{Inf}_{H/N}^H \psi(g^x) = \frac{1}{|H|} \sum_{\substack{x \in G: \\ g^x \in H}} \psi(g^x N)$$

$$= \frac{|N|}{|H|} \sum_{\substack{xN \in G/N: \\ g^x N \in H/N}} \psi(g^x N) = \operatorname{Ind}_{H/N}^{G/N} \psi(gN) = (\operatorname{Inf}_{G/N}^{G} \operatorname{Ind}_{H/N}^{G/N} \psi)(g).$$

Since g is arbitrary in G, the claim follows. □

We recall some classical results that link the representation theory of G with the representation theory of some normal subgroup N of G. A result of major importance for the development of our work is *Clifford's theorem*, as in [Is, Theorem 6.2].

Theorem 1.9 (Clifford's theorem)**.** *Let $N \trianglelefteq G$. Let $\chi \in \operatorname{Irr}(G)$ and $\eta \in \operatorname{Irr}(N)$ be such that $\chi \in \operatorname{Irr}(G \mid \eta)$. Let $\{\eta_1, \eta_2, \ldots, \eta_s\}$, with $\eta_1 = \eta$, be the distinct G-conjugates of η. Then we have that $\chi|_N = t \sum_{i=1}^{s} \eta_i$, with $t = \langle \chi|_N, \eta \rangle$.*

We also recall one of the main consequences of Clifford's theorem, which is used in the sequel.

Theorem 1.10 ([Is], Theorem 6.11(b))**.** *Let $N \trianglelefteq G$. Let $\eta \in \operatorname{Irr}(N)$, and let*

$$I_G(\eta) = \{g \in G \mid {}^g\eta = \eta\}$$

be the inertia group *of η in G. Then induction gives a bijection between $\operatorname{Irr}(I_G(\eta) \mid \eta)$ and $\operatorname{Irr}(G \mid \eta)$.*

We state the following result, which we use in the sequel.

Proposition 1.11. *Let T be a normal subgroup of G, and let Z be a subgroup of $Z(G)$ such that $Z \cap T = 1$. Let $\lambda \in \operatorname{Irr}(Z)$, and let $\tilde{\lambda}$ denote its inflation to ZT. Then we have*

(i) $\lambda^G = \sum_{\chi \in \operatorname{Irr}(G|\lambda)} \chi(1)\chi$.

(ii) $T \subseteq \ker(\chi)$ *for every* $\chi \in \operatorname{Irr}(G \mid \tilde{\lambda})$.

(iii) Let us identify Z as a subgroup of G/T. Then the inflation map

$$\mathrm{Irr}(G/T \mid \lambda) \longrightarrow \mathrm{Irr}(G \mid \tilde{\lambda}) \tag{1.2.3}$$

is a bijection.

Proof. Of course, since $\lambda \in \mathrm{Irr}(Z)$, then $\lambda^g = \lambda$ for every $g \in G$. Thus if $\chi \in \mathrm{Irr}(G \mid \lambda)$, by Clifford's theorem we have that $\chi|_Z = c\lambda$ for some integer c. Now evaluating at 1 gives $c = c\lambda(1) = \chi|_Z(1) = \chi(1)$. Then by Frobenius reciprocity we have that

$$\chi(1) = \langle \chi|_Z, \lambda \rangle = \langle \chi, \lambda^G \rangle.$$

Since χ was arbitrary in $\mathrm{Irr}(G \mid \lambda)$, this proves (i).

Now let $t \in T$. Since T is normal in G, we have

$$\tilde{\lambda}^G(t) = \frac{1}{|ZT|} \sum_{\substack{x \in G: \\ t^x \in ZT}} \tilde{\lambda}(t^x) = \frac{1}{|ZT|} \sum_{x \in G} \tilde{\lambda}(t^x) = \frac{|G|}{|ZT|} = \tilde{\lambda}^G(1).$$

Then $T \subseteq \ker(\tilde{\lambda}^G)$, hence $T \subseteq \ker(\chi)$ for every χ summand of $\tilde{\lambda}^G$. This proves (ii).

Finally, for $\chi \in \mathrm{Irr}(G \mid \lambda)$ the map

$$\hat{\chi} : G/T \longrightarrow \mathbb{C}, \qquad \hat{\chi}(gT) = \chi(g),$$

is well-defined by (ii), and one easily checks that the function

$$\Psi : \mathrm{Irr}(G \mid \tilde{\lambda}) \to \mathrm{Irr}(G/T \mid \lambda)$$

$$\chi \mapsto \hat{\chi}$$

is an inverse to Inf_T^G, proving (iii). □

The next lemma is key to developing Algorithm 2.6 in the sequel. This result was proved in [HLM15, Lemma 2.1] and we refer to it as the reduction lemma. We note that a similar result in the context of algebra groups was previously proved by Evseev in [Ev11, Lemma 2.1].

Lemma 1.12 (Reduction lemma)**.** *Let G be a finite group, let $H \leq G$ and let X be a transversal of H in G. Suppose that Y and Z are subgroups of H, and λ is an irreducible character of Z, such that*

(i) $Z \subseteq Z(G)$,

(ii) $Y \trianglelefteq H$,

(iii) $Z \cap Y = 1$,

(iv) $ZY \trianglelefteq G$,

(v) for the inflation $\tilde{\lambda} \in \operatorname{Irr}(ZY)$ of λ, we have that ${}^{x_1}\tilde{\lambda} \neq {}^{x_2}\tilde{\lambda}$ for all $x_1, x_2 \in X$ with $x_1 \neq x_2$.

Then we have a bijection

$$\Psi : \operatorname{Irr}(H/Y \mid \lambda) \to \operatorname{Irr}(G \mid \lambda) \cap \operatorname{Irr}(G \mid 1_Y)$$
$$\chi \mapsto \tilde{\chi}^G$$

given by inflating over Y, then inducing from H to G.
Moreover, if $|X| = |Y|$, then $\operatorname{Irr}(G \mid \lambda) \cap \operatorname{Irr}(G \mid 1_Y) = \operatorname{Irr}(G \mid \lambda)$.

Proof. We have that $g \in I_G(\tilde{\lambda})$ if and only if ${}^{g}\tilde{\lambda}(yz) = \tilde{\lambda}(yz)$ for all $y \in Y$ and $z \in Z$. Let then $y \in Y$ and $z \in Z$. Then for every $h \in H$, we have that

$${}^{h}\tilde{\lambda}(yz) = \tilde{\lambda}(y^h z^h) = \tilde{\lambda}(y^h)\tilde{\lambda}(z) = \tilde{\lambda}(z) = \tilde{\lambda}(yz),$$

where we used the fact that Y is normalized by H, and $\tilde{\lambda}$ is trivial on Y. Then we have that $H \subseteq I_G(\tilde{\lambda})$, and in fact $H = I_G(\tilde{\lambda})$ by (v). By Theorem 1.10, this implies that the induction map is a bijection between $\mathrm{Irr}(H \mid \tilde{\lambda})$ and $\mathrm{Irr}(G \mid \tilde{\lambda})$. By part (iii) of Proposition 1.11, the inflation map is a bijection between $\mathrm{Irr}(H \mid \tilde{\lambda})$ and $\mathrm{Irr}(H/Y \mid \lambda)$.

Let us now prove that

$$\mathrm{Irr}(G \mid \tilde{\lambda}) = \mathrm{Irr}(G \mid \lambda) \cap \mathrm{Irr}(G \mid 1_Y).$$

For $\mathrm{Irr}(G \mid \tilde{\lambda}) \subseteq \mathrm{Irr}(G \mid \lambda) \cap \mathrm{Irr}(G \mid 1_Y)$, it is enough to prove, by Frobenius reciprocity, that

$$\lambda^{YZ} = \tilde{\lambda} + \psi_1 \quad \text{and} \quad 1_Y^{YZ} = \tilde{\lambda} + \psi_2$$

for some characters ψ_1, ψ_2 of YZ. The second equality follows from Equation (1.2.1). For the first one, by Lemma 1.7 and Equation (1.2.1) we note that

$$\lambda^{YZ} = (\tilde{\lambda}|_Z)^{YZ} = (\tilde{\lambda}|_Z \otimes 1_Z)^{YZ} = \tilde{\lambda} \otimes 1_Z^{YZ} = \tilde{\lambda} + \sum_{\substack{\chi \in \mathrm{Irr}(Y): \\ \chi \neq 1_Y}} \tilde{\lambda} \otimes \tilde{\chi}.$$

Then the first inclusion is proved.

Now let $\chi \in \mathrm{Irr}(G \mid \lambda) \cap \mathrm{Irr}(G \mid 1_Y)$. Then we decompose

$$\chi|_{YZ} = \sum_{i=1}^{m} \mu_i \otimes \lambda_i$$

for some $m \geq 1$, and not necessarily distinct $\mu_i \in \mathrm{Irr}(Y)$ and $\lambda_i \in \mathrm{Irr}(Z)$ for $i = 1, \dots, m$. Since $YZ \trianglelefteq G$, by Clifford's theorem we have that the $\mu_i \otimes \lambda_i$'s lie in a single G-orbit. Since $\chi \in \mathrm{Irr}(G \mid \lambda)$, we have that $\lambda_i = \lambda$ for some $i \in \{1, \dots, m\}$. But then since $\lambda \in \mathrm{Irr}(Z)$ and $Z \subseteq Z(G)$, we have that $\lambda_i = \lambda$ for all $i = 1, \dots, m$. Also, since $\chi \in \mathrm{Irr}(G \mid 1_Y)$, we have that $\mu_i = 1_Y$ for some $i \in \{1, \dots, m\}$. This proves $\mathrm{Irr}(G \mid \lambda) \cap \mathrm{Irr}(G \mid 1_Y) \subseteq \mathrm{Irr}(G \mid \tilde{\lambda})$.

Combining this with what previously obtained, we finally get that

$$\operatorname{Ind}_H^G \operatorname{Inf}_{H/Y}^H : \operatorname{Irr}(H/Y \mid \lambda) \longrightarrow \operatorname{Irr}(G \mid \lambda) \cap \operatorname{Irr}(G \mid 1_Y)$$

is a bijective map. This proves the first claim.

Assume now $|X| = |Y| = m$. Let $X = \{x_1, \dots, x_m\}$. Then by assumption (v) we have that $\tilde{\lambda}^{x_1}, \dots, \tilde{\lambda}^{x_m}$ are different irreducible characters that restrict to λ, that is, by Frobenius reciprocity, $\langle \lambda^{YZ}, \tilde{\lambda}^{x_i} \rangle \neq 0$ for $i = 1, \dots, m$. Then since $|X| = |Y| = \lambda^{YZ}(1)$, we have that $\tilde{\lambda}^{x_1}, \dots, \tilde{\lambda}^{x_m}$ are all irreducible components of λ^{YZ}. In fact, by part (i) of Proposition 1.11, we have that

$$\lambda^{YZ} = \sum_{\psi \in \operatorname{Irr}(YZ|\lambda)} \psi(1)\psi = \sum_{i=1}^{m} \tilde{\lambda}^{x_i},$$

in particular $\lambda^{YZ} = \tilde{\lambda} + \varphi$ for φ some character of YZ with $\langle \varphi, \tilde{\lambda} \rangle = 0$. The second claim follows. □

We end this section by studying in detail the behavior of the elements of $\operatorname{Irr}(\mathbb{F}_q)$, where $q = p^e$ is as in Section 1.1, and $(\mathbb{F}_q, +)$ is regarded as an abelian group. Denote now by $\operatorname{Tr} : \mathbb{F}_q \to \mathbb{F}_p$ the field trace map, that is,

$$\operatorname{Tr}(t) = t + t^p + \cdots + t^{p^{e-1}}$$

for every $t \in \mathbb{F}_q$, and define $\phi : \mathbb{F}_q \to \mathbb{C}^\times$ by $\phi(t) = e^{\frac{i2\pi \operatorname{Tr}(t)}{p}}$ for $t \in \mathbb{F}_q$, so that ϕ is a nontrivial ordinary character of the additive group $\mathbb{F}_q$. For $a \in \mathbb{F}_q$, we define $\phi_a \in \operatorname{Irr}(\mathbb{F}_q)$ by $\phi_a(t) = \phi(at)$. It is easy to see that $\phi_a \neq \phi_b$ if a and b are distinct in $\mathbb{F}_q$. Therefore we get $\operatorname{Irr}(\mathbb{F}_q) = \{\phi_a \mid a \in \mathbb{F}_q\}$.

For a fixed $m \in \mathbb{Z}_{\geq 1}$, it is straightforward to see that $\phi(a_1 s_1 + \cdots + a_m s_m) = 1$ for every $s_1, \dots, s_m \in \mathbb{F}_q$ holds if and only if $a_1 = \cdots = a_m = 0$. Moreover, since the map sending t to t^p is an automorphism in $\mathbb{F}_q$, we have that the equality $\phi(at^p) = 1$ for every $t \in \mathbb{F}_q$ holds if and only if $a = 0$.

The next lemma is of major importance for the analysis in Chapter 3.

Lemma 1.13. *For a fixed $a \in \mathbb{F}_q^\times$, let $\mathbb{T}_a = \{t^p - a^{p-1}t \mid t \in \mathbb{F}_q\}$. Then*

$$a^{-p}\,\mathbb{T}_a = \ker(\phi).$$

Proof. We have that

$$a^{-p}\,\mathbb{T}_a = \{a^{-p}(t^p - a^{p-1}t) \mid t \in \mathbb{F}_q\} = \{\big(a^{-1}t\big)^p - a^{-1}t \mid t \in \mathbb{F}_q\} = \{u^p - u \mid u \in \mathbb{F}_q\}.$$

Now, we also have that

$$\mathrm{Tr}(t^p - t) = \mathrm{Tr}(t^p) - \mathrm{Tr}(t) = \mathrm{Tr}(t) - \mathrm{Tr}(t) = 0,$$

therefore

$$\{t^p - t \mid t \in \mathbb{F}_q\} \subseteq \{t \in \mathbb{F}_q \mid \mathrm{Tr}(t) = 0\} = \ker(\phi),$$

and all those sets have same cardinality q/p, therefore $\ker(\phi) = \{t^p - t \mid t \in \mathbb{F}_q\} = a^{-p}\,\mathbb{T}_a$. □

1.3 Pattern and quattern groups and antichains

In this section, we recall the notion of patterns and pattern groups, and then introduce *quatterns* and *quattern groups*. These objects were studied deeply in [HLM15]. The advantage of working with patterns and quatterns is that these provide a natural "dictionary" between subsets of positive roots and the structure of certain subquotients of the Sylow p-subgroup U of G. By working in Φ^+, we then consider groups of fixed type and rank for every q simultaneously. We end this section by recalling the notion of antichains; these are in bijection with the normal subsets of Φ^+, as explained later.

Definition 1.14. A subset $\mathcal{P}$ of Φ^+ is said to be a *pattern* (or *closed*) if for $\alpha, \beta \in \mathcal{P}$, we have that $\alpha+\beta \in \mathcal{P}$ whenever $\alpha+\beta \in \Phi^+$.

Under the assumption that $\mathcal{P} \subseteq \Phi^+$ is closed, we can associate to $\mathcal{P}$ a subgroup $X_{\mathcal{P}}$ of U.

Proposition 1.15. *Let $\mathcal{P} = \{\beta_1, \ldots, \beta_m\}$ be a pattern. Then*

$$X_{\mathcal{P}} := X_{\beta_1} \ldots X_{\beta_m}$$

is a subgroup of U.

Proof. We prove this by induction on the cardinality of a pattern. Without loss of generality, we can assume that $\beta_1 = \alpha_{i_1}, \ldots, \beta_m = \alpha_{i_m}$, such that $1 \leq j < k \leq m$ implies $i_j < i_k$. Then it is easy to see that $\mathcal{P}' := \mathcal{P} \setminus \{\beta_1\}$ is also a pattern, and $X_{\mathcal{P}'}$ is a subgroup by the inductive hypothesis. Let us define

$$x_{\beta_1,\ldots,\beta_m}(t_1, \ldots, t_m) = x_{\beta_1}(t_1) \cdots x_{\beta_m}(t_m),$$

For $\underline{t}, \underline{s} \in \mathbb{F}_q^m$, let us put $x(\underline{t}) = x_{\beta_1,\ldots,\beta_m}(t_1, \ldots, t_m)$, and let us put

$$x'(\underline{t}, \underline{s}) = x_{\beta_2,\ldots,\beta_m}(t_2, \ldots, t_m) x_{\beta_m,\ldots,\beta_2}(-s_m, \ldots, -s_2).$$

Notice that $x'(\underline{t}, \underline{s}) \in X_{\mathcal{P}'}$. Then we have that

$$\begin{aligned} x(\underline{t})x(\underline{s})^{-1} &= x_{\beta_1,\ldots,\beta_m}(t_1, \ldots, t_m) x_{\beta_m,\ldots,\beta_1}(-s_m, \ldots, -s_1) \\ &= x_{\beta_1}(t_1 - s_1) x'(\underline{t}, \underline{s}) \left[x'(\underline{t}, \underline{s}), x_{\beta_1}(-s_1)\right]. \end{aligned}$$

By Equation (1.1.1), we have that $[x'(\underline{t},\underline{s}), x_{\beta_1}(-s_1)] \in X_{\mathcal{P}'}$. The claim follows. $\square$

We can then just write

$$X_{\mathcal{P}} = \prod_{\alpha \in \mathcal{P}} X_\alpha,$$

and we call $X_{\mathcal{P}}$ the *pattern subgroup* corresponding to the pattern $\mathcal{P}$. In general, it is not true that if $\beta_1, \ldots, \beta_m \in \Phi^+$ and $X_{\beta_1} \cdots X_{\beta_m}$ is a subgroup of U, then $\{\beta_1, \ldots, \beta_m\}$ is a pattern. In fact, let us consider $U(\mathrm{B}_2(2^e))$. Let $\alpha_1 \in \Phi^+$ be the long simple root, and let $\alpha_2 \in \Phi^+$ be the short simple root. In this case, we have that $[x_{\alpha_2}(s), x_{\alpha_1+\alpha_2}(t)] = 0$ for every $s, t \in \mathbb{F}_q$, and of course it is still true that $\alpha_1 + 2\alpha_2 \in \Phi^+$. Therefore, $X_{\alpha_2}X_{\alpha_1+\alpha_2}$ is a subgroup of $U(\mathrm{B}_2(q))$, without $\{\alpha_2, \alpha_1 + \alpha_2\}$ being a pattern.

However, the converse of Proposition 1.15 does hold *when p is not a very bad prime for G.*

Proposition 1.16. *Assume that p is not a very bad prime for G. If $\alpha, \beta \in \Phi^+$, then*

$$[X_\alpha, X_\beta] = \prod_{i,j>0 \mid i\alpha + j\beta \in \Phi^+} X_{i\alpha+j\beta}.$$

We refer to [HLM15, Lemma 3.5] for the proof of the result. The inclusion "$\subseteq$" follows from Equation (1.1.1). The other inclusion follows from a straightforward case-by-case check, and that is where the hypothesis of p being not very bad is crucial. The following is the announced consequence of Proposition 1.16.

Corollary 1.17. *Let $\mathcal{P} \subseteq \Phi^+$, and assume that p is not a very bad prime for G. Then $\mathcal{P} := \{\beta_1, \ldots, \beta_s\} \subseteq \Phi^+$ is a pattern if and only if $X_{\mathcal{P}}$ is a subgroup.*

We now introduce the notion of normality in patterns.

Definition 1.18. Let $\mathcal{P}$ be a pattern. We say that a subset $\mathcal{K}$ is *normal* in $\mathcal{P}$, and we write $\mathcal{K} \trianglelefteq \mathcal{P}$, if for every $\alpha \in \mathcal{P}$ and every $\delta \in \mathcal{K}$, we have that $\alpha + \delta \in \mathcal{K}$ whenever $\alpha + \delta \in \Phi^+$.

We would like to associate a normal subgroup of $X_\mathcal{P}$ to a normal subset of $\mathcal{P}$. In general, even without the assumption on p being not very bad, by Equation (1.1.1) we have that $\mathcal{K} \trianglelefteq \mathcal{P}$ implies $X_\mathcal{K} \trianglelefteq X_\mathcal{P}$. If p is not a very bad prime, then we also get the converse of this.

Proposition 1.19. *Let $\mathcal{P}$ be a pattern, and let $\mathcal{K} \subseteq \mathcal{P}$. Assume that p is not a very bad prime for U. Then $\mathcal{K} \trianglelefteq \mathcal{P}$ if and only if $X_\mathcal{K} \trianglelefteq X_\mathcal{P}$*

Proof. We just need to prove that $X_\mathcal{K} \trianglelefteq X_\mathcal{P}$ implies $\mathcal{K} \trianglelefteq \mathcal{P}$. Assume that $X_\mathcal{K} \trianglelefteq X_\mathcal{P}$, and that $\alpha \in \mathcal{P}$ and $\delta \in \mathcal{K}$ are such that $\alpha + \delta \in \Phi^+$. By Proposition 1.16, we have that $X_{\alpha+\delta} \subseteq [X_\alpha, X_\delta]$. Of course $[X_\alpha, X_\delta] \subseteq [X_\alpha, X_\mathcal{K}]$, and by normality of $X_\mathcal{K}$ in $X_\mathcal{P}$ we have $[X_\alpha, X_\mathcal{K}] \subseteq X_\mathcal{K}$. This implies $X_{\alpha+\delta} \subseteq X_\mathcal{K}$, which by definition of $X_\mathcal{K}$ gives $\alpha + \delta \in \mathcal{K}$, as required. □

The case $U(\mathrm{B}_2(2^e))$ again provides a counterexample to the fact that if $\mathcal{K} \subseteq \mathcal{P}$ with $\mathcal{P}$ a pattern, then $X_\mathcal{K} \trianglelefteq X_\mathcal{P}$ implies $\mathcal{K} \trianglelefteq \mathcal{P}$. It is easy to check that $X_{\alpha_1+\alpha_2}$ is normal in $U(\mathrm{B}_2(q))$, but of course $\{\alpha_1 + \alpha_2\}$ is not a normal subset of Φ^+.

From now on, we assume that p is not a very bad prime for G, that is, $p \neq 2$ in types B_r, C_r and F_4, and $p \neq 2, 3$ in type G_2. We are now ready to define quatterns.

Definition 1.20. We say that a subset of Φ^+ of the form $\mathcal{S} := \mathcal{P} \setminus \mathcal{K}$, with $\mathcal{P}$ a pattern and $\mathcal{K}$ normal in $\mathcal{P}$, is a *quattern* with respect to $\mathcal{P}$ and $\mathcal{K}$.

The explanation of this name is to remind that these sets correspond to suitable quotients of some subgroups of U. In fact, we define

$$X_\mathcal{S} = X_{\mathcal{P}\setminus\mathcal{K}} = X_\mathcal{P}/X_\mathcal{K},$$

which we call a *quattern group* corresponding to $\mathcal{S}$. Although the definition of $X_\mathcal{S}$ depends on $\mathcal{P}$ and $\mathcal{K}$, the quattern group $X_\mathcal{S}$ just depends on $\mathcal{S}$ up to isomorphism. In fact, it is easy to check that if $\mathcal{P}'$ is another pattern with $\mathcal{K}' \trianglelefteq \mathcal{P}'$, such that $\mathcal{P}' \setminus \mathcal{K}' = \mathcal{S}$, then we have that $X_{\mathcal{P}}/X_{\mathcal{K}}. \cong X_{\mathcal{P}'}/X_{\mathcal{K}'}$. There is then no ambiguity in the notation $X_{\mathcal{S}}$. We often write $\mathcal{S} = \mathcal{P} \setminus \mathcal{K}$ for a quattern, where we are implicitly assuming that $\mathcal{P}$ and $\mathcal{K}$ are such a choice. Given $\alpha \in \mathcal{S}$, by a mild abuse of notation we identify X_α with its image in $X_{\mathcal{S}}$ for the remainder of this work.

Let $\mathcal{S} \subseteq \Phi^+$ be a quattern and let $X_{\mathcal{S}}$ be the corresponding quattern group. We define

$$\mathcal{Z}(\mathcal{S}) = \{\gamma \in \mathcal{S} \mid \gamma + \alpha \notin \mathcal{S} \text{ for all } \alpha \in \mathcal{S}\}$$

the set of *central roots* with respect to $\mathcal{S}$. In fact, using the commutator relations and the assumption that p is not very bad for G, it can be shown, as in [HLM15, Lemma 3.8], that

$$Z(X_{\mathcal{S}}) = X_{\mathcal{Z}(\mathcal{S})}.$$

Similarly, we define

$$\mathcal{D}(\mathcal{S}) = \{\gamma \in \mathcal{Z}(\mathcal{S}) \mid \alpha + \beta \neq \gamma \text{ for all } \alpha, \beta \in \mathcal{S}\}.$$

We call $\mathcal{D}(\mathcal{S})$ the set of *direct product roots* corresponding to $\mathcal{S}$. By the commutator relations and Proposition 1.16, we have that $X_{\mathcal{D}(\mathcal{S})}$ is a normal subgroup of $X_{\mathcal{S}}$ isomorphic to the direct product of its root subgroups. Moreover, $\mathcal{S} \setminus \mathcal{D}(\mathcal{S})$ is a quattern, and $X_{\mathcal{S} \setminus \mathcal{D}(\mathcal{S})}$ is centralized by $X_{\mathcal{D}(\mathcal{S})}$. So we have

$$X_{\mathcal{S}} = X_{\mathcal{S} \setminus \mathcal{D}(\mathcal{S})} \times X_{\mathcal{D}(\mathcal{S})}.$$

Let $\mathcal{S}$ be a quattern and let $\mathcal{Z} \subseteq \mathcal{Z}(\mathcal{S})$. We define

$$\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}} = \{\chi \in \mathrm{Irr}(X_{\mathcal{S}}) \mid X_\alpha \nsubseteq \ker(\chi) \text{ for all } \alpha \in \mathcal{Z}\}. \tag{1.3.1}$$

These sets of irreducible characters are very important to developing Algorithm 2.6 presented in the next chapter.

We now introduce some key objects for our study.

Definition 1.21. A subset Σ of Φ^+ is called an *antichain* if for all $\alpha, \beta \in \Sigma$ with $\alpha \neq \beta$, we have $\alpha \not< \beta$ and $\beta \not< \alpha$, i.e. α and β are incomparable in the partial order defined on Φ^+.

We have the following result, also presented in [CP02, Section 4].

Proposition 1.22. *The antichains in Φ^+ are in bijection with the normal subsets of Φ^+.*

Proof. Let $\Sigma \in \Phi^+$ an antichain. We define

$$\mathcal{K}_\Sigma = \{\beta \in \Phi^+ \mid \beta \not\leq \gamma \text{ for all } \gamma \in \Sigma\}. \tag{1.3.2}$$

Let $\alpha \in \Phi^+$ and $\delta \in \mathcal{K}_\Sigma$, and assume that $\alpha + \delta \in \Phi^+$. We cannot have $\alpha + \delta \leq \gamma$ for some $\gamma \in \Sigma$, since in this case we would have $\delta \leq \gamma$, which contradicts the fact that $\delta \in \mathcal{K}_\Sigma$. Then $\alpha + \delta \in \mathcal{K}_\Sigma$, that is, $\mathcal{K}_\Sigma$ is a normal subset of Φ^+.

Conversely, let $\mathcal{K} \trianglelefteq \Phi^+$. We define

$$\Sigma_\mathcal{K} = \{\gamma \in \Phi^+ \setminus \mathcal{K} \mid \gamma \not\leq \alpha \text{ for all } \alpha \in \Phi^+ \setminus (\mathcal{K} \cup \{\gamma\})\}.$$

By definition, $\Sigma_\mathcal{K}$ is an antichain.

Let $\eta \in \mathcal{K}$. If $\eta \leq \delta$ for some $\delta \in \Phi^+$, then either $\eta = \delta$ or, by Proposition 1.5, we can write

$$\delta - \eta = \epsilon_1 + \cdots + \epsilon_m$$

for some $m \geq 1$, with $\epsilon_i \in \Pi$ and $\eta + \epsilon_1 + \cdots + \epsilon_i \in \Phi^+$ for every $i = 1, \ldots, m$, and this implies $\delta \in \mathcal{K}$. Then we have $\eta \not\leq \gamma$ for every $\gamma \in \Sigma_\mathcal{K}$, that is, $\mathcal{K} \subseteq \mathcal{K}_{\Sigma_\mathcal{K}}$. Finally, let $\beta \in \Phi^+$, such that $\beta \not\leq \gamma$ for every γ maximal element in $\Phi^+ \setminus \mathcal{K}$. Then we have $\beta \in \mathcal{K}$. This implies

$\mathcal{K}_{\Sigma_{\mathcal{K}}} \subseteq \mathcal{K}$, then $\mathcal{K} = \mathcal{K}_{\Sigma_{\mathcal{K}}}$. It is then easy to see that $\Sigma_{\mathcal{K}_\Sigma} = \Sigma$. This means that $\mathcal{K} \mapsto \Sigma_{\mathcal{K}}$ and $\Sigma \mapsto \mathcal{K}_\Sigma$ are inverse maps, which give the desired bijection. $\square$

For an antichain Σ in Φ^+, we define the quattern $\mathcal{S}_\Sigma = \Phi^+ \setminus \mathcal{K}_\Sigma$, for $\mathcal{K}_\Sigma$ as defined in Equation (1.3.2). Then it is an easy consequence of the definitions that $\mathcal{Z}(\mathcal{S}_\Sigma) = \Sigma$.

We recall that the number of antichains has been determined in each type and rank, for example in [FR05, Theorem 5.1]. Moreover, it is straightforward to get an algorithm to determine all antichains in a root system of small rank. In particular, we have done this in GAP3 [GAP3] using the package CHEVIE [CHEVIE] for every group of Lie type examined in this thesis. All the antichains in these cases are given by the indices of the families of characters in the first column of the tables in Appendix D.

A_r	B_r, C_r	D_r	E_6	E_7	E_8	F_4	G_2
$\frac{1}{r+2}\binom{2r+2}{r+1}$	$\binom{2r}{r}$	$\frac{3r-2}{r}\binom{2r-2}{r-1}$	833	4160	25080	105	8

Table 1.2: The number of antichains in a root system.

Now let $\chi \in \mathrm{Irr}(U)$. We define $\mathcal{R}(\chi) = \{\alpha \in \Phi^+ \mid X_\alpha \subseteq \ker \chi\}$. If we let $\delta \in \mathcal{R}(\chi)$ and ρ be the representation corresponding to χ, then $\rho(x_\delta(t)) = 1$ for every $t \in \mathbb{F}_q$. Let now $\alpha \in \Phi^+$ be such that $\alpha + \delta \in \Phi^+$. By Proposition 1.16, for every $t \in \mathbb{F}_q$ we have that

$$x_{\alpha+\delta}(t) = \prod_{i=1}^{m} [x_\alpha(s_i), x_\delta(t_i)]$$

for some $m \geq 1$ and $s_1, \dots, s_m, t_1 \dots, t_m \in \mathbb{F}_q$, so that

$$\rho(x_{\alpha+\delta}(t)) = \rho\left(\prod_{i=1}^{m} [x_\alpha(s_i), x_\delta(t_i)]\right) = \prod_{i=1}^{m} [\rho(x_\alpha(s_i)), \rho(x_\delta(t_i))] = 1.$$

Therefore, we have that $\rho(x_{\alpha+\delta}(t)) = 1$ for every $t \in \mathbb{F}_q$. This means that $\mathcal{R}(\chi)$ is a normal subset of Φ^+. In particular, $\Sigma_{\mathcal{R}(\chi)}$ is an antichain in Φ^+.

Finally, we provide the partition of $\mathrm{Irr}(U)$ into families of irreducible characters parametrized by antichains. For an antichain $\Sigma \in \Phi^+$, we define $\mathrm{Irr}(U)_\Sigma = \{\chi \in \mathrm{Irr}(U) \mid \Sigma_{\mathcal{R}(\chi)} = \Sigma\}$. Then we have the partition

$$\mathrm{Irr}(U) = \bigsqcup_{\Sigma} \mathrm{Irr}(U)_\Sigma,$$

where the union is taken over all antichains Σ in Φ^+. Moreover, we have that any character in $\mathrm{Irr}(U)_\Sigma$ is the inflation of an irreducible character in $\mathrm{Irr}(X_{\mathcal{S}_\Sigma})_\Sigma$, namely obtained by inflating over $X_{\mathcal{K}_\Sigma}$.

CHAPTER 2

PARAMETRIZING Irr(U) UP TO NONABELIAN CORES

The goal of this chapter is to develop an algorithm, namely Algorithm 2.6, in order to get a generic parametrization of a large part of the irreducible characters of a Sylow p-subgroup U of a split finite group of Lie type G defined over $\mathbb{F}_q$, where $q = p^e$ and p is not a very bad prime for U. We do this via successive reductions to smaller subquotients of U corresponding to quatterns in Φ^+. The irreducible characters parametrized in this way arise from *abelian cores*, as explained in the sequel. The focus of the following chapters is to deal with the cases we are left with, namely *nonabelian cores*.

We go in more detail into the structure of this chapter. In Section 2.1, we prove some preliminary results that determine the reductions of the algorithm. We give an idea of how the algorithm works in Section 2.2 through an example in type F_4, while in Section 2.3 we give a formal outline of it. Every character arising from an abelian core is given by an inflation followed by an induction of a linear character of an abelian quattern group, as explained in Section 2.4. To finish, we present the output of the algorithm in Section 2.5. The whole of Irr(U) is parametrized in this way when the rank of G is at most 3 or $G = \mathrm{C}_4(q)$, as well as "most" of the characters from Irr(U) for G of rank at most 5.

2.1 Lemmas required for the algorithm

Let $\mathcal{S} = \mathcal{P} \setminus \mathcal{K}$ be a quattern in Φ^+ as in Definition 1.20, and let $\mathcal{Z} \subseteq \mathcal{Z}(\mathcal{S})$ be a subset of its set of central roots. We fix some notation for inflation and induction of characters, that we frequently use in the sequel. Let $\mathcal{S}' = \mathcal{P}' \setminus \mathcal{K}'$ be another quattern, and let ψ be a character of $X_{\mathcal{S}'}$. If $\mathcal{P}' = \mathcal{P}$ and $\mathcal{K}' \supseteq \mathcal{K}$, then we let $\mathcal{L} = \mathcal{K}' \setminus \mathcal{K}$ and we write $\operatorname{Inf}_{\mathcal{L}} \psi$ for the inflation of ψ from $X_{\mathcal{S}'}$ to $X_{\mathcal{S}}$; in case $\mathcal{L} = \{\alpha\}$ has one element, we write $\operatorname{Inf}_{\alpha} \psi = \operatorname{Inf}_{\mathcal{L}} \psi$. If $\mathcal{K}' = \mathcal{K}$ and $\mathcal{P}' \subseteq \mathcal{P}$, then we let $\mathcal{T} = \mathcal{P} \setminus \mathcal{P}'$ and we write $\operatorname{Ind}^{\mathcal{T}} \psi$ for $\psi^{X_{\mathcal{S}}}$; in case $\mathcal{T} = \{\alpha\}$ has one element, we write $\operatorname{Ind}^{\alpha} \psi = \operatorname{Ind}^{\mathcal{T}} \psi$.

The following result, which is part of our reduction procedure, is a consequence of the reduction lemma, that is, Lemma 1.12. It provides us with a natural method to reduce our investigation from $\operatorname{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}$, defined in Equation (1.3.1), to $\operatorname{Irr}(X_{\mathcal{S}'})_{\mathcal{Z}}$, where $\mathcal{S}'$ contains two fewer roots than $\mathcal{S}$, and $\mathcal{Z} \subseteq \mathcal{Z}(\mathcal{S}')$. In this lemma, we have $\mathcal{S}' = \mathcal{S} \setminus \{\delta, \beta\}$, where δ and β are positive roots satisfying certain assumptions. We can transfer this information to determine the behavior of the corresponding root subgroups as explained in Section 1.3. In particular, we immediately check that assumptions (i) to (iv) of the reduction lemma are satisfied by $Y := X_\delta$, $X := X_\beta$ and $Z := X_{\delta+\beta}$ in $X_{\mathcal{S}}$. With some more work, we show that with such a choice of Y and X assumption (v) is also satisfied.

Lemma 2.1. *Let $\mathcal{S} = \mathcal{P} \setminus \mathcal{K}$ be a quattern, let $\mathcal{Z} \subseteq \mathcal{Z}(\mathcal{S})$ and let $\gamma \in \mathcal{Z}$. Suppose that there exist $\delta, \beta \in \mathcal{S} \setminus \{\gamma\}$, with $\delta + \beta = \gamma$, such that for all $\alpha, \alpha' \in \mathcal{S}$ we have $\alpha + \alpha' \neq \beta$, and that for all $\alpha \in \mathcal{S} \setminus \{\beta\}$ we have $\delta + \alpha \notin \mathcal{S}$. Let $\mathcal{P}' = \mathcal{P} \setminus \{\beta\}$ and $\mathcal{K}' = \mathcal{K} \cup \{\delta\}$. Then we have that $\mathcal{S}' = \mathcal{P}' \setminus \mathcal{K}'$ is a quattern with $X_{\mathcal{S}'} \cong X_{\mathcal{P}'}/X_{\mathcal{K}'}$, and we have a bijection*

$$\begin{aligned} \operatorname{Irr}(X_{\mathcal{S}'})_{\mathcal{Z}} &\to \operatorname{Irr}(X_{\mathcal{S}})_{\mathcal{Z}} \\ \chi &\mapsto \operatorname{Ind}^{\beta} \operatorname{Inf}_{\delta} \chi \end{aligned}$$

by inflating over X_δ and inducing to $X_{\mathcal{S}}$ over X_β.

Proof. Let $\alpha, \alpha' \in \mathcal{P}'$. If $\alpha \in \mathcal{K}$ or $\alpha' \in \mathcal{K}$, then it cannot be that $\alpha + \alpha' = \beta$, since in that case we would get $\beta \in \mathcal{K}$, a contradiction with $\beta \in \mathcal{S}$. If $\alpha, \alpha' \in \mathcal{S}'$, by assumption the equality $\alpha + \alpha' = \beta$ cannot hold as well. Since $\mathcal{P}' = \mathcal{S}' \cup \mathcal{K}$, this proves that $\mathcal{P}'$ is closed.

Let now $\alpha \in \mathcal{P}'$, and $\alpha' \in \mathcal{K}'$. If $\alpha' \in \mathcal{K}$, then $\alpha + \alpha' \in \mathcal{K}'$ whenever $\alpha + \alpha' \in \Phi^+$, since $\mathcal{K} \trianglelefteq \mathcal{P}$. Otherwise, $\alpha' = \delta$, and by assumption $\alpha + \delta \notin \mathcal{S}$ since $\alpha \neq \beta$, therefore if $\alpha + \delta \in \Phi^+$ then $\alpha + \delta \in \mathcal{K}'$. Therefore $\mathcal{K}' \trianglelefteq \mathcal{P}'$, and $\mathcal{S}' = \mathcal{P}' \setminus \mathcal{K}'$ is a quattern.

Let us put $G = X_{\mathcal{S}}$, $Z = X_\gamma$, $H = X_{\mathcal{S} \setminus \{\beta\}}$, $X = X_\beta$ and $Y = X_\delta$. It is immediate to check that conditions (i) and (iii) of the reduction lemma hold. By assumption, for every $\alpha \in \mathcal{S} \setminus \{\beta\}$, Equation (1.1.1) implies $[X_\alpha, X_\delta] = 1$ in G, hence (ii) also holds. On the other hand, we have $[X_\beta, X_\delta] \subseteq Z$, therefore (iv) is satisfied.

We are left to prove (v). Let $\lambda \in \mathrm{Irr}(Z)$, such that $\lambda(x_\gamma(s)) = \phi(as)$ for some $a \in \mathbb{F}_q^\times$, and let us define $\tilde{\lambda} = \mathrm{Inf}_\delta \lambda$. Then for $s_1, s_2 \in \mathbb{F}_q$, we have

$$ {}^{x_\beta(s_1)}\tilde{\lambda} = {}^{x_\beta(s_2)}\tilde{\lambda} \Longrightarrow \tilde{\lambda}([x_\beta(s_1), x_\delta(t)]) = \tilde{\lambda}([x_\beta(s_2), x_\delta(t)]) \text{ for all } t \in \mathbb{F}_q. $$

For $i = 1, 2$, we have that $[x_\beta(s_i), x_\delta(t)] \in Z$, hence $\tilde{\lambda}([x_\beta(s_i), x_\delta(t)]) = \lambda([x_\beta(s_i), x_\delta(t)])$, and by Equation (1.1.1) we have $[x_\beta(s_i), x_\delta(t)] = x_\gamma(c_{1,1}^{\beta,\delta} s_i t)$, with $c_{1,1}^{\beta,\delta} \neq 0$, since p does not divide $c_{1,1}^{\beta,\delta}$. Then

$$ {}^{x_\beta(s_1)}\tilde{\lambda} = {}^{x_\beta(s_2)}\tilde{\lambda} \Longrightarrow \phi(ac_{1,1}^{\beta,\delta} t(s_1 - s_2)) = 1 \text{ for every } t \in \mathbb{F}_q. \tag{2.1.1} $$

As remarked at the end of Section 1.2, this implies that $s_1 = s_2$. Then (v) is also satisfied, and of course $|X| = |Y| = q$, so the lemma follows. □

An application of the lemma in the following example gives the main idea of the "Type R" reduction, later described in Algorithm 2.6.

Example 2.2. Let $U = U_{\mathrm{A}_2}$. Lemma 2.1 easily applies to give a parametrization of $\mathrm{Irr}(U)$ for every prime p. Let us first consider $\mathrm{Irr}(U)_{\mathcal{Z}}$ with $\mathcal{Z} = \{\alpha_3\}$. It is clear that Lemma 2.1 applies with $\mathcal{S} = \Phi^+$, $\gamma = \alpha_3$, $\delta = \alpha_2$ and $\beta = \alpha_1$. Notice that $\mathcal{S}' = \{\gamma\} = \mathcal{Z}$, therefore

$$\mathrm{Irr}(X_{\mathcal{S}'})_{\mathcal{Z}} = \{\lambda^{a_3} : X_3 \to \mathbb{C} \mid a_3 \in \mathbb{F}_q^\times \text{ and } \lambda^{a_3}(x_3(t)) = \phi(a_3 t) \text{ for all } t \in \mathbb{F}_q\} =: \mathrm{Irr}(X_3)^\times.$$

Then we have a bijection

$$\begin{aligned} \mathrm{Irr}(X_3)^\times &\to \mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}} \\ \lambda^{a_3} &\mapsto \mathrm{Ind}^{\alpha_1} \mathrm{Inf}_{\alpha_2} \lambda^{a_3} =: \chi^{a_3}. \end{aligned}$$

The set $\mathcal{C}_1 := \{\chi^{a_3} \mid a_3 \in \mathbb{F}_q^\times\}$ consists of $q-1$ irreducible characters of U of degree q.

We notice that the linear characters are obtained by inflating the irreducible characters of $U/[U,U] \cong X_1 \times X_2$ over X_3, that is, they are given by the set

$$\mathcal{C}_2 := \{\chi_{b_1,b_2} \mid b_1, b_2 \in \mathbb{F}_q\},$$

with $\chi_{b_1,b_2} = \mathrm{Inf}_{\alpha_3} \lambda_{b_1,b_2}$, and $\lambda_{b_1,b_2} \in \mathrm{Irr}(X_1 \times X_2)$ is such that

$$\lambda_{b_1,b_2}(x_1(t_1)x_2(t_2)) = \phi(b_1 t_1 + b_2 t_2) \text{ for every } t_1, t_2 \in \mathbb{F}_q.$$

Finally, we note that

$$\mathrm{Irr}(U) = \mathcal{C}_1 \sqcup \mathcal{C}_2,$$

therefore our parametrization is complete.

Remark 2.3. The assumption of Y and X consisting each of just one root subgroup is crucial in the proof of Lemma 2.1. In fact, we will see in Chapters 4 and 5 several examples where Y and X are products of two or more root subgroups and satisfy assumptions (i)–(iv) of the reduction lemma, but in these cases Equation (2.1.1) gives rise to a system of linear equations with nontrivial space of solutions. Correspondingly, assumption (v) of the reduction lemma is not satisfied with this choice of Y and X. We would need to apply the reduction lemma by choosing a subgroup $\tilde{Y}$ of Y and a subset $\tilde{X}$ of X which are *not* products of root subgroups.

Our second lemma is an immediate consequence of the definitions.

Lemma 2.4. *Let $\mathcal{S} = \mathcal{P} \setminus \mathcal{K}$ be a quattern and $\alpha \in \mathcal{Z}(\mathcal{S})$. Let $\mathcal{S} \setminus \{\alpha\} := \mathcal{P} \setminus (\mathcal{K} \cup \alpha)$ be regarded as a quattern. Then there is a bijection* $\mathrm{Irr}(X_{\mathcal{S}}) \to \mathrm{Irr}(X_{\mathcal{S}})_{\{\alpha\}} \sqcup \mathrm{Irr}(X_{\mathcal{S}\setminus\{\alpha\}})$.

Proof. Let us put

$$\mathcal{C}_1 := \{\chi \in \mathrm{Irr}(X_{\mathcal{S}}) \mid X_\alpha \nsubseteq \ker(\chi)\} \quad \text{and} \quad \mathcal{C}_2 := \{\chi \in \mathrm{Irr}(X_{\mathcal{S}}) \mid X_\alpha \subseteq \ker(\chi)\}.$$

We have that $\mathcal{C}_1 = \mathrm{Irr}(X_{\mathcal{S}})_{\{\alpha\}}$. If $\mathcal{S} = \mathcal{P} \setminus \mathcal{K}$, then certainly $\mathcal{K} \cup \{\alpha\} \trianglelefteq \mathcal{P}$ since $\alpha \in \mathcal{Z}(\mathcal{S})$, then $\mathcal{S} \setminus \{\alpha\}$ is a well-defined quattern. We then observe that every $\chi \in \mathcal{C}_2$ can be written as $\chi = \overline{\chi} \circ \pi$, where π is the projection from $X_{\mathcal{S}}$ to $X_{\mathcal{S}\setminus\{\alpha\}}$ and $\overline{\chi} \in \mathrm{Irr}(X_{\mathcal{S}\setminus\{\alpha\}})$, and in fact the map $\chi \mapsto \overline{\chi}$ is a bijection from $\mathcal{C}_2$ to $\mathrm{Irr}(X_{\mathcal{S}\setminus\{\alpha\}})$.

Noting that $\mathrm{Irr}(X_{\mathcal{S}}) = \mathcal{C}_1 \sqcup \mathcal{C}_2$, we conclude that the map

$$\begin{aligned} \Psi : \mathrm{Irr}(X_{\mathcal{S}}) &\to \mathrm{Irr}(X_{\mathcal{S}})_{\{\alpha\}} \sqcup \mathrm{Irr}(X_{\mathcal{S}\setminus\{\alpha\}}) \\ \chi &\mapsto \begin{cases} \chi & \text{if } \chi \in \mathcal{C}_1 \\ \overline{\chi} & \text{if } \chi \in \mathcal{C}_2 \end{cases} \end{aligned}$$

is the desired bijection. □

2.2 An example of the algorithm

The algorithm which will be defined in Section 2.3 is based on a reduction procedure, by checking at each step, for a given quattern, if Lemma 2.1 or Lemma 2.4 can be applied for a certain choice of roots. Before we give a formal outline of the algorithm, we illustrate how it works in more detail in a particular case. We refer to Table A.4 for the root numbering in type F_4.

Example 2.5. Let $U = U_{\mathrm{F}_4}$. We want to compute $\mathrm{Irr}(U)_\Sigma$, where $\Sigma = \{\alpha_{12}\}$. We let $\mathcal{S} = \mathcal{S}_\Sigma = \Phi^+ \setminus \mathcal{K}_\Sigma$, where $\mathcal{K}_\Sigma$ is as defined in Equation (1.3.2), so $\mathcal{S} = \{\alpha_1, \dots, \alpha_8\} \cup \{\alpha_{10}, \alpha_{12}\}$. Also we let $\mathcal{Z} = \Sigma = \{\alpha_{12}\}$. So we want to compute $\mathrm{Irr}(X_\mathcal{S})_\mathcal{Z}$.

Let us define

$$(\beta_1, \delta_1) = (\alpha_1, \alpha_{10}), \qquad (\beta_2, \delta_2) = (\alpha_4, \alpha_8), \qquad (\beta_3, \delta_3) = (\alpha_5, \alpha_7), \qquad (\beta_4, \delta_4) = (\alpha_2, \alpha_3).$$

An application of Lemma 2.1 for $(\beta, \delta) = (\beta_1, \delta_1)$ gives a bijection

$$\begin{aligned} \mathrm{Irr}(X_{\mathcal{S}^1})_\mathcal{Z} &\to \mathrm{Irr}(X_\mathcal{S})_\mathcal{Z} \\ \chi &\mapsto \mathrm{Ind}^{\beta_1} \mathrm{Inf}_{\delta_1} \chi \end{aligned}$$

where $\mathcal{S}^1 = \mathcal{S} \setminus \{\beta_1, \delta_1\}$. Two further applications of Lemma 2.1 for $(\beta, \delta) = (\beta_i, \delta_i)$, $i = 2, 3$ give bijections

$$\begin{aligned} \mathrm{Irr}(X_{\mathcal{S}^2})_\mathcal{Z} &\to \mathrm{Irr}(X_{\mathcal{S}^1})_\mathcal{Z} \\ \chi &\mapsto \mathrm{Ind}^{\beta_2} \mathrm{Inf}_{\delta_2} \chi \end{aligned}$$

where $\mathcal{S}^2 = \mathcal{S}^1 \setminus \{\beta_2, \delta_2\}$, and

$$\begin{aligned}\operatorname{Irr}(X_{\mathcal{S}^3})_{\mathcal{Z}} &\to \operatorname{Irr}(X_{\mathcal{S}^2})_{\mathcal{Z}}\\ \chi &\mapsto \operatorname{Ind}^{\beta_3}\operatorname{Inf}_{\delta_3}\chi\end{aligned}$$

with $\mathcal{S}^3 = \mathcal{S}^2 \setminus \{\beta_3, \delta_3\}$. We record the sets $\mathcal{A} = \{\beta_1, \beta_2, \beta_3\}$ and $\mathcal{L} = \{\delta_1, \delta_2, \delta_3\}$ to remind us which reductions were performed. We also define $\mathcal{K} = \mathcal{K}_\Sigma \cup \mathcal{L}$. These three reductions are all instances of TYPE R reductions (the capitalized R stands for "reduction lemma") in Algorithm 2.6 in Section 2.3.

Now we can see that $\alpha_{12} \in \mathcal{D}(\mathcal{S}^3)$, so that $X_{\mathcal{S}^3} \cong X_{\mathcal{S}^3\setminus\{\alpha_{12}\}} \times X_{12}$. In particular, this means there is no possibility to apply Lemma 2.1 with $\gamma \in \mathcal{Z} = \{\alpha_{12}\}$.

We find that $\mathcal{Z}(\mathcal{S}^3) \setminus \mathcal{D}(\mathcal{S}^3) = \{\alpha_6\}$. We can apply Lemma 2.4 to obtain a bijection

$$\operatorname{Irr}(X_{\mathcal{S}^3})_{\mathcal{Z}} \to \operatorname{Irr}(X_{\mathcal{S}^3})_{\mathcal{Z}\cup\{\alpha_6\}} \sqcup \operatorname{Irr}(X_{\mathcal{S}^3\setminus\{\alpha_6\}})_{\mathcal{Z}}$$

$$\chi \mapsto \begin{cases}\chi & \text{if } X_6 \not\subseteq \ker\chi\\ \overline{\chi} & \text{if } X_6 \subseteq \ker\chi,\end{cases}$$

where $\chi = \overline{\chi} \circ \pi$ and π is the projection from $X_{\mathcal{S}^3}$ to $X_{\mathcal{S}^3\setminus\{\alpha_6\}}$. We now split the two cases and consider them in turn. We note that this is an example of a TYPE S reduction (the capitalized S stands for "split") as defined in our algorithm in Section 2.3.

First we consider

$$\operatorname{Irr}(X_{\mathcal{S}^3})_{\mathcal{Z}^3},$$

where $\mathcal{S}^3 = \{\alpha_2, \alpha_3, \alpha_6, \alpha_{12}\}$ and $\mathcal{Z}^3 = \{\alpha_6, \alpha_{12}\}$. Lemma 2.1 applies with $(\beta, \delta) = (\beta_4, \delta_4)$ and $\gamma = \alpha_6$. We then get a bijection

$$\operatorname{Irr}(X_{\mathcal{S}^4})_{\mathcal{Z}^3} \to \operatorname{Irr}(X_{\mathcal{S}^3})_{\mathcal{Z}^3}$$

$$\chi \mapsto \operatorname{Ind}^{\beta_4} \operatorname{Inf}_{\delta_4} \chi$$

where $\mathcal{S}^4 = \mathcal{S}^3 \setminus \{\alpha_2, \alpha_3\} = \{\alpha_6, \alpha_{12}\}$. This is another reduction of TYPE R as defined in Section 2.3. We record this reduction by adjoining α_2 to $\mathcal{A}$ to obtain $\mathcal{A}' = \{\alpha_1, \alpha_4, \alpha_5, \alpha_2\}$ and adjoining α_3 to $\mathcal{L}$ to obtain $\mathcal{L}' = \{\alpha_{10}, \alpha_8, \alpha_7, \alpha_3\}$. Moreover, we put $\mathcal{K}' = \mathcal{K}_\Sigma \cup \mathcal{L}'$.

We note that $X_{\mathcal{S}^4} = X_6 \times X_{12}$, so we can parametrize $\operatorname{Irr}(X_{\mathcal{S}^4})_{\mathcal{Z}^3}$ as $\{\lambda^{a_6,a_{12}} \mid a_6, a_{12} \in \mathbb{F}_q^\times\}$, where $\lambda^{a_6,a_{12}}(x_i(t)) = \phi(a_i t)$ for $i = 6, 12$. Through the bijections given by Lemma 2.1, we obtain characters of U forming part of $\operatorname{Irr}(U)_\Sigma$ by a process of successive inflation and induction of the characters $\lambda^{a_6,a_{12}}$. These characters are

$$\chi^{a_6,a_{12}} = \operatorname{Inf}_{\mathcal{K}_\Sigma} \operatorname{Ind}^{\alpha_1} \operatorname{Inf}_{\alpha_{10}} \operatorname{Ind}^{\alpha_4} \operatorname{Inf}_{\alpha_8} \operatorname{Ind}^{\alpha_5} \operatorname{Inf}_{\alpha_7} \operatorname{Ind}^{\alpha_2} \operatorname{Inf}_{\alpha_3} \lambda^{a_6,a_{12}}.$$

However, it turns out that these characters can be obtained by a single inflation and then induction, thanks to Theorem 2.11 proved in Section 2.4, and we have

$$\chi^{a_6,a_{12}} = \operatorname{Ind}^{\mathcal{A}'} \operatorname{Inf}_{\mathcal{K}'} \lambda^{a_6,a_{12}}.$$

The characters $\chi^{a_6,a_{12}}$ have degree q^4.

Next we move on to consider the characters in $\operatorname{Irr}(X_{\mathcal{S}^5})_{\mathcal{Z}}$ where $\mathcal{S}^5 = \mathcal{S}^3 \setminus \{\alpha_6\} = \{\alpha_2, \alpha_3, \alpha_{12}\}$, and $\mathcal{Z} = \{\alpha_{12}\}$. We record that we have put α_6 in the kernel by adjoining it to $\mathcal{K}$ to obtain $\mathcal{K}'' = \mathcal{K} \cup \{\alpha_6\}$. We see that $X_{\mathcal{S}^5}$ is abelian, so that

$$\operatorname{Irr}(X_{\mathcal{S}^5}) = \{\lambda^{a_{12}}_{b_2,b_3} \mid a_{12} \in \mathbb{F}_q^\times, b_2, b_3 \in \mathbb{F}_q\},$$

where $\lambda^{a_{12}}_{b_2,b_3}(x_i(t)) = \phi(b_i t)$ for $i = 2, 3$, and $\lambda^{a_{12}}_{b_2,b_3}(x_{12}(t)) = \phi(a_{12} t)$.

Now through the bijections previously obtained in Lemma 2.1, we obtain characters $\chi^{a_{12}}_{b_2,b_3}$ of U forming part of $\operatorname{Irr}(U)_\Sigma$ from the characters $\lambda^{a_{12}}_{b_2,b_3}$ by a process of successive inflation

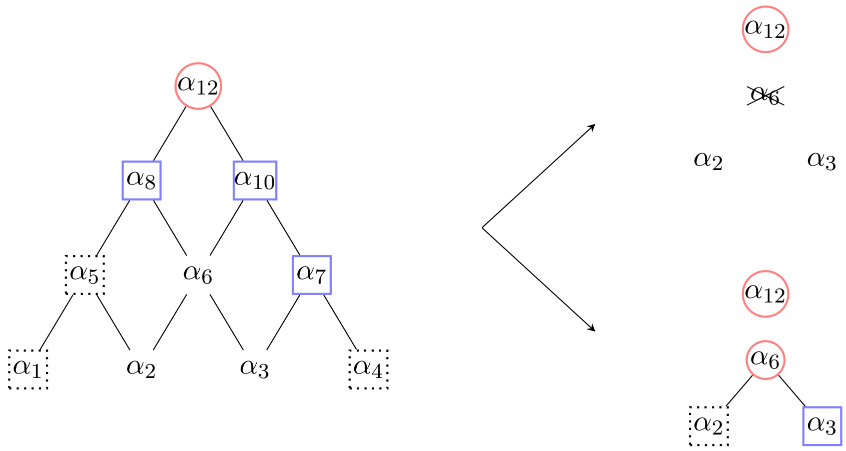

Figure 2.1: A pictorial representation of the calculation of the characters in $\mathrm{Irr}(U)_{\{\alpha_{12}\}}$ for G of type F_4.

and induction. We have

$$\chi^{a_{12}}_{b_2,b_3} = \mathrm{Inf}_{\mathcal{K}_\Sigma} \mathrm{Ind}^{\alpha_1} \mathrm{Inf}_{\alpha_{10}} \mathrm{Ind}^{\alpha_4} \mathrm{Inf}_{\alpha_8} \mathrm{Ind}^{\alpha_5} \mathrm{Inf}_{\alpha_7} \mathrm{Inf}_{\alpha_6} \lambda^{a_{12}}_{b_2,b_3},$$

and note that by using Theorem 2.11, we can write these characters as

$$\chi^{a_{12}}_{b_2,b_3} = \mathrm{Ind}^{\mathcal{A}} \mathrm{Inf}_{\mathcal{K}''} \lambda^{a_6,a_{12}}.$$

These characters have degree q^3.

Putting this together, we have that

$$\mathrm{Irr}(U)_{\{\alpha_{12}\}} = \{\chi^{a_6,a_{12}} \mid a_6, a_{12} \in \mathbb{F}_q^\times\} \sqcup \{\chi^{a_{12}}_{b_2,b_3} \mid b_2, b_3 \in \mathbb{F}_q, a_{12} \in \mathbb{F}_q^\times\}.$$

Therefore, $\mathrm{Irr}(U)_{\{\alpha_{12}\}}$ consists of:

- $(q-1)^2$ characters of degree q^4, and
- $q^2(q-1)$ characters of degree q^3.

We illustrate in Figure 2.1 how we have calculated these characters. The vertices in the picture are the roots involved in the quatterns that we study. The partial order on Φ^+ naturally gives them a poset structure; we join the vertices accordingly. The roots in a circle are in the set $\mathcal{Z}$; the roots in a straight box are in $\mathcal{L}$ and the roots in a dotted box are in $\mathcal{A}$. A cross on α_6 means that the corresponding root subgroup is in the kernel of the character examined at that stage.

2.3 A formal outline of the algorithm

We now describe the algorithm, which is used to calculate $\mathrm{Irr}(U)_\Sigma$ for each antichain Σ in Φ^+. We explain the algorithm below, which is written in a sort of pseudocode; the comments in *italics* aim to make it easier to understand.

Algorithm 2.6. *Reduction procedure for* $\mathrm{Irr}(U)_\Sigma$

INPUT:

- $\Phi^+ = \{\alpha_1, \ldots, \alpha_N\}$, the set of positive roots of a root system with a fixed enumeration such that $i \leq j$ whenever $\alpha_i \leq \alpha_j$.
- Σ, an antichain in Φ^+.

VARIABLES:

- $\mathcal{S} \subseteq \Phi^+$ is a quattern.
- $\mathcal{Z}$ is a subset of $\mathcal{Z}(\mathcal{S})$.
- $\mathcal{A} \subseteq \Phi^+$ keeps a record of the roots β used in a TYPE R reduction.

- $\mathcal{L} \subseteq \Phi^+$ keeps a record of the roots δ used in a TYPE R reduction.
- $\mathcal{K} \subseteq \Phi^+$ keeps a record of the roots indexing root subgroups in the quotient of the associated quattern group.
- $\mathfrak{S}$ is a stack of tuples of the form $(\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ as above to be considered later in the algorithm.
- $\mathfrak{O} = (\mathfrak{O}_1, \mathfrak{O}_2)$ is the output. Each of $\mathfrak{O}_1$ and $\mathfrak{O}_2$ is a set consisting of tuples of the form $(\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ as above corresponding to abelian and nonabelian cores respectively, as defined later and described in the algorithm.

INITIALIZATION:

- $\mathcal{K} := \mathcal{K}_\Sigma$.
- $\mathcal{S} := \Phi^+ \setminus \mathcal{K}_\Sigma$.
- $\mathcal{Z} := \Sigma$.
- $\mathcal{A} := \varnothing$.
- $\mathcal{L} := \varnothing$.
- $\mathfrak{S} := \varnothing$.
- $\mathfrak{O} := (\varnothing, \varnothing)$.

During the algorithm we consider $\mathrm{Irr}(X_\mathcal{S})_\mathcal{Z}$*, going into four subroutines called "ABELIAN CORE", "TYPE R", "TYPE S" and "NONABELIAN CORE".*

ABELIAN CORE.

if $\mathcal{S} = \mathcal{Z}(\mathcal{S})$ **then**

Adjoin $(\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ to $\mathfrak{O}_1$.

In this case $X_{\mathcal{S}}$ is abelian and we can parametrize the characters in $\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}$.

if $\mathfrak{S} = \varnothing$ **then**

Finish and **output** $\mathfrak{O}$.

In this case we have no more characters to consider, so we are done.

else

Remove the tuple at the top of the stack $\mathfrak{S}$ and replace $(\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ with it, and go to ABELIAN CORE.

end if

else

Go to TYPE R.

end if

TYPE R.

Look for pairs $(\beta, \delta) = (\alpha_i, \alpha_j)$ that satisfy the conditions of Lemma 2.1 for some $\gamma \in \mathcal{Z}$.

if such a pair (α_i, α_j) exists **then**

Choose the pair with j maximal, and update the variables as follows.

- $\mathcal{S} := \mathcal{S} \setminus \{\alpha_i, \alpha_j\}$.
- $\mathcal{A} := \mathcal{A} \cup \{\alpha_i\}$.
- $\mathcal{L} := \mathcal{L} \cup \{\alpha_j\}$.
- $\mathcal{K} := \mathcal{K} \cup \{\alpha_j\}$.

We are replacing $\mathcal{S}$ with $\mathcal{S}'$ as in Lemma 2.1, and recording this in $\mathcal{A}$, $\mathcal{L}$ and $\mathcal{K}$.

Go to ABELIAN CORE.

else

Go to TYPE S.

end if

TYPE S.

if $\mathcal{Z}(\mathcal{S}) \setminus (\mathcal{Z} \cup \mathcal{D}(\mathcal{S})) \neq \varnothing$ **then**

Let i be maximal such that $\alpha_i \in \mathcal{Z}(\mathcal{S}) \setminus (\mathcal{Z} \cup \mathcal{D}(\mathcal{S}))$, and update as follows.

- We add $(\mathcal{S} \setminus \{\alpha_i\}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K} \cup \{\alpha_i\})$ at the top of the stack, that is, we put $\mathfrak{S} := \mathfrak{S} \cup \{(\mathcal{S} \setminus \{\alpha_i\}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K} \cup \{\alpha_i\})\}$.
- We update $\mathcal{Z} := \mathcal{Z} \cup \{\alpha_i\}$.

Here we are using Lemma 2.4. We first add $(\mathcal{S} \setminus \{\alpha_i\}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K} \cup \{\alpha_i\})$ *to the stack to be considered later, recording that* X_i *is in the kernel of these characters by adding* α_i *to* $\mathcal{K}$*. Then we replace* $(\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ *with* $(\mathcal{S}, \mathcal{Z} \cup \{\alpha_i\}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ *for the current run.*

Go to ABELIAN CORE.

else

Go to NONABELIAN CORE

end if

NONABELIAN CORE.

Adjoin $(\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ to $\mathfrak{O}_2$.

We are no longer able to apply reductions of TYPE R or of TYPE S, and $X_{\mathcal{S}}$ *is not abelian, so the algorithm gives up, and this case is output as a nonabelian core as discussed further later.*

if $\mathfrak{S} = \varnothing$ **then**

Output $\mathfrak{O}$ and **finish**.

In this case we have no more characters to consider, so we are done.

else

Remove the tuple at the top of the stack $\mathfrak{S}$ and replace $(\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ with it, and go to ABELIAN CORE.

end if

Let us explain the conventions for reduction types. We recall from Section 2.2 that the letter R in the TYPE R reduction corresponds to "reduction lemma", to record when Lemma 2.1 is applied. The letter S in the TYPE S reduction stands for "split", and this records instead the iteration at which we have the branching determined in Lemma 2.4. We notice that at any stage, it takes a finite number of iterations to either remove one or more roots from $\mathcal{S}$, or add one root to $\mathcal{Z}$. Then we see that this algorithm does in turn terminate in a finite number of steps.

We move on to discuss how we interpret the output. We begin by defining what we mean by a *core*, which is an element of the output of our algorithm.

Definition 2.7. Let us suppose that Algorithm 2.6 has run with input (Φ^+, Σ) and given output $\mathfrak{O}$.

- An element $(\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ of $\mathfrak{O}_1$ is called an *abelian core* for $\mathrm{Irr}(U)_\Sigma$.
- An element $(\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ of $\mathfrak{O}_2$ is called a *nonabelian core* for $\mathrm{Irr}(U)_\Sigma$.

In the rest of Chapter 2, we discuss how we can determine the characters in $\mathrm{Irr}(U)_\Sigma$ corresponding to a core $\mathfrak{C} = (\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ in $\mathfrak{O}_1 \cup \mathfrak{O}_2$. In particular, when $\mathfrak{C} \in \mathfrak{O}_1$ is an abelian core, we give a complete description of the irreducible characters, however for nonabelian cores there is more work required. Before we move onto this, we require some more notation.

We obtain $\mathfrak{C}$ through a sequence of reductions of TYPE R and of TYPE S applied in Algorithm 2.6. So we consider the sequence of reductions where in each one either:

- a pair of roots β and δ is taken from $\mathcal{S}$ in a TYPE R reduction, and β is added to $\mathcal{A}$ and δ is added to $\mathcal{L}$ and $\mathcal{K}$; or
- a root γ is taken from $\mathcal{S}$ and added to $\mathcal{K}$.

We let $\ell = \ell_{\mathfrak{C}}$ be the number of these reductions, and define the sequence $T(\mathfrak{C}) = (t_1, \ldots, t_\ell)$, where $t_i = \mathrm{R}$ if the ith reduction is a TYPE R reduction and $t_i = \mathrm{S}$ if the ith reduction is a TYPE S reduction. We let $I(\mathrm{R}, \mathfrak{C})$ be the set of i such that $t_i = \mathrm{R}$ and $I(\mathrm{S}, \mathfrak{C})$ be the set of i such that $t_i = \mathrm{S}$. For $i \in I(\mathrm{R}, \mathfrak{C})$ we write (β_i, δ_i) for the pair of roots used in the TYPE R reduction, and for $i \in I(\mathrm{S}, \mathfrak{C})$, we write γ_i for the root added to $\mathcal{K}$ in the TYPE S reduction. Thus we have $\mathcal{A} = \{\beta_i \mid i \in I(\mathrm{R}, \mathfrak{C})\}$, $\mathcal{L} = \{\delta_i \mid i \in I(\mathrm{R}, \mathfrak{C})\}$ and $\mathcal{K} \setminus \mathcal{K}_\Sigma = \mathcal{L} \cup \{\gamma_i \mid i \in I(\mathrm{S}, \mathfrak{C})\}$.

2.4 Compacting sequences of inflations and inductions

This section is of major importance in order to understand how the parametrization of "most" irreducible characters in $\mathrm{Irr}(U)$ follows from Algorithm 2.6. The reductions are all of types R and S; as a consequence, we obtain the characters of $\mathrm{Irr}(U)_\Sigma$ corresponding to some antichain Σ as a sequence of inflations and inductions from a character corresponding to a core. The main goal is to show that this corresponds to a *single* inflation, followed by a *single* induction of such a character.

We first prove the character-theoretical results that we need for this. Let us suppose that $\mathfrak{C} = (\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ is a core corresponding to an antichain Σ, and that β_i, δ_i and γ_i correspond to type R and type S reductions respectively, as in Section 2.3. We define the subsets $\mathcal{P}^0, \mathcal{P}^1, \ldots, \mathcal{P}^\ell$ and $\mathcal{K}^0, \mathcal{K}^1, \ldots, \mathcal{K}^\ell$ of Φ^+ recursively by

$$\mathcal{P}^0 = \Phi^+ \text{ and } \mathcal{K}^0 = \mathcal{K}_\Sigma;$$

$$\mathcal{P}^i = \begin{cases} \mathcal{P}^{i-1} \setminus \{\beta_i\} & \text{if } t_i = \mathrm{R} \\ \mathcal{P}^{i-1} & \text{if } t_i = \mathrm{S} \end{cases}$$

$$\mathcal{K}^i = \begin{cases} \mathcal{K}^{i-1} \cup \{\delta_i\} & \text{if } t_i = \mathrm{R} \\ \mathcal{K}^{i-1} \cup \{\gamma_i\} & \text{if } t_i = \mathrm{S} \end{cases}$$

We have the following lemma about these sets.

Lemma 2.8. *For each $i, j = 0, 1, \ldots, \ell$ with $i \leq j$, we have that $\mathcal{P}^j$ is a closed set, and $\mathcal{K}^i$ is normal in $\mathcal{P}^j$. In particular, $\mathcal{S}^{i,j} = \mathcal{P}^j \setminus \mathcal{K}^i$ are quatterns.*

Proof. Of course, $\mathcal{P}^0 = \Phi^+$ is closed. Let us assume that $\mathcal{P}^{i-1}$ is closed. If $\mathcal{P}^i = \mathcal{P}^{i-1}$, then there is nothing to prove. Let then $\mathcal{P}^i = \mathcal{P}^{i-1} \setminus \{\beta_i\}$. For $\alpha, \alpha' \in \mathcal{P}^i$, it cannot be that $\alpha + \alpha' = \beta_i$ by construction of $\mathcal{P}^i$. Also, by inductive assumption, we have that $\alpha + \alpha' \in \mathcal{P}^{i-1}$ if $\alpha + \alpha'$ is a positive root. This implies $\alpha + \alpha' \in \mathcal{P}^i$ or $\alpha + \alpha' \notin \Phi^+$, that is, $\mathcal{P}^i$ is closed.

To prove that $\mathcal{K}^i$ is normal in $\mathcal{P}^j$ for $i \leq j$, it is enough to prove that $\mathcal{K}^i$ is normal in $\mathcal{P}^i$, since $\mathcal{K}^i \subseteq \mathcal{P}^j \subseteq \mathcal{P}^i$. Let $\alpha \in \mathcal{P}^i$ and $\eta \in \mathcal{K}^i$. Recall that $\eta \in \mathcal{K}_\Sigma$ or η is of the form γ_k or δ_k as above for some $k \leq i$. If $\eta \in \mathcal{K}_\Sigma$, then since $\mathcal{K}_\Sigma \trianglelefteq \Phi^+$ we have that $\alpha + \eta \in \mathcal{K}_\Sigma \subseteq \mathcal{K}^i$ whenever $\alpha + \eta \in \Phi^+$. If $\eta = \gamma_k$ for some $k \leq i$, then η is a central root in $\mathcal{S}^{k-1,k-1} \supseteq \mathcal{S}^{i,i}$, therefore since $\alpha \in \mathcal{P}^i$ we have that $\alpha + \eta \in \mathcal{K}^{k-1} \subseteq \mathcal{K}^i$ or $\alpha + \eta \notin \Phi^+$. If $\eta = \delta_k$, then we notice that $\beta_k \notin \mathcal{P}^k$, thus $\beta_k \notin \mathcal{P}^i$, therefore if $\alpha \in \mathcal{P}^i$ then $\alpha + \eta \in \mathcal{K}^{k-1}$ or $\alpha + \eta \notin \Phi^+$. This implies that $\mathcal{K}^i$ is normal in $\mathcal{P}^i$. $\square$

For $i = 0, \ldots, \ell$, we define $\mathcal{S}^i := \mathcal{S}^{i,i}$. Then we have that $\mathcal{S}^0 = \Phi^+ \setminus \Sigma$, and $\mathcal{S}^\ell = \mathcal{S}$. Now let $\psi \in \mathrm{Irr}(X_\mathcal{S})$. We define characters $\overline{\psi}_i \in \mathrm{Irr}(X_{\mathcal{S}^i})$ for $i = \ell, \ell - 1, \ldots, 1, 0$ recursively by the following sequence of inflations and inductions.

$$\overline{\psi}_\ell = \psi$$

$$\overline{\psi}_{i-1} = \begin{cases} \mathrm{Ind}^{\beta_i} \mathrm{Inf}_{\delta_i} \overline{\psi}_i & \text{if } t_i = \mathrm{R} \\ \mathrm{Inf}_{\gamma_i} \overline{\psi}_i & \text{if } t_i = \mathrm{S} \end{cases}$$

Finally, we let $\overline{\psi} = \operatorname{Inf}_{\mathcal{K}_\Sigma} \overline{\psi}_0 \in \operatorname{Irr}(U)$. In the statement of the following proposition about sequences of inflations and inductions, we use the notation

$$\mathcal{A}_i = \{\beta_j \mid j \geq i\}, \quad \mathcal{L}_i = \{\delta_j \mid j \geq i\} \quad \text{and} \quad \mathcal{K}_i = \{\gamma_j \mid j \geq i\}.$$

Proposition 2.9. *For every $i = 1, \ldots, \ell$, we have that*

$$\left(\operatorname{Ind}_{X_{\mathcal{S}^{i-1,i}}}^{X_{\mathcal{S}^{i-1}}} \operatorname{Inf}_{X_{\mathcal{S}^i}}^{X_{\mathcal{S}^{i-1,i}}}\right) \cdots \left(\operatorname{Ind}_{X_{\mathcal{S}^{\ell-1,\ell}}}^{X_{\mathcal{S}^{\ell-1}}} \operatorname{Inf}_{X_{\mathcal{S}}}^{X_{\mathcal{S}^{\ell-1,\ell}}}\right) = \operatorname{Ind}^{\mathcal{A}_i} \operatorname{Inf}_{\mathcal{L}_i \cup \mathcal{K}_i}$$

as maps from $\mathbb{Z}_{\geq 0} \operatorname{Irr}(X_{\mathcal{S}})$ to $\mathbb{Z}_{\geq 0} \operatorname{Irr}(X_{\mathcal{S}^{i-1}})$. In particular, the image of $\operatorname{Irr}(X_{\mathcal{S}})$ under $\operatorname{Ind}^{\mathcal{A}_i} \operatorname{Inf}_{\mathcal{L}_i \cup \mathcal{K}_i}$ lies in $\operatorname{Irr}(X_{\mathcal{S}^{i-1}})$, and

$$\overline{\psi}_{i-1} = \operatorname{Ind}^{\mathcal{A}_i} \operatorname{Inf}_{\mathcal{L}_i \cup \mathcal{K}_i} \psi.$$

Proof. We prove this by reverse induction on i, the case of $i = \ell$ being trivial.

The inductive step boils down to showing the following claim for $i < \ell$,

$$\left(\operatorname{Ind}_{X_{\mathcal{S}^{i-1,i}}}^{X_{\mathcal{S}^{i-1}}} \operatorname{Inf}_{X_{\mathcal{S}^i}}^{X_{\mathcal{S}^{i-1,i}}}\right) \left(\operatorname{Ind}^{\mathcal{A}_{i+1}} \operatorname{Inf}_{\mathcal{L}_{i+1} \cup \mathcal{K}_{i+1}}\right) = \operatorname{Ind}^{\mathcal{A}_i} \operatorname{Inf}_{\mathcal{L}_i \cup \mathcal{K}_i}.$$

Let us assume that $\mathcal{S}^i$ is constructed from $\mathcal{S}^{i-1}$ by a type R reduction. Then we have that $\mathcal{P}^i = \mathcal{P}^{i-1} \setminus \{\beta^i\}$ and $\mathcal{K}^i = \mathcal{K}^{i-1} \cup \{\delta^i\}$. The claim in this case is equivalent to

$$\left(\operatorname{Ind}^{\beta_i} \operatorname{Inf}_{\delta_i}\right) \left(\operatorname{Ind}^{\mathcal{A}_{i+1}} \operatorname{Inf}_{\mathcal{L}_{i+1} \cup \mathcal{K}_{i+1}}\right) = \operatorname{Ind}^{\mathcal{A}_i} \operatorname{Inf}_{\mathcal{L}_i \cup \mathcal{K}_i}.$$

Let us put

$$N := X_{\delta_i}, \qquad H := X_{\mathcal{S}^{i-1,\ell}}, \qquad G := X_{\mathcal{S}^{i-1,i}}.$$

It is clear that $N \leq H \leq G$, and $N \trianglelefteq G$. Then Lemma 1.8 applies, and we get

$$\operatorname{Inf}_{\delta_i} \operatorname{Ind}^{\mathcal{A}_{i+1}} = \operatorname{Ind}^{\mathcal{A}_{i+1}} \operatorname{Inf}_{\delta_i} .$$

Therefore we get

$$\begin{aligned}
\left(\operatorname{Ind}^{\beta_i} \operatorname{Inf}_{\delta_i}\right) \left(\operatorname{Ind}^{\mathcal{A}_{i+1}} \operatorname{Inf}_{\mathcal{L}_{i+1} \cup \mathcal{K}_{i+1}}\right) &= \operatorname{Ind}^{\beta_i} \left(\operatorname{Inf}_{\delta_i} \operatorname{Ind}^{\mathcal{A}_{i+1}}\right) \operatorname{Inf}_{\mathcal{L}_{i+1} \cup \mathcal{K}_{i+1}} \\
&= \operatorname{Ind}^{\beta_i} \left(\operatorname{Ind}^{\mathcal{A}_{i+1}} \operatorname{Inf}_{\delta_i}\right) \operatorname{Inf}_{\mathcal{L}_{i+1} \cup \mathcal{K}_{i+1}} \\
&= \left(\operatorname{Ind}^{\beta_i} \operatorname{Ind}^{\mathcal{A}_{i+1}}\right) \left(\operatorname{Inf}_{\delta_i} \operatorname{Inf}_{\mathcal{L}_{i+1} \cup \mathcal{K}_{i+1}}\right) \\
&= \operatorname{Ind}^{\mathcal{A}_i} \operatorname{Inf}_{\mathcal{L}_i \cup \mathcal{K}_i},
\end{aligned}$$

as required.

