THE RANDOM CONTINUED FRACTION TRANSFORMATION

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ABSTRACT. We introduce a random dynamical system related to continued fraction expansions. It uses random combination of the Gauss map and the Rényi (or backwards) continued fraction map. We explore the continued fraction expansions that this system produces as well as the dynamical properties of the system.

1. Introduction

In 1913 ([Per50]) Perron described an algorithm producing finite and infinite continued fraction expansions of real numbers of the form

$$x = d_0 + \cfrac{\epsilon_0}{d_1 + \cfrac{\epsilon_1}{d_2 + \ddots + \cfrac{\epsilon_{n-1}}{d_n + \ddots}}},$$

where $d_0 \in \mathbb{Z}$ and for each $n \geq 1$, $\epsilon_{n-1} \in \{-1, 1\}$, $d_n \in \mathbb{N}$ and $d_n + \epsilon_n \geq 1$. Moreover, in case the continued fraction is infinite, the algorithm guarantees that $d_n + \epsilon_{n+1} \geq 1$ infinitely often. Perron called these expansions semi-regular continued fractions.

Within this framework one can see a number of more familiar systems of continued fractions, each of which can be studied by a corresponding dynamical system. Regular continued fractions, which correspond to letting each $\epsilon_n = 1$, are generated by the Gauss map $Tx = \frac{1}{x} \pmod 1$. Backwards continued fractions, which were introduced by Rényi in [Rén57], correspond to $\epsilon_n = -1$. Odd and even continued fractions ([HK02]) and α-continued fractions ([Nak81]) can also be seen within this framework. In this article we define a random dynamical system which allows one to generate all semi-regular continued fraction expansions of the form (1) for any given x and to study their dynamical and ergodic properties. We do not require the condition that $d_n + \epsilon_{n+1} \geq 1$ infinitely often, which makes our set-up slightly more general.

Over the last decade there has been a great deal of work on random dynamical systems. A random system is given by a finite family of maps defined on the same state space and a probabilistic regime for choosing one of these maps at each time step. The study of conditions that guarantee the existence of an invariant measure for such systems was initiated by Morita ([Mor85]) and Pelikan ([Pel84]). They consider the case where each map in the random system is piecewise smooth with respect to some finite partition and expanding on average. In [GB03] and [BG05] these results are extended to when the probabilistic regime is position dependent. These results were further generalised by Inoue ([Ino12]) to more general underlying partitions, including ones with countably many elements. In [ANV] limit theorems are studied for random systems consisting of countably many maps. Their examples include families of maps that are piecewise smooth with respect to some finite partition and are expanding on average. Rousseau and Todd studied hitting time statistics for random maps [RT15].

Random dynamical systems have also been used in relation to representations of numbers, see for example [Mor]. In particular, the introduction of the random β-transformation has opened up new approaches to studying the dynamical and ergodic properties of β-expansions and Bernoulli convolutions, see [DdV05, DK13, Kem13]. A key part of this approach was to study the invariant measures and ergodic properties of the random β-transformation, as was done in [DdV05, DdV07, Kem14].

In this article we first introduce the random continued fraction map K. We show that it generates convergent continued fraction expansions for all points in its domain and we explore some of the properties of these expansions. The map K, and a related map R which is easier to analyse, present several challenges which are interesting from a purely dynamical point of view. In particular, R has countably many discontinuities and one of the interval maps defining R has an indifferent fixed point. To overcome these difficultieswe build on the work of Inoue [Ino12], who studied transfer operators for countably branched skew product systems that are expanding on average. We can then use the results from [ANV] to obtain limit theorems. Other specific examples of random intermittent systems have been recently considered in [BBD14]. Here Bahsoun, Bose and Duan studied limit theorems and mixing rates for random combinations of maps that are variations of the Manneville-Pomeau map.

The paper is outlined as follows. In Section 2 we first show that every expansion of the form (1) is generated by K, and study further dynamical properties of random continued fractions. In Section 3 we prove that there exists an absolutely continuous invariant measure for R. The density of this measure is of bounded variation. A key part of our approach here is to study the transfer operator associated with R. We show that R satisfies conditions of Inoue [Ino12] which gives that the transfer operator is quasi-compact. In Section 4 we use this to show that the invariant measure is fully supported. We can then employ results from [ANV] to obtain that R is mixing and that the Central Limit Theorem and Large Deviation Principle hold. Numerical evidence seems to suggest that the density in

fact is quite smooth. In the last section we discuss this in more detail and also mention some open questions and future directions.