In the case when a type S reduction occurs, we have $\mathcal{P}^i = \mathcal{P}^{i-1}$ and $\mathcal{K}^i = \mathcal{K}^{i-1} \cup \{\gamma^i\}$, and the claim is equivalent to prove

$$\operatorname{Inf}_{\gamma_i} \left(\operatorname{Ind}^{\mathcal{A}_{i+1}} \operatorname{Inf}_{\mathcal{L}_{i+1} \cup \mathcal{K}_{i+1}}\right) = \operatorname{Ind}^{\mathcal{A}_i} \operatorname{Inf}_{\mathcal{L}_i \cup \mathcal{K}_i} .$$

Now if we put

$$N := X_{\gamma_i}, \qquad H := X_{\mathcal{S}^{i-1,\ell}}, \qquad G := X_{\mathcal{S}^{i-1,i}},$$

then again Lemma 1.8 applies. We get

$$\operatorname{Inf}_{\gamma_i} \operatorname{Ind}^{\mathcal{A}_{i+1}} = \operatorname{Ind}^{\mathcal{A}_{i+1}} \operatorname{Inf}_{\gamma_i} .$$

A computation as above yields

$$\begin{aligned}
\operatorname{Inf}_{\gamma_i} \left(\operatorname{Ind}^{\mathcal{A}_{i+1}} \operatorname{Inf}_{\mathcal{L}_{i+1} \cup \mathcal{K}_{i+1}}\right) &= \left(\operatorname{Inf}_{\gamma_i} \operatorname{Ind}^{\mathcal{A}_{i+1}}\right) \operatorname{Inf}_{\mathcal{L}_{i+1} \cup \mathcal{K}_{i+1}} \\
&= \left(\operatorname{Ind}^{\mathcal{A}_{i+1}} \operatorname{Inf}_{\gamma_i}\right) \operatorname{Inf}_{\mathcal{L}_{i+1} \cup \mathcal{K}_{i+1}} \\
&= \operatorname{Ind}^{\mathcal{A}_{i+1}} \left(\operatorname{Inf}_{\gamma_i} \operatorname{Inf}_{\mathcal{L}_{i+1} \cup \mathcal{K}_{i+1}}\right)
\end{aligned}$$

$$= \operatorname{Ind}^{\mathcal{A}_i} \operatorname{Inf}_{\mathcal{L}_i \cup \mathcal{K}_i},$$

which proves the claim. □

We can finally obtain any $\overline{\psi}$ arising from Σ by the above construction as an inflation, followed by an induction of a character of $X_{\mathcal{S}}$.

Corollary 2.10. *Let $\psi \in \operatorname{Irr}(X_{\mathcal{S}})$ and $\overline{\psi}$ as above. We have that*

$$\overline{\psi} = \operatorname{Ind}^{\mathcal{A}} \operatorname{Inf}_{\mathcal{K} \cup \mathcal{K}_\Sigma} \psi.$$

Proof. By Lemma 2.8 and Proposition 2.9 substituting $i = 1$, we have that

$$\overline{\psi}_0 = \operatorname{Ind}^{\mathcal{A}} \operatorname{Inf}_{\mathcal{K}} \psi.$$

Lemma 1.8 applies to the groups

$$N := X_{\mathcal{K}_\Sigma}, \qquad H := X_{\mathcal{P}^\ell}, \qquad G := U,$$

We then have

$$\begin{aligned} \operatorname{Inf}_{\mathcal{K}_\Sigma}(\operatorname{Ind}^{\mathcal{A}} \operatorname{Inf}_{\mathcal{K}}) &= (\operatorname{Inf}_{\mathcal{K}_\Sigma} \operatorname{Ind}^{\mathcal{A}}) \operatorname{Inf}_{\mathcal{K}} \\ &= (\operatorname{Ind}^{\mathcal{A}} \operatorname{Inf}_{\mathcal{K}_\Sigma}) \operatorname{Inf}_{\mathcal{K}} \\ &= \operatorname{Ind}^{\mathcal{A}}(\operatorname{Inf}_{\mathcal{K}_\Sigma} \operatorname{Inf}_{\mathcal{K}}) \\ &= \operatorname{Ind}^{\mathcal{A}} \operatorname{Inf}_{\mathcal{K} \cup \mathcal{K}_\Sigma}, \end{aligned}$$

therefore

$$\overline{\psi} = \operatorname{Inf}_{\mathcal{K}_\Sigma}(\operatorname{Ind}^{\mathcal{A}} \operatorname{Inf}_{\mathcal{K}} \psi) = \operatorname{Ind}^{\mathcal{A}} \operatorname{Inf}_{\mathcal{K} \cup \mathcal{K}_\Sigma} \psi,$$

as claimed. $\square$

We now distinguish two cases, according to the output of Algorithm 2.6. Suppose that $\mathfrak{C} = (\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K}) \in \mathfrak{O}_1$ is an abelian core. We let $\mathcal{Z} = \{\alpha_{i_1}, \dots, \alpha_{i_m}\}$ and $\mathcal{S} \setminus \mathcal{Z} = \{\alpha_{j_1}, \dots, \alpha_{j_n}\}$. Then we have

$$\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}} = \{\lambda_{\underline{b}}^{\underline{a}} \mid \underline{a} = (a_{i_1}, \dots, a_{i_m}) \in (\mathbb{F}_q^{\times})^m,\ \underline{b} = (b_{j_1}, \dots, b_{j_n}) \in \mathbb{F}_q^n\},$$

where $\lambda_{\underline{b}}^{\underline{a}}$ is defined by

$$\lambda_{\underline{b}}^{\underline{a}}(x_{\alpha_{i_k}}(t)) = \phi(a_{i_k} t) \quad \text{ and } \quad \lambda_{\underline{b}}^{\underline{a}}(x_{\alpha_{j_h}}(t)) = \phi(b_{j_h} t)$$

for every $k = 1, \dots, m$ and $h = 1, \dots, n$. We define $\chi_{\underline{b}}^{\underline{a}} = \overline{\lambda_{\underline{b}}^{\underline{a}}}$ and

$$\mathrm{Irr}(U)_{\mathfrak{C}} = \{\chi_{\underline{b}}^{\underline{a}} \mid \underline{a} = (a_{i_1}, \dots, a_{i_m}) \in (\mathbb{F}_q^{\times})^m, \underline{b} = (b_{j_1}, \dots, b_{j_n}) \in \mathbb{F}_q^n\}.$$

Through the bijections given by Lemmas 2.1 and 2.4, this is precisely the set of characters in $\mathrm{Irr}(U)_{\Sigma}$ corresponding to $\mathfrak{C}$.

We move on to consider a nonabelian core $\mathfrak{C} = (\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K}) \in \mathfrak{O}_2$. In this case $X_{\mathcal{S}}$ is not abelian, so we do not immediately have a parametrization of $\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}$, and it is necessary for us to determine a parametrization by hand. We suppose this has been done and we have

$$\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}} = \{\psi_{\underline{c}} \mid \underline{c} \in J_{\mathfrak{C}}\},$$

where $J_{\mathfrak{C}}$ is some indexing set. We define $\chi_{\underline{c}} = \overline{\psi_{\underline{c}}}$ and

$$\mathrm{Irr}(U)_{\mathfrak{C}} = \{\chi_{\underline{c}} \mid \underline{c} \in J_{\mathfrak{C}}\}.$$

The aim of Chapters 3, 4 and 5 of this thesis is to develop a method for determining the set $J_{\mathfrak{C}}$ for split finite groups of Lie type of rank at most 5.

In principle, the characters $\chi_{\underline{b}}^{\underline{a}}$ and $\chi_{\underline{c}}$ of U are defined as potentially very long sequences of inflations and inductions. But Proposition 2.9 and Corollary 2.10 applied to $\mathcal{S}^\ell, \mathcal{S}^{\ell-1}, \dots, \mathcal{S}^0$ and then Φ^+ allow us to express $\chi_{\underline{b}}^{\underline{a}}$ and $\chi_{\underline{c}}$ as a single inflation, followed by just one induction of the corresponding character of $X_{\mathcal{S}^\ell} = X_{\mathcal{S}}$.

We summarize the discussion of this section in the following theorem.

Theorem 2.11. *Let $\mathfrak{C} \in (\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ be a core.*

(a) Suppose that $\mathfrak{C} \in \mathfrak{O}_1$ is abelian, and let $\chi_{\underline{b}}^{\underline{a}} \in \mathrm{Irr}(U)_{\mathfrak{C}}$ be defined as above. Then

$$\chi_{\underline{b}}^{\underline{a}} = \mathrm{Ind}^{\mathcal{A}} \mathrm{Inf}_{\mathcal{K}} \lambda_{\underline{b}}^{\underline{a}}.$$

In particular, $\chi_{\underline{b}}^{\underline{a}}$ is induced from a linear character of $X_{\mathcal{S}\cup\mathcal{K}}$.

(b) Suppose that $\mathfrak{C} \in \mathfrak{O}_2$ is nonabelian, and let $\chi_{\underline{c}} \in \mathrm{Irr}(U)_{\mathfrak{C}}$ be defined as above. Then

$$\chi_{\underline{c}} = \mathrm{Ind}^{\mathcal{A}} \mathrm{Inf}_{\mathcal{K}} \psi_{\underline{c}}.$$

From the comments given within Algorithm 2.6 and the discussion above, we deduce the following theorem regarding the validity of our algorithm.

Theorem 2.12. *Suppose that Algorithm 2.6 has run with input (Φ^+, Σ) and given output $\mathfrak{O} = (\mathfrak{O}_1, \mathfrak{O}_2)$. Then we have*

$$\mathrm{Irr}(U)_\Sigma = \bigsqcup_{\mathfrak{C}\in\mathfrak{O}_1} \mathrm{Irr}(U)_{\mathfrak{C}} \sqcup \bigsqcup_{\mathfrak{C}\in\mathfrak{O}_2} \mathrm{Irr}(U)_{\mathfrak{C}}.$$

Remark 2.13. We make a slight abuse in the notation $\chi_{\underline{b}}^{\underline{a}}$. In fact, each a_i and b_j is supposed to record not just a value in $\mathbb{F}_q^\times$ and $\mathbb{F}_q$ respectively, but also i and j, so that $\chi_{\underline{b}}^{\underline{a}}$ should strictly read $\chi^{((i_1,a_{i_1}),\dots,(i_m,a_{i_m}))}_{((j_1,b_{j_1}),\dots,(j_n,b_{j_n}))}$, for corresponding choices of $i_1,\dots,i_m$ and $j_1,\dots,j_n$ indexing positive roots.

Remark 2.14. The choice of total order on $\Phi^+ = \{\alpha_1,\dots,\alpha_N\}$ has a significant effect on how the algorithm runs, as this is used to determine which reductions to make when there may be a choice. The resulting parametrization of $\mathrm{Irr}(U)_\Sigma$ consequently depends on this choice of enumeration.

2.5 Results of the algorithm and parametrization of $\mathrm{Irr}(U_{\mathrm{C}_4})$

We have implemented Algorithm 2.6 in the algebra system GAP3, using the CHEVIE package. The algorithm requires us to just work with Φ^+ and the GAP commands for root systems allow us to do this. We use the enumeration of Φ^+ as given in CHEVIE.

We have run the GAP program for G of rank less than or equal to 7. We present its output in Table 2.2, including the number of nonabelian cores. For G of rank 3 or less, or G of type C_4, there are no nonabelian cores; the algorithm provides straight away a parametrization of $\mathrm{Irr}(U)$ in these cases. The parametrization of $\mathrm{Irr}(U_{\mathrm{C}_4})$ is provided in Table D.2.

For every split finite group of Lie type G, we denote by $k(U,D)$ the number of irreducible characters of $U = U(G)$ of degree D. The expressions for $k(U,D)$ are given as polynomials in $v := q-1$ in [GMR15] for every character degree D and every split finite group of Lie type of rank 8 or less, except E_8. These are given for $p \geq h$ by the number of coadjoint orbits, computed in [GMR15] via the Kirillov orbit method.

Our methods allow in principle to compute $k(U,D)$ for every p which is not a very bad prime for G. The expressions we obtain for $k(U_{\mathrm{C}_4},D)$ for every D agree with, hence extend, the ones previously obtained for $p \geq h$ where h is the Coxeter number of C_4, namely $h = 8$. We collect in Table 2.1 the numbers $k(U_{\mathrm{C}_4},D)$ for every q and $p \neq 2$. We notice that D is always a power of q.

D	$k(U_{C_4}, D)$
1	$v^4 + 4v^3 + 6v^2 + 4v + 1$
q	$v^5 + 6v^4 + 13v^3 + 12v^2 + 4v$
q^2	$2v^5 + 11v^4 + 20v^3 + 14v^2 + 3v$
q^3	$v^6 + 6v^5 + 17v^4 + 24v^3 + 14v^2 + 3v$
q^4	$2v^5 + 9v^4 + 15v^3 + 9v^2 + v$
q^5	$3v^4 + 8v^3 + 6v^2 + v$
q^6	$v^4 + 4v^3 + 3v^2$

Table 2.1: Numbers of irreducible characters of U_{C_4} of fixed degree, for $v = q - 1$ and $p \neq 2$.

For other split finite groups of Lie type of rank 4 or 5, the algorithm does not provide a full parametrization of $\mathrm{Irr}(U)$. Nevertheless, we obtain labels for characters arising from abelian cores, collected in plain font in Appendix D. In this sense, we say that we get the parametrization of "most" of $\mathrm{Irr}(U)$, namely up to irreducible characters arising from nonabelian cores. For the types B_4 and D_4 there is one nonabelian core each, and in type F_4 we find six nonabelian cores. These will be dealt with in Chapter 4, via the analysis presented in Chapter 3. A parametrization of $\mathrm{Irr}(U)$ in these cases is collected in Tables D.1, D.3 and D.4 respectively.

For G of type B_5, C_5 and D_5 there are 10, 1 and 7 nonabelian cores respectively. The analysis in Chapter 3, along with a method to deal with several nonabelian cores at the same time described in Section 5.1, will lead us to obtain a full parametrization of $\mathrm{Irr}(U)$ in these cases. This is given in Chapter 5; we refer to Tables D.5, D.6 and D.7 for character labels. We currently do not have a method to study all of the nonabelian cores of any group of rank 6 or higher; this is a topic for further research.

We note that the parametrization of irreducible characters for U_{B_3} can be read off from that for U_{B_4}, as U_{B_3} is a quotient of U_{B_4}. Similarly, the parametrization of irreducible

Type	Antichains	Abelian cores	Nonabelian cores	Running time
B_4	70	80	1 (1.23%)	$T \ll 1$ sec
C_4	70	90	0 (0%)	$T \ll 1$ sec
D_4	50	52	1 (1.88%)	$T \ll 1$ sec
F_4	105	177	6 (3.28%)	$T \sim 1$ sec
B_5	252	358	10 (2.72%)	$T \sim 3$ sec
C_5	252	417	1 (0.24%)	$T \sim 3$ sec
D_5	182	214	7 (3.17%)	$T \sim 1$ sec
B_6	924	1842	95 (4.90%)	$T \sim 30$ sec
C_6	924	2254	22 (0.97%)	$T \sim 30$ sec
D_6	672	991	55 (5.26%)	$T \sim 10$ sec
E_6	833	1656	156 (8.61%)	$T \sim 30$ sec
B_7	3432	11240	969 (7.94%)	$T \sim 7$ min
C_7	3432	14216	294 (2.03%)	$T \sim 7$ min
D_7	2508	5479	531 (8.84%)	$T \sim 2.5$ min
E_7	4160	33594	7798 (18.84%)	$T \sim 45$ min

Table 2.2: Results of Algorithm 2.6 in types $\mathrm{B}_r, \mathrm{C}_r$ and D_r, $r = 4, 5, 6, 7$ and F_4, E_i, $i = 6, 7$.

characters of U_{C_3} can be read off from that for U_{C_4}. We remark, on the one hand, that our parametrization of $\mathrm{Irr}(U_{\mathrm{D}_4})$ agrees with the one determined in [HLM11]. On the other hand, similar parametrizations of $\mathrm{Irr}(U_{\mathrm{B}_4})$, $\mathrm{Irr}(U_{\mathrm{C}_4})$ and $\mathrm{Irr}(U_{\mathrm{F}_4})$ do not seem to be given explicitly in previous literature, as well as parametrizations of $\mathrm{Irr}(U_{\mathrm{B}_5})$, $\mathrm{Irr}(U_{\mathrm{C}_5})$ and $\mathrm{Irr}(U_{\mathrm{D}_5})$.

CHAPTER 3

A METHOD FOR ANALYSING NONABELIAN CORES

In this chapter we explain the methods employed to analyse the nonabelian cores arising from our analysis in Chapter 2. Our approach is based on a direct examination of these cases. It is helpful for us to first deal with certain 3-dimensional groups over $\mathbb{F}_q$ that frequently appear in our analysis; this is developed in Section 3.1. We then outline in Section 3.2 our general methods for dealing with nonabelian cores.

We remark that we do not assert that these methods are guaranteed to work for every nonabelian core. Nevertheless, we will see in Chapters 4 and 5 that this is enough to complete the parametrization of $\mathrm{Irr}(U)$ for every split finite group of Lie type of rank at most 5. Moreover, the methods outlined in this chapter lay the groundwork towards a parametrization of $\mathrm{Irr}(U)$ in higher ranks.

3.1 Some 3-dimensional groups

Let $f : \mathbb{F}_q \times \mathbb{F}_q \to \mathbb{F}_q$ be an $\mathbb{F}_p$-bilinear map, which we assume to be surjective. We define the group $V = V_f$ to be generated by subgroups

$$X_1 = \{x_1(t) \mid t \in \mathbb{F}_q\} \cong \mathbb{F}_q, \quad X_2 = \{x_2(t) \mid t \in \mathbb{F}_q\} \cong \mathbb{F}_q, \quad Z = \{z(t) \mid t \in \mathbb{F}_q\} \cong \mathbb{F}_q,$$

subject to $Z \subseteq Z(V)$, and

$$[x_1(s), x_2(t)] = z(f(s,t)).$$

In particular, throughout this section, X_1 will *not* denote the root subgroup X_{α_1}, similarly for X_2.

Since Z is central in V, we have that X_2Z is a subgroup of V. By definition of V, we have that $[X_1, X_2Z] \subseteq X_2Z$, therefore X_1X_2Z is also a subgroup, which implies $V = X_1X_2Z$. The assumption of f being surjective implies that $[V,V] = Z$.

The focus of this section is on constructing the irreducible characters of V. Let us fix $a \in \mathbb{F}_q^\times$. We define the linear character $\lambda^a \in \mathrm{Irr}(Z)$ by $\lambda^a(z(t)) = \phi(at)$. We define

$$X_1' = \{x_1(s) \in X_1 \mid \mathrm{Tr}(af(s,t)) = 0 \text{ for all } t \in \mathbb{F}_q\}$$

and

$$X_2' = \{x_2(t) \in X_2 \mid \mathrm{Tr}(af(s,t)) = 0 \text{ for all } s \in \mathbb{F}_q\}.$$

Note that in general X_1' and X_2' may depend on a.

We first note that the linear characters of V are given by inflating over Z the characters of $V/Z \cong X_1 \times X_2$. For $b_1, b_2 \in \mathbb{F}_q$, we define $\chi_{b_1,b_2} \in \mathrm{Irr}(V)$ by

$$\chi_{b_1,b_2}(x_1(s_1)x_2(s_2)z(t)) = \phi(b_1s_1 + b_2s_2)$$

for every $s_1, s_2, t \in \mathbb{F}_q$, thus there are q^2 linear characters of V of the form $\{\chi_{b_1,b_2} \mid b_1, b_2 \in \mathbb{F}_q\}$.

We now analyse the characters in $\mathrm{Irr}(V \mid \lambda^a)$ for $a \in \mathbb{F}_q^\times$ using the reduction lemma, that is, Lemma 1.12. The map

$$\begin{aligned} B : \mathbb{F}_q \times \mathbb{F}_q &\to \mathbb{F}_p \\ (s,t) &\mapsto \mathrm{Tr}(f(s,t)) \end{aligned}$$

is $\mathbb{F}_p$-bilinear. For $i = 1, 2$, let V_i be the image of X_i' under the natural isomorphism $X_i \cong \mathbb{F}_q$. Then V_1, V_2 are $\mathbb{F}_p$-subspaces of $\mathbb{F}_q$, and the bilinear form B induces a non-degenerate bilinear form

$$\bar{B} : (\mathbb{F}_q/V_1) \times (\mathbb{F}_q/V_2) \to \mathbb{F}_p,$$

hence

$$\dim_{\mathbb{F}_p}(\mathbb{F}_q/V_1) = \dim_{\mathbb{F}_p}(\mathbb{F}_q/V_2),$$

which implies $|X_1'| = |X_2'|$. Thus for $i = 1, 2$ we can choose a complement $\tilde{X}_i$ of X_i' in X_i, such that $|\tilde{X}_1| = |\tilde{X}_2|$.

Now by Lemma 1.12 (with $X = \tilde{X}_1$ and $Y = \tilde{Y}_1$), we see that the map

$$\begin{aligned} \mathrm{Irr}(X_1'X_2Z/(\tilde{X}_2 \ker \lambda^a) \mid \lambda^a) &\to \mathrm{Irr}(V \mid \lambda^a) \\ \psi &\mapsto \mathrm{Ind}_{X_1'X_2Z}^{V} \mathrm{Inf}_{X_1'X_2Z/(\tilde{X}_2 \ker \lambda^a)}^{X_1'X_2Z} \psi \end{aligned}$$

is a bijection.

Finally, we observe that $X_1'X_2Z/(\tilde{X}_2 \ker \lambda^a)$ is abelian, so $\mathrm{Irr}(X_1'X_2Z/\tilde{X}_2 \mid \lambda^a)$ is in bijection with $\mathrm{Irr}(X_1' \times X_2')$. We then get

$$\mathrm{Irr}(V \mid \lambda^a) = \{\mathrm{Ind}_{X_1'X_2Z}^{V} \mathrm{Inf}_{X_1'X_2Z/(\tilde{X}_2 \ker \lambda^a)}^{X_1'X_2Z} \psi \mid \psi \in \mathrm{Irr}(X_1' \times X_2')\},$$

with each $\psi \in \mathrm{Irr}(X_1' \times X_2')$ linear. Since

$$\mathrm{Irr}(V) = \{\chi_{b_1,b_2} \mid b_1, b_2 \in \mathbb{F}_q\} \sqcup \bigsqcup_{a \in \mathbb{F}_q^\times} \mathrm{Irr}(V \mid \lambda^a),$$

this completes our general description of $\mathrm{Irr}(V)$.

We are going to examine in more detail $\mathrm{Irr}(V_f)$ for certain choices of f. We first recall the following general result. Let V_f be as defined above, and let φ be an automorphism of $\mathbb{F}_q$ regarded as an abelian group. We write $V_f = \{x_1(s)x_2(t)z(u) \mid s, t, u \in \mathbb{F}_q\}$, and since $\varphi \circ f$ is surjective it makes sense to define $V_{\varphi\circ f} = \{x_1'(s)x_2'(t)z'(u) \mid s, t, u \in \mathbb{F}_q\}$ similarly. Suppose that there exists a group homomorphism $\Psi : V_f \to V_{\varphi\circ f}$ such that $\Psi(x_1(s)) = x_1'(s)$ and $\Psi(x_2(t)) = x_2'(t)$ for all $s, t, \in \mathbb{F}_q$. For every $u \in \mathbb{F}_q$, we can write $u = f(s,t)$ for some $s, t \in \mathbb{F}_q$. We then get

$$\begin{aligned}\Psi(z(u)) = \Psi(z(f(s,t))) = \Psi([x_1(s), x_2(t)]) &= [\Psi(x_1(s)), \Psi(x_2(t))]) \\ &= [x_1'(s), x_2'(t)] = z'(\varphi \circ f(s,t)) = z'(\varphi(f(s,t))) = z'(\varphi(u)).\end{aligned}$$

Then for such a group homomorphism Ψ we have $\Psi(z(u)) = z'(\varphi(u))$ for every $u \in \mathbb{F}_q$. But now it is immediate to check that a map Ψ defined as

$$\Psi(x_1(s)) = x_1'(s), \qquad \Psi(x_2(t)) = x_2'(t), \qquad \Psi(z(u)) = z'(\varphi(u))$$

for every $s, t, u \in \mathbb{F}_q$ defines a group homomorphism, and in fact this is a group isomorphism since φ is an automorphism of $\mathbb{F}_q$ as an abelian group. Then $V_f \cong V_{\varphi\circ f}$. In particular, we get that $V_f \cong V_{cf}$ for every $c \in \mathbb{F}_q^\times$.

To finish, we determine V_f explicitly for some choices of f. For $f(s,t) = st$, we have that V_f is isomorphic to U_{A_2}. Clearly we get $X_1' = Y_1' = 1$, and then

$$\operatorname{Irr}(V) = \{\chi_{b_1,b_2} \mid b_1, b_2 \in \mathbb{F}_q\} \cup \{\chi^a \mid a \in \mathbb{F}_q^\times\},$$

where $\chi^a = \operatorname{Ind}_{X_2Z}^V \operatorname{Inf}_Z^{X_2Z} \lambda^a$. For $f(s,t) = s^p t$ or $f(s,t) = (s^p - ds)t$ where $d \in \mathbb{F}_q^\times$ is not a $(p-1)$th power, applying the argument of Lemma 1.13, we have that

$$\operatorname{Im} af \cap \ker \operatorname{Tr} = \varnothing, \text{ that is, for every } s,t \in \mathbb{F}_q,\ \operatorname{Tr}(af(s,t)) = 0 \text{ implies } af(s,t) = 0.$$

Again, we see that $X_1' = Y_1' = 1$, and $\operatorname{Irr}(V)$ is given as above. In these three cases, we get

- $q-1$ characters of degree q, and
- q^2 linear characters.

The case of major interest to us here is $f(s,t) = (s^p - ds)t$ where $d \in \mathbb{F}_q^\times$ is a $(p-1)$-th power, say $d = e^{p-1}$. Then by using Lemma 1.13, we find that

$$X_1' = \{x_1(s) \mid s^p - e^{p-1}s = 0\} = \{x_1(s) \mid s \in e\mathbb{F}_p\}$$

and

$$X_2' = \{x_2(t) \mid \operatorname{Tr}(at\mathbb{T}_e) = 0\} = \{x_2(t) \mid t \in (e^{-p}/a)\mathbb{F}_p\}.$$

Now for $c_1, c_2 \in \mathbb{F}_p$ we define the characters $\lambda^a_{c_1,c_2} \in \operatorname{Irr}(X_1'X_2Z/(\tilde{X}_2 \ker \lambda^a))$ by

$$\lambda^a_{c_1,c_2}(x_1(es_1)x_2((e^{-p}/a)s_2)z(t)) = \phi(c_1s_1 + c_2s_2 + at)$$

for every $s_1, s_2 \in \mathbb{F}_p$ and $t \in \mathbb{F}_q$. Then we have

$$\operatorname{Irr}(V) = \{\chi_{b_1,b_2} \mid b_1, b_2 \in \mathbb{F}_q\} \cup \{\chi^a_{c_1,c_2} \mid a \in \mathbb{F}_q^\times, c_1, c_2 \in \mathbb{F}_p\},$$

where $\chi^a_{c_1,c_2} = \operatorname{Ind}_{X_1'X_2Z}^{V} \operatorname{Inf}_{X_1'X_2Z/(\tilde{X}_2 \ker \lambda^a)}^{X_1'X_2Z} \lambda^a_{c_1,c_2}$. In this case, we get

- $p^2(q-1)$ characters of degree q/p, and
- q^2 linear characters.

3.2 Adapting the reduction lemma for nonabelian cores

Let $\mathfrak{C} = (\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ be a nonabelian core. The set $\mathcal{S}$ is a quattern corresponding to the pattern $\Phi^+ \setminus \mathcal{A}$ and its normal subset $\mathcal{K}$. Further, we have $\mathcal{Z} = \mathcal{Z}(\mathcal{S}) \setminus \mathcal{D}(\mathcal{S})$ as $\mathfrak{C}$ is a nonabelian core, and we let $\mathcal{Z} = \{\alpha_{i_1}, \dots, \alpha_{i_m}\}$. For each $\underline{a} = (a_{i_1}, \dots, a_{i_m}) \in (\mathbb{F}_q^\times)^m$, we define the map $\mu = \mu^{\underline{a}} : X_{\mathcal{Z}} \to \mathbb{F}_q$ by $\mu(x_{i_h}(t)) = a_{i_h} t$ for $h = 1, \dots, m$. Then

$$\lambda = \lambda^{\underline{a}} = \phi \circ \mu^{\underline{a}} : X_{\mathcal{Z}} \longrightarrow \mathbb{C}^\times$$

is a linear character of $X_{\mathcal{Z}}$.

We give a method to analyse the characters in $\operatorname{Irr}(X_{\mathcal{S}} \mid \lambda)$. We note that the nature of the resulting parametrization and construction of the characters may depend on the choice of $\underline{a}$, and we see instances of this dependence in most examples in Section 4.3. Further we remark that we do not assert that this method is guaranteed to work for every nonabelian core, though it does apply for all the cores that we consider in Chapters 4 and 5.

The setting throughout this section is as follows. We define $V = X_{\mathcal{S}}/\ker \mu$, and we let $Z = X_{\mathcal{Z}}/\ker \mu$. Since $\ker \mu \subseteq \ker \lambda$, we have that λ factors through Z and we also write λ for this character of Z. Then we have a bijection between $\operatorname{Irr}(V \mid \lambda)$ and $\operatorname{Irr}(X_{\mathcal{S}} \mid \lambda)$ by inflating over $\ker \mu$, and we work in $\mathrm{Irr}(V \mid \lambda)$ rather than in $\mathrm{Irr}(X_{\mathcal{S}} \mid \lambda)$. Given $\alpha \in \mathcal{S} \setminus \mathcal{Z}$ we identify X_α with its image in V.

We aim to find subsets $\mathcal{I}$ and $\mathcal{J}$ of $\mathcal{S} \setminus \mathcal{Z}$ such that the following hold.

- $|\mathcal{I}| = |\mathcal{J}|$,
- $H = X_{\mathcal{S}\setminus(\mathcal{I}\cup\mathcal{Z})}Z$ is a subgroup of V,
- $Y = X_{\mathcal{J}} \leq Z(H)$, and
- YZ is a normal subgroup of V.

We note that this implies that

- $X = X_{\mathcal{I}}$ is a transversal of H in V.

Under these assumptions, we notice that the following property holds, namely

$$\text{for every } \alpha \in \mathcal{J} \text{ and } \beta \in \mathcal{S}, \text{ we have that } 2\alpha + \beta \notin \mathcal{S}. \tag{3.2.1}$$

In fact, if $\alpha \in \mathcal{J}$ and $\beta \in \mathcal{S}$, we have $\alpha + \beta \in \mathcal{S} \setminus \mathcal{I}$ since $YZ \trianglelefteq V$. If $\alpha + \beta \in \mathcal{Z}$, then it is clear that $2\alpha+\beta \notin \mathcal{S}$. Otherwise, $\alpha+\beta \in \mathcal{S}\setminus(\mathcal{I}\cup\mathcal{Z})$, and since $Y \leq Z(H)$ then $2\alpha+\beta \notin \mathcal{S}$ in this case as well.

We would like to apply the reduction lemma in this case. Conditions (i)–(iv) do hold, but condition (v) may not be satisfied, so we aim to adapt the situation slightly.

We consider the inflation $\hat{\mu}$ of μ to YZ and let $\hat{\lambda} = \phi \circ \hat{\mu}$ be the inflation of λ to YZ. For $v \in V$, we consider the map $\psi_v : Y \to \mathbb{F}_q$ given by $\psi_v(y) = \hat{\mu}([v, y])$. For $v \in V$ and $y_1, y_2 \in Y$, we have that

$$\hat{\mu}([v, y_1y_2]) = \hat{\mu}([v, y_2][v, y_1]^{y_2}) = \hat{\mu}([v, y_2][v, y_1]) = \hat{\mu}([v, y_1][v, y_2]) = \hat{\mu}([v, y_1]) + \hat{\mu}([v, y_2]),$$

where we used the facts that YZ is abelian and that $[V, Y] \subseteq YZ$, since $YZ \trianglelefteq V$. Moreover, if $\mathcal{S} = \{\beta_1, \ldots, \beta_m\}$ and $\mathcal{J} = \{\alpha_1, \ldots, \alpha_k\}$, then by what is observed in (3.2.1) we have that

$$[x_{\beta_1}(s_1) \cdots x_{\beta_m}(s_m), x_{\alpha_1}(t_1) \cdots x_{\alpha_k}(t_k)] = \prod_{\substack{i_1,\ldots,i_m>0,\, j=1,\ldots,k: \\ i_1\beta_1+\cdots+i_m\beta_m+\alpha_j \in \Phi^+}} x_{i_1\beta_1+\cdots+i_m\beta_m+\alpha_j}(d^{\beta_1,\ldots,\beta_m,\alpha_j}_{i_1,\ldots,i_m,j} s_1^{i_1} \cdots s_m^{i_m} t_j)$$

for some $d^{\beta_1,\ldots,\beta_m,\alpha_j}_{i_1,\ldots,i_m,j} \in \mathbb{Z}$. We see by the right hand side of the above equality that this expression is linear in $t_1, \ldots, t_k$ once we apply $\hat{\mu}$. Therefore ψ_v is $\mathbb{F}_q$-linear, that is, ψ_v belongs to the dual space $Y^* = \mathrm{Hom}(Y, \mathbb{F}_q)$ of Y. We let

$$Y' = \bigcap_{v \in V} \ker(\psi_v) = \{y \in Y \mid {}^v\hat{\mu}(y) = \hat{\mu}(y) \text{ for all } v \in V\}.$$

Then Y' is an $\mathbb{F}_q$-subspace of $Y \cong \mathbb{F}_q^{|\mathcal{J}|}$. Also, we define

$$\tilde{H} = \mathrm{Stab}_V(\hat{\mu}) = \{v \in V \mid {}^v\hat{\mu} = \hat{\mu}\}.$$

Then $\tilde{H}$ is a subgroup of V and $\tilde{H} = X'H$ for $X' = \{x \in X \mid {}^x\hat{\mu} = \hat{\mu}\}$.

To prove that X' and Y' have the same cardinality we assume, for the rest of this chapter, that

$$W := \{\psi_v \mid v \in V\} \text{ is an } \mathbb{F}_q\text{-subspace of } Y^*.$$

This condition is easily checked to hold for all nonabelian cores that we examine when G is of rank 5 or less, by looking at the form of Equation (3.2.3) defined below in each of these cases.

Lemma 3.1. $|X'| = |Y'|$.

Proof. We have that the annihilator $\mathrm{Ann}_Y(W)$ of W is Y' by definition. Hence we have $\dim Y = \dim Y' + \dim W$, that is, $|Y|/|Y'| = |W|$.

Now let us call $\hat{\mu}^V$ the V-orbit of $\hat{\mu}$ in $\mathrm{Hom}(YZ, \mathbb{F}_q)$. For $v, v' \in V$ we have that

$$\psi_v = \psi_{v'} \Longleftrightarrow \hat{\mu}([v,y]) = \hat{\mu}([v',y]) \text{ for all } y \in Y \Longleftrightarrow \hat{\mu}(y^v) = \hat{\mu}(y^{v'}) \text{ for all } y \in Y,$$

then the map

$$\hat{\mu}^V \longrightarrow W$$
$$\hat{\mu}^v \mapsto \psi_v$$

is well-defined and injective. It is clear that it is also surjective. Therefore, we have $|W| = |\hat{\mu}^V|$. Now by the orbit-stabilizer theorem [Is2, Theorem 1.4], we have that

$$|\hat{\mu}^V| = |V|/\mathrm{Stab}_V(\hat{\mu}) = |V|/|\tilde{H}| = |X|/|X'|.$$

Combining the above equalities, we get

$$|Y|/|Y'| = |W| = |\hat{\mu}^V| = |X|/|X'|.$$

Since $|Y| = |X|$, the claim follows. □

Moreover, we have the following property about X'.

Lemma 3.2. *Let $x \in X$ be such that ${}^x\hat{\lambda} = \hat{\lambda}$. Then $x \in X'$.*

Proof. We show that for such x we have ${}^x\hat{\mu} = \hat{\mu}$. The hypothesis is equivalent to

$$\phi \circ {}^x\hat{\mu} = \phi \circ \hat{\mu}, \text{ that is, } \phi \circ ({}^x\hat{\mu} - \hat{\mu}) = 1. \tag{3.2.2}$$

For $y \in Y$ and $z \in Z$, we have

$$^{x}\hat{\mu}(yz) = \hat{\mu}(y^{x}z^{x}) = {}^{x}\hat{\mu}(y) + \hat{\mu}(z) = \hat{\mu}([y,x]) + \mu(z) = -\psi_{x}(y) + \mu(z),$$

then by what observed in (3.2.1), the same argument used to prove that ψ_v is $\mathbb{F}_q$-linear applies to prove that ${}^{x}\hat{\mu}$ is $\mathbb{F}_q$-linear. Hence ${}^{x}\hat{\mu} - \hat{\mu}$ is also $\mathbb{F}_q$-linear. Therefore the image of ${}^{x}\hat{\mu} - \hat{\mu}$ is either 0 or $\mathbb{F}_q$. But if it were $\mathbb{F}_q$, then Equation (3.2.2) would imply $\phi(c) = 1$ for every $c \in \mathbb{F}_q$, which is a contradiction. Then we have ${}^{x}\hat{\mu} = \hat{\mu}$, that is, $x \in X'$. □

We write $\mathcal{I} = \{\alpha_{i_1}, \cdots, \alpha_{i_m}\}$ and $\mathcal{J} = \{\alpha_{j_1}, \cdots, \alpha_{j_m}\}$, such that $i_1 \leq \cdots \leq i_m$ and $j_1 \leq \cdots \leq j_m$. In general, Y' and X' can be determined by the following equation,

$$\hat{\mu}([x_{\alpha_{j_1}}(s_{j_1}) \cdots x_{\alpha_{j_m}}(s_{j_m}), x_{\alpha_{i_1}}(t_{i_1}) \cdots x_{\alpha_{i_m}}(t_{i_m})]) = 0. \tag{3.2.3}$$

We note that as the map ψ_x for $x \in X$ is $\mathbb{F}_q$-linear, the left hand side of Equation (3.2.3) is linear in $s_{j_1}, \ldots, s_{j_m}$. Therefore, the values of $s_{j_1}, \ldots, s_{j_m}$ such that Equation (3.2.3) holds for every $t_{i_1}, \ldots, t_{i_m}$ form an $\mathbb{F}_q$-subspace of Y, which determines Y'.

Under an additional assumption on Y, we are able to apply the reduction lemma in the following proposition. We define $\bar{H}$ to be the preimage of $\tilde{H}$ in $X_{\mathcal{S}}$.

Proposition 3.3. *Suppose that there exists a subgroup $\tilde{Y}$ of Y such that $Y = Y' \times \tilde{Y}$ and $[X, \tilde{Y}] \subseteq \tilde{Y}Z$. Then we have a bijection*

$$\begin{aligned} \operatorname{Irr}(\tilde{H}/\tilde{Y} \mid \lambda) &\to \operatorname{Irr}(V \mid \lambda) \\ \chi &\mapsto \operatorname{Ind}_{\tilde{H}}^{V} \operatorname{Inf}_{\tilde{H}/\tilde{Y}}^{\tilde{H}} \chi \end{aligned}$$

Consequently we have a bijection

$$\begin{aligned} \operatorname{Irr}(\tilde{H}/\tilde{Y} \mid \lambda) &\to \operatorname{Irr}(X_{\mathcal{S}} \mid \lambda) \\ \chi &\mapsto \operatorname{Ind}_{\bar{H}}^{X_{\mathcal{S}}} \operatorname{Inf}_{\tilde{H}/\tilde{Y}}^{\bar{H}} \chi \end{aligned}$$

Proof. We want to check that $\tilde{H}, \tilde{X}, \tilde{Y}$ and Z satisfy all the assumptions of the reduction lemma as subgroups of V with respect to $\lambda \in \mathrm{Irr}(Z)$. Clearly we have that $Z \leq Z(V)$ and $\tilde{Y} \cap Z = 1$. By assumption, we have that X normalizes $\tilde{Y}Z$, and we have that H centralizes $\tilde{Y}Z$, so $\tilde{Y}Z \trianglelefteq V$. Since $\tilde{Y} \leq Y \leq Z(H)$, we have that $\tilde{Y}$ is normalized by H. Moreover, if $x' \in X'$ and $y \in Y$, by definition of X' we have that

$$\hat{\mu}(y^{-1}y^{x'}) = \hat{\mu}(y^{-1}) + \hat{\mu}(y^{x'}) = 0,$$

and since $\ker \hat{\mu} = Y \ker \mu$ we have that X' normalizes Y. Along with the assumption that $[X, \tilde{Y}] \subseteq \tilde{Y}Z$, we deduce that X' normalizes $\tilde{Y}$. Hence $\tilde{Y} \trianglelefteq \tilde{H}$.

Now we are left to check condition (v) of the reduction lemma. We write $\tilde{\lambda} \in \mathrm{Irr}(\tilde{Y}Z)$ for the inflation of λ to $\tilde{Y}Z$, and note that $\tilde{\lambda} = \hat{\lambda}|_{\tilde{Y}Z}$. Let $\tilde{X}$ be a transversal of $\tilde{H}$ in V. Assume that ${}^{\tilde{x}_1}\tilde{\lambda} = {}^{\tilde{x}_2}\tilde{\lambda}$ for $\tilde{x}_1, \tilde{x}_2 \in \tilde{X}$. Let $y \in Y$ and $z \in Z$ and write $y = y'\tilde{y}$, where $y' \in Y'$ and $\tilde{y} \in \tilde{Y}$. We have

$$\begin{aligned} {}^{\tilde{x}_1}\hat{\lambda}(y'\tilde{y}z) &= \hat{\lambda}(y'^{\tilde{x}_1})\tilde{\lambda}(\tilde{y}^{\tilde{x}_1})\lambda(z) \\ &= \hat{\lambda}(y')({}^{\tilde{x}_1}\tilde{\lambda})(\tilde{y})\lambda(z) \\ &= \hat{\lambda}(y'^{\tilde{x}_2})({}^{\tilde{x}_2}\tilde{\lambda})(\tilde{y})\lambda(z) \\ &= \hat{\lambda}(y'^{\tilde{x}_2})\hat{\lambda}(\tilde{y}^{\tilde{x}_2})\lambda(z) \\ &= {}^{\tilde{x}_2}\hat{\lambda}(y'\tilde{y}z). \end{aligned}$$

In the above sequence of equalities we use that $\hat{\lambda}(y'^{\tilde{x}_1}) = \hat{\lambda}(y') = \hat{\lambda}(y'^{\tilde{x}_2})$ by definition of Y', that $\tilde{y}^{\tilde{x}_1}, \tilde{y}^{\tilde{x}_2} \in \tilde{Y}Z$ since $[X, \tilde{Y}] \subseteq \tilde{Y}Z$, and that ${}^{\tilde{x}_1}\tilde{\lambda} = {}^{\tilde{x}_2}\tilde{\lambda}$ by assumption. Hence we have ${}^{\tilde{x}_1\tilde{x}_2^{-1}}\hat{\lambda} = \hat{\lambda}$. By Lemma 3.2, this implies $\tilde{x}_1\tilde{x}_2^{-1} \in X'$ and thus $\tilde{x}_1 = \tilde{x}_2$ as $\tilde{X}$ is a transversal of $\tilde{H}$ in V. By Lemma 3.1, we have that $|X'| = |Y'|$. We can then apply the reduction lemma to deduce the first bijection of the claim.

Now we notice that the assumptions of Lemma 1.8 apply with $N = \ker\mu$, $H = \bar{H}$ and $G = X_{\mathcal{S}}$. Then we have

$$\operatorname{Inf}_{V}^{X_{\mathcal{S}}} \operatorname{Ind}_{\tilde{H}}^{V} = \operatorname{Ind}_{\bar{H}}^{X_{\mathcal{S}}} \operatorname{Inf}_{\tilde{H}}^{\bar{H}},$$

which yields the second bijection in the claim. □

Remark 3.4. Let us suppose that $[X, Y] \subseteq Z$. Then we may take an arbitrary complement $\tilde{Y}$ of Y' in Y, and the assumption $[X, \tilde{Y}] \subseteq \tilde{Y}Z$ in Proposition 3.3 is obviously satisfied.

If we assume that Y' is central in $\tilde{H}/\tilde{Y}$, then we can extend $\lambda \in \operatorname{Irr}(Z)$ to Y'. This is very useful to apply again the reduction lemma in $\tilde{H}/\tilde{Y}$ with respect to such an extension of λ, as we will see in type D_4 in Section 4.2 and in type F_4 in Section 4.3.

Remark 3.5. Suppose that Proposition 3.3 applies and let $\psi \in \operatorname{Irr}(\tilde{H}/\tilde{Y} \mid \lambda)$. Then we have that $\operatorname{Ind}_{\bar{H}}^{X_{\mathcal{S}}} \operatorname{Inf}_{\tilde{H}/\tilde{Y}}^{\bar{H}} \psi \in \operatorname{Irr}(X_{\mathcal{S}})$, and

$$\overline{\psi} = \operatorname{Ind}_{X_{\mathcal{S}\cup\mathcal{K}}}^{U} \operatorname{Inf}_{X_{\mathcal{S}}}^{X_{\mathcal{S}\cup\mathcal{K}}} \operatorname{Ind}_{\bar{H}}^{X_{\mathcal{S}}} \operatorname{Inf}_{\tilde{H}/\tilde{Y}}^{\bar{H}} \psi \in \operatorname{Irr}(U)_{\mathfrak{C}}$$

by Theorem 2.11. Since $X_{\mathcal{K}} \trianglelefteq U$, we have that $\bar{H}X_{\mathcal{K}}$ is a subgroup of $X_{\mathcal{S}\cup\mathcal{K}}$, and of course $X_{\mathcal{K}} \trianglelefteq \bar{H}X_{\mathcal{K}}$. Then $X_{\mathcal{K}}$, $\bar{H}X_{\mathcal{K}}$ and $X_{\mathcal{S}\cup\mathcal{K}}$ play the roles of N, H and G respectively as defined in Lemma 1.8. We can change the order of inflation and induction accordingly, and we get

$$\overline{\psi} = \operatorname{Ind}_{\bar{H}X_{\mathcal{K}}}^{U} \operatorname{Inf}_{\tilde{H}/\tilde{Y}}^{\bar{H}X_{\mathcal{K}}} \psi.$$

In Chapters 4 and 5, we apply this argument (sometimes iteratively) to show that each irreducible character considered there can be obtained as an induced character of a linear character.

A particular case in which Proposition 3.3 applies repeatedly occurs in the sequel. Suppose that $Y' = 1$ for all choices of λ, and Y is normal in $\bar{H}$. We have $\bar{H}/Y = X_{\mathcal{S}\setminus(\mathcal{I}\cup\mathcal{J})}$.

Defining

$$\overline{\psi} = \operatorname{Ind}^{\mathcal{A}\cup\mathcal{I}} \operatorname{Inf}_{\mathcal{K}\cup\mathcal{J}} \psi$$

for $\psi \in \operatorname{Irr}(X_{\mathcal{S}\setminus(\mathcal{I}\cup\mathcal{J})})_{\mathcal{Z}}$ sets up a bijection from $\operatorname{Irr}(X_{\mathcal{S}\setminus(\mathcal{I}\cup\mathcal{J})})_{\mathcal{Z}}$ to $\operatorname{Irr}(U)_{\mathfrak{C}}$.

CHAPTER 4

PARAMETRIZATION OF Irr(U) IN RANK 4

In this chapter we apply the technique presented in Chapter 3 to split finite groups of Lie type of rank 4 in order to determine the irreducible characters of U. This technique, together with some more computations detailed here, allows us to give a full parametrization of Irr(U) in all these cases. We remark that for each of the types B_4 and D_4 we have just one nonabelian core, and for type F_4 we have six nonabelian cores. The rest of this chapter is split into sections according to the examination of such nonabelian cores. For each core $\mathfrak{C} = (\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ examined in Chapters 4 and 5, we give explicitly $\mathcal{S}$, $\mathcal{Z}$, $\mathcal{A}$ and $\mathcal{L}$; we note that $\mathcal{K}$ can then easily be determined.

The nonabelian core of U_{B_4} is dealt with in Section 4.1. Under the assumption $p \geq 3$, it is easy to understand the behavior of such a core. There is some more work to do for the nonabelian core of U_{D_4} when $p = 2$; the analysis of this core is contained in Section 4.2. We recall that this case has already been discussed in [HLM11], and analysed with similar methods. Finally, we will be able to deal with all six nonabelian cores in type F_4; this is done in Section 4.3. We remark the different behavior of the prime $p = 3$ in the parametrization of Irr(U_{F_4}); in particular, this shows up in the investigation of the fifth core of F_4 examined in this chapter.

4.1 Parametrization of $\mathrm{Irr}(U_{\mathrm{B}_4})$

We denote the only nonabelian core of B_4 by $\mathfrak{C}^{\mathrm{B}_4}$. We describe in this section the representation theory of the corresponding quattern group. We can then complete the parametrization of $\mathrm{Irr}(U_{\mathrm{B}_4})$, given in Table D.1. In particular, we can determine the expressions of the numbers of irreducible characters of U_{B_4} of a fixed degree as polynomials in $v := q - 1$. These are collected in Table 4.1.

D	$k(U_{\mathrm{B}_4}, D)$
1	$v^4 + 4v^3 + 6v^2 + 4v + 1$
q	$3v^4 + 10v^3 + 11v^2 + 4v$
q^2	$v^6 + 6v^5 + 16v^4 + 23v^3 + 15v^2 + 3v$
q^3	$2v^5 + 10v^4 + 18v^3 + 13v^2 + 3v$
q^4	$2v^5 + 11v^4 + 19v^3 + 11v^2 + v$
q^5	$v^5 + 6v^4 + 11v^3 + 6v^2 + v$
q^6	$v^4 + 3v^3 + 2v^2$

Table 4.1: Numbers of irreducible characters of U_{B_4} of fixed degree, for $v = q - 1$ and $p \neq 2$.

The core $\mathfrak{C}^{\mathrm{B}_4}$ occurs for $\Sigma = \{\alpha_{13}, \alpha_{14}\}$. Correspondingly, we have

- $\mathcal{S} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_6, \alpha_7, \alpha_8, \alpha_{10}, \alpha_{11}, \alpha_{13}, \alpha_{14}\}$,
- $\mathcal{Z} = \{\alpha_{10}, \alpha_{13}, \alpha_{14}\}$,
- $\mathcal{A} = \{\alpha_1, \alpha_5\}$ and
- $\mathcal{L} = \{\alpha_9, \alpha_{12}\}$.

We see that the method of Section 3.2 applies, by taking

- $Y = X_{\mathcal{J}}$, where $\mathcal{J} = \{\alpha_6, \alpha_7, \alpha_{11}\}$,

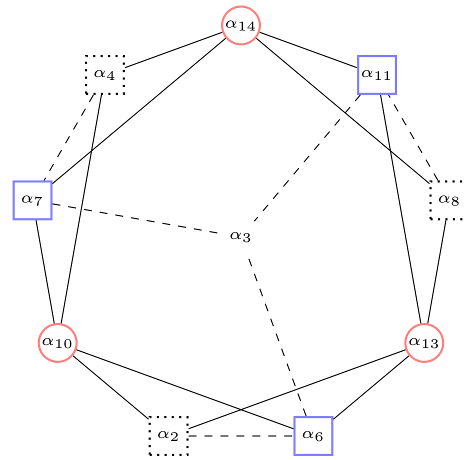

Figure 4.1: A picture representing the configuration of $\mathfrak{C}^{\mathrm{B}_4}$.

- $X = X_{\mathcal{I}}$, where $\mathcal{I} = \{\alpha_2, \alpha_4, \alpha_8\}$, then
- $H = X_3 Y Z$.

We explain in this discussion the meaning of Figure 4.1 and of the following pictures that represent the structure of a fixed nonabelian core, and how to take advantage of them to find candidates for X and Y in our analysis. The vertices of the graph are labelled by roots in $\mathcal{S}$. We join two roots $\alpha, \beta \in \mathcal{S}$ with $\gamma \in \mathcal{S}$ if $\alpha + \beta = \gamma$; when a vertex has degree at least 3, the relations between roots collected in Appendix B determine how to relate multiple edges from a vertex. The roots in $\mathcal{Z}$ are in circles.

We now determine the roots in $\mathcal{J}$ and $\mathcal{I}$, which are in straight and dashed boxes in Figure 4.1. For our choice of Y, we decide to put the highest root of $\mathcal{S} \setminus \mathcal{Z}$ in $\mathcal{J}$. We will see that this choice works to find suitable X and Y in most of the nonabelian cores examined in this work; in the other cases, as in the case of the core $\mathfrak{C}^6$ in F_4, this will work by adding the highest root of $\mathcal{S} \setminus \mathcal{Z}$ to $\mathcal{I}$ instead. We then put α_{11} in $\mathcal{J}$ in this case. We notice that

$$\alpha_2 + \alpha_{11} = \alpha_{13} \in \mathcal{Z} \quad \text{and} \quad \alpha_4 + \alpha_{11} = \alpha_{14} \in \mathcal{Z},$$

then we put α_2 and α_4 in $\mathcal{I}$. Moreover, we have that

$$\alpha_2 + \alpha_7 = \alpha_{10} \in \mathcal{Z} \quad \text{and} \quad \alpha_4 + \alpha_6 = \alpha_{10} \in \mathcal{Z},$$

and we put α_6 and α_7 in $\mathcal{J}$. Finally, we notice that

$$\alpha_6 + \alpha_8 = \alpha_{13} \in \mathcal{Z} \quad \text{and} \quad \alpha_7 + \alpha_8 = \alpha_{14} \in \mathcal{Z},$$

and we add α_8 to $\mathcal{I}$. Along with the other three relations in $\mathcal{S}$, namely

$$\alpha_2 + \alpha_3 = \alpha_6, \qquad \alpha_3 + \alpha_4 = \alpha_7 \quad \text{and} \quad \alpha_3 + \alpha_8 = \alpha_{11},$$

we now easily check that assumptions (i)–(iv) of the reduction lemma do hold with $Y = X_{\mathcal{J}}$ and $X = X_{\mathcal{I}}$, where $\mathcal{J} = \{\alpha_6, \alpha_7, \alpha_{11}\}$ and $\mathcal{I} = \{\alpha_2, \alpha_4, \alpha_8\}$.

Let us examine Equation (3.2.3) in this case. It gives

$$s_6(a_{10}t_2 + a_{14}t_8) + s_7(-a_{10}t_4 - a_{13}t_8) + s_{11}(-a_{13}t_2 - a_{14}t_4) = 0.$$

The prime p is not very bad for B_4, hence $p \neq 2$. In this case, the above equation is satisfied for every $s_6, s_7, s_{11} \in \mathbb{F}_q$ if and only if $t_2 = t_4 = t_8 = 0$, and vice versa. This means that we get $Y' = 1$, and that Y is normal in $\bar{H}$. Moreover, we have that $\bar{H}/Y \cong X_3X_{\mathcal{Z}}$ is abelian.

For $b_3 \in \mathbb{F}_q$ we denote by $\lambda_{b_3}^{a_{10},a_{13},a_{14}} \in \mathrm{Irr}(X_{\mathcal{S}\setminus(\mathcal{I}\cup\mathcal{J})})_{\mathcal{Z}}$ the linear character as in Section 2.4. Then the assumption of Remark 3.5 holds. Hence we obtain

$$\mathrm{Irr}(U)_{\mathfrak{C}^{\mathrm{B}_4}} = \{\chi_{b_3}^{a_{10},a_{13},a_{14}} \mid a_{10}, a_{13}, a_{14} \in \mathbb{F}_q^\times, b_3 \in \mathbb{F}_q\},$$

where $\chi_{b_3}^{a_{10},a_{13},a_{14}} = \mathrm{Ind}^{\mathcal{A}\cup\mathcal{I}} \mathrm{Inf}_{\mathcal{K}\cup\mathcal{J}} \lambda_{b_3}^{a_{10},a_{13},a_{14}}$. The family $\mathrm{Irr}(U)_{\mathfrak{C}^{\mathrm{B}_4}}$ consists of $q(q-1)^3$ characters of degree q^5.

4.2 Parametrization of $\mathrm{Irr}(U_{\mathrm{D}_4})$

We have only one nonabelian core in type D_4, which we denote by $\mathfrak{C}^{\mathrm{D}_4}$. The examination of this core allows us to complete the parametrization of $\mathrm{Irr}(U_{\mathrm{D}_4})$; this is given in Table D.3. The parametrization of $\mathrm{Irr}(U_{\mathrm{D}_4})$ was already given in [HLM11], and the one provided here agrees with it. We provide in Table 4.2 the number of irreducible characters of every fixed degree in $\mathrm{UD}_4(q)$ for every p, expressed as a polynomial in $v = q-1$. We notice that such expressions are uniform for $p \geq 3$, but the different behavior of the core $\mathfrak{C}^{\mathrm{D}_4}$ for $p = 2$ yields in this case a different formula for $k(\mathrm{UD}_4(q), q^4)$, as well as irreducible characters of degree $q^3/2$.

D	$k(U_{\mathrm{D}_4}, D)$
1	$v^4 + 4v^3 + 6v^2 + 4v + 1$
q	$v^5 + 5v^4 + 10v^3 + 9v^2 + 3v$
q^2	$3v^4 + 9v^3 + 9v^2 + 3v$
$q^3/2$	0, if $p \geq 3$ $4v^4$, if $p = 2$
q^3	$v^5 + 5v^4 + 10v^3 + 7v^2 + v$, if $p \geq 3$ $v^5 + 4v^4 + 10v^3 + 7v^2 + v$, if $p = 2$
q^4	$v^4 + 3v^3 + 3v^2 + v$

Table 4.2: Numbers of irreducible characters of U_{D_4} of fixed degree, for $v = q - 1$.

The core $\mathfrak{C}^{\mathrm{D}_4}$ occurs for $\Sigma = \{\alpha_8, \alpha_9, \alpha_{10}\}$, and we have that

- $\mathcal{S} = \{\alpha_1, \ldots, \alpha_{10}\}$,
- $\mathcal{Z} = \Sigma = \{\alpha_8, \alpha_9, \alpha_{10}\}$,
- $\mathcal{A} = \varnothing$ and
- $\mathcal{L} = \varnothing$.

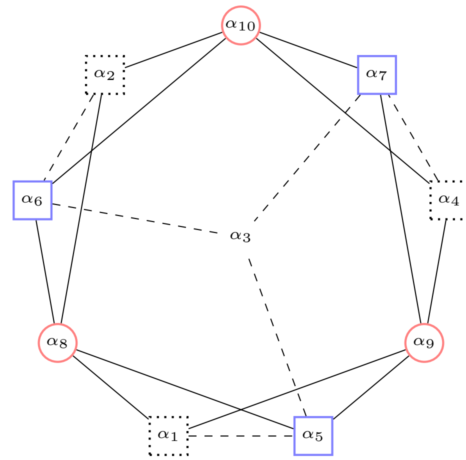

Figure 4.2: A picture representing the configuration of $\mathfrak{C}^{\mathrm{D}_4}$.

Using the method of Section 3.2, we take

- $Y = X_{\mathcal{J}}$, where $\mathcal{J} = \{\alpha_5, \alpha_6, \alpha_7\}$,
- $X = X_{\mathcal{I}}$, where $\mathcal{I} = \{\alpha_1, \alpha_2, \alpha_4\}$, then
- $H = X_3 Y Z$.

In this case Equation (3.2.3) is

$$s_5(a_8t_2 + a_9t_4) + s_6(a_8t_1 + a_{10}t_4) + s_7(-a_9t_1 - a_{10}t_2) = 0.$$

Let $p \geq 3$. Then we get that $Y' = 1$, and that Y is normal in $\bar{H}$. Furthermore, $\bar{H}/Y \cong X_3 X_{\mathcal{Z}}$ is abelian. For $b_3 \in \mathbb{F}_q$ we let $\lambda_{b_3}^{a_8,a_9,a_{10}} \in \mathrm{Irr}(X_{\mathcal{S}\setminus(\mathcal{I}\cup\mathcal{J})})_{\mathcal{Z}}$ be the linear character as in the notation of Section 2.4. Then as explained in Remark 3.5 we obtain

$$\mathrm{Irr}(U)_{\mathfrak{C}^{\mathrm{D}_4}}^{p\geq 3} = \{\chi_{b_3}^{a_8,a_9,a_{10}} \mid a_8, a_9, a_{10} \in \mathbb{F}_q^{\times}, b_3 \in \mathbb{F}_q\},$$

where $\chi_{b_3}^{a_8,a_9,a_{10}} = \mathrm{Ind}^{\mathcal{A}\cup\mathcal{I}} \mathrm{Inf}_{\mathcal{K}\cup\mathcal{J}} \lambda_{b_3}^{a_8,a_9,a_{10}}$. We have that $\mathrm{Irr}(U)_{\mathfrak{C}^{\mathrm{D}_4}}^{p\geq 3}$ is a family of $q(q-1)^3$ characters of degree q^3.

Now suppose $p = 2$. In this case, we have that $X' = \{x_{1,2,4}(t) \mid t \in \mathbb{F}_q\}$ and $Y' = \{x_{5,6,7}(s) \mid s \in \mathbb{F}_q\}$, where

$$x_{1,2,4}(t) = x_1(a_{10}t)x_2(a_9t)x_4(a_8t), \text{ and}$$

$$x_{5,6,7}(s) = x_5(a_{10}s)x_6(a_9s)x_7(a_8s).$$

We can take $\tilde{Y} = X_6X_7$, and we have $\tilde{H}/\tilde{Y} = X_3X'YZ/\tilde{Y}$. By Proposition 3.3, we have that $\mathrm{Irr}(V \mid \lambda)$ is in bijection with $\mathrm{Irr}(\tilde{H}/\tilde{Y} \mid \lambda)$.

Let us then consider $\mathrm{Irr}(\tilde{H}/\tilde{Y} \mid \lambda)$. We note that Y' lies in the centre of $\tilde{H}/\tilde{Y}$. For every $a_{5,6,7} \in \mathbb{F}_q$, we let $\mu^{a_{5,6,7}} : Y'Z \to \mathbb{F}_q$ be the extension of μ to $Y'Z$ with $\mu^{a_{5,6,7}}(x_{5,6,7}(t)) = a_{5,6,7}t$, and we let $\lambda^{a_{5,6,7}}$ be defined by $\lambda^{a_{5,6,7}} = \phi \circ \mu_{a_{5,6,7}}$ Then we have that $\mathrm{Irr}(\tilde{H}/\tilde{Y} \mid \lambda)$ is the disjoint union of $\mathrm{Irr}(\tilde{H}/\tilde{Y} \mid \lambda^{a_{5,6,7}})$ over $a_{5,6,7} \in \mathbb{F}_q^\times$, along with $\mathrm{Irr}(\tilde{H}/Y \mid \lambda)$.

A computation in $\tilde{H}/\tilde{Y}$ gives

$$\begin{aligned}[x_3(t), x_{1,2,4}(s)] &= x_5(a_{10}st)x_6(a_9st)x_7(a_8st)x_8(a_9a_{10}s^2t)x_9(a_8a_{10}s^2t)x_{10}(a_8a_9s^2t)\\ &= x_{5,6,7}(st)x_8(a_9a_{10}s^2t)x_9(a_8a_{10}s^2t)x_{10}(a_8a_9s^2t).\end{aligned}$$

Applying $\mu^{a_{5,6,7}}$ to this equality, we get

$$\mu^{a_{5,6,7}}([x_3(t), x_{1,2,4}(s)]) = t(3a_8a_9a_{10}s^2 + a_{5,6,7}s) = t(a_8a_9a_{10}s^2 + a_{5,6,7}s).$$

We notice that the quotient $\tilde{H}/(\tilde{Y} \ker \mu^{a_{5,6,7}}) = X_3X'YZ/(\tilde{Y} \ker \mu^{a_{5,6,7}})$ is isomorphic to the three-dimensional group V_f, where

$$f(s,t) = t(a_8a_9a_{10}s^2 + a_{5,6,7}s)$$

is as defined at the end of Section 3.1. We have that $\mathrm{Irr}(\tilde{H}/\tilde{Y} \mid \lambda^{a_{5,6,7}})$ is in bijection with

$\mathrm{Irr}(\tilde{H}/(\tilde{Y}\ker\mu^{a_{5,6,7}}) \mid \lambda^{a_{5,6,7}})$. Then we can analyse the latter as explained in Section 3.1.

Let $a_{5,6,7} \neq 0$. We let

$$W_1 := \{x_{1,2,4}(a_{5,6,7}s/a_8a_9a_{10}) \mid s \in \mathbb{F}_2\}, \qquad W_2 := \{x_3(a_8a_9a_{10}t/a_{5,6,7}^2) \mid t \in \mathbb{F}_2\}.$$

We define the characters $\lambda_{c_{1,2,4},c_3}^{a_{5,6,7},a_8,a_9,a_{10}}$ for $c_{1,2,4}, c_3 \in \mathbb{F}_2$ of $W_1W_2YZ/(\tilde{Y}\ker\lambda^{a_{5,6,7},a_8,a_9,a_{10}})$ as discussed at the end of Section 3.1. Then we get the family of characters

$$\mathrm{Irr}(U)_{\mathfrak{C}^{\mathrm{D}_4}}^{1,p=2} = \{\chi_{c_{1,2,4},c_3}^{a_{5,6,7},a_8,a_9,a_{10}} \mid a_{5,6,7}, a_8, a_9, a_{10} \in \mathbb{F}_q^\times, c_{1,2,4}, c_3 \in \mathbb{F}_2\},$$

where

$$\chi_{c_{1,2,4},c_3}^{a_{5,6,7},a_8,a_9,a_{10}} = \mathrm{Ind}_{X'W_2YX_{\mathcal{Z}}X_{\mathcal{K}}}^{U} \mathrm{Inf}_{W_1W_2YZ/(\tilde{Y}\ker\lambda^{a_{5,6,7},a_8,a_9,a_{10}})}^{X'W_2YX_{\mathcal{Z}}X_{\mathcal{K}}} \lambda_{c_{1,2,4},c_3}^{a_{5,6,7},a_8,a_9,a_{10}}.$$

We have that $\mathrm{Irr}(U)_{\mathfrak{C}^{\mathrm{D}_4}}^{1,p=2}$ consists of $4(q-1)^4$ characters of degree $q^3/2$.

Finally, let $a_{5,6,7} = 0$. We analyse in this case the set $\mathrm{Irr}(\tilde{H}/Y \mid \lambda)$ by using the arguments for the three-dimensional group V_f where $f(s,t) = a_8a_9a_{10}s^2t$. Therefore, we get the family of characters

$$\mathrm{Irr}(U)_{\mathfrak{C}^{\mathrm{D}_4}}^{2,p=2} = \{\chi^{a_8,a_9,a_{10}} \mid a_8, a_9, a_{10} \in \mathbb{F}_q^\times\},$$

where

$$\chi^{a_8,a_9,a_{10}} = \mathrm{Ind}_{X'YX_{\mathcal{Z}}X_{\mathcal{K}}}^{U} \mathrm{Inf}_{Z}^{X'YX_{\mathcal{Z}}X_{\mathcal{K}}} \lambda^{a_8,a_9,a_{10}}.$$

We have that $\mathrm{Irr}(U)_{\mathfrak{C}^{\mathrm{D}_4}}^{2,p=2}$ consists of $(q-1)^3$ characters of degree q^3.

We are done also with the case $p = 2$, since

$$\mathrm{Irr}(U)_{\mathfrak{C}^{\mathrm{D}_4}}^{p=2} = \mathrm{Irr}(U)_{\mathfrak{C}^{\mathrm{D}_4}}^{1,p=2} \cup \mathrm{Irr}(U)_{\mathfrak{C}^{\mathrm{D}_4}}^{2,p=2}.$$

4.3 Parametrization of $\mathrm{Irr}(U_{\mathrm{F}_4})$

Below we consider the remaining nonabelian cores in $U = U_{\mathrm{F}_4}$. We denote these cores by $\mathfrak{C}^1$, $\mathfrak{C}^2$, $\mathfrak{C}^3$, $\mathfrak{C}^4$, $\mathfrak{C}^5$ and $\mathfrak{C}^6$. For each of them, we analyse $\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}$ before explaining how this parametrizes $\mathrm{Irr}(U)_{\mathfrak{C}^i}$ for $i = 1, \dots, 6$ and how these characters can be obtained by inducing linear characters using Proposition 3.3 and Remark 3.5. We notice that if $\mathfrak{C} = (\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ and $\mathfrak{C}' = (\mathcal{S}', \mathcal{Z}', \mathcal{A}', \mathcal{L}', \mathcal{K}')$ are cores of U_{F_4}, then $(|\mathcal{S}|, |\mathcal{Z}|) \neq (|\mathcal{S}'|, |\mathcal{Z}'|)$. In particular, $X_{\mathcal{S}}$ is not isomorphic to $X_{\mathcal{S}'}$.

After the study of $\mathfrak{C}^1, \dots, \mathfrak{C}^6$, we can finally describe in detail the set $\mathrm{Irr}(U_{\mathrm{F}_4})$. The parametrization of the irreducible characters is given in Table D.4. On the one hand, the expressions of numbers of irreducible characters of fixed degrees of U_{F_4} for $p \geq 5$, collected in Table 4.3, are uniform as polynomials in $v = q - 1$. On the other hand, as in the case of $\mathrm{UD}_4(2^e)$, we note from Table 4.3 the difference between the expressions as polynomials in $v = q-1$ of the numbers $k(\mathrm{UF}_4(p^e), D)$ for $p \geq 5$ and the ones of the numbers $k(\mathrm{UF}_4(3^e), D)$; in particular, the expression of $k(\mathrm{UF}_4(q), q^4)$ does *not* have integer coefficients for $p = 3$. In this case we also get some irreducible characters of degree $q^4/3$. The different behavior of $p = 3$ with respect to every $p \geq 5$ shows up in the study of the cores $\mathfrak{C}^4$, $\mathfrak{C}^5$ and $\mathfrak{C}^6$.

The nonabelian core $\mathfrak{C}^1$. This core occurs for $\Sigma = \{\alpha_{22}\}$. In this case, we have

- $\mathcal{S} = \{\alpha_1, \alpha_2, \alpha_5, \alpha_9, \alpha_{11}, \alpha_{14}, \alpha_{16}, \alpha_{18}, \alpha_{20}, \alpha_{22}\}$,
- $\mathcal{Z} = \{\alpha_{14}, \alpha_{20}, \alpha_{22}\}$,
- $\mathcal{A} = \{\alpha_3, \alpha_4, \alpha_6, \alpha_7, \alpha_{10}, \alpha_{13}\}$ and
- $\mathcal{L} = \{\alpha_8, \alpha_{12}, \alpha_{15}, \alpha_{17}, \alpha_{19}, \alpha_{21}\}$.

We follow the discussion in Section 3.2. We have

- $Y = X_{\mathcal{J}}$, where $\mathcal{J} = \{\alpha_5, \alpha_{11}, \alpha_{18}\}$,

D	$k(U_{\mathrm{F}_4}, D)$
1	$v^4 + 4v^3 + 6v^2 + 4v + 1$
q	$v^5 + 6v^4 + 13v^3 + 12v^2 + 4v$
q^2	$v^6 + 7v^5 + 20v^4 + 28v^3 + 18v^2 + 4v$
q^3	$4v^5 + 20v^4 + 33v^3 + 21v^2 + 4v$
$q^4/3$	$0, \text{ if } p \geq 5$ $9v^4/2, \text{ if } p = 3$
q^4	$v^8 + 8v^7 + 28v^6 + 58v^5 + 79v^4 + 66v^3 + 24v^2 + 2v, \text{ if } p \geq 5$ $v^8 + 8v^7 + 28v^6 + 59v^5 + 161v^4/2 + 67v^3 + 24v^2 + 2v, \text{ if } p = 3$
q^5	$v^7 + 7v^6 + 22v^5 + 39v^4 + 37v^3 + 15v^2 + 2v, \text{ if } p \geq 5$ $v^7 + 7v^6 + 23v^5 + 41v^4 + 37v^3 + 15v^2 + 2v, \text{ if } p = 3$
q^6	$2v^6 + 14v^5 + 36v^4 + 40v^3 + 17v^2 + 2v, \text{ if } p \geq 5$ $2v^6 + 14v^5 + 36v^4 + 39v^3 + 17v^2 + 2v, \text{ if } p = 3$
q^7	$2v^6 + 13v^5 + 32v^4 + 34v^3 + 13v^2 + 2v$
q^8	$4v^5 + 15v^4 + 19v^3 + 8v^2$
q^9	$v^5 + 7v^4 + 11v^3 + 5v^2$
q^{10}	$v^4 + 3v^3 + v^2$

Table 4.3: Numbers of irreducible characters of U_{F_4} of fixed degree, for $v = q - 1$ and $p \neq 2$.

- $X = X_{\mathcal{I}}$, where $\mathcal{I} = \{\alpha_2, \alpha_9, \alpha_{16}\}$, then
- $H = X_1 Y Z$.

Equation (3.2.3) now yields

$$s_5(-a_{14}t_9 - a_{20}t_{16}) + s_{11}(-a_{14}t_2 + a_{22}t_{16}) + s_{18}(-a_{20}t_2 + a_{22}t_9) = 0.$$

We recall that $p \neq 2$ since p is not a very bad prime for F_4. We then easily get $Y' = X' = 1$, and Y is normal in $\bar{H}$. Moreover, we have that $\bar{H}/Y \cong X_1 X_{\mathcal{Z}}$ is abelian.

For $b_1 \in \mathbb{F}_q$ we denote by $\lambda_{b_1}^{a_{14}, a_{20}, a_{22}} \in \mathrm{Irr}(X_{\mathcal{S} \setminus (\mathcal{I} \cup \mathcal{J})})_{\mathcal{Z}}$ the linear character as defined in

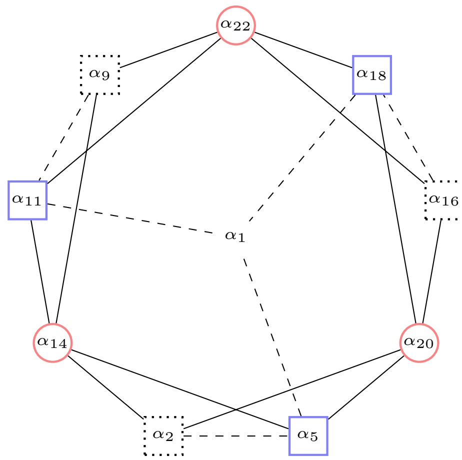

Figure 4.3: A picture representing the configuration of $\mathfrak{C}^1$ of F_4.

Section 2.4. The conditions of Remark 3.5 are then satisfied. Hence we obtain

$$\mathrm{Irr}(U)_{\mathfrak{C}^{\mathrm{B}_4}} = \{\chi_{b_1}^{a_{14},a_{20},a_{22}} \mid a_{14}, a_{20}, a_{22} \in \mathbb{F}_q^{\times}, b_1 \in \mathbb{F}_q\},$$

where $\chi_{b_1}^{a_{14},a_{20},a_{22}} = \mathrm{Ind}^{\mathcal{A}\cup\mathcal{I}} \mathrm{Inf}_{\mathcal{K}\cup\mathcal{J}} \lambda_{b_1}^{a_{14},a_{20},a_{22}}$. The family $\mathrm{Irr}(U)_{\mathfrak{C}^1}$ consists of $q(q-1)^3$ characters of degree q^9.

The nonabelian core $\mathfrak{C}^2$. This core occurs for $\Sigma = \{\alpha_{11}, \alpha_{13}\}$, and we have

- $\mathcal{S} = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{13}\}$,
- $\mathcal{Z} = \{\alpha_5, \alpha_{10}, \alpha_{11}, \alpha_{13}\}$,
- $\mathcal{A} = \{\alpha_3\}$ and
- $\mathcal{L} = \{\alpha_8\}$.

Using the method of Section 3.2, we take

- $Y = X_{\mathcal{J}}$, where $\mathcal{J} = \{\alpha_2, \alpha_6, \alpha_9\}$,
- $X = X_{\mathcal{I}}$, where $\mathcal{I} = \{\alpha_1, \alpha_4, \alpha_7\}$ and then we have that

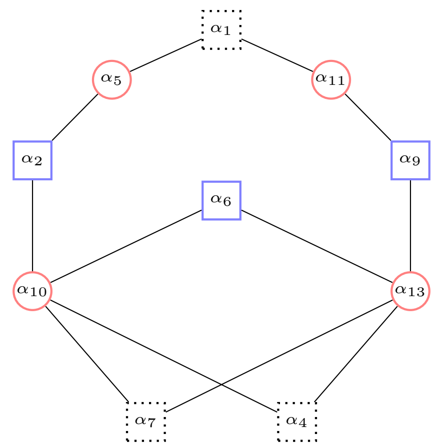

Figure 4.4: A picture representing the configuration of $\mathfrak{C}^2$ of F_4.

- $H = YZ$.

In this case Equation (3.2.3) is

$$s_2(-a_5t_1 + a_{10}t_7) + s_6(a_{10}t_4 - a_{13}t_7) + s_9(-a_{11}t_1 + a_{13}t_4) = 0.$$

For $a_{11} \neq a_5a_{13}^2/a_{10}^2$, we have $Y' = 1$ and Y is normal in $\bar{H}$. Then as explained in Remark 3.5 we get the family of characters

$$\mathrm{Irr}(U)^1_{\mathfrak{C}^2} = \{\chi^{a_5,a_{10},a_{11}^*,a_{13}} \mid a_5, a_{10}, a_{11}^*, a_{13} \in \mathbb{F}_q^\times, a_{11}^* \neq a_5(a_{13}/a_{10})^2\} \subseteq \mathrm{Irr}(U)_{\mathfrak{C}^2},$$

where

$$\chi^{a_5,a_{10},a_{11}^*,a_{13}} = \mathrm{Ind}^{\mathcal{A}\cup\mathcal{I}} \mathrm{Inf}_{\mathcal{K}\cup\mathcal{J}}\, \lambda^{a_5,a_{10},a_{11}^*,a_{13}}.$$

We have that $\mathrm{Irr}(U)^1_{\mathfrak{C}^2}$ consists of $(q-1)^3(q-2)$ characters of degree q^4.

For $a_{11} = a_5a_{13}^2/a_{10}^2$, we have $X' = X_{1,4,7} = \{x_{1,4,7}(t) \mid t \in \mathbb{F}_q\}$ and $Y' = X_{2,6,9} = \{x_{2,6,9}(s) \mid s \in \mathbb{F}_q\}$, where

$$x_{1,4,7}(t) = x_1(a_{10}^2t)x_4(a_5a_{13}t)x_7(a_5a_{10}t) \text{ and } x_{2,6,9}(s) = x_2(a_{13}^2s)x_6(a_{10}a_{13}s)x_9(-a_{10}^2s).$$

We can take any complement of Y' in Y, and we choose $\tilde{Y} = X_2X_9$. Then we have $\tilde{H}/\tilde{Y} = X'Y'Z$, which is abelian. We denote by $\lambda^{a_5,a_{10},a_{13}}$ the character $\lambda^{a_5,a_{10},a_{11},a_{13}}$ with $a_{11} = a_5a_{13}^2/a_{10}^2$. For $b_{1,4,7}, b_{2,6,9} \in \mathbb{F}_q$, we define $\lambda^{a_5,a_{10},a_{13}}_{b_{1,4,7},b_{2,6,9}} \in \mathrm{Irr}(X'Y'Z)$ by extending $\lambda^{a_5,a_{10},a_{13}}$, and setting $\lambda^{a_5,a_{10},a_{13}}_{b_{1,4,7},b_{2,6,9}}(x_{1,4,7}(t)) = \phi(b_{1,4,7}t)$ and $\lambda^{a_5,a_{10},a_{13}}_{b_{1,4,7},b_{2,6,9}}(x_{2,6,9}(t)) = \phi(b_{2,6,9}t)$ for every $t \in \mathbb{F}_q$. Then as explained in Remark 3.5 we get the family of characters

$$\mathrm{Irr}(U)^2_{\mathfrak{C}^2} = \{\chi^{a_5,a_{10},a_{13}}_{b_{1,4,7},b_{2,6,9}} \mid a_5, a_{10}, a_{13} \in \mathbb{F}_q^\times, b_{1,4,7}, b_{2,6,9} \in \mathbb{F}_q\},$$

where

$$\chi^{a_5,a_{10},a_{13}}_{b_{1,4,7},b_{2,6,9}} = \mathrm{Ind}^U_{\bar{H}X_\mathcal{K}} \mathrm{Inf}^{\bar{H}X_\mathcal{K}}_{\tilde{H}/\tilde{Y}} \lambda^{a_5,a_{10},a_{13}}_{b_{1,4,7},b_{2,6,9}}.$$

We have that $\mathrm{Irr}(U)^2_{\mathfrak{C}^2}$ consists of $q^2(q-1)^3$ characters of degree q^3.

We have that $\mathrm{Irr}(U)_{\mathfrak{C}^2} = \mathrm{Irr}(U)^1_{\mathfrak{C}^2} \cup \mathrm{Irr}(U)^2_{\mathfrak{C}^2}$ and this gives all the irreducible characters corresponding to $\mathfrak{C}^2$.

The nonabelian core $\mathfrak{C}^3$. This core occurs for $\Sigma = \{\alpha_{12}, \alpha_{16}\}$, and we have

- $\mathcal{S} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{12}, \alpha_{16}\}$,
- $\mathcal{Z} = \{\alpha_8, \alpha_9, \alpha_{12}, \alpha_{16}\}$,
- $\mathcal{A} = \{\alpha_4\}$ and
- $\mathcal{L} = \{\alpha_{13}\}$.

Using the method of Section 3.2, we take

- $Y = X_\mathcal{J}$, where $\mathcal{J} = \{\alpha_5, \alpha_6, \alpha_{10}\}$,
- $X = X_\mathcal{I}$, where $\mathcal{I} = \{\alpha_1, \alpha_3, \alpha_7\}$ and then we have that
- $H = X_2YZ$.

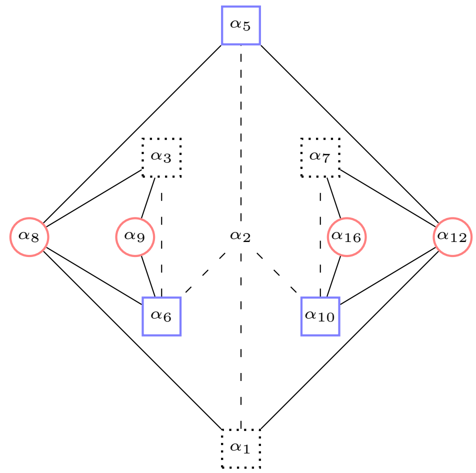

Figure 4.5: A picture representing the configuration of $\mathfrak{C}^3$ of F_4.

In this case Equation (3.2.3) is

$$s_5(a_8t_3 + a_{12}t_7) + s_6(-a_8t_1 - 2a_9t_3) + s_{10}(-a_{12}t_1 + 2a_{16}t_7) = 0.$$

For $a_{16} \neq a_9a_{12}^2/a_8^2$, we have $Y' = 1$ and Y is normal in $\bar{H}$. Further, $\bar{H}/Y = X_2X_{\mathcal{Z}}$, so as explained in Remark 3.5 we get the family of characters

$$\mathrm{Irr}(U)^1_{\mathfrak{C}^3} = \{\chi_{b_2}^{a_8,a_9,a_{12},a_{16}^*} \mid a_8, a_9, a_{12}, a_{16}^* \in \mathbb{F}_q^\times, a_{16}^* \neq a_9(a_{12}/a_8)^2, b_2 \in \mathbb{F}_q\} \subseteq \mathrm{Irr}(U)_{\mathfrak{C}^3},$$

where

$$\chi_{b_2}^{a_8,a_9,a_{12},a_{16}^*} = \mathrm{Ind}^{\mathcal{A}\cup\mathcal{I}} \mathrm{Inf}_{\mathcal{K}\cup\mathcal{J}} \lambda_{b_2}^{a_8,a_9,a_{12},a_{16}^*},$$

and $\lambda_{b_2}^{a_8,a_9,a_{12},a_{16}^*} \in \mathrm{Irr}(\bar{H}/Y)$ is defined in the usual way. We have that $\mathrm{Irr}(U)^1_{\mathfrak{C}^3}$ consists of $q(q-1)^3(q-2)$ characters of degree q^4.

For $a_{16} = a_9a_{12}^2/a_8^2$, we have $X' = \{x_{1,3,7}(t) \mid t \in \mathbb{F}_q\}$ and $Y' = \{x_{5,6,10}(s) \mid s \in \mathbb{F}_q\}$, where

$$x_{1,3,7}(t) = x_1(2a_9a_{12}t)x_3(-a_8a_{12}t)x_7(a_8^2t) \quad \text{and} \quad x_{5,6,10}(s) = x_5(2a_9a_{12}s)x_6(a_8a_{12}s)x_{10}(-a_8^2s).$$

We can take any complement of Y' in Y and we choose $\tilde{Y} = X_5X_{10}$. Then we have $\tilde{H}/\tilde{Y} = X_2X'YZ/\tilde{Y}$ and $Y' \subseteq Z(\tilde{H}/\tilde{Y})$. From now on, we denote by $\lambda^{a_8,a_9,a_{12}}$ the character $\lambda^{a_8,a_9,a_{12},a_{16}}$ with $a_{16} = a_9a_{12}^2/a_8^2$.

A computation in $\tilde{H}/\tilde{Y}$ gives

$$[x_2(s), x_{1,3,7}(t)] = x_{5,6,10}(-st).$$

Therefore, $\tilde{H}/\tilde{Y}$ is the direct product of Z and $X_2X'Y/\tilde{Y}$. Further $X_2X'Y/\tilde{Y}$ is isomorphic to the three-dimensional group V_f for $f(s,t) = -st$ from Section 3.1.

We label the linear characters of $X_2X'Y/\tilde{Y}$ by $\chi_{b_2,b_{1,3,7}}$. By tensoring these characters with $\lambda^{a_8,a_9,a_{12}}$ and then applying $\mathrm{Ind}_{\bar{H}X_{\mathcal{K}}}^{U} \mathrm{Inf}_{\tilde{H}/\tilde{Y}}^{\bar{H}X_{\mathcal{K}}}$ we obtain the family of characters

$$\mathrm{Irr}(U)^2_{\mathfrak{C}^3} = \{\chi^{a_8,a_9,a_{12}}_{b_2,b_{1,3,7}} \mid a_8, a_9, a_{12} \in \mathbb{F}_q^\times, b_2, b_{1,3,7} \in \mathbb{F}_q\},$$

which consists of $q^2(q-1)^3$ characters of degree q^3.

Let $a_{5,6,10} \in \mathbb{F}_q^\times$. We write $\lambda^{a_8,a_9,a_{12},a_{5,6,10}}$ for the linear character of $Y'Z$ defined by extending $\lambda^{a_8,a_9,a_{12}}$ to Y' in the usual way. By applying $\mathrm{Ind}_{X'YX_{\mathcal{Z}}X_{\mathcal{K}}}^{U} \mathrm{Inf}_{YZ/\tilde{Y}}^{X'YX_{\mathcal{Z}}X_{\mathcal{K}}}$ to these linear characters we obtain the family of characters

$$\mathrm{Irr}(U)^3_{\mathfrak{C}^3} = \{\chi^{a_8,a_9,a_{12},a_{5,6,10}} \mid a_8, a_9, a_{12}, a_{5,6,10} \in \mathbb{F}_q^\times\},$$

which consists of $(q-1)^4$ characters of degree q^4.

We have $\mathrm{Irr}(U)_{\mathfrak{C}^3} = \mathrm{Irr}(U)^1_{\mathfrak{C}^3} \cup \mathrm{Irr}(U)^2_{\mathfrak{C}^3} \cup \mathrm{Irr}(U)^3_{\mathfrak{C}^3}$ and this gives all the irreducible characters corresponding to $\mathfrak{C}^3$.

The nonabelian core $\mathfrak{C}^4$. This core occurs for $\Sigma = \{\alpha_{14}, \alpha_{15}\}$, and we have

- $\mathcal{S} = \{\alpha_2, \alpha_4, \alpha_6, \alpha_7, \alpha_8, \alpha_{10}, \alpha_{11}, \alpha_{14}, \alpha_{15}\}$,

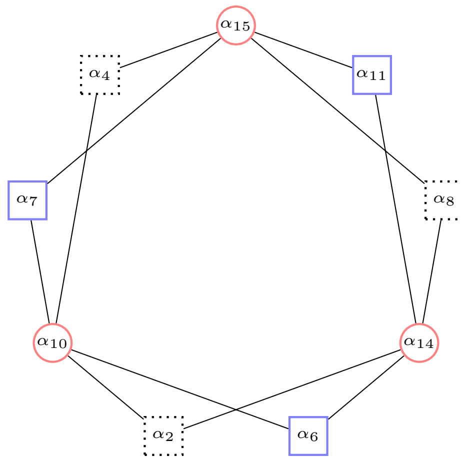

Figure 4.6: A picture representing the configuration of $\mathfrak{C}^4$ of F_4.

- $\mathcal{Z} = \{\alpha_{10}, \alpha_{14}, \alpha_{15}\}$,
- $\mathcal{A} = \{\alpha_1, \alpha_3, \alpha_5\}$ and
- $\mathcal{L} = \{\alpha_9, \alpha_{12}, \alpha_{13}\}$.

Using the method of Section 3.2, we take

- $Y = X_{\mathcal{J}}$, where $\mathcal{J} = \{\alpha_6, \alpha_7, \alpha_{11}\}$,
- $X = X_{\mathcal{I}}$, where $\mathcal{I} = \{\alpha_2, \alpha_4, \alpha_8\}$ and then we have that
- $H = YZ$.

In this case Equation (3.2.3) is

$$s_6(a_{10}t_4 + 2a_{14}t_8) + s_7(-a_{10}t_2 + a_{15}t_8) + s_{11}(-a_{14}t_2 + a_{15}t_4) = 0.$$

For $p \geq 5$, we have $Y' = 1$ and Y is normal in $\bar{H}$. So as explained in Remark 3.5 we obtain

$$\mathrm{Irr}(U)_{\mathfrak{C}^4}^{p\geq 5} = \{\chi^{a_{10},a_{14},a_{15}} \mid a_{10}, a_{14}, a_{15} \in \mathbb{F}_q^\times\}$$

by applying $\operatorname{Ind}^{\mathcal{A}\cup\mathcal{I}}\operatorname{Inf}_{\mathcal{K}\cup\mathcal{J}}$ to the characters in $\operatorname{Irr}(X_{\mathcal{S}\setminus(\mathcal{I}\cup\mathcal{J})})_{\mathcal{Z}}$. We have that $\operatorname{Irr}(U)_{\mathfrak{C}^4}^{p\geq 5}$ consists of $(q-1)^3$ characters of degree q^6.

Now suppose $p=3$. We have $X' = \{x_{2,4,8}(t) \mid t \in \mathbb{F}_q\}$ and $Y' = \{x_{6,7,11}(s) \mid s \in \mathbb{F}_q\}$, where

$$x_{2,4,8}(t) = x_2(a_{15}t)x_4(a_{14}t)x_8(a_{10}t) \quad \text{and} \quad x_{6,7,11}(s) = x_6(a_{15}s)x_7(a_{14}s)x_{11}(-a_{10}s).$$

We can take $\tilde{Y} = X_6X_{11}$, and we have that $\tilde{H}/\tilde{Y} \cong X'Y'Z$ is abelian. This yields

$$\operatorname{Irr}(U)_{\mathfrak{C}^4}^{p=3} = \{\chi_{b_{2,4,8},b_{6,7,11}}^{a_{10},a_{14},a_{15}} \mid a_{10},a_{14},a_{15} \in \mathbb{F}_q^\times, b_{2,4,8}, b_{6,7,11} \in \mathbb{F}_q\},$$

where these characters are obtained by applying $\operatorname{Ind}_{\bar{H}X_{\mathcal{K}}}^{U}\operatorname{Inf}_{\tilde{H}/\tilde{Y}}^{\bar{H}X_{\mathcal{K}}}$ to the linear characters $\lambda_{b_{2,4,8},b_{6,7,11}}^{a_{10},a_{14},a_{15}}$ of $\tilde{H}/\tilde{Y}$, which are labelled in the usual way. We have that $\operatorname{Irr}(U)_{\mathfrak{C}^4}^{p=3}$ consists of $q^2(q-1)^3$ characters of degree q^5.

The nonabelian core $\mathfrak{C}^5$. This core occurs for $\Sigma = \{\alpha_{11},\alpha_{12},\alpha_{13}\}$, and we have

- $\mathcal{S} = \{\alpha_1,\dots,\alpha_{13}\}$,
- $\mathcal{Z} = \Sigma = \{\alpha_{11},\alpha_{12},\alpha_{13}\}$,
- $\mathcal{A} = \varnothing$ and
- $\mathcal{L} = \varnothing$.

Using the method of Section 3.2, we take

- $Y = X_{\mathcal{J}}$, where $\mathcal{J} = \{\alpha_5,\alpha_8,\alpha_9,\alpha_{10}\}$,
- $X = X_{\mathcal{I}}$, where $\mathcal{I} = \{\alpha_1,\alpha_3,\alpha_4,\alpha_7\}$ and then we have that
- $H = X_2X_6YZ$.

In this case, Equation (3.2.3) is

$$s_5(-a_{11}t_3^2+a_{12}t_7)+s_8(-2a_{11}t_3+a_{12}t_4)+s_9(-a_{11}t_1+a_{13}t_4)+s_{10}(-a_{12}t_1-a_{13}t_3)=0. \quad (4.3.1)$$

Let $p \geq 5$. We want to compute X'. By choosing $s_5 = 0$, we must have, for every s_8, s_9, s_{10} in $\mathbb{F}_q$,

$$s_8(-2a_{11}t_3+a_{12}t_4)+s_9(-a_{11}t_1+a_{13}t_4)+s_{10}(-a_{12}t_1-a_{13}t_3)=0. \quad (4.3.2)$$

This yields a linear system of equations in t_1, t_3, t_4 given by setting each expression in the brackets equal to zero. The determinant of the associated matrix is $3a_{11}a_{12}a_{13} \neq 0$. Therefore, this implies $t_1 = t_3 = t_4 = 0$. Equation (4.3.1) then becomes $a_{12}s_5t_7 = 0$ in this case, which of course holds for every $s_5 \in \mathbb{F}_q$ if and only if $t_7 = 0$. Hence $X' = 1$.

We also have that $Y' = 1$. Namely, we can rewrite Equation (4.3.1) as

$$t_1(-a_{11}s_9-a_{12}s_{10})+t_4(a_{12}s_8+a_{13}s_9)+t_7(a_{12}s_5)-2a_{11}s_8t_3-a_{13}s_{10}t_3-a_{11}s_5t_3^2=0.$$

For this equality to be satisfied for every $t_1, t_3, t_4, t_7 \in \mathbb{F}_q$, we notice first that $s_5 = 0$ by choosing $t_1 = t_3 = t_4 = 0$ and $t_7 = 1$. Then we can rewrite the above equation as

$$t_1(-a_{11}s_9-a_{12}s_{10})+t_4(a_{12}s_8+a_{13}s_9)+t_3(-2a_{11}s_8-a_{13}s_{10})=0. \quad (4.3.3)$$

This gives rise to a system of equations in s_8, s_9, s_{10}, and the determinant of the associated matrix is again $3a_{11}a_{12}a_{13} \neq 0$. Hence $s_8 = s_9 = s_{10} = 0$, and $Y' = 1$.

Then $X' = Y' = 1$, and Y is normal in $\bar{H}$. Also we have $\bar{H}/Y \cong X_2X_6X_{\mathcal{Z}}$ is abelian. For $b_2, b_6 \in \mathbb{F}_q$ we let $\lambda_{b_2,b_6}^{a_{11},a_{12},a_{13}} \in \mathrm{Irr}(X_{\mathcal{S}\setminus(\mathcal{I}\cup\mathcal{J})})_{\mathcal{Z}}$ be the linear character with the usual notation.

Then as explained in Remark 3.5 we obtain

$$\mathrm{Irr}(U)_{\mathfrak{C}^5}^{p\geq 5} = \{\chi_{b_2,b_6}^{a_{11},a_{12},a_{13}} \mid a_{11}, a_{12}, a_{13} \in \mathbb{F}_q^\times, b_2, b_6 \in \mathbb{F}_q\},$$

where $\chi_{b_2,b_6}^{a_{11},a_{12},a_{13}} = \mathrm{Ind}^{\mathcal{A}\cup\mathcal{I}} \mathrm{Inf}_{\mathcal{K}\cup\mathcal{J}} \lambda_{b_2,b_6}^{a_{11},a_{12},a_{13}}$. We have that $\mathrm{Irr}(U)_{\mathfrak{C}^5}^{p\geq 5}$ is a family of $q^2(q-1)^3$ characters of degree q^4.

Now suppose $p = 3$. To compute X' (respectively Y'), we see by Equation (4.3.1) that we get a system of linear equations as in Equation (4.3.2) (respectively Equation (4.3.3)) in t_1, t_3, t_4 (respectively s_8, s_9, s_{10}), but now this gives a nontrivial space of solutions. In fact, we notice that $3a_{11}a_{12}a_{13} = 0$ in this case. Correspondingly, we get $X' = \{x_{1,3,4,7}(t) \mid t \in \mathbb{F}_q\}$ and $Y' = \{x_{8,9,10}(s) \mid s \in \mathbb{F}_q\}$, where

$$x_{1,3,4,7}(t) = x_1(a_{13}t)x_3(-a_{12}t)x_4(a_{11}t)x_7(-a_{11}a_{12}t^2) \text{ and}$$

$$x_{8,9,10}(s) = x_8(a_{13}s)x_9(-a_{12}s)x_{10}(a_{11}s).$$

We can take $\tilde{Y} = X_5X_8X_9$, and we have $\tilde{H}/\tilde{Y} = X_2X_6X'YZ/\tilde{Y}$. By Proposition 3.3, we have that $\mathrm{Irr}(V \mid \lambda)$ is in bijection with $\mathrm{Irr}(\tilde{H}/\tilde{Y} \mid \lambda)$.

We continue by considering $\mathrm{Irr}(\tilde{H}/\tilde{Y} \mid \lambda)$ and note that Y' lies in the centre of $\tilde{H}/\tilde{Y}$. For $a_{8,9,10} \in \mathbb{F}_q^\times$, we let $\lambda^{a_{8,9,10}}$ be the extension of λ to $Y'Z$ with $\lambda^{a_{8,9,10}}(x_{8,9,10}(t)) = \phi(a_{8,9,10}t)$ for every $t \in \mathbb{F}_q$. Then $\mathrm{Irr}(\tilde{H}/\tilde{Y} \mid \lambda)$ decomposes as the union of $\mathrm{Irr}(\tilde{H}/\tilde{Y} \mid \lambda^{a_{8,9,10}})$ over $a_{8,9,10} \in \mathbb{F}_q^\times$ along with $\mathrm{Irr}(\tilde{H}/Y \mid \lambda)$.

A computation in $\tilde{H}/\tilde{Y}$ gives

$$[x_6(s), x_{1,3,4,7}(t)] = x_{8,9,10}(st).$$

By Lemma 1.12, we have that $\mathrm{Irr}(\tilde{H}/\tilde{Y} \mid \lambda^{a_{8,9,10}})$ is in bijection with $\mathrm{Irr}(X_2YZ/\tilde{Y} \mid \lambda^{a_{8,9,10}})$. Further, we have that $X_2YZ/\tilde{Y} \cong X_2Y'Z$ is abelian, and we label the linear characters in

$\mathrm{Irr}(X_2YZ/\tilde{Y} \mid \lambda^{a_{8,9,10}})$ as $\lambda_{b_2}^{a_{11},a_{12},a_{13},a_{8,9,10}}$ in the usual way. This gives the family of characters

$$\mathrm{Irr}(U)_{\mathfrak{C}_5}^{1,p=3} = \{\chi_{b_2}^{a_{11},a_{12},a_{13},a_{8,9,10}} \mid a_{11}, a_{12}, a_{13}, a_{8,9,10} \in \mathbb{F}_q^\times, b_2 \in \mathbb{F}_q\},$$

where by Remark 3.5 we have $\chi_{b_2}^{a_{11},a_{12},a_{13},a_{8,9,10}} = \mathrm{Ind}_{X_2X'YX_{\mathcal{Z}}X_{\mathcal{K}}}^{U} \mathrm{Inf}_{X_2YZ/\tilde{Y}}^{X_2X'YX_{\mathcal{Z}}X_{\mathcal{K}}} \lambda_{b_2}^{a_{11},a_{12},a_{13},a_{8,9,10}}$. We have that $\mathrm{Irr}(U)_{\mathfrak{C}^5}^{1,p=3}$ consists of $q(q-1)^4$ irreducible characters of degree q^4.

It remains to consider $\mathrm{Irr}(\tilde{H}/Y \mid \lambda)$. We have $\tilde{H}/Y = X_2X'X_6YZ/Y$ and X_6 is central in $\tilde{H}/Y$. For $a_6 \in \mathbb{F}_q^\times$, we let $\mu^{a_6} : X_6Z \to \mathbb{F}_q$ be the extension of $\mu : Z \to \mathbb{F}_q$ to X_6 defined as usual, and $\lambda^{a_6} \in \mathrm{Irr}(X_6Z)$ be such that $\lambda^{a_6} = \phi \circ \mu^{a_6}$. Then $\mathrm{Irr}(\tilde{H}/Y \mid \lambda)$ decomposes as the union of $\mathrm{Irr}(\tilde{H}/Y \mid \lambda^{a_6})$ over $a_6 \in \mathbb{F}_q^\times$ along with $\mathrm{Irr}(\tilde{H}/X_6Y \mid \lambda)$.

A computation in $\tilde{H}/Y$ gives

$$[x_2(t), x_{1,3,4,7}(s)] = x_6(-a_{12}st)x_{11}(a_{12}^2a_{13}s^3t).$$

We note that the quotient $\tilde{H}/(Y \ker \mu^{a_6}) = X_2X'X_6YZ/(Y \ker \mu^{a_6})$ is isomorphic to the three-dimensional group V_f where $f(s,t) = a_{12}t(a_{11}a_{12}a_{13}s^3 - a_6s)$ is as given in Section 3.1, and we have that $\mathrm{Irr}(\tilde{H}/Y \mid \lambda^{a_6})$ is in bijection with $\mathrm{Irr}(\tilde{H}/(Y \ker \mu^{a_6}) \mid \lambda^{a_6})$. Thus we can apply the analysis of $\mathrm{Irr}(V_f)$ in Section 3.1. We let $d = a_6/a_{11}a_{12}a_{13}$.

Suppose first that d is a square in $\mathbb{F}_q$. In this case we write $a_{1,6}$ for a_6, and we define $e \in \mathbb{F}_q$ such that $e^2 = d$. We let

$$W_1 = \{x_{1,3,4,7}(es) \mid s \in \mathbb{F}_3\} \quad \text{and} \quad W_2 = \{x_2((e^{-2}/a_{11}a_{12}^2a_{13})t) \mid t \in \mathbb{F}_3\},$$

and we define $\lambda_{c_{1,3,4,7},c_2}^{a_{11},a_{12},a_{13},a_{1,6}}$ for $c_{1,3,4,7}, c_2 \in \mathbb{F}_3$ of $W_1W_2X_6YZ/(Y \ker \lambda^{a_{11},a_{12},a_{13},a_{1,6}})$ as in Section 3.1. Then we get the family of characters

$$\mathrm{Irr}(U)_{\mathfrak{C}^5}^{2,1,p=3} = \{\chi_{c_{1,3,4,7},c_2}^{a_{11},a_{12},a_{13},a_{1,6}} \mid a_{11}, a_{12}, a_{13} \in \mathbb{F}_q^\times, a_{1,6} \in a_{11}a_{12}a_{13}S_q, c_{1,3,4,7}, c_2 \in \mathbb{F}_3\},$$

where

$$\chi_{c_{1,3,4,7},c_2}^{a_{11},a_{12},a_{13},a_{1,6}} = \operatorname{Ind}_{X'W_2X_6YX_{\mathcal{Z}}X_{\mathcal{K}}}^{U} \operatorname{Inf}_{W_1W_2X_6YZ/(Y\ker\lambda^{a_{11},a_{12},a_{13},a_{1,6}})}^{X'W_2X_6YX_{\mathcal{Z}}X_{\mathcal{K}}} \lambda_{c_{1,3,4,7},c_2}^{a_{11},a_{12},a_{13},a_{1,6}}$$

and S_q denotes the set of nonzero squares in $\mathbb{F}_q$. We have that $\operatorname{Irr}(U)_{\mathfrak{C}^5}^{2,1,p=3}$ consists of $9(q-1)^4/2$ characters of degree $q^4/3$.

Suppose now that d is a nonsquare in $\mathbb{F}_q$. In this case we write $a_{2,6}$ for a_6. We write $\lambda^{a_{11},a_{12},a_{13},a_{2,6}}$ for the linear characters of $X_6YZ/(Y\ker\lambda^{a_{11},a_{12},a_{13},a_{2,6}})$ in the usual notation. As explained in Section 3.1, we get in this case the family of characters

$$\operatorname{Irr}(U)_{\mathfrak{C}^5}^{2,2,p=3} = \{\chi^{a_{11},a_{12},a_{13},a_{2,6}} \mid a_{2,6} \in \mathbb{F}_q^\times \setminus (a_{11}a_{12}a_{13}S_q), a_{11}, a_{12}, a_{13} \in \mathbb{F}_q^\times\},$$

where

$$\chi^{a_{11},a_{12},a_{13},a_{2,6}} = \operatorname{Ind}_{X'X_6YX_{\mathcal{Z}}X_{\mathcal{K}}}^{U} \operatorname{Inf}_{X_6YZ/(Y\ker\lambda^{a_{11},a_{12},a_{13},a_{2,6}})}^{X'X_6YX_{\mathcal{Z}}X_{\mathcal{K}}} \lambda^{a_{11},a_{12},a_{13},a_{2,6}},$$

and S_q is as defined above. We have that $\operatorname{Irr}(U)_{\mathfrak{C}^5}^{2,2,p=3}$ consists of $(q-1)^4/2$ characters of degree q^4.

Similarly, we can analyse $\operatorname{Irr}(\tilde{H}/X_6Y \mid \lambda)$ using the arguments for the three-dimensional group V_f where $f(s,t) = a_{11}a_{12}^2a_{13}s^3t$. Therefore, we get the family of characters

$$\operatorname{Irr}(U)_{\mathfrak{C}^5}^{3,p=3} = \{\chi^{a_{11},a_{12},a_{13}} \mid a_{11}, a_{12}, a_{13} \in \mathbb{F}_q^\times\},$$

where the characters are given by

$$\chi^{a_{11},a_{12},a_{13}} = \operatorname{Ind}_{X'X_6YX_{\mathcal{Z}}X_{\mathcal{K}}}^{U} \operatorname{Inf}_{X_6YZ/X_6Y}^{X'X_6YX_{\mathcal{Z}}X_{\mathcal{K}}} \lambda^{a_{11},a_{12},a_{13}}.$$

We have that $\operatorname{Irr}(U)_{\mathfrak{C}^5}^{3,p=3}$ consists of $(q-1)^3$ characters of degree q^4.

Putting this together we obtain

$$\mathrm{Irr}(U)_{\mathfrak{C}^5}^{p=3} = \mathrm{Irr}(U)_{\mathfrak{C}^5}^{1,p=3} \cup \mathrm{Irr}(U)_{\mathfrak{C}^5}^{2,1,p=3} \cup \mathrm{Irr}(U)_{\mathfrak{C}^5}^{2,2,p=3} \cup \mathrm{Irr}(U)_{\mathfrak{C}^5}^{3,p=3}.$$

The nonabelian core $\mathfrak{C}^6$. This core occurs for $\Sigma = \{\alpha_{12}, \alpha_{13}, \alpha_{14}\}$, and we have

- $\mathcal{S} = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{12}, \alpha_{13}, \alpha_{14}\}$,
- $\mathcal{Z} = \Sigma = \{\alpha_{12}, \alpha_{13}, \alpha_{14}\}$,
- $\mathcal{A} = \{\alpha_2\}$ and
- $\mathcal{L} = \{\alpha_{11}\}$.

Using the method of Section 3.2, we take

- $Y = X_{\mathcal{J}}$, where $\mathcal{J} = \{\alpha_1, \alpha_7, \alpha_8, \alpha_9\}$,
- $X = X_{\mathcal{I}}$, where $\mathcal{I} = \{\alpha_4, \alpha_5, \alpha_6, \alpha_{10}\}$ and then we have that
- $H = X_3 Y Z$.

In this case Equation (3.2.3) is

$$s_1(-a_{14}t_4^2 + a_{12}t_{10}) + s_7(-a_{12}t_5 + a_{13}t_6) + s_8(a_{12}t_4 - 2a_{14}t_6) + s_9(a_{13}t_4 + a_{14}t_5) = 0.$$

The computations in order to determine X' and Y' in this case are very similar to the ones detailed in the case of core $\mathfrak{C}^5$. For $p \geq 5$, we have that $Y' = 1$, and Y is normal in $\bar{H}$. Also we have $\bar{H}/Y \cong X_3 X_{\mathcal{Z}}$ is abelian. For $b_3 \in \mathbb{F}_q$ we let $\lambda_{b_3}^{a_{12},a_{13},a_{14}} \in \mathrm{Irr}(X_{\mathcal{S}\setminus(\mathcal{I}\cup\mathcal{J})})_{\mathcal{Z}}$ be the linear character with the usual notation. Then we obtain

$$\mathrm{Irr}(U)_{\mathfrak{C}^6}^{p\geq 5} = \{\chi_{b_3}^{a_{12},a_{13},a_{14}} \mid a_{12}, a_{13}, a_{14} \in \mathbb{F}_q^\times, b_3 \in \mathbb{F}_q\},$$

where $\chi_{b_3}^{a_{12},a_{13},a_{14}} = \mathrm{Ind}^{\mathcal{A}\cup\mathcal{I}} \mathrm{Inf}_{\mathcal{K}\cup\mathcal{J}} \lambda_{b_3}^{a_{12},a_{13},a_{14}}$. We have that $\mathrm{Irr}(U)_{\mathfrak{C}^6}^{p\geq 5}$ is a family of $q(q-1)^3$ characters of degree q^5.

Now suppose $p = 3$. We have $X' = \{x_{4,5,6,10}(t) \mid t \in \mathbb{F}_q\}$ and $Y' = \{x_{7,8,9}(s) \mid s \in \mathbb{F}_q\}$, where

$$x_{4,5,6,10} = x_4(-a_{14}t)x_5(a_{13}t)x_6(a_{12}t)x_{10}(a_{12}a_{14}t^2) \text{ and } x_{7,8,9} = x_7(a_{14}s)x_8(-a_{13}s)x_9(a_{12}s).$$

We can take $\tilde{Y} = X_1X_7X_8$, and we have $\tilde{H}/\tilde{Y} = X_3X'YZ/\tilde{Y}$. By Proposition 3.3, we have that $\mathrm{Irr}(V \mid \lambda)$ is in bijection with $\mathrm{Irr}(\tilde{H}/\tilde{Y} \mid \lambda)$.

A computation in $\tilde{H}/\tilde{Y}$ gives

$$[x_3(s), x_{4,5,6,10}(t)] = x_{7,8,9}(-st).$$

We notice that $\tilde{H}/\tilde{Y}$ is the direct product of Z and the group $X_3X'Y/\tilde{Y} \cong X_3X'Y'$, which is 3-dimensional. Then the analysis in Section 3.1 applies with $f(s,t) = -st$.

We label the linear characters of $X_3X'Y/\tilde{Y}$ by $\chi_{b_3,b_{4,5,6,10}}$. By tensoring these characters with $\lambda^{a_{12},a_{13},a_{14}}$ and then applying $\mathrm{Ind}_{\bar{H}X_{\mathcal{K}}}^{U} \mathrm{Inf}_{\tilde{H}/\tilde{Y}}^{\bar{H}X_{\mathcal{K}}}$ we obtain the family of characters

$$\mathrm{Irr}(U)_{\mathfrak{C}^6}^{1,p=3} = \{\chi_{b_3,b_{4,5,6,10}}^{a_{12},a_{13},a_{14}} \mid a_{12}, a_{13}, a_{14} \in \mathbb{F}_q^\times, b_3, b_{4,5,6,10} \in \mathbb{F}_q\},$$

which consists of $q^2(q-1)^3$ characters of degree q^4.

Let us fix $a_{7,8,9} \in \mathbb{F}_q^\times$. We write $\lambda^{a_{12},a_{13},a_{14},a_{7,8,9}}$ for the linear character of $Y'Z$ defined in the usual way. By applying $\mathrm{Ind}_{X'YX_{\mathcal{Z}}X_{\mathcal{K}}}^{U} \mathrm{Inf}_{YZ/\tilde{Y}}^{X'YX_{\mathcal{Z}}X_{\mathcal{K}}}$ to these linear characters we obtain the family of characters

$$\mathrm{Irr}(U)_{\mathfrak{C}^6}^{2,p=3} = \{\chi^{a_{12},a_{13},a_{14}a_{7,8,9}} \mid a_{12}, a_{13}, a_{14}, a_{7,8,9} \in \mathbb{F}_q^\times\},$$

which consists of $(q-1)^4$ characters of degree q^5.

We have $\mathrm{Irr}(U)_{\mathfrak{C}^6}^{p=3} = \mathrm{Irr}(U)_{\mathfrak{C}^6}^{1,p=3} \cup \mathrm{Irr}(U)_{\mathfrak{C}^6}^{2,p=3}$.

CHAPTER 5

PARAMETRIZATION OF $\mathrm{Irr}(U)$ IN RANK 5 AND HIGHER

In this last chapter, we generalize the ideas of the previous chapters to get a parametrization of the irreducible characters of a Sylow p-subgroup U of a split finite group of Lie type G of rank 5. As we see in Table 2.2, the total number of nonabelian cores in this case gets higher, so we would like to develop a method to deal with larger classes of nonabelian cores with a similar structure. For example, we observe that the cores $\mathfrak{C}^{\mathrm{B}_4}$, $\mathfrak{C}^{\mathrm{D}_4}$ and the core $\mathfrak{C}^4$ of F_4 studied in Chapter 4 have a similar behavior, and in fact they can be simultaneously investigated, as we explain in Section 5.1. In order to do this, we define a triple of invariants associated to a nonabelian core.

Definition 5.1. Let $\mathfrak{C} = (\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ be a nonabelian core. We say that $\mathfrak{C}$ is a (z, m, c)-*core* if

- $|\mathcal{Z} \setminus \mathcal{D}(\mathcal{S})| = z$,
- $|\mathcal{S} \setminus \mathcal{D}(\mathcal{S})| = m$, and
- there are c triples (i, j, k), with $i < j$ and $\alpha_i, \alpha_j, \alpha_k \in \mathcal{S}$, such that $\alpha_i + \alpha_j = \alpha_k$.

Section 5.1 contains the main original feature of this chapter, that is, to prove that the $(3, 10, 9)$-cores that occur in $\mathrm{B}_5, \mathrm{C}_5$ and D_5 can be studied simultaneously. In fact, the study of these cores turns out to be very similar to the study of $\mathfrak{C}^{\mathrm{D}_4}$ in Section 4.2. We apply this result, along with the analysis in Chapter 3, to parametrize $\mathrm{Irr}(U)$ for G of type B_5, C_5 and D_5 in Sections 5.2, 5.3 and 5.4 respectively. Finally, we collect in Section 5.5 all triples (z, m, c) for nonabelian cores in rank 6, and some remarks towards a parametrization of $\mathrm{Irr}(U)$ in ranks higher than 5.

5.1 $(3, 10, 9)$-cores in rank 5 or less

In this section, $\mathfrak{C} = (\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ denotes a $(3, 10, 9)$-core such that $\mathcal{S} = \{\beta_1, \ldots, \beta_{10}\}$ and $\mathcal{Z} = \{\beta_8, \beta_9, \beta_{10}\}$, and the relations between roots of $\mathcal{S}$, for each $1 \leq i < j \leq 10$, are

$$\beta_i + \beta_j = \beta_k \text{ if and only if } (i, j, k) \in \mathcal{T} \text{ for some } k \in \{1, \ldots, 10\},$$

where

$$\mathcal{T} = \{(1,3,5), (1,6,8), (1,7,9), (2,3,6), (2,5,8), (2,7,10), (3,4,7), (4,5,9), (4,6,10)\}.$$

Notice that $\mathcal{D}(\mathcal{S}) = \varnothing$. The configuration is then as in Figure 5.1, with notation for pictures as explained in Section 4.1. Furthermore, we assume that

$$[x_{\beta_i}(t_i), x_{\beta_j}(t_j)] = x_{\beta_k}(\epsilon_{i,j} t_i t_j), \tag{5.1.1}$$

where if $i < j$, then $\epsilon_{i,j} \in \{\pm 1\}$ if $(i, j, k) \in \mathcal{T}$ for some k, and $\epsilon_{i,j} = 0$ if $(i, j, k) \notin \mathcal{T}$ for any k. For the relations in the case $i > j$, we recall that $\epsilon_{i,j} = -\epsilon_{j,i}$.

Remark 5.2. By studying $\mathfrak{C}$, we can completely determine the irreducible characters of U arising from every $(3, 10, 9)$-core for G of rank at most 5. In fact, let $\mathfrak{C}' = (\mathcal{S}', \mathcal{Z}', \mathcal{A}', \mathcal{L}', \mathcal{K}')$

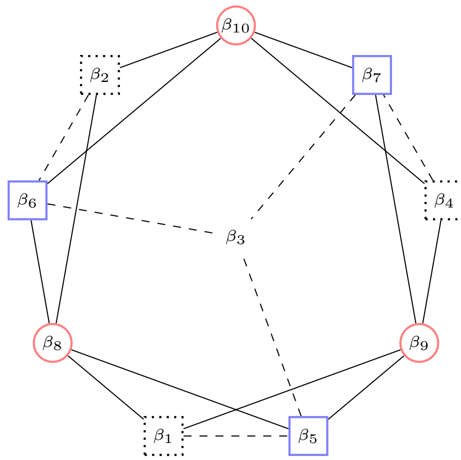

Figure 5.1: A picture representing the $(3, 10, 9)$-core $\mathfrak{C}$.

be an arbitrary $(3, 10, 9)$-core in this case. Then we have that

$$\mathrm{Irr}(X_{\mathcal{S}'})_{\mathcal{Z}'} = \mathrm{Irr}(X_{\mathcal{S}'\setminus\mathcal{D}(\mathcal{S}')})_{\mathcal{Z}'\setminus\mathcal{D}(\mathcal{S}')} \times \mathrm{Irr}(X_{\mathcal{D}(\mathcal{S}')})_{\mathcal{Z}'\cap\mathcal{D}(\mathcal{S}')},$$

and we can check, using CHEVIE, that

$$\overline{\mathfrak{C}} := (\mathcal{S}' \setminus \mathcal{D}(\mathcal{S}'), \mathcal{Z}' \setminus \mathcal{D}(\mathcal{S}'), \mathcal{A}', \mathcal{L}', \mathcal{K}')$$

has the same structure as $\mathfrak{C}$. We can determine character labels for $\mathrm{Irr}(U)_{\overline{\mathfrak{C}}}$ through the procedure described in Section 2.4. Assume that the irreducible characters of $\mathrm{Irr}(U)_{\overline{\mathfrak{C}}}$ are labelled as $\chi^{\underline{a}}_{\underline{b}}$ for $\underline{a} \in (\mathbb{F}_q^\times)^k$ and $\underline{b} \in \mathbb{F}_q^h$ for some $h, k \geq 0$. Then the labels for the irreducible characters in $\mathrm{Irr}(U)_{\mathfrak{C}'}$ are of the form $\chi^{\underline{a},\underline{a}'}_{\underline{b},\underline{b}'}$, where

$$\underline{a}' = (a'_{i_1}, \dots, a'_{i_{k'}}) \in (\mathbb{F}_q^\times)^{k'} \quad \text{and} \quad \underline{b}' = (b'_{j_1}, \dots, b'_{j_{h'}}) \in \mathbb{F}_q^{h'},$$

where the indices are determined by

$$\mathcal{D}(\mathcal{S}') \cap \mathcal{Z}' = \{\alpha_{i_1}, \dots, \alpha_{i_{k'}}\} \quad \text{and} \quad \mathcal{D}(\mathcal{S}') \setminus \mathcal{Z}' = \{\alpha_{j_1}, \dots, \alpha_{j_{h'}}\}.$$

The goal of this section is to prove that even without knowing the exact value of the constants $\epsilon_{i,j} \in \{\pm 1\}$ for $(i,j,k) \in \mathcal{T}$ previously defined, we can still determine the representation theory of $X_{\mathcal{S}}$ in a uniform way. The following result allows us to determine some dependence among the $\epsilon_{i,j}$'s.

Lemma 5.3. *Let $\beta_i, \beta_j, \beta_k \in \mathcal{S}$. Then*

(i) $[x_{\beta_i}(t_i), x_{\beta_j}(t_j)] = [x_{\beta_j}(t_j), x_{\beta_i}(-t_i)]$.

(ii) $[[x_{\beta_i}(t_i), x_{\beta_j}(-t_j)], x_{\beta_k}(t_k)][[x_{\beta_j}(t_j), x_{\beta_k}(-t_k)], x_{\beta_i}(t_i)][[x_{\beta_k}(t_k), x_{\beta_i}(-t_i)], x_{\beta_j}(t_j)] = 1$.

Proof. Using Equation (5.1.1), we get

$$[x_{\beta_i}(t_i), x_{\beta_j}(t_j)] = x_{\beta_k}(\epsilon_{i,j} t_i t_j) = x_{\beta_k}(-\epsilon_{j,i} t_i t_j) = [x_{\beta_j}(t_j), x_{\beta_i}(-t_i)],$$

which proves (i).

For (ii), we recall the Hall–Witt identity [Is2, Section 4B], valid for every group G and every $x, y, z \in G$,

$$[[x, y^{-1}], z]^y [[y, z^{-1}], x]^z [[z, x^{-1}], y]^x = 1.$$

Now we just put $x = x_{\beta_i}(t_i)$, $y = x_{\beta_j}(t_j)$ and $z = x_{\beta_k}(t_k)$, and the claim follows by observing that $[[X_{\mathcal{S}}, X_{\mathcal{S}}], X_{\mathcal{S}}] \subseteq Z(X_{\mathcal{S}})$. □

In order to be precise for our further computations, we need to determine how to write $\epsilon_{2,5}$, $\epsilon_{4,5}$, $\epsilon_{4,6}$ as functions of $\epsilon_{1,3}$, $\epsilon_{1,6}$, $\epsilon_{1,7}$, $\epsilon_{2,3}$, $\epsilon_{2,7}$ and $\epsilon_{3,4}$. We repeatedly use Lemma 5.3(i) in the following computations. We get

$$\begin{aligned} [x_{\beta_2}(t_2), x_{\beta_5}(t_5)] &= [x_{\beta_5}(t_5), x_{\beta_2}(-t_2)] \\ &= [[x_{\beta_1}(t_5), x_{\beta_3}(\epsilon_{1,3})], x_{\beta_2}(-t_2)] \\ &= [[x_{\beta_3}(\epsilon_{1,3}), x_{\beta_2}(t_2)], x_{\beta_1}(t_5)] \end{aligned}$$

$$= [x_{\beta_6}(-\epsilon_{1,3}\epsilon_{2,3}t_2), x_{\beta_1}(t_5)]$$

$$= x_{\beta_8}(\epsilon_{1,3}\epsilon_{2,3}\epsilon_{1,6}t_2t_5),$$

where between lines 2 and 3 we used Lemma 5.3(ii), along with the fact that X_{β_i} and X_{β_j} centralize each other if $i, j \in \{1, 2, 4\}$. Then $\epsilon_{2,5} = \epsilon_{1,3}\epsilon_{2,3}\epsilon_{1,6}$. Similarly, we have

$$\begin{aligned}
[x_{\beta_4}(t_4), x_{\beta_5}(t_5)] &= [x_{\beta_5}(t_5), x_{\beta_4}(-t_4)] \\
&= [[x_{\beta_1}(t_5), x_{\beta_3}(\epsilon_{1,3})], x_{\beta_4}(-t_4)] \\
&= [[x_{\beta_3}(\epsilon_{1,3}), x_{\beta_4}(t_4)], x_{\beta_1}(t_5)] \\
&= [x_{\beta_7}(\epsilon_{1,3}\epsilon_{3,4}t_4), x_{\beta_1}(t_5)] \\
&= x_{\beta_9}(-\epsilon_{1,3}\epsilon_{3,4}\epsilon_{1,7}t_4t_6),
\end{aligned}$$

that means $\epsilon_{4,5} = -\epsilon_{1,3}\epsilon_{3,4}\epsilon_{1,7}$, and

$$\begin{aligned}
[x_{\beta_4}(t_4), x_{\beta_6}(t_6)] &= [x_{\beta_6}(t_6), x_{\beta_4}(-t_4)] \\
&= [[x_{\beta_2}(t_6), x_{\beta_3}(\epsilon_{2,3})], x_{\beta_4}(-t_4)] \\
&= [[x_{\beta_3}(\epsilon_{2,3}), x_{\beta_4}(t_4)], x_{\beta_2}(t_6)] \\
&= [x_{\beta_7}(\epsilon_{2,3}\epsilon_{3,4}t_4), x_{\beta_2}(t_6)] \\
&= x_{\beta_{10}}(-\epsilon_{2,3}\epsilon_{3,4}\epsilon_{2,7}t_4t_6),
\end{aligned}$$

which finally gives $\epsilon_{4,6} = -\epsilon_{2,3}\epsilon_{3,4}\epsilon_{2,7}$. We are ready to prove what was previously claimed.

Proposition 5.4. *Let $\mathfrak{C} = (\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ be as above, and let us define $j_1, \ldots, j_{10}$ such that $\beta_i = \alpha_{j_i}$, for $i = 1, \ldots, 10$. Let $\mathcal{I} = \{j_1, j_2, j_4\}$, $\mathcal{J} = \{j_5, j_6, j_7\}$ and $X = X_{\mathcal{I}}$, $Y = X_{\mathcal{J}}$, and let $H = X_{\beta_3}YX_{\mathcal{Z}}$.*

- *If $p \geq 3$, then we have that*

$$\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}^{p\geq 3} = \{\chi_{b_{j_3}}^{a_{j_8},a_{j_9},a_{j_{10}}} \mid a_{j_8}, a_{j_9}, a_{j_{10}} \in \mathbb{F}_q^\times, b_{j_3} \in \mathbb{F}_q\},$$

where

$$\chi_{b_{j_3}}^{a_{j_8},a_{j_9},a_{j_{10}}} = \mathrm{Ind}^{\mathcal{A}\cup\mathcal{I}} \mathrm{Inf}_{\mathcal{K}\cup\mathcal{J}} \lambda_{b_{j_3}}^{a_{j_8},a_{j_9},a_{j_{10}}},$$

and $\lambda_{b_{j_3}}^{a_{j_8},a_{j_9},a_{j_{10}}} \in \mathrm{Irr}(X_{\beta_3}Z)$ *is as in the notation of Section 2.4.*

- *If* $p = 2$*, then we have*

$$\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}^{p=2} = \mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}^{1,p=2} \cup \mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}^{2,p=2},$$

for

$$\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}^{1,p=2} = \{\chi_{c_{j_1,j_2,j_4},c_{j_3}}^{a_{j_5,j_6,j_7},a_{j_8},a_{j_9},a_{j_{10}}} \mid a_{j_5,j_6,j_7}, a_{j_8}, a_{j_9}, a_{j_{10}} \in \mathbb{F}_q^\times, c_{j_1,j_2,j_4}, c_{j_3} \in \mathbb{F}_2\},$$

and

$$\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}^{2,p=2} = \{\chi^{a_{j_8},a_{j_9},a_{j_{10}}} \mid a_{j_8}, a_{j_9}, a_{j_{10}} \in \mathbb{F}_q^\times\},$$

where

$$\chi_{c_{j_1,j_2,j_4},c_{j_3}}^{a_{j_5,j_6,j_7},a_{j_8},a_{j_9},a_{j_{10}}} = \mathrm{Ind}_{X'W_2YX_{\mathcal{Z}}X_{\mathcal{K}}}^{U} \mathrm{Inf}_{W_1W_2YZ/(\tilde{Y}\ker\lambda^{a_{j_5,j_6,j_7},a_{j_8},a_{j_9},a_{j_{10}}})}^{X'W_2YX_{\mathcal{Z}}X_{\mathcal{K}}} \lambda_{c_{j_1,j_2,j_4},c_{j_3}}^{a_{j_5,j_6,j_7},a_{j_8},a_{j_9},a_{j_{10}}},$$

and

$$\chi^{a_{j_8},a_{j_9},a_{j_{10}}} = \mathrm{Ind}_{X'YX_{\mathcal{Z}}X_{\mathcal{K}}}^{U} \mathrm{Inf}_{Z}^{X'YX_{\mathcal{Z}}X_{\mathcal{K}}} \lambda^{a_{j_8},a_{j_9},a_{j_{10}}},$$

for $X' = \{x_{\beta_1,\beta_2,\beta_4}(t) \mid t \in \mathbb{F}_q\}$ *and* $Y' = \{x_{\beta_5,\beta_6,\beta_7}(s) \mid s \in \mathbb{F}_q\}$*, with*

$$x_{\beta_1,\beta_2,\beta_4}(t) = x_{\beta_1}(a_{j_{10}}t)x_{\beta_2}(a_{j_9}t)x_{\beta_4}(a_{j_8}t), \quad x_{\beta_5,\beta_6,\beta_7}(s) = x_{\beta_5}(a_{j_{10}}s)x_{\beta_6}(a_{j_9}s)x_{\beta_7}(a_{j_8}s)$$

and $\tilde{Y} = X_{\beta_6}X_{\beta_7}$, $Z = X_{\mathcal{Z}}/\ker(\mu^{a_{j_8},a_{j_9},a_{10}})$, *and*

$$W_1 = \{1, x_{\beta_1,\beta_2,\beta_4}(a_{j_5,j_6,j_7}/a_{j_8}a_{j_9}a_{j_{10}})\}, \quad W_2 = \{1, x_{\beta_3}(a_{j_8}a_{j_9}a_{j_{10}}/a^2_{j_5,j_6,j_7})\},$$

with the usual notation for $\lambda^{a_{j_5,j_6,j_7},a_{j_8},a_{j_9},a_{j_{10}}}_{c_{j_1,j_2,j_4},c_{j_3}} \in \mathrm{Irr}(W_1W_2YZ/(\tilde{Y}\ker\lambda^{a_{j_5,j_6,j_7},a_{j_8},a_{j_9},a_{j_{10}}}))$, $\mu^{a_{j_8},a_{j_9},a_{j_{10}}} : Z \to \mathbb{F}_q$ *and* $\lambda^{a_{j_8},a_{j_9},a_{j_{10}}} \in \mathrm{Irr}(Z)$.

Proof. Let $\lambda = \lambda^{a_{j_8},a_{j_9},a_{j_{10}}}$ be such that $\lambda(x_{\beta_i}(t)) = \phi(a_{j_i}t)$, for $i = 8, 9, 10$. We want to apply the reduction lemma with Y and X as previously stated. Then we only need to check assumption (v). Expanding

$$[x_{\beta_1}(t_1)x_{\beta_2}(t_2)x_{\beta_4}(t_4), x_{\beta_5}(s_5)x_{\beta_6}(s_6)x_{\beta_7}(s_7)] = 1,$$

we get

$$x_{\beta_8}(\epsilon_{1,6}s_6t_1 + \epsilon_{2,5}s_5t_2)x_{\beta_9}(\epsilon_{1,7}s_7t_1 + \epsilon_{4,5}s_5t_4)x_{\beta_{10}}(\epsilon_{2,7}s_7t_2 + \epsilon_{4,6}s_6t_4) = 1.$$

By applying μ, this gives

$$s_5(\epsilon_{2,5}a_{j_8}t_2 + \epsilon_{4,5}a_{j_9}t_4) + s_6(\epsilon_{1,6}a_{j_8}t_1 + \epsilon_{4,6}a_{j_{10}}t_4) + s_7(\epsilon_{1,7}a_{j_9}t_1 + \epsilon_{2,7}a_{j_{10}}t_2) = 0.$$

Let us find X' and Y' as defined in Section 3.2. In order to find X', we want the above equation to be satisfied for all $s_5, s_6, s_7 \in \mathbb{F}_q$. The linear system of equations in the t_1, t_2, t_4 and associated matrix are

$$\begin{cases} \epsilon_{2,5}a_{j_8}t_2 + \epsilon_{4,5}a_{j_9}t_4 = 0 \\ \epsilon_{1,6}a_{j_8}t_1 + \epsilon_{4,6}a_{j_{10}}t_4 = 0 \\ \epsilon_{1,7}a_{j_9}t_1 + \epsilon_{2,7}a_{j_{10}}t_2 = 0 \end{cases} \quad \text{and} \quad M = \begin{pmatrix} 0 & \epsilon_{2,5}a_{j_8} & \epsilon_{4,5}a_{j_9} \\ \epsilon_{1,6}a_{j_8} & 0 & \epsilon_{4,6}a_{j_{10}} \\ \epsilon_{1,7}a_{j_9} & \epsilon_{2,7}a_{j_{10}} & 0 \end{pmatrix}.$$

By using the expressions for $\epsilon_{2,5}$, $\epsilon_{4,5}$ and $\epsilon_{4,6}$ previously obtained, one gets

$$\det M = (\epsilon_{1,6}\epsilon_{2,7}\epsilon_{4,5} + \epsilon_{1,7}\epsilon_{2,5}\epsilon_{4,6})a_{j_8}a_{j_9}a_{j_{10}} = -2\epsilon_{1,3}\epsilon_{1,6}\epsilon_{1,7}\epsilon_{2,7}\epsilon_{3,4}a_{j_8}a_{j_9}a_{j_{10}}.$$

Let $p \neq 2$. Then clearly $\det M \neq 0$ and $X' = 1$. Similarly, we can check that $Y' = 1$. Therefore, the reduction lemma applies. Repeating the discussion as in $\mathfrak{C}^{\mathrm{D}_4}$ in Section 4.2 with β_i in place of α_i for $i = 1, \dots, 10$, we get the claim for $p \neq 2$.

Now if $p = 2$, then $\epsilon_{i,j} = 1$ for every i and j with $(i,j,k) \in \mathcal{T}$. The claim now again follows from the computations in the case of $\mathfrak{C}^{\mathrm{D}_4}$. □

An important consequence of Proposition 5.4 is that we can also parametrize $(3, 9, 6)$-cores that arise from $(3, 10, 9)$-cores with the same structure as $\mathfrak{C}$ by removing β_3 from $\mathcal{S}$. We state the result in the case when no roots in the corresponding quattern are in direct product; the argument in Remark 5.2 applies to generalize the result to every $(3, 9, 6)$-core arising from a $(3, 10, 9)$-core with the same structure as the core $\mathfrak{C}'$ defined in the remark, by removing the root that plays the role of β_3 in $\mathfrak{C}'$.

Proposition 5.5. *Let $\mathfrak{C} = (\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ be a $(3, 10, 9)$-core as above, and let us suppose that $\mathfrak{D} = (\mathcal{S}', \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ is a $(3, 9, 6)$-core such that $\mathcal{S}' = \mathcal{S} \setminus \{\beta_3\}$.*

- *If $p \geq 3$, then we have that*

$$\mathrm{Irr}(X_{\mathcal{S}'})_{\mathcal{Z}}^{p\geq 3} = \{\chi^{a_{j_8}, a_{j_9}, a_{j_{10}}} \mid a_{j_8}, a_{j_9}, a_{j_{10}} \in \mathbb{F}_q^\times\},$$

  where

$$\chi^{a_{j_8}, a_{j_9}, a_{j_{10}}} = \mathrm{Ind}^{\mathcal{A}\cup\mathcal{I}} \mathrm{Inf}_{\mathcal{K}\cup\mathcal{J}} \lambda^{a_{j_8}, a_{j_9}, a_{j_{10}}},$$

  and $\mathcal{I}$, $\mathcal{J}$ are as in Proposition 5.4, and $\lambda^{a_{j_8}, a_{j_9}, a_{j_{10}}} \in \mathrm{Irr}(Z)$ is as in the notation of Section 2.4.

- *If $p = 2$, then we have*

$$\mathrm{Irr}(X_{\mathcal{S}'})_{\mathcal{Z}}^{2,p=2} = \{\chi_{b_{j_1,j_2,j_4},b_{j_5,j_6,j_7}}^{a_{j_8},a_{j_9},a_{j_{10}}} \mid a_{j_8}, a_{j_9}, a_{j_{10}} \in \mathbb{F}_q^{\times}, b_{j_1,j_2,j_4}, b_{j_5,j_6,j_7} \in \mathbb{F}_q\},$$

where

$$\chi_{b_{j_1,j_2,j_4},b_{j_5,j_6,j_7}}^{a_{j_8},a_{j_9},a_{j_{10}}} = \mathrm{Ind}_{X'YX_{\mathcal{Z}}X_{\mathcal{K}}}^{U} \mathrm{Inf}_{X'Y'Z}^{X'YX_{\mathcal{Z}}X_{\mathcal{K}}} \lambda_{b_{j_1,j_2,j_4},b_{j_5,j_6,j_7}}^{a_{j_8},a_{j_9},a_{j_{10}}},$$

and X, Y, X', Y' and Z are as in Proposition 5.4, and $\lambda_{b_{j_1,j_2,j_4},b_{j_5,j_6,j_7}}^{a_{j_8},a_{j_9},a_{j_{10}}} \in \mathrm{Irr}(X'Y'Z)$ is defined as usual.

The proof of this result follows by applying the reduction lemma to $X_{\mathcal{S}}$ inflating over Y and inducing over X in the case $p \geq 3$, and inflating over $\tilde{Y} = X_{\beta_6}X_{\beta_7}$ and inducing over any transversal $\tilde{X}$ of X' if $p = 2$. The claim then immediately follows from the computations in Proposition 5.4.

5.2 Parametrization of $\mathrm{Irr}(U_{\mathrm{B}_5})$

In the case of type B_5 we get 10 nonabelian cores, as we see from Table 2.2. Namely, we have eight $(3, 10, 9)$-cores, one $(3, 9, 6)$-core, and one $(5, 16, 16)$-core. The $(3, 10, 9)$-cores and the $(3, 9, 6)$-core are of the form considered in Section 5.1; we can then determine at once the parametrization of the irreducible characters arising from these classes of cores. The $(5, 16, 16)$-core is dealt with via the method outlined in Section 3.2.

The study of these cores allows us to complete the parametrization of $\mathrm{Irr}(U_{\mathrm{B}_5})$. The labels for irreducible characters can be found in Table D.5. As a consequence, we obtain the formulas for $k(U_{\mathrm{B}_5}(q), q^d)$ for $d \geq 0$ as polynomials in v. They are collected in Table 5.1.

The eight $(3, 10, 9)$**-cores.** We collect in Table C.1 the relevant information for the $(3, 10, 9)$-cores in type B_5, which we call $\mathfrak{C}_1^{\mathrm{B}_5}, \ldots, \mathfrak{C}_8^{\mathrm{B}_5}$. From Table C.1, we see that no commutator

D	$k(U_{\mathrm{B}_5}, D)$
1	$v^5 + 5v^4 + 10v^3 + 10v^2 + 5v + 1$
q	$4v^5 + 17v^4 + 27v^3 + 19v^2 + 5v$
q^2	$v^7 + 8v^6 + 29v^5 + 56v^4 + 56v^3 + 26v^2 + 4v$
q^3	$2v^7 + 15v^6 + 49v^5 + 83v^4 + 72v^3 + 29v^2 + 4v$
q^4	$v^8 + 9v^7 + 37v^6 + 87v^5 + 119v^4 + 87v^3 + 29v^2 + 3v$
q^5	$3v^7 + 23v^6 + 70v^5 + 105v^4 + 77v^3 + 24v^2 + 2v$
q^6	$9v^6 + 48v^5 + 90v^4 + 71v^3 + 21v^2 + v$
q^7	$v^8 + 8v^7 + 31v^6 + 73v^5 + 96v^4 + 57v^3 + 11v^2 + v$
q^8	$v^7 + 7v^6 + 23v^5 + 37v^4 + 26v^3 + 7v^2$
q^9	$2v^6 + 10v^5 + 19v^4 + 14v^3 + 3v^2$
q^{10}	$v^5 + 3v^4 + 3v^3 + v^2$

Table 5.1: Numbers of irreducible characters of U_{B_5} of fixed degree, for $v = q - 1$ and $p \neq 2$.

relations $[x_i(t_i), x_j(t_j)] = x_k(\pm 2t_i t_j)$ are involved. Therefore Proposition 5.4 applies. Labels for irreducible characters are obtained as in the statement of Proposition 5.4; these are collected in bold font in Table D.5.

The $(3, 9, 6)$**-core.** This core, which we denote by $\mathfrak{C}_0^{\mathrm{B}_5}$, occurs for $\Sigma = \{\alpha_{17}, \alpha_{21}\}$, and we have

- $\mathcal{S} = \{\alpha_2, \alpha_7, \alpha_9, \alpha_{10}, \alpha_{13}, \alpha_{14}, \alpha_{16}, \alpha_{17}, \alpha_{21}\} \cup \mathcal{D}(\mathcal{S})$,
- $\mathcal{Z} = \{\alpha_{16}, \alpha_{17}, \alpha_{21}\} \cup \mathcal{D}(\mathcal{S})$,
- $\mathcal{A} = \{\alpha_1, \alpha_3, \alpha_5, \alpha_6\}$ and
- $\mathcal{L} = \{\alpha_4, \alpha_{11}, \alpha_{18}, \alpha_{19}\}$,

and $\mathcal{D}(\mathcal{S}) = \{\alpha_8\}$. We see that $\mathcal{S} \setminus \mathcal{D}(\mathcal{S})$ is contained in the quattern corresponding to $\mathfrak{C}_2^{\mathrm{B}_5}$ or $\mathfrak{C}_3^{\mathrm{B}_5}$, namely by removing the root α_3, playing the role of β_3 in Proposition 5.5. Therefore, Proposition 5.5 applies. The corresponding family of characters is given in bold font in Table D.5.

The $(5, 16, 16)$**-core.** In order to study this core we apply the analysis outlined in Section 3.2. Since this is a distinguished nonabelian core of B_5, we call it $\mathfrak{C}^{\mathrm{B}_5}$. It occurs for $\Sigma = \{\alpha_{17}, \alpha_{21}\}$, and we have

- $\mathcal{S} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15}, \alpha_{16}, \alpha_{17}, \alpha_{21}\}$,
- $\mathcal{Z} = \{\alpha_{12}, \alpha_{15}, \alpha_{16}, \alpha_{17}, \alpha_{21}\}$,
- $\mathcal{A} = \{\alpha_1, \alpha_5\}$ and
- $\mathcal{L} = \{\alpha_{18}, \alpha_{19}\}$.

We take

- $Y = X_{\mathcal{J}}$, where $\mathcal{J} = \{\alpha_7, \alpha_8, \alpha_{11}, \alpha_{13}, \alpha_{14}\}$,
- $X = X_{\mathcal{I}}$, where $\mathcal{I} = \{\alpha_2, \alpha_4, \alpha_6, \alpha_9, \alpha_{10}\}$ and then we have that
- $H = X_3 Y Z$.

In this case Equation (3.2.3) is

$$s_7(-a_{12}t_4 - a_{16}t_9 - a_{17}t_{10}) + s_8(a_{12}t_2 + a_{15}t_6) + s_{11}(-a_{15}t_4 - 2a_{17}t_6) \\ + s_{13}(a_{16}t_2 + a_{21}t_{10}) + s_{14}(-a_{17}t_2 - a_{21}t_9) = 0.$$

Let $a_{21} \neq a_{16}(a_{15}/a_{12})^2$. We have that $Y' = 1$ and Y is normal in $\bar{H}$. Moreover, $\bar{H}/Y = X_3 X_{\mathcal{Z}}$. As explained in Remark 3.5, we get the family of characters

$$\mathrm{Irr}(U)^1_{\mathfrak{C}^{\mathrm{B}_5}} = \{\chi_{b_3}^{a_{12},a_{15},a_{16},a_{17},a_{21}^*} \mid a_{12}, a_{15}, a_{16}, a_{17}, a_{21}^* \in \mathbb{F}_q^\times, a_{21} \neq a_{16}(a_{15}/a_{12})^2, b_3 \in \mathbb{F}_q\},$$

where

$$\chi_{b_3}^{a_{12},a_{15},a_{16},a_{17},a_{21}^*} = \operatorname{Ind}^{\mathcal{A}\cup\mathcal{I}} \operatorname{Inf}_{\mathcal{K}\cup\mathcal{J}} \lambda_{b_3}^{a_{12},a_{15},a_{16},a_{17},a_{21}^*},$$

and $\lambda_{b_3}^{a_{12},a_{15},a_{16},a_{17},a_{21}^*} \in \operatorname{Irr}(\bar{H}/Y)$ is defined in the usual way. We have that $\operatorname{Irr}(U)_{\mathfrak{C}^{\mathrm{B}_5}}^1$ consists of $q(q-1)^4(q-2)$ characters of degree q^7.

Assume now that $a_{21} = a_{16}(a_{15}/a_{12})^2$. We have $X' = \{x_{2,4,6,9,10}(t) \mid t \in \mathbb{F}_q\}$ and $Y' = \{x_{7,8,11,13,14}(s) \mid s \in \mathbb{F}_q\}$, where

$$x_{2,4,6,9,10}(t) = x_2(-a_{15}^2 a_{21} t)x_4(-2a_{12}a_{17}a_{21}t)x_6(a_{12}a_{15}a_{21}t)x_9(a_{15}^2 a_{17}t)x_{10}(a_{15}^2 a_{16}t),$$

and

$$x_{7,8,11,13,14}(s) = x_7(-a_{15}^2 a_{21}s)x_8(2a_{12}a_{17}a_{21}s)x_{11}(a_{12}a_{15}a_{21}s)x_{13}(-a_{15}^2 a_{17}s)x_{14}(a_{15}^2 a_{16}s).$$

We can take any complement of Y' in Y, for example we fix $\tilde{Y} = X_7X_8X_{11}X_{13}$. Then we have $\tilde{H}/\tilde{Y} = X_3X'YZ/\tilde{Y}$ and $Y' \subseteq Z(\tilde{H}/\tilde{Y})$. From now on, we denote by $\lambda^{a_{12},a_{15},a_{16},a_{17}}$ the character $\lambda^{a_{12},a_{15},a_{16},a_{17},a_{21}}$ of $\operatorname{Irr}(Z)$ such that $a_{21} = a_{16}(a_{15}/a_{12})^2$.

A computation in $\tilde{H}/\tilde{Y}$ gives

$$[x_3(s), x_{2,4,6,9,10}(t)] = x_{7,8,11,13,14}(st).$$

The subquotient $\tilde{H}/\tilde{Y}$ can then be decomposed as a direct product of Z with the subquotient $X_3X'Y/\tilde{Y}$, which is isomorphic to the three-dimensional group V_f for $f(s,t) = st$ as defined in Section 3.1.

We label the linear characters of $X_1X'Y/\tilde{Y}$ by $\chi_{b_3,b_{2,4,6,9,10}}$. By tensoring these characters with $\lambda^{a_{12},a_{15},a_{16},a_{17}}$, and then applying $\operatorname{Ind}_{\tilde{H}X_{\mathcal{K}}}^{U} \operatorname{Inf}_{\tilde{H}/\tilde{Y}}^{\tilde{H}X_{\mathcal{K}}}$ we obtain the family of characters

$$\operatorname{Irr}(U)_{\mathfrak{C}^{\mathrm{B}_5}}^2 = \{\chi_{b_3,b_{2,4,6,9,10}}^{a_{12},a_{15},a_{16},a_{17}} \mid a_{12},a_{15},a_{16},a_{17} \in \mathbb{F}_q^\times, b_3, b_{2,4,6,9,10} \in \mathbb{F}_q\},$$

which consists of $q^2(q-1)^4$ characters of degree q^6.

We write $\lambda^{a_{12},a_{15},a_{16},a_{17},a_{7,8,11,13,14}}$ for the linear character of $Y'Z$ defined by extending $\lambda^{a_{12},a_{15},a_{16},a_{17}}$ to Y' nontrivially in the usual way. We apply $\mathrm{Ind}_{X'YX_{\mathcal{Z}}X_{\mathcal{K}}}^{U} \mathrm{Inf}_{Y'Z}^{X'YX_{\mathcal{Z}}X_{\mathcal{K}}}$ to these linear characters, and we finally obtain the family

$$\mathrm{Irr}(U)^3_{\mathfrak{C}^{\mathrm{B}_5}} = \{\chi^{a_{12},a_{15},a_{16},a_{17},a_{7,8,11,13,14}} \mid a_{12}, a_{15}, a_{16}, a_{17}, a_{7,8,11,13,14} \in \mathbb{F}_q^\times\},$$

which consists of $(q-1)^5$ characters of degree q^7.

We have $\mathrm{Irr}(U)_{\mathfrak{C}^{\mathrm{B}_5}} = \mathrm{Irr}(U)^1_{\mathfrak{C}^{\mathrm{B}_5}} \cup \mathrm{Irr}(U)^2_{\mathfrak{C}^{\mathrm{B}_5}} \cup \mathrm{Irr}(U)^3_{\mathfrak{C}^{\mathrm{B}_5}}$ and this gives all the irreducible characters corresponding to $\mathfrak{C}^{\mathrm{B}_5}$.

5.3 Parametrization of $\mathrm{Irr}(U_{\mathrm{C}_5})$

We get only one nonabelian core in type C_5, which we denote by $\mathfrak{C}^{\mathrm{C}_5}$. This is a $(3,8,6)$-core, which is studied by applying the method outlined in Section 3.2. We obtain the missing character labels arising from the antichain $\Sigma = \{\alpha_{13}, \alpha_{22}\}$. This completes the parametrization of $\mathrm{Irr}(U_{\mathrm{C}_5})$. The labels for irreducible characters are collected in Table D.6. The expressions of $k(U_{\mathrm{C}_5}, q^d)$ for $d = 0, \dots, 10$ as polynomials in v are collected in Table 5.2.

The $(3,8,6)$-core. This core occurs for $\Sigma = \{\alpha_{13}, \alpha_{22}\}$, and we have

- $\mathcal{S} = \{\alpha_1, \alpha_2, \alpha_6, \alpha_7, \alpha_{10}, \alpha_{11}, \alpha_{14}, \alpha_{17}\} \cup \mathcal{D}(\mathcal{S})$,
- $\mathcal{Z} = \{\alpha_{10}, \alpha_{14}, \alpha_{17}\} \cup \mathcal{D}_1(\mathcal{S})$,
- $\mathcal{A} = \{\alpha_3, \alpha_4, \alpha_8, \alpha_{12}\}$ and
- $\mathcal{L} = \{\alpha_9, \alpha_{15}, \alpha_{18}, \alpha_{20}\}$,

with $\mathcal{D}(\mathcal{S}) = \mathcal{D}_1(\mathcal{S}) \sqcup \mathcal{D}_2(\mathcal{S})$, for $\mathcal{D}_1(\mathcal{S}) = \{\alpha_{13}, \alpha_{22}\}$ and $\mathcal{D}_2(\mathcal{S}) = \{\alpha_5\}$. We take

D	$k(U_{\mathrm{C}_5}, D)$
1	$v^5 + 5v^4 + 10v^3 + 10v^2 + 5v + 1$
q	$v^6 + 8v^5 + 23v^4 + 31v^3 + 20v^2 + 5v$
q^2	$v^7 + 8v^6 + 29v^5 + 55v^4 + 54v^3 + 25v^2 + 4v$
q^3	$v^7 + 12v^6 + 48v^5 + 88v^4 + 78v^3 + 31v^2 + 4v$
q^4	$v^8 + 9v^7 + 38v^6 + 91v^5 + 123v^4 + 86v^3 + 27v^2 + 3v$
q^5	$3v^7 + 24v^6 + 74v^5 + 110v^4 + 80v^3 + 25v^2 + 2v$
q^6	$v^8 + 8v^7 + 32v^6 + 78v^5 + 105v^4 + 69v^3 + 18v^2 + v$
q^7	$2v^7 + 14v^6 + 44v^5 + 68v^4 + 46v^3 + 12v^2 + v$
q^8	$3v^6 + 17v^5 + 34v^4 + 26v^3 + 7v^2$
q^9	$4v^5 + 14v^4 + 14v^3 + 4v^2$
q^{10}	$v^5 + 5v^4 + 6v^3 + v^2$

Table 5.2: Numbers of irreducible characters of U_{C_5} of fixed degree, for $v = q - 1$ and $p \neq 2$.

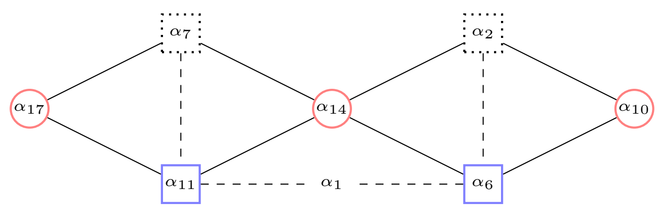

Figure 5.2: A picture representing the configuration of $\mathfrak{C}^{\mathrm{C}_5}$.

- $Y = X_{\mathcal{J}}$, where $\mathcal{J} = \{\alpha_6, \alpha_{11}\}$,
- $X = X_{\mathcal{I}}$, where $\mathcal{I} = \{\alpha_2, \alpha_7\}$ and then we have that
- $H = X_1 Y Z$.

Equation (3.2.3) now gives

$$s_6(-2a_{10}t_2 - a_{14}t_7) + s_{11}(-a_{14}t_2 - 2a_{17}t_7) = 0.$$

Let $a_{17} \neq a_{14}^2/(4a_{10})$. We have $Y' = 1$ and $Y \trianglelefteq \bar{H}$. Moreover, $\bar{H}/Y = X_1 X_{\mathcal{Z}}$. Tensoring with characters in $X_{\mathcal{D}(\mathcal{S})}$ and applying Remark 3.5, we get the family of characters

$$\mathrm{Irr}(U)^1_{\mathfrak{C}^{\mathrm{C}_5}} = \{\chi^{a_{10},a_{13},a_{14},a^*_{17},a_{22}}_{b_1,b_5} \mid a_{10}, a_{13}, a_{14}, a^*_{17}, a_{22} \in \mathbb{F}_q^\times, a_{17} \neq a_{14}^2/(4a_{10}), b_1, b_5 \in \mathbb{F}_q\}$$

of $\mathrm{Irr}(U)_{\mathfrak{C}^{\mathrm{C}_5}}$, where

$$\chi^{a_{10},a_{13},a_{14},a^*_{17},a_{22}}_{b_1,b_5} = \mathrm{Ind}^{\mathcal{A}\cup\mathcal{I}} \mathrm{Inf}_{\mathcal{K}\cup\mathcal{J}}(\lambda^{a_{10},a_{14},a^*_{17}}_{b_1} \otimes \lambda^{a_{13},a_{22}}_{b_5}),$$

and $\lambda^{a_{10},a_{14},a^*_{17}}_{b_1} \in \mathrm{Irr}(\bar{H}/Y)$ and $\lambda^{a_{13},a_{22}}_{b_5} \in \mathrm{Irr}(X_{\mathcal{D}(\mathcal{S})})_{\mathcal{D}_1(\mathcal{S})}$ are defined in the usual way. We have that $\mathrm{Irr}(U)^1_{\mathfrak{C}^{\mathrm{C}_5}}$ consists of $q^2(q-1)^4(q-2)$ characters of degree q^6.

We now examine the case $a_{17} = a_{14}^2/(4a_{10})$. Here we have that $X' = \{x_{2,7}(t) \mid t \in \mathbb{F}_q\}$ and $Y' = \{x_{6,11}(s) \mid s \in \mathbb{F}_q\}$, where

$$x_{2,7}(t) = x_2(a_{14}t)x_7(-2a_{10}t) \quad \text{and} \quad x_{6,11}(s) = x_6(a_{14}s)x_{11}(-2a_{10}s).$$

We can take $\tilde{Y} = X_6$ as a complement for Y' in Y. This gives $\tilde{H}/\tilde{Y} = X_1 X' Y Z X_{\mathcal{D}(\mathcal{S})}/\tilde{Y}$ and $Y' \subseteq Z(\tilde{H}/\tilde{Y})$. We denote by $\lambda^{a_{10},a_{14}}$ the character $\lambda^{a_{10},a_{14},a_{17}} \in \mathrm{Irr}(Z)$ such that $a_{17} = a_{14}^2/(4a_{10})$.

A computation in $\tilde{H}/\tilde{Y}$ gives

$$[x_1(s), x_{2,7}(t)] = x_{6,11}(-st).$$

We then get $\tilde{H}/\tilde{Y}$ as the direct product of the three subgroups Z, $X_1X'Y/\tilde{Y}$ and $X_{\mathcal{D}(\mathcal{S})}$. Moreover, $X_1X'Y/\tilde{Y}$ is isomorphic to the three-dimensional group V_f for $f(s,t) = -st$ as in Section 3.1.

We label the linear characters of $X_1X'Y/\tilde{Y}$ by $\chi_{b_1,b_{2,7}}$. By tensoring these characters with $\lambda^{a_{10},a_{14}} \in \mathrm{Irr}(Z)$ and $\lambda_{b_5}^{a_{13},a_{22}} \in \mathrm{Irr}(X_{\mathcal{D}(\mathcal{S})})_{\mathcal{D}_1(\mathcal{S})}$, and then applying $\mathrm{Ind}_{\bar{H}X_{\mathcal{K}}}^{U} \mathrm{Inf}_{\tilde{H}/\tilde{Y}}^{\bar{H}X_{\mathcal{K}}}$ we obtain the family of characters

$$\mathrm{Irr}(U)^2_{\mathfrak{C}^3} = \{\chi_{b_1,b_5,b_{2,7}}^{a_{10},a_{13},a_{14},a_{22}} \mid a_{10}, a_{13}, a_{14}, a_{22} \in \mathbb{F}_q^\times, b_1, b_5, b_{2,7} \in \mathbb{F}_q\},$$

which consists of $q^3(q-1)^4$ characters of degree q^5.

Let us fix $a_{6,11} \in \mathbb{F}_q^\times$. We write $\lambda^{a_{10},a_{14},a_{6,11}}$ for the linear character of $Y'Z$ defined in the usual way. If we apply $\mathrm{Ind}_{X'YX_{\mathcal{Z}}X_{\mathcal{K}}}^{U} \mathrm{Inf}_{Y'Z}^{X'YX_{\mathcal{Z}}X_{\mathcal{K}}}$ to the characters $\lambda^{a_{10},a_{14},a_{6,11}} \otimes \lambda_{b_5}^{a_{13},a_{22}}$, for $a_{13}, a_{22} \in \mathbb{F}_q^\times$ and $b_5 \in \mathbb{F}_q$, we obtain the family

$$\mathrm{Irr}(U)^3_{\mathfrak{C}^3} = \{\chi_{b_5}^{a_{10},a_{13},a_{14},a_{22},a_{6,11}} \mid a_{10}, a_{13}, a_{14}, a_{22}, a_{6,11} \in \mathbb{F}_q^\times, b_5 \in \mathbb{F}_q\},$$

which consists of $q(q-1)^5$ characters of degree q^6.

We have $\mathrm{Irr}(U)_{\mathfrak{C}^{\mathrm{C}_5}} = \mathrm{Irr}(U)^1_{\mathfrak{C}^{\mathrm{C}_5}} \cup \mathrm{Irr}(U)^2_{\mathfrak{C}^{\mathrm{C}_5}} \cup \mathrm{Irr}(U)^3_{\mathfrak{C}^{\mathrm{C}_5}}$, which gives all the irreducible characters corresponding to $\mathfrak{C}^{\mathrm{C}_5}$.

5.4 Parametrization of $\mathrm{Irr}(U_{\mathrm{D}_5})$

We get 7 nonabelian cores in type D_5, namely six $(3, 10, 9)$-cores and one $(3, 9, 6)$-core. As in the case of type B_5, these cores are of the form considered in Section 5.1. We now proceed to investigate them, so we can fully parametrize $\mathrm{Irr}(U_{\mathrm{D}_5})$. The labels for irreducible characters are collected in Table D.7. We obtain new formulas for $k(\mathrm{UD}_5(q), D)$ when $q = 2^e$, for $D = 1, q, \dots, q^8$ and $D = q^3/2, \dots, q^7/2$, which are collected in Table 5.3 along with each $k(\mathrm{UD}_5(q), q^d)$ for $p \geq 3$.

The six $(3, 10, 9)$-cores. We label these cores as $\mathfrak{C}_1^{\mathrm{D}_5}, \dots, \mathfrak{C}_6^{\mathrm{D}_5}$. As in the case of type B_5,

D	$k(\mathrm{UD}_5(q), D)$
1	$v^5 + 5v^4 + 10v^3 + 10v^2 + 5v + 1$
q	$v^6 + 7v^5 + 19v^4 + 25v^3 + 16v^2 + 4v$
q^2	$2v^6 + 14v^5 + 35v^4 + 40v^3 + 21v^2 + 4v$
$q^3/2$	0, if $p \geq 3$ $4v^5 + 4v^4$, if $p = 2$
q^3	$v^7 + 8v^6 + 29v^5 + 54v^4 + 50v^3 + 21v^2 + 3v$, if $p \geq 3$ $v^7 + 8v^6 + 28v^5 + 53v^4 + 50v^3 + 21v^2 + 3v$, if $p = 2$
$q^4/2$	0, if $p \geq 3$ $4v^5 + 4v^4$, if $p = 2$
q^4	$2v^6 + 17v^5 + 42v^4 + 42v^3 + 17v^2 + 2v$, if $p \geq 3$ $3v^6 + 18v^5 + 42v^4 + 42v^3 + 17v^2 + 2v$, if $p = 2$
$q^5/2$	0, if $p \geq 3$ $4v^5 + 4v^4$, if $p = 2$
q^5	$v^7 + 8v^6 + 29v^5 + 53v^4 + 43v^3 + 14v^2 + v$, if $p \geq 3$ $v^7 + 8v^6 + 28v^5 + 51v^4 + 43v^3 + 14v^2 + v$, if $p = 2$
$q^6/2$	0, if $p \geq 3$ $4v^5 + 4v^4$, if $p = 2$
q^6	$v^6 + 7v^5 + 18v^4 + 18v^3 + 7v^2 + v$, if $p \geq 3$ $v^6 + 6v^5 + 17v^4 + 18v^3 + 7v^2 + v$, if $p = 2$
$q^7/2$	0, if $p \geq 3$ $4v^4$, if $p = 2$
q^7	$2v^5 + 8v^4 + 10v^3 + 3v^2$, if $p \geq 3$ $2v^5 + 7v^4 + 10v^3 + 3v^2$, if $p = 2$
q^8	$v^4 + 2v^3 + v^2$

Table 5.3: Numbers of irreducible characters of $\mathrm{UD}_5(q)$ of fixed degree, $v = q - 1$.

we collect in Table C.2 the relevant information for them. We are in a simply laced case, so each of the constants $c_{i,j}^{\alpha,\beta}$ as in Equation (1.1.1) is either 1 or -1. The analysis in Section 5.1 then also applies in this case. The corresponding labels for irreducible characters can be found in Table D.7; they are collected in bold font.

The $(3, 9, 6)$-core. We denote by $\mathfrak{C}_0^{\mathrm{D}_5}$ the unique $(3, 9, 6)$-core in D_5. It occurs for $\Sigma = \{\alpha_{10}, \alpha_{15}, \alpha_{16}\}$, and we have

- $\mathcal{S} = \{\alpha_1, \alpha_2, \alpha_6, \alpha_7, \alpha_9, \alpha_{10}, \alpha_{13}, \alpha_{15}, \alpha_{16}\} \cup \mathcal{D}(\mathcal{S})$,

- $\mathcal{Z} = \{\alpha_{10}, \alpha_{15}, \alpha_{16}\} \cup \mathcal{D}(\mathcal{S})$,

- $\mathcal{A} = \{\alpha_3, \alpha_5\}$ and

- $\mathcal{L} = \{\alpha_4, \alpha_{11}\}$,

and $\mathcal{D}(\mathcal{S}) = \{\alpha_8\}$. We note that $\mathcal{S} \setminus \mathcal{D}(\mathcal{S})$ lies into the quattern corresponding to $\mathfrak{C}_3^{\mathrm{D}_5}$ by removing the root α_3, which plays the role of the root β_3 in Proposition 5.5. As done in the case of type B_5, Proposition 5.5 applies. As usual, the labels for the characters corresponding to $\mathfrak{C}_0^{\mathrm{D}_5}$ are given in bold font in Table D.7.

5.5 Towards a parametrization of $\mathrm{Irr}(U_{\mathrm{Y}_r})$ for $r \geq 6$

For types up to rank 7, we can determine all triples (z, m, c) associated to cores. We collect the numbers of (z, m, c)-cores in types $\mathrm{B}_6, \mathrm{C}_6, \mathrm{D}_6$ and E_6 in Table 5.4. We omit all triples (z, m, c) in rank 7. We just mention that in type B_7 (respectively C_7, D_7, E_7) there are 65 (respectively $49, 36, 382$) different triples of the form (z, m, c) associated to cores. We observe that many cores in rank 6, in fact most of them, are $(3, 10, 9)$-cores and $(3, 9, 6)$-cores.

By using CHEVIE, it is easy to check that all $(3, 10, 9)$-cores and all $(3, 9, 6)$-cores up to rank 6 are of the same form as the ones considered in Section 5.1; in particular, each $(3, 9, 6)$-core derives from a $(3, 10, 9)$-core in the sense of Proposition 5.5. Using Remark 5.2 and Propositions 5.4 and 5.5, we completely determine the representation theory in these cases.

However, one has to deal with the other cases. On the one hand, although most nonabelian cores can be dealt with on a case-by-case check, one should formulate a more general method that can be applied, or even implemented in CHEVIE or MAGMA, for a given class of nonabelian cores. On the other hand, it is still not known, through the methods outlined so far, how to parametrize the irreducible characters arising from each of the $(4, 24, 23)$-cores

B_6	
(z, m, c)	#
$(2, 8, 7)$	1
$(3, 9, 6)$	14
$(3, 10, 9)$	64
$(4, 8, 4)$	1
$(4, 18, 18)$	1
$(4, 21, 28)$	1
$(4, 24, 43)$	1
$(5, 15, 11)$	1
$(5, 16, 16)$	8
$(5, 18, 18)$	1
$(5, 21, 25)$	1
$(6, 19, 20)$	1

C_6	
(z, m, c)	#
$(3, 7, 4)$	1
$(3, 8, 6)$	9
$(3, 10, 9)$	1
$(3, 15, 22)$	1
$(4, 11, 9)$	1
$(4, 11, 10)$	1
$(5, 10, 5)$	1
$(5, 12, 10)$	1
$(5, 18, 19)$	1
$(5, 21, 24)$	1
$(6, 12, 6)$	1
$(6, 13, 7)$	1
$(6, 16, 10)$	1
$(6, 19, 19)$	1

D_6	
(z, m, c)	#
$(3, 9, 6)$	11
$(3, 10, 9)$	39
$(4, 18, 18)$	1
$(4, 21, 28)$	1
$(4, 24, 43)$	1
$(5, 18, 18)$	1
$(6, 19, 20)$	1

E_6	
(z, m, c)	#
$(3, 9, 6)$	31
$(3, 10, 9)$	88
$(4, 8, 4)$	13
$(5, 10, 5)$	1
$(5, 12, 8)$	2
$(5, 15, 11)$	3
$(5, 20, 25)$	1
$(5, 21, 30)$	1
$(6, 12, 6)$	5
$(6, 13, 7)$	1
$(6, 14, 8)$	3
$(6, 15, 12)$	2
$(6, 16, 12)$	1
$(6, 17, 17)$	1
$(7, 15, 9)$	3

Table 5.4: The numbers of (z, m, c)-cores in rank 6.

that appear in types B_6 and D_6. These look to be the most complicated nonabelian cores to examine up to rank 6.

The reason for this, which points us to the first problem to consider in future research, is the following. Let $\mathfrak{C}$ be a (z, m, c)-core. Then the analysis outlined in previous chapters provides a method for applying the reduction lemma in a quattern group, *except* for finding a choice of X and Y as in the statement. For small values of c, it is easy to produce candidates for $X = X_{\mathcal{I}}$ and $Y = X_{\mathcal{J}}$ as explained in the case of $\mathfrak{C}^{B_4}$ in Section 4.1, namely by drawing a picture corresponding to the structure of $\mathcal{S}$, and then setting $\delta \in \mathcal{J}$ for a suitable $\delta \in \mathcal{S} \setminus \mathcal{Z}$ and adding roots to $\mathcal{J}$ and $\mathcal{I}$ by following successive neighbors of δ via edges corresponding to relations of the form $\alpha + \beta = \gamma$, with $\alpha, \beta \in \mathcal{S}$ and $\gamma \in \mathcal{Z}$. Then our programs in CHEVIE and MAGMA can easily check whether assumptions (i) to (iv) of the reduction lemma are satisfied in this case. However, for a more complicated structure of $\mathcal{S}$ it is in general difficult to produce such candidates X and Y. A priori, checking all possibilities for X and Y is not efficient, even for $(4,24,43)$-cores. Moreover, if c is big enough, then the size of X and Y might be small in comparison to m, and we would have to apply this argument several times, as we get smaller subquotients.

The second problem for future research is the following. In the case of $(3,10,9)$-cores in rank 6 or less, by applying the argument in Proposition 5.4 we can deal with all (z,m,c)-cores simultaneously. While this is true for $(3,9,6)$-cores arising from such $(3,10,9)$-cores, this is not true for all $(3,9,6)$-cores in general, as the following example in a simply laced type in rank 7 points out.

Example 5.6. We get the following nonabelian core in type E_7. It occurs for $\Sigma = \{\alpha_{47}, \alpha_{49}\}$, and we have

- $\mathcal{S} = \{\alpha_1, \alpha_5, \alpha_{14}, \alpha_{17}, \alpha_{20}, \alpha_{21}, \alpha_{22}, \alpha_{26}, \alpha_{37}\} \cup \mathcal{D}(\mathcal{S})$,
- $\mathcal{Z} = \{\alpha_{21}, \alpha_{26}, \alpha_{37}\} \cup \mathcal{D}_1(\mathcal{S})$,
- $\mathcal{A} = \{\alpha_2, \alpha_3, \alpha_6, \alpha_7, \alpha_8, \alpha_{10}, \alpha_{12}, \alpha_{13}, \alpha_{15}, \alpha_{24}, \alpha_{29}, \alpha_{31}, \alpha_{35}, \alpha_{36}\}$ and
- $\mathcal{L} = \{\alpha_{18}, \alpha_{19}, \alpha_{23}, \alpha_{25}, \alpha_{27}, \alpha_{28}, \alpha_{30}, \alpha_{34}, \alpha_{39}, \alpha_{40}, \alpha_{41}, \alpha_{42}, \alpha_{44}, \alpha_{45}\}$,

with $\mathcal{D}(\mathcal{S}) = \mathcal{D}_1(\mathcal{S}) \sqcup \mathcal{D}_2(\mathcal{S})$, for $\mathcal{D}_1(\mathcal{S}) = \{\alpha_{33}, \alpha_{47}, \alpha_{49}\}$ and $\mathcal{D}_2(\mathcal{S}) = \{\alpha_4, \alpha_9\}$. We take

- $Y = X_{\mathcal{J}}$, where $\mathcal{J} = \{\alpha_5, \alpha_{17}, \alpha_{22}\}$,
- $X = X_{\mathcal{I}}$, where $\mathcal{I} = \{\alpha_1, \alpha_{14}, \alpha_{20}\}$ and then we have that
- $H = YZ$.

Equation (3.2.3) yields

$$s_5(-a_{21}t_{14} - a_{26}t_{20}) + s_{17}(-a_{21}t_1 - a_{37}t_{20}) + s_{22}(-a_{26}t_1 + a_{37}t_{14}) = 0.$$

In this case, we see that *for every choice of* p we get

$$X' = \{x_{1,14,20}(t) \mid t \in \mathbb{F}_q\} \quad \text{and} \quad Y' = \{x_{5,17,22}(s) \mid s \in \mathbb{F}_q\},$$

where

$$x_{1,14,20}(t) = x_1(a_{37}t)x_{14}(a_{26}t)x_{20}(-a_{21}t) \text{ and } x_{5,17,22}(t) = x_5(a_{37}s)x_{17}(-a_{26}s)x_{22}(a_{21}s).$$

This case is then different from the ones described in Section 5.1.

Of course, the core in Example 5.6 is not contained in any of the $(3, 10, 9)$-cores examined in Section 5.1. We would like to determine the numbers of classes of (z, m, c)-cores, fixed z, m and c, that could be dealt with simultaneously.

Finally, a more ambitious problem for future research is towards type E_8. This case does not appear in Table 2.2, and in fact it is not yet even known how many cores we get in this type, since the program we have used for this work does not terminate in a reasonable time. With some amount of computational work, we can improve our program by detecting whether a quattern appearing in our analysis occurs in a proper irreducible subsystem of E_8, storing that information and using later the data collected for other types to determine the corresponding representation theory. Provided we manage to do this, the two research problems previously pointed out have of course to be overcome as well, in order to have a full parametrization of $\mathrm{Irr}(U_{\mathrm{E}_8})$. Progress in these three directions would determine a large amount of useful information towards a determination of $\mathrm{Irr}(U_{\mathrm{E}_8})$.

APPENDIX A

ROOT LABELLING

Height	Roots			
1	$\alpha_1 = 1\ 0\ 0\ 0$	$\alpha_2 = 0\ 1\ 0\ 0$	$\alpha_3 = 0\ 0\ 1\ 0$	$\alpha_4 = 0\ 0\ 0\ 1$
2	$\alpha_5 = 1\ 1\ 0\ 0$	$\alpha_6 = 0\ 1\ 1\ 0$	$\alpha_7 = 0\ 0\ 1\ 1$	
3	$\alpha_8 = 2\ 1\ 0\ 0$	$\alpha_9 = 1\ 1\ 1\ 0$	$\alpha_{10} = 0\ 1\ 1\ 1$	
4	$\alpha_{11} = 2\ 1\ 1\ 0$	$\alpha_{12} = 1\ 1\ 1\ 1$		
5	$\alpha_{13} = 2\ 2\ 1\ 0$	$\alpha_{14} = 2\ 1\ 1\ 1$		
6	$\alpha_{15} = 2\ 2\ 1\ 1$			
7	$\alpha_{16} = 2\ 2\ 2\ 1$			

Table A.1: Positive roots in a root system of type B_4.

Height	Roots			
1	$\alpha_1 = 1\ 0\ 0\ 0$	$\alpha_2 = 0\ 1\ 0\ 0$	$\alpha_3 = 0\ 0\ 1\ 0$	$\alpha_4 = 0\ 0\ 0\ 1$
2	$\alpha_5 = 1\ 1\ 0\ 0$	$\alpha_6 = 0\ 1\ 1\ 0$	$\alpha_7 = 0\ 0\ 1\ 1$	
3	$\alpha_8 = 1\ 2\ 0\ 0$	$\alpha_9 = 1\ 1\ 1\ 0$	$\alpha_{10} = 0\ 1\ 1\ 1$	
4	$\alpha_{11} = 1\ 2\ 1\ 0$	$\alpha_{12} = 1\ 1\ 1\ 1$		
5	$\alpha_{13} = 1\ 2\ 2\ 0$	$\alpha_{14} = 1\ 2\ 1\ 1$		
6	$\alpha_{15} = 1\ 2\ 2\ 1$			
7	$\alpha_{16} = 1\ 2\ 2\ 2$			

Table A.2: Positive roots in a root system of type C_4.

Height	Roots			
1	$\alpha_1 = \begin{matrix}1\\0\end{matrix}\ \ 0\ \ 0$	$\alpha_2 = \begin{matrix}0\\1\end{matrix}\ \ 0\ \ 0$	$\alpha_3 = \begin{matrix}0\\0\end{matrix}\ \ 1\ \ 0$	$\alpha_4 = \begin{matrix}0\\0\end{matrix}\ \ 0\ \ 1$
2	$\alpha_5 = \begin{matrix}1\\0\end{matrix}\ \ 1\ \ 0$	$\alpha_6 = \begin{matrix}0\\1\end{matrix}\ \ 1\ \ 0$	$\alpha_7 = \begin{matrix}0\\0\end{matrix}\ \ 1\ \ 1$	
3	$\alpha_8 = \begin{matrix}1\\1\end{matrix}\ \ 1\ \ 0$	$\alpha_9 = \begin{matrix}1\\0\end{matrix}\ \ 1\ \ 1$	$\alpha_{10} = \begin{matrix}0\\1\end{matrix}\ \ 1\ \ 1$	
4	$\alpha_{11} = \begin{matrix}1\\1\end{matrix}\ \ 1\ \ 1$			
5	$\alpha_{12} = \begin{matrix}1\\1\end{matrix}\ \ 2\ \ 1$			

Table A.3: Positive roots in a root system of type D_4.

Height	Roots			
1	$\alpha_1 = 1\ \ 0\ \ 0\ \ 0$	$\alpha_2 = 0\ \ 1\ \ 0\ \ 0$	$\alpha_3 = 0\ \ 0\ \ 1\ \ 0$	$\alpha_4 = 0\ \ 0\ \ 0\ \ 1$
2	$\alpha_5 = 1\ \ 1\ \ 0\ \ 0$	$\alpha_6 = 0\ \ 1\ \ 1\ \ 0$	$\alpha_7 = 0\ \ 0\ \ 1\ \ 1$	
3	$\alpha_8 = 1\ \ 1\ \ 1\ \ 0$	$\alpha_9 = 0\ \ 1\ \ 2\ \ 0$	$\alpha_{10} = 0\ \ 1\ \ 1\ \ 1$	
4	$\alpha_{11} = 1\ \ 1\ \ 2\ \ 0$	$\alpha_{12} = 1\ \ 1\ \ 1\ \ 1$	$\alpha_{13} = 0\ \ 1\ \ 2\ \ 1$	
5	$\alpha_{14} = 1\ \ 2\ \ 2\ \ 0$	$\alpha_{15} = 1\ \ 1\ \ 2\ \ 1$	$\alpha_{16} = 0\ \ 1\ \ 2\ \ 2$	
6	$\alpha_{17} = 1\ \ 2\ \ 2\ \ 1$	$\alpha_{18} = 1\ \ 1\ \ 2\ \ 2$		
7	$\alpha_{19} = 1\ \ 2\ \ 3\ \ 1$	$\alpha_{20} = 1\ \ 2\ \ 2\ \ 2$		
8	$\alpha_{21} = 1\ \ 2\ \ 3\ \ 2$			
9	$\alpha_{22} = 1\ \ 2\ \ 4\ \ 2$			
10	$\alpha_{23} = 1\ \ 3\ \ 4\ \ 2$			
11	$\alpha_{24} = 2\ \ 3\ \ 4\ \ 2$			

Table A.4: Positive roots in a root system of type F_4.

Height	Roots				
1	α_1	α_2	α_3	α_4	α_5
2	$\alpha_6 = 1\ \ 1\ \ 0\ \ 0\ \ 0$	$\alpha_7 = 0\ \ 1\ \ 1\ \ 0\ \ 0$	$\alpha_8 = 0\ \ 0\ \ 1\ \ 1\ \ 0$	$\alpha_9 = 0\ \ 0\ \ 0\ \ 1\ \ 1$	
3	$\alpha_{10} = 2\ \ 1\ \ 0\ \ 0\ \ 0$	$\alpha_{11} = 1\ \ 1\ \ 1\ \ 0\ \ 0$	$\alpha_{12} = 0\ \ 1\ \ 1\ \ 1\ \ 0$	$\alpha_{13} = 0\ \ 0\ \ 1\ \ 1\ \ 1$	
4	$\alpha_{14} = 2\ \ 1\ \ 1\ \ 0\ \ 0$	$\alpha_{15} = 1\ \ 1\ \ 1\ \ 1\ \ 0$	$\alpha_{16} = 0\ \ 1\ \ 1\ \ 1\ \ 1$		
5	$\alpha_{17} = 2\ \ 2\ \ 1\ \ 0\ \ 0$	$\alpha_{18} = 2\ \ 1\ \ 1\ \ 1\ \ 0$	$\alpha_{19} = 1\ \ 1\ \ 1\ \ 1\ \ 1$		
6	$\alpha_{20} = 2\ \ 2\ \ 1\ \ 1\ \ 0$	$\alpha_{21} = 2\ \ 1\ \ 1\ \ 1\ \ 1$			
7	$\alpha_{22} = 2\ \ 2\ \ 2\ \ 1\ \ 0$	$\alpha_{23} = 2\ \ 2\ \ 1\ \ 1\ \ 1$			
8	$\alpha_{24} = 2\ \ 2\ \ 2\ \ 1\ \ 1$				
9	$\alpha_{25} = 2\ \ 2\ \ 2\ \ 2\ \ 1$				

Table A.5: Positive roots in a root system of type B_5.

Height	Roots				
1	α_1	α_2	α_3	α_4	α_5
2	$\alpha_6 = 1\ \ 1\ \ 0\ \ 0\ \ 0$	$\alpha_7 = 0\ \ 1\ \ 1\ \ 0\ \ 0$	$\alpha_8 = 0\ \ 0\ \ 1\ \ 1\ \ 0$	$\alpha_9 = 0\ \ 0\ \ 0\ \ 1\ \ 1$	
3	$\alpha_{10} = 1\ \ 2\ \ 0\ \ 0\ \ 0$	$\alpha_{11} = 1\ \ 1\ \ 1\ \ 0\ \ 0$	$\alpha_{12} = 0\ \ 1\ \ 1\ \ 1\ \ 0$	$\alpha_{13} = 0\ \ 0\ \ 1\ \ 1\ \ 1$	
4	$\alpha_{14} = 1\ \ 2\ \ 1\ \ 0\ \ 0$	$\alpha_{15} = 1\ \ 1\ \ 1\ \ 1\ \ 0$	$\alpha_{16} = 0\ \ 1\ \ 1\ \ 1\ \ 1$		
5	$\alpha_{17} = 1\ \ 2\ \ 2\ \ 0\ \ 0$	$\alpha_{18} = 1\ \ 2\ \ 1\ \ 1\ \ 0$	$\alpha_{19} = 1\ \ 1\ \ 1\ \ 1\ \ 1$		
6	$\alpha_{20} = 1\ \ 2\ \ 2\ \ 1\ \ 0$	$\alpha_{21} = 1\ \ 2\ \ 1\ \ 1\ \ 1$			
7	$\alpha_{22} = 1\ \ 2\ \ 2\ \ 2\ \ 0$	$\alpha_{23} = 1\ \ 2\ \ 2\ \ 1\ \ 1$			
8	$\alpha_{24} = 1\ \ 2\ \ 2\ \ 2\ \ 1$				
9	$\alpha_{25} = 1\ \ 2\ \ 2\ \ 2\ \ 2$				

Table A.6: Positive roots in a root system of type C_5.

Height	Roots				
1	α_1	α_2	α_3	α_4	α_5
2	$\alpha_6 = \begin{smallmatrix}1\\0\end{smallmatrix}\ \ 1\ \ 0\ \ 0$	$\alpha_7 = \begin{smallmatrix}0\\1\end{smallmatrix}\ \ 1\ \ 0\ \ 0$	$\alpha_8 = \begin{smallmatrix}0\\0\end{smallmatrix}\ \ 1\ \ 1\ \ 0$	$\alpha_9 = \begin{smallmatrix}0\\0\end{smallmatrix}\ \ 0\ \ 1\ \ 1$	
3	$\alpha_{10} = \begin{smallmatrix}1\\1\end{smallmatrix}\ \ 1\ \ 0\ \ 0$	$\alpha_{11} = \begin{smallmatrix}1\\0\end{smallmatrix}\ \ 1\ \ 1\ \ 0$	$\alpha_{12} = \begin{smallmatrix}0\\1\end{smallmatrix}\ \ 1\ \ 1\ \ 0$	$\alpha_{13} = \begin{smallmatrix}0\\0\end{smallmatrix}\ \ 1\ \ 1\ \ 1$	
4	$\alpha_{14} = \begin{smallmatrix}1\\1\end{smallmatrix}\ \ 1\ \ 1\ \ 0$	$\alpha_{15} = \begin{smallmatrix}1\\0\end{smallmatrix}\ \ 1\ \ 1\ \ 1$	$\alpha_{16} = \begin{smallmatrix}0\\1\end{smallmatrix}\ \ 1\ \ 1\ \ 1$		
5	$\alpha_{17} = \begin{smallmatrix}1\\1\end{smallmatrix}\ \ 2\ \ 1\ \ 0$	$\alpha_{18} = \begin{smallmatrix}1\\1\end{smallmatrix}\ \ 1\ \ 1\ \ 1$			
6	$\alpha_{19} = \begin{smallmatrix}1\\1\end{smallmatrix}\ \ 2\ \ 1\ \ 1$				
7	$\alpha_{20} = \begin{smallmatrix}1\\1\end{smallmatrix}\ \ 2\ \ 2\ \ 1$				

Table A.7: Positive roots in a root system of type D_5.

APPENDIX B

COMMUTATOR RELATIONS

$[x_1(s), x_2(r)] = x_5(-rs)x_8(rs^2)$	$[x_1(s), x_5(r)] = x_8(2rs)$
$[x_1(s), x_6(r)] = x_9(-rs)x_{11}(rs^2)$	$[x_1(s), x_9(r)] = x_{11}(2rs)$
$[x_1(s), x_{10}(r)] = x_{12}(-rs)x_{14}(rs^2)$	$[x_1(s), x_{12}(r)] = x_{14}(2rs)$
$[x_2(s), x_3(r)] = x_6(-rs)$	$[x_2(s), x_7(r)] = x_{10}(-rs)$
$[x_2(s), x_{11}(r)] = x_{13}(rs)$	$[x_2(s), x_{14}(r)] = x_{15}(rs)$
$[x_3(s), x_4(r)] = x_7(-rs)$	$[x_3(s), x_5(r)] = x_9(rs)x_{13}(-r^2s)$
$[x_3(s), x_8(r)] = x_{11}(rs)$	$[x_3(s), x_{15}(r)] = x_{16}(rs)$
$[x_4(s), x_6(r)] = x_{10}(rs)$	$[x_4(s), x_9(r)] = x_{12}(rs)x_{16}(-r^2s)$
$[x_4(s), x_{11}(r)] = x_{14}(rs)$	$[x_4(s), x_{13}(r)] = x_{15}(rs)$
$[x_5(s), x_7(r)] = x_{12}(-rs)x_{15}(rs^2)$	$[x_5(s), x_9(r)] = x_{13}(2rs)$
$[x_5(s), x_{12}(r)] = x_{15}(2rs)$	$[x_6(s), x_8(r)] = x_{13}(-rs)$
$[x_6(s), x_{14}(r)] = x_{16}(rs)$	$[x_7(s), x_8(r)] = x_{14}(rs)$
$[x_7(s), x_{13}(r)] = x_{16}(-rs)$	$[x_8(s), x_{10}(r)] = x_{15}(rs)$
$[x_9(s), x_{12}(r)] = x_{16}(2rs)$	$[x_{10}(s), x_{11}(r)] = x_{16}(-rs)$

Table B.1: Commutator relations for U for G of type B_4.

$[x_1(s), x_2(r)] = x_5(-rs)x_8(r^2s)$	$[x_1(s), x_6(r)] = x_9(-rs)x_{13}(r^2s)$
$[x_1(s), x_{10}(r)] = x_{12}(-rs)x_{16}(r^2s)$	$[x_2(s), x_3(r)] = x_6(-rs)$
$[x_2(s), x_5(r)] = x_8(2rs)$	$[x_2(s), x_7(r)] = x_{10}(-rs)$
$[x_2(s), x_9(r)] = x_{11}(rs)$	$[x_2(s), x_{12}(r)] = x_{14}(rs)$
$[x_3(s), x_4(r)] = x_7(-rs)$	$[x_3(s), x_5(r)] = x_9(rs)$
$[x_3(s), x_8(r)] = x_{11}(rs)x_{13}(-rs^2)$	$[x_3(s), x_{11}(r)] = x_{13}(2rs)$
$[x_3(s), x_{14}(r)] = x_{15}(rs)$	$[x_4(s), x_6(r)] = x_{10}(rs)$
$[x_4(s), x_9(r)] = x_{12}(rs)$	$[x_4(s), x_{11}(r)] = x_{14}(rs)$
$[x_4(s), x_{13}(r)] = x_{15}(rs)x_{16}(-rs^2)$	$[x_4(s), x_{15}(r)] = x_{16}(2rs)$
$[x_5(s), x_6(r)] = x_{11}(-rs)$	$[x_5(s), x_7(r)] = x_{12}(-rs)$
$[x_5(s), x_{10}(r)] = x_{14}(-rs)$	$[x_6(s), x_9(r)] = x_{13}(2rs)$
$[x_6(s), x_{12}(r)] = x_{15}(rs)$	$[x_7(s), x_8(r)] = x_{14}(rs)x_{16}(-rs^2)$
$[x_7(s), x_{11}(r)] = x_{15}(rs)$	$[x_7(s), x_{14}(r)] = x_{16}(2rs)$
$[x_9(s), x_{10}(r)] = x_{15}(-rs)$	$[x_{10}(s), x_{12}(r)] = x_{16}(2rs)$

Table B.2: Commutator relations for U for G of type C_4.

$[x_1(s), x_3(r)] = x_5(rs)$	$[x_1(s), x_6(r)] = x_8(-rs)$
$[x_1(s), x_7(r)] = x_9(rs)$	$[x_1(s), x_{10}(r)] = x_{11}(-rs)$
$[x_2(s), x_3(r)] = x_6(rs)$	$[x_2(s), x_5(r)] = x_8(-rs)$
$[x_2(s), x_7(r)] = x_{10}(rs)$	$[x_2(s), x_9(r)] = x_{11}(-rs)$
$[x_3(s), x_4(r)] = x_7(rs)$	$[x_3(s), x_{11}(r)] = x_{12}(-rs)$
$[x_4(s), x_5(r)] = x_9(-rs)$	$[x_4(s), x_6(r)] = x_{10}(-rs)$
$[x_4(s), x_8(r)] = x_{11}(-rs)$	$[x_5(s), x_{10}(r)] = x_{12}(-rs)$
$[x_6(s), x_9(r)] = x_{12}(-rs)$	$[x_7(s), x_8(r)] = x_{12}(rs)$

Table B.3: Commutator relations for U for G of type D_4.

$[x_1(s), x_2(r)] = x_5(rs)$	$[x_1(s), x_6(r)] = x_8(rs)x_{14}(-r^2s)$
$[x_1(s), x_9(r)] = x_{11}(rs)$	$[x_1(s), x_{10}(r)] = x_{12}(rs)x_{20}(r^2s)$
$[x_1(s), x_{13}(r)] = x_{15}(rs)x_{22}(r^2s)$	$[x_1(s), x_{16}(r)] = x_{18}(rs)$
$[x_1(s), x_{23}(r)] = x_{24}(rs)$	$[x_2(s), x_3(r)] = x_6(rs)x_9(-r^2s)$
$[x_2(s), x_7(r)] = x_{10}(rs)x_{16}(r^2s)$	$[x_2(s), x_{11}(r)] = x_{14}(rs)$
$[x_2(s), x_{15}(r)] = x_{17}(rs)x_{24}(r^2s)$	$[x_2(s), x_{18}(r)] = x_{20}(rs)$
$[x_2(s), x_{22}(r)] = x_{23}(rs)$	$[x_3(s), x_4(r)] = x_7(rs)$
$[x_3(s), x_5(r)] = x_8(-rs)x_{11}(rs^2)$	$[x_3(s), x_6(r)] = x_9(2rs)$
$[x_3(s), x_8(r)] = x_{11}(2rs)$	$[x_3(s), x_{10}(r)] = x_{13}(rs)$
$[x_3(s), x_{12}(r)] = x_{15}(rs)$	$[x_3(s), x_{17}(r)] = x_{19}(rs)$
$[x_3(s), x_{20}(r)] = x_{21}(rs)x_{22}(-rs^2)$	$[x_3(s), x_{21}(r)] = x_{22}(2rs)$
$[x_4(s), x_6(r)] = x_{10}(-rs)$	$[x_4(s), x_8(r)] = x_{12}(-rs)$
$[x_4(s), x_9(r)] = x_{13}(-rs)x_{16}(rs^2)$	$[x_4(s), x_{11}(r)] = x_{15}(-rs)x_{18}(rs^2)$
$[x_4(s), x_{13}(r)] = x_{16}(2rs)$	$[x_4(s), x_{14}(r)] = x_{17}(-rs)x_{20}(rs^2)$
$[x_4(s), x_{15}(r)] = x_{18}(2rs)$	$[x_4(s), x_{17}(r)] = x_{20}(2rs)$
$[x_4(s), x_{19}(r)] = x_{21}(rs)$	$[x_5(s), x_7(r)] = x_{12}(rs)x_{18}(r^2s)$
$[x_5(s), x_9(r)] = x_{14}(-rs)$	$[x_5(s), x_{13}(r)] = x_{17}(-rs)x_{23}(-r^2s)$
$[x_5(s), x_{16}(r)] = x_{20}(-rs)$	$[x_5(s), x_{22}(r)] = x_{24}(rs)$
$[x_6(s), x_7(r)] = x_{13}(-rs)$	$[x_6(s), x_8(r)] = x_{14}(2rs)$
$[x_6(s), x_{12}(r)] = x_{17}(rs)$	$[x_6(s), x_{15}(r)] = x_{19}(-rs)$
$[x_6(s), x_{18}(r)] = x_{21}(-rs)x_{23}(rs^2)$	$[x_6(s), x_{21}(r)] = x_{23}(2rs)$
$[x_7(s), x_8(r)] = x_{15}(rs)$	$[x_7(s), x_{10}(r)] = x_{16}(-2rs)$
$[x_7(s), x_{12}(r)] = x_{18}(-2rs)$	$[x_7(s), x_{14}(r)] = x_{19}(-rs)x_{22}(rs^2)$
$[x_7(s), x_{17}(r)] = x_{21}(rs)$	$[x_7(s), x_{19}(r)] = x_{22}(2rs)$
$[x_8(s), x_{10}(r)] = x_{17}(-rs)$	$[x_8(s), x_{13}(r)] = x_{19}(rs)$
$[x_8(s), x_{16}(r)] = x_{21}(rs)x_{24}(-rs^2)$	$[x_8(s), x_{21}(r)] = x_{24}(2rs)$
$[x_9(s), x_{12}(r)] = x_{19}(rs)x_{24}(-r^2s)$	$[x_9(s), x_{18}(r)] = x_{22}(-rs)$
$[x_9(s), x_{20}(r)] = x_{23}(-rs)$	$[x_{10}(s), x_{11}(r)] = x_{19}(rs)x_{23}(-rs^2)$
$[x_{10}(s), x_{12}(r)] = x_{20}(-2rs)$	$[x_{10}(s), x_{15}(r)] = x_{21}(-rs)$
$[x_{10}(s), x_{19}(r)] = x_{23}(2rs)$	$[x_{11}(s), x_{16}(r)] = x_{22}(rs)$
$[x_{11}(s), x_{20}(r)] = x_{24}(-rs)$	$[x_{12}(s), x_{13}(r)] = x_{21}(rs)$
$[x_{12}(s), x_{19}(r)] = x_{24}(2rs)$	$[x_{13}(s), x_{15}(r)] = x_{22}(-2rs)$
$[x_{13}(s), x_{17}(r)] = x_{23}(-2rs)$	$[x_{14}(s), x_{16}(r)] = x_{23}(rs)$
$[x_{14}(s), x_{18}(r)] = x_{24}(rs)$	$[x_{15}(s), x_{17}(r)] = x_{24}(-2rs)$

Table B.4: Commutator relations for U for G of type F_4.

$[x_1(s), x_2(r)] = x_6(-rs)x_{10}(rs^2)$	$[x_1(s), x_6(r)] = x_{10}(2rs)$
$[x_1(s), x_7(r)] = x_{11}(-rs)x_{14}(rs^2)$	$[x_1(s), x_{11}(r)] = x_{14}(2rs)$
$[x_1(s), x_{12}(r)] = x_{15}(-rs)x_{18}(rs^2)$	$[x_1(s), x_{15}(r)] = x_{18}(2rs)$
$[x_1(s), x_{16}(r)] = x_{19}(-rs)x_{21}(rs^2)$	$[x_1(s), x_{19}(r)] = x_{21}(2rs)$
$[x_2(s), x_3(r)] = x_7(-rs)$	$[x_2(s), x_8(r)] = x_{12}(-rs)$
$[x_2(s), x_{13}(r)] = x_{16}(-rs)$	$[x_2(s), x_{14}(r)] = x_{17}(rs)$
$[x_2(s), x_{18}(r)] = x_{20}(rs)$	$[x_2(s), x_{21}(r)] = x_{23}(rs)$
$[x_3(s), x_4(r)] = x_8(-rs)$	$[x_3(s), x_6(r)] = x_{11}(rs)x_{17}(-r^2s)$
$[x_3(s), x_9(r)] = x_{13}(-rs)$	$[x_3(s), x_{10}(r)] = x_{14}(rs)$
$[x_3(s), x_{20}(r)] = x_{22}(rs)$	$[x_3(s), x_{23}(r)] = x_{24}(rs)$
$[x_4(s), x_5(r)] = x_9(-rs)$	$[x_4(s), x_7(r)] = x_{12}(rs)$
$[x_4(s), x_{11}(r)] = x_{15}(rs)x_{22}(-r^2s)$	$[x_4(s), x_{14}(r)] = x_{18}(rs)$
$[x_4(s), x_{17}(r)] = x_{20}(rs)$	$[x_4(s), x_{24}(r)] = x_{25}(rs)$
$[x_5(s), x_8(r)] = x_{13}(rs)$	$[x_5(s), x_{12}(r)] = x_{16}(rs)$
$[x_5(s), x_{15}(r)] = x_{19}(rs)x_{25}(-r^2s)$	$[x_5(s), x_{18}(r)] = x_{21}(rs)$
$[x_5(s), x_{20}(r)] = x_{23}(rs)$	$[x_5(s), x_{22}(r)] = x_{24}(rs)$
$[x_6(s), x_8(r)] = x_{15}(-rs)x_{20}(rs^2)$	$[x_6(s), x_{11}(r)] = x_{17}(2rs)$
$[x_6(s), x_{13}(r)] = x_{19}(-rs)x_{23}(rs^2)$	$[x_6(s), x_{15}(r)] = x_{20}(2rs)$
$[x_6(s), x_{19}(r)] = x_{23}(2rs)$	$[x_7(s), x_9(r)] = x_{16}(-rs)$
$[x_7(s), x_{10}(r)] = x_{17}(-rs)$	$[x_7(s), x_{18}(r)] = x_{22}(rs)$
$[x_7(s), x_{21}(r)] = x_{24}(rs)$	$[x_8(s), x_{10}(r)] = x_{18}(rs)$
$[x_8(s), x_{17}(r)] = x_{22}(-rs)$	$[x_8(s), x_{23}(r)] = x_{25}(rs)$
$[x_9(s), x_{11}(r)] = x_{19}(rs)x_{24}(-r^2s)$	$[x_9(s), x_{14}(r)] = x_{21}(rs)$
$[x_9(s), x_{17}(r)] = x_{23}(rs)$	$[x_9(s), x_{22}(r)] = x_{25}(-rs)$
$[x_{10}(s), x_{12}(r)] = x_{20}(rs)$	$[x_{10}(s), x_{13}(r)] = x_{21}(-rs)$
$[x_{10}(s), x_{16}(r)] = x_{23}(rs)$	$[x_{11}(s), x_{15}(r)] = x_{22}(2rs)$
$[x_{11}(s), x_{19}(r)] = x_{24}(2rs)$	$[x_{12}(s), x_{14}(r)] = x_{22}(-rs)$
$[x_{12}(s), x_{21}(r)] = x_{25}(rs)$	$[x_{13}(s), x_{17}(r)] = x_{24}(-rs)$
$[x_{13}(s), x_{20}(r)] = x_{25}(-rs)$	$[x_{14}(s), x_{16}(r)] = x_{24}(rs)$
$[x_{15}(s), x_{19}(r)] = x_{25}(2rs)$	$[x_{16}(s), x_{18}(r)] = x_{25}(-rs)$

Table B.5: Commutator relations for U for G of type B_5.

$[x_1(s), x_2(r)] = x_6(-rs)x_{10}(r^2s)$	$[x_1(s), x_7(r)] = x_{11}(-rs)x_{17}(r^2s)$
$[x_1(s), x_{12}(r)] = x_{15}(-rs)x_{22}(r^2s)$	$[x_1(s), x_{16}(r)] = x_{19}(-rs)x_{25}(r^2s)$
$[x_2(s), x_3(r)] = x_7(-rs)$	$[x_2(s), x_6(r)] = x_{10}(2rs)$
$[x_2(s), x_8(r)] = x_{12}(-rs)$	$[x_2(s), x_{11}(r)] = x_{14}(rs)$
$[x_2(s), x_{13}(r)] = x_{16}(-rs)$	$[x_2(s), x_{15}(r)] = x_{18}(rs)$
$[x_2(s), x_{19}(r)] = x_{21}(rs)$	$[x_3(s), x_4(r)] = x_8(-rs)$
$[x_3(s), x_6(r)] = x_{11}(rs)$	$[x_3(s), x_9(r)] = x_{13}(-rs)$
$[x_3(s), x_{10}(r)] = x_{14}(rs)x_{17}(-rs^2)$	$[x_3(s), x_{14}(r)] = x_{17}(2rs)$
$[x_3(s), x_{18}(r)] = x_{20}(rs)$	$[x_3(s), x_{21}(r)] = x_{23}(rs)$
$[x_4(s), x_5(r)] = x_9(-rs)$	$[x_4(s), x_7(r)] = x_{12}(rs)$
$[x_4(s), x_{11}(r)] = x_{15}(rs)$	$[x_4(s), x_{14}(r)] = x_{18}(rs)$
$[x_4(s), x_{17}(r)] = x_{20}(rs)x_{22}(-rs^2)$	$[x_4(s), x_{20}(r)] = x_{22}(2rs)$
$[x_4(s), x_{23}(r)] = x_{24}(rs)$	$[x_5(s), x_8(r)] = x_{13}(rs)$
$[x_5(s), x_{12}(r)] = x_{16}(rs)$	$[x_5(s), x_{15}(r)] = x_{19}(rs)$
$[x_5(s), x_{18}(r)] = x_{21}(rs)$	$[x_5(s), x_{20}(r)] = x_{23}(rs)$
$[x_5(s), x_{22}(r)] = x_{24}(rs)x_{25}(-rs^2)$	$[x_5(s), x_{24}(r)] = x_{25}(2rs)$
$[x_6(s), x_7(r)] = x_{14}(-rs)$	$[x_6(s), x_8(r)] = x_{15}(-rs)$
$[x_6(s), x_{12}(r)] = x_{18}(-rs)$	$[x_6(s), x_{13}(r)] = x_{19}(-rs)$
$[x_6(s), x_{16}(r)] = x_{21}(-rs)$	$[x_7(s), x_9(r)] = x_{16}(-rs)$
$[x_7(s), x_{11}(r)] = x_{17}(2rs)$	$[x_7(s), x_{15}(r)] = x_{20}(rs)$
$[x_7(s), x_{19}(r)] = x_{23}(rs)$	$[x_8(s), x_{10}(r)] = x_{18}(rs)x_{22}(-rs^2)$
$[x_8(s), x_{14}(r)] = x_{20}(rs)$	$[x_8(s), x_{18}(r)] = x_{22}(2rs)$
$[x_8(s), x_{21}(r)] = x_{24}(rs)$	$[x_9(s), x_{11}(r)] = x_{19}(rs)$
$[x_9(s), x_{14}(r)] = x_{21}(rs)$	$[x_9(s), x_{17}(r)] = x_{23}(rs)x_{25}(-rs^2)$
$[x_9(s), x_{20}(r)] = x_{24}(rs)$	$[x_9(s), x_{23}(r)] = x_{25}(2rs)$
$[x_{10}(s), x_{13}(r)] = x_{21}(-rs)x_{25}(r^2s)$	$[x_{11}(s), x_{12}(r)] = x_{20}(-rs)$
$[x_{11}(s), x_{16}(r)] = x_{23}(-rs)$	$[x_{12}(s), x_{15}(r)] = x_{22}(2rs)$
$[x_{12}(s), x_{19}(r)] = x_{24}(rs)$	$[x_{13}(s), x_{14}(r)] = x_{23}(rs)$
$[x_{13}(s), x_{18}(r)] = x_{24}(rs)$	$[x_{13}(s), x_{21}(r)] = x_{25}(2rs)$
$[x_{15}(s), x_{16}(r)] = x_{24}(-rs)$	$[x_{16}(s), x_{19}(r)] = x_{25}(2rs)$

Table B.6: Commutator relations for U for G of type C_5.

APPENDIX C

NONABELIAN CORES IN TYPES B_5 AND D_5

We provide relevant information about each $(3, 10, 9)$-core of the form $\mathfrak{C} = (\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ in types B_5 and D_5. We recall that the argument in Proposition 5.4 applies in these cases to give the corresponding irreducible characters of U. We define

$$I_\Sigma := \{i \in \{1, \dots, |\Phi^+|\} \mid \alpha_i \in \Sigma\},$$

and similarly we define $I_{\mathcal{S}\setminus\mathcal{D}(\mathcal{S})}$, $I_{\mathcal{Z}}$, $I_{\mathcal{D}(\mathcal{S})}$, $I_{\mathcal{A}}$ and $I_{\mathcal{L}}$. Moreover, $\mathcal{I}$ and $\mathcal{J}$ are defined as in Proposition 5.4.

Core	I_Σ	$I_{\mathcal{S}\setminus\mathcal{D}(\mathcal{S})}$	$I_{\mathcal{Z}}$	$I_{\mathcal{D}(\mathcal{S})}$	$I_{\mathcal{A}}$	$I_{\mathcal{L}}$	$\mathcal{I}$	$\mathcal{J}$
$\mathfrak{C}_1^{\mathrm{B}_5}$	$\{17, 18\}$	$\{2, 3, 4, 7, 8, 10, 12, 14, 17, 18\}$	$\{12, 17, 18\}$	$\varnothing$	$\{1, 6\}$	$\{11, 15\}$	$\{2, 4, 10\}$	$\{7, 8, 14\}$
$\mathfrak{C}_2^{\mathrm{B}_5}$	$\{17, 21\}$	$\{2, 3, 7, 9, 10, 13, 14, 16, 17, 21\}$	$\{15, 16, 17, 21\}$	$\{15\}$	$\{1, 4, 5, 6\}$	$\{8, 11, 18, 19\}$	$\{2, 9, 10\}$	$\{7, 13, 14\}$
$\mathfrak{C}_3^{\mathrm{B}_5}$	$\{17, 21\}$	$\{2, 3, 7, 9, 10, 13, 14, 16, 17, 21\}$	$\{16, 17, 21\}$	$\{4\}$	$\{1, 5, 6\}$	$\{11, 18, 19\}$	$\{2, 9, 10\}$	$\{7, 13, 14\}$
$\mathfrak{C}_4^{\mathrm{B}_5}$	$\{20, 21\}$	$\{2, 5, 8, 10, 12, 13, 16, 18, 20, 21\}$	$\{16, 20, 21\}$	$\{3, 11\}$	$\{1, 4, 6, 9\}$	$\{14, 15, 17, 19\}$	$\{2, 5, 10\}$	$\{12, 13, 18\}$
$\mathfrak{C}_5^{\mathrm{B}_5}$	$\{21, 22\}$	$\{4, 5, 7, 9, 12, 14, 16, 18, 21, 22\}$	$\{16, 21, 22\}$	$\{2, 6\}$	$\{1, 3, 8, 11, 13\}$	$\{10, 15, 17, 19, 20\}$	$\{5, 7, 14\}$	$\{9, 12, 18\}$
$\mathfrak{C}_6^{\mathrm{B}_5}$	$\{22, 23\}$	$\{3, 4, 5, 8, 9, 13, 17, 20, 22, 23\}$	$\{13, 22, 23\}$	$\{1\}$	$\{2, 6, 7, 10, 11, 12\}$	$\{14, 15, 16, 18, 19, 21\}$	$\{3, 5, 17\}$	$\{8, 9, 20\}$
$\mathfrak{C}_7^{\mathrm{B}_5}$	$\{5, 17, 18\}$	$\{2, 3, 4, 7, 8, 10, 12, 14, 17, 18\}$	$\{5, 12, 17, 18\}$	$\{5\}$	$\{1, 6\}$	$\{11, 15\}$	$\{2, 4, 10\}$	$\{7, 8, 14\}$
$\mathfrak{C}_8^{\mathrm{B}_5}$	$\{17, 18, 19\}$	$\{2, 3, 4, 7, 8, 10, 12, 14, 17, 18\}$	$\{12, 17, 18, 19\}$	$\{19\}$	$\{1, 5, 6, 9\}$	$\{11, 13, 15, 16\}$	$\{2, 4, 10\}$	$\{7, 8, 14\}$

Figure C.1: The $(3, 10, 9)$-cores of U_{B_5}.

Core	I_Σ	$I_{\mathcal{S}\setminus\mathcal{D}(\mathcal{S})}$	$I_{\mathcal{Z}}$	$I_{\mathcal{D}(\mathcal{S})}$	$I_{\mathcal{A}}$	$I_{\mathcal{L}}$	$\mathcal{I}$	$\mathcal{J}$
$\mathfrak{C}_1^{\mathrm{D}_5}$	$\{17,18\}$	$\{3,4,5,8,9,10,13,14,17,18\}$	$\{13,17,18\}$	$\varnothing$	$\{1,2,6,7\}$	$\{11,12,15,16\}$	$\{3,5,10\}$	$\{8,9,14\}$
$\mathfrak{C}_2^{\mathrm{D}_5}$	$\{10,11,12\}$	$\{1,2,3,4,6,7,8,10,11,12\}$	$\{10,11,12\}$	$\varnothing$	$\varnothing$	$\varnothing$	$\{1,2,4\}$	$\{6,7,8\}$
$\mathfrak{C}_3^{\mathrm{D}_5}$	$\{10,15,16\}$	$\{1,2,3,6,7,9,10,13,15,16\}$	$\{10,15,16\}$	$\{4\}$	$\{5\}$	$\{11\}$	$\{1,2,9\}$	$\{6,7,13\}$
$\mathfrak{C}_4^{\mathrm{D}_5}$	$\{14,15,16\}$	$\{1,2,5,8,11,12,13,14,15,16\}$	$\{14,15,16\}$	$\{3\}$	$\{4,9\}$	$\{6,10\}$	$\{1,2,5\}$	$\{11,12,13\}$
$\mathfrak{C}_5^{\mathrm{D}_5}$	$\{15,16,17\}$	$\{4,5,6,7,9,11,12,15,16,17\}$	$\{15,16,17\}$	$\{2\}$	$\{3,8,13\}$	$\{1,10,14\}$	$\{5,6,7\}$	$\{9,11,12\}$
$\mathfrak{C}_6^{\mathrm{D}_5}$	$\{5,10,11,12\}$	$\{1,2,3,4,6,7,8,10,11,12\}$	$\{5,10,11,12\}$	$\{5\}$	$\varnothing$	$\varnothing$	$\{1,2,4\}$	$\{6,7,8\}$

Figure C.2: The $(3,10,9)$-cores of U_{D_5}.

APPENDIX D

PARAMETRIZATION OF CHARACTERS

We now present the parametrization of the irreducible characters of U when G is a split finite group of Lie type B_4, C_4, D_4, F_4, B_5, C_5 or D_5 and p is not a very bad prime for U, that is, $p \neq 2$ in all types except D_4 and D_5.

The notation in the tables is as follows. The first column corresponds to the families of the form $\mathcal{F}_\Sigma$, where $\mathcal{F}_\Sigma$ is the family of irreducible characters of U arising from an antichain Σ. The second column contains character labels for the families determined in the previous chapters. For a fixed core $(\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$, we define

$$I_\mathcal{A} = \{i \in \{1, \ldots, |\Phi^+|\} \mid \alpha_i \in \mathcal{A}\},$$

and define $I_\mathcal{L}$ similarly. In case of nonabelian cores, $I_\mathcal{I}$ and $I_\mathcal{J}$ are also defined in the same fashion. The third column contains $I_\mathcal{A}$ and $I_\mathcal{L}$. We note that $\mathcal{K}$ can be determined from $\mathcal{A}$, $\mathcal{L}$ and the labels of the characters. For the abelian cores, we recall that the irreducible characters are obtained by applying $\mathrm{Ind}^\mathcal{A} \mathrm{Inf}_\mathcal{K}$ to the linear characters in $\mathrm{Irr}(X_\mathcal{S})_\mathcal{Z}$. We use the **bold** font to identify nonabelian cores. In these cases, we also use the second column to give any relation between the indices and the third column to give some information on the construction of these characters. In the case where we have $Y' = 1$ and Y is normal in $\bar{H}$, we give $I_{\mathcal{I}}$ and $I_{\mathcal{J}}$ in the third column, as in this case the irreducible characters are given by applying $\mathrm{Ind}^{\mathcal{A}\cup\mathcal{I}}\,\mathrm{Inf}_{\mathcal{K}\cup\mathcal{J}}$ to linear characters in $\mathrm{Irr}(X_{\mathcal{S}\setminus(\mathcal{I}\cup\mathcal{J})})_{\mathcal{Z}}$. In other cases, we refer the reader to the relevant parts of Chapters 4 and 5. Finally, the fourth column records the number of irreducible characters in a family corresponding to some character labels, and the fifth column records their degree.

Parametrization of the irreducible characters of U_{B_4}

$\mathcal{F}$	χ	I	Number	Degree
$\mathcal{F}_{\mathrm{lin}}$	χ_{b_1,b_2,b_3,b_4}		q^4	1
$\mathcal{F}_{\mathbf{5}}$	χ^{a_5}	$I_{\mathcal{A}} = \{1\}$, $I_{\mathcal{L}} = \{2\}$,	$q-1$	q
$\mathcal{F}_{\mathbf{6}}$	χ^{a_6}	$I_{\mathcal{A}} = \{2\}$, $I_{\mathcal{L}} = \{3\}$,	$q-1$	q
$\mathcal{F}_{\mathbf{7}}$	χ^{a_7}	$I_{\mathcal{A}} = \{3\}$, $I_{\mathcal{L}} = \{4\}$,	$q-1$	q
$\mathcal{F}_{\mathbf{8}}$	$\chi^{a_8}_{b_2}$	$I_{\mathcal{A}} = \{1\}$, $I_{\mathcal{L}} = \{5\}$,	$q(q-1)$	q
$\mathcal{F}_{\mathbf{9}}$	$\chi^{a_9}_{b_2}$	$I_{\mathcal{A}} = \{1,3\}$, $I_{\mathcal{L}} = \{5,6\}$,	$q(q-1)$	q^2
$\mathcal{F}_{\mathbf{10}}$	$\chi^{a_{10}}_{b_3}$	$I_{\mathcal{A}} = \{2,4\}$, $I_{\mathcal{L}} = \{6,7\}$,	$q(q-1)$	q^2
$\mathcal{F}_{\mathbf{11}}$	$\chi^{a_{11}}_{b_2,b_5,b_6}$	$I_{\mathcal{A}} = \{1,3\}$, $I_{\mathcal{L}} = \{8,9\}$,	$q^3(q-1)$	q^2
$\mathcal{F}_{\mathbf{12}}$	$\chi^{a_6,a_{12}}$	$I_{\mathcal{A}} = \{1,3,4,7\}$,	$(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{2,5,9,10\}$,		
	$\chi^{a_{12}}_{b_2,b_3}$	$I_{\mathcal{A}} = \{1,4,7\}$, $I_{\mathcal{L}} = \{5,9,10\}$,	$q^2(q-1)$	q^3
$\mathcal{F}_{\mathbf{13}}$	$\chi^{a_{13}}_{b_1,b_3}$	$I_{\mathcal{A}} = \{2,5,6\}$, $I_{\mathcal{L}} = \{8,9,11\}$,	$q^2(q-1)$	q^3
$\mathcal{F}_{\mathbf{14}}$	$\chi^{a_9,a_{14}}_{b_2,b_6,b_{10}}$	$I_{\mathcal{A}} = \{1,3,4,7\}$,	$q^3(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{5,8,11,12\}$,		
	$\chi^{a_6,a_{14}}_{b_5,b_{10}}$	$I_{\mathcal{A}} = \{1,3,4,7\}$,	$q^2(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{2,8,11,12\}$,		
	$\chi^{a_{14}}_{b_2,b_3,b_5,b_{10}}$	$I_{\mathcal{A}} = \{1,4,7\}$, $I_{\mathcal{L}} = \{8,11,12\}$,	$q^4(q-1)$	q^3
$\mathcal{F}_{\mathbf{15}}$	$\chi^{a_{11},a_{15}}_{b_3,b_6,b_7}$	$I_{\mathcal{A}} = \{1,2,4,5,8\}$,	$q^3(q-1)^2$	q^5
		$I_{\mathcal{L}} = \{9,10,12,13,14\}$,		
	$\chi^{a_9,a_{15}}_{b_3,b_7}$	$I_{\mathcal{A}} = \{1,2,4,5,8\}$,	$q^2(q-1)^2$	q^5
		$I_{\mathcal{L}} = \{6,10,12,13,14\}$,		
	$\chi^{a_{15}}_{b_1,b_3,b_6,b_7}$	$I_{\mathcal{A}} = \{2,4,5,8\}$,	$q^4(q-1)$	q^4
		$I_{\mathcal{L}} = \{10,12,13,14\}$,		
$\mathcal{F}_{\mathbf{16}}$	$\chi^{a_8,a_{16}}_{b_2,b_4}$	$I_{\mathcal{A}} = \{1,3,6,7,9,10\}$,	$q^2(q-1)^2$	q^6
		$I_{\mathcal{L}} = \{5,11,12,13,14,15\}$,		
	$\chi^{a_5,a_{16}}_{b_4}$	$I_{\mathcal{A}} = \{2,3,6,7,9,10\}$,	$q(q-1)^2$	q^6
		$I_{\mathcal{L}} = \{1,11,12,13,14,15\}$,		
	$\chi^{a_{16}}_{b_1,b_2,b_4}$	$I_{\mathcal{A}} = \{3,6,7,9,10\}$,	$q^3(q-1)$	q^5
		$I_{\mathcal{L}} = \{11,12,13,14,15\}$,		
$\mathcal{F}_{\mathbf{1,6}}$	χ^{a_1,a_6}	$I_{\mathcal{A}} = \{2\}$, $I_{\mathcal{L}} = \{3\}$,	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{1,7}}$	χ^{a_1,a_7}	$I_{\mathcal{A}} = \{3\}$, $I_{\mathcal{L}} = \{4\}$,	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{1,10}}$	$\chi^{a_1,a_{10}}_{b_3}$	$I_{\mathcal{A}} = \{2,4\}$, $I_{\mathcal{L}} = \{6,7\}$,	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{2,7}}$	χ^{a_2,a_7}	$I_{\mathcal{A}} = \{3\}$, $I_{\mathcal{L}} = \{4\}$,	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{3,5}}$	χ^{a_3,a_5}	$I_{\mathcal{A}} = \{1\}$, $I_{\mathcal{L}} = \{2\}$,	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{3,8}}$	$\chi^{a_3,a_8}_{b_2}$	$I_{\mathcal{A}} = \{1\}$, $I_{\mathcal{L}} = \{5\}$,	$q(q-1)^2$	q
$\mathcal{F}_{\mathbf{4,5}}$	χ^{a_4,a_5}	$I_{\mathcal{A}} = \{1\}$, $I_{\mathcal{L}} = \{2\}$,	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{4,6}}$	χ^{a_4,a_6}	$I_{\mathcal{A}} = \{2\}$, $I_{\mathcal{L}} = \{3\}$,	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{4,8}}$	$\chi^{a_4,a_8}_{b_2}$	$I_{\mathcal{A}} = \{1\}$, $I_{\mathcal{L}} = \{5\}$,	$q(q-1)^2$	q
$\mathcal{F}_{\mathbf{4,9}}$	$\chi^{a_4,a_9}_{b_2}$	$I_{\mathcal{A}} = \{1,3\}$, $I_{\mathcal{L}} = \{5,6\}$,	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{4,11}}$	$\chi^{a_4,a_{11}}_{b_2,b_5,b_6}$	$I_{\mathcal{A}} = \{1,3\}$, $I_{\mathcal{L}} = \{8,9\}$,	$q^3(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{4,13}}$	$\chi^{a_4,a_{13}}_{b_1,b_3}$	$I_{\mathcal{A}} = \{2,5,6\}$, $I_{\mathcal{L}} = \{8,9,11\}$,	$q^2(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{5,6}}$	$\chi^{a_5,a_6}_{b_3}$	$I_{\mathcal{A}} = \{1\}$, $I_{\mathcal{L}} = \{2\}$,	$q(q-1)^2$	q
$\mathcal{F}_{\mathbf{5,7}}$	χ^{a_5,a_7}	$I_{\mathcal{A}} = \{1,3\}$, $I_{\mathcal{L}} = \{2,4\}$,	$(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{5,10}}$	$\chi^{a_5,a_{10}}_{b_1,b_3}$	$I_{\mathcal{A}} = \{2,4\}$, $I_{\mathcal{L}} = \{6,7\}$,	$q^2(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{6,7}}$	$\chi^{a_6,a_7}_{b_4}$	$I_{\mathcal{A}} = \{2\}$, $I_{\mathcal{L}} = \{3\}$,	$q(q-1)^2$	q
$\mathcal{F}_{\mathbf{6,8}}$	χ^{a_6,a_8}	$I_{\mathcal{A}} = \{1,3\}$, $I_{\mathcal{L}} = \{2,5\}$,	$(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{7,8}}$	$\chi^{a_7,a_8}_{b_2}$	$I_{\mathcal{A}} = \{1,4\}$, $I_{\mathcal{L}} = \{3,5\}$,	$q(q-1)^2$	q^2

$\mathcal{F}$	χ	I	Number	Degree
$\mathcal{F}_{\mathbf{7,9}}$	$\chi_{b_2,b_4}^{a_7,a_9}$	$I_{\mathcal{A}} = \{1,3\}$, $I_{\mathcal{L}} = \{5,6\}$,	$q^2(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{7,11}}$	$\chi_{b_2,b_4,b_5,b_6}^{a_7,a_{11}}$	$I_{\mathcal{A}} = \{1,3\}$, $I_{\mathcal{L}} = \{8,9\}$,	$q^4(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{7,13}}$	$\chi_{b_1}^{a_7,a_{13}}$	$I_{\mathcal{A}} = \{2,4,5,6\}$, $I_{\mathcal{L}} = \{3,8,9,11\}$,	$q(q-1)^2$	q^4
$\mathcal{F}_{\mathbf{8,9}}$	$\chi_{b_2}^{a_8,a_9}$	$I_{\mathcal{A}} = \{1,3\}$, $I_{\mathcal{L}} = \{5,6\}$,	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{8,10}}$	$\chi_{b_3}^{a_8,a_{10}}$	$I_{\mathcal{A}} = \{1,2,4\}$, $I_{\mathcal{L}} = \{5,6,7\}$,	$q(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{8,12}}$	$\chi^{a_6,a_8,a_{12}}$	$I_{\mathcal{A}} = \{1,3,4,7\}$, $I_{\mathcal{L}} = \{2,5,9,10\}$,	$(q-1)^3$	q^4
	$\chi_{b_2,b_3}^{a_8,a_{12}}$	$I_{\mathcal{A}} = \{1,4,7\}$, $I_{\mathcal{L}} = \{5,9,10\}$,	$q^2(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{9,10}}$	$\chi_{b_4}^{a_9,a_{10}}$	$I_{\mathcal{A}} = \{2,3,6\}$, $I_{\mathcal{L}} = \{1,5,7\}$,	$q(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{10,11}}$	$\chi_{b_5}^{a_{10},a_{11}}$	$I_{\mathcal{A}} = \{1,2,3,4\}$, $I_{\mathcal{L}} = \{6,7,8,9\}$,	$q(q-1)^2$	q^4
$\mathcal{F}_{\mathbf{10,13}}$	$\chi_{b_1}^{a_7,a_{10},a_{13}}$	$I_{\mathcal{A}} = \{2,4,5,6\}$, $I_{\mathcal{L}} = \{3,8,9,11\}$,	$q(q-1)^3$	q^4
	$\chi_{b_1,b_3,b_4}^{a_{10},a_{13}}$	$I_{\mathcal{A}} = \{2,5,6\}$, $I_{\mathcal{L}} = \{8,9,11\}$,	$q^3(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{11,12}}$	$\chi_{b_2,b_6}^{a_{11},a_{12}}$	$I_{\mathcal{A}} = \{1,3,4,7\}$, $I_{\mathcal{L}} = \{5,8,9,10\}$,	$q^2(q-1)^2$	q^4
$\mathcal{F}_{\mathbf{12,13}}$	$\chi_{b_3}^{a_{12},a_{13}}$	$I_{\mathcal{A}} = \{1,2,4,5,6\}$, $I_{\mathcal{L}} = \{7,8,9,10,11\}$,	$q(q-1)^2$	q^5
$\mathcal{F}_{\mathbf{13,14}}$	$\boldsymbol{\chi_{b_3}^{a_{10},a_{13},a_{14}}}$	$\boldsymbol{I_{\mathcal{A}} = \{1,5\},\ I_{\mathcal{I}} = \{2,4,8\},\ I_{\mathcal{L}} = \{9,12\},\ I_{\mathcal{J}} = \{6,7,11\}.}$	$\boldsymbol{q(q-1)^3}$	$\boldsymbol{q^5}$
	$\chi^{a_7,a_{13},a_{14}}$	$I_{\mathcal{A}} = \{1,4,5,8,11\}$, $I_{\mathcal{L}} = \{2,3,6,9,12\}$,	$(q-1)^3$	q^5
	$\chi_{b_3,b_4}^{a_{13},a_{14}}$	$I_{\mathcal{A}} = \{1,5,8,11\}$, $I_{\mathcal{L}} = \{2,6,9,12\}$,	$q^2(q-1)^2$	q^4
$\mathcal{F}_{\mathbf{1,2,7}}$	χ^{a_1,a_2,a_7}	$I_{\mathcal{A}} = \{3\}$, $I_{\mathcal{L}} = \{4\}$,	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{1,4,6}}$	χ^{a_1,a_4,a_6}	$I_{\mathcal{A}} = \{2\}$, $I_{\mathcal{L}} = \{3\}$,	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{1,6,7}}$	$\chi_{b_4}^{a_1,a_6,a_7}$	$I_{\mathcal{A}} = \{2\}$, $I_{\mathcal{L}} = \{3\}$,	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{3,4,5}}$	χ^{a_3,a_4,a_5}	$I_{\mathcal{A}} = \{1\}$, $I_{\mathcal{L}} = \{2\}$,	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{3,4,8}}$	$\chi_{b_2}^{a_3,a_4,a_8}$	$I_{\mathcal{A}} = \{1\}$, $I_{\mathcal{L}} = \{5\}$,	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{4,5,6}}$	$\chi_{b_3}^{a_4,a_5,a_6}$	$I_{\mathcal{A}} = \{1\}$, $I_{\mathcal{L}} = \{2\}$,	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{4,6,8}}$	χ^{a_4,a_6,a_8}	$I_{\mathcal{A}} = \{1,3\}$, $I_{\mathcal{L}} = \{2,5\}$,	$(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{4,8,9}}$	$\chi_{b_2}^{a_4,a_8,a_9}$	$I_{\mathcal{A}} = \{1,3\}$, $I_{\mathcal{L}} = \{5,6\}$,	$q(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{5,6,7}}$	χ^{a_5,a_6,a_7}	$I_{\mathcal{A}} = \{1,3\}$, $I_{\mathcal{L}} = \{2,4\}$,	$(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{6,7,8}}$	$\chi_{b_4}^{a_6,a_7,a_8}$	$I_{\mathcal{A}} = \{1,3\}$, $I_{\mathcal{L}} = \{2,5\}$,	$q(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{7,8,9}}$	$\chi_{b_2,b_4}^{a_7,a_8,a_9}$	$I_{\mathcal{A}} = \{1,3\}$, $I_{\mathcal{L}} = \{5,6\}$,	$q^2(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{8,9,10}}$	$\chi_{b_4}^{a_8,a_9,a_{10}}$	$I_{\mathcal{A}} = \{2,5,6\}$, $I_{\mathcal{L}} = \{1,3,7\}$,	$q(q-1)^3$	q^3

Table D.1: The parametrization of the irreducible characters of $\mathrm{UB}_4(q)$, where $q = p^e$ and $p \geq 3$.

Parametrization of the irreducible characters of U_{C_4}

$\mathcal{F}$	χ	I	Number	Degree
$\mathcal{F}_{\mathrm{lin}}$	χ_{b_1,b_2,b_3,b_4}		q^4	1
$\mathcal{F}_{\mathbf{5}}$	χ^{a_5}	$I_{\mathcal{A}} = \{1\}$, $I_{\mathcal{L}} = \{2\}$,	$q-1$	q
$\mathcal{F}_{\mathbf{6}}$	χ^{a_6}	$I_{\mathcal{A}} = \{2\}$, $I_{\mathcal{L}} = \{3\}$,	$q-1$	q
$\mathcal{F}_{\mathbf{7}}$	χ^{a_7}	$I_{\mathcal{A}} = \{3\}$, $I_{\mathcal{L}} = \{4\}$,	$q-1$	q
$\mathcal{F}_{\mathbf{8}}$	$\chi^{a_8}_{b_1}$	$I_{\mathcal{A}} = \{2\}$, $I_{\mathcal{L}} = \{5\}$,	$q(q-1)$	q
$\mathcal{F}_{\mathbf{9}}$	$\chi^{a_9}_{b_2}$	$I_{\mathcal{A}} = \{1,3\}$, $I_{\mathcal{L}} = \{5,6\}$,	$q(q-1)$	q^2
$\mathcal{F}_{\mathbf{10}}$	$\chi^{a_{10}}_{b_3}$	$I_{\mathcal{A}} = \{2,4\}$, $I_{\mathcal{L}} = \{6,7\}$,	$q(q-1)$	q^2
$\mathcal{F}_{\mathbf{11}}$	$\chi^{a_{11}}_{b_1}$	$I_{\mathcal{A}} = \{2,3,6\}$, $I_{\mathcal{L}} = \{5,8,9\}$,	$q(q-1)$	q^3
$\mathcal{F}_{\mathbf{12}}$	$\chi^{a_6,a_{12}}$	$I_{\mathcal{A}} = \{1,3,4,7\}$,	$(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{2,5,9,10\}$,		
	$\chi^{a_{12}}_{b_2,b_3}$	$I_{\mathcal{A}} = \{1,4,7\}$, $I_{\mathcal{L}} = \{5,9,10\}$,	$q^2(q-1)$	q^3
$\mathcal{F}_{\mathbf{13}}$	$\chi^{a_8,a_{13}}_{b_1}$	$I_{\mathcal{A}} = \{2,3,6\}$, $I_{\mathcal{L}} = \{5,9,11\}$,	$q(q-1)^2$	q^3
	$\chi^{a_5,a_{13}}$	$I_{\mathcal{A}} = \{2,3,6\}$, $I_{\mathcal{L}} = \{1,9,11\}$,	$(q-1)^2$	q^3
	$\chi^{a_{13}}_{b_1,b_2}$	$I_{\mathcal{A}} = \{3,6\}$, $I_{\mathcal{L}} = \{9,11\}$,	$q^2(q-1)$	q^2
$\mathcal{F}_{\mathbf{14}}$	$\chi^{a_9,a_{14}}_{b_3}$	$I_{\mathcal{A}} = \{1,2,4,5,7\}$,	$q(q-1)^2$	q^5
		$I_{\mathcal{L}} = \{6,8,10,11,12\}$,		
	$\chi^{a_{14}}_{b_1,b_3,b_6}$	$I_{\mathcal{A}} = \{2,4,5,7\}$,	$q^3(q-1)$	q^4
		$I_{\mathcal{L}} = \{8,10,11,12\}$,		
$\mathcal{F}_{\mathbf{15}}$	$\chi^{a_8,a_{15}}_{b_1}$	$I_{\mathcal{A}} = \{2,3,4,6,7,10\}$,	$q(q-1)^2$	q^6
		$I_{\mathcal{L}} = \{5,9,11,12,13,14\}$,		
	$\chi^{a_5,a_{15}}$	$I_{\mathcal{A}} = \{2,3,4,6,7,10\}$,	$(q-1)^2$	q^6
		$I_{\mathcal{L}} = \{1,9,11,12,13,14\}$,		
	$\chi^{a_{15}}_{b_1,b_2}$	$I_{\mathcal{A}} = \{3,4,6,7,10\}$,	$q^2(q-1)$	q^5
		$I_{\mathcal{L}} = \{9,11,12,13,14\}$,		
$\mathcal{F}_{\mathbf{16}}$	$\chi^{a_8,a_{13},a_{16}}_{b_1}$	$I_{\mathcal{A}} = \{2,3,4,6,7,10\}$,	$q(q-1)^3$	q^6
		$I_{\mathcal{L}} = \{5,9,11,12,14,15\}$,		
	$\chi^{a_5,a_{13},a_{16}}$	$I_{\mathcal{A}} = \{2,3,4,6,7,10\}$,	$(q-1)^3$	q^6
		$I_{\mathcal{L}} = \{1,9,11,12,14,15\}$,		
	$\chi^{a_{13},a_{16}}_{b_1,b_2}$	$I_{\mathcal{A}} = \{3,4,6,7,10\}$,	$q^2(q-1)^2$	q^5
		$I_{\mathcal{L}} = \{9,11,12,14,15\}$,		
	$\chi^{a_{11},a_{16}}_{b_1}$	$I_{\mathcal{A}} = \{2,3,4,6,7,10\}$,	$q(q-1)^2$	q^6
		$I_{\mathcal{L}} = \{5,8,9,12,14,15\}$,		
	$\chi^{a_8,a_9,a_{16}}_{b_3}$	$I_{\mathcal{A}} = \{1,4,5,7,10\}$,	$q(q-1)^3$	q^5
		$I_{\mathcal{L}} = \{2,6,12,14,15\}$,		
	$\chi^{a_8,a_{16}}_{b_1,b_3,b_6}$	$I_{\mathcal{A}} = \{2,4,7,10\}$,	$q^3(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{5,12,14,15\}$,		
	$\chi^{a_9,a_{16}}_{b_2}$	$I_{\mathcal{A}} = \{1,3,4,7,10\}$,	$q(q-1)^2$	q^5
		$I_{\mathcal{L}} = \{5,6,12,14,15\}$,		
	$\chi^{a_6,a_{16}}_{b_1,b_5}$	$I_{\mathcal{A}} = \{2,4,7,10\}$,	$q^2(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{3,12,14,15\}$,		
	$\chi^{a_5,a_{16}}_{b_3}$	$I_{\mathcal{A}} = \{2,4,7,10\}$,	$q(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{1,12,14,15\}$,		
	$\chi^{a_{16}}_{b_1,b_2,b_3}$	$I_{\mathcal{A}} = \{4,7,10\}$, $I_{\mathcal{L}} = \{12,14,15\}$,	$q^3(q-1)$	q^3
$\mathcal{F}_{\mathbf{1,6}}$	χ^{a_1,a_6}	$I_{\mathcal{A}} = \{2\}$, $I_{\mathcal{L}} = \{3\}$,	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{1,7}}$	χ^{a_1,a_7}	$I_{\mathcal{A}} = \{3\}$, $I_{\mathcal{L}} = \{4\}$,	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{1,10}}$	$\chi^{a_1,a_{10}}_{b_3}$	$I_{\mathcal{A}} = \{2,4\}$, $I_{\mathcal{L}} = \{6,7\}$,	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{2,7}}$	χ^{a_2,a_7}	$I_{\mathcal{A}} = \{3\}$, $I_{\mathcal{L}} = \{4\}$,	$(q-1)^2$	q

$\mathcal{F}$	χ	I	Number	Degree
$\mathcal{F}_{\mathbf{3,5}}$	χ^{a_3,a_5}	$I_{\mathcal{A}} = \{1\}$, $I_{\mathcal{L}} = \{2\}$,	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{3,8}}$	$\chi_{b_1}^{a_3,a_8}$	$I_{\mathcal{A}} = \{2\}$, $I_{\mathcal{L}} = \{5\}$,	$q(q-1)^2$	q
$\mathcal{F}_{\mathbf{4,5}}$	χ^{a_4,a_5}	$I_{\mathcal{A}} = \{1\}$, $I_{\mathcal{L}} = \{2\}$,	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{4,6}}$	χ^{a_4,a_6}	$I_{\mathcal{A}} = \{2\}$, $I_{\mathcal{L}} = \{3\}$,	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{4,8}}$	$\chi_{b_1}^{a_4,a_8}$	$I_{\mathcal{A}} = \{2\}$, $I_{\mathcal{L}} = \{5\}$,	$q(q-1)^2$	q
$\mathcal{F}_{\mathbf{4,9}}$	$\chi_{b_2}^{a_4,a_9}$	$I_{\mathcal{A}} = \{1,3\}$, $I_{\mathcal{L}} = \{5,6\}$,	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{4,11}}$	$\chi_{b_1}^{a_4,a_{11}}$	$I_{\mathcal{A}} = \{2,3,6\}$, $I_{\mathcal{L}} = \{5,8,9\}$,	$q(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{4,13}}$	$\chi_{b_1}^{a_4,a_8,a_{13}}$	$I_{\mathcal{A}} = \{2,3,6\}$, $I_{\mathcal{L}} = \{5,9,11\}$,	$q(q-1)^3$	q^3
	$\chi^{a_4,a_5,a_{13}}$	$I_{\mathcal{A}} = \{2,3,6\}$, $I_{\mathcal{L}} = \{1,9,11\}$,	$(q-1)^3$	q^3
	$\chi_{b_1,b_2}^{a_4,a_{13}}$	$I_{\mathcal{A}} = \{3,6\}$, $I_{\mathcal{L}} = \{9,11\}$,	$q^2(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{5,6}}$	$\chi_{b_3}^{a_5,a_6}$	$I_{\mathcal{A}} = \{1\}$, $I_{\mathcal{L}} = \{2\}$,	$q(q-1)^2$	q
$\mathcal{F}_{\mathbf{5,7}}$	χ^{a_5,a_7}	$I_{\mathcal{A}} = \{1,3\}$, $I_{\mathcal{L}} = \{2,4\}$,	$(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{5,10}}$	$\chi_{b_1,b_3}^{a_5,a_{10}}$	$I_{\mathcal{A}} = \{2,4\}$, $I_{\mathcal{L}} = \{6,7\}$,	$q^2(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{6,7}}$	$\chi_{b_4}^{a_6,a_7}$	$I_{\mathcal{A}} = \{2\}$, $I_{\mathcal{L}} = \{3\}$,	$q(q-1)^2$	q
$\mathcal{F}_{\mathbf{6,8}}$	$\chi_{b_1,b_3}^{a_6,a_8}$	$I_{\mathcal{A}} = \{2\}$, $I_{\mathcal{L}} = \{5\}$,	$q^2(q-1)^2$	q
$\mathcal{F}_{\mathbf{7,8}}$	$\chi_{b_1}^{a_7,a_8}$	$I_{\mathcal{A}} = \{2,4\}$, $I_{\mathcal{L}} = \{3,5\}$,	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{7,9}}$	$\chi_{b_2,b_4}^{a_7,a_9}$	$I_{\mathcal{A}} = \{1,3\}$, $I_{\mathcal{L}} = \{5,6\}$,	$q^2(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{7,11}}$	$\chi_{b_1,b_4}^{a_7,a_{11}}$	$I_{\mathcal{A}} = \{2,3,6\}$, $I_{\mathcal{L}} = \{5,8,9\}$,	$q^2(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{7,13}}$	$\chi_{b_1,b_4}^{a_7,a_8,a_{13}}$	$I_{\mathcal{A}} = \{2,3,6\}$, $I_{\mathcal{L}} = \{5,9,11\}$,	$q^2(q-1)^3$	q^3
	$\chi_{b_4}^{a_5,a_7,a_{13}}$	$I_{\mathcal{A}} = \{2,3,6\}$, $I_{\mathcal{L}} = \{1,9,11\}$,	$q(q-1)^3$	q^3
	$\chi_{b_1,b_2,b_4}^{a_7,a_{13}}$	$I_{\mathcal{A}} = \{3,6\}$, $I_{\mathcal{L}} = \{9,11\}$,	$q^3(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{8,9}}$	$\chi_{b_3}^{a_8,a_9}$	$I_{\mathcal{A}} = \{1,5\}$, $I_{\mathcal{L}} = \{2,6\}$,	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{8,10}}$	$\chi_{b_1,b_3,b_7}^{a_8,a_{10}}$	$I_{\mathcal{A}} = \{2,4\}$, $I_{\mathcal{L}} = \{5,6\}$,	$q^3(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{8,12}}$	$\chi^{a_6,a_8,a_{12}}$	$I_{\mathcal{A}} = \{1,3,4,5\}$,	$(q-1)^3$	q^4
		$I_{\mathcal{L}} = \{2,7,9,10\}$,		
	$\chi_{b_2,b_3}^{a_8,a_{12}}$	$I_{\mathcal{A}} = \{1,4,5\}$, $I_{\mathcal{L}} = \{7,9,10\}$,	$q^2(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{9,10}}$	$\chi_{b_4}^{a_9,a_{10}}$	$I_{\mathcal{A}} = \{2,3,6\}$, $I_{\mathcal{L}} = \{1,5,7\}$,	$q(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{10,11}}$	$\chi_{b_1,b_4,b_7}^{a_{10},a_{11}}$	$I_{\mathcal{A}} = \{2,3,6\}$, $I_{\mathcal{L}} = \{5,8,9\}$,	$q^3(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{10,13}}$	$\chi_{b_1,b_4,b_5,b_8}^{a_{10},a_{13}}$	$I_{\mathcal{A}} = \{2,3,6\}$, $I_{\mathcal{L}} = \{7,9,11\}$,	$q^4(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{11,12}}$	$\chi_{b_4,b_7}^{a_{11},a_{12}}$	$I_{\mathcal{A}} = \{1,3,5,9\}$,	$q^2(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{2,6,8,10\}$,		
$\mathcal{F}_{\mathbf{12,13}}$	$\chi_{b_2,b_4,b_8}^{a_{12},a_{13}}$	$I_{\mathcal{A}} = \{1,3,5,9\}$,	$q^3(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{6,7,10,11\}$,		
$\mathcal{F}_{\mathbf{13,14}}$	$\chi_{b_1,b_4}^{a_{13},a_{14}}$	$I_{\mathcal{A}} = \{2,5,6,8,11\}$,	$q^2(q-1)^2$	q^5
		$I_{\mathcal{L}} = \{3,7,9,10,12\}$,		
$\mathcal{F}_{\mathbf{1,2,7}}$	χ^{a_1,a_2,a_7}	$I_{\mathcal{A}} = \{3\}$, $I_{\mathcal{L}} = \{4\}$,	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{1,4,6}}$	χ^{a_1,a_4,a_6}	$I_{\mathcal{A}} = \{2\}$, $I_{\mathcal{L}} = \{3\}$,	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{1,6,7}}$	$\chi_{b_4}^{a_1,a_6,a_7}$	$I_{\mathcal{A}} = \{2\}$, $I_{\mathcal{L}} = \{3\}$,	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{3,4,5}}$	χ^{a_3,a_4,a_5}	$I_{\mathcal{A}} = \{1\}$, $I_{\mathcal{L}} = \{2\}$,	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{3,4,8}}$	$\chi_{b_1}^{a_3,a_4,a_8}$	$I_{\mathcal{A}} = \{2\}$, $I_{\mathcal{L}} = \{5\}$,	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{4,5,6}}$	$\chi_{b_3}^{a_4,a_5,a_6}$	$I_{\mathcal{A}} = \{1\}$, $I_{\mathcal{L}} = \{2\}$,	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{4,6,8}}$	$\chi_{b_1,b_3}^{a_4,a_6,a_8}$	$I_{\mathcal{A}} = \{2\}$, $I_{\mathcal{L}} = \{5\}$,	$q^2(q-1)^3$	q
$\mathcal{F}_{\mathbf{4,8,9}}$	$\chi_{b_3}^{a_4,a_8,a_9}$	$I_{\mathcal{A}} = \{1,5\}$, $I_{\mathcal{L}} = \{2,6\}$,	$q(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{5,6,7}}$	χ^{a_5,a_6,a_7}	$I_{\mathcal{A}} = \{1,3\}$, $I_{\mathcal{L}} = \{2,4\}$,	$(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{6,7,8}}$	$\chi_{b_1}^{a_6,a_7,a_8}$	$I_{\mathcal{A}} = \{2,4\}$, $I_{\mathcal{L}} = \{3,5\}$,	$q(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{7,8,9}}$	χ^{a_7,a_8,a_9}	$I_{\mathcal{A}} = \{1,4,5\}$, $I_{\mathcal{L}} = \{2,3,6\}$,	$(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{8,9,10}}$	$\chi_{b_4}^{a_8,a_9,a_{10}}$	$I_{\mathcal{A}} = \{2,3,6\}$, $I_{\mathcal{L}} = \{1,5,7\}$,	$q(q-1)^3$	q^3

Table D.2: The parametrization of the irreducible characters of $\mathrm{UC}_4(q)$, where $q = p^e$ and $p \geq 3$.

Parametrization of the irreducible characters of U_{D_4}

$\mathcal{F}$	χ	I	Number	Degree
$\mathcal{F}_{\mathrm{lin}}$	χ_{b_1,b_2,b_3,b_4}		q^4	1
$\mathcal{F}_{\mathbf{5}}$	χ^{a_5}	$I_{\mathcal{A}} = \{1\}$, $I_{\mathcal{L}} = \{3\}$,	$q-1$	q
$\mathcal{F}_{\mathbf{6}}$	χ^{a_6}	$I_{\mathcal{A}} = \{2\}$, $I_{\mathcal{L}} = \{3\}$,	$q-1$	q
$\mathcal{F}_{\mathbf{7}}$	χ^{a_7}	$I_{\mathcal{A}} = \{3\}$, $I_{\mathcal{L}} = \{4\}$,	$q-1$	q
$\mathcal{F}_{\mathbf{8}}$	$\chi_{b_3}^{a_8}$	$I_{\mathcal{A}} = \{1,2\}$, $I_{\mathcal{L}} = \{5,6\}$,	$q(q-1)$	q^2
$\mathcal{F}_{\mathbf{9}}$	$\chi_{b_3}^{a_9}$	$I_{\mathcal{A}} = \{1,4\}$, $I_{\mathcal{L}} = \{5,7\}$,	$q(q-1)$	q^2
$\mathcal{F}_{\mathbf{10}}$	$\chi_{b_3}^{a_{10}}$	$I_{\mathcal{A}} = \{2,4\}$, $I_{\mathcal{L}} = \{6,7\}$,	$q(q-1)$	q^2
$\mathcal{F}_{\mathbf{11}}$	$\chi_{b_3,b_5,b_6,b_7}^{a_{11}}$	$I_{\mathcal{A}} = \{1,2,4\}$, $I_{\mathcal{L}} = \{8,9,10\}$,	$q^4(q-1)$	q^3
$\mathcal{F}_{\mathbf{12}}$	$\chi_{b_1,b_2,b_4}^{a_{12}}$	$I_{\mathcal{A}} = \{3,5,6,7\}$, $I_{\mathcal{L}} = \{8,9,10,11\}$,	$q^3(q-1)$	q^4
$\mathcal{F}_{\mathbf{1,6}}$	χ^{a_1,a_6}	$I_{\mathcal{A}} = \{2\}$, $I_{\mathcal{L}} = \{3\}$,	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{1,7}}$	χ^{a_1,a_7}	$I_{\mathcal{A}} = \{3\}$, $I_{\mathcal{L}} = \{4\}$,	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{1,10}}$	$\chi_{b_3}^{a_1,a_{10}}$	$I_{\mathcal{A}} = \{2,4\}$, $I_{\mathcal{L}} = \{6,7\}$,	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{2,5}}$	χ^{a_2,a_5}	$I_{\mathcal{A}} = \{1\}$, $I_{\mathcal{L}} = \{3\}$,	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{2,7}}$	χ^{a_2,a_7}	$I_{\mathcal{A}} = \{3\}$, $I_{\mathcal{L}} = \{4\}$,	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{2,9}}$	$\chi_{b_3}^{a_2,a_9}$	$I_{\mathcal{A}} = \{1,4\}$, $I_{\mathcal{L}} = \{5,7\}$,	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{4,5}}$	χ^{a_4,a_5}	$I_{\mathcal{A}} = \{1\}$, $I_{\mathcal{L}} = \{3\}$,	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{4,6}}$	χ^{a_4,a_6}	$I_{\mathcal{A}} = \{2\}$, $I_{\mathcal{L}} = \{3\}$,	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{4,8}}$	$\chi_{b_3}^{a_4,a_8}$	$I_{\mathcal{A}} = \{1,2\}$, $I_{\mathcal{L}} = \{5,6\}$,	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{5,6}}$	$\chi_{b_2}^{a_5,a_6}$	$I_{\mathcal{A}} = \{1\}$, $I_{\mathcal{L}} = \{3\}$,	$q(q-1)^2$	q
$\mathcal{F}_{\mathbf{5,7}}$	$\chi_{b_4}^{a_5,a_7}$	$I_{\mathcal{A}} = \{1\}$, $I_{\mathcal{L}} = \{3\}$,	$q(q-1)^2$	q
$\mathcal{F}_{\mathbf{5,10}}$	$\chi^{a_5,a_{10}}$	$I_{\mathcal{A}} = \{2,3,4\}$, $I_{\mathcal{L}} = \{1,6,7\}$,	$(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{6,7}}$	$\chi_{b_4}^{a_6,a_7}$	$I_{\mathcal{A}} = \{2\}$, $I_{\mathcal{L}} = \{3\}$,	$q(q-1)^2$	q
$\mathcal{F}_{\mathbf{6,9}}$	χ^{a_6,a_9}	$I_{\mathcal{A}} = \{1,3,4\}$, $I_{\mathcal{L}} = \{2,5,7\}$,	$(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{7,8}}$	χ^{a_7,a_8}	$I_{\mathcal{A}} = \{1,2,4\}$, $I_{\mathcal{L}} = \{3,5,6\}$,	$(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{8,9}}$	χ^{a_7,a_8,a_9} $\chi_{b_3,b_4}^{a_8,a_9}$	$I_{\mathcal{A}} = \{1,4,5\}$, $I_{\mathcal{L}} = \{2,3,6\}$, $I_{\mathcal{A}} = \{1,5\}$, $I_{\mathcal{L}} = \{2,6\}$,	$(q-1)^3$ $q^2(q-1)^2$	q^3 q^2
$\mathcal{F}_{\mathbf{8,10}}$	$\chi^{a_7,a_8,a_{10}}$ $\chi_{b_3,b_4}^{a_8,a_{10}}$	$I_{\mathcal{A}} = \{1,3,5\}$, $I_{\mathcal{L}} = \{2,4,6\}$, $I_{\mathcal{A}} = \{1,5\}$, $I_{\mathcal{L}} = \{2,6\}$,	$(q-1)^3$ $q^2(q-1)^2$	q^3 q^2
$\mathcal{F}_{\mathbf{9,10}}$	$\chi^{a_6,a_9,a_{10}}$ $\chi_{b_2,b_3}^{a_9,a_{10}}$	$I_{\mathcal{A}} = \{1,2,5\}$, $I_{\mathcal{L}} = \{3,4,7\}$, $I_{\mathcal{A}} = \{1,5\}$, $I_{\mathcal{L}} = \{4,7\}$,	$(q-1)^3$ $q^2(q-1)^2$	q^3 q^2
$\mathcal{F}_{\mathbf{1,2,7}}$	χ^{a_1,a_2,a_7}	$I_{\mathcal{A}} = \{3\}$, $I_{\mathcal{L}} = \{4\}$,	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{1,4,6}}$	χ^{a_1,a_4,a_6}	$I_{\mathcal{A}} = \{2\}$, $I_{\mathcal{L}} = \{3\}$,	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{1,6,7}}$	$\chi_{b_4}^{a_1,a_6,a_7}$	$I_{\mathcal{A}} = \{2\}$, $I_{\mathcal{L}} = \{3\}$,	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{2,4,5}}$	χ^{a_2,a_4,a_5}	$I_{\mathcal{A}} = \{1\}$, $I_{\mathcal{L}} = \{3\}$,	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{2,5,7}}$	$\chi_{b_4}^{a_2,a_5,a_7}$	$I_{\mathcal{A}} = \{1\}$, $I_{\mathcal{L}} = \{3\}$,	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{4,5,6}}$	$\chi_{b_2}^{a_4,a_5,a_6}$	$I_{\mathcal{A}} = \{1\}$, $I_{\mathcal{L}} = \{3\}$,	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{5,6,7}}$	$\chi_{b_2,b_4}^{a_5,a_6,a_7}$	$I_{\mathcal{A}} = \{1\}$, $I_{\mathcal{L}} = \{3\}$,	$q^2(q-1)^3$	q
$\mathcal{F}_{\mathbf{8,9,10}}^{\boldsymbol{p\geq 3}}$	$\boldsymbol{\chi_{b_3}^{a_8,a_9,a_{10}}}$	$\boldsymbol{I_{\mathcal{I}} = \{1,2,4\}}$ $\boldsymbol{I_{\mathcal{J}} = \{5,6,7\}}$	$\boldsymbol{q(q-1)^3}$	$\boldsymbol{q^3}$
$\mathcal{F}_{\mathbf{8,9,10}}^{\boldsymbol{p=2}}$	$\boldsymbol{\chi^{a_8,a_9,a_{10}}}$ $\boldsymbol{\chi_{c_{1,2,4},c_3}^{a_{5,6,7},a_8,a_9,a_{10}}}$	**See** $\mathfrak{C}^{\mathrm{D}_4}$ **in Section 4.2**	$\boldsymbol{(q-1)^3}$ $\boldsymbol{4(q-1)^4}$	$\boldsymbol{q^3}$ $\boldsymbol{q^3/2}$

Table D.3: The parametrization of the irreducible characters of $\mathrm{UD}_4(q)$ for every $q = p^e$.

Parametrization of the irreducible characters of U_{F_4}

$\mathcal{F}$	χ	I	Number	Degree
$\mathcal{F}_{\mathrm{lin}}$	χ_{b_1,b_2,b_3,b_4}		q^4	1
$\mathcal{F}_{\mathbf{5}}$	χ^{a_5}	$I_{\mathcal{A}} = \{1\}$, $I_{\mathcal{L}} = \{2\}$,	$q-1$	q^1
$\mathcal{F}_{\mathbf{6}}$	χ^{a_6}	$I_{\mathcal{A}} = \{2\}$, $I_{\mathcal{L}} = \{3\}$,	$q-1$	q^1
$\mathcal{F}_{\mathbf{7}}$	χ^{a_7}	$I_{\mathcal{A}} = \{3\}$, $I_{\mathcal{L}} = \{4\}$,	$q-1$	q^1
$\mathcal{F}_{\mathbf{8}}$	$\chi^{a_8}_{b_2}$	$I_{\mathcal{A}} = \{1,3\}$, $I_{\mathcal{L}} = \{5,6\}$,	$q(q-1)$	q^2
$\mathcal{F}_{\mathbf{9}}$	$\chi^{a_9}_{b_2}$	$I_{\mathcal{A}} = \{3\}$, $I_{\mathcal{L}} = \{6\}$,	$q(q-1)$	q^1
$\mathcal{F}_{\mathbf{10}}$	$\chi^{a_{10}}_{b_3}$	$I_{\mathcal{A}} = \{2,4\}$, $I_{\mathcal{L}} = \{6,7\}$,	$q(q-1)$	q^2
$\mathcal{F}_{\mathbf{11}}$	$\chi^{a_{11}}_{b_2,b_5,b_6}$	$I_{\mathcal{A}} = \{1,3\}$, $I_{\mathcal{L}} = \{8,9\}$,	$q^3(q-1)$	q^2
$\mathcal{F}_{\mathbf{12}}$	$\chi^{a_6,a_{12}}$	$I_{\mathcal{A}} = \{1,3,4,7\}$,	$(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{2,5,8,10\}$,		
	$\chi^{a_{12}}_{b_2,b_3}$	$I_{\mathcal{A}} = \{1,4,7\}$, $I_{\mathcal{L}} = \{5,8,10\}$,	$q^2(q-1)$	q^3
$\mathcal{F}_{\mathbf{13}}$	$\chi^{a_{13}}_{b_2}$	$I_{\mathcal{A}} = \{3,4,7\}$, $I_{\mathcal{L}} = \{6,9,10\}$,	$q(q-1)$	q^3
$\mathcal{F}_{\mathbf{14}}$	$\chi^{a_{14}}_{b_1,b_3}$	$I_{\mathcal{A}} = \{2,5,6\}$, $I_{\mathcal{L}} = \{8,9,11\}$,	$q^2(q-1)$	q^3
$\mathcal{F}_{\mathbf{15}}$	$\chi^{a_{15}}_{b_2,b_5,b_6,b_9,b_{10}}$	$I_{\mathcal{A}} = \{1,3,4,7\}$,	$q^5(q-1)$	q^4
		$I_{\mathcal{L}} = \{8,11,12,13\}$,		
$\mathcal{F}_{\mathbf{16}}$	$\chi^{a_9,a_{16}}_{b_2}$	$I_{\mathcal{A}} = \{3,4,7\}$, $I_{\mathcal{L}} = \{6,10,13\}$,	$q(q-1)^2$	q^3
	$\chi^{a_6,a_{16}}$	$I_{\mathcal{A}} = \{3,4,7\}$, $I_{\mathcal{L}} = \{2,10,13\}$,	$(q-1)^2$	q^3
	$\chi^{a_{16}}_{b_2,b_3}$	$I_{\mathcal{A}} = \{4,7\}$, $I_{\mathcal{L}} = \{10,13\}$,	$q^2(q-1)$	q^2
$\mathcal{F}_{\mathbf{17}}$	$\chi^{a_{11},a_{17}}_{b_3,b_7}$	$I_{\mathcal{A}} = \{1,2,4,5,6,8\}$,	$q^2(q-1)^2$	q^6
		$I_{\mathcal{L}} = \{9,10,12,13,14,15\}$,		
	$\chi^{a_{17}}_{b_1,b_3,b_7,b_9}$	$I_{\mathcal{A}} = \{2,4,5,6,8\}$,	$q^4(q-1)$	q^5
		$I_{\mathcal{L}} = \{10,12,13,14,15\}$,		
$\mathcal{F}_{\mathbf{18}}$	$\chi^{a_{11},a_{18}}_{b_2,b_5,b_6,b_9,b_{10},b_{13}}$	$I_{\mathcal{A}} = \{1,3,4,7\}$,	$q^6(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{8,12,15,16\}$,		
	$\chi^{a_{13},a_{18}}_{b_2,b_5,b_6,b_8,b_9}$	$I_{\mathcal{A}} = \{1,3,4,7\}$,	$q^5(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{10,12,15,16\}$,		
	$\chi^{a_8,a_{18}}_{b_2,b_6,b_9,b_{10}}$	$I_{\mathcal{A}} = \{1,3,4,7\}$,	$q^4(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{5,12,15,16\}$,		
	$\chi^{a_9,a_{18}}_{b_2,b_5,b_{10}}$	$I_{\mathcal{A}} = \{1,3,4,7\}$,	$q^3(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{6,12,15,16\}$,		
	$\chi^{a_6,a_{18}}_{b_5,b_{10}}$	$I_{\mathcal{A}} = \{1,3,4,7\}$,	$q^2(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{2,12,15,16\}$,		
	$\chi^{a_{18}}_{b_2,b_3,b_5,b_{10}}$	$I_{\mathcal{A}} = \{1,4,7\}$, $I_{\mathcal{L}} = \{12,15,16\}$,	$q^4(q-1)$	q^3
$\mathcal{F}_{\mathbf{19}}$	$\chi^{a_5,a_{19}}_{b_4}$	$I_{\mathcal{A}} = \{2,3,6,7,8,9,10\}$,	$q(q-1)^2$	q^7
		$I_{\mathcal{L}} = \{1,11,12,13,14,15,17\}$,		
	$\chi^{a_{19}}_{b_1,b_2,b_4}$	$I_{\mathcal{A}} = \{3,6,7,8,9,10\}$,	$q^3(q-1)$	q^6
		$I_{\mathcal{L}} = \{11,12,13,14,15,17\}$,		
$\mathcal{F}_{\mathbf{20}}$	$\chi^{a_9,a_{14},a_{15},a_{20}}_{b_{11}}$	$I_{\mathcal{A}} = \{1,2,3,4,5,8,12\}$,	$q(q-1)^4$	q^7
		$I_{\mathcal{L}} = \{6,7,10,13,16,17,18\}$,		
	$\chi^{a_{14},a_{15},a_{20}}_{b_3,b_6,b_{11}}$	$I_{\mathcal{A}} = \{1,2,4,5,8,12\}$,	$q^3(q-1)^3$	q^6
		$I_{\mathcal{L}} = \{7,10,13,16,17,18\}$,		
	$\chi^{a_{11},a_{13},a_{14},a_{20}}$	$I_{\mathcal{A}} = \{1,2,3,4,5,6,10\}$,	$(q-1)^4$	q^7
		$I_{\mathcal{L}} = \{7,8,9,12,16,17,18\}$,		
	$\chi^{a_{13},a_{14},a_{20}}_{b_1,b_3,b_8,b_9}$	$I_{\mathcal{A}} = \{2,4,5,6,10\}$,	$q^4(q-1)^3$	q^5

$\mathcal{F}$	χ	I	Number	Degree
		$I_{\mathcal{L}} = \{7, 12, 16, 17, 18\}$,		
	$\chi_{b_3,b_7}^{a_{11},a_{14},a_{20}}$	$I_{\mathcal{A}} = \{1, 2, 4, 5, 8, 10\}$,	$q^2(q-1)^3$	q^6
		$I_{\mathcal{L}} = \{6, 9, 12, 16, 17, 18\}$,		
	$\chi_{b_1,b_3,b_7,b_9}^{a_{14},a_{20}}$	$I_{\mathcal{A}} = \{2, 4, 5, 6, 10\}$,	$q^4(q-1)^2$	q^5
		$I_{\mathcal{L}} = \{8, 12, 16, 17, 18\}$,		
	$\chi_{b_{11}}^{a_9,a_{15},a_{20}}$	$I_{\mathcal{A}} = \{1, 2, 3, 4, 5, 8, 12\}$,	$q(q-1)^3$	q^7
		$I_{\mathcal{L}} = \{6, 7, 10, 13, 16, 17, 18\}$,		
	$\chi_{b_3,b_6,b_{11}}^{a_{15},a_{20}}$	$I_{\mathcal{A}} = \{1, 2, 4, 5, 8, 12\}$,	$q^3(q-1)^2$	q^6
		$I_{\mathcal{L}} = \{7, 10, 13, 16, 17, 18\}$,		
	$\chi^{a_{11},a_{13},a_{20}}$	$I_{\mathcal{A}} = \{1, 2, 3, 4, 5, 6, 10\}$,	$(q-1)^3$	q^7
		$I_{\mathcal{L}} = \{7, 8, 9, 12, 16, 17, 18\}$,		
	$\chi_{b_1,b_3,b_8,b_9}^{a_{13},a_{20}}$	$I_{\mathcal{A}} = \{2, 4, 5, 6, 10\}$,	$q^4(q-1)^2$	q^5
		$I_{\mathcal{L}} = \{7, 12, 16, 17, 18\}$,		
	$\chi_{b_6,b_7}^{a_{11},a_{20}}$	$I_{\mathcal{A}} = \{1, 2, 3, 4, 5, 10\}$,	$q^2(q-1)^2$	q^6
		$I_{\mathcal{L}} = \{8, 9, 12, 16, 17, 18\}$,		
	$\chi_{b_1,b_7,b_8}^{a_9,a_{20}}$	$I_{\mathcal{A}} = \{2, 4, 5, 6, 10\}$,	$q^3(q-1)^2$	q^5
		$I_{\mathcal{L}} = \{3, 12, 16, 17, 18\}$,		
	$\chi_{b_3,b_7}^{a_8,a_{20}}$	$I_{\mathcal{A}} = \{1, 2, 4, 5, 10\}$,	$q^2(q-1)^2$	q^5
		$I_{\mathcal{L}} = \{6, 12, 16, 17, 18\}$,		
	$\chi_{b_1,b_3,b_6,b_7}^{a_{20}}$	$I_{\mathcal{A}} = \{2, 4, 5, 10\}$,	$q^4(q-1)$	q^4
		$I_{\mathcal{L}} = \{12, 16, 17, 18\}$,		
$\mathcal{F}_{\mathbf{21}}$	$\chi_{b_1}^{a_{14},a_{21}}$	$I_{\mathcal{A}} = \{2, 3, 4, 5, 6, 7, 8, 10, 13\}$,	$q(q-1)^2$	q^9
		$I_{\mathcal{L}} = \{9, 11, 12, 15, 16, 17, 18, 19, 20\}$,		
	$\chi_{b_2,b_5}^{a_{11},a_{21}}$	$I_{\mathcal{A}} = \{1, 3, 4, 6, 7, 8, 10, 13\}$,	$q^2(q-1)^2$	q^8
		$I_{\mathcal{L}} = \{9, 12, 15, 16, 17, 18, 19, 20\}$,		
	$\chi_{b_9}^{a_5,a_{21}}$	$I_{\mathcal{A}} = \{2, 3, 4, 6, 7, 8, 10, 13\}$,	$q(q-1)^2$	q^8
		$I_{\mathcal{L}} = \{1, 12, 15, 16, 17, 18, 19, 20\}$,		
	$\chi_{b_1,b_2,b_9}^{a_{21}}$	$I_{\mathcal{A}} = \{3, 4, 6, 7, 8, 10, 13\}$,	$q^3(q-1)$	q^7
		$I_{\mathcal{L}} = \{12, 15, 16, 17, 18, 19, 20\}$,		
$\mathcal{F}_{\mathbf{22}}$	$\boldsymbol{\chi_{b_1}^{a_{14},a_{20},a_{22}}}$	$\boldsymbol{I_{\mathcal{A}} = \{3, 4, 6, 7, 10, 13\},\ I_{\mathcal{I}} = \{2, 9, 16\}.}$	$\boldsymbol{q(q-1)^3}$	$\boldsymbol{q^9}$
		$\boldsymbol{I_{\mathcal{L}} = \{8, 12, 15, 17, 19, 21\},\ I_{\mathcal{J}} = \{5, 11, 18\},}$		
	$\chi_{b_2,b_5}^{a_8,a_{20},a_{22}}$	$I_{\mathcal{A}} = \{1, 3, 4, 7, 10, 13, 16, 18\}$,	$q^2(q-1)^3$	q^8
		$I_{\mathcal{L}} = \{6, 9, 11, 12, 15, 17, 19, 21\}$,		
	$\chi_{b_6}^{a_5,a_{20},a_{22}}$	$I_{\mathcal{A}} = \{2, 3, 4, 7, 10, 13, 16, 18\}$,	$q(q-1)^3$	q^8
		$I_{\mathcal{L}} = \{1, 9, 11, 12, 15, 17, 19, 21\}$,		
	$\chi_{b_1,b_2,b_6}^{a_{20},a_{22}}$	$I_{\mathcal{A}} = \{3, 4, 7, 10, 13, 16, 18\}$,	$q^3(q-1)^2$	q^7
		$I_{\mathcal{L}} = \{9, 11, 12, 15, 17, 19, 21\}$,		
	$\chi_{b_1}^{a_{17},a_{22}}$	$I_{\mathcal{A}} = \{2, 3, 4, 6, 7, 9, 10, 11, 13\}$,	$q(q-1)^2$	q^9
		$I_{\mathcal{L}} = \{5, 8, 12, 14, 15, 16, 18, 19, 21\}$,		
	$\chi_{b_2,b_4,b_5}^{a_{12},a_{14},a_{22}}$	$I_{\mathcal{A}} = \{1, 3, 7, 8, 9, 11, 13\}$,	$q^3(q-1)^3$	q^7
		$I_{\mathcal{L}} = \{6, 10, 15, 16, 18, 19, 21\}$,		
	$\chi_{b_4,b_{10}}^{a_5,a_{14},a_{22}}$	$I_{\mathcal{A}} = \{2, 3, 6, 7, 9, 11, 13\}$,	$q^2(q-1)^3$	q^7
		$I_{\mathcal{L}} = \{1, 8, 15, 16, 18, 19, 21\}$,		
	$\chi_{b_1,b_2,b_4,b_{10}}^{a_{14},a_{22}}$	$I_{\mathcal{A}} = \{3, 6, 7, 9, 11, 13\}$,	$q^4(q-1)^2$	q^6
		$I_{\mathcal{L}} = \{8, 15, 16, 18, 19, 21\}$,		
	$\chi_{b_2,b_5,b_6}^{a_{12},a_{22}}$	$I_{\mathcal{A}} = \{1, 3, 4, 7, 9, 11, 13\}$,	$q^3(q-1)^2$	q^7
		$I_{\mathcal{L}} = \{8, 10, 15, 16, 18, 19, 21\}$,		
	$\chi_{b_8}^{a_5,a_{10},a_{22}}$	$I_{\mathcal{A}} = \{2, 3, 6, 7, 9, 11, 13\}$,	$q(q-1)^3$	q^7
		$I_{\mathcal{L}} = \{1, 4, 15, 16, 18, 19, 21\}$,		
	$\chi_{b_1,b_2,b_8}^{a_{10},a_{22}}$	$I_{\mathcal{A}} = \{3, 6, 7, 9, 11, 13\}$,	$q^3(q-1)^2$	q^6
		$I_{\mathcal{L}} = \{4, 15, 16, 18, 19, 21\}$,		

$\mathcal{F}$	χ	I	Number	Degree
	$\chi^{a_8,a_{22}}_{b_2,b_4,b_5}$	$I_{\mathcal{A}} = \{1,3,7,9,11,13\}$,	$q^3(q-1)^2$	q^6
		$I_{\mathcal{L}} = \{6,15,16,18,19,21\}$,		
	$\chi^{a_5,a_{22}}_{b_4,b_6}$	$I_{\mathcal{A}} = \{2,3,7,9,11,13\}$,	$q^2(q-1)^2$	q^6
		$I_{\mathcal{L}} = \{1,15,16,18,19,21\}$,		
	$\chi^{a_{22}}_{b_1,b_2,b_4,b_6}$	$I_{\mathcal{A}} = \{3,7,9,11,13\}$,	$q^4(q-1)$	q^5
		$I_{\mathcal{L}} = \{15,16,18,19,21\}$,		
$\mathcal{F}_{\mathbf{23}}$	$\chi^{a_{11},a_{18},a_{23}}_{b_1,b_5}$	$I_{\mathcal{A}} = \{2,3,4,6,7,9,10,13,16\}$,	$q^2(q-1)^3$	q^9
		$I_{\mathcal{L}} = \{8,12,14,15,17,19,20,21,22\}$,		
	$\chi^{a_8,a_{18},a_{23}}_{b_1}$	$I_{\mathcal{A}} = \{2,3,4,6,7,9,10,13,16\}$,	$q(q-1)^3$	q^9
		$I_{\mathcal{L}} = \{5,12,14,15,17,19,20,21,22\}$,		
	$\chi^{a_{18},a_{23}}_{b_1,b_3,b_5}$	$I_{\mathcal{A}} = \{2,4,6,7,9,10,13,16\}$,	$q^3(q-1)^2$	q^8
		$I_{\mathcal{L}} = \{12,14,15,17,19,20,21,22\}$,		
	$\chi^{a_{15},a_{23}}_{b_1,b_5}$	$I_{\mathcal{A}} = \{2,3,4,6,7,9,10,13,16\}$,	$q^2(q-1)^2$	q^9
		$I_{\mathcal{L}} = \{8,11,12,14,17,19,20,21,22\}$,		
	$\chi^{a_{11},a_{12},a_{23}}_{b_1,b_4}$	$I_{\mathcal{A}} = \{2,5,6,8,9,10,13,16\}$,	$q^2(q-1)^3$	q^8
		$I_{\mathcal{L}} = \{3,7,14,17,19,20,21,22\}$,		
	$\chi^{a_{11},a_{23}}_{b_1,b_4,b_5,b_7}$	$I_{\mathcal{A}} = \{2,3,6,9,10,13,16\}$,	$q^4(q-1)^2$	q^7
		$I_{\mathcal{L}} = \{8,14,17,19,20,21,22\}$,		
	$\chi^{a_{12},a_{23}}_{b_1,b_3}$	$I_{\mathcal{A}} = \{2,4,6,7,9,10,13,16\}$,	$q^2(q-1)^2$	q^8
		$I_{\mathcal{L}} = \{5,8,14,17,19,20,21,22\}$,		
	$\chi^{a_8,a_{23}}_{b_1,b_4,b_7}$	$I_{\mathcal{A}} = \{2,3,6,9,10,13,16\}$,	$q^3(q-1)^2$	q^7
		$I_{\mathcal{L}} = \{5,14,17,19,20,21,22\}$,		
	$\chi^{a_7,a_{23}}_{b_1,b_5}$	$I_{\mathcal{A}} = \{2,4,6,9,10,13,16\}$,	$q^2(q-1)^2$	q^7
		$I_{\mathcal{L}} = \{3,14,17,19,20,21,22\}$,		
	$\chi^{a_{23}}_{b_1,b_3,b_4,b_5}$	$I_{\mathcal{A}} = \{2,6,9,10,13,16\}$,	$q^4(q-1)$	q^6
		$I_{\mathcal{L}} = \{14,17,19,20,21,22\}$,		
$\mathcal{F}_{\mathbf{24}}$	$\chi^{a_9,a_{16},a_{24}}_{b_2}$	$I_{\mathcal{A}} = \{1,3,4,5,7,8,11,12,14,15\}$,	$q(q-1)^3$	q^{10}
		$I_{\mathcal{L}} = \{6,10,13,17,18,19,20,21,22,23\}$,		
	$\chi^{a_6,a_{16},a_{24}}$	$I_{\mathcal{A}} = \{1,3,4,5,7,8,11,12,14,15\}$,	$(q-1)^3$	q^{10}
		$I_{\mathcal{L}} = \{2,10,13,17,18,19,20,21,22,23\}$,		
	$\chi^{a_{16},a_{24}}_{b_2,b_3}$	$I_{\mathcal{A}} = \{1,4,5,7,8,11,12,14,15\}$,	$q^2(q-1)^2$	q^9
		$I_{\mathcal{L}} = \{10,13,17,18,19,20,21,22,23\}$,		
	$\chi^{a_{13},a_{24}}_{b_2}$	$I_{\mathcal{A}} = \{1,3,4,5,7,8,11,12,14,15\}$,	$q(q-1)^2$	q^{10}
		$I_{\mathcal{L}} = \{6,9,10,17,18,19,20,21,22,23\}$,		
	$\chi^{a_{10},a_{24}}_{b_3,b_9}$	$I_{\mathcal{A}} = \{1,2,5,6,8,11,12,14,15\}$,	$q^2(q-1)^2$	q^9
		$I_{\mathcal{L}} = \{4,7,17,18,19,20,21,22,23\}$,		
	$\chi^{a_9,a_{24}}_{b_2,b_4,b_7}$	$I_{\mathcal{A}} = \{1,3,5,8,11,12,14,15\}$,	$q^3(q-1)^2$	q^8
		$I_{\mathcal{L}} = \{6,17,18,19,20,21,22,23\}$,		
	$\chi^{a_6,a_{24}}_{b_4,b_7}$	$I_{\mathcal{A}} = \{1,3,5,8,11,12,14,15\}$,	$q^2(q-1)^2$	q^8
		$I_{\mathcal{L}} = \{2,17,18,19,20,21,22,23\}$,		
	$\chi^{a_7,a_{24}}_{b_2}$	$I_{\mathcal{A}} = \{1,4,5,8,11,12,14,15\}$,	$q(q-1)^2$	q^8
		$I_{\mathcal{L}} = \{3,17,18,19,20,21,22,23\}$,		
	$\chi^{a_{24}}_{b_2,b_3,b_4}$	$I_{\mathcal{A}} = \{1,5,8,11,12,14,15\}$,	$q^3(q-1)$	q^7
		$I_{\mathcal{L}} = \{17,18,19,20,21,22,23\}$,		
$\mathcal{F}_{\mathbf{1,6}}$	χ^{a_1,a_6}	$I_{\mathcal{A}} = \{2\}$, $I_{\mathcal{L}} = \{3\}$,	$(q-1)^2$	q^1
$\mathcal{F}_{\mathbf{1,7}}$	χ^{a_1,a_7}	$I_{\mathcal{A}} = \{3\}$, $I_{\mathcal{L}} = \{4\}$,	$(q-1)^2$	q^1
$\mathcal{F}_{\mathbf{1,9}}$	$\chi^{a_1,a_9}_{b_2}$	$I_{\mathcal{A}} = \{3\}$, $I_{\mathcal{L}} = \{6\}$,	$q(q-1)^2$	q^1
$\mathcal{F}_{\mathbf{1,10}}$	$\chi^{a_1,a_{10}}_{b_3}$	$I_{\mathcal{A}} = \{2,4\}$, $I_{\mathcal{L}} = \{6,7\}$,	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{1,13}}$	$\chi^{a_1,a_{13}}_{b_2}$	$I_{\mathcal{A}} = \{3,4,7\}$, $I_{\mathcal{L}} = \{6,9,10\}$,	$q(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{1,16}}$	$\chi^{a_1,a_9,a_{16}}_{b_2}$	$I_{\mathcal{A}} = \{3,4,7\}$, $I_{\mathcal{L}} = \{6,10,13\}$,	$q(q-1)^3$	q^3
	$\chi^{a_1,a_6,a_{16}}$	$I_{\mathcal{A}} = \{3,4,7\}$, $I_{\mathcal{L}} = \{2,10,13\}$,	$(q-1)^3$	q^3

$\mathcal{F}$	χ	I	Number	Degree
	$\chi_{b_2,b_3}^{a_1,a_{16}}$	$I_{\mathcal{A}} = \{4,7\}$, $I_{\mathcal{L}} = \{10,13\}$,	$q^2(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{2,7}}$	χ^{a_2,a_7}	$I_{\mathcal{A}} = \{3\}$, $I_{\mathcal{L}} = \{4\}$,	$(q-1)^2$	q^1
$\mathcal{F}_{\mathbf{3,5}}$	χ^{a_3,a_5}	$I_{\mathcal{A}} = \{1\}$, $I_{\mathcal{L}} = \{2\}$,	$(q-1)^2$	q^1
$\mathcal{F}_{\mathbf{4,5}}$	χ^{a_4,a_5}	$I_{\mathcal{A}} = \{1\}$, $I_{\mathcal{L}} = \{2\}$,	$(q-1)^2$	q^1
$\mathcal{F}_{\mathbf{4,6}}$	χ^{a_4,a_6}	$I_{\mathcal{A}} = \{2\}$, $I_{\mathcal{L}} = \{3\}$,	$(q-1)^2$	q^1
$\mathcal{F}_{\mathbf{4,8}}$	$\chi_{b_2}^{a_4,a_8}$	$I_{\mathcal{A}} = \{1,3\}$, $I_{\mathcal{L}} = \{5,6\}$,	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{4,9}}$	$\chi_{b_2}^{a_4,a_9}$	$I_{\mathcal{A}} = \{3\}$, $I_{\mathcal{L}} = \{6\}$,	$q(q-1)^2$	q^1
$\mathcal{F}_{\mathbf{4,11}}$	$\chi_{b_2,b_5,b_6}^{a_4,a_{11}}$	$I_{\mathcal{A}} = \{1,3\}$, $I_{\mathcal{L}} = \{8,9\}$,	$q^3(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{4,14}}$	$\chi_{b_1,b_3}^{a_4,a_{14}}$	$I_{\mathcal{A}} = \{2,5,6\}$, $I_{\mathcal{L}} = \{8,9,11\}$,	$q^2(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{5,6}}$	$\chi_{b_3}^{a_5,a_6}$	$I_{\mathcal{A}} = \{1\}$, $I_{\mathcal{L}} = \{2\}$,	$q(q-1)^2$	q^1
$\mathcal{F}_{\mathbf{5,7}}$	χ^{a_5,a_7}	$I_{\mathcal{A}} = \{1,3\}$, $I_{\mathcal{L}} = \{2,4\}$,	$(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{5,9}}$	χ^{a_5,a_9}	$I_{\mathcal{A}} = \{2,3\}$, $I_{\mathcal{L}} = \{1,6\}$,	$(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{5,10}}$	$\chi_{b_1,b_3}^{a_5,a_{10}}$	$I_{\mathcal{A}} = \{2,4\}$, $I_{\mathcal{L}} = \{6,7\}$,	$q^2(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{5,13}}$	$\chi^{a_5,a_{13}}$	$I_{\mathcal{A}} = \{2,3,4,7\}$, $I_{\mathcal{L}} = \{1,6,9,10\}$,	$(q-1)^2$	q^4
$\mathcal{F}_{\mathbf{5,16}}$	$\chi^{a_5,a_9,a_{16}}$	$I_{\mathcal{A}} = \{2,3,4,7\}$, $I_{\mathcal{L}} = \{1,6,10,13\}$,	$(q-1)^3$	q^4
	$\chi_{b_3,b_6}^{a_5,a_{16}}$	$I_{\mathcal{A}} = \{2,4,7\}$, $I_{\mathcal{L}} = \{1,10,13\}$,	$q^2(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{6,7}}$	$\chi_{b_4}^{a_6,a_7}$	$I_{\mathcal{A}} = \{2\}$, $I_{\mathcal{L}} = \{3\}$,	$q(q-1)^2$	q^1
$\mathcal{F}_{\mathbf{7,8}}$	$\chi_{b_2,b_4}^{a_7,a_8}$	$I_{\mathcal{A}} = \{1,3\}$, $I_{\mathcal{L}} = \{5,6\}$,	$q^2(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{7,9}}$	$\chi_{b_2,b_4}^{a_7,a_9}$	$I_{\mathcal{A}} = \{3\}$, $I_{\mathcal{L}} = \{6\}$,	$q^2(q-1)^2$	q^1
$\mathcal{F}_{\mathbf{7,11}}$	$\chi_{b_2,b_4,b_5,b_6}^{a_7,a_{11}}$	$I_{\mathcal{A}} = \{1,3\}$, $I_{\mathcal{L}} = \{8,9\}$,	$q^4(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{7,14}}$	$\chi_{b_1}^{a_7,a_{14}}$	$I_{\mathcal{A}} = \{2,4,5,6\}$, $I_{\mathcal{L}} = \{3,8,9,11\}$,	$q(q-1)^2$	q^4
$\mathcal{F}_{\mathbf{8,9}}$	$\chi_{b_2}^{a_8,a_9}$	$I_{\mathcal{A}} = \{1,3\}$, $I_{\mathcal{L}} = \{5,6\}$,	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{8,10}}$	$\chi_{b_4}^{a_8,a_{10}}$	$I_{\mathcal{A}} = \{2,3,6\}$, $I_{\mathcal{L}} = \{1,5,7\}$,	$q(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{8,13}}$	$\chi^{a_5,a_8,a_{13}}$	$I_{\mathcal{A}} = \{2,3,4,6\}$, $I_{\mathcal{L}} = \{1,7,9,10\}$,	$(q-1)^3$	q^4
	$\chi_{b_1,b_2}^{a_8,a_{13}}$	$I_{\mathcal{A}} = \{3,4,6\}$, $I_{\mathcal{L}} = \{7,9,10\}$,	$q^2(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{8,16}}$	$\chi_{b_2,b_9}^{a_8,a_{16}}$	$I_{\mathcal{A}} = \{1,3,4,7\}$, $I_{\mathcal{L}} = \{5,6,10,13\}$,	$q^2(q-1)^2$	q^4
$\mathcal{F}_{\mathbf{9,10}}$	$\chi_{b_4}^{a_9,a_{10}}$	$I_{\mathcal{A}} = \{2,6\}$, $I_{\mathcal{L}} = \{3,7\}$,	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{9,12}}$	$\chi_{b_2}^{a_9,a_{12}}$	$I_{\mathcal{A}} = \{1,3,4,7\}$, $I_{\mathcal{L}} = \{5,6,8,10\}$,	$q(q-1)^2$	q^4
$\mathcal{F}_{\mathbf{10,11}}$	$\chi_{b_5}^{a_{10},a_{11}}$	$I_{\mathcal{A}} = \{1,2,3,4\}$, $I_{\mathcal{L}} = \{6,7,8,9\}$,	$q(q-1)^2$	q^4
$\mathcal{F}_{\mathbf{10,14}}$	$\chi_{b_1}^{a_7,a_{10},a_{14}}$	$I_{\mathcal{A}} = \{2,4,5,6\}$, $I_{\mathcal{L}} = \{3,8,9,11\}$,	$q(q-1)^3$	q^4
	$\chi_{b_1,b_3,b_4}^{a_{10},a_{14}}$	$I_{\mathcal{A}} = \{2,5,6\}$, $I_{\mathcal{L}} = \{8,9,11\}$,	$q^3(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{11,12}}$	$\chi_{b_2}^{a_{10},a_{11},a_{12}}$	$I_{\mathcal{A}} = \{1,3,4,7\}$, $I_{\mathcal{L}} = \{5,6,8,9\}$,	$q(q-1)^3$	q^4
	$\chi^{a_6,a_{11},a_{12}}$	$I_{\mathcal{A}} = \{1,3,5,8\}$, $I_{\mathcal{L}} = \{2,4,7,9\}$,	$(q-1)^3$	q^4
	$\chi_{b_2,b_3}^{a_{11},a_{12}}$	$I_{\mathcal{A}} = \{1,5,8\}$, $I_{\mathcal{L}} = \{4,7,9\}$,	$q^2(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{11,13}}$	$\boldsymbol{\chi^{a_5,a_{10},a_{11}^*,a_{13}}}$ $(a_{11}^* \neq a_5 a_{13}^2/a_{10}^2)$	$\boldsymbol{I_{\mathcal{A}} = \{3\}}$, $\boldsymbol{I_{\mathcal{I}} = \{1,4,7\}}$, $\boldsymbol{I_{\mathcal{L}} = \{8\}}$, $\boldsymbol{I_{\mathcal{J}} = \{2,6,9\}}$.	$\boldsymbol{(q-1)^3(q-2)}$	$\boldsymbol{q^4}$
	$\boldsymbol{\chi_{b_{1,4,7},b_{2,6,9}}^{a_5,a_{10},a_{13}}}$	**See $\mathfrak{C}^2$ in Section 4.3**	$\boldsymbol{q^2(q-1)^3}$	$\boldsymbol{q^3}$
	$\chi^{a_{10},a_{11},a_{13}}$	$I_{\mathcal{A}} = \{1,4,7,8\}$, $I_{\mathcal{L}} = \{2,3,6,9\}$,	$(q-1)^3$	q^4
	$\chi^{a_5,a_{11},a_{13}}$	$I_{\mathcal{A}} = \{1,4,6,8\}$,	$(q-1)^3$	q^4

$\mathcal{F}$	χ	I	Number	Degree
		$I_{\mathcal{L}} = \{2, 3, 7, 9\}$,		
	$\chi_{b_1,b_2}^{a_{11},a_{13}}$	$I_{\mathcal{A}} = \{4, 6, 8\}$, $I_{\mathcal{L}} = \{3, 7, 9\}$,	$q^2(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{11,16}}$	$\chi_{b_2,b_5,b_6}^{a_{11},a_{16}}$	$I_{\mathcal{A}} = \{1, 3, 4, 7\}$, $I_{\mathcal{L}} = \{8, 9, 10, 13\}$,	$q^3(q-1)^2$	q^4
$\mathcal{F}_{\mathbf{12,13}}$	$\chi_{b_2}^{a_9,a_{12},a_{13}}$	$I_{\mathcal{A}} = \{1, 5, 6, 8\}$, $I_{\mathcal{L}} = \{3, 4, 7, 10\}$,	$q(q-1)^3$	q^4
	$\chi^{a_6,a_{12},a_{13}}$	$I_{\mathcal{A}} = \{1, 2, 5, 8\}$, $I_{\mathcal{L}} = \{3, 4, 7, 10\}$,	$(q-1)^3$	q^4
	$\chi_{b_2,b_3}^{a_{12},a_{13}}$	$I_{\mathcal{A}} = \{1, 5, 8\}$, $I_{\mathcal{L}} = \{4, 7, 10\}$.	$q^2(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{12,14}}$	$\chi^{a_7,a_{12},a_{14}}$	$I_{\mathcal{A}} = \{1, 2, 4, 5, 8\}$, $I_{\mathcal{L}} = \{3, 6, 9, 10, 11\}$,	$(q-1)^3$	q^5
	$\chi_{b_3,b_4}^{a_{12},a_{14}}$	$I_{\mathcal{A}} = \{1, 2, 5, 8\}$, $I_{\mathcal{L}} = \{6, 9, 10, 11\}$,	$q^2(q-1)^2$	q^4
$\mathcal{F}_{\mathbf{12,16}}$	$\boldsymbol{\chi_{b_2}^{a_8,a_9,a_{12},a_{16}^*}}$ $(a_{16}^* \neq a_9 a_{12}^2/a_8^2)$	$\boldsymbol{I_{\mathcal{A}} = \{4\},\ I_{\mathcal{I}} = \{1, 3, 7\}}$ $\boldsymbol{I_{\mathcal{L}} = \{13\},\ I_{\mathcal{J}} = \{5, 6, 10\}}$	$\boldsymbol{q(q-1)^3(q-2)}$	$\boldsymbol{q^4}$
	$\boldsymbol{\chi^{a_8,a_9,a_{12},a_{5,6,10}}}$	**See $\mathfrak{C}^3$ in Section 4.3**	$\boldsymbol{(q-1)^4}$	$\boldsymbol{q^4}$
	$\boldsymbol{\chi_{b_2,b_{1,3,7}}^{a_8,a_9,a_{12}}}$	**See $\mathfrak{C}^3$ in Section 4.3**	$\boldsymbol{q^2(q-1)^3}$	$\boldsymbol{q^3}$
	$\chi_{b_2}^{a_8,a_{12},a_{16}}$	$I_{\mathcal{A}} = \{1, 3, 4, 7\}$, $I_{\mathcal{L}} = \{5, 6, 10, 13\}$,	$q(q-1)^3$	q^4
	$\chi_{b_2}^{a_9,a_{12},a_{16}}$	$I_{\mathcal{A}} = \{1, 3, 4, 7\}$, $I_{\mathcal{L}} = \{5, 6, 10, 13\}$,	$q(q-1)^3$	q^4
	$\chi^{a_6,a_{12},a_{16}}$	$I_{\mathcal{A}} = \{1, 3, 4, 7\}$, $I_{\mathcal{L}} = \{2, 5, 10, 13\}$,	$(q-1)^3$	q^4
	$\chi_{b_2,b_3}^{a_{12},a_{16}}$	$I_{\mathcal{A}} = \{1, 4, 7\}$, $I_{\mathcal{L}} = \{5, 10, 13\}$,	$q^2(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{13,14}}$	$\chi_{b_1,b_4,b_7}^{a_{13},a_{14}}$	$I_{\mathcal{A}} = \{2, 3, 6, 9\}$, $I_{\mathcal{L}} = \{5, 8, 10, 11\}$,	$q^3(q-1)^2$	q^4
$\mathcal{F}_{\mathbf{14,15}}^{\boldsymbol{p \geq 5}}$	$\boldsymbol{\chi^{a_{10},a_{14},a_{15}}}$	$\boldsymbol{I_{\mathcal{A}} = \{1, 3, 5\},\ I_{\mathcal{I}} = \{2, 4, 8\}}$ $\boldsymbol{I_{\mathcal{L}} = \{9, 12, 13\},\ I_{\mathcal{J}} = \{6, 7, 11\}}$	$\boldsymbol{(q-1)^3}$	$\boldsymbol{q^6}$
	$\chi_{b_4,b_7}^{a_{14},a_{15}}$	$I_{\mathcal{A}} = \{1, 3, 5, 8, 11\}$, $I_{\mathcal{L}} = \{2, 6, 9, 12, 13\}$,	$q^2(q-1)^2$	q^5
$\mathcal{F}_{\mathbf{14,15}}^{\boldsymbol{p=3}}$	$\boldsymbol{\chi_{b_{2,4,8},b_{6,7,11}}^{a_{10},a_{14},a_{15}}}$	**See $\mathfrak{C}^4$ in Section 4.3**	$\boldsymbol{q^2(q-1)^3}$	$\boldsymbol{q^5}$
	$\chi_{b_4,b_7}^{a_{14},a_{15}}$	$I_{\mathcal{A}} = \{1, 3, 5, 8, 11\}$, $I_{\mathcal{L}} = \{2, 6, 9, 12, 13\}$,	$q^2(q-1)^2$	q^5
$\mathcal{F}_{\mathbf{14,16}}$	$\chi_{b_1,b_3}^{a_{14},a_{16}}$	$I_{\mathcal{A}} = \{2, 4, 5, 6, 7\}$, $I_{\mathcal{L}} = \{8, 9, 10, 11, 13\}$,	$q^2(q-1)^2$	q^5
$\mathcal{F}_{\mathbf{14,18}}$	$\chi^{a_{13},a_{14},a_{18}}$	$I_{\mathcal{A}} = \{1, 2, 3, 4, 5, 6, 7\}$, $I_{\mathcal{L}} = \{8, 9, 10, 11, 12, 15, 16\}$,	$(q-1)^3$	q^7
	$\chi_{b_3,b_{10}}^{a_{14},a_{18}}$	$I_{\mathcal{A}} = \{1, 2, 4, 5, 6, 7\}$, $I_{\mathcal{L}} = \{8, 9, 11, 12, 15, 16\}$,	$q^2(q-1)^2$	q^6
$\mathcal{F}_{\mathbf{15,16}}$	$\chi_{b_2,b_5,b_6,b_9,b_{10}}^{a_{15},a_{16}}$	$I_{\mathcal{A}} = \{1, 3, 4, 7\}$, $I_{\mathcal{L}} = \{8, 11, 12, 13\}$,	$q^5(q-1)^2$	q^4
$\mathcal{F}_{\mathbf{16,17}}$	$\chi^{a_{11},a_{16},a_{17}}$	$I_{\mathcal{A}} = \{1, 2, 3, 4, 5, 6, 10\}$, $I_{\mathcal{L}} = \{7, 8, 9, 12, 13, 14, 15\}$,	$(q-1)^3$	q^7
	$\chi_{b_1,b_3,b_8,b_9}^{a_{16},a_{17}}$	$I_{\mathcal{A}} = \{2, 4, 5, 6, 10\}$, $I_{\mathcal{L}} = \{7, 12, 13, 14, 15\}$,	$q^4(q-1)^2$	q^5
$\mathcal{F}_{\mathbf{16,19}}$	$\chi_{b_4}^{a_5,a_{16},a_{19}}$	$I_{\mathcal{A}} = \{2, 3, 6, 7, 9, 10, 13\}$, $I_{\mathcal{L}} = \{1, 8, 11, 12, 14, 15, 17\}$,	$q(q-1)^3$	q^7
	$\chi_{b_1,b_2,b_4}^{a_{16},a_{19}}$	$I_{\mathcal{A}} = \{3, 6, 7, 9, 10, 13\}$, $I_{\mathcal{L}} = \{8, 11, 12, 14, 15, 17\}$,	$q^3(q-1)^2$	q^6
$\mathcal{F}_{\mathbf{17,18}}$	$\chi_{b_{11}}^{a_9,a_{17},a_{18}}$	$I_{\mathcal{A}} = \{1, 2, 3, 4, 5, 8, 12\}$,	$q(q-1)^3$	q^7

$\mathcal{F}$	χ	I	Number	Degree
	$\chi^{a_{17},a_{18}}_{b_3,b_6,b_{11}}$	$I_{\mathcal{L}} = \{6,7,10,13,14,15,16\}$, $I_{\mathcal{A}} = \{1,2,4,5,8,12\}$, $I_{\mathcal{L}} = \{7,10,13,14,15,16\}$,	$q^3(q-1)^2$	q^6
$\mathcal{F}_{\mathbf{18},\mathbf{19}}$	$\chi^{a_{18},a_{19}}_{b_2,b_4,b_5}$	$I_{\mathcal{A}} = \{1,3,7,8,9,11,15\}$, $I_{\mathcal{L}} = \{6,10,12,13,14,16,17\}$,	$q^3(q-1)^2$	q^7
$\mathcal{F}_{\mathbf{19},\mathbf{20}}$	$\chi^{a_{19},a_{20}}_{b_1,b_4}$	$I_{\mathcal{A}} = \{2,5,6,8,9,10,14,17\}$, $I_{\mathcal{L}} = \{3,7,11,12,13,15,16,18\}$,	$q^2(q-1)^2$	q^8
$\mathcal{F}_{\mathbf{1},\mathbf{2},\mathbf{7}}$	χ^{a_1,a_2,a_7}	$I_{\mathcal{A}} = \{3\}$, $I_{\mathcal{L}} = \{4\}$,	$(q-1)^3$	q^1
$\mathcal{F}_{\mathbf{1},\mathbf{4},\mathbf{6}}$	χ^{a_1,a_4,a_6}	$I_{\mathcal{A}} = \{2\}$, $I_{\mathcal{L}} = \{3\}$,	$(q-1)^3$	q^1
$\mathcal{F}_{\mathbf{1},\mathbf{4},\mathbf{9}}$	$\chi^{a_1,a_4,a_9}_{b_2}$	$I_{\mathcal{A}} = \{3\}$, $I_{\mathcal{L}} = \{6\}$,	$q(q-1)^3$	q^1
$\mathcal{F}_{\mathbf{1},\mathbf{6},\mathbf{7}}$	$\chi^{a_1,a_6,a_7}_{b_4}$	$I_{\mathcal{A}} = \{2\}$, $I_{\mathcal{L}} = \{3\}$,	$q(q-1)^3$	q^1
$\mathcal{F}_{\mathbf{1},\mathbf{7},\mathbf{9}}$	$\chi^{a_1,a_7,a_9}_{b_2,b_4}$	$I_{\mathcal{A}} = \{3\}$, $I_{\mathcal{L}} = \{6\}$,	$q^2(q-1)^3$	q^1
$\mathcal{F}_{\mathbf{1},\mathbf{9},\mathbf{10}}$	$\chi^{a_1,a_9,a_{10}}_{b_4}$	$I_{\mathcal{A}} = \{2,6\}$, $I_{\mathcal{L}} = \{3,7\}$,	$q(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{3},\mathbf{4},\mathbf{5}}$	χ^{a_3,a_4,a_5}	$I_{\mathcal{A}} = \{1\}$, $I_{\mathcal{L}} = \{2\}$,	$(q-1)^3$	q^1
$\mathcal{F}_{\mathbf{4},\mathbf{5},\mathbf{6}}$	$\chi^{a_4,a_5,a_6}_{b_3}$	$I_{\mathcal{A}} = \{1\}$, $I_{\mathcal{L}} = \{2\}$,	$q(q-1)^3$	q^1
$\mathcal{F}_{\mathbf{4},\mathbf{5},\mathbf{9}}$	χ^{a_4,a_5,a_9}	$I_{\mathcal{A}} = \{2,3\}$, $I_{\mathcal{L}} = \{1,6\}$,	$(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{4},\mathbf{8},\mathbf{9}}$	$\chi^{a_4,a_8,a_9}_{b_2}$	$I_{\mathcal{A}} = \{1,3\}$, $I_{\mathcal{L}} = \{5,6\}$,	$q(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{5},\mathbf{6},\mathbf{7}}$	χ^{a_5,a_6,a_7}	$I_{\mathcal{A}} = \{1,3\}$, $I_{\mathcal{L}} = \{2,4\}$,	$(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{5},\mathbf{7},\mathbf{9}}$	$\chi^{a_5,a_7,a_9}_{b_4}$	$I_{\mathcal{A}} = \{2,3\}$, $I_{\mathcal{L}} = \{1,6\}$,	$q(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{5},\mathbf{9},\mathbf{10}}$	$\chi^{a_5,a_9,a_{10}}_{b_1,b_4}$	$I_{\mathcal{A}} = \{2,6\}$, $I_{\mathcal{L}} = \{3,7\}$,	$q^2(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{7},\mathbf{8},\mathbf{9}}$	$\chi^{a_7,a_8,a_9}_{b_2,b_4}$	$I_{\mathcal{A}} = \{1,3\}$, $I_{\mathcal{L}} = \{5,6\}$,	$q^2(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{8},\mathbf{9},\mathbf{10}}$	$\chi^{a_8,a_9,a_{10}}_{b_4}$	$I_{\mathcal{A}} = \{2,3,6\}$, $I_{\mathcal{L}} = \{1,5,7\}$,	$q(q-1)^3$	q^3
$\mathcal{F}^{\boldsymbol{p\geq 5}}_{\mathbf{11},\mathbf{12},\mathbf{13}}$	$\boldsymbol{\chi^{a_{11},a_{12},a_{13}}_{b_2,b_6}}$	$\boldsymbol{I_{\mathcal{I}} = \{1,3,4,7\}}$ $\boldsymbol{I_{\mathcal{J}} = \{5,8,9,10\}}$	$\boldsymbol{q^2(q-1)^3}$	$\boldsymbol{q^4}$
$\mathcal{F}^{\boldsymbol{p=3}}_{\mathbf{11},\mathbf{12},\mathbf{13}}$	$\boldsymbol{\chi^{a_{11},a_{12},a_{13},a_{8,9,10}}_{b_2}}$ $\boldsymbol{\chi^{a_{11},a_{12},a_{13},a_{2,6}}}$ $\boldsymbol{\chi^{a_{11},a_{12},a_{13}}}$ $\boldsymbol{\chi^{a_{11},a_{12},a_{13},a_{1,6}}_{c_{1,3,4,7},c_2}}$	**See $\mathfrak{C}^5$ in Section 4.3** **See $\mathfrak{C}^5$ in Section 4.3** **See $\mathfrak{C}^5$ in Section 4.3** **See $\mathfrak{C}^5$ in Section 4.3**	$\boldsymbol{q(q-1)^4}$ $\boldsymbol{(q-1)^4/2}$ $\boldsymbol{(q-1)^3}$ $\boldsymbol{9(q-1)^4/2}$	$\boldsymbol{q^4}$ $\boldsymbol{q^4}$ $\boldsymbol{q^4}$ $\boldsymbol{q^4/3}$
$\mathcal{F}_{\mathbf{11},\mathbf{12},\mathbf{16}}$	$\chi^{a_{11},a_{12},a_{16}}_{b_2,b_5,b_6}$	$I_{\mathcal{A}} = \{1,3,4,7\}$, $I_{\mathcal{L}} = \{8,9,10,13\}$,	$q^3(q-1)^3$	q^4
$\mathcal{F}^{\boldsymbol{p\geq 5}}_{\mathbf{12},\mathbf{13},\mathbf{14}}$	$\boldsymbol{\chi^{a_{12},a_{13},a_{14}}_{b_3}}$	$\boldsymbol{I_{\mathcal{A}} = \{2\},\ I_{\mathcal{I}} = \{4,5,6,10\}}$ $\boldsymbol{I_{\mathcal{L}} = \{11\},\ I_{\mathcal{J}} = \{1,7,8,9\}}$	$\boldsymbol{q(q-1)^3}$	$\boldsymbol{q^5}$
$\mathcal{F}^{\boldsymbol{p=3}}_{\mathbf{12},\mathbf{13},\mathbf{14}}$	$\boldsymbol{\chi^{a_{12},a_{13},a_{14},a_{7,8,9}}}$ $\boldsymbol{\chi^{a_{12},a_{13},a_{14}}_{b_3,b_{4,5,6,10}}}$	**See $\mathfrak{C}^6$ in Section 4.3** **See $\mathfrak{C}^6$ in Section 4.3**	$\boldsymbol{(q-1)^4}$ $\boldsymbol{q^2(q-1)^3}$	$\boldsymbol{q^5}$ $\boldsymbol{q^4}$
$\mathcal{F}_{\mathbf{12},\mathbf{14},\mathbf{16}}$	$\chi^{a_{12},a_{14},a_{16}}_{b_1,b_3}$	$I_{\mathcal{A}} = \{2,4,5,6,10\}$, $I_{\mathcal{L}} = \{7,8,9,11,13\}$,	$q^2(q-1)^3$	q^5
$\mathcal{F}_{\mathbf{14},\mathbf{15},\mathbf{16}}$	$\chi^{a_{14},a_{15},a_{16}}_{b_4}$	$I_{\mathcal{A}} = \{3,6,7,9,11,13\}$, $I_{\mathcal{L}} = \{1,2,5,8,10,12\}$,	$q(q-1)^3$	q^6

Table D.4: The parametrization of the irreducible characters of $\mathrm{UF}_4(q)$, where $q = p^e$ and $p \geq 3$.

Parametrization of the irreducible characters of U_{B_5}

$\mathcal{F}$	χ	I	Number	Degree
$\mathcal{F}_{\mathrm{lin}}$	$\chi_{b_1,b_2,b_3,b_4,b_5}$		q^5	1
$\mathcal{F}_{\mathbf{6}}$	χ^{a_6}	$I_{\mathcal{A}}=\{1\}, I_{\mathcal{L}}=\{2\}$	$q-1$	q
$\mathcal{F}_{\mathbf{7}}$	χ^{a_7}	$I_{\mathcal{A}}=\{2\}, I_{\mathcal{L}}=\{3\}$	$q-1$	q
$\mathcal{F}_{\mathbf{8}}$	χ^{a_8}	$I_{\mathcal{A}}=\{3\}, I_{\mathcal{L}}=\{4\}$	$q-1$	q
$\mathcal{F}_{\mathbf{9}}$	χ^{a_9}	$I_{\mathcal{A}}=\{4\}, I_{\mathcal{L}}=\{5\}$	$q-1$	q
$\mathcal{F}_{\mathbf{10}}$	$\chi^{a_{10}}_{b_2}$	$I_{\mathcal{A}}=\{1\}, I_{\mathcal{L}}=\{6\}$	$q(q-1)$	q
$\mathcal{F}_{\mathbf{11}}$	$\chi^{a_{11}}_{b_2}$	$I_{\mathcal{A}}=\{1,3\}, I_{\mathcal{L}}=\{6,7\}$	$q(q-1)$	q^2
$\mathcal{F}_{\mathbf{12}}$	$\chi^{a_{12}}_{b_3}$	$I_{\mathcal{A}}=\{2,4\}, I_{\mathcal{L}}=\{7,8\}$	$q(q-1)$	q^2
$\mathcal{F}_{\mathbf{13}}$	$\chi^{a_{13}}_{b_4}$	$I_{\mathcal{A}}=\{3,5\}, I_{\mathcal{L}}=\{8,9\}$	$q(q-1)$	q^2
$\mathcal{F}_{\mathbf{14}}$	$\chi^{a_{14}}_{b_2,b_6,b_7}$	$I_{\mathcal{A}}=\{1,3\}, I_{\mathcal{L}}=\{10,11\}$	$q^3(q-1)$	q^2
$\mathcal{F}_{\mathbf{15}}$	$\chi^{a_7,a_{15}}$	$I_{\mathcal{A}}=\{1,3,4,8\}$	$(q-1)^2$	q^4
		$I_{\mathcal{L}}=\{2,6,11,12\}$		
	$\chi^{a_{15}}_{b_2,b_3}$	$I_{\mathcal{A}}=\{1,4,8\}, I_{\mathcal{L}}=\{6,11,12\}$	$q^2(q-1)$	q^3
$\mathcal{F}_{\mathbf{16}}$	$\chi^{a_8,a_{16}}$	$I_{\mathcal{A}}=\{2,4,5,9\}$	$(q-1)^2$	q^4
		$I_{\mathcal{L}}=\{3,7,12,13\}$		
	$\chi^{a_{16}}_{b_3,b_4}$	$I_{\mathcal{A}}=\{2,5,9\}, I_{\mathcal{L}}=\{7,12,13\}$	$q^2(q-1)$	q^3
$\mathcal{F}_{\mathbf{17}}$	$\chi^{a_{17}}_{b_1,b_3}$	$I_{\mathcal{A}}=\{2,6,7\}, I_{\mathcal{L}}=\{10,11,14\}$	$q^2(q-1)$	q^3
$\mathcal{F}_{\mathbf{18}}$	$\chi^{a_{11},a_{18}}_{b_2,b_7,b_{12}}$	$I_{\mathcal{A}}=\{1,3,4,8\}$	$q^3(q-1)^2$	q^4
		$I_{\mathcal{L}}=\{6,10,14,15\}$		
	$\chi^{a_7,a_{18}}_{b_6,b_{12}}$	$I_{\mathcal{A}}=\{1,3,4,8\}$	$q^2(q-1)^2$	q^4
		$I_{\mathcal{L}}=\{2,10,14,15\}$		
	$\chi^{a_{18}}_{b_2,b_3,b_6,b_{12}}$	$I_{\mathcal{A}}=\{1,4,8\}, I_{\mathcal{L}}=\{10,14,15\}$	$q^4(q-1)$	q^3
$\mathcal{F}_{\mathbf{19}}$	$\chi^{a_{12},a_{19}}_{b_3}$	$I_{\mathcal{A}}=\{1,2,4,5,6,9\}$	$q(q-1)^2$	q^6
		$I_{\mathcal{L}}=\{7,8,11,13,15,16\}$		
	$\chi^{a_8,a_{19}}_{b_2,b_7}$	$I_{\mathcal{A}}=\{1,3,5,6,9\}$	$q^2(q-1)^2$	q^5
		$I_{\mathcal{L}}=\{4,11,13,15,16\}$		
	$\chi^{a_7,a_{19}}_{b_4}$	$I_{\mathcal{A}}=\{1,3,5,6,9\}$	$q(q-1)^2$	q^5
		$I_{\mathcal{L}}=\{2,11,13,15,16\}$		
	$\chi^{a_{19}}_{b_2,b_3,b_4}$	$I_{\mathcal{A}}=\{1,5,6,9\}$	$q^3(q-1)$	q^4
		$I_{\mathcal{L}}=\{11,13,15,16\}$		
$\mathcal{F}_{\mathbf{20}}$	$\chi^{a_{14},a_{20}}_{b_3,b_7,b_8}$	$I_{\mathcal{A}}=\{1,2,4,6,10\}$	$q^3(q-1)^2$	q^5
		$I_{\mathcal{L}}=\{11,12,15,17,18\}$		
	$\chi^{a_{11},a_{20}}_{b_3,b_8}$	$I_{\mathcal{A}}=\{1,2,4,6,10\}$	$q^2(q-1)^2$	q^5
		$I_{\mathcal{L}}=\{7,12,15,17,18\}$		
	$\chi^{a_{20}}_{b_1,b_3,b_7,b_8}$	$I_{\mathcal{A}}=\{2,4,6,10\}$	$q^4(q-1)$	q^4
		$I_{\mathcal{L}}=\{12,15,17,18\}$		
$\mathcal{F}_{\mathbf{21}}$	$\chi^{a_7,a_{15},a_{21}}_{b_{12},b_{16}}$	$I_{\mathcal{A}}=\{1,3,4,5,8,9,13\}$	$q^2(q-1)^3$	q^7
		$I_{\mathcal{L}}=\{2,6,10,11,14,18,19\}$		
	$\chi^{a_{15},a_{21}}_{b_2,b_3,b_{12},b_{16}}$	$I_{\mathcal{A}}=\{1,4,5,8,9,13\}$	$q^4(q-1)^2$	q^6
		$I_{\mathcal{L}}=\{6,10,11,14,18,19\}$		
	$\chi^{a_{11},a_{12},a_{21}}_{b_{16}}$	$I_{\mathcal{A}}=\{1,2,3,4,5,9,13\}$	$q(q-1)^3$	q^7
		$I_{\mathcal{L}}=\{6,7,8,10,14,18,19\}$		
	$\chi^{a_{11},a_{21}}_{b_2,b_4,b_7,b_8,b_{16}}$	$I_{\mathcal{A}}=\{1,3,5,9,13\}$	$q^5(q-1)^2$	q^5
		$I_{\mathcal{L}}=\{6,10,14,18,19\}$		
	$\chi^{a_{12},a_{21}}_{b_3,b_6,b_{16}}$	$I_{\mathcal{A}}=\{1,2,4,5,9,13\}$	$q^3(q-1)^2$	q^6
		$I_{\mathcal{L}}=\{7,8,10,14,18,19\}$		
	$\chi^{a_7,a_{21}}_{b_4,b_6,b_8,b_{16}}$	$I_{\mathcal{A}}=\{1,3,5,9,13\}$	$q^4(q-1)^2$	q^5
		$I_{\mathcal{L}}=\{2,10,14,18,19\}$		

$\mathcal{F}$	χ	I	Number	Degree
	$\chi^{a_8,a_{21}}_{b_2,b_6,b_{16}}$	$I_{\mathcal{A}} = \{1,4,5,9,13\}$	$q^3(q-1)^2$	q^5
		$I_{\mathcal{L}} = \{3,10,14,18,19\}$		
	$\chi^{a_{21}}_{b_2,b_3,b_4,b_6,b_{16}}$	$I_{\mathcal{A}} = \{1,5,9,13\}$	$q^5(q-1)$	q^4
		$I_{\mathcal{L}} = \{10,14,18,19\}$		
$\mathcal{F}_{\mathbf{22}}$	$\chi^{a_{10},a_{22}}_{b_2,b_4}$	$I_{\mathcal{A}} = \{1,3,7,8,11,12\}$	$q^2(q-1)^2$	q^6
		$I_{\mathcal{L}} = \{6,14,15,17,18,20\}$		
	$\chi^{a_6,a_{22}}_{b_4}$	$I_{\mathcal{A}} = \{2,3,7,8,11,12\}$	$q(q-1)^2$	q^6
		$I_{\mathcal{L}} = \{1,14,15,17,18,20\}$		
	$\chi^{a_{22}}_{b_1,b_2,b_4}$	$I_{\mathcal{A}} = \{3,7,8,11,12\}$	$q^3(q-1)$	q^5
		$I_{\mathcal{L}} = \{14,15,17,18,20\}$		
$\mathcal{F}_{\mathbf{23}}$	$\chi^{a_{18},a_{23}}_{b_3,b_7,b_8,b_{11},b_{12},b_{13}}$	$I_{\mathcal{A}} = \{1,2,4,5,6,9,10\}$	$q^6(q-1)^2$	q^7
		$I_{\mathcal{L}} = \{14,15,16,17,19,20,21\}$		
	$\chi^{a_{14},a_{15},a_{23}}_{b_3,b_7,b_8,b_{13}}$	$I_{\mathcal{A}} = \{1,2,4,5,6,9,10\}$	$q^4(q-1)^3$	q^7
		$I_{\mathcal{L}} = \{11,12,16,17,19,20,21\}$		
	$\chi^{a_{12},a_{14},a_{23}}_{b_3,b_8,b_{13}}$	$I_{\mathcal{A}} = \{1,2,4,5,6,9,10\}$	$q^3(q-1)^3$	q^7
		$I_{\mathcal{L}} = \{7,11,16,17,19,20,21\}$		
	$\chi^{a_8,a_{14},a_{23}}_{b_7,b_{13}}$	$I_{\mathcal{A}} = \{1,2,4,5,6,9,10\}$	$q^2(q-1)^3$	q^7
		$I_{\mathcal{L}} = \{3,11,16,17,19,20,21\}$		
	$\chi^{a_{14},a_{23}}_{b_3,b_4,b_7,b_{13}}$	$I_{\mathcal{A}} = \{1,2,5,6,9,10\}$	$q^4(q-1)^2$	q^6
		$I_{\mathcal{L}} = \{11,16,17,19,20,21\}$		
	$\chi^{a_{15},a_{23}}_{b_3,b_7,b_8,b_{13}}$	$I_{\mathcal{A}} = \{1,2,4,5,6,9,10\}$	$q^4(q-1)^2$	q^7
		$I_{\mathcal{L}} = \{11,12,16,17,19,20,21\}$		
	$\chi^{a_8,a_{11},a_{23}}_{b_{12},b_{13}}$	$I_{\mathcal{A}} = \{2,4,5,6,7,9,10\}$	$q^2(q-1)^3$	q^7
		$I_{\mathcal{L}} = \{1,3,16,17,19,20,21\}$		
	$\chi^{a_{11},a_{23}}_{b_3,b_4,b_{12},b_{13}}$	$I_{\mathcal{A}} = \{2,5,6,7,9,10\}$	$q^4(q-1)^2$	q^6
		$I_{\mathcal{L}} = \{1,16,17,19,20,21\}$		
	$\chi^{a_{12},a_{23}}_{b_1,b_3,b_8,b_{13}}$	$I_{\mathcal{A}} = \{2,4,5,6,9,10\}$	$q^4(q-1)^2$	q^6
		$I_{\mathcal{L}} = \{7,16,17,19,20,21\}$		
	$\chi^{a_8,a_{23}}_{b_1,b_7,b_{13}}$	$I_{\mathcal{A}} = \{2,4,5,6,9,10\}$	$q^3(q-1)^2$	q^6
		$I_{\mathcal{L}} = \{3,16,17,19,20,21\}$		
	$\chi^{a_{23}}_{b_1,b_3,b_4,b_7,b_{13}}$	$I_{\mathcal{A}} = \{2,5,6,9,10\}$	$q^5(q-1)$	q^5
		$I_{\mathcal{L}} = \{16,17,19,20,21\}$		
$\mathcal{F}_{\mathbf{24}}$	$\chi^{a_{20},a_{24}}_{b_1,b_4,b_8,b_9}$	$I_{\mathcal{A}} = \{2,3,5,6,7,11,12,14,17\}$	$q^4(q-1)^2$	q^9
		$I_{\mathcal{L}} = \{10,13,15,16,18,19,21,22,23\}$		
	$\chi^{a_{18},a_{24}}_{b_2,b_4,b_6,b_9,b_{12}}$	$I_{\mathcal{A}} = \{1,3,5,7,8,11,14,17\}$	$q^5(q-1)^2$	q^8
		$I_{\mathcal{L}} = \{10,13,15,16,19,21,22,23\}$		
	$\chi^{a_{15},a_{24}}_{b_2,b_4,b_9,b_{10}}$	$I_{\mathcal{A}} = \{1,3,5,7,8,11,14,17\}$	$q^4(q-1)^2$	q^8
		$I_{\mathcal{L}} = \{6,12,13,16,19,21,22,23\}$		
	$\chi^{a_{10},a_{12},a_{24}}_{b_4,b_9}$	$I_{\mathcal{A}} = \{1,2,3,5,7,11,14,17\}$	$q^2(q-1)^3$	q^8
		$I_{\mathcal{L}} = \{6,8,13,16,19,21,22,23\}$		
	$\chi^{a_{12},a_{24}}_{b_1,b_4,b_6,b_9}$	$I_{\mathcal{A}} = \{2,3,5,7,11,14,17\}$	$q^4(q-1)^2$	q^7
		$I_{\mathcal{L}} = \{8,13,16,19,21,22,23\}$		
	$\chi^{a_{10},a_{24}}_{b_2,b_4,b_8,b_9}$	$I_{\mathcal{A}} = \{1,3,5,7,11,14,17\}$	$q^4(q-1)^2$	q^7
		$I_{\mathcal{L}} = \{6,13,16,19,21,22,23\}$		
	$\chi^{a_6,a_{24}}_{b_4,b_8,b_9}$	$I_{\mathcal{A}} = \{2,3,5,7,11,14,17\}$	$q^3(q-1)^2$	q^7
		$I_{\mathcal{L}} = \{1,13,16,19,21,22,23\}$		
	$\chi^{a_{24}}_{b_1,b_2,b_4,b_8,b_9}$	$I_{\mathcal{A}} = \{3,5,7,11,14,17\}$	$q^5(q-1)$	q^6
		$I_{\mathcal{L}} = \{13,16,19,21,22,23\}$		
$\mathcal{F}_{\mathbf{25}}$	$\chi^{a_{17},a_{25}}_{b_1,b_3,b_5}$	$I_{\mathcal{A}} = \{2,4,6,7,8,9,12,13,15,16\}$	$q^3(q-1)^2$	q^{10}
		$I_{\mathcal{L}} = \{10,11,14,18,19,20,\dots,24\}$		
	$\chi^{a_{14},a_{25}}_{b_2,b_5,b_6,b_7}$	$I_{\mathcal{A}} = \{1,3,4,8,9,12,13,15,16\}$	$q^4(q-1)^2$	q^9

$\mathcal{F}$	χ	I	Number	Degree
		$I_{\mathcal{L}} = \{10, 11, 18, 19, 20, \dots, 24\}$		
	$\chi^{a_{10},a_{11},a_{25}}_{b_2,b_5}$	$I_{\mathcal{A}} = \{1, 3, 4, 8, 9, 12, 13, 15, 16\}$	$q^2(q-1)^3$	q^9
		$I_{\mathcal{L}} = \{6, 7, 18, 19, 20, 21, 22, 23, 24\}$		
	$\chi^{a_7,a_{10},a_{25}}_{b_5}$	$I_{\mathcal{A}} = \{1, 3, 4, 8, 9, 12, 13, 15, 16\}$	$q(q-1)^3$	q^9
		$I_{\mathcal{L}} = \{2, 6, 18, 19, 20, 21, 22, 23, 24\}$		
	$\chi^{a_{10},a_{25}}_{b_2,b_3,b_5}$	$I_{\mathcal{A}} = \{1, 4, 8, 9, 12, 13, 15, 16\}$	$q^3(q-1)^2$	q^8
		$I_{\mathcal{L}} = \{6, 18, 19, 20, 21, 22, 23, 24\}$		
	$\chi^{a_{11},a_{25}}_{b_2,b_5}$	$I_{\mathcal{A}} = \{1, 3, 4, 8, 9, 12, 13, 15, 16\}$	$q^2(q-1)^2$	q^9
		$I_{\mathcal{L}} = \{6, 7, 18, 19, 20, 21, 22, 23, 24\}$		
	$\chi^{a_6,a_{25}}_{b_3,b_5,b_7}$	$I_{\mathcal{A}} = \{2, 4, 8, 9, 12, 13, 15, 16\}$	$q^3(q-1)^2$	q^8
		$I_{\mathcal{L}} = \{1, 18, 19, 20, 21, 22, 23, 24\}$		
	$\chi^{a_7,a_{25}}_{b_1,b_5}$	$I_{\mathcal{A}} = \{3, 4, 8, 9, 12, 13, 15, 16\}$	$q^2(q-1)^2$	q^8
		$I_{\mathcal{L}} = \{2, 18, 19, 20, 21, 22, 23, 24\}$		
	$\chi^{a_{25}}_{b_1,b_2,b_3,b_5}$	$I_{\mathcal{A}} = \{4, 8, 9, 12, 13, 15, 16\}$	$q^4(q-1)$	q^7
		$I_{\mathcal{L}} = \{18, 19, 20, 21, 22, 23, 24\}$		
$\mathcal{F}_{\mathbf{1,7}}$	χ^{a_1,a_7}	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{3\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{1,8}}$	χ^{a_1,a_8}	$I_{\mathcal{A}} = \{3\}, I_{\mathcal{L}} = \{4\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{1,9}}$	χ^{a_1,a_9}	$I_{\mathcal{A}} = \{4\}, I_{\mathcal{L}} = \{5\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{1,12}}$	$\chi^{a_1,a_{12}}_{b_3}$	$I_{\mathcal{A}} = \{2, 4\}, I_{\mathcal{L}} = \{7, 8\}$	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{1,13}}$	$\chi^{a_1,a_{13}}_{b_4}$	$I_{\mathcal{A}} = \{3, 5\}, I_{\mathcal{L}} = \{8, 9\}$	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{1,16}}$	$\chi^{a_1,a_8,a_{16}}$	$I_{\mathcal{A}} = \{2, 4, 5, 9\}$	$(q-1)^3$	q^4
		$I_{\mathcal{L}} = \{3, 7, 12, 13\}$		
	$\chi^{a_1,a_{16}}_{b_3,b_4}$	$I_{\mathcal{A}} = \{2, 5, 9\}, I_{\mathcal{L}} = \{7, 12, 13\}$	$q^2(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{2,8}}$	χ^{a_2,a_8}	$I_{\mathcal{A}} = \{3\}, I_{\mathcal{L}} = \{4\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{2,9}}$	χ^{a_2,a_9}	$I_{\mathcal{A}} = \{4\}, I_{\mathcal{L}} = \{5\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{2,13}}$	$\chi^{a_2,a_{13}}_{b_4}$	$I_{\mathcal{A}} = \{3, 5\}, I_{\mathcal{L}} = \{8, 9\}$	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{3,6}}$	χ^{a_3,a_6}	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{2\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{3,9}}$	χ^{a_3,a_9}	$I_{\mathcal{A}} = \{4\}, I_{\mathcal{L}} = \{5\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{3,10}}$	$\chi^{a_3,a_{10}}_{b_2}$	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{6\}$	$q(q-1)^2$	q
$\mathcal{F}_{\mathbf{4,6}}$	χ^{a_4,a_6}	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{2\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{4,7}}$	χ^{a_4,a_7}	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{3\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{4,10}}$	$\chi^{a_4,a_{10}}_{b_2}$	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{6\}$	$q(q-1)^2$	q
$\mathcal{F}_{\mathbf{4,11}}$	$\chi^{a_4,a_{11}}_{b_2}$	$I_{\mathcal{A}} = \{1, 3\}, I_{\mathcal{L}} = \{6, 7\}$	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{4,14}}$	$\chi^{a_4,a_{14}}_{b_2,b_6,b_7}$	$I_{\mathcal{A}} = \{1, 3\}, I_{\mathcal{L}} = \{10, 11\}$	$q^3(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{4,17}}$	$\chi^{a_4,a_{17}}_{b_1,b_3}$	$I_{\mathcal{A}} = \{2, 6, 7\}, I_{\mathcal{L}} = \{10, 11, 14\}$	$q^2(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{5,6}}$	χ^{a_5,a_6}	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{2\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{5,7}}$	χ^{a_5,a_7}	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{3\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{5,8}}$	χ^{a_5,a_8}	$I_{\mathcal{A}} = \{3\}, I_{\mathcal{L}} = \{4\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{5,10}}$	$\chi^{a_5,a_{10}}_{b_2}$	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{6\}$	$q(q-1)^2$	q
$\mathcal{F}_{\mathbf{5,11}}$	$\chi^{a_5,a_{11}}_{b_2}$	$I_{\mathcal{A}} = \{1, 3\}, I_{\mathcal{L}} = \{6, 7\}$	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{5,12}}$	$\chi^{a_5,a_{12}}_{b_3}$	$I_{\mathcal{A}} = \{2, 4\}, I_{\mathcal{L}} = \{7, 8\}$	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{5,14}}$	$\chi^{a_5,a_{14}}_{b_2,b_6,b_7}$	$I_{\mathcal{A}} = \{1, 3\}, I_{\mathcal{L}} = \{10, 11\}$	$q^3(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{5,15}}$	$\chi^{a_5,a_7,a_{15}}$	$I_{\mathcal{A}} = \{1, 3, 4, 8\}$	$(q-1)^3$	q^4
		$I_{\mathcal{L}} = \{2, 6, 11, 12\}$		
	$\chi^{a_5,a_{15}}_{b_2,b_3}$	$I_{\mathcal{A}} = \{1, 4, 8\}, I_{\mathcal{L}} = \{6, 11, 12\}$	$q^2(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{5,17}}$	$\chi^{a_5,a_{17}}_{b_1,b_3}$	$I_{\mathcal{A}} = \{2, 6, 7\}, I_{\mathcal{L}} = \{10, 11, 14\}$	$q^2(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{5,18}}$	$\chi^{a_5,a_{11},a_{18}}_{b_2,b_7,b_{12}}$	$I_{\mathcal{A}} = \{1, 3, 4, 8\}$	$q^3(q-1)^3$	q^4
		$I_{\mathcal{L}} = \{6, 10, 14, 15\}$		
	$\chi^{a_5,a_7,a_{18}}_{b_6,b_{12}}$	$I_{\mathcal{A}} = \{1, 3, 4, 8\}$	$q^2(q-1)^3$	q^4
		$I_{\mathcal{L}} = \{2, 10, 14, 15\}$		

$\mathcal{F}$	χ	I	Number	Degree
	$\chi^{a_5,a_{18}}_{b_2,b_3,b_6,b_{12}}$	$I_{\mathcal{A}}=\{1,4,8\}, I_{\mathcal{L}}=\{10,14,15\}$	$q^4(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{5,20}}$	$\chi^{a_5,a_{14},a_{20}}_{b_3,b_7,b_8}$	$I_{\mathcal{A}}=\{1,2,4,6,10\}$	$q^3(q-1)^3$	q^5
		$I_{\mathcal{L}}=\{11,12,15,17,18\}$		
	$\chi^{a_5,a_{11},a_{20}}_{b_3,b_8}$	$I_{\mathcal{A}}=\{1,2,4,6,10\}$	$q^2(q-1)^3$	q^5
		$I_{\mathcal{L}}=\{7,12,15,17,18\}$		
	$\chi^{a_5,a_{20}}_{b_1,b_3,b_7,b_8}$	$I_{\mathcal{A}}=\{2,4,6,10\}$	$q^4(q-1)^2$	q^4
		$I_{\mathcal{L}}=\{12,15,17,18\}$		
$\mathcal{F}_{\mathbf{5,22}}$	$\chi^{a_5,a_{10},a_{22}}_{b_2,b_4}$	$I_{\mathcal{A}}=\{1,3,7,8,11,12\}$	$q^2(q-1)^3$	q^6
		$I_{\mathcal{L}}=\{6,14,15,17,18,20\}$		
	$\chi^{a_5,a_6,a_{22}}_{b_4}$	$I_{\mathcal{A}}=\{2,3,7,8,11,12\}$	$q(q-1)^3$	q^6
		$I_{\mathcal{L}}=\{1,14,15,17,18,20\}$		
	$\chi^{a_5,a_{22}}_{b_1,b_2,b_4}$	$I_{\mathcal{A}}=\{3,7,8,11,12\}$	$q^3(q-1)^2$	q^5
		$I_{\mathcal{L}}=\{14,15,17,18,20\}$		
$\mathcal{F}_{\mathbf{6,7}}$	$\chi^{a_6,a_7}_{b_3}$	$I_{\mathcal{A}}=\{1\}, I_{\mathcal{L}}=\{2\}$	$q(q-1)^2$	q
$\mathcal{F}_{\mathbf{6,8}}$	χ^{a_6,a_8}	$I_{\mathcal{A}}=\{1,3\}, I_{\mathcal{L}}=\{2,4\}$	$(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{6,9}}$	χ^{a_6,a_9}	$I_{\mathcal{A}}=\{1,4\}, I_{\mathcal{L}}=\{2,5\}$	$(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{6,12}}$	$\chi^{a_6,a_{12}}_{b_1,b_3}$	$I_{\mathcal{A}}=\{2,4\}, I_{\mathcal{L}}=\{7,8\}$	$q^2(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{6,13}}$	$\chi^{a_6,a_{13}}_{b_4}$	$I_{\mathcal{A}}=\{2,3,5\}, I_{\mathcal{L}}=\{1,8,9\}$	$q(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{6,16}}$	$\chi^{a_6,a_8,a_{16}}_{b_1}$	$I_{\mathcal{A}}=\{2,4,5,9\}$	$q(q-1)^3$	q^4
		$I_{\mathcal{L}}=\{3,7,12,13\}$		
	$\chi^{a_6,a_{16}}_{b_1,b_3,b_4}$	$I_{\mathcal{A}}=\{2,5,9\}, I_{\mathcal{L}}=\{7,12,13\}$	$q^3(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{7,8}}$	$\chi^{a_7,a_8}_{b_4}$	$I_{\mathcal{A}}=\{2\}, I_{\mathcal{L}}=\{3\}$	$q(q-1)^2$	q
$\mathcal{F}_{\mathbf{7,9}}$	χ^{a_7,a_9}	$I_{\mathcal{A}}=\{2,4\}, I_{\mathcal{L}}=\{3,5\}$	$(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{7,10}}$	$\chi^{a_7,a_{10}}$	$I_{\mathcal{A}}=\{1,3\}, I_{\mathcal{L}}=\{2,6\}$	$(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{7,13}}$	$\chi^{a_7,a_{13}}_{b_2,b_4}$	$I_{\mathcal{A}}=\{3,5\}, I_{\mathcal{L}}=\{8,9\}$	$q^2(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{8,9}}$	$\chi^{a_8,a_9}_{b_5}$	$I_{\mathcal{A}}=\{3\}, I_{\mathcal{L}}=\{4\}$	$q(q-1)^2$	q
$\mathcal{F}_{\mathbf{8,10}}$	$\chi^{a_8,a_{10}}_{b_2}$	$I_{\mathcal{A}}=\{1,4\}, I_{\mathcal{L}}=\{3,6\}$	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{8,11}}$	$\chi^{a_8,a_{11}}_{b_2,b_4}$	$I_{\mathcal{A}}=\{1,3\}, I_{\mathcal{L}}=\{6,7\}$	$q^2(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{8,14}}$	$\chi^{a_8,a_{14}}_{b_2,b_4,b_6,b_7}$	$I_{\mathcal{A}}=\{1,3\}, I_{\mathcal{L}}=\{10,11\}$	$q^4(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{8,17}}$	$\chi^{a_8,a_{17}}_{b_1}$	$I_{\mathcal{A}}=\{2,4,6,7\}$	$q(q-1)^2$	q^4
		$I_{\mathcal{L}}=\{3,10,11,14\}$		
$\mathcal{F}_{\mathbf{9,10}}$	$\chi^{a_9,a_{10}}_{b_2}$	$I_{\mathcal{A}}=\{1,5\}, I_{\mathcal{L}}=\{4,6\}$	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{9,11}}$	$\chi^{a_9,a_{11}}_{b_2}$	$I_{\mathcal{A}}=\{1,3,5\}, I_{\mathcal{L}}=\{4,6,7\}$	$q(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{9,12}}$	$\chi^{a_9,a_{12}}_{b_3,b_5}$	$I_{\mathcal{A}}=\{2,4\}, I_{\mathcal{L}}=\{7,8\}$	$q^2(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{9,14}}$	$\chi^{a_9,a_{14}}_{b_2,b_6,b_7}$	$I_{\mathcal{A}}=\{1,3,5\}, I_{\mathcal{L}}=\{4,10,11\}$	$q^3(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{9,15}}$	$\chi^{a_7,a_9,a_{15}}_{b_5}$	$I_{\mathcal{A}}=\{1,3,4,8\}$	$q(q-1)^3$	q^4
		$I_{\mathcal{L}}=\{2,6,11,12\}$		
	$\chi^{a_9,a_{15}}_{b_2,b_3,b_5}$	$I_{\mathcal{A}}=\{1,4,8\}, I_{\mathcal{L}}=\{6,11,12\}$	$q^3(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{9,17}}$	$\chi^{a_9,a_{17}}_{b_1,b_3}$	$I_{\mathcal{A}}=\{2,5,6,7\}$	$q^2(q-1)^2$	q^4
		$I_{\mathcal{L}}=\{4,10,11,14\}$		
$\mathcal{F}_{\mathbf{9,18}}$	$\chi^{a_9,a_{11},a_{18}}_{b_2,b_5,b_7,b_{12}}$	$I_{\mathcal{A}}=\{1,3,4,8\}$	$q^4(q-1)^3$	q^4
		$I_{\mathcal{L}}=\{6,10,14,15\}$		
	$\chi^{a_7,a_9,a_{18}}_{b_5,b_6,b_{12}}$	$I_{\mathcal{A}}=\{1,3,4,8\}$	$q^3(q-1)^3$	q^4
		$I_{\mathcal{L}}=\{2,10,14,15\}$		
	$\chi^{a_9,a_{18}}_{b_2,b_3,b_5,b_6,b_{12}}$	$I_{\mathcal{A}}=\{1,4,8\}, I_{\mathcal{L}}=\{10,14,15\}$	$q^5(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{9,20}}$	$\chi^{a_9,a_{14},a_{20}}_{b_3,b_5,b_7,b_8}$	$I_{\mathcal{A}}=\{1,2,4,6,10\}$	$q^4(q-1)^3$	q^5
		$I_{\mathcal{L}}=\{11,12,15,17,18\}$		
	$\chi^{a_9,a_{11},a_{20}}_{b_3,b_5,b_8}$	$I_{\mathcal{A}}=\{1,2,4,6,10\}$	$q^3(q-1)^3$	q^5
		$I_{\mathcal{L}}=\{7,12,15,17,18\}$		
	$\chi^{a_9,a_{20}}_{b_1,b_3,b_5,b_7,b_8}$	$I_{\mathcal{A}}=\{2,4,6,10\}$	$q^5(q-1)^2$	q^4

$\mathcal{F}$	χ	I	Number	Degree
		$I_{\mathcal{L}} = \{12, 15, 17, 18\}$		
$\mathcal{F}_{\mathbf{9,22}}$	$\chi_{b_2}^{a_9,a_{10},a_{22}}$	$I_{\mathcal{A}} = \{1, 3, 5, 7, 8, 11, 12\}$	$q(q-1)^3$	q^7
		$I_{\mathcal{L}} = \{4, 6, 14, 15, 17, 18, 20\}$		
	$\chi^{a_6,a_9,a_{22}}$	$I_{\mathcal{A}} = \{2, 3, 5, 7, 8, 11, 12\}$	$(q-1)^3$	q^7
		$I_{\mathcal{L}} = \{1, 4, 14, 15, 17, 18, 20\}$		
	$\chi_{b_1,b_2}^{a_9,a_{22}}$	$I_{\mathcal{A}} = \{3, 5, 7, 8, 11, 12\}$	$q^2(q-1)^2$	q^6
		$I_{\mathcal{L}} = \{4, 14, 15, 17, 18, 20\}$		
$\mathcal{F}_{\mathbf{10,11}}$	$\chi_{b_2}^{a_{10},a_{11}}$	$I_{\mathcal{A}} = \{1, 3\}, I_{\mathcal{L}} = \{6, 7\}$	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{10,12}}$	$\chi_{b_3}^{a_{10},a_{12}}$	$I_{\mathcal{A}} = \{1, 2, 4\}, I_{\mathcal{L}} = \{6, 7, 8\}$	$q(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{10,13}}$	$\chi_{b_2,b_4}^{a_{10},a_{13}}$	$I_{\mathcal{A}} = \{1, 3, 5\}, I_{\mathcal{L}} = \{6, 8, 9\}$	$q^2(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{10,15}}$	$\chi^{a_7,a_{10},a_{15}}$	$I_{\mathcal{A}} = \{1, 3, 4, 8\}$	$(q-1)^3$	q^4
		$I_{\mathcal{L}} = \{2, 6, 11, 12\}$		
	$\chi_{b_2,b_3}^{a_{10},a_{15}}$	$I_{\mathcal{A}} = \{1, 4, 8\}, I_{\mathcal{L}} = \{6, 11, 12\}$	$q^2(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{10,16}}$	$\chi^{a_8,a_{10},a_{16}}$	$I_{\mathcal{A}} = \{1, 2, 4, 5, 9\}$	$(q-1)^3$	q^5
		$I_{\mathcal{L}} = \{3, 6, 7, 12, 13\}$		
	$\chi_{b_3,b_4}^{a_{10},a_{16}}$	$I_{\mathcal{A}} = \{1, 2, 5, 9\}$	$q^2(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{6, 7, 12, 13\}$		
$\mathcal{F}_{\mathbf{10,19}}$	$\chi_{b_3}^{a_{10},a_{12},a_{19}}$	$I_{\mathcal{A}} = \{1, 2, 4, 5, 6, 9\}$	$q(q-1)^3$	q^6
		$I_{\mathcal{L}} = \{7, 8, 11, 13, 15, 16\}$		
	$\chi_{b_2,b_7}^{a_8,a_{10},a_{19}}$	$I_{\mathcal{A}} = \{1, 3, 5, 6, 9\}$	$q^2(q-1)^3$	q^5
		$I_{\mathcal{L}} = \{4, 11, 13, 15, 16\}$		
	$\chi_{b_4}^{a_7,a_{10},a_{19}}$	$I_{\mathcal{A}} = \{1, 3, 5, 6, 9\}$	$q(q-1)^3$	q^5
		$I_{\mathcal{L}} = \{2, 11, 13, 15, 16\}$		
	$\chi_{b_2,b_3,b_4}^{a_{10},a_{19}}$	$I_{\mathcal{A}} = \{1, 5, 6, 9\}$	$q^3(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{11, 13, 15, 16\}$		
$\mathcal{F}_{\mathbf{11,12}}$	$\chi_{b_4}^{a_{11},a_{12}}$	$I_{\mathcal{A}} = \{2, 3, 7\}, I_{\mathcal{L}} = \{1, 6, 8\}$	$q(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{11,13}}$	$\chi_{b_2,b_4,b_9}^{a_{11},a_{13}}$	$I_{\mathcal{A}} = \{1, 3, 5\}, I_{\mathcal{L}} = \{6, 7, 8\}$	$q^3(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{11,16}}$	$\chi_{b_1,b_4,b_8}^{a_{11},a_{16}}$	$I_{\mathcal{A}} = \{2, 3, 5, 7\}$	$q^3(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{6, 9, 12, 13\}$		
$\mathcal{F}_{\mathbf{12,13}}$	$\chi_{b_5}^{a_{12},a_{13}}$	$I_{\mathcal{A}} = \{3, 4, 8\}, I_{\mathcal{L}} = \{2, 7, 9\}$	$q(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{12,14}}$	$\chi_{b_6}^{a_{12},a_{14}}$	$I_{\mathcal{A}} = \{1, 2, 3, 4\}$	$q(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{7, 8, 10, 11\}$		
$\mathcal{F}_{\mathbf{12,17}}$	$\chi_{b_1}^{a_8,a_{12},a_{17}}$	$I_{\mathcal{A}} = \{2, 4, 6, 7\}$	$q(q-1)^3$	q^4
		$I_{\mathcal{L}} = \{3, 10, 11, 14\}$		
	$\chi_{b_1,b_3,b_4}^{a_{12},a_{17}}$	$I_{\mathcal{A}} = \{2, 6, 7\}, I_{\mathcal{L}} = \{10, 11, 14\}$	$q^3(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{13,14}}$	$\chi_{b_2,b_4,b_6,b_7,b_9}^{a_{13},a_{14}}$	$I_{\mathcal{A}} = \{1, 3, 5\}, I_{\mathcal{L}} = \{8, 10, 11\}$	$q^5(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{13,15}}$	$\chi_{b_2,b_5,b_7}^{a_{13},a_{15}}$	$I_{\mathcal{A}} = \{1, 3, 4, 8\}$	$q^3(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{6, 9, 11, 12\}$		
$\mathcal{F}_{\mathbf{13,17}}$	$\chi_{b_1,b_4}^{a_{13},a_{17}}$	$I_{\mathcal{A}} = \{2, 3, 5, 6, 7\}$	$q^2(q-1)^2$	q^5
		$I_{\mathcal{L}} = \{8, 9, 10, 11, 14\}$		
$\mathcal{F}_{\mathbf{13,18}}$	$\chi_{b_2,b_5,b_6,b_7,b_{11},b_{12}}^{a_{13},a_{18}}$	$I_{\mathcal{A}} = \{1, 3, 4, 8\}$	$q^6(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{9, 10, 14, 15\}$		
$\mathcal{F}_{\mathbf{13,20}}$	$\chi_{b_7}^{a_{13},a_{14},a_{20}}$	$I_{\mathcal{A}} = \{1, 2, 3, 4, 5, 6, 10\}$	$q(q-1)^3$	q^7
		$I_{\mathcal{L}} = \{8, 9, 11, 12, 15, 17, 18\}$		
	$\chi^{a_{11},a_{13},a_{20}}$	$I_{\mathcal{A}} = \{1, 2, 3, 4, 5, 6, 10\}$	$(q-1)^3$	q^7
		$I_{\mathcal{L}} = \{7, 8, 9, 12, 15, 17, 18\}$		
	$\chi_{b_1,b_7}^{a_{13},a_{20}}$	$I_{\mathcal{A}} = \{2, 3, 4, 5, 6, 10\}$	$q^2(q-1)^2$	q^6
		$I_{\mathcal{L}} = \{8, 9, 12, 15, 17, 18\}$		
$\mathcal{F}_{\mathbf{13,22}}$	$\chi_{b_2}^{a_9,a_{10},a_{13},a_{22}}$	$I_{\mathcal{A}} = \{1, 3, 5, 7, 8, 11, 12\}$	$q(q-1)^4$	q^7
		$I_{\mathcal{L}} = \{4, 6, 14, 15, 17, 18, 20\}$		

$\mathcal{F}$	χ	I	Number	Degree
	$\chi_{b_2,b_4,b_5}^{a_{10},a_{13},a_{22}}$	$I_{\mathcal{A}} = \{1,3,7,8,11,12\}$	$q^3(q-1)^3$	q^6
		$I_{\mathcal{L}} = \{6,14,15,17,18,20\}$		
	$\chi^{a_6,a_9,a_{13},a_{22}}$	$I_{\mathcal{A}} = \{2,3,5,7,8,11,12\}$	$(q-1)^4$	q^7
		$I_{\mathcal{L}} = \{1,4,14,15,17,18,20\}$		
	$\chi_{b_1,b_2}^{a_9,a_{13},a_{22}}$	$I_{\mathcal{A}} = \{3,5,7,8,11,12\}$	$q^2(q-1)^3$	q^6
		$I_{\mathcal{L}} = \{4,14,15,17,18,20\}$		
	$\chi_{b_4,b_5}^{a_6,a_{13},a_{22}}$	$I_{\mathcal{A}} = \{2,3,7,8,11,12\}$	$q^2(q-1)^3$	q^6
		$I_{\mathcal{L}} = \{1,14,15,17,18,20\}$		
	$\chi_{b_1,b_2,b_4,b_5}^{a_{13},a_{22}}$	$I_{\mathcal{A}} = \{3,7,8,11,12\}$	$q^4(q-1)^2$	q^5
		$I_{\mathcal{L}} = \{14,15,17,18,20\}$		
$\mathcal{F}_{\mathbf{14,15}}$	$\chi_{b_2,b_7}^{a_{14},a_{15}}$	$I_{\mathcal{A}} = \{1,3,4,8\}$	$q^2(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{6,10,11,12\}$		
$\mathcal{F}_{\mathbf{14,16}}$	$\chi_{b_4,b_6,b_8}^{a_{14},a_{16}}$	$I_{\mathcal{A}} = \{1,2,3,5,9\}$	$q^3(q-1)^2$	q^5
		$I_{\mathcal{L}} = \{7,10,11,12,13\}$		
$\mathcal{F}_{\mathbf{14,19}}$	$\chi^{a_{12},a_{14},a_{19}}$	$I_{\mathcal{A}} = \{1,2,3,4,5,6,9\}$	$(q-1)^3$	q^7
		$I_{\mathcal{L}} = \{7,8,10,11,13,15,16\}$		
	$\chi_{b_2,b_4,b_7,b_8}^{a_{14},a_{19}}$	$I_{\mathcal{A}} = \{1,3,5,6,9\}$	$q^4(q-1)^2$	q^5
		$I_{\mathcal{L}} = \{10,11,13,15,16\}$		
$\mathcal{F}_{\mathbf{15,16}}$	$\chi_{b_3,b_5}^{a_{15},a_{16}}$	$I_{\mathcal{A}} = \{2,4,8,9,12\}$	$q^2(q-1)^2$	q^5
		$I_{\mathcal{L}} = \{1,6,7,11,13\}$		
$\mathcal{F}_{\mathbf{15,17}}$	$\chi_{b_3}^{a_{15},a_{17}}$	$I_{\mathcal{A}} = \{1,2,4,6,7\}$	$q(q-1)^2$	q^5
		$I_{\mathcal{L}} = \{8,10,11,12,14\}$		
$\mathcal{F}_{\mathbf{16,17}}$	$\chi_{b_1,b_4,b_8}^{a_{13},a_{16},a_{17}}$	$I_{\mathcal{A}} = \{2,3,5,6,7\}$	$q^3(q-1)^3$	q^5
		$I_{\mathcal{L}} = \{9,10,11,12,14\}$		
	$\chi_{b_1,b_9}^{a_8,a_{16},a_{17}}$	$I_{\mathcal{A}} = \{2,4,5,6,7\}$	$q^2(q-1)^3$	q^5
		$I_{\mathcal{L}} = \{3,10,11,12,14\}$		
	$\chi_{b_1,b_3,b_4,b_9}^{a_{16},a_{17}}$	$I_{\mathcal{A}} = \{2,5,6,7\}$	$q^4(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{10,11,12,14\}$		
$\mathcal{F}_{\mathbf{16,18}}$	$\chi^{a_{11},a_{16},a_{18}}$	$I_{\mathcal{A}} = \{1,2,3,4,5,8,9\}$	$(q-1)^3$	q^7
		$I_{\mathcal{L}} = \{6,7,10,12,13,14,15\}$		
	$\chi_{b_3,b_6}^{a_{16},a_{18}}$	$I_{\mathcal{A}} = \{1,2,4,5,8,9\}$	$q^2(q-1)^2$	q^6
		$I_{\mathcal{L}} = \{7,10,12,13,14,15\}$		
$\mathcal{F}_{\mathbf{16,20}}$	$\chi_{b_8,b_{13}}^{a_{14},a_{16},a_{20}}$	$I_{\mathcal{A}} = \{1,2,4,5,6,9,10\}$	$q^2(q-1)^3$	q^7
		$I_{\mathcal{L}} = \{3,7,11,12,15,17,18\}$		
	$\chi^{a_{11},a_{13},a_{16},a_{20}}$	$I_{\mathcal{A}} = \{2,3,4,5,6,7,12\}$	$(q-1)^4$	q^7
		$I_{\mathcal{L}} = \{1,8,9,10,15,17,18\}$		
	$\chi_{b_3,b_5,b_8,b_9}^{a_{11},a_{16},a_{20}}$	$I_{\mathcal{A}} = \{2,4,6,7,12\}$	$q^4(q-1)^3$	q^5
		$I_{\mathcal{L}} = \{1,10,15,17,18\}$		
	$\chi_{b_1,b_3}^{a_{13},a_{16},a_{20}}$	$I_{\mathcal{A}} = \{2,4,5,6,9,12\}$	$q^2(q-1)^3$	q^6
		$I_{\mathcal{L}} = \{7,8,10,15,17,18\}$		
	$\chi_{b_1,b_3,b_5,b_8}^{a_{16},a_{20}}$	$I_{\mathcal{A}} = \{2,4,6,9,12\}$	$q^4(q-1)^2$	q^5
		$I_{\mathcal{L}} = \{7,10,15,17,18\}$		
$\mathcal{F}_{\mathbf{16,22}}$	$\chi^{a_9,a_{10},a_{16},a_{22}}$	$I_{\mathcal{A}} = \{1,2,3,5,7,8,11,12\}$	$(q-1)^4$	q^8
		$I_{\mathcal{L}} = \{4,6,13,14,15,17,18,20\}$		
	$\chi_{b_4,b_5}^{a_{10},a_{16},a_{22}}$	$I_{\mathcal{A}} = \{1,2,3,7,8,11,12\}$	$q^2(q-1)^3$	q^7
		$I_{\mathcal{L}} = \{6,13,14,15,17,18,20\}$		
	$\chi_{b_1,b_6}^{a_9,a_{16},a_{22}}$	$I_{\mathcal{A}} = \{2,3,5,7,8,11,12\}$	$q^2(q-1)^3$	q^7
		$I_{\mathcal{L}} = \{4,13,14,15,17,18,20\}$		
	$\chi_{b_1,b_4,b_5,b_6}^{a_{16},a_{22}}$	$I_{\mathcal{A}} = \{2,3,7,8,11,12\}$	$q^4(q-1)^2$	q^6
		$I_{\mathcal{L}} = \{13,14,15,17,18,20\}$		
$\mathcal{F}_{\mathbf{17,18}}$	$\chi^{a_8,a_{17},a_{18}}$	$I_{\mathcal{A}} = \{1,4,6,10,14\}$	$(q-1)^3$	q^5

$\mathcal{F}$	χ	I	Number	Degree
		$I_{\mathcal{L}} = \{2, 3, 7, 11, 15\}$		
	$\chi_{b_3,b_4}^{a_{17},a_{18}}$	$I_{\mathcal{A}} = \{1, 6, 10, 14\}$ $I_{\mathcal{L}} = \{2, 7, 11, 15\}$	$q^2(q-1)^2$	q^4
	$\boldsymbol{\chi_{b_3}^{a_{12},a_{17},a_{18}}}$	$\boldsymbol{I_{\mathcal{A}} = \{1, 6\},\ I_{\mathcal{I}} = \{2, 4, 10\}.}$ $\boldsymbol{I_{\mathcal{L}} = \{11, 15\},\ I_{\mathcal{J}} = \{7, 8, 14\},}$	$\boldsymbol{q(q-1)^3}$	$\boldsymbol{q^5}$
$\mathcal{F}_{\mathbf{17},\mathbf{19}}$	$\chi_{b_{12}}^{a_8,a_{17},a_{19}}$	$I_{\mathcal{A}} = \{1, 2, 4, 5, 6, 7, 9\}$ $I_{\mathcal{L}} = \{3, 10, 11, 13, 14, 15, 16\}$	$q(q-1)^3$	q^7
	$\chi_{b_3,b_4,b_{12}}^{a_{17},a_{19}}$	$I_{\mathcal{A}} = \{1, 2, 5, 6, 7, 9\}$ $I_{\mathcal{L}} = \{10, 11, 13, 14, 15, 16\}$	$q^3(q-1)^2$	q^6
$\mathcal{F}_{\mathbf{17},\mathbf{21}}$	$\chi_{b_3}^{a_{12},a_{16},a_{17},a_{21}}$	$I_{\mathcal{A}} = \{1, 2, 4, 5, 6, 9, 10\}$ $I_{\mathcal{L}} = \{7, 8, 11, 13, 14, 18, 19\}$	$q(q-1)^4$	q^7
	$\chi_{b_3}^{a_{12},a_{15},a_{17},a_{21}}$	$I_{\mathcal{A}} = \{1, 2, 4, 5, 6, 10, 14\}$ $I_{\mathcal{L}} = \{7, 8, 9, 11, 13, 18, 19\}$	$q(q-1)^4$	q^7
	$\chi^{a_7,a_{15},a_{17},a_{21}}$	$I_{\mathcal{A}} = \{1, 3, 4, 5, 6, 10, 14\}$ $I_{\mathcal{L}} = \{2, 8, 9, 11, 13, 18, 19\}$	$(q-1)^4$	q^7
	$\chi_{b_2,b_3}^{a_{15},a_{17},a_{21}}$	$I_{\mathcal{A}} = \{1, 4, 5, 6, 10, 14\}$ $I_{\mathcal{L}} = \{8, 9, 11, 13, 18, 19\}$	$q^2(q-1)^3$	q^6
	$\chi_{b_3}^{a_{12},a_{17},a_{21}}$	$I_{\mathcal{A}} = \{1, 2, 4, 5, 6, 10, 14\}$ $I_{\mathcal{L}} = \{7, 8, 9, 11, 13, 18, 19\}$	$q(q-1)^3$	q^7
	$\chi_{b_4,b_8}^{a_7,a_{17},a_{21}}$	$I_{\mathcal{A}} = \{1, 3, 5, 6, 10, 14\}$ $I_{\mathcal{L}} = \{2, 9, 11, 13, 18, 19\}$	$q^2(q-1)^3$	q^6
	$\chi_{b_2}^{a_8,a_{17},a_{21}}$	$I_{\mathcal{A}} = \{1, 4, 5, 6, 10, 14\}$ $I_{\mathcal{L}} = \{3, 9, 11, 13, 18, 19\}$	$q(q-1)^3$	q^6
	$\chi_{b_2,b_3,b_4}^{a_{17},a_{21}}$	$I_{\mathcal{A}} = \{1, 5, 6, 10, 14\}$ $I_{\mathcal{L}} = \{9, 11, 13, 18, 19\}$	$q^3(q-1)^2$	q^5
	$\boldsymbol{\chi_{b_3}^{a_{15},a_{16},a_{17},a_{21}}}$	$\boldsymbol{I_{\mathcal{A}} = \{1, 4, 5, 6\},\ I_{\mathcal{I}} = \{2, 9, 10\}.}$ $\boldsymbol{I_{\mathcal{L}} = \{8, 11, 18, 19\},\ I_{\mathcal{J}} = \{7, 13, 14\},}$	$\boldsymbol{q(q-1)^4}$	$\boldsymbol{q^7}$
	$\boldsymbol{\chi^{a_8,a_{16},a_{17},a_{21}}}$	$\boldsymbol{I_{\mathcal{A}} = \{1, 3, 5, 6\},\ I_{\mathcal{I}} = \{2, 9, 10\}.}$ $\boldsymbol{I_{\mathcal{L}} = \{4, 11, 18, 19\},\ I_{\mathcal{J}} = \{7, 13, 14\},}$	$\boldsymbol{(q-1)^4}$	$\boldsymbol{q^7}$
	$\boldsymbol{\chi_{b_3,b_4}^{a_{16},a_{17},a_{21}}}$	$\boldsymbol{I_{\mathcal{A}} = \{1, 5, 6\},\ I_{\mathcal{I}} = \{2, 9, 10\}.}$ $\boldsymbol{I_{\mathcal{L}} = \{11, 18, 19\},\ I_{\mathcal{J}} = \{7, 13, 14\},}$	$\boldsymbol{q^2(q-1)^3}$	$\boldsymbol{q^6}$
	$\boldsymbol{\chi_{b_3}^{a_{12},a_{15},a_{16},a_{17},a_{21}^*}}$ $(a_{21}^* \neq a_{16}(a_{15}/a_{12})^2)$	$\boldsymbol{I_{\mathcal{A}} = \{1, 5\},\ I_{\mathcal{I}} = \{2, 4, 6, 9, 10\}}$ $\boldsymbol{I_{\mathcal{L}} = \{18, 19\},\ I_{\mathcal{J}} = \{7, 8, 11, 13, 14\}}$	$\boldsymbol{q(q-1)^4(q-2)}$	$\boldsymbol{q^7}$
	$\boldsymbol{\chi^{a_{12},a_{15},a_{16},a_{17},a_{7,\dots,14}}}$	**See $\mathfrak{C}^{\mathrm{B}_5}$ in Section 5.2**	$\boldsymbol{(q-1)^5}$	$\boldsymbol{q^7}$
	$\boldsymbol{\chi_{b_3,b_{2,4,6,9,10}}^{a_{12},a_{15},a_{16},a_{17}}}$	**See $\mathfrak{C}^{\mathrm{B}_5}$ in Section 5.2**	$\boldsymbol{q^2(q-1)^4}$	$\boldsymbol{q^6}$
$\mathcal{F}_{\mathbf{18},\mathbf{19}}$	$\chi_{b_{12}}^{a_7,a_{18},a_{19}}$	$I_{\mathcal{A}} = \{1, 3, 4, 5, 6, 8, 9\}$ $I_{\mathcal{L}} = \{2, 10, 11, 13, 14, 15, 16\}$	$q(q-1)^3$	q^7
	$\chi_{b_2,b_3,b_{12}}^{a_{18},a_{19}}$	$I_{\mathcal{A}} = \{1, 4, 5, 6, 8, 9\}$ $I_{\mathcal{L}} = \{10, 11, 13, 14, 15, 16\}$	$q^3(q-1)^2$	q^6
$\mathcal{F}_{\mathbf{19},\mathbf{20}}$	$\chi_{b_3,b_7,b_8,b_{14}}^{a_{19},a_{20}}$	$I_{\mathcal{A}} = \{1, 2, 4, 5, 6, 9, 10\}$ $I_{\mathcal{L}} = \{11, 12, 13, 15, 16, 17, 18\}$	$q^4(q-1)^2$	q^7
$\mathcal{F}_{\mathbf{19},\mathbf{22}}$	$\chi_{b_2,b_4,b_{10}}^{a_{19},a_{22}}$	$I_{\mathcal{A}} = \{1, 3, 5, 6, 7, 8, 11, 12\}$ $I_{\mathcal{L}} = \{9, 13, 14, 15, 16, 17, 18, 20\}$	$q^3(q-1)^2$	q^8
$\mathcal{F}_{\mathbf{20},\mathbf{21}}$	$\chi_{b_8,b_{11},b_{13}}^{a_7,a_{16},a_{20},a_{21}}$	$I_{\mathcal{A}} = \{1, 2, 4, 5, 6, 9, 10\}$ $I_{\mathcal{L}} = \{3, 12, 14, 15, 17, 18, 19\}$	$q^3(q-1)^4$	q^7
	$\chi_{b_3,b_7,b_{11}}^{a_{13},a_{20},a_{21}}$	$I_{\mathcal{A}} = \{1, 2, 4, 5, 6, 9, 10\}$ $I_{\mathcal{L}} = \{8, 12, 14, 15, 17, 18, 19\}$	$q^3(q-1)^3$	q^7
	$\chi_{b_8,b_{11}}^{a_7,a_{20},a_{21}}$	$I_{\mathcal{A}} = \{1, 3, 4, 6, 9, 10, 18\}$ $I_{\mathcal{L}} = \{2, 5, 12, 14, 15, 17, 19\}$	$q^2(q-1)^3$	q^7
	$\chi_{b_2,b_3,b_8,b_{11}}^{a_{20},a_{21}}$	$I_{\mathcal{A}} = \{1, 4, 6, 9, 10, 18\}$ $I_{\mathcal{L}} = \{5, 12, 14, 15, 17, 19\}$	$q^4(q-1)^2$	q^6

$\mathcal{F}$	χ	I	Number	Degree
	$\boldsymbol{\chi_{b_3,b_8,b_{11}}^{a_{16},a_{20},a_{21}}}$	$\boldsymbol{I_{\mathcal{A}} = \{1,4,6,9\}}$, $\boldsymbol{I_{\mathcal{L}} = \{14,15,17,19\}}$, $\boldsymbol{I_{\mathcal{I}} = \{2,5,10\}, I_{\mathcal{J}} = \{12,13,18\}}$	$\boldsymbol{q^3(q-1)^3}$	$\boldsymbol{q^7}$
$\mathcal{F}_{\mathbf{21},\mathbf{22}}$	$\chi_{b_2,b_6}^{a_9,a_{21},a_{22}}$	$I_{\mathcal{A}} = \{1,3,5,8,11,13,14,18\}$ $I_{\mathcal{L}} = \{4,7,10,12,15,17,19,20\}$	$q^2(q-1)^3$	q^8
	$\chi_{b_2,b_4,b_5,b_6}^{a_{21},a_{22}}$	$I_{\mathcal{A}} = \{1,3,8,11,13,14,18\}$ $I_{\mathcal{L}} = \{7,10,12,15,17,19,20\}$	$q^4(q-1)^2$	q^7
	$\boldsymbol{\chi_{b_2,b_4,b_6}^{a_{16},a_{21},a_{22}}}$	$\boldsymbol{I_{\mathcal{A}} = \{1,3,8,11,13\}}$, $\boldsymbol{I_{\mathcal{L}} = \{10,15,17,19,20\}}$, $\boldsymbol{I_{\mathcal{I}} = \{5,7,14\}, I_{\mathcal{J}} = \{9,12,18\}}$	$\boldsymbol{q^3(q-1)^3}$	$\boldsymbol{q^8}$
$\mathcal{F}_{\mathbf{22},\mathbf{23}}$	$\chi_{b_1}^{a_9,a_{22},a_{23}}$	$I_{\mathcal{A}} = \{2,5,6,7,10,11,12,17,20\}$ $I_{\mathcal{L}} = \{3,4,8,14,15,16,18,19,21\}$	$q(q-1)^3$	q^9
	$\chi_{b_1,b_4,b_5}^{a_{22},a_{23}}$	$I_{\mathcal{A}} = \{2,6,7,10,11,12,17,20\}$ $I_{\mathcal{L}} = \{3,8,14,15,16,18,19,21\}$	$q^3(q-1)^2$	q^8
	$\boldsymbol{\chi_{b_1,b_4}^{a_{13},a_{22},a_{23}}}$	$\boldsymbol{I_{\mathcal{A}} = \{2,6,7,10,11,12\}}$, $\boldsymbol{I_{\mathcal{L}} = \{14,15,16,18,19,21\}}$, $\boldsymbol{I_{\mathcal{I}} = \{3,5,17\}, I_{\mathcal{J}} = \{8,9,20\}}$	$\boldsymbol{q^2(q-1)^3}$	$\boldsymbol{q^9}$
$\mathcal{F}_{\mathbf{1},\mathbf{2},\mathbf{8}}$	χ^{a_1,a_2,a_8}	$I_{\mathcal{A}} = \{3\}, I_{\mathcal{L}} = \{4\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{1},\mathbf{2},\mathbf{9}}$	χ^{a_1,a_2,a_9}	$I_{\mathcal{A}} = \{4\}, I_{\mathcal{L}} = \{5\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{1},\mathbf{2},\mathbf{13}}$	$\chi_{b_4}^{a_1,a_2,a_{13}}$	$I_{\mathcal{A}} = \{3,5\}, I_{\mathcal{L}} = \{8,9\}$	$q(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{1},\mathbf{3},\mathbf{9}}$	χ^{a_1,a_3,a_9}	$I_{\mathcal{A}} = \{4\}, I_{\mathcal{L}} = \{5\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{1},\mathbf{4},\mathbf{7}}$	χ^{a_1,a_4,a_7}	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{3\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{1},\mathbf{5},\mathbf{7}}$	χ^{a_1,a_5,a_7}	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{3\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{1},\mathbf{5},\mathbf{8}}$	χ^{a_1,a_5,a_8}	$I_{\mathcal{A}} = \{3\}, I_{\mathcal{L}} = \{4\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{1},\mathbf{5},\mathbf{12}}$	$\chi_{b_3}^{a_1,a_5,a_{12}}$	$I_{\mathcal{A}} = \{2,4\}, I_{\mathcal{L}} = \{7,8\}$	$q(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{1},\mathbf{7},\mathbf{8}}$	$\chi_{b_4}^{a_1,a_7,a_8}$	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{3\}$	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{1},\mathbf{7},\mathbf{9}}$	χ^{a_1,a_7,a_9}	$I_{\mathcal{A}} = \{2,4\}, I_{\mathcal{L}} = \{3,5\}$	$(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{1},\mathbf{7},\mathbf{13}}$	$\chi_{b_2,b_4}^{a_1,a_7,a_{13}}$	$I_{\mathcal{A}} = \{3,5\}, I_{\mathcal{L}} = \{8,9\}$	$q^2(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{1},\mathbf{8},\mathbf{9}}$	$\chi_{b_5}^{a_1,a_8,a_9}$	$I_{\mathcal{A}} = \{3\}, I_{\mathcal{L}} = \{4\}$	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{1},\mathbf{9},\mathbf{12}}$	$\chi_{b_3,b_5}^{a_1,a_9,a_{12}}$	$I_{\mathcal{A}} = \{2,4\}, I_{\mathcal{L}} = \{7,8\}$	$q^2(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{1},\mathbf{12},\mathbf{13}}$	$\chi_{b_5}^{a_1,a_{12},a_{13}}$	$I_{\mathcal{A}} = \{3,4,8\}, I_{\mathcal{L}} = \{2,7,9\}$	$q(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{2},\mathbf{3},\mathbf{9}}$	χ^{a_2,a_3,a_9}	$I_{\mathcal{A}} = \{4\}, I_{\mathcal{L}} = \{5\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{2},\mathbf{5},\mathbf{8}}$	χ^{a_2,a_5,a_8}	$I_{\mathcal{A}} = \{3\}, I_{\mathcal{L}} = \{4\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{2},\mathbf{8},\mathbf{9}}$	$\chi_{b_5}^{a_2,a_8,a_9}$	$I_{\mathcal{A}} = \{3\}, I_{\mathcal{L}} = \{4\}$	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{3},\mathbf{4},\mathbf{6}}$	χ^{a_3,a_4,a_6}	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{2\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{3},\mathbf{4},\mathbf{10}}$	$\chi_{b_2}^{a_3,a_4,a_{10}}$	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{6\}$	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{3},\mathbf{5},\mathbf{6}}$	χ^{a_3,a_5,a_6}	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{2\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{3},\mathbf{5},\mathbf{10}}$	$\chi_{b_2}^{a_3,a_5,a_{10}}$	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{6\}$	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{3},\mathbf{6},\mathbf{9}}$	χ^{a_3,a_6,a_9}	$I_{\mathcal{A}} = \{1,4\}, I_{\mathcal{L}} = \{2,5\}$	$(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{3},\mathbf{9},\mathbf{10}}$	$\chi_{b_2}^{a_3,a_9,a_{10}}$	$I_{\mathcal{A}} = \{1,5\}, I_{\mathcal{L}} = \{4,6\}$	$q(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{4},\mathbf{5},\mathbf{6}}$	χ^{a_4,a_5,a_6}	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{2\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{4},\mathbf{5},\mathbf{7}}$	χ^{a_4,a_5,a_7}	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{3\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{4},\mathbf{5},\mathbf{10}}$	$\chi_{b_2}^{a_4,a_5,a_{10}}$	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{6\}$	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{4},\mathbf{5},\mathbf{11}}$	$\chi_{b_2}^{a_4,a_5,a_{11}}$	$I_{\mathcal{A}} = \{1,3\}, I_{\mathcal{L}} = \{6,7\}$	$q(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{4},\mathbf{5},\mathbf{14}}$	$\chi_{b_2,b_6,b_7}^{a_4,a_5,a_{14}}$	$I_{\mathcal{A}} = \{1,3\}, I_{\mathcal{L}} = \{10,11\}$	$q^3(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{4},\mathbf{5},\mathbf{17}}$	$\chi_{b_1,b_3}^{a_4,a_5,a_{17}}$	$I_{\mathcal{A}} = \{2,6,7\}, I_{\mathcal{L}} = \{10,11,14\}$	$q^2(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{4},\mathbf{6},\mathbf{7}}$	$\chi_{b_3}^{a_4,a_6,a_7}$	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{2\}$	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{4},\mathbf{7},\mathbf{10}}$	$\chi^{a_4,a_7,a_{10}}$	$I_{\mathcal{A}} = \{1,3\}, I_{\mathcal{L}} = \{2,6\}$	$(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{4},\mathbf{10},\mathbf{11}}$	$\chi_{b_2}^{a_4,a_{10},a_{11}}$	$I_{\mathcal{A}} = \{1,3\}, I_{\mathcal{L}} = \{6,7\}$	$q(q-1)^3$	q^2

$\mathcal{F}$	χ	I	Number	Degree
$\mathcal{F}_{\mathbf{5,6,7}}$	$\chi_{b_3}^{a_5,a_6,a_7}$	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{2\}$	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{5,6,8}}$	χ^{a_5,a_6,a_8}	$I_{\mathcal{A}} = \{1,3\}, I_{\mathcal{L}} = \{2,4\}$	$(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{5,6,12}}$	$\chi_{b_1,b_3}^{a_5,a_6,a_{12}}$	$I_{\mathcal{A}} = \{2,4\}, I_{\mathcal{L}} = \{7,8\}$	$q^2(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{5,7,8}}$	$\chi_{b_4}^{a_5,a_7,a_8}$	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{3\}$	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{5,7,10}}$	$\chi^{a_5,a_7,a_{10}}$	$I_{\mathcal{A}} = \{1,3\}, I_{\mathcal{L}} = \{2,6\}$	$(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{5,8,10}}$	$\chi_{b_2}^{a_5,a_8,a_{10}}$	$I_{\mathcal{A}} = \{1,4\}, I_{\mathcal{L}} = \{3,6\}$	$q(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{5,8,11}}$	$\chi_{b_2,b_4}^{a_5,a_8,a_{11}}$	$I_{\mathcal{A}} = \{1,3\}, I_{\mathcal{L}} = \{6,7\}$	$q^2(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{5,8,14}}$	$\chi_{b_2,b_4,b_6,b_7}^{a_5,a_8,a_{14}}$	$I_{\mathcal{A}} = \{1,3\}, I_{\mathcal{L}} = \{10,11\}$	$q^4(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{5,8,17}}$	$\chi_{b_1}^{a_5,a_8,a_{17}}$	$I_{\mathcal{A}} = \{2,4,6,7\}$	$q(q-1)^3$	q^4
		$I_{\mathcal{L}} = \{3,10,11,14\}$		
$\mathcal{F}_{\mathbf{5,10,11}}$	$\chi_{b_2}^{a_5,a_{10},a_{11}}$	$I_{\mathcal{A}} = \{1,3\}, I_{\mathcal{L}} = \{6,7\}$	$q(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{5,10,12}}$	$\chi_{b_3}^{a_5,a_{10},a_{12}}$	$I_{\mathcal{A}} = \{1,2,4\}, I_{\mathcal{L}} = \{6,7,8\}$	$q(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{5,10,15}}$	$\chi^{a_5,a_7,a_{10},a_{15}}$	$I_{\mathcal{A}} = \{1,3,4,8\}$	$(q-1)^4$	q^4
		$I_{\mathcal{L}} = \{2,6,11,12\}$		
	$\chi_{b_2,b_3}^{a_5,a_{10},a_{15}}$	$I_{\mathcal{A}} = \{1,4,8\}, I_{\mathcal{L}} = \{6,11,12\}$	$q^2(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{5,11,12}}$	$\chi_{b_4}^{a_5,a_{11},a_{12}}$	$I_{\mathcal{A}} = \{2,3,7\}, I_{\mathcal{L}} = \{1,6,8\}$	$q(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{5,12,14}}$	$\chi_{b_6}^{a_5,a_{12},a_{14}}$	$I_{\mathcal{A}} = \{1,2,3,4\}$	$q(q-1)^3$	q^4
		$I_{\mathcal{L}} = \{7,8,10,11\}$		
$\mathcal{F}_{\mathbf{5,12,17}}$	$\chi_{b_1}^{a_5,a_8,a_{12},a_{17}}$	$I_{\mathcal{A}} = \{2,4,6,7\}$	$q(q-1)^4$	q^4
		$I_{\mathcal{L}} = \{3,10,11,14\}$		
	$\chi_{b_1,b_3,b_4}^{a_5,a_{12},a_{17}}$	$I_{\mathcal{A}} = \{2,6,7\}, I_{\mathcal{L}} = \{10,11,14\}$	$q^3(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{5,14,15}}$	$\chi_{b_2,b_7}^{a_5,a_{14},a_{15}}$	$I_{\mathcal{A}} = \{1,3,4,8\}$	$q^2(q-1)^3$	q^4
		$I_{\mathcal{L}} = \{6,10,11,12\}$		
$\mathcal{F}_{\mathbf{5,15,17}}$	$\chi_{b_3}^{a_5,a_{15},a_{17}}$	$I_{\mathcal{A}} = \{1,2,4,6,7\}$	$q(q-1)^3$	q^5
		$I_{\mathcal{L}} = \{8,10,11,12,14\}$		
$\mathcal{F}_{\mathbf{5,17,18}}$	$\chi^{a_5,a_8,a_{17},a_{18}}$	$I_{\mathcal{A}} = \{1,4,6,10,14\}$	$(q-1)^4$	q^5
		$I_{\mathcal{L}} = \{2,3,7,11,15\}$		
	$\chi_{b_3,b_4}^{a_5,a_{17},a_{18}}$	$I_{\mathcal{A}} = \{1,6,10,14\}$	$q^2(q-1)^3$	q^4
		$I_{\mathcal{L}} = \{2,7,11,15\}$		
	$\chi_{b_3}^{a_5,a_{12},a_{17},a_{18}}$	**$I_{\mathcal{A}} = \{1,6\},\ I_{\mathcal{I}} = \{2,4,10\}.$**	**$q(q-1)^4$**	**q^5**
		$I_{\mathcal{L}} = \{11,15\},\ I_{\mathcal{J}} = \{7,8,14\},$		
$\mathcal{F}_{\mathbf{6,7,8}}$	χ^{a_6,a_7,a_8}	$I_{\mathcal{A}} = \{1,3\}, I_{\mathcal{L}} = \{2,4\}$	$(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{6,7,9}}$	$\chi_{b_3}^{a_6,a_7,a_9}$	$I_{\mathcal{A}} = \{1,4\}, I_{\mathcal{L}} = \{2,5\}$	$q(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{6,7,13}}$	$\chi_{b_4}^{a_6,a_7,a_{13}}$	$I_{\mathcal{A}} = \{2,3,5\}, I_{\mathcal{L}} = \{1,8,9\}$	$q(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{6,8,9}}$	$\chi_{b_5}^{a_6,a_8,a_9}$	$I_{\mathcal{A}} = \{1,3\}, I_{\mathcal{L}} = \{2,4\}$	$q(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{6,9,12}}$	$\chi_{b_1,b_3,b_5}^{a_6,a_9,a_{12}}$	$I_{\mathcal{A}} = \{2,4\}, I_{\mathcal{L}} = \{7,8\}$	$q^3(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{6,12,13}}$	$\chi^{a_6,a_{12},a_{13}}$	$I_{\mathcal{A}} = \{1,3,4,8\}$	$(q-1)^3$	q^4
		$I_{\mathcal{L}} = \{2,5,7,9\}$		
$\mathcal{F}_{\mathbf{7,8,9}}$	χ^{a_7,a_8,a_9}	$I_{\mathcal{A}} = \{2,4\}, I_{\mathcal{L}} = \{3,5\}$	$(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{7,8,10}}$	$\chi_{b_4}^{a_7,a_8,a_{10}}$	$I_{\mathcal{A}} = \{1,3\}, I_{\mathcal{L}} = \{2,6\}$	$q(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{7,9,10}}$	$\chi^{a_7,a_9,a_{10}}$	$I_{\mathcal{A}} = \{1,3,5\}, I_{\mathcal{L}} = \{2,4,6\}$	$(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{7,10,13}}$	$\chi_{b_2,b_4}^{a_7,a_{10},a_{13}}$	$I_{\mathcal{A}} = \{1,3,5\}, I_{\mathcal{L}} = \{6,8,9\}$	$q^2(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{8,9,10}}$	$\chi_{b_2,b_5}^{a_8,a_9,a_{10}}$	$I_{\mathcal{A}} = \{1,4\}, I_{\mathcal{L}} = \{3,6\}$	$q^2(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{8,9,11}}$	$\chi_{b_2}^{a_8,a_9,a_{11}}$	$I_{\mathcal{A}} = \{1,3,5\}, I_{\mathcal{L}} = \{4,6,7\}$	$q(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{8,9,14}}$	$\chi_{b_2,b_6,b_7}^{a_8,a_9,a_{14}}$	$I_{\mathcal{A}} = \{1,3,5\}, I_{\mathcal{L}} = \{4,10,11\}$	$q^3(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{8,9,17}}$	$\chi_{b_1,b_5}^{a_8,a_9,a_{17}}$	$I_{\mathcal{A}} = \{2,4,6,7\}$	$q^2(q-1)^3$	q^4
		$I_{\mathcal{L}} = \{3,10,11,14\}$		
$\mathcal{F}_{\mathbf{8,10,11}}$	$\chi_{b_2,b_4}^{a_8,a_{10},a_{11}}$	$I_{\mathcal{A}} = \{1,3\}, I_{\mathcal{L}} = \{6,7\}$	$q^2(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{9,10,11}}$	$\chi_{b_2}^{a_9,a_{10},a_{11}}$	$I_{\mathcal{A}} = \{1,3,5\}, I_{\mathcal{L}} = \{4,6,7\}$	$q(q-1)^3$	q^3

$\mathcal{F}$	χ	I	Number	Degree
$\mathcal{F}_{\mathbf{9,10,12}}$	$\chi^{a_9,a_{10},a_{12}}_{b_3,b_5}$	$I_{\mathcal{A}}=\{1,2,4\}, I_{\mathcal{L}}=\{6,7,8\}$	$q^2(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{9,10,15}}$	$\chi^{a_7,a_9,a_{10},a_{15}}_{b_5}$	$I_{\mathcal{A}}=\{1,3,4,8\}$	$q(q-1)^4$	q^4
		$I_{\mathcal{L}}=\{2,6,11,12\}$		
	$\chi^{a_9,a_{10},a_{15}}_{b_2,b_3,b_5}$	$I_{\mathcal{A}}=\{1,4,8\}, I_{\mathcal{L}}=\{6,11,12\}$	$q^3(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{9,11,12}}$	$\chi^{a_9,a_{11},a_{12}}$	$I_{\mathcal{A}}=\{2,3,5,7\}$	$(q-1)^3$	q^4
		$I_{\mathcal{L}}=\{1,4,6,8\}$		
$\mathcal{F}_{\mathbf{9,12,14}}$	$\chi^{a_9,a_{12},a_{14}}_{b_5,b_6}$	$I_{\mathcal{A}}=\{1,2,3,4\}$	$q^2(q-1)^3$	q^4
		$I_{\mathcal{L}}=\{7,8,10,11\}$		
$\mathcal{F}_{\mathbf{9,12,17}}$	$\chi^{a_9,a_{12},a_{17}}_{b_1,b_3,b_8}$	$I_{\mathcal{A}}=\{2,4,6,7\}$	$q^3(q-1)^3$	q^4
		$I_{\mathcal{L}}=\{5,10,11,14\}$		
$\mathcal{F}_{\mathbf{9,14,15}}$	$\chi^{a_9,a_{14},a_{15}}_{b_2,b_5,b_7}$	$I_{\mathcal{A}}=\{1,3,4,8\}$	$q^3(q-1)^3$	q^4
		$I_{\mathcal{L}}=\{6,10,11,12\}$		
$\mathcal{F}_{\mathbf{9,15,17}}$	$\chi^{a_9,a_{15},a_{17}}_{b_3,b_5}$	$I_{\mathcal{A}}=\{1,2,4,6,7\}$	$q^2(q-1)^3$	q^5
		$I_{\mathcal{L}}=\{8,10,11,12,14\}$		
$\mathcal{F}_{\mathbf{9,17,18}}$	$\chi^{a_9,a_{17},a_{18}}_{b_3,b_8,b_{12}}$	$I_{\mathcal{A}}=\{1,2,4,6,10\}$	$q^3(q-1)^3$	q^5
		$I_{\mathcal{L}}=\{5,7,11,14,15\}$		
$\mathcal{F}_{\mathbf{10,11,12}}$	$\chi^{a_{10},a_{11},a_{12}}_{b_4}$	$I_{\mathcal{A}}=\{2,6,7\}, I_{\mathcal{L}}=\{1,3,8\}$	$q(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{10,11,13}}$	$\chi^{a_{10},a_{11},a_{13}}_{b_2,b_4,b_9}$	$I_{\mathcal{A}}=\{1,3,5\}, I_{\mathcal{L}}=\{6,7,8\}$	$q^3(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{10,11,16}}$	$\chi^{a_8,a_{10},a_{11},a_{16}}$	$I_{\mathcal{A}}=\{2,4,5,6,7\}$	$(q-1)^4$	q^5
		$I_{\mathcal{L}}=\{1,3,9,12,13\}$		
	$\chi^{a_{10},a_{11},a_{16}}_{b_3,b_4}$	$I_{\mathcal{A}}=\{2,5,6,7\}$	$q^2(q-1)^3$	q^4
		$I_{\mathcal{L}}=\{1,9,12,13\}$		
$\mathcal{F}_{\mathbf{10,12,13}}$	$\chi^{a_{10},a_{12},a_{13}}_{b_5}$	$I_{\mathcal{A}}=\{1,3,4,8\}$	$q(q-1)^3$	q^4
		$I_{\mathcal{L}}=\{2,6,7,9\}$		
$\mathcal{F}_{\mathbf{10,13,15}}$	$\chi^{a_{10},a_{13},a_{15}}_{b_2,b_5,b_7}$	$I_{\mathcal{A}}=\{1,3,4,8\}$	$q^3(q-1)^3$	q^4
		$I_{\mathcal{L}}=\{6,9,11,12\}$		
$\mathcal{F}_{\mathbf{10,15,16}}$	$\chi^{a_{10},a_{15},a_{16}}_{b_3,b_5}$	$I_{\mathcal{A}}=\{2,4,6,9,12\}$	$q^2(q-1)^3$	q^5
		$I_{\mathcal{L}}=\{1,7,8,11,13\}$		
$\mathcal{F}_{\mathbf{11,12,13}}$	$\chi^{a_9,a_{11},a_{12},a_{13}}$	$I_{\mathcal{A}}=\{1,2,4,6\}$	$(q-1)^4$	q^4
		$I_{\mathcal{L}}=\{3,5,7,8\}$		
	$\chi^{a_{11},a_{12},a_{13}}_{b_4,b_5}$	$I_{\mathcal{A}}=\{1,2,6\}, I_{\mathcal{L}}=\{3,7,8\}$	$q^2(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{12,13,14}}$	$\chi^{a_{12},a_{13},a_{14}}_{b_5,b_6,b_9}$	$I_{\mathcal{A}}=\{1,3,4,8\}$	$q^3(q-1)^3$	q^4
		$I_{\mathcal{L}}=\{2,7,10,11\}$		
$\mathcal{F}_{\mathbf{12,13,17}}$	$\chi^{a_{12},a_{13},a_{17}}_{b_1,b_4}$	$I_{\mathcal{A}}=\{2,3,5,6,7\}$	$q^2(q-1)^3$	q^5
		$I_{\mathcal{L}}=\{8,9,10,11,14\}$		
$\mathcal{F}_{\mathbf{13,14,15}}$	$\chi^{a_{13},a_{14},a_{15}}_{b_2,b_5,b_7,b_9}$	$I_{\mathcal{A}}=\{1,3,4,8\}$	$q^4(q-1)^3$	q^4
		$I_{\mathcal{L}}=\{6,10,11,12\}$		
$\mathcal{F}_{\mathbf{13,15,17}}$	$\chi^{a_{13},a_{15},a_{17}}_{b_5}$	$I_{\mathcal{A}}=\{1,2,3,7,8,11\}$	$q(q-1)^3$	q^6
		$I_{\mathcal{L}}=\{4,6,9,10,12,14\}$		
$\mathcal{F}_{\mathbf{13,17,18}}$	$\chi^{a_{13},a_{17},a_{18}}_{b_4,b_{12}}$	$I_{\mathcal{A}}=\{1,3,6,7,8,14\}$	$q^2(q-1)^3$	q^6
		$I_{\mathcal{L}}=\{2,5,9,10,11,15\}$		
$\mathcal{F}_{\mathbf{14,15,16}}$	$\chi^{a_{14},a_{15},a_{16}}_{b_5}$	$I_{\mathcal{A}}=\{2,3,7,8,11,12\}$	$q(q-1)^3$	q^6
		$I_{\mathcal{L}}=\{1,4,6,9,10,13\}$		
$\mathcal{F}_{\mathbf{15,16,17}}$	$\chi^{a_{13},a_{15},a_{16},a_{17}}_{b_5}$	$I_{\mathcal{A}}=\{2,3,7,8,11,12\}$	$q(q-1)^4$	q^6
		$I_{\mathcal{L}}=\{1,4,6,9,10,14\}$		
	$\chi^{a_{15},a_{16},a_{17}}_{b_3,b_5,b_9}$	$I_{\mathcal{A}}=\{2,4,6,7,12\}$	$q^3(q-1)^3$	q^5
		$I_{\mathcal{L}}=\{1,8,10,11,14\}$		
$\mathcal{F}_{\mathbf{16,17,18}}$	$\chi^{a_{16},a_{17},a_{18}}_{b_3}$	$I_{\mathcal{A}}=\{1,2,4,5,6,9,10\}$	$q(q-1)^3$	q^7
		$I_{\mathcal{L}}=\{7,8,11,12,13,14,15\}$		
$\mathcal{F}_{\mathbf{17,18,19}}$	$\chi^{a_8,a_{17},a_{18},a_{19}}$	$I_{\mathcal{A}}=\{1,4,5,6,9,10,14\}$	$(q-1)^4$	q^7

$\mathcal{F}$	χ	I	Number	Degree
	$\chi_{b_3,b_4}^{a_{17},a_{18},a_{19}}$	$I_{\mathcal{L}} = \{2,3,7,11,13,15,16\}$ $I_{\mathcal{A}} = \{1,5,6,9,10,14\}$ $I_{\mathcal{L}} = \{2,7,11,13,15,16\}$	$q^2(q-1)^3$	q^6
	$\boldsymbol{\chi_{b_3}^{a_{12},a_{17},a_{18},a_{19}}}$	$\boldsymbol{I_{\mathcal{A}} = \{1,5,6,9\},}$ $\boldsymbol{I_{\mathcal{L}} = \{11,13,15,16\},}$ $\boldsymbol{I_{\mathcal{I}} = \{2,4,10\}, I_{\mathcal{J}} = \{7,8,14\}}$	$\boldsymbol{q(q-1)^4}$	$\boldsymbol{q^7}$
$\mathcal{F}_{\mathbf{1,2,3,9}}$	χ^{a_1,a_2,a_3,a_9}	$I_{\mathcal{A}} = \{4\}, I_{\mathcal{L}} = \{5\}$	$(q-1)^4$	q
$\mathcal{F}_{\mathbf{1,2,5,8}}$	χ^{a_1,a_2,a_5,a_8}	$I_{\mathcal{A}} = \{3\}, I_{\mathcal{L}} = \{4\}$	$(q-1)^4$	q
$\mathcal{F}_{\mathbf{1,2,8,9}}$	$\chi_{b_5}^{a_1,a_2,a_8,a_9}$	$I_{\mathcal{A}} = \{3\}, I_{\mathcal{L}} = \{4\}$	$q(q-1)^4$	q
$\mathcal{F}_{\mathbf{1,4,5,7}}$	χ^{a_1,a_4,a_5,a_7}	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{3\}$	$(q-1)^4$	q
$\mathcal{F}_{\mathbf{1,5,7,8}}$	$\chi_{b_4}^{a_1,a_5,a_7,a_8}$	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{3\}$	$q(q-1)^4$	q
$\mathcal{F}_{\mathbf{1,7,8,9}}$	χ^{a_1,a_7,a_8,a_9}	$I_{\mathcal{A}} = \{2,4\}, I_{\mathcal{L}} = \{3,5\}$	$(q-1)^4$	q^2
$\mathcal{F}_{\mathbf{3,4,5,6}}$	χ^{a_3,a_4,a_5,a_6}	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{2\}$	$(q-1)^4$	q
$\mathcal{F}_{\mathbf{3,4,5,10}}$	$\chi_{b_2}^{a_3,a_4,a_5,a_{10}}$	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{6\}$	$q(q-1)^4$	q
$\mathcal{F}_{\mathbf{4,5,6,7}}$	$\chi_{b_3}^{a_4,a_5,a_6,a_7}$	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{2\}$	$q(q-1)^4$	q
$\mathcal{F}_{\mathbf{4,5,7,10}}$	$\chi^{a_4,a_5,a_7,a_{10}}$	$I_{\mathcal{A}} = \{1,3\}, I_{\mathcal{L}} = \{2,6\}$	$(q-1)^4$	q^2
$\mathcal{F}_{\mathbf{4,5,10,11}}$	$\chi_{b_2}^{a_4,a_5,a_{10},a_{11}}$	$I_{\mathcal{A}} = \{1,3\}, I_{\mathcal{L}} = \{6,7\}$	$q(q-1)^4$	q^2
$\mathcal{F}_{\mathbf{5,6,7,8}}$	χ^{a_5,a_6,a_7,a_8}	$I_{\mathcal{A}} = \{1,3\}, I_{\mathcal{L}} = \{2,4\}$	$(q-1)^4$	q^2
$\mathcal{F}_{\mathbf{5,7,8,10}}$	$\chi_{b_4}^{a_5,a_7,a_8,a_{10}}$	$I_{\mathcal{A}} = \{1,3\}, I_{\mathcal{L}} = \{2,6\}$	$q(q-1)^4$	q^2
$\mathcal{F}_{\mathbf{5,8,10,11}}$	$\chi_{b_2,b_4}^{a_5,a_8,a_{10},a_{11}}$	$I_{\mathcal{A}} = \{1,3\}, I_{\mathcal{L}} = \{6,7\}$	$q^2(q-1)^4$	q^2
$\mathcal{F}_{\mathbf{5,10,11,12}}$	$\chi_{b_4}^{a_5,a_{10},a_{11},a_{12}}$	$I_{\mathcal{A}} = \{2,6,7\}, I_{\mathcal{L}} = \{1,3,8\}$	$q(q-1)^4$	q^3
$\mathcal{F}_{\mathbf{6,7,8,9}}$	$\chi_{b_5}^{a_6,a_7,a_8,a_9}$	$I_{\mathcal{A}} = \{1,3\}, I_{\mathcal{L}} = \{2,4\}$	$q(q-1)^4$	q^2
$\mathcal{F}_{\mathbf{7,8,9,10}}$	$\chi^{a_7,a_8,a_9,a_{10}}$	$I_{\mathcal{A}} = \{1,3,5\}, I_{\mathcal{L}} = \{2,4,6\}$	$(q-1)^4$	q^3
$\mathcal{F}_{\mathbf{8,9,10,11}}$	$\chi_{b_2}^{a_8,a_9,a_{10},a_{11}}$	$I_{\mathcal{A}} = \{1,3,5\}, I_{\mathcal{L}} = \{4,6,7\}$	$q(q-1)^4$	q^3
$\mathcal{F}_{\mathbf{9,10,11,12}}$	$\chi^{a_9,a_{10},a_{11},a_{12}}$	$I_{\mathcal{A}} = \{2,5,6,7\}$ $I_{\mathcal{L}} = \{1,3,4,8\}$	$(q-1)^4$	q^4
$\mathcal{F}_{\mathbf{10,11,12,13}}$	$\chi_{b_5}^{a_{10},a_{11},a_{12},a_{13}}$	$I_{\mathcal{A}} = \{1,3,4,8\}$ $I_{\mathcal{L}} = \{2,6,7,9\}$	$q(q-1)^4$	q^4

Table D.5: The parametrization of the irreducible characters of $\mathrm{UB}_5(q)$, where $q = p^e$ and $p \geq 3$.

Parametrization of the irreducible characters of U_{C_5}

$\mathcal{F}$	χ	I	Number	Degree
$\mathcal{F}_{\mathrm{lin}}$	$\chi_{b_1,b_2,b_3,b_4,b_5}$		q^5	1
$\mathcal{F}_{\mathbf{6}}$	χ^{a_6}	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{2\}$	$q-1$	q
$\mathcal{F}_{\mathbf{7}}$	χ^{a_7}	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{3\}$	$q-1$	q
$\mathcal{F}_{\mathbf{8}}$	χ^{a_8}	$I_{\mathcal{A}} = \{3\}, I_{\mathcal{L}} = \{4\}$	$q-1$	q
$\mathcal{F}_{\mathbf{9}}$	χ^{a_9}	$I_{\mathcal{A}} = \{4\}, I_{\mathcal{L}} = \{5\}$	$q-1$	q
$\mathcal{F}_{\mathbf{10}}$	$\chi^{a_{10}}_{b_1}$	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{6\}$	$q(q-1)$	q
$\mathcal{F}_{\mathbf{11}}$	$\chi^{a_{11}}_{b_2}$	$I_{\mathcal{A}} = \{1,3\}, I_{\mathcal{L}} = \{6,7\}$	$q(q-1)$	q^2
$\mathcal{F}_{\mathbf{12}}$	$\chi^{a_{12}}_{b_3}$	$I_{\mathcal{A}} = \{2,4\}, I_{\mathcal{L}} = \{7,8\}$	$q(q-1)$	q^2
$\mathcal{F}_{\mathbf{13}}$	$\chi^{a_{13}}_{b_4}$	$I_{\mathcal{A}} = \{3,5\}, I_{\mathcal{L}} = \{8,9\}$	$q(q-1)$	q^2
$\mathcal{F}_{\mathbf{14}}$	$\chi^{a_{14}}_{b_1}$	$I_{\mathcal{A}} = \{2,3,7\}, I_{\mathcal{L}} = \{6,10,11\}$	$q(q-1)$	q^3
$\mathcal{F}_{\mathbf{15}}$	$\chi^{a_7,a_{15}}$	$I_{\mathcal{A}} = \{1,3,4,8\}$	$(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{2,6,11,12\}$		
	$\chi^{a_{15}}_{b_2,b_3}$	$I_{\mathcal{A}} = \{1,4,8\}, I_{\mathcal{L}} = \{6,11,12\}$	$q^2(q-1)$	q^3
$\mathcal{F}_{\mathbf{16}}$	$\chi^{a_8,a_{16}}$	$I_{\mathcal{A}} = \{2,4,5,9\}$	$(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{3,7,12,13\}$		
	$\chi^{a_{16}}_{b_3,b_4}$	$I_{\mathcal{A}} = \{2,5,9\}, I_{\mathcal{L}} = \{7,12,13\}$	$q^2(q-1)$	q^3
$\mathcal{F}_{\mathbf{17}}$	$\chi^{a_{10},a_{17}}_{b_1}$	$I_{\mathcal{A}} = \{2,3,7\}, I_{\mathcal{L}} = \{6,11,14\}$	$q(q-1)^2$	q^3
	$\chi^{a_6,a_{17}}$	$I_{\mathcal{A}} = \{2,3,7\}, I_{\mathcal{L}} = \{1,11,14\}$	$(q-1)^2$	q^3
	$\chi^{a_{17}}_{b_1,b_2}$	$I_{\mathcal{A}} = \{3,7\}, I_{\mathcal{L}} = \{11,14\}$	$q^2(q-1)$	q^2
$\mathcal{F}_{\mathbf{18}}$	$\chi^{a_{11},a_{18}}_{b_3}$	$I_{\mathcal{A}} = \{1,2,4,6,8\}$	$q(q-1)^2$	q^5
		$I_{\mathcal{L}} = \{7,10,12,14,15\}$		
	$\chi^{a_{18}}_{b_1,b_3,b_7}$	$I_{\mathcal{A}} = \{2,4,6,8\}$	$q^3(q-1)$	q^4
		$I_{\mathcal{L}} = \{10,12,14,15\}$		
$\mathcal{F}_{\mathbf{19}}$	$\chi^{a_{12},a_{19}}_{b_3}$	$I_{\mathcal{A}} = \{1,2,4,5,6,9\}$	$q(q-1)^2$	q^6
		$I_{\mathcal{L}} = \{7,8,11,13,15,16\}$		
	$\chi^{a_8,a_{19}}_{b_2,b_7}$	$I_{\mathcal{A}} = \{1,3,5,6,9\}$	$q^2(q-1)^2$	q^5
		$I_{\mathcal{L}} = \{4,11,13,15,16\}$		
	$\chi^{a_7,a_{19}}_{b_4}$	$I_{\mathcal{A}} = \{1,3,5,6,9\}$	$q(q-1)^2$	q^5
		$I_{\mathcal{L}} = \{2,11,13,15,16\}$		
	$\chi^{a_{19}}_{b_2,b_3,b_4}$	$I_{\mathcal{A}} = \{1,5,6,9\}$	$q^3(q-1)$	q^4
		$I_{\mathcal{L}} = \{11,13,15,16\}$		
$\mathcal{F}_{\mathbf{20}}$	$\chi^{a_{10},a_{20}}_{b_1}$	$I_{\mathcal{A}} = \{2,3,4,7,8,12\}$	$q(q-1)^2$	q^6
		$I_{\mathcal{L}} = \{6,11,14,15,17,18\}$		
	$\chi^{a_6,a_{20}}$	$I_{\mathcal{A}} = \{2,3,4,7,8,12\}$	$(q-1)^2$	q^6
		$I_{\mathcal{L}} = \{1,11,14,15,17,18\}$		
	$\chi^{a_{20}}_{b_1,b_2}$	$I_{\mathcal{A}} = \{3,4,7,8,12\}$	$q^2(q-1)$	q^5
		$I_{\mathcal{L}} = \{11,14,15,17,18\}$		
$\mathcal{F}_{\mathbf{21}}$	$\chi^{a_{15},a_{21}}_{b_3,b_7,b_8}$	$I_{\mathcal{A}} = \{1,2,4,5,6,9,13\}$	$q^3(q-1)^2$	q^7
		$I_{\mathcal{L}} = \{10,11,12,14,16,18,19\}$		
	$\chi^{a_8,a_{11},a_{21}}_{b_{12}}$	$I_{\mathcal{A}} = \{2,4,5,6,7,9,13\}$	$q(q-1)^3$	q^7
		$I_{\mathcal{L}} = \{1,3,10,14,16,18,19\}$		
	$\chi^{a_{11},a_{21}}_{b_3,b_4,b_{12}}$	$I_{\mathcal{A}} = \{2,5,6,7,9,13\}$	$q^3(q-1)^2$	q^6
		$I_{\mathcal{L}} = \{1,10,14,16,18,19\}$		
	$\chi^{a_{12},a_{21}}_{b_1,b_3,b_8}$	$I_{\mathcal{A}} = \{2,4,5,6,9,13\}$	$q^3(q-1)^2$	q^6
		$I_{\mathcal{L}} = \{7,10,14,16,18,19\}$		
	$\chi^{a_8,a_{21}}_{b_1,b_7}$	$I_{\mathcal{A}} = \{2,4,5,6,9,13\}$	$q^2(q-1)^2$	q^6
		$I_{\mathcal{L}} = \{3,10,14,16,18,19\}$		
	$\chi^{a_{21}}_{b_1,b_3,b_4,b_7}$	$I_{\mathcal{A}} = \{2,5,6,9,13\}$	$q^4(q-1)$	q^5

$\mathcal{F}$	χ	I	Number	Degree
		$I_{\mathcal{L}} = \{10, 14, 16, 18, 19\}$		
$\mathcal{F}_{\mathbf{22}}$	$\chi_{b_1}^{a_{10},a_{17},a_{22}}$	$I_{\mathcal{A}} = \{2, 3, 4, 7, 8, 12\}$	$q(q-1)^3$	q^6
		$I_{\mathcal{L}} = \{6, 11, 14, 15, 18, 20\}$		
	$\chi^{a_6,a_{17},a_{22}}$	$I_{\mathcal{A}} = \{2, 3, 4, 7, 8, 12\}$	$(q-1)^3$	q^6
		$I_{\mathcal{L}} = \{1, 11, 14, 15, 18, 20\}$		
	$\chi_{b_1,b_2}^{a_{17},a_{22}}$	$I_{\mathcal{A}} = \{3, 4, 7, 8, 12\}$	$q^2(q-1)^2$	q^5
		$I_{\mathcal{L}} = \{11, 14, 15, 18, 20\}$		
	$\chi_{b_1}^{a_{14},a_{22}}$	$I_{\mathcal{A}} = \{2, 3, 4, 7, 8, 12\}$	$q(q-1)^2$	q^6
		$I_{\mathcal{L}} = \{6, 10, 11, 15, 18, 20\}$		
	$\chi_{b_3}^{a_{10},a_{11},a_{22}}$	$I_{\mathcal{A}} = \{1, 4, 6, 8, 12\}$	$q(q-1)^3$	q^5
		$I_{\mathcal{L}} = \{2, 7, 15, 18, 20\}$		
	$\chi_{b_1,b_3,b_7}^{a_{10},a_{22}}$	$I_{\mathcal{A}} = \{2, 4, 8, 12\}$	$q^3(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{6, 15, 18, 20\}$		
	$\chi_{b_2}^{a_{11},a_{22}}$	$I_{\mathcal{A}} = \{1, 3, 4, 8, 12\}$	$q(q-1)^2$	q^5
		$I_{\mathcal{L}} = \{6, 7, 15, 18, 20\}$		
	$\chi_{b_1,b_6}^{a_7,a_{22}}$	$I_{\mathcal{A}} = \{2, 4, 8, 12\}$	$q^2(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{3, 15, 18, 20\}$		
	$\chi_{b_3}^{a_6,a_{22}}$	$I_{\mathcal{A}} = \{2, 4, 8, 12\}$	$q(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{1, 15, 18, 20\}$		
	$\chi_{b_1,b_2,b_3}^{a_{22}}$	$I_{\mathcal{A}} = \{4, 8, 12\}, I_{\mathcal{L}} = \{15, 18, 20\}$	$q^3(q-1)$	q^3
$\mathcal{F}_{\mathbf{23}}$	$\chi_{b_1,b_4}^{a_{18},a_{23}}$	$I_{\mathcal{A}} = \{2, 3, 5, 6, 7, 8, 9, 11, 14\}$	$q^2(q-1)^2$	q^9
		$I_{\mathcal{L}} = \{10, 12, 13, 15, 16, 17, 19, 20, 21\}$		
	$\chi_{b_2,b_4,b_{10}}^{a_{15},a_{23}}$	$I_{\mathcal{A}} = \{1, 3, 5, 6, 7, 9, 11, 14\}$	$q^3(q-1)^2$	q^8
		$I_{\mathcal{L}} = \{8, 12, 13, 16, 17, 19, 20, 21\}$		
	$\chi_{b_1,b_4,b_6,b_{10}}^{a_{12},a_{23}}$	$I_{\mathcal{A}} = \{2, 3, 5, 7, 9, 11, 14\}$	$q^4(q-1)^2$	q^7
		$I_{\mathcal{L}} = \{8, 13, 16, 17, 19, 20, 21\}$		
	$\chi_{b_1,b_4,b_8}^{a_{10},a_{23}}$	$I_{\mathcal{A}} = \{2, 3, 5, 7, 9, 11, 14\}$	$q^3(q-1)^2$	q^7
		$I_{\mathcal{L}} = \{6, 13, 16, 17, 19, 20, 21\}$		
	$\chi_{b_4,b_8}^{a_6,a_{23}}$	$I_{\mathcal{A}} = \{2, 3, 5, 7, 9, 11, 14\}$	$q^2(q-1)^2$	q^7
		$I_{\mathcal{L}} = \{1, 13, 16, 17, 19, 20, 21\}$		
	$\chi_{b_1,b_2,b_4,b_8}^{a_{23}}$	$I_{\mathcal{A}} = \{3, 5, 7, 9, 11, 14\}$	$q^4(q-1)$	q^6
		$I_{\mathcal{L}} = \{13, 16, 17, 19, 20, 21\}$		
$\mathcal{F}_{\mathbf{24}}$	$\chi_{b_1}^{a_{10},a_{17},a_{24}}$	$I_{\mathcal{A}} = \{2, 3, 4, 5, 7, 8, 9, 12, 13, 16\}$	$q(q-1)^3$	q^{10}
		$I_{\mathcal{L}} = \{6, 11, 14, 15, 18, 19, 20, 21, 22, 23\}$		
	$\chi^{a_6,a_{17},a_{24}}$	$I_{\mathcal{A}} = \{2, 3, 4, 5, 7, 8, 9, 12, 13, 16\}$	$(q-1)^3$	q^{10}
		$I_{\mathcal{L}} = \{1, 11, 14, 15, 18, 19, 20, 21, 22, 23\}$		
	$\chi_{b_1,b_2}^{a_{17},a_{24}}$	$I_{\mathcal{A}} = \{3, 4, 5, 7, 8, 9, 12, 13, 16\}$	$q^2(q-1)^2$	q^9
		$I_{\mathcal{L}} = \{11, 14, 15, 18, 19, 20, 21, 22, 23\}$		
	$\chi_{b_1}^{a_{14},a_{24}}$	$I_{\mathcal{A}} = \{2, 3, 4, 5, 7, 8, 9, 12, 13, 16\}$	$q(q-1)^2$	q^{10}
		$I_{\mathcal{L}} = \{6, 10, 11, 15, 18, 19, 20, 21, 22, 23\}$		
	$\chi_{b_3}^{a_{10},a_{11},a_{24}}$	$I_{\mathcal{A}} = \{1, 4, 5, 6, 8, 9, 12, 13, 16\}$	$q(q-1)^3$	q^9
		$I_{\mathcal{L}} = \{2, 7, 15, 18, 19, 20, 21, 22, 23\}$		
	$\chi_{b_1,b_3,b_7}^{a_{10},a_{24}}$	$I_{\mathcal{A}} = \{2, 4, 5, 8, 9, 12, 13, 16\}$	$q^3(q-1)^2$	q^8
		$I_{\mathcal{L}} = \{6, 15, 18, 19, 20, 21, 22, 23\}$		
	$\chi_{b_2}^{a_{11},a_{24}}$	$I_{\mathcal{A}} = \{1, 3, 4, 5, 8, 9, 12, 13, 16\}$	$q(q-1)^2$	q^9
		$I_{\mathcal{L}} = \{6, 7, 15, 18, 19, 20, 21, 22, 23\}$		
	$\chi_{b_1,b_6}^{a_7,a_{24}}$	$I_{\mathcal{A}} = \{2, 4, 5, 8, 9, 12, 13, 16\}$	$q^2(q-1)^2$	q^8
		$I_{\mathcal{L}} = \{3, 15, 18, 19, 20, 21, 22, 23\}$		
	$\chi_{b_3}^{a_6,a_{24}}$	$I_{\mathcal{A}} = \{2, 4, 5, 8, 9, 12, 13, 16\}$	$q(q-1)^2$	q^8
		$I_{\mathcal{L}} = \{1, 15, 18, 19, 20, 21, 22, 23\}$		
	$\chi_{b_1,b_2,b_3}^{a_{24}}$	$I_{\mathcal{A}} = \{4, 5, 8, 9, 12, 13, 16\}$	$q^3(q-1)$	q^7

$\mathcal{F}$	χ	I	Number	Degree
		$I_{\mathcal{L}} = \{15, 18, 19, 20, 21, 22, 23\}$		
$\mathcal{F}_{\mathbf{25}}$	$\chi_{b_1}^{a_{10},a_{17},a_{22},a_{25}}$	$I_{\mathcal{A}} = \{2, 3, 4, 5, 7, 8, 9, 12, 13, 16\}$	$q(q-1)^4$	q^{10}
		$I_{\mathcal{L}} = \{6, 11, 14, 15, 18, 19, 20, 21, 23, 24\}$		
	$\chi^{a_6,a_{17},a_{22},a_{25}}$	$I_{\mathcal{A}} = \{2, 3, 4, 5, 7, 8, 9, 12, 13, 16\}$	$(q-1)^4$	q^{10}
		$I_{\mathcal{L}} = \{1, 11, 14, 15, 18, 19, 20, 21, 23, 24\}$		
	$\chi_{b_1,b_2}^{a_{17},a_{22},a_{25}}$	$I_{\mathcal{A}} = \{3, 4, 5, 7, 8, 9, 12, 13, 16\}$	$q^2(q-1)^3$	q^9
		$I_{\mathcal{L}} = \{11, 14, 15, 18, 19, 20, 21, 23, 24\}$		
	$\chi_{b_1}^{a_{14},a_{22},a_{25}}$	$I_{\mathcal{A}} = \{2, 3, 4, 5, 7, 8, 9, 12, 13, 16\}$	$q(q-1)^3$	q^{10}
		$I_{\mathcal{L}} = \{6, 10, 11, 15, 18, 19, 20, 21, 23, 24\}$		
	$\chi_{b_2,b_{10}}^{a_{11},a_{22},a_{25}}$	$I_{\mathcal{A}} = \{1, 4, 5, 6, 8, 9, 12, 13, 16\}$	$q^2(q-1)^3$	q^9
		$I_{\mathcal{L}} = \{3, 7, 15, 18, 19, 20, 21, 23, 24\}$		
	$\chi_{b_1,b_3,b_7}^{a_{10},a_{22},a_{25}}$	$I_{\mathcal{A}} = \{2, 4, 5, 8, 9, 12, 13, 16\}$	$q^3(q-1)^3$	q^8
		$I_{\mathcal{L}} = \{6, 15, 18, 19, 20, 21, 23, 24\}$		
	$\chi_{b_1,b_6}^{a_7,a_{22},a_{25}}$	$I_{\mathcal{A}} = \{2, 4, 5, 8, 9, 12, 13, 16\}$	$q^2(q-1)^3$	q^8
		$I_{\mathcal{L}} = \{3, 15, 18, 19, 20, 21, 23, 24\}$		
	$\chi_{b_3}^{a_6,a_{22},a_{25}}$	$I_{\mathcal{A}} = \{2, 4, 5, 8, 9, 12, 13, 16\}$	$q(q-1)^3$	q^8
		$I_{\mathcal{L}} = \{1, 15, 18, 19, 20, 21, 23, 24\}$		
	$\chi_{b_1,b_2,b_3}^{a_{22},a_{25}}$	$I_{\mathcal{A}} = \{4, 5, 8, 9, 12, 13, 16\}$	$q^3(q-1)^2$	q^7
		$I_{\mathcal{L}} = \{15, 18, 19, 20, 21, 23, 24\}$		
	$\chi_{b_1}^{a_{10},a_{20},a_{25}}$	$I_{\mathcal{A}} = \{2, 3, 4, 5, 7, 8, 9, 12, 13, 16\}$	$q(q-1)^3$	q^{10}
		$I_{\mathcal{L}} = \{6, 11, 14, 15, 17, 18, 19, 21, 23, 24\}$		
	$\chi^{a_6,a_{20},a_{25}}$	$I_{\mathcal{A}} = \{2, 3, 4, 5, 7, 8, 9, 12, 13, 16\}$	$(q-1)^3$	q^{10}
		$I_{\mathcal{L}} = \{1, 11, 14, 15, 17, 18, 19, 21, 23, 24\}$		
	$\chi_{b_1,b_2}^{a_{20},a_{25}}$	$I_{\mathcal{A}} = \{3, 4, 5, 7, 8, 9, 12, 13, 16\}$	$q^2(q-1)^2$	q^9
		$I_{\mathcal{L}} = \{11, 14, 15, 17, 18, 19, 21, 23, 24\}$		
	$\chi_{b_1,b_4}^{a_{17},a_{18},a_{25}}$	$I_{\mathcal{A}} = \{2, 5, 6, 7, 9, 10, 13, 14, 16\}$	$q^2(q-1)^3$	q^9
		$I_{\mathcal{L}} = \{3, 8, 11, 12, 15, 19, 21, 23, 24\}$		
	$\chi_{b_2,b_4,b_{10}}^{a_{15},a_{17},a_{25}}$	$I_{\mathcal{A}} = \{1, 3, 5, 6, 9, 11, 13, 16\}$	$q^3(q-1)^3$	q^8
		$I_{\mathcal{L}} = \{7, 8, 12, 14, 19, 21, 23, 24\}$		
	$\chi_{b_1,b_4,b_6,b_{10}}^{a_{12},a_{17},a_{25}}$	$I_{\mathcal{A}} = \{2, 3, 5, 7, 9, 13, 16\}$	$q^4(q-1)^3$	q^7
		$I_{\mathcal{L}} = \{8, 11, 14, 19, 21, 23, 24\}$		
	$\chi_{b_1,b_4,b_8}^{a_{10},a_{17},a_{25}}$	$I_{\mathcal{A}} = \{2, 3, 5, 7, 9, 13, 16\}$	$q^3(q-1)^3$	q^7
		$I_{\mathcal{L}} = \{6, 11, 14, 19, 21, 23, 24\}$		
	$\chi_{b_4,b_8}^{a_6,a_{17},a_{25}}$	$I_{\mathcal{A}} = \{2, 3, 5, 7, 9, 13, 16\}$	$q^2(q-1)^3$	q^7
		$I_{\mathcal{L}} = \{1, 11, 14, 19, 21, 23, 24\}$		
	$\chi_{b_1,b_2,b_4,b_8}^{a_{17},a_{25}}$	$I_{\mathcal{A}} = \{3, 5, 7, 9, 13, 16\}$	$q^4(q-1)^2$	q^6
		$I_{\mathcal{L}} = \{11, 14, 19, 21, 23, 24\}$		
	$\chi_{b_3}^{a_{11},a_{18},a_{25}}$	$I_{\mathcal{A}} = \{1, 2, 4, 5, 6, 8, 9, 13, 16\}$	$q(q-1)^3$	q^9
		$I_{\mathcal{L}} = \{7, 10, 12, 14, 15, 19, 21, 23, 24\}$		
	$\chi_{b_1,b_3,b_7}^{a_{18},a_{25}}$	$I_{\mathcal{A}} = \{2, 4, 5, 6, 8, 9, 13, 16\}$	$q^3(q-1)^2$	q^8
		$I_{\mathcal{L}} = \{10, 12, 14, 15, 19, 21, 23, 24\}$		
	$\chi_{b_2,b_7}^{a_{14},a_{15},a_{25}}$	$I_{\mathcal{A}} = \{1, 3, 5, 6, 9, 11, 13, 16\}$	$q^2(q-1)^3$	q^8
		$I_{\mathcal{L}} = \{4, 8, 10, 12, 19, 21, 23, 24\}$		
	$\chi_{b_{10}}^{a_7,a_{15},a_{25}}$	$I_{\mathcal{A}} = \{1, 3, 5, 6, 9, 11, 13, 16\}$	$q(q-1)^3$	q^8
		$I_{\mathcal{L}} = \{2, 4, 8, 12, 19, 21, 23, 24\}$		
	$\chi_{b_2,b_3,b_{10}}^{a_{15},a_{25}}$	$I_{\mathcal{A}} = \{1, 5, 6, 9, 11, 13, 16\}$	$q^3(q-1)^2$	q^7
		$I_{\mathcal{L}} = \{4, 8, 12, 19, 21, 23, 24\}$		
	$\chi_{b_1,b_4,b_8,b_{12}}^{a_{14},a_{25}}$	$I_{\mathcal{A}} = \{2, 3, 5, 7, 9, 13, 16\}$	$q^4(q-1)^2$	q^7
		$I_{\mathcal{L}} = \{6, 10, 11, 19, 21, 23, 24\}$		
	$\chi_{b_4,b_{10}}^{a_{11},a_{12},a_{25}}$	$I_{\mathcal{A}} = \{2, 3, 5, 7, 9, 13, 16\}$	$q^2(q-1)^3$	q^7
		$I_{\mathcal{L}} = \{1, 6, 8, 19, 21, 23, 24\}$		

$\mathcal{F}$	χ	I	Number	Degree
	$\chi^{a_8,a_{10},a_{11},a_{25}}$	$I_{\mathcal{A}}=\{1,4,5,6,9,13,16\}$	$(q-1)^4$	q^7
		$I_{\mathcal{L}}=\{2,3,7,19,21,23,24\}$		
	$\chi^{a_{10},a_{11},a_{25}}_{b_3,b_4}$	$I_{\mathcal{A}}=\{1,5,6,9,13,16\}$	$q^2(q-1)^3$	q^6
		$I_{\mathcal{L}}=\{2,7,19,21,23,24\}$		
	$\chi^{a_{11},a_{25}}_{b_2,b_4,b_8}$	$I_{\mathcal{A}}=\{1,3,5,9,13,16\}$	$q^3(q-1)^2$	q^6
		$I_{\mathcal{L}}=\{6,7,19,21,23,24\}$		
	$\chi^{a_{12},a_{25}}_{b_1,b_3,b_6,b_{10}}$	$I_{\mathcal{A}}=\{2,4,5,9,13,16\}$	$q^4(q-1)^2$	q^6
		$I_{\mathcal{L}}=\{7,8,19,21,23,24\}$		
	$\chi^{a_8,a_{10},a_{25}}_{b_1,b_7}$	$I_{\mathcal{A}}=\{2,4,5,9,13,16\}$	$q^2(q-1)^3$	q^6
		$I_{\mathcal{L}}=\{3,6,19,21,23,24\}$		
	$\chi^{a_{10},a_{25}}_{b_1,b_3,b_4,b_7}$	$I_{\mathcal{A}}=\{2,5,9,13,16\}$	$q^4(q-1)^2$	q^5
		$I_{\mathcal{L}}=\{6,19,21,23,24\}$		
	$\chi^{a_6,a_7,a_8,a_{25}}$	$I_{\mathcal{A}}=\{2,3,5,9,13,16\}$	$(q-1)^4$	q^6
		$I_{\mathcal{L}}=\{1,4,19,21,23,24\}$		
	$\chi^{a_7,a_8,a_{25}}_{b_1,b_2}$	$I_{\mathcal{A}}=\{3,5,9,13,16\}$	$q^2(q-1)^3$	q^5
		$I_{\mathcal{L}}=\{4,19,21,23,24\}$		
	$\chi^{a_7,a_{25}}_{b_1,b_4,b_6}$	$I_{\mathcal{A}}=\{2,5,9,13,16\}$	$q^3(q-1)^2$	q^5
		$I_{\mathcal{L}}=\{3,19,21,23,24\}$		
	$\chi^{a_6,a_8,a_{25}}$	$I_{\mathcal{A}}=\{2,4,5,9,13,16\}$	$(q-1)^3$	q^6
		$I_{\mathcal{L}}=\{1,3,19,21,23,24\}$		
	$\chi^{a_8,a_{25}}_{b_1,b_2}$	$I_{\mathcal{A}}=\{4,5,9,13,16\}$	$q^2(q-1)^2$	q^5
		$I_{\mathcal{L}}=\{3,19,21,23,24\}$		
	$\chi^{a_6,a_{25}}_{b_3,b_4}$	$I_{\mathcal{A}}=\{2,5,9,13,16\}$	$q^2(q-1)^2$	q^5
		$I_{\mathcal{L}}=\{1,19,21,23,24\}$		
	$\chi^{a_{25}}_{b_1,b_2,b_3,b_4}$	$I_{\mathcal{A}}=\{5,9,13,16\}$	$q^4(q-1)$	q^4
		$I_{\mathcal{L}}=\{19,21,23,24\}$		
$\mathcal{F}_{\mathbf{1,7}}$	χ^{a_1,a_7}	$I_{\mathcal{A}}=\{2\}, I_{\mathcal{L}}=\{3\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{1,8}}$	χ^{a_1,a_8}	$I_{\mathcal{A}}=\{3\}, I_{\mathcal{L}}=\{4\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{1,9}}$	χ^{a_1,a_9}	$I_{\mathcal{A}}=\{4\}, I_{\mathcal{L}}=\{5\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{1,12}}$	$\chi^{a_1,a_{12}}_{b_3}$	$I_{\mathcal{A}}=\{2,4\}, I_{\mathcal{L}}=\{7,8\}$	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{1,13}}$	$\chi^{a_1,a_{13}}_{b_4}$	$I_{\mathcal{A}}=\{3,5\}, I_{\mathcal{L}}=\{8,9\}$	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{1,16}}$	$\chi^{a_1,a_8,a_{16}}$	$I_{\mathcal{A}}=\{2,4,5,9\}$	$(q-1)^3$	q^4
		$I_{\mathcal{L}}=\{3,7,12,13\}$		
	$\chi^{a_1,a_{16}}_{b_3,b_4}$	$I_{\mathcal{A}}=\{2,5,9\}, I_{\mathcal{L}}=\{7,12,13\}$	$q^2(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{2,8}}$	χ^{a_2,a_8}	$I_{\mathcal{A}}=\{3\}, I_{\mathcal{L}}=\{4\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{2,9}}$	χ^{a_2,a_9}	$I_{\mathcal{A}}=\{4\}, I_{\mathcal{L}}=\{5\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{2,13}}$	$\chi^{a_2,a_{13}}_{b_4}$	$I_{\mathcal{A}}=\{3,5\}, I_{\mathcal{L}}=\{8,9\}$	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{3,6}}$	χ^{a_3,a_6}	$I_{\mathcal{A}}=\{1\}, I_{\mathcal{L}}=\{2\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{3,9}}$	χ^{a_3,a_9}	$I_{\mathcal{A}}=\{4\}, I_{\mathcal{L}}=\{5\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{3,10}}$	$\chi^{a_3,a_{10}}_{b_1}$	$I_{\mathcal{A}}=\{2\}, I_{\mathcal{L}}=\{6\}$	$q(q-1)^2$	q
$\mathcal{F}_{\mathbf{4,6}}$	χ^{a_4,a_6}	$I_{\mathcal{A}}=\{1\}, I_{\mathcal{L}}=\{2\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{4,7}}$	χ^{a_4,a_7}	$I_{\mathcal{A}}=\{2\}, I_{\mathcal{L}}=\{3\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{4,10}}$	$\chi^{a_4,a_{10}}_{b_1}$	$I_{\mathcal{A}}=\{2\}, I_{\mathcal{L}}=\{6\}$	$q(q-1)^2$	q
$\mathcal{F}_{\mathbf{4,11}}$	$\chi^{a_4,a_{11}}_{b_2}$	$I_{\mathcal{A}}=\{1,3\}, I_{\mathcal{L}}=\{6,7\}$	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{4,14}}$	$\chi^{a_4,a_{14}}_{b_1}$	$I_{\mathcal{A}}=\{2,3,7\}, I_{\mathcal{L}}=\{6,10,11\}$	$q(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{4,17}}$	$\chi^{a_4,a_{10},a_{17}}_{b_1}$	$I_{\mathcal{A}}=\{2,3,7\}, I_{\mathcal{L}}=\{6,11,14\}$	$q(q-1)^3$	q^3
	$\chi^{a_4,a_6,a_{17}}$	$I_{\mathcal{A}}=\{2,3,7\}, I_{\mathcal{L}}=\{1,11,14\}$	$(q-1)^3$	q^3
	$\chi^{a_4,a_{17}}_{b_1,b_2}$	$I_{\mathcal{A}}=\{3,7\}, I_{\mathcal{L}}=\{11,14\}$	$q^2(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{5,6}}$	χ^{a_5,a_6}	$I_{\mathcal{A}}=\{1\}, I_{\mathcal{L}}=\{2\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{5,7}}$	χ^{a_5,a_7}	$I_{\mathcal{A}}=\{2\}, I_{\mathcal{L}}=\{3\}$	$(q-1)^2$	q

$\mathcal{F}$	χ	I	Number	Degree
$\mathcal{F}_{\mathbf{5,8}}$	χ^{a_5,a_8}	$I_{\mathcal{A}}=\{3\}, I_{\mathcal{L}}=\{4\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{5,10}}$	$\chi_{b_1}^{a_5,a_{10}}$	$I_{\mathcal{A}}=\{2\}, I_{\mathcal{L}}=\{6\}$	$q(q-1)^2$	q
$\mathcal{F}_{\mathbf{5,11}}$	$\chi_{b_2}^{a_5,a_{11}}$	$I_{\mathcal{A}}=\{1,3\}, I_{\mathcal{L}}=\{6,7\}$	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{5,12}}$	$\chi_{b_3}^{a_5,a_{12}}$	$I_{\mathcal{A}}=\{2,4\}, I_{\mathcal{L}}=\{7,8\}$	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{5,14}}$	$\chi_{b_1}^{a_5,a_{14}}$	$I_{\mathcal{A}}=\{2,3,7\}, I_{\mathcal{L}}=\{6,10,11\}$	$q(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{5,15}}$	$\chi^{a_5,a_7,a_{15}}$	$I_{\mathcal{A}}=\{1,3,4,8\}$	$(q-1)^3$	q^4
		$I_{\mathcal{L}}=\{2,6,11,12\}$		
	$\chi_{b_2,b_3}^{a_5,a_{15}}$	$I_{\mathcal{A}}=\{1,4,8\}, I_{\mathcal{L}}=\{6,11,12\}$	$q^2(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{5,17}}$	$\chi_{b_1}^{a_5,a_{10},a_{17}}$	$I_{\mathcal{A}}=\{2,3,7\}, I_{\mathcal{L}}=\{6,11,14\}$	$q(q-1)^3$	q^3
	$\chi^{a_5,a_6,a_{17}}$	$I_{\mathcal{A}}=\{2,3,7\}, I_{\mathcal{L}}=\{1,11,14\}$	$(q-1)^3$	q^3
	$\chi_{b_1,b_2}^{a_5,a_{17}}$	$I_{\mathcal{A}}=\{3,7\}, I_{\mathcal{L}}=\{11,14\}$	$q^2(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{5,18}}$	$\chi_{b_3}^{a_5,a_{11},a_{18}}$	$I_{\mathcal{A}}=\{1,2,4,6,8\}$	$q(q-1)^3$	q^5
		$I_{\mathcal{L}}=\{7,10,12,14,15\}$		
	$\chi_{b_1,b_3,b_7}^{a_5,a_{18}}$	$I_{\mathcal{A}}=\{2,4,6,8\}$	$q^3(q-1)^2$	q^4
		$I_{\mathcal{L}}=\{10,12,14,15\}$		
$\mathcal{F}_{\mathbf{5,20}}$	$\chi_{b_1}^{a_5,a_{10},a_{20}}$	$I_{\mathcal{A}}=\{2,3,4,7,8,12\}$	$q(q-1)^3$	q^6
		$I_{\mathcal{L}}=\{6,11,14,15,17,18\}$		
	$\chi^{a_5,a_6,a_{20}}$	$I_{\mathcal{A}}=\{2,3,4,7,8,12\}$	$(q-1)^3$	q^6
		$I_{\mathcal{L}}=\{1,11,14,15,17,18\}$		
	$\chi_{b_1,b_2}^{a_5,a_{20}}$	$I_{\mathcal{A}}=\{3,4,7,8,12\}$	$q^2(q-1)^2$	q^5
		$I_{\mathcal{L}}=\{11,14,15,17,18\}$		
$\mathcal{F}_{\mathbf{5,22}}$	$\chi_{b_1}^{a_5,a_{10},a_{17},a_{22}}$	$I_{\mathcal{A}}=\{2,3,4,7,8,12\}$	$q(q-1)^4$	q^6
		$I_{\mathcal{L}}=\{6,11,14,15,18,20\}$		
	$\chi^{a_5,a_6,a_{17},a_{22}}$	$I_{\mathcal{A}}=\{2,3,4,7,8,12\}$	$(q-1)^4$	q^6
		$I_{\mathcal{L}}=\{1,11,14,15,18,20\}$		
	$\chi_{b_1,b_2}^{a_5,a_{17},a_{22}}$	$I_{\mathcal{A}}=\{3,4,7,8,12\}$	$q^2(q-1)^3$	q^5
		$I_{\mathcal{L}}=\{11,14,15,18,20\}$		
	$\chi_{b_1}^{a_5,a_{14},a_{22}}$	$I_{\mathcal{A}}=\{2,3,4,7,8,12\}$	$q(q-1)^3$	q^6
		$I_{\mathcal{L}}=\{6,10,11,15,18,20\}$		
	$\chi_{b_2,b_{10}}^{a_5,a_{11},a_{22}}$	$I_{\mathcal{A}}=\{1,4,6,8,12\}$	$q^2(q-1)^3$	q^5
		$I_{\mathcal{L}}=\{3,7,15,18,20\}$		
	$\chi_{b_1,b_3,b_7}^{a_5,a_{10},a_{22}}$	$I_{\mathcal{A}}=\{2,4,8,12\}$	$q^3(q-1)^3$	q^4
		$I_{\mathcal{L}}=\{6,15,18,20\}$		
	$\chi_{b_3,b_7}^{a_5,a_6,a_{22}}$	$I_{\mathcal{A}}=\{2,4,8,12\}$	$q^2(q-1)^3$	q^4
		$I_{\mathcal{L}}=\{1,15,18,20\}$		
	$\chi_{b_1}^{a_5,a_7,a_{22}}$	$I_{\mathcal{A}}=\{3,4,8,12\}$	$q(q-1)^3$	q^4
		$I_{\mathcal{L}}=\{2,15,18,20\}$		
	$\chi_{b_1,b_2,b_3}^{a_5,a_{22}}$	$I_{\mathcal{A}}=\{4,8,12\}, I_{\mathcal{L}}=\{15,18,20\}$	$q^3(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{6,7}}$	$\chi_{b_3}^{a_6,a_7}$	$I_{\mathcal{A}}=\{1\}, I_{\mathcal{L}}=\{2\}$	$q(q-1)^2$	q
$\mathcal{F}_{\mathbf{6,8}}$	χ^{a_6,a_8}	$I_{\mathcal{A}}=\{1,3\}, I_{\mathcal{L}}=\{2,4\}$	$(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{6,9}}$	χ^{a_6,a_9}	$I_{\mathcal{A}}=\{1,4\}, I_{\mathcal{L}}=\{2,5\}$	$(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{6,12}}$	$\chi_{b_1,b_3}^{a_6,a_{12}}$	$I_{\mathcal{A}}=\{2,4\}, I_{\mathcal{L}}=\{7,8\}$	$q^2(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{6,13}}$	$\chi_{b_4}^{a_6,a_{13}}$	$I_{\mathcal{A}}=\{2,3,5\}, I_{\mathcal{L}}=\{1,8,9\}$	$q(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{6,16}}$	$\chi_{b_1}^{a_6,a_8,a_{16}}$	$I_{\mathcal{A}}=\{2,4,5,9\}$	$q(q-1)^3$	q^4
		$I_{\mathcal{L}}=\{3,7,12,13\}$		
	$\chi_{b_1,b_3,b_4}^{a_6,a_{16}}$	$I_{\mathcal{A}}=\{2,5,9\}, I_{\mathcal{L}}=\{7,12,13\}$	$q^3(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{7,8}}$	$\chi_{b_4}^{a_7,a_8}$	$I_{\mathcal{A}}=\{2\}, I_{\mathcal{L}}=\{3\}$	$q(q-1)^2$	q
$\mathcal{F}_{\mathbf{7,9}}$	χ^{a_7,a_9}	$I_{\mathcal{A}}=\{2,4\}, I_{\mathcal{L}}=\{3,5\}$	$(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{7,10}}$	$\chi_{b_1,b_3}^{a_7,a_{10}}$	$I_{\mathcal{A}}=\{2\}, I_{\mathcal{L}}=\{6\}$	$q^2(q-1)^2$	q
$\mathcal{F}_{\mathbf{7,13}}$	$\chi_{b_2,b_4}^{a_7,a_{13}}$	$I_{\mathcal{A}}=\{3,5\}, I_{\mathcal{L}}=\{8,9\}$	$q^2(q-1)^2$	q^2

$\mathcal{F}$	χ	I	Number	Degree
$\mathcal{F}_{\mathbf{8,9}}$	$\chi_{b_5}^{a_8,a_9}$	$I_{\mathcal{A}}=\{3\}, I_{\mathcal{L}}=\{4\}$	$q(q-1)^2$	q
$\mathcal{F}_{\mathbf{8,10}}$	$\chi_{b_1}^{a_8,a_{10}}$	$I_{\mathcal{A}}=\{2,4\}, I_{\mathcal{L}}=\{3,6\}$	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{8,11}}$	$\chi_{b_2,b_4}^{a_8,a_{11}}$	$I_{\mathcal{A}}=\{1,3\}, I_{\mathcal{L}}=\{6,7\}$	$q^2(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{8,14}}$	$\chi_{b_1,b_4}^{a_8,a_{14}}$	$I_{\mathcal{A}}=\{2,3,7\}, I_{\mathcal{L}}=\{6,10,11\}$	$q^2(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{8,17}}$	$\chi_{b_1,b_4}^{a_8,a_{10},a_{17}}$ $\chi_{b_4}^{a_6,a_8,a_{17}}$ $\chi_{b_1,b_2,b_4}^{a_8,a_{17}}$	$I_{\mathcal{A}}=\{2,3,7\}, I_{\mathcal{L}}=\{6,11,14\}$ $I_{\mathcal{A}}=\{2,3,7\}, I_{\mathcal{L}}=\{1,11,14\}$ $I_{\mathcal{A}}=\{3,7\}, I_{\mathcal{L}}=\{11,14\}$	$q^2(q-1)^3$ $q(q-1)^3$ $q^3(q-1)^2$	q^3 q^3 q^2
$\mathcal{F}_{\mathbf{9,10}}$	$\chi_{b_1}^{a_9,a_{10}}$	$I_{\mathcal{A}}=\{2,5\}, I_{\mathcal{L}}=\{4,6\}$	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{9,11}}$	$\chi_{b_2}^{a_9,a_{11}}$	$I_{\mathcal{A}}=\{1,3,5\}, I_{\mathcal{L}}=\{4,6,7\}$	$q(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{9,12}}$	$\chi_{b_3,b_5}^{a_9,a_{12}}$	$I_{\mathcal{A}}=\{2,4\}, I_{\mathcal{L}}=\{7,8\}$	$q^2(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{9,14}}$	$\chi_{b_1}^{a_9,a_{14}}$	$I_{\mathcal{A}}=\{2,3,5,7\}$ $I_{\mathcal{L}}=\{4,6,10,11\}$	$q(q-1)^2$	q^4
$\mathcal{F}_{\mathbf{9,15}}$	$\chi_{b_5}^{a_7,a_9,a_{15}}$ $\chi_{b_2,b_3,b_5}^{a_9,a_{15}}$	$I_{\mathcal{A}}=\{1,3,4,8\}$ $I_{\mathcal{L}}=\{2,6,11,12\}$ $I_{\mathcal{A}}=\{1,4,8\}, I_{\mathcal{L}}=\{6,11,12\}$	$q(q-1)^3$ $q^3(q-1)^2$	q^4 q^3
$\mathcal{F}_{\mathbf{9,17}}$	$\chi_{b_1}^{a_9,a_{10},a_{17}}$ $\chi^{a_6,a_9,a_{17}}$ $\chi_{b_1,b_2}^{a_9,a_{17}}$	$I_{\mathcal{A}}=\{2,3,5,7\}$ $I_{\mathcal{L}}=\{4,6,11,14\}$ $I_{\mathcal{A}}=\{2,3,5,7\}$ $I_{\mathcal{L}}=\{1,4,11,14\}$ $I_{\mathcal{A}}=\{3,5,7\}, I_{\mathcal{L}}=\{4,11,14\}$	$q(q-1)^3$ $(q-1)^3$ $q^2(q-1)^2$	q^4 q^4 q^3
$\mathcal{F}_{\mathbf{9,18}}$	$\chi_{b_3,b_5}^{a_9,a_{11},a_{18}}$ $\chi_{b_1,b_3,b_5,b_7}^{a_9,a_{18}}$	$I_{\mathcal{A}}=\{1,2,4,6,8\}$ $I_{\mathcal{L}}=\{7,10,12,14,15\}$ $I_{\mathcal{A}}=\{2,4,6,8\}$ $I_{\mathcal{L}}=\{10,12,14,15\}$	$q^2(q-1)^3$ $q^4(q-1)^2$	q^5 q^4
$\mathcal{F}_{\mathbf{9,20}}$	$\chi_{b_1,b_5}^{a_9,a_{10},a_{20}}$ $\chi_{b_5}^{a_6,a_9,a_{20}}$ $\chi_{b_1,b_2,b_5}^{a_9,a_{20}}$	$I_{\mathcal{A}}=\{2,3,4,7,8,12\}$ $I_{\mathcal{L}}=\{6,11,14,15,17,18\}$ $I_{\mathcal{A}}=\{2,3,4,7,8,12\}$ $I_{\mathcal{L}}=\{1,11,14,15,17,18\}$ $I_{\mathcal{A}}=\{3,4,7,8,12\}$ $I_{\mathcal{L}}=\{11,14,15,17,18\}$	$q^2(q-1)^3$ $q(q-1)^3$ $q^3(q-1)^2$	q^6 q^6 q^5
$\mathcal{F}_{\mathbf{9,22}}$	$\chi_{b_1,b_5}^{a_9,a_{10},a_{17},a_{22}}$ $\chi_{b_5}^{a_6,a_9,a_{17},a_{22}}$ $\chi_{b_1,b_2,b_5}^{a_9,a_{17},a_{22}}$ $\chi_{b_1,b_5}^{a_9,a_{14},a_{22}}$ $\chi_{b_2,b_5,b_{10}}^{a_9,a_{11},a_{22}}$ $\chi_{b_1,b_3,b_5,b_7}^{a_9,a_{10},a_{22}}$ $\chi_{b_3,b_5,b_7}^{a_6,a_9,a_{22}}$ $\chi_{b_1,b_5}^{a_7,a_9,a_{22}}$ $\chi_{b_1,b_2,b_3,b_5}^{a_9,a_{22}}$	$I_{\mathcal{A}}=\{2,3,4,7,8,12\}$ $I_{\mathcal{L}}=\{6,11,14,15,18,20\}$ $I_{\mathcal{A}}=\{2,3,4,7,8,12\}$ $I_{\mathcal{L}}=\{1,11,14,15,18,20\}$ $I_{\mathcal{A}}=\{3,4,7,8,12\}$ $I_{\mathcal{L}}=\{11,14,15,18,20\}$ $I_{\mathcal{A}}=\{2,3,4,7,8,12\}$ $I_{\mathcal{L}}=\{6,10,11,15,18,20\}$ $I_{\mathcal{A}}=\{1,4,6,8,12\}$ $I_{\mathcal{L}}=\{3,7,15,18,20\}$ $I_{\mathcal{A}}=\{2,4,8,12\}$ $I_{\mathcal{L}}=\{6,15,18,20\}$ $I_{\mathcal{A}}=\{2,4,8,12\}$ $I_{\mathcal{L}}=\{1,15,18,20\}$ $I_{\mathcal{A}}=\{3,4,8,12\}$ $I_{\mathcal{L}}=\{2,15,18,20\}$ $I_{\mathcal{A}}=\{4,8,12\}, I_{\mathcal{L}}=\{15,18,20\}$	$q^2(q-1)^4$ $q(q-1)^4$ $q^3(q-1)^3$ $q^2(q-1)^3$ $q^3(q-1)^3$ $q^4(q-1)^3$ $q^3(q-1)^3$ $q^2(q-1)^3$ $q^4(q-1)^2$	q^6 q^6 q^5 q^6 q^5 q^4 q^4 q^4 q^3
$\mathcal{F}_{\mathbf{10,11}}$	$\chi_{b_3}^{a_{10},a_{11}}$	$I_{\mathcal{A}}=\{1,6\}, I_{\mathcal{L}}=\{2,7\}$	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{10,12}}$	$\chi_{b_1,b_3,b_8}^{a_{10},a_{12}}$	$I_{\mathcal{A}}=\{2,4\}, I_{\mathcal{L}}=\{6,7\}$	$q^3(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{10,13}}$	$\chi_{b_1,b_4}^{a_{10},a_{13}}$	$I_{\mathcal{A}}=\{2,3,5\}, I_{\mathcal{L}}=\{6,8,9\}$	$q^2(q-1)^2$	q^3

$\mathcal{F}$	χ	I	Number	Degree
$\mathcal{F}_{\mathbf{10,15}}$	$\chi^{a_7,a_{10},a_{15}}$	$I_{\mathcal{A}} = \{1,3,4,6\}$	$(q-1)^3$	q^4
		$I_{\mathcal{L}} = \{2,8,11,12\}$		
	$\chi^{a_{10},a_{15}}_{b_2,b_3}$	$I_{\mathcal{A}} = \{1,4,6\}, I_{\mathcal{L}} = \{8,11,12\}$	$q^2(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{10,16}}$	$\chi^{a_8,a_{10},a_{16}}_{b_1,b_6}$	$I_{\mathcal{A}} = \{2,4,5,9\}$	$q^2(q-1)^3$	q^4
		$I_{\mathcal{L}} = \{3,7,12,13\}$		
	$\chi^{a_{10},a_{16}}_{b_1,b_3,b_4,b_6}$	$I_{\mathcal{A}} = \{2,5,9\}, I_{\mathcal{L}} = \{7,12,13\}$	$q^4(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{10,19}}$	$\chi^{a_{10},a_{12},a_{19}}_{b_3}$	$I_{\mathcal{A}} = \{1,2,4,5,6,9\}$	$q(q-1)^3$	q^6
		$I_{\mathcal{L}} = \{7,8,11,13,15,16\}$		
	$\chi^{a_8,a_{10},a_{19}}_{b_2,b_7}$	$I_{\mathcal{A}} = \{1,3,5,6,9\}$	$q^2(q-1)^3$	q^5
		$I_{\mathcal{L}} = \{4,11,13,15,16\}$		
	$\chi^{a_7,a_{10},a_{19}}_{b_4}$	$I_{\mathcal{A}} = \{1,3,5,6,9\}$	$q(q-1)^3$	q^5
		$I_{\mathcal{L}} = \{2,11,13,15,16\}$		
	$\chi^{a_{10},a_{19}}_{b_2,b_3,b_4}$	$I_{\mathcal{A}} = \{1,5,6,9\}$	$q^3(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{11,13,15,16\}$		
$\mathcal{F}_{\mathbf{11,12}}$	$\chi^{a_{11},a_{12}}_{b_4}$	$I_{\mathcal{A}} = \{2,3,7\}, I_{\mathcal{L}} = \{1,6,8\}$	$q(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{11,13}}$	$\chi^{a_{11},a_{13}}_{b_2,b_4,b_9}$	$I_{\mathcal{A}} = \{1,3,5\}, I_{\mathcal{L}} = \{6,7,8\}$	$q^3(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{11,16}}$	$\chi^{a_{11},a_{16}}_{b_1,b_4,b_8}$	$I_{\mathcal{A}} = \{2,3,5,7\}$	$q^3(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{6,9,12,13\}$		
$\mathcal{F}_{\mathbf{12,13}}$	$\chi^{a_{12},a_{13}}_{b_5}$	$I_{\mathcal{A}} = \{3,4,8\}, I_{\mathcal{L}} = \{2,7,9\}$	$q(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{12,14}}$	$\chi^{a_{12},a_{14}}_{b_1,b_4,b_8}$	$I_{\mathcal{A}} = \{2,3,7\}, I_{\mathcal{L}} = \{6,10,11\}$	$q^3(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{12,17}}$	$\chi^{a_{12},a_{17}}_{b_1,b_4,b_6,b_{10}}$	$I_{\mathcal{A}} = \{2,3,7\}, I_{\mathcal{L}} = \{8,11,14\}$	$q^4(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{13,14}}$	$\chi^{a_{13},a_{14}}_{b_1,b_4,b_9}$	$I_{\mathcal{A}} = \{2,3,5,7\}$	$q^3(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{6,8,10,11\}$		
$\mathcal{F}_{\mathbf{13,15}}$	$\chi^{a_{13},a_{15}}_{b_2,b_5,b_7}$	$I_{\mathcal{A}} = \{1,3,4,8\}$	$q^3(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{6,9,11,12\}$		
$\mathcal{F}_{\mathbf{13,17}}$	$\chi^{a_{10},a_{13},a_{17}}_{b_1,b_4,b_9}$	$I_{\mathcal{A}} = \{2,3,5,7\}$	$q^3(q-1)^3$	q^4
		$I_{\mathcal{L}} = \{6,8,11,14\}$		
	$\chi^{a_6,a_{13},a_{17}}_{b_4,b_9}$	$I_{\mathcal{A}} = \{2,3,5,7\}$	$q^2(q-1)^3$	q^4
		$I_{\mathcal{L}} = \{1,8,11,14\}$		
	$\chi^{a_{13},a_{17}}_{b_1,b_2,b_4,b_9}$	$I_{\mathcal{A}} = \{3,5,7\}, I_{\mathcal{L}} = \{8,11,14\}$	$q^4(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{13,18}}$	$\chi^{a_{11},a_{13},a_{18}}_{b_5}$	$I_{\mathcal{A}} = \{1,2,3,4,6,8\}$	$q(q-1)^3$	q^6
		$I_{\mathcal{L}} = \{7,9,10,12,14,15\}$		
	$\chi^{a_{13},a_{18}}_{b_1,b_5,b_7}$	$I_{\mathcal{A}} = \{2,3,4,6,8\}$	$q^3(q-1)^2$	q^5
		$I_{\mathcal{L}} = \{9,10,12,14,15\}$		
$\mathcal{F}_{\mathbf{13,20}}$	$\chi^{a_{10},a_{13},a_{20}}_{b_1,b_5,b_9}$	$I_{\mathcal{A}} = \{2,3,4,7,8,12\}$	$q^3(q-1)^3$	q^6
		$I_{\mathcal{L}} = \{6,11,14,15,17,18\}$		
	$\chi^{a_6,a_{13},a_{20}}_{b_5,b_9}$	$I_{\mathcal{A}} = \{2,3,4,7,8,12\}$	$q^2(q-1)^3$	q^6
		$I_{\mathcal{L}} = \{1,11,14,15,17,18\}$		
	$\chi^{a_{13},a_{20}}_{b_1,b_2,b_5,b_9}$	$I_{\mathcal{A}} = \{3,4,7,8,12\}$	$q^4(q-1)^2$	q^5
		$I_{\mathcal{L}} = \{11,14,15,17,18\}$		
$\mathcal{F}_{\mathbf{13,22}}$	$\chi^{a_{13},a_{14},a_{17},a_{22}}_{b_1,b_5}$	$I_{\mathcal{A}} = \{2,3,4,7,8,12\}$	$q^2(q-1)^4$	q^6
		$I_{\mathcal{L}} = \{6,9,11,15,18,20\}$		
	$\chi^{a_{10},a_{13},a_{17},a_{22}}_{b_1,b_5}$	$I_{\mathcal{A}} = \{2,3,4,7,8,12\}$	$q^2(q-1)^4$	q^6
		$I_{\mathcal{L}} = \{6,9,11,15,18,20\}$		
	$\chi^{a_6,a_{13},a_{17},a_{22}}_{b_5}$	$I_{\mathcal{A}} = \{2,3,4,7,8,12\}$	$q(q-1)^4$	q^6
		$I_{\mathcal{L}} = \{1,9,11,15,18,20\}$		
	$\chi^{a_{13},a_{17},a_{22}}_{b_1,b_2,b_5}$	$I_{\mathcal{A}} = \{3,4,7,8,12\}$	$q^3(q-1)^3$	q^5
		$I_{\mathcal{L}} = \{9,11,15,18,20\}$		
	$\chi^{a_{13},a_{14},a_{22}}_{b_1,b_5,b_{10}}$	$I_{\mathcal{A}} = \{2,3,4,7,8,12\}$	$q^3(q-1)^3$	q^6
		$I_{\mathcal{L}} = \{6,9,11,15,18,20\}$		

$\mathcal{F}$	χ	I	Number	Degree
	$\chi_{b_5}^{a_{10},a_{11},a_{13},a_{22}}$	$I_{\mathcal{A}} = \{1, 2, 3, 4, 8, 12\}$ $I_{\mathcal{L}} = \{6, 7, 9, 15, 18, 20\}$	$q(q-1)^4$	q^6
	$\chi_{b_2,b_5,b_6}^{a_{11},a_{13},a_{22}}$	$I_{\mathcal{A}} = \{1, 3, 4, 8, 12\}$ $I_{\mathcal{L}} = \{7, 9, 15, 18, 20\}$	$q^3(q-1)^3$	q^5
	$\chi_{b_1,b_5,b_7}^{a_{10},a_{13},a_{22}}$	$I_{\mathcal{A}} = \{2, 3, 4, 8, 12\}$ $I_{\mathcal{L}} = \{6, 9, 15, 18, 20\}$	$q^3(q-1)^3$	q^5
	$\chi_{b_5,b_7}^{a_6,a_{13},a_{22}}$	$I_{\mathcal{A}} = \{2, 3, 4, 8, 12\}$ $I_{\mathcal{L}} = \{1, 9, 15, 18, 20\}$	$q^2(q-1)^3$	q^5
	$\chi_{b_1,b_2,b_5,b_7}^{a_{13},a_{22}}$	$I_{\mathcal{A}} = \{3, 4, 8, 12\}$ $I_{\mathcal{L}} = \{9, 15, 18, 20\}$	$q^4(q-1)^2$	q^4
	$\boldsymbol{\chi_{b_1,b_5}^{a_{10},a_{13},a_{14},a_{17}^*,a_{22}}}$ $(a_{17}^* \neq a_{14}^2/(4a_{10}))$	$\boldsymbol{I_{\mathcal{A}} = \{3, 4, 8, 12\},\ I_{\mathcal{I}} = \{2, 7\}}$ $\boldsymbol{I_{\mathcal{L}} = \{9, 15, 18, 20\},\ I_{\mathcal{J}} = \{6, 11\}}$	$\boldsymbol{q^2(q-1)^4(q-2)}$	$\boldsymbol{q^6}$
	$\boldsymbol{\chi_{b_5}^{a_{10},a_{13},a_{14},a_{22},a_{6,11}}}$	**See $\mathfrak{C}^{C5}$ in Section 5.3**	$\boldsymbol{q(q-1)^5}$	$\boldsymbol{q^6}$
	$\boldsymbol{\chi_{b_1,b_5,b_{2,7}}^{a_{10},a_{13},a_{14},a_{22}}}$	**See $\mathfrak{C}^{C5}$ in Section 5.3**	$\boldsymbol{q^3(q-1)^4}$	$\boldsymbol{q^5}$
$\mathcal{F}_{\mathbf{14,15}}$	$\chi_{b_4,b_8}^{a_{14},a_{15}}$	$I_{\mathcal{A}} = \{1, 3, 6, 11\}$ $I_{\mathcal{L}} = \{2, 7, 10, 12\}$	$q^2(q-1)^2$	q^4
$\mathcal{F}_{\mathbf{14,16}}$	$\chi_{b_1,b_4,b_8,b_9,b_{13}}^{a_{14},a_{16}}$	$I_{\mathcal{A}} = \{2, 3, 5, 7\}$ $I_{\mathcal{L}} = \{6, 10, 11, 12\}$	$q^5(q-1)^2$	q^4
$\mathcal{F}_{\mathbf{14,19}}$	$\chi^{a_{12},a_{14},a_{19}}$	$I_{\mathcal{A}} = \{1, 2, 3, 4, 5, 6, 11\}$ $I_{\mathcal{L}} = \{7, 8, 9, 10, 13, 15, 16\}$	$(q-1)^3$	q^7
	$\chi_{b_2,b_4,b_7,b_8}^{a_{14},a_{19}}$	$I_{\mathcal{A}} = \{1, 3, 5, 6, 11\}$ $I_{\mathcal{L}} = \{9, 10, 13, 15, 16\}$	$q^4(q-1)^2$	q^5
$\mathcal{F}_{\mathbf{15,16}}$	$\chi_{b_3,b_5}^{a_{15},a_{16}}$	$I_{\mathcal{A}} = \{2, 4, 8, 9, 12\}$ $I_{\mathcal{L}} = \{1, 6, 7, 11, 13\}$	$q^2(q-1)^2$	q^5
$\mathcal{F}_{\mathbf{15,17}}$	$\chi_{b_2,b_4,b_{10}}^{a_{15},a_{17}}$	$I_{\mathcal{A}} = \{1, 3, 6, 11\}$ $I_{\mathcal{L}} = \{7, 8, 12, 14\}$	$q^3(q-1)^2$	q^4
$\mathcal{F}_{\mathbf{16,17}}$	$\chi_{b_1,b_4,b_6,b_8,b_9,b_{10}}^{a_{16},a_{17}}$	$I_{\mathcal{A}} = \{2, 3, 5, 7\}$ $I_{\mathcal{L}} = \{11, 12, 13, 14\}$	$q^6(q-1)^2$	q^4
$\mathcal{F}_{\mathbf{16,18}}$	$\chi_{b_1}^{a_{11},a_{13},a_{16},a_{18}}$	$I_{\mathcal{A}} = \{2, 3, 4, 7, 8, 12\}$ $I_{\mathcal{L}} = \{5, 6, 9, 10, 14, 15\}$	$q(q-1)^4$	q^6
	$\chi_{b_1,b_3,b_6}^{a_{13},a_{16},a_{18}}$	$I_{\mathcal{A}} = \{2, 4, 8, 9, 12\}$ $I_{\mathcal{L}} = \{5, 7, 10, 14, 15\}$	$q^3(q-1)^3$	q^5
	$\chi_{b_1}^{a_{11},a_{16},a_{18}}$	$I_{\mathcal{A}} = \{2, 3, 4, 7, 8, 12\}$ $I_{\mathcal{L}} = \{5, 6, 9, 10, 14, 15\}$	$q(q-1)^3$	q^6
	$\chi_{b_1,b_3,b_6}^{a_{16},a_{18}}$	$I_{\mathcal{A}} = \{2, 4, 7, 8, 12\}$ $I_{\mathcal{L}} = \{5, 9, 10, 14, 15\}$	$q^3(q-1)^2$	q^5
$\mathcal{F}_{\mathbf{16,20}}$	$\chi_{b_1,b_5,b_6,b_9,b_{10}}^{a_{16},a_{20}}$	$I_{\mathcal{A}} = \{2, 3, 4, 7, 8, 12\}$ $I_{\mathcal{L}} = \{11, 13, 14, 15, 17, 18\}$	$q^5(q-1)^2$	q^6
$\mathcal{F}_{\mathbf{16,22}}$	$\chi_{b_1,b_5,b_6,b_{10},b_{11}}^{a_{16},a_{17},a_{22}}$	$I_{\mathcal{A}} = \{2, 3, 4, 7, 8, 12\}$ $I_{\mathcal{L}} = \{9, 13, 14, 15, 18, 20\}$	$q^5(q-1)^3$	q^6
	$\chi_{b_1,b_5,b_6,b_{11}}^{a_{14},a_{16},a_{22}}$	$I_{\mathcal{A}} = \{2, 3, 4, 7, 8, 12\}$ $I_{\mathcal{L}} = \{9, 10, 13, 15, 18, 20\}$	$q^4(q-1)^3$	q^6
	$\chi_{b_1,b_5,b_{10}}^{a_{11},a_{16},a_{22}}$	$I_{\mathcal{A}} = \{2, 3, 4, 7, 8, 12\}$ $I_{\mathcal{L}} = \{6, 9, 13, 15, 18, 20\}$	$q^3(q-1)^3$	q^6
	$\chi_{b_1,b_3,b_5,b_6,b_{10}}^{a_{16},a_{22}}$	$I_{\mathcal{A}} = \{2, 4, 7, 8, 12\}$ $I_{\mathcal{L}} = \{9, 13, 15, 18, 20\}$	$q^5(q-1)^2$	q^5
$\mathcal{F}_{\mathbf{17,18}}$	$\chi_{b_1,b_4}^{a_{17},a_{18}}$	$I_{\mathcal{A}} = \{2, 6, 7, 10, 14\}$ $I_{\mathcal{L}} = \{3, 8, 11, 12, 15\}$	$q^2(q-1)^2$	q^5
$\mathcal{F}_{\mathbf{17,19}}$	$\chi_{b_{10}}^{a_{12},a_{17},a_{19}}$	$I_{\mathcal{A}} = \{1, 2, 3, 4, 5, 6, 11\}$ $I_{\mathcal{L}} = \{7, 8, 9, 13, 14, 15, 16\}$	$q(q-1)^3$	q^7

$\mathcal{F}$	χ	I	Number	Degree
	$\chi^{a_{17},a_{19}}_{b_2,b_4,b_7,b_8,b_{10}}$	$I_{\mathcal{A}} = \{1,3,5,6,11\}$ $I_{\mathcal{L}} = \{9,13,14,15,16\}$	$q^5(q-1)^2$	q^5
$\mathcal{F}_{\mathbf{17,21}}$	$\chi^{a_8,a_{15},a_{17},a_{21}}$	$I_{\mathcal{A}} = \{1,2,4,5,6,10,11,14\}$ $I_{\mathcal{L}} = \{3,7,9,12,13,16,18,19\}$	$(q-1)^4$	q^8
	$\chi^{a_{15},a_{17},a_{21}}_{b_3,b_4}$	$I_{\mathcal{A}} = \{1,2,5,6,10,11,14\}$ $I_{\mathcal{L}} = \{7,9,12,13,16,18,19\}$	$q^2(q-1)^3$	q^7
	$\chi^{a_8,a_{17},a_{21}}_{b_1,b_{12}}$	$I_{\mathcal{A}} = \{2,4,5,6,7,10,14\}$ $I_{\mathcal{L}} = \{3,9,11,13,16,18,19\}$	$q^2(q-1)^3$	q^7
	$\chi^{a_{17},a_{21}}_{b_1,b_3,b_4,b_{12}}$	$I_{\mathcal{A}} = \{2,5,6,7,10,14\}$ $I_{\mathcal{L}} = \{9,11,13,16,18,19\}$	$q^4(q-1)^2$	q^6
$\mathcal{F}_{\mathbf{18,19}}$	$\chi^{a_7,a_{18},a_{19}}_{b_{13}}$	$I_{\mathcal{A}} = \{1,3,4,6,8,9,15\}$ $I_{\mathcal{L}} = \{2,5,10,11,12,14,16\}$	$q(q-1)^3$	q^7
	$\chi^{a_{18},a_{19}}_{b_2,b_3,b_{13}}$	$I_{\mathcal{A}} = \{1,4,6,8,9,15\}$ $I_{\mathcal{L}} = \{5,10,11,12,14,16\}$	$q^3(q-1)^2$	q^6
$\mathcal{F}_{\mathbf{19,20}}$	$\chi^{a_{19},a_{20}}_{b_2,b_5,b_9,b_{10}}$	$I_{\mathcal{A}} = \{1,3,4,6,8,11,15\}$ $I_{\mathcal{L}} = \{7,12,13,14,16,17,18\}$	$q^4(q-1)^2$	q^7
$\mathcal{F}_{\mathbf{19,22}}$	$\chi^{a_{17},a_{19},a_{22}}_{b_2,b_5,b_7,b_{10}}$	$I_{\mathcal{A}} = \{1,3,4,6,8,11,15\}$ $I_{\mathcal{L}} = \{9,12,13,14,16,18,20\}$	$q^4(q-1)^3$	q^7
	$\chi^{a_{14},a_{19},a_{22}}_{b_2,b_5,b_7}$	$I_{\mathcal{A}} = \{1,3,4,6,8,11,15\}$ $I_{\mathcal{L}} = \{9,10,12,13,16,18,20\}$	$q^3(q-1)^3$	q^7
	$\chi^{a_7,a_{19},a_{22}}_{b_5,b_{10}}$	$I_{\mathcal{A}} = \{1,3,4,6,8,11,15\}$ $I_{\mathcal{L}} = \{2,9,12,13,16,18,20\}$	$q^2(q-1)^3$	q^7
	$\chi^{a_{19},a_{22}}_{b_2,b_3,b_5,b_{10}}$	$I_{\mathcal{A}} = \{1,4,6,8,11,15\}$ $I_{\mathcal{L}} = \{9,12,13,16,18,20\}$	$q^4(q-1)^2$	q^6
$\mathcal{F}_{\mathbf{20,21}}$	$\chi^{a_{20},a_{21}}_{b_1,b_5,b_9}$	$I_{\mathcal{A}} = \{2,4,6,7,10,12,14,18\}$ $I_{\mathcal{L}} = \{3,8,11,13,15,16,17,19\}$	$q^3(q-1)^2$	q^8
$\mathcal{F}_{\mathbf{21,22}}$	$\chi^{a_{17},a_{21},a_{22}}_{b_1,b_3,b_5}$	$I_{\mathcal{A}} = \{2,4,6,7,10,12,14,18\}$ $I_{\mathcal{L}} = \{8,9,11,13,15,16,19,20\}$	$q^3(q-1)^3$	q^8
	$\chi^{a_{11},a_{21},a_{22}}_{b_3,b_5}$	$I_{\mathcal{A}} = \{1,2,4,6,10,12,14,18\}$ $I_{\mathcal{L}} = \{7,8,9,13,15,16,19,20\}$	$q^2(q-1)^3$	q^8
	$\chi^{a_{21},a_{22}}_{b_1,b_3,b_5,b_7}$	$I_{\mathcal{A}} = \{2,4,6,10,12,14,18\}$ $I_{\mathcal{L}} = \{8,9,13,15,16,19,20\}$	$q^4(q-1)^2$	q^7
$\mathcal{F}_{\mathbf{22,23}}$	$\chi^{a_{10},a_{22},a_{23}}_{b_1,b_5}$	$I_{\mathcal{A}} = \{2,3,7,8,11,12,13,17,20\}$ $I_{\mathcal{L}} = \{4,6,9,14,15,16,18,19,21\}$	$q^2(q-1)^3$	q^9
	$\chi^{a_6,a_{22},a_{23}}_{b_5}$	$I_{\mathcal{A}} = \{2,3,7,8,11,12,13,17,20\}$ $I_{\mathcal{L}} = \{1,4,9,14,15,16,18,19,21\}$	$q(q-1)^3$	q^9
	$\chi^{a_{22},a_{23}}_{b_1,b_2,b_5}$	$I_{\mathcal{A}} = \{3,7,8,11,12,13,17,20\}$ $I_{\mathcal{L}} = \{4,9,14,15,16,18,19,21\}$	$q^3(q-1)^2$	q^8
$\mathcal{F}_{\mathbf{1,2,8}}$	χ^{a_1,a_2,a_8}	$I_{\mathcal{A}} = \{3\}, I_{\mathcal{L}} = \{4\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{1,2,9}}$	χ^{a_1,a_2,a_9}	$I_{\mathcal{A}} = \{4\}, I_{\mathcal{L}} = \{5\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{1,2,13}}$	$\chi^{a_1,a_2,a_{13}}_{b_4}$	$I_{\mathcal{A}} = \{3,5\}, I_{\mathcal{L}} = \{8,9\}$	$q(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{1,3,9}}$	χ^{a_1,a_3,a_9}	$I_{\mathcal{A}} = \{4\}, I_{\mathcal{L}} = \{5\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{1,4,7}}$	χ^{a_1,a_4,a_7}	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{3\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{1,5,7}}$	χ^{a_1,a_5,a_7}	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{3\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{1,5,8}}$	χ^{a_1,a_5,a_8}	$I_{\mathcal{A}} = \{3\}, I_{\mathcal{L}} = \{4\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{1,5,12}}$	$\chi^{a_1,a_5,a_{12}}_{b_3}$	$I_{\mathcal{A}} = \{2,4\}, I_{\mathcal{L}} = \{7,8\}$	$q(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{1,7,8}}$	$\chi^{a_1,a_7,a_8}_{b_4}$	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{3\}$	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{1,7,9}}$	χ^{a_1,a_7,a_9}	$I_{\mathcal{A}} = \{2,4\}, I_{\mathcal{L}} = \{3,5\}$	$(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{1,7,13}}$	$\chi^{a_1,a_7,a_{13}}_{b_2,b_4}$	$I_{\mathcal{A}} = \{3,5\}, I_{\mathcal{L}} = \{8,9\}$	$q^2(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{1,8,9}}$	$\chi^{a_1,a_8,a_9}_{b_5}$	$I_{\mathcal{A}} = \{3\}, I_{\mathcal{L}} = \{4\}$	$q(q-1)^3$	q

$\mathcal{F}$	χ	I	Number	Degree
$\mathcal{F}_{\mathbf{1,9,12}}$	$\chi_{b_3,b_5}^{a_1,a_9,a_{12}}$	$I_{\mathcal{A}} = \{2,4\}, I_{\mathcal{L}} = \{7,8\}$	$q^2(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{1,12,13}}$	$\chi_{b_5}^{a_1,a_{12},a_{13}}$	$I_{\mathcal{A}} = \{3,4,8\}, I_{\mathcal{L}} = \{2,7,9\}$	$q(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{2,3,9}}$	χ^{a_2,a_3,a_9}	$I_{\mathcal{A}} = \{4\}, I_{\mathcal{L}} = \{5\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{2,5,8}}$	χ^{a_2,a_5,a_8}	$I_{\mathcal{A}} = \{3\}, I_{\mathcal{L}} = \{4\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{2,8,9}}$	$\chi_{b_5}^{a_2,a_8,a_9}$	$I_{\mathcal{A}} = \{3\}, I_{\mathcal{L}} = \{4\}$	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{3,4,6}}$	χ^{a_3,a_4,a_6}	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{2\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{3,4,10}}$	$\chi_{b_1}^{a_3,a_4,a_{10}}$	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{6\}$	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{3,5,6}}$	χ^{a_3,a_5,a_6}	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{2\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{3,5,10}}$	$\chi_{b_1}^{a_3,a_5,a_{10}}$	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{6\}$	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{3,6,9}}$	χ^{a_3,a_6,a_9}	$I_{\mathcal{A}} = \{1,4\}, I_{\mathcal{L}} = \{2,5\}$	$(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{3,9,10}}$	$\chi_{b_1}^{a_3,a_9,a_{10}}$	$I_{\mathcal{A}} = \{2,5\}, I_{\mathcal{L}} = \{4,6\}$	$q(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{4,5,6}}$	χ^{a_4,a_5,a_6}	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{2\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{4,5,7}}$	χ^{a_4,a_5,a_7}	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{3\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{4,5,10}}$	$\chi_{b_1}^{a_4,a_5,a_{10}}$	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{6\}$	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{4,5,11}}$	$\chi_{b_2}^{a_4,a_5,a_{11}}$	$I_{\mathcal{A}} = \{1,3\}, I_{\mathcal{L}} = \{6,7\}$	$q(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{4,5,14}}$	$\chi_{b_1}^{a_4,a_5,a_{14}}$	$I_{\mathcal{A}} = \{2,3,7\}, I_{\mathcal{L}} = \{6,10,11\}$	$q(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{4,5,17}}$	$\chi_{b_1}^{a_4,a_5,a_{10},a_{17}}$	$I_{\mathcal{A}} = \{2,3,7\}, I_{\mathcal{L}} = \{6,11,14\}$	$q(q-1)^4$	q^3
	$\chi^{a_4,a_5,a_6,a_{17}}$	$I_{\mathcal{A}} = \{2,3,7\}, I_{\mathcal{L}} = \{1,11,14\}$	$(q-1)^4$	q^3
	$\chi_{b_1,b_2}^{a_4,a_5,a_{17}}$	$I_{\mathcal{A}} = \{3,7\}, I_{\mathcal{L}} = \{11,14\}$	$q^2(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{4,6,7}}$	$\chi_{b_3}^{a_4,a_6,a_7}$	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{2\}$	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{4,7,10}}$	$\chi_{b_1,b_3}^{a_4,a_7,a_{10}}$	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{6\}$	$q^2(q-1)^3$	q
$\mathcal{F}_{\mathbf{4,10,11}}$	$\chi_{b_3}^{a_4,a_{10},a_{11}}$	$I_{\mathcal{A}} = \{1,6\}, I_{\mathcal{L}} = \{2,7\}$	$q(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{5,6,7}}$	$\chi_{b_3}^{a_5,a_6,a_7}$	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{2\}$	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{5,6,8}}$	χ^{a_5,a_6,a_8}	$I_{\mathcal{A}} = \{1,3\}, I_{\mathcal{L}} = \{2,4\}$	$(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{5,6,12}}$	$\chi_{b_1,b_3}^{a_5,a_6,a_{12}}$	$I_{\mathcal{A}} = \{2,4\}, I_{\mathcal{L}} = \{7,8\}$	$q^2(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{5,7,8}}$	$\chi_{b_4}^{a_5,a_7,a_8}$	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{3\}$	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{5,7,10}}$	$\chi_{b_1,b_3}^{a_5,a_7,a_{10}}$	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{6\}$	$q^2(q-1)^3$	q
$\mathcal{F}_{\mathbf{5,8,10}}$	$\chi_{b_1}^{a_5,a_8,a_{10}}$	$I_{\mathcal{A}} = \{2,4\}, I_{\mathcal{L}} = \{3,6\}$	$q(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{5,8,11}}$	$\chi_{b_2,b_4}^{a_5,a_8,a_{11}}$	$I_{\mathcal{A}} = \{1,3\}, I_{\mathcal{L}} = \{6,7\}$	$q^2(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{5,8,14}}$	$\chi_{b_1,b_4}^{a_5,a_8,a_{14}}$	$I_{\mathcal{A}} = \{2,3,7\}, I_{\mathcal{L}} = \{6,10,11\}$	$q^2(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{5,8,17}}$	$\chi_{b_1,b_4}^{a_5,a_8,a_{10},a_{17}}$	$I_{\mathcal{A}} = \{2,3,7\}, I_{\mathcal{L}} = \{6,11,14\}$	$q^2(q-1)^4$	q^3
	$\chi_{b_4}^{a_5,a_6,a_8,a_{17}}$	$I_{\mathcal{A}} = \{2,3,7\}, I_{\mathcal{L}} = \{1,11,14\}$	$q(q-1)^4$	q^3
	$\chi_{b_1,b_2,b_4}^{a_5,a_8,a_{17}}$	$I_{\mathcal{A}} = \{3,7\}, I_{\mathcal{L}} = \{11,14\}$	$q^3(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{5,10,11}}$	$\chi_{b_3}^{a_5,a_{10},a_{11}}$	$I_{\mathcal{A}} = \{1,6\}, I_{\mathcal{L}} = \{2,7\}$	$q(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{5,10,12}}$	$\chi_{b_1,b_3,b_8}^{a_5,a_{10},a_{12}}$	$I_{\mathcal{A}} = \{2,4\}, I_{\mathcal{L}} = \{6,7\}$	$q^3(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{5,10,15}}$	$\chi^{a_5,a_7,a_{10},a_{15}}$	$I_{\mathcal{A}} = \{1,3,4,6\}$ $I_{\mathcal{L}} = \{2,8,11,12\}$	$(q-1)^4$	q^4
	$\chi_{b_2,b_3}^{a_5,a_{10},a_{15}}$	$I_{\mathcal{A}} = \{1,4,6\}, I_{\mathcal{L}} = \{8,11,12\}$	$q^2(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{5,11,12}}$	$\chi_{b_4}^{a_5,a_{11},a_{12}}$	$I_{\mathcal{A}} = \{2,3,7\}, I_{\mathcal{L}} = \{1,6,8\}$	$q(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{5,12,14}}$	$\chi_{b_1,b_4,b_8}^{a_5,a_{12},a_{14}}$	$I_{\mathcal{A}} = \{2,3,7\}, I_{\mathcal{L}} = \{6,10,11\}$	$q^3(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{5,12,17}}$	$\chi_{b_1,b_4,b_6,b_{10}}^{a_5,a_{12},a_{17}}$	$I_{\mathcal{A}} = \{2,3,7\}, I_{\mathcal{L}} = \{8,11,14\}$	$q^4(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{5,14,15}}$	$\chi_{b_4,b_8}^{a_5,a_{14},a_{15}}$	$I_{\mathcal{A}} = \{1,3,6,11\}$ $I_{\mathcal{L}} = \{2,7,10,12\}$	$q^2(q-1)^3$	q^4
$\mathcal{F}_{\mathbf{5,15,17}}$	$\chi_{b_2,b_4,b_{10}}^{a_5,a_{15},a_{17}}$	$I_{\mathcal{A}} = \{1,3,6,11\}$ $I_{\mathcal{L}} = \{7,8,12,14\}$	$q^3(q-1)^3$	q^4
$\mathcal{F}_{\mathbf{5,17,18}}$	$\chi_{b_1,b_4}^{a_5,a_{17},a_{18}}$	$I_{\mathcal{A}} = \{2,6,7,10,14\}$ $I_{\mathcal{L}} = \{3,8,11,12,15\}$	$q^2(q-1)^3$	q^5
$\mathcal{F}_{\mathbf{6,7,8}}$	χ^{a_6,a_7,a_8}	$I_{\mathcal{A}} = \{1,3\}, I_{\mathcal{L}} = \{2,4\}$	$(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{6,7,9}}$	$\chi_{b_3}^{a_6,a_7,a_9}$	$I_{\mathcal{A}} = \{1,4\}, I_{\mathcal{L}} = \{2,5\}$	$q(q-1)^3$	q^2

$\mathcal{F}$	χ	I	Number	Degree
$\mathcal{F}_{\mathbf{6,7,13}}$	$\chi_{b_4}^{a_6,a_7,a_{13}}$	$I_{\mathcal{A}} = \{2,3,5\}, I_{\mathcal{L}} = \{1,8,9\}$	$q(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{6,8,9}}$	$\chi_{b_5}^{a_6,a_8,a_9}$	$I_{\mathcal{A}} = \{1,3\}, I_{\mathcal{L}} = \{2,4\}$	$q(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{6,9,12}}$	$\chi_{b_1,b_3,b_5}^{a_6,a_9,a_{12}}$	$I_{\mathcal{A}} = \{2,4\}, I_{\mathcal{L}} = \{7,8\}$	$q^3(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{6,12,13}}$	$\chi^{a_6,a_{12},a_{13}}$	$I_{\mathcal{A}} = \{1,3,4,8\}$ $I_{\mathcal{L}} = \{2,5,7,9\}$	$(q-1)^3$	q^4
$\mathcal{F}_{\mathbf{7,8,9}}$	χ^{a_7,a_8,a_9}	$I_{\mathcal{A}} = \{2,4\}, I_{\mathcal{L}} = \{3,5\}$	$(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{7,8,10}}$	$\chi_{b_1}^{a_7,a_8,a_{10}}$	$I_{\mathcal{A}} = \{2,4\}, I_{\mathcal{L}} = \{3,6\}$	$q(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{7,9,10}}$	$\chi_{b_1,b_3}^{a_7,a_9,a_{10}}$	$I_{\mathcal{A}} = \{2,5\}, I_{\mathcal{L}} = \{4,6\}$	$q^2(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{7,10,13}}$	$\chi_{b_1,b_4}^{a_7,a_{10},a_{13}}$	$I_{\mathcal{A}} = \{2,3,5\}, I_{\mathcal{L}} = \{6,8,9\}$	$q^2(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{8,9,10}}$	$\chi_{b_1,b_5}^{a_8,a_9,a_{10}}$	$I_{\mathcal{A}} = \{2,4\}, I_{\mathcal{L}} = \{3,6\}$	$q^2(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{8,9,11}}$	$\chi_{b_2}^{a_8,a_9,a_{11}}$	$I_{\mathcal{A}} = \{1,3,5\}, I_{\mathcal{L}} = \{4,6,7\}$	$q(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{8,9,14}}$	$\chi_{b_1}^{a_8,a_9,a_{14}}$	$I_{\mathcal{A}} = \{2,3,5,7\}$ $I_{\mathcal{L}} = \{4,6,10,11\}$	$q(q-1)^3$	q^4
$\mathcal{F}_{\mathbf{8,9,17}}$	$\chi_{b_1}^{a_8,a_9,a_{10},a_{17}}$	$I_{\mathcal{A}} = \{2,3,5,7\}$ $I_{\mathcal{L}} = \{4,6,11,14\}$	$q(q-1)^4$	q^4
	$\chi^{a_6,a_8,a_9,a_{17}}$	$I_{\mathcal{A}} = \{2,3,5,7\}$ $I_{\mathcal{L}} = \{1,4,11,14\}$	$(q-1)^4$	q^4
	$\chi_{b_1,b_2}^{a_8,a_9,a_{17}}$	$I_{\mathcal{A}} = \{3,5,7\}, I_{\mathcal{L}} = \{4,11,14\}$	$q^2(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{8,10,11}}$	$\chi^{a_8,a_{10},a_{11}}$	$I_{\mathcal{A}} = \{1,4,6\}, I_{\mathcal{L}} = \{2,3,7\}$	$(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{9,10,11}}$	$\chi_{b_3}^{a_9,a_{10},a_{11}}$	$I_{\mathcal{A}} = \{1,5,6\}, I_{\mathcal{L}} = \{2,4,7\}$	$q(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{9,10,12}}$	$\chi_{b_1,b_3,b_5,b_8}^{a_9,a_{10},a_{12}}$	$I_{\mathcal{A}} = \{2,4\}, I_{\mathcal{L}} = \{6,7\}$	$q^4(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{9,10,15}}$	$\chi_{b_5}^{a_7,a_9,a_{10},a_{15}}$	$I_{\mathcal{A}} = \{1,3,4,6\}$ $I_{\mathcal{L}} = \{2,8,11,12\}$	$q(q-1)^4$	q^4
	$\chi_{b_2,b_3,b_5}^{a_9,a_{10},a_{15}}$	$I_{\mathcal{A}} = \{1,4,6\}, I_{\mathcal{L}} = \{8,11,12\}$	$q^3(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{9,11,12}}$	$\chi^{a_9,a_{11},a_{12}}$	$I_{\mathcal{A}} = \{2,3,5,7\}$ $I_{\mathcal{L}} = \{1,4,6,8\}$	$(q-1)^3$	q^4
$\mathcal{F}_{\mathbf{9,12,14}}$	$\chi_{b_1,b_8}^{a_9,a_{12},a_{14}}$	$I_{\mathcal{A}} = \{2,3,5,7\}$ $I_{\mathcal{L}} = \{4,6,10,11\}$	$q^2(q-1)^3$	q^4
$\mathcal{F}_{\mathbf{9,12,17}}$	$\chi_{b_1,b_6,b_{10}}^{a_9,a_{12},a_{17}}$	$I_{\mathcal{A}} = \{2,3,5,7\}$ $I_{\mathcal{L}} = \{4,8,11,14\}$	$q^3(q-1)^3$	q^4
$\mathcal{F}_{\mathbf{9,14,15}}$	$\chi_{b_8}^{a_9,a_{14},a_{15}}$	$I_{\mathcal{A}} = \{1,3,5,6,11\}$ $I_{\mathcal{L}} = \{2,4,7,10,12\}$	$q(q-1)^3$	q^5
$\mathcal{F}_{\mathbf{9,15,17}}$	$\chi_{b_2,b_{10}}^{a_9,a_{15},a_{17}}$	$I_{\mathcal{A}} = \{1,3,5,6,11\}$ $I_{\mathcal{L}} = \{4,7,8,12,14\}$	$q^2(q-1)^3$	q^5
$\mathcal{F}_{\mathbf{9,17,18}}$	$\chi_{b_1}^{a_9,a_{17},a_{18}}$	$I_{\mathcal{A}} = \{2,5,6,7,10,14\}$ $I_{\mathcal{L}} = \{3,4,8,11,12,15\}$	$q(q-1)^3$	q^6
$\mathcal{F}_{\mathbf{10,11,12}}$	$\chi_{b_4}^{a_{10},a_{11},a_{12}}$	$I_{\mathcal{A}} = \{2,3,7\}, I_{\mathcal{L}} = \{1,6,8\}$	$q(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{10,11,13}}$	$\chi_{b_4}^{a_{10},a_{11},a_{13}}$	$I_{\mathcal{A}} = \{1,2,3,5\}$ $I_{\mathcal{L}} = \{6,7,8,9\}$	$q(q-1)^3$	q^4
$\mathcal{F}_{\mathbf{10,11,16}}$	$\chi_{b_1,b_4,b_8}^{a_{10},a_{11},a_{16}}$	$I_{\mathcal{A}} = \{2,3,5,7\}$ $I_{\mathcal{L}} = \{6,9,12,13\}$	$q^3(q-1)^3$	q^4
$\mathcal{F}_{\mathbf{10,12,13}}$	$\chi_{b_1}^{a_{10},a_{12},a_{13}}$	$I_{\mathcal{A}} = \{2,3,4,5\}$ $I_{\mathcal{L}} = \{6,7,8,9\}$	$q(q-1)^3$	q^4
$\mathcal{F}_{\mathbf{10,13,15}}$	$\chi_{b_7}^{a_{10},a_{13},a_{15}}$	$I_{\mathcal{A}} = \{1,3,4,5,6\}$ $I_{\mathcal{L}} = \{2,8,9,11,12\}$	$q(q-1)^3$	q^5
$\mathcal{F}_{\mathbf{10,15,16}}$	$\chi_{b_3,b_5}^{a_{10},a_{15},a_{16}}$	$I_{\mathcal{A}} = \{2,4,8,9,12\}$ $I_{\mathcal{L}} = \{1,6,7,11,13\}$	$q^2(q-1)^3$	q^5
$\mathcal{F}_{\mathbf{11,12,13}}$	$\chi^{a_9,a_{11},a_{12},a_{13}}$	$I_{\mathcal{A}} = \{1,2,4,6\}$ $I_{\mathcal{L}} = \{3,5,7,8\}$	$(q-1)^4$	q^4

$\mathcal{F}$	χ	I	Number	Degree
	$\chi^{a_{11},a_{12},a_{13}}_{b_4,b_5}$	$I_{\mathcal{A}} = \{1,2,6\}, I_{\mathcal{L}} = \{3,7,8\}$	$q^2(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{12,13,14}}$	$\chi^{a_{12},a_{13},a_{14}}_{b_1,b_4,b_9}$	$I_{\mathcal{A}} = \{2,3,5,7\}$ $I_{\mathcal{L}} = \{6,8,10,11\}$	$q^3(q-1)^3$	q^4
$\mathcal{F}_{\mathbf{12,13,17}}$	$\chi^{a_{10},a_{12},a_{13},a_{17}}_{b_1,b_4,b_9}$	$I_{\mathcal{A}} = \{2,3,5,7\}$ $I_{\mathcal{L}} = \{6,8,11,14\}$	$q^3(q-1)^4$	q^4
	$\chi^{a_9,a_{12},a_{13},a_{17}}_{b_1,b_6}$	$I_{\mathcal{A}} = \{2,3,5,7\}$ $I_{\mathcal{L}} = \{4,8,11,14\}$	$q^2(q-1)^4$	q^4
	$\chi^{a_6,a_{12},a_{13},a_{17}}_{b_4}$	$I_{\mathcal{A}} = \{2,3,7,8\}$ $I_{\mathcal{L}} = \{1,5,11,14\}$	$q(q-1)^4$	q^4
	$\chi^{a_{12},a_{13},a_{17}}_{b_1,b_2,b_4}$	$I_{\mathcal{A}} = \{3,7,8\}, I_{\mathcal{L}} = \{5,11,14\}$	$q^3(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{13,14,15}}$	$\chi^{a_{13},a_{14},a_{15}}_{b_4,b_9}$	$I_{\mathcal{A}} = \{1,3,5,6,11\}$ $I_{\mathcal{L}} = \{2,7,8,10,12\}$	$q^2(q-1)^3$	q^5
$\mathcal{F}_{\mathbf{13,15,17}}$	$\chi^{a_{10},a_{13},a_{15},a_{17}}_{b_4,b_9}$	$I_{\mathcal{A}} = \{1,3,5,6,11\}$ $I_{\mathcal{L}} = \{2,7,8,12,14\}$	$q^2(q-1)^4$	q^5
	$\chi^{a_9,a_{13},a_{15},a_{17}}_{b_2}$	$I_{\mathcal{A}} = \{1,3,5,8,11\}$ $I_{\mathcal{L}} = \{4,6,7,12,14\}$	$q(q-1)^4$	q^5
	$\chi^{a_{13},a_{15},a_{17}}_{b_2,b_4,b_5}$	$I_{\mathcal{A}} = \{1,3,8,11\}$ $I_{\mathcal{L}} = \{6,7,12,14\}$	$q^3(q-1)^3$	q^4
$\mathcal{F}_{\mathbf{13,17,18}}$	$\chi^{a_{13},a_{17},a_{18}}_{b_1,b_5}$	$I_{\mathcal{A}} = \{2,3,4,6,7,8\}$ $I_{\mathcal{L}} = \{9,10,11,12,14,15\}$	$q^2(q-1)^3$	q^6
$\mathcal{F}_{\mathbf{14,15,16}}$	$\chi^{a_{14},a_{15},a_{16}}_{b_5}$	$I_{\mathcal{A}} = \{2,3,4,6,9,12\}$ $I_{\mathcal{L}} = \{1,7,8,10,11,13\}$	$q(q-1)^3$	q^6
$\mathcal{F}_{\mathbf{15,16,17}}$	$\chi^{a_{15},a_{16},a_{17}}_{b_5,b_{10}}$	$I_{\mathcal{A}} = \{2,3,4,7,8,12\}$ $I_{\mathcal{L}} = \{1,6,9,11,13,14\}$	$q^2(q-1)^3$	q^6
$\mathcal{F}_{\mathbf{16,17,18}}$	$\chi^{a_{13},a_{16},a_{17},a_{18}}_{b_1,b_5}$	$I_{\mathcal{A}} = \{2,3,4,7,8,12\}$ $I_{\mathcal{L}} = \{6,9,10,11,14,15\}$	$q^2(q-1)^4$	q^6
	$\chi^{a_9,a_{16},a_{17},a_{18}}_{b_1}$	$I_{\mathcal{A}} = \{2,5,7,10,12,14\}$ $I_{\mathcal{L}} = \{3,4,6,8,11,15\}$	$q(q-1)^4$	q^6
	$\chi^{a_{16},a_{17},a_{18}}_{b_1,b_4,b_5}$	$I_{\mathcal{A}} = \{2,7,10,12,14\}$ $I_{\mathcal{L}} = \{3,6,8,11,15\}$	$q^3(q-1)^3$	q^5
$\mathcal{F}_{\mathbf{17,18,19}}$	$\chi^{a_{13},a_{17},a_{18},a_{19}}_{b_5}$	$I_{\mathcal{A}} = \{1,3,4,6,8,11,15\}$ $I_{\mathcal{L}} = \{2,7,9,10,12,14,16\}$	$q(q-1)^4$	q^7
	$\chi^{a_9,a_{17},a_{18},a_{19}}$	$I_{\mathcal{A}} = \{1,5,6,10,11,14,15\}$ $I_{\mathcal{L}} = \{2,3,4,7,8,12,16\}$	$(q-1)^4$	q^7
	$\chi^{a_{17},a_{18},a_{19}}_{b_4,b_5}$	$I_{\mathcal{A}} = \{1,6,10,11,14,15\}$ $I_{\mathcal{L}} = \{2,3,7,8,12,16\}$	$q^2(q-1)^3$	q^6
$\mathcal{F}_{\mathbf{1,2,3,9}}$	χ^{a_1,a_2,a_3,a_9}	$I_{\mathcal{A}} = \{4\}, I_{\mathcal{L}} = \{5\}$	$(q-1)^4$	q
$\mathcal{F}_{\mathbf{1,2,5,8}}$	χ^{a_1,a_2,a_5,a_8}	$I_{\mathcal{A}} = \{3\}, I_{\mathcal{L}} = \{4\}$	$(q-1)^4$	q
$\mathcal{F}_{\mathbf{1,2,8,9}}$	$\chi^{a_1,a_2,a_8,a_9}_{b_5}$	$I_{\mathcal{A}} = \{3\}, I_{\mathcal{L}} = \{4\}$	$q(q-1)^4$	q
$\mathcal{F}_{\mathbf{1,4,5,7}}$	χ^{a_1,a_4,a_5,a_7}	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{3\}$	$(q-1)^4$	q
$\mathcal{F}_{\mathbf{1,5,7,8}}$	$\chi^{a_1,a_5,a_7,a_8}_{b_4}$	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{3\}$	$q(q-1)^4$	q
$\mathcal{F}_{\mathbf{1,7,8,9}}$	χ^{a_1,a_7,a_8,a_9}	$I_{\mathcal{A}} = \{2,4\}, I_{\mathcal{L}} = \{3,5\}$	$(q-1)^4$	q^2
$\mathcal{F}_{\mathbf{3,4,5,6}}$	χ^{a_3,a_4,a_5,a_6}	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{2\}$	$(q-1)^4$	q
$\mathcal{F}_{\mathbf{3,4,5,10}}$	$\chi^{a_3,a_4,a_5,a_{10}}_{b_1}$	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{6\}$	$q(q-1)^4$	q
$\mathcal{F}_{\mathbf{4,5,6,7}}$	$\chi^{a_4,a_5,a_6,a_7}_{b_3}$	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{2\}$	$q(q-1)^4$	q
$\mathcal{F}_{\mathbf{4,5,7,10}}$	$\chi^{a_4,a_5,a_7,a_{10}}_{b_1,b_3}$	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{6\}$	$q^2(q-1)^4$	q
$\mathcal{F}_{\mathbf{4,5,10,11}}$	$\chi^{a_4,a_5,a_{10},a_{11}}_{b_3}$	$I_{\mathcal{A}} = \{1,6\}, I_{\mathcal{L}} = \{2,7\}$	$q(q-1)^4$	q^2
$\mathcal{F}_{\mathbf{5,6,7,8}}$	χ^{a_5,a_6,a_7,a_8}	$I_{\mathcal{A}} = \{1,3\}, I_{\mathcal{L}} = \{2,4\}$	$(q-1)^4$	q^2
$\mathcal{F}_{\mathbf{5,7,8,10}}$	$\chi^{a_5,a_7,a_8,a_{10}}_{b_1}$	$I_{\mathcal{A}} = \{2,4\}, I_{\mathcal{L}} = \{3,6\}$	$q(q-1)^4$	q^2
$\mathcal{F}_{\mathbf{5,8,10,11}}$	$\chi^{a_5,a_8,a_{10},a_{11}}$	$I_{\mathcal{A}} = \{1,4,6\}, I_{\mathcal{L}} = \{2,3,7\}$	$(q-1)^4$	q^3

$\mathcal{F}$	χ	I	Number	Degree
$\mathcal{F}_{\mathbf{5,10,11,12}}$	$\chi_{b_4}^{a_5,a_{10},a_{11},a_{12}}$	$I_{\mathcal{A}} = \{2,3,7\}, I_{\mathcal{L}} = \{1,6,8\}$	$q(q-1)^4$	q^3
$\mathcal{F}_{\mathbf{6,7,8,9}}$	$\chi_{b_5}^{a_6,a_7,a_8,a_9}$	$I_{\mathcal{A}} = \{1,3\}, I_{\mathcal{L}} = \{2,4\}$	$q(q-1)^4$	q^2
$\mathcal{F}_{\mathbf{7,8,9,10}}$	$\chi_{b_1,b_5}^{a_7,a_8,a_9,a_{10}}$	$I_{\mathcal{A}} = \{2,4\}, I_{\mathcal{L}} = \{3,6\}$	$q^2(q-1)^4$	q^2
$\mathcal{F}_{\mathbf{8,9,10,11}}$	$\chi_{b_5}^{a_8,a_9,a_{10},a_{11}}$	$I_{\mathcal{A}} = \{1,4,6\}, I_{\mathcal{L}} = \{2,3,7\}$	$q(q-1)^4$	q^3
$\mathcal{F}_{\mathbf{9,10,11,12}}$	$\chi^{a_9,a_{10},a_{11},a_{12}}$	$I_{\mathcal{A}} = \{2,3,5,7\}$ $I_{\mathcal{L}} = \{1,4,6,8\}$	$(q-1)^4$	q^4
$\mathcal{F}_{\mathbf{10,11,12,13}}$	$\chi_{b_4}^{a_{10},a_{11},a_{12},a_{13}}$	$I_{\mathcal{A}} = \{2,3,5,7\}$ $I_{\mathcal{L}} = \{1,6,8,9\}$	$q(q-1)^4$	q^4

Table D.6: The parametrization of the irreducible characters of $\mathrm{UC}_5(q)$, where $q = p^e$ and $p \geq 3$.

Parametrization of the irreducible characters of U_{D_5}

$\mathcal{F}$	χ	I	Number	Degree
$\mathcal{F}_{\mathrm{lin}}$	$\chi_{b_1,b_2,b_3,b_4,b_5}$		q^5	1
$\mathcal{F}_{\mathbf{6}}$	χ^{a_6}	$I_{\mathcal{A}}=\{1\}, I_{\mathcal{L}}=\{3\}$	$q-1$	q
$\mathcal{F}_{\mathbf{7}}$	χ^{a_7}	$I_{\mathcal{A}}=\{2\}, I_{\mathcal{L}}=\{3\}$	$q-1$	q
$\mathcal{F}_{\mathbf{8}}$	χ^{a_8}	$I_{\mathcal{A}}=\{3\}, I_{\mathcal{L}}=\{4\}$	$q-1$	q
$\mathcal{F}_{\mathbf{9}}$	χ^{a_9}	$I_{\mathcal{A}}=\{4\}, I_{\mathcal{L}}=\{5\}$	$q-1$	q
$\mathcal{F}_{\mathbf{10}}$	$\chi^{a_{10}}_{b_3}$	$I_{\mathcal{A}}=\{1,2\}, I_{\mathcal{L}}=\{6,7\}$	$q(q-1)$	q^2
$\mathcal{F}_{\mathbf{11}}$	$\chi^{a_{11}}_{b_3}$	$I_{\mathcal{A}}=\{1,4\}, I_{\mathcal{L}}=\{6,8\}$	$q(q-1)$	q^2
$\mathcal{F}_{\mathbf{12}}$	$\chi^{a_{12}}_{b_3}$	$I_{\mathcal{A}}=\{2,4\}, I_{\mathcal{L}}=\{7,8\}$	$q(q-1)$	q^2
$\mathcal{F}_{\mathbf{13}}$	$\chi^{a_{13}}_{b_4}$	$I_{\mathcal{A}}=\{3,5\}, I_{\mathcal{L}}=\{8,9\}$	$q(q-1)$	q^2
$\mathcal{F}_{\mathbf{14}}$	$\chi^{a_{14}}_{b_3,b_6,b_7,b_8}$	$I_{\mathcal{A}}=\{1,2,4\}, I_{\mathcal{L}}=\{10,11,12\}$	$q^4(q-1)$	q^3
$\mathcal{F}_{\mathbf{15}}$	$\chi^{a_8,a_{15}}$	$I_{\mathcal{A}}=\{1,4,5,9\}$	$(q-1)^2$	q^4
		$I_{\mathcal{L}}=\{3,6,11,13\}$		
	$\chi^{a_{15}}_{b_3,b_4}$	$I_{\mathcal{A}}=\{1,5,9\}, I_{\mathcal{L}}=\{6,11,13\}$	$q^2(q-1)$	q^3
$\mathcal{F}_{\mathbf{16}}$	$\chi^{a_8,a_{16}}$	$I_{\mathcal{A}}=\{2,4,5,9\}$	$(q-1)^2$	q^4
		$I_{\mathcal{L}}=\{3,7,12,13\}$		
	$\chi^{a_{16}}_{b_3,b_4}$	$I_{\mathcal{A}}=\{2,5,9\}, I_{\mathcal{L}}=\{7,12,13\}$	$q^2(q-1)$	q^3
$\mathcal{F}_{\mathbf{17}}$	$\chi^{a_{17}}_{b_1,b_2,b_4}$	$I_{\mathcal{A}}=\{3,6,7,8\}$	$q^3(q-1)$	q^4
		$I_{\mathcal{L}}=\{10,11,12,14\}$		
$\mathcal{F}_{\mathbf{18}}$	$\chi^{a_{11},a_{18}}_{b_3,b_7,b_8,b_{12},b_{13}}$	$I_{\mathcal{A}}=\{1,2,4,5,9\}$	$q^5(q-1)^2$	q^5
		$I_{\mathcal{L}}=\{6,10,14,15,16\}$		
	$\chi^{a_{12},a_{18}}_{b_3,b_6,b_8,b_{13}}$	$I_{\mathcal{A}}=\{1,2,4,5,9\}$	$q^4(q-1)^2$	q^5
		$I_{\mathcal{L}}=\{7,10,14,15,16\}$		
	$\chi^{a_8,a_{18}}_{b_6,b_7,b_{13}}$	$I_{\mathcal{A}}=\{1,2,4,5,9\}$	$q^3(q-1)^2$	q^5
		$I_{\mathcal{L}}=\{3,10,14,15,16\}$		
	$\chi^{a_{18}}_{b_3,b_4,b_6,b_7,b_{13}}$	$I_{\mathcal{A}}=\{1,2,5,9\}$	$q^5(q-1)$	q^4
		$I_{\mathcal{L}}=\{10,14,15,16\}$		
$\mathcal{F}_{\mathbf{19}}$	$\chi^{a_{14},a_{19}}_{b_4,b_8,b_9}$	$I_{\mathcal{A}}=\{1,2,3,5,6,7,10\}$	$q^3(q-1)^2$	q^7
		$I_{\mathcal{L}}=\{11,12,13,15,16,17,18\}$		
	$\chi^{a_{12},a_{19}}_{b_1,b_4,b_9,b_{11}}$	$I_{\mathcal{A}}=\{3,5,6,7,8,10\}$	$q^4(q-1)^2$	q^6
		$I_{\mathcal{L}}=\{2,13,15,16,17,18\}$		
	$\chi^{a_{11},a_{19}}_{b_2,b_4,b_9}$	$I_{\mathcal{A}}=\{1,3,5,6,7,10\}$	$q^3(q-1)^2$	q^6
		$I_{\mathcal{L}}=\{8,13,15,16,17,18\}$		
	$\chi^{a_{19}}_{b_1,b_2,b_4,b_8,b_9}$	$I_{\mathcal{A}}=\{3,5,6,7,10\}$	$q^5(q-1)$	q^5
		$I_{\mathcal{L}}=\{13,15,16,17,18\}$		
$\mathcal{F}_{\mathbf{20}}$	$\chi^{a_{10},a_{20}}_{b_3,b_5}$	$I_{\mathcal{A}}=\{1,2,4,8,9,11,12,13\}$	$q^2(q-1)^2$	q^8
		$I_{\mathcal{L}}=\{6,7,14,15,16,17,18,19\}$		
	$\chi^{a_7,a_{20}}_{b_1,b_5,b_6}$	$I_{\mathcal{A}}=\{3,4,8,9,11,12,13\}$	$q^3(q-1)^2$	q^7
		$I_{\mathcal{L}}=\{2,14,15,16,17,18,19\}$		
	$\chi^{a_6,a_{20}}_{b_2,b_5}$	$I_{\mathcal{A}}=\{3,4,8,9,11,12,13\}$	$q^2(q-1)^2$	q^7
		$I_{\mathcal{L}}=\{1,14,15,16,17,18,19\}$		
	$\chi^{a_{20}}_{b_1,b_2,b_3,b_5}$	$I_{\mathcal{A}}=\{4,8,9,11,12,13\}$	$q^4(q-1)$	q^6
		$I_{\mathcal{L}}=\{14,15,16,17,18,19\}$		
$\mathcal{F}_{\mathbf{1,7}}$	χ^{a_1,a_7}	$I_{\mathcal{A}}=\{2\}, I_{\mathcal{L}}=\{3\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{1,8}}$	χ^{a_1,a_8}	$I_{\mathcal{A}}=\{3\}, I_{\mathcal{L}}=\{4\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{1,9}}$	χ^{a_1,a_9}	$I_{\mathcal{A}}=\{4\}, I_{\mathcal{L}}=\{5\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{1,12}}$	$\chi^{a_1,a_{12}}_{b_3}$	$I_{\mathcal{A}}=\{2,4\}, I_{\mathcal{L}}=\{7,8\}$	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{1,13}}$	$\chi^{a_1,a_{13}}_{b_4}$	$I_{\mathcal{A}}=\{3,5\}, I_{\mathcal{L}}=\{8,9\}$	$q(q-1)^2$	q^2

$\mathcal{F}$	χ	I	Number	Degree
$\mathcal{F}_{\mathbf{1,16}}$	$\chi^{a_1,a_8,a_{16}}$	$I_{\mathcal{A}} = \{2,4,5,9\}$	$(q-1)^3$	q^4
		$I_{\mathcal{L}} = \{3,7,12,13\}$		
	$\chi^{a_1,a_{16}}_{b_3,b_4}$	$I_{\mathcal{A}} = \{2,5,9\}, I_{\mathcal{L}} = \{7,12,13\}$	$q^2(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{2,6}}$	χ^{a_2,a_6}	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{3\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{2,8}}$	χ^{a_2,a_8}	$I_{\mathcal{A}} = \{3\}, I_{\mathcal{L}} = \{4\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{2,9}}$	χ^{a_2,a_9}	$I_{\mathcal{A}} = \{4\}, I_{\mathcal{L}} = \{5\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{2,11}}$	$\chi^{a_2,a_{11}}_{b_3}$	$I_{\mathcal{A}} = \{1,4\}, I_{\mathcal{L}} = \{6,8\}$	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{2,13}}$	$\chi^{a_2,a_{13}}_{b_4}$	$I_{\mathcal{A}} = \{3,5\}, I_{\mathcal{L}} = \{8,9\}$	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{2,15}}$	$\chi^{a_2,a_8,a_{15}}$	$I_{\mathcal{A}} = \{1,4,5,9\}$	$(q-1)^3$	q^4
		$I_{\mathcal{L}} = \{3,6,11,13\}$		
	$\chi^{a_2,a_{15}}_{b_3,b_4}$	$I_{\mathcal{A}} = \{1,5,9\}, I_{\mathcal{L}} = \{6,11,13\}$	$q^2(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{3,9}}$	χ^{a_3,a_9}	$I_{\mathcal{A}} = \{4\}, I_{\mathcal{L}} = \{5\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{4,6}}$	χ^{a_4,a_6}	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{3\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{4,7}}$	χ^{a_4,a_7}	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{3\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{4,10}}$	$\chi^{a_4,a_{10}}_{b_3}$	$I_{\mathcal{A}} = \{1,2\}, I_{\mathcal{L}} = \{6,7\}$	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{5,6}}$	χ^{a_5,a_6}	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{3\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{5,7}}$	χ^{a_5,a_7}	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{3\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{5,8}}$	χ^{a_5,a_8}	$I_{\mathcal{A}} = \{3\}, I_{\mathcal{L}} = \{4\}$	$(q-1)^2$	q
$\mathcal{F}_{\mathbf{5,10}}$	$\chi^{a_5,a_{10}}_{b_3}$	$I_{\mathcal{A}} = \{1,2\}, I_{\mathcal{L}} = \{6,7\}$	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{5,11}}$	$\chi^{a_5,a_{11}}_{b_3}$	$I_{\mathcal{A}} = \{1,4\}, I_{\mathcal{L}} = \{6,8\}$	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{5,12}}$	$\chi^{a_5,a_{12}}_{b_3}$	$I_{\mathcal{A}} = \{2,4\}, I_{\mathcal{L}} = \{7,8\}$	$q(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{5,14}}$	$\chi^{a_5,a_{14}}_{b_3,b_6,b_7,b_8}$	$I_{\mathcal{A}} = \{1,2,4\}, I_{\mathcal{L}} = \{10,11,12\}$	$q^4(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{5,17}}$	$\chi^{a_5,a_{17}}_{b_1,b_2,b_4}$	$I_{\mathcal{A}} = \{3,6,7,8\}$	$q^3(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{10,11,12,14\}$		
$\mathcal{F}_{\mathbf{6,7}}$	$\chi^{a_6,a_7}_{b_2}$	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{3\}$	$q(q-1)^2$	q
$\mathcal{F}_{\mathbf{6,8}}$	$\chi^{a_6,a_8}_{b_4}$	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{3\}$	$q(q-1)^2$	q
$\mathcal{F}_{\mathbf{6,9}}$	χ^{a_6,a_9}	$I_{\mathcal{A}} = \{1,4\}, I_{\mathcal{L}} = \{3,5\}$	$(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{6,12}}$	$\chi^{a_6,a_{12}}$	$I_{\mathcal{A}} = \{2,3,4\}, I_{\mathcal{L}} = \{1,7,8\}$	$(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{6,13}}$	$\chi^{a_6,a_{13}}_{b_1,b_4}$	$I_{\mathcal{A}} = \{3,5\}, I_{\mathcal{L}} = \{8,9\}$	$q^2(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{6,16}}$	$\chi^{a_6,a_{16}}_{b_4,b_8}$	$I_{\mathcal{A}} = \{2,3,5,9\}$	$q^2(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{1,7,12,13\}$		
$\mathcal{F}_{\mathbf{7,8}}$	$\chi^{a_7,a_8}_{b_4}$	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{3\}$	$q(q-1)^2$	q
$\mathcal{F}_{\mathbf{7,9}}$	χ^{a_7,a_9}	$I_{\mathcal{A}} = \{2,4\}, I_{\mathcal{L}} = \{3,5\}$	$(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{7,11}}$	$\chi^{a_7,a_{11}}$	$I_{\mathcal{A}} = \{1,3,4\}, I_{\mathcal{L}} = \{2,6,8\}$	$(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{7,13}}$	$\chi^{a_7,a_{13}}_{b_2,b_4}$	$I_{\mathcal{A}} = \{3,5\}, I_{\mathcal{L}} = \{8,9\}$	$q^2(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{7,15}}$	$\chi^{a_7,a_{15}}_{b_4,b_8}$	$I_{\mathcal{A}} = \{1,3,5,9\}$	$q^2(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{2,6,11,13\}$		
$\mathcal{F}_{\mathbf{8,9}}$	$\chi^{a_8,a_9}_{b_5}$	$I_{\mathcal{A}} = \{3\}, I_{\mathcal{L}} = \{4\}$	$q(q-1)^2$	q
$\mathcal{F}_{\mathbf{8,10}}$	$\chi^{a_8,a_{10}}$	$I_{\mathcal{A}} = \{1,2,4\}, I_{\mathcal{L}} = \{3,6,7\}$	$(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{9,10}}$	$\chi^{a_9,a_{10}}_{b_3}$	$I_{\mathcal{A}} = \{1,2,5\}, I_{\mathcal{L}} = \{4,6,7\}$	$q(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{9,11}}$	$\chi^{a_9,a_{11}}_{b_3,b_5}$	$I_{\mathcal{A}} = \{1,4\}, I_{\mathcal{L}} = \{6,8\}$	$q^2(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{9,12}}$	$\chi^{a_9,a_{12}}_{b_3,b_5}$	$I_{\mathcal{A}} = \{2,4\}, I_{\mathcal{L}} = \{7,8\}$	$q^2(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{9,14}}$	$\chi^{a_9,a_{14}}_{b_3,b_5,b_6,b_7,b_8}$	$I_{\mathcal{A}} = \{1,2,4\}, I_{\mathcal{L}} = \{10,11,12\}$	$q^5(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{9,17}}$	$\chi^{a_9,a_{17}}_{b_1,b_2}$	$I_{\mathcal{A}} = \{3,5,6,7,8\}$	$q^2(q-1)^2$	q^5
		$I_{\mathcal{L}} = \{4,10,11,12,14\}$		
$\mathcal{F}_{\mathbf{10,11}}$	$\chi^{a_8,a_{10},a_{11}}$	$I_{\mathcal{A}} = \{1,4,6\}, I_{\mathcal{L}} = \{2,3,7\}$	$(q-1)^3$	q^3
	$\chi^{a_{10},a_{11}}_{b_3,b_4}$	$I_{\mathcal{A}} = \{1,6\}, I_{\mathcal{L}} = \{2,7\}$	$q^2(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{10,12}}$	$\chi^{a_8,a_{10},a_{12}}$	$I_{\mathcal{A}} = \{1,3,6\}, I_{\mathcal{L}} = \{2,4,7\}$	$(q-1)^3$	q^3
	$\chi^{a_{10},a_{12}}_{b_3,b_4}$	$I_{\mathcal{A}} = \{1,6\}, I_{\mathcal{L}} = \{2,7\}$	$q^2(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{10,13}}$	$\chi^{a_{10},a_{13}}_{b_4}$	$I_{\mathcal{A}} = \{1,2,3,5\}$	$q(q-1)^2$	q^4

$\mathcal{F}$	χ	I	Number	Degree
		$I_{\mathcal{L}} = \{6, 7, 8, 9\}$		
$\mathcal{F}_{\mathbf{10,15}}$	$\chi_{b_4,b_8}^{a_7,a_{10},a_{15}}$	$I_{\mathcal{A}} = \{1, 3, 5, 6\}$	$q^2(q-1)^3$	q^4
		$I_{\mathcal{L}} = \{2, 9, 11, 13\}$		
	$\chi_{b_2}^{a_8,a_{10},a_{15}}$	$I_{\mathcal{A}} = \{1, 4, 5, 6\}$	$q(q-1)^3$	q^4
		$I_{\mathcal{L}} = \{3, 9, 11, 13\}$		
	$\chi_{b_2,b_3,b_4}^{a_{10},a_{15}}$	$I_{\mathcal{A}} = \{1, 5, 6\}, I_{\mathcal{L}} = \{9, 11, 13\}$	$q^3(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{10,16}}$	$\chi_{b_4,b_8}^{a_6,a_{10},a_{16}}$	$I_{\mathcal{A}} = \{2, 3, 5, 7\}$	$q^2(q-1)^3$	q^4
		$I_{\mathcal{L}} = \{1, 9, 12, 13\}$		
	$\chi_{b_1}^{a_8,a_{10},a_{16}}$	$I_{\mathcal{A}} = \{2, 4, 5, 7\}$	$q(q-1)^3$	q^4
		$I_{\mathcal{L}} = \{3, 9, 12, 13\}$		
	$\chi_{b_1,b_3,b_4}^{a_{10},a_{16}}$	$I_{\mathcal{A}} = \{2, 5, 7\}, I_{\mathcal{L}} = \{9, 12, 13\}$	$q^3(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{11,12}}$	$\chi^{a_7,a_{11},a_{12}}$	$I_{\mathcal{A}} = \{1, 2, 6\}, I_{\mathcal{L}} = \{3, 4, 8\}$	$(q-1)^3$	q^3
	$\chi_{b_2,b_3}^{a_{11},a_{12}}$	$I_{\mathcal{A}} = \{1, 6\}, I_{\mathcal{L}} = \{4, 8\}$	$q^2(q-1)^2$	q^2
$\mathcal{F}_{\mathbf{11,13}}$	$\chi_{b_5}^{a_{11},a_{13}}$	$I_{\mathcal{A}} = \{3, 4, 8\}, I_{\mathcal{L}} = \{1, 6, 9\}$	$q(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{11,16}}$	$\chi_{b_3}^{a_{11},a_{16}}$	$I_{\mathcal{A}} = \{1, 2, 4, 5, 9\}$	$q(q-1)^2$	q^5
		$I_{\mathcal{L}} = \{6, 7, 8, 12, 13\}$		
$\mathcal{F}_{\mathbf{12,13}}$	$\chi_{b_5}^{a_{12},a_{13}}$	$I_{\mathcal{A}} = \{3, 4, 8\}, I_{\mathcal{L}} = \{2, 7, 9\}$	$q(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{12,15}}$	$\chi_{b_3}^{a_{12},a_{15}}$	$I_{\mathcal{A}} = \{1, 2, 4, 5, 9\}$	$q(q-1)^2$	q^5
		$I_{\mathcal{L}} = \{6, 7, 8, 11, 13\}$		
$\mathcal{F}_{\mathbf{13,14}}$	$\chi_{b_6,b_7}^{a_{13},a_{14}}$	$I_{\mathcal{A}} = \{1, 2, 3, 4, 5\}$	$q^2(q-1)^2$	q^5
		$I_{\mathcal{L}} = \{8, 9, 10, 11, 12\}$		
$\mathcal{F}_{\mathbf{13,17}}$	$\chi_{b_1,b_2}^{a_9,a_{13},a_{17}}$	$I_{\mathcal{A}} = \{3, 5, 6, 7, 8\}$	$q^2(q-1)^3$	q^5
		$I_{\mathcal{L}} = \{4, 10, 11, 12, 14\}$		
	$\chi_{b_1,b_2,b_4,b_5}^{a_{13},a_{17}}$	$I_{\mathcal{A}} = \{3, 6, 7, 8\}$	$q^4(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{10, 11, 12, 14\}$		
$\mathcal{F}_{\mathbf{14,15}}$	$\chi_{b_3,b_7}^{a_{13},a_{14},a_{15}}$	$I_{\mathcal{A}} = \{1, 2, 4, 5, 9\}$	$q^2(q-1)^3$	q^5
		$I_{\mathcal{L}} = \{6, 8, 10, 11, 12\}$		
	$\chi_{b_8}^{a_7,a_{14},a_{15}}$	$I_{\mathcal{A}} = \{1, 3, 4, 9, 11\}$	$q(q-1)^3$	q^5
		$I_{\mathcal{L}} = \{2, 5, 6, 10, 12\}$		
	$\chi_{b_2,b_3,b_8}^{a_{14},a_{15}}$	$I_{\mathcal{A}} = \{1, 4, 9, 11\}$	$q^3(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{5, 6, 10, 12\}$		
$\mathcal{F}_{\mathbf{14,16}}$	$\chi_{b_3,b_6}^{a_{13},a_{14},a_{16}}$	$I_{\mathcal{A}} = \{1, 2, 4, 5, 9\}$	$q^2(q-1)^3$	q^5
		$I_{\mathcal{L}} = \{7, 8, 10, 11, 12\}$		
	$\chi_{b_8}^{a_6,a_{14},a_{16}}$	$I_{\mathcal{A}} = \{2, 3, 4, 9, 12\}$	$q(q-1)^3$	q^5
		$I_{\mathcal{L}} = \{1, 5, 7, 10, 11\}$		
	$\chi_{b_1,b_3,b_8}^{a_{14},a_{16}}$	$I_{\mathcal{A}} = \{2, 4, 9, 12\}$	$q^3(q-1)^2$	q^4
		$I_{\mathcal{L}} = \{5, 7, 10, 11\}$		
$\mathcal{F}_{\mathbf{15,16}}$	$\chi_{b_3}^{a_{12},a_{15},a_{16}}$	$I_{\mathcal{A}} = \{1, 6, 7, 8, 11\}$	$q(q-1)^3$	q^5
		$I_{\mathcal{L}} = \{2, 4, 5, 9, 13\}$		
	$\chi_{b_4,b_8}^{a_7,a_{15},a_{16}}$	$I_{\mathcal{A}} = \{1, 2, 6, 11\}$	$q^2(q-1)^3$	q^4
		$I_{\mathcal{L}} = \{3, 5, 9, 13\}$		
	$\chi_{b_2}^{a_8,a_{15},a_{16}}$	$I_{\mathcal{A}} = \{1, 3, 6, 11\}$	$q(q-1)^3$	q^4
		$I_{\mathcal{L}} = \{4, 5, 9, 13\}$		
	$\chi_{b_2,b_3,b_4}^{a_{15},a_{16}}$	$I_{\mathcal{A}} = \{1, 6, 11\}, I_{\mathcal{L}} = \{5, 9, 13\}$	$q^3(q-1)^2$	q^3
$\mathcal{F}_{\mathbf{15,17}}$	$\chi_{b_2}^{a_9,a_{15},a_{17}}$	$I_{\mathcal{A}} = \{1, 3, 5, 6, 8, 11\}$	$q(q-1)^3$	q^6
		$I_{\mathcal{L}} = \{4, 7, 10, 12, 13, 14\}$		
	$\chi_{b_2,b_4,b_5}^{a_{15},a_{17}}$	$I_{\mathcal{A}} = \{1, 3, 6, 8, 11\}$	$q^3(q-1)^2$	q^5
		$I_{\mathcal{L}} = \{7, 10, 12, 13, 14\}$		
$\mathcal{F}_{\mathbf{16,17}}$	$\chi_{b_1}^{a_9,a_{16},a_{17}}$	$I_{\mathcal{A}} = \{2, 3, 5, 7, 8, 12\}$	$q(q-1)^3$	q^6
		$I_{\mathcal{L}} = \{4, 6, 10, 11, 13, 14\}$		

$\mathcal{F}$	χ	I	Number	Degree
	$\chi_{b_1,b_4,b_5}^{a_{16},a_{17}}$	$I_{\mathcal{A}} = \{2,3,7,8,12\}$	$q^3(q-1)^2$	q^5
		$I_{\mathcal{L}} = \{6,10,11,13,14\}$		
$\mathcal{F}_{\mathbf{17,18}}^{\boldsymbol{p\geq 3}}$	$\chi^{a_9,a_{17},a_{18}}$	$I_{\mathcal{A}} = \{1,2,5,6,7,10,14\}$	$(q-1)^3$	q^7
		$I_{\mathcal{L}} = \{3,4,8,11,12,15,16\}$		
	$\chi_{b_4,b_5}^{a_{17},a_{18}}$	$I_{\mathcal{A}} = \{1,2,6,7,10,14\}$	$q^2(q-1)^2$	q^6
		$I_{\mathcal{L}} = \{3,8,11,12,15,16\}$		
	$\boldsymbol{\chi_{b_4}^{a_{13},a_{17},a_{18}}}$	$\boldsymbol{I_{\mathcal{A}} = \{1,2,6,7\},\ I_{\mathcal{I}} = \{3,5,10\}}$	$\boldsymbol{q(q-1)^3}$	$\boldsymbol{q^7}$
		$\boldsymbol{I_{\mathcal{L}} = \{11,12,15,16\},\ I_{\mathcal{J}} = \{8,9,14\}}$		
$\mathcal{F}_{\mathbf{17,18}}^{\boldsymbol{p=2}}$	$\chi^{a_9,a_{17},a_{18}}$	$I_{\mathcal{A}} = \{1,2,5,6,7,10,14\}$	$(q-1)^3$	q^7
		$I_{\mathcal{L}} = \{3,4,8,11,12,15,16\}$		
	$\chi_{b_4,b_5}^{a_{17},a_{18}}$	$I_{\mathcal{A}} = \{1,2,6,7,10,14\}$	$q^2(q-1)^2$	q^6
		$I_{\mathcal{L}} = \{3,8,11,12,15,16\}$		
	$\boldsymbol{\chi^{a_{13},a_{17},a_{18}}}$	**See $\mathfrak{C}_1^{D_5}$ in Section 5.4**	$\boldsymbol{(q-1)^3}$	$\boldsymbol{q^7}$
	$\boldsymbol{\chi_{c_4,c_{3,5,10}}^{a_{13},a_{17},a_{18},a_{8,9,14}}}$	**See $\mathfrak{C}_1^{D_5}$ in Section 5.4**	$\boldsymbol{4(q-1)^4}$	$\boldsymbol{q^7/2}$
$\mathcal{F}_{\mathbf{1,2,8}}$	χ^{a_1,a_2,a_8}	$I_{\mathcal{A}} = \{3\}, I_{\mathcal{L}} = \{4\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{1,2,9}}$	χ^{a_1,a_2,a_9}	$I_{\mathcal{A}} = \{4\}, I_{\mathcal{L}} = \{5\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{1,2,13}}$	$\chi_{b_4}^{a_1,a_2,a_{13}}$	$I_{\mathcal{A}} = \{3,5\}, I_{\mathcal{L}} = \{8,9\}$	$q(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{1,3,9}}$	χ^{a_1,a_3,a_9}	$I_{\mathcal{A}} = \{4\}, I_{\mathcal{L}} = \{5\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{1,4,7}}$	χ^{a_1,a_4,a_7}	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{3\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{1,5,7}}$	χ^{a_1,a_5,a_7}	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{3\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{1,5,8}}$	χ^{a_1,a_5,a_8}	$I_{\mathcal{A}} = \{3\}, I_{\mathcal{L}} = \{4\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{1,5,12}}$	$\chi_{b_3}^{a_1,a_5,a_{12}}$	$I_{\mathcal{A}} = \{2,4\}, I_{\mathcal{L}} = \{7,8\}$	$q(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{1,7,8}}$	$\chi_{b_4}^{a_1,a_7,a_8}$	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{3\}$	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{1,7,9}}$	χ^{a_1,a_7,a_9}	$I_{\mathcal{A}} = \{2,4\}, I_{\mathcal{L}} = \{3,5\}$	$(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{1,7,13}}$	$\chi_{b_2,b_4}^{a_1,a_7,a_{13}}$	$I_{\mathcal{A}} = \{3,5\}, I_{\mathcal{L}} = \{8,9\}$	$q^2(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{1,8,9}}$	$\chi_{b_5}^{a_1,a_8,a_9}$	$I_{\mathcal{A}} = \{3\}, I_{\mathcal{L}} = \{4\}$	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{1,9,12}}$	$\chi_{b_3,b_5}^{a_1,a_9,a_{12}}$	$I_{\mathcal{A}} = \{2,4\}, I_{\mathcal{L}} = \{7,8\}$	$q^2(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{1,12,13}}$	$\chi_{b_5}^{a_1,a_{12},a_{13}}$	$I_{\mathcal{A}} = \{3,4,8\}, I_{\mathcal{L}} = \{2,7,9\}$	$q(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{2,3,9}}$	χ^{a_2,a_3,a_9}	$I_{\mathcal{A}} = \{4\}, I_{\mathcal{L}} = \{5\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{2,4,6}}$	χ^{a_2,a_4,a_6}	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{3\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{2,5,6}}$	χ^{a_2,a_5,a_6}	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{3\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{2,5,8}}$	χ^{a_2,a_5,a_8}	$I_{\mathcal{A}} = \{3\}, I_{\mathcal{L}} = \{4\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{2,5,11}}$	$\chi_{b_3}^{a_2,a_5,a_{11}}$	$I_{\mathcal{A}} = \{1,4\}, I_{\mathcal{L}} = \{6,8\}$	$q(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{2,6,8}}$	$\chi_{b_4}^{a_2,a_6,a_8}$	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{3\}$	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{2,6,9}}$	χ^{a_2,a_6,a_9}	$I_{\mathcal{A}} = \{1,4\}, I_{\mathcal{L}} = \{3,5\}$	$(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{2,6,13}}$	$\chi_{b_1,b_4}^{a_2,a_6,a_{13}}$	$I_{\mathcal{A}} = \{3,5\}, I_{\mathcal{L}} = \{8,9\}$	$q^2(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{2,8,9}}$	$\chi_{b_5}^{a_2,a_8,a_9}$	$I_{\mathcal{A}} = \{3\}, I_{\mathcal{L}} = \{4\}$	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{2,9,11}}$	$\chi_{b_3,b_5}^{a_2,a_9,a_{11}}$	$I_{\mathcal{A}} = \{1,4\}, I_{\mathcal{L}} = \{6,8\}$	$q^2(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{2,11,13}}$	$\chi_{b_5}^{a_2,a_{11},a_{13}}$	$I_{\mathcal{A}} = \{3,4,8\}, I_{\mathcal{L}} = \{1,6,9\}$	$q(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{4,5,6}}$	χ^{a_4,a_5,a_6}	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{3\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{4,5,7}}$	χ^{a_4,a_5,a_7}	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{3\}$	$(q-1)^3$	q
$\mathcal{F}_{\mathbf{4,5,10}}$	$\chi_{b_3}^{a_4,a_5,a_{10}}$	$I_{\mathcal{A}} = \{1,2\}, I_{\mathcal{L}} = \{6,7\}$	$q(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{4,6,7}}$	$\chi_{b_2}^{a_4,a_6,a_7}$	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{3\}$	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{5,6,7}}$	$\chi_{b_2}^{a_5,a_6,a_7}$	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{3\}$	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{5,6,8}}$	$\chi_{b_4}^{a_5,a_6,a_8}$	$I_{\mathcal{A}} = \{1\}, I_{\mathcal{L}} = \{3\}$	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{5,6,12}}$	$\chi^{a_5,a_6,a_{12}}$	$I_{\mathcal{A}} = \{2,3,4\}, I_{\mathcal{L}} = \{1,7,8\}$	$(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{5,7,8}}$	$\chi_{b_4}^{a_5,a_7,a_8}$	$I_{\mathcal{A}} = \{2\}, I_{\mathcal{L}} = \{3\}$	$q(q-1)^3$	q
$\mathcal{F}_{\mathbf{5,7,11}}$	$\chi^{a_5,a_7,a_{11}}$	$I_{\mathcal{A}} = \{1,3,4\}, I_{\mathcal{L}} = \{2,6,8\}$	$(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{5,8,10}}$	$\chi^{a_5,a_8,a_{10}}$	$I_{\mathcal{A}} = \{1,2,4\}, I_{\mathcal{L}} = \{3,6,7\}$	$(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{5,10,11}}$	$\chi^{a_5,a_8,a_{10},a_{11}}$	$I_{\mathcal{A}} = \{1,4,6\}, I_{\mathcal{L}} = \{2,3,7\}$	$(q-1)^4$	q^3

$\mathcal{F}$	χ	I	Number	Degree
	$\chi_{b_3,b_4}^{a_5,a_{10},a_{11}}$	$I_{\mathcal{A}}=\{1,6\}, I_{\mathcal{L}}=\{2,7\}$	$q^2(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{5,10,12}}$	$\chi^{a_5,a_8,a_{10},a_{12}}$	$I_{\mathcal{A}}=\{1,3,6\}, I_{\mathcal{L}}=\{2,4,7\}$	$(q-1)^4$	q^3
	$\chi_{b_3,b_4}^{a_5,a_{10},a_{12}}$	$I_{\mathcal{A}}=\{1,6\}, I_{\mathcal{L}}=\{2,7\}$	$q^2(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{5,11,12}}$	$\chi^{a_5,a_7,a_{11},a_{12}}$	$I_{\mathcal{A}}=\{1,2,6\}, I_{\mathcal{L}}=\{3,4,8\}$	$(q-1)^4$	q^3
	$\chi_{b_2,b_3}^{a_5,a_{11},a_{12}}$	$I_{\mathcal{A}}=\{1,6\}, I_{\mathcal{L}}=\{4,8\}$	$q^2(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{6,7,8}}$	$\chi_{b_2,b_4}^{a_6,a_7,a_8}$	$I_{\mathcal{A}}=\{1\}, I_{\mathcal{L}}=\{3\}$	$q^2(q-1)^3$	q
$\mathcal{F}_{\mathbf{6,7,9}}$	$\chi_{b_2}^{a_6,a_7,a_9}$	$I_{\mathcal{A}}=\{1,4\}, I_{\mathcal{L}}=\{3,5\}$	$q(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{6,7,13}}$	$\chi_{b_1,b_2,b_4}^{a_6,a_7,a_{13}}$	$I_{\mathcal{A}}=\{3,5\}, I_{\mathcal{L}}=\{8,9\}$	$q^3(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{6,8,9}}$	χ^{a_6,a_8,a_9}	$I_{\mathcal{A}}=\{1,4\}, I_{\mathcal{L}}=\{3,5\}$	$(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{6,9,12}}$	$\chi_{b_5}^{a_6,a_9,a_{12}}$	$I_{\mathcal{A}}=\{2,3,4\}, I_{\mathcal{L}}=\{1,7,8\}$	$q(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{6,12,13}}$	$\chi_{b_1,b_5}^{a_6,a_{12},a_{13}}$	$I_{\mathcal{A}}=\{3,4,8\}, I_{\mathcal{L}}=\{2,7,9\}$	$q^2(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{7,8,9}}$	χ^{a_7,a_8,a_9}	$I_{\mathcal{A}}=\{2,4\}, I_{\mathcal{L}}=\{3,5\}$	$(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{7,9,11}}$	$\chi_{b_5}^{a_7,a_9,a_{11}}$	$I_{\mathcal{A}}=\{1,3,4\}, I_{\mathcal{L}}=\{2,6,8\}$	$q(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{7,11,13}}$	$\chi_{b_2,b_5}^{a_7,a_{11},a_{13}}$	$I_{\mathcal{A}}=\{3,4,8\}, I_{\mathcal{L}}=\{1,6,9\}$	$q^2(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{8,9,10}}$	$\chi_{b_5}^{a_8,a_9,a_{10}}$	$I_{\mathcal{A}}=\{1,2,4\}, I_{\mathcal{L}}=\{3,6,7\}$	$q(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{9,10,11}}$	$\chi_{b_3,b_8}^{a_9,a_{10},a_{11}}$	$I_{\mathcal{A}}=\{1,4,6\}, I_{\mathcal{L}}=\{2,5,7\}$	$q^2(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{9,10,12}}$	$\chi_{b_3,b_8}^{a_9,a_{10},a_{12}}$	$I_{\mathcal{A}}=\{2,4,7\}, I_{\mathcal{L}}=\{1,5,6\}$	$q^2(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{9,11,12}}$	$\chi_{b_5}^{a_7,a_9,a_{11},a_{12}}$	$I_{\mathcal{A}}=\{1,2,6\}, I_{\mathcal{L}}=\{3,4,8\}$	$q(q-1)^4$	q^3
	$\chi_{b_2,b_3,b_5}^{a_9,a_{11},a_{12}}$	$I_{\mathcal{A}}=\{1,6\}, I_{\mathcal{L}}=\{4,8\}$	$q^3(q-1)^3$	q^2
$\mathcal{F}_{\mathbf{10,11,12}}^{\boldsymbol{p\geq 3}}$	$\boldsymbol{\chi_{b_3}^{a_{10},a_{11},a_{12}}}$	$\boldsymbol{I_{\mathcal{I}}=\{1,2,4\}, I_{\mathcal{J}}=\{6,7,8\}}$	$\boldsymbol{q(q-1)^3}$	$\boldsymbol{q^3}$
$\mathcal{F}_{\mathbf{10,11,12}}^{\boldsymbol{p=2}}$	$\boldsymbol{\chi^{a_{10},a_{11},a_{12}}}$	**See $\mathfrak{C}_2^{\mathrm{D}_5}$ in Section 5.4**	$\boldsymbol{(q-1)^3}$	$\boldsymbol{q^3}$
	$\boldsymbol{\chi_{c_3,c_{1,2,4}}^{a_{10},a_{11},a_{12},a_{6,7,8}}}$	**See $\mathfrak{C}_2^{\mathrm{D}_5}$ in Section 5.4**	$\boldsymbol{4(q-1)^4}$	$\boldsymbol{q^3/2}$
$\mathcal{F}_{\mathbf{10,11,13}}$	$\chi_{b_4}^{a_{10},a_{11},a_{13}}$	$I_{\mathcal{A}}=\{1,3,5,6\}$	$q(q-1)^3$	q^4
		$I_{\mathcal{L}}=\{2,7,8,9\}$		
$\mathcal{F}_{\mathbf{10,11,16}}$	$\chi_{b_3}^{a_{10},a_{11},a_{16}}$	$I_{\mathcal{A}}=\{1,2,4,5,7\}$	$q(q-1)^3$	q^5
		$I_{\mathcal{L}}=\{6,8,9,12,13\}$		
$\mathcal{F}_{\mathbf{10,12,13}}$	$\chi_{b_4}^{a_{10},a_{12},a_{13}}$	$I_{\mathcal{A}}=\{2,3,5,7\}$	$q(q-1)^3$	q^4
		$I_{\mathcal{L}}=\{1,6,8,9\}$		
$\mathcal{F}_{\mathbf{10,12,15}}$	$\chi_{b_3}^{a_{10},a_{12},a_{15}}$	$I_{\mathcal{A}}=\{1,2,4,5,6\}$	$q(q-1)^3$	q^5
		$I_{\mathcal{L}}=\{7,8,9,11,13\}$		
$\mathcal{F}_{\mathbf{10,15,16}}^{\boldsymbol{p\geq 3}}$	$\chi_{b_3}^{a_{10},a_{12},a_{15},a_{16}}$	$I_{\mathcal{A}}=\{1,2,4,5,9\}$	$q(q-1)^4$	q^5
		$I_{\mathcal{L}}=\{6,7,8,11,13\}$		
	$\boldsymbol{\chi^{a_8,a_{10},a_{15},a_{16}}}$	$\boldsymbol{I_{\mathcal{A}}=\{3,5\},\ I_{\mathcal{I}}=\{1,2,9\}}$	$\boldsymbol{(q-1)^4}$	$\boldsymbol{q^5}$
		$\boldsymbol{I_{\mathcal{L}}=\{4,11\},\ I_{\mathcal{J}}=\{6,7,13\}}$		
	$\boldsymbol{\chi_{b_3,b_4}^{a_{10},a_{15},a_{16}}}$	$\boldsymbol{I_{\mathcal{A}}=\{5\},\ I_{\mathcal{I}}=\{1,2,9\}}$	$\boldsymbol{q^2(q-1)^3}$	$\boldsymbol{q^4}$
		$\boldsymbol{I_{\mathcal{L}}=\{11\},\ I_{\mathcal{J}}=\{6,7,13\}}$		
$\mathcal{F}_{\mathbf{10,15,16}}^{\boldsymbol{p=2}}$	$\chi_{b_3}^{a_{10},a_{12},a_{15},a_{16}}$	$I_{\mathcal{A}}=\{1,2,4,5,9\}$	$q(q-1)^4$	q^5
		$I_{\mathcal{L}}=\{6,7,8,11,13\}$		
	$\boldsymbol{\chi_{b_{1,2,9},b_{6,7,13}}^{a_8,a_{10},a_{15},a_{16}}}$	**See $\mathfrak{C}_0^{\mathrm{D}_5}$ in Section 5.4**	$\boldsymbol{q^2(q-1)^4}$	$\boldsymbol{q^4}$
	$\boldsymbol{\chi_{b_4}^{a_{10},a_{15},a_{16}}}$	**See $\mathfrak{C}_3^{\mathrm{D}_5}$ in Section 5.4**	$\boldsymbol{q(q-1)^3}$	$\boldsymbol{q^4}$
	$\boldsymbol{\chi_{b_4,c_3,c_{1,2,9}}^{a_{10},a_{15},a_{16},a_{6,7,13}}}$	**See $\mathfrak{C}_3^{\mathrm{D}_5}$ in Section 5.4**	$\boldsymbol{4q(q-1)^4}$	$\boldsymbol{q^4/2}$
$\mathcal{F}_{\mathbf{11,12,13}}$	$\chi_{b_2,b_5,b_7}^{a_{11},a_{12},a_{13}}$	$I_{\mathcal{A}}=\{3,4,8\}, I_{\mathcal{L}}=\{1,6,9\}$	$q^3(q-1)^3$	q^3
$\mathcal{F}_{\mathbf{14,15,16}}^{\boldsymbol{p\geq 3}}$	$\chi_{b_8,b_{13}}^{a_7,a_{14},a_{15},a_{16}}$	$I_{\mathcal{A}}=\{1,2,4,5,9\}$	$q^2(q-1)^4$	q^5
		$I_{\mathcal{L}}=\{3,6,10,11,12\}$		
	$\boldsymbol{\chi_{b_3,b_8}^{a_{14},a_{15},a_{16}}}$	$\boldsymbol{I_{\mathcal{A}}=\{4,9\},\ I_{\mathcal{I}}=\{1,2,5\}}$	$\boldsymbol{q^2(q-1)^3}$	$\boldsymbol{q^5}$
		$\boldsymbol{I_{\mathcal{L}}=\{6,10\},\ I_{\mathcal{J}}=\{11,12,13\}}$		
$\mathcal{F}_{\mathbf{14,15,16}}^{\boldsymbol{p=2}}$	$\chi_{b_8,b_{13}}^{a_7,a_{14},a_{15},a_{16}}$	$I_{\mathcal{A}}=\{1,2,4,5,9\}$	$q^2(q-1)^4$	q^5
		$I_{\mathcal{L}}=\{3,6,10,11,12\}$		
	$\boldsymbol{\chi_{b_3}^{a_{14},a_{15},a_{16}}}$	**See $\mathfrak{C}_4^{\mathrm{D}_5}$ in Section 5.4**	$\boldsymbol{q(q-1)^3}$	$\boldsymbol{q^5}$

$\mathcal{F}$	χ	I	Number	Degree
	$\boldsymbol{\chi_{b_3,c_8,c_{1,2,5}}^{a_{14},a_{15},a_{16},a_{11,12,13}}}$	**See $\mathfrak{c}_4^{\mathrm{D}5}$ in Section 5.4**	$\boldsymbol{4q(q-1)^4}$	$\boldsymbol{q^5/2}$
$\mathcal{F}_{15,16,17}^{p\geq3}$	$\boldsymbol{\chi_{b_2,b_4}^{a_{15},a_{16},a_{17}}}$	$\boldsymbol{I_{\mathcal{A}}=\{3,8,13\},\ I_{\mathcal{I}}=\{5,6,7\}}$ $\boldsymbol{I_{\mathcal{L}}=\{1,10,14\},\ I_{\mathcal{J}}=\{9,11,12\}}$	$\boldsymbol{q^2(q-1)^3}$	$\boldsymbol{q^6}$
$\mathcal{F}_{15,16,17}^{p=2}$	$\boldsymbol{\chi_{b_2}^{a_{15},a_{16},a_{17}}}$ $\boldsymbol{\chi_{b_2,c_4,c_{5,6,7}}^{a_{15},a_{16},a_{17},a_{9,11,12}}}$	**See $\mathfrak{c}_5^{\mathrm{D}5}$ in Section 5.4** **See $\mathfrak{c}_5^{\mathrm{D}5}$ in Section 5.4**	$\boldsymbol{q(q-1)^3}$ $\boldsymbol{4q(q-1)^4}$	$\boldsymbol{q^6}$ $\boldsymbol{q^6/2}$
$\mathcal{F}_{1,2,3,9}$	χ^{a_1,a_2,a_3,a_9}	$I_{\mathcal{A}}=\{4\}, I_{\mathcal{L}}=\{5\}$	$(q-1)^4$	q
$\mathcal{F}_{1,2,5,8}$	χ^{a_1,a_2,a_5,a_8}	$I_{\mathcal{A}}=\{3\}, I_{\mathcal{L}}=\{4\}$	$(q-1)^4$	q
$\mathcal{F}_{1,2,8,9}$	$\chi_{b_5}^{a_1,a_2,a_8,a_9}$	$I_{\mathcal{A}}=\{3\}, I_{\mathcal{L}}=\{4\}$	$q(q-1)^4$	q
$\mathcal{F}_{1,4,5,7}$	χ^{a_1,a_4,a_5,a_7}	$I_{\mathcal{A}}=\{2\}, I_{\mathcal{L}}=\{3\}$	$(q-1)^4$	q
$\mathcal{F}_{1,5,7,8}$	$\chi_{b_4}^{a_1,a_5,a_7,a_8}$	$I_{\mathcal{A}}=\{2\}, I_{\mathcal{L}}=\{3\}$	$q(q-1)^4$	q
$\mathcal{F}_{1,7,8,9}$	χ^{a_1,a_7,a_8,a_9}	$I_{\mathcal{A}}=\{2,4\}, I_{\mathcal{L}}=\{3,5\}$	$(q-1)^4$	q^2
$\mathcal{F}_{2,4,5,6}$	χ^{a_2,a_4,a_5,a_6}	$I_{\mathcal{A}}=\{1\}, I_{\mathcal{L}}=\{3\}$	$(q-1)^4$	q
$\mathcal{F}_{2,5,6,8}$	$\chi_{b_4}^{a_2,a_5,a_6,a_8}$	$I_{\mathcal{A}}=\{1\}, I_{\mathcal{L}}=\{3\}$	$q(q-1)^4$	q
$\mathcal{F}_{2,6,8,9}$	χ^{a_2,a_6,a_8,a_9}	$I_{\mathcal{A}}=\{1,4\}, I_{\mathcal{L}}=\{3,5\}$	$(q-1)^4$	q^2
$\mathcal{F}_{4,5,6,7}$	$\chi_{b_2}^{a_4,a_5,a_6,a_7}$	$I_{\mathcal{A}}=\{1\}, I_{\mathcal{L}}=\{3\}$	$q(q-1)^4$	q
$\mathcal{F}_{5,6,7,8}$	$\chi_{b_2,b_4}^{a_5,a_6,a_7,a_8}$	$I_{\mathcal{A}}=\{1\}, I_{\mathcal{L}}=\{3\}$	$q^2(q-1)^4$	q
$\mathcal{F}_{5,10,11,12}^{p\geq3}$	$\boldsymbol{\chi_{b_3}^{a_5,a_{10},a_{11},a_{12}}}$	$\boldsymbol{I_{\mathcal{I}}=\{1,2,4\}, I_{\mathcal{J}}=\{6,7,8\}}$	$\boldsymbol{q(q-1)^4}$	$\boldsymbol{q^3}$
$\mathcal{F}_{5,10,11,12}^{p=2}$	$\boldsymbol{\chi^{a_5,a_{10},a_{11},a_{12}}}$ $\boldsymbol{\chi_{c_3,c_{1,2,4}}^{a_5,a_{10},a_{11},a_{12},a_{6,7,8}}}$	**See $\mathfrak{c}_6^{\mathrm{D}5}$ in Section 5.4** **See $\mathfrak{c}_6^{\mathrm{D}5}$ in Section 5.4**	$\boldsymbol{(q-1)^4}$ $\boldsymbol{4(q-1)^5}$	$\boldsymbol{q^3}$ $\boldsymbol{q^3/2}$
$\mathcal{F}_{6,7,8,9}$	$\chi_{b_2}^{a_6,a_7,a_8,a_9}$	$I_{\mathcal{A}}=\{1,4\}, I_{\mathcal{L}}=\{3,5\}$	$q(q-1)^4$	q^2
$\mathcal{F}_{9,10,11,12}$	$\chi_{b_3,b_8}^{a_9,a_{10},a_{11},a_{12}}$	$I_{\mathcal{A}}=\{1,2,4\}, I_{\mathcal{L}}=\{5,6,7\}$	$q^2(q-1)^4$	q^3
$\mathcal{F}_{10,11,12,13}$	$\chi_{b_4}^{a_{10},a_{11},a_{12},a_{13}}$	$I_{\mathcal{A}}=\{3,6,7,8\}$ $I_{\mathcal{L}}=\{1,2,5,9\}$	$q(q-1)^4$	q^4

Table D.7: The parametrization of the irreducible characters of $\mathrm{UD}_5(q)$ for every $q=p^e$.

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