2. The random map

2.1. **Definition of the map.** It is clear that $x \in \mathbb{R}$ has an expansion of the form

$$
(1) \qquad x = \cfrac{\epsilon_0}{d_1 + \cfrac{\epsilon_1}{d_2 + \cdots + \cfrac{\epsilon_{n-1}}{d_n + \cdots}}},
$$

where $\epsilon_{n-1} \in \{-1, 1\}$, $d_n \in \mathbb{N}$ and $d_n + \epsilon_n \geq 1$ for all $n \in \mathbb{N}$ if and only if $x \in [-1, 1] \backslash \{0\}$. If $|x| > 1$, we can subtract a suitable integer d_0 from x and use the representation from (1) for $x - d_0$ to obtain a continued fraction expansion of x.

To find the right definition of the random dynamical system K, we first ask which $x \in [-1, 1]$ have expansions that begin with a given choice of ϵ_0 and d_1. Let ϵ_0 and $d_1 > 1$ be given. Then x can be written in the form (1) if and only if $x = \frac{\epsilon_0}{d_1 + y}$ for some $y \in [-1, 1] \setminus \{0\}$. This can be satisfied if and only if $\epsilon_0 = sgn(x)$ and

$$
|x| = \epsilon_0 x \in \left(\frac{1}{d_1 + 1}, \frac{1}{d_1 - 1} \right).
$$

This typically gives two choices of the digit d_1. We see that

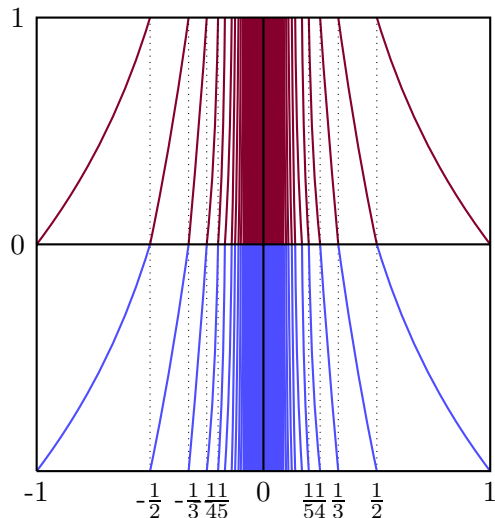$$
y = \frac{\epsilon_0}{x} - d_1 = \frac{1}{|x|} - d_1.
$$

Similarly, if $d_1 = 1$ then we require $\epsilon_1 = 1$ and have $x = \frac{\epsilon_0}{d_1 + y}$ for some $y \in (0, 1]$. This can be satisfied for $|x| \in (\frac{1}{2}, 1]$ and we have

$$
y = \frac{1}{|x|} - 1.
$$

As is standard with dynamical constructions of expansions of real numbers, the possible values of ϵ_1, d_2 are equal to the values of ϵ_0, d_1 associated with $y = \frac{1}{|x|} - d_1$. Thus we can generate all expansions of the form (1) using the following transformation.

Let $\Omega = \{(\omega_k)_{k \geq 1} : \omega_k \in \{0, 1\}\} = \{0, 1\}^{\mathbb{N}}$ and let $\sigma : \Omega \to \Omega$ be the left shift. We define the random continued fraction map $K : \Omega \times [-1, 1] \to \Omega \times [-1, 1]$ by setting $K(\omega, 0) = (\sigma(\omega), 0)$ and for $|x| \in \left(\frac{1}{k+1}, \frac{1}{k} \right]$,

$$
K(\omega, x) = \left(\sigma(\omega), \left| \frac{1}{x} \right| - (k + \omega_1) \right).
$$

FIGURE 1. The map T_0 in red and T_1 in blue.

One can think of the map K as follows. Let maps $T_0, T_1 : [0, 1] \to [0, 1]$ be the Gauss and Rényi maps respectively, given by

$$(2) \quad T_0 x = \begin{cases} 0, & \text{if } x = 0, \\ \dfrac{1}{x} \ (\text{mod } 1), & \text{otherwise,} \end{cases} \quad \text{and} \quad T_1 x = \begin{cases} 0, & \text{if } x = 1, \\ \dfrac{1}{1-x} \ (\text{mod } 1), & \text{otherwise.} \end{cases}$$

Let $\pi : \Omega \times [-1, 1] \to [-1, 1]$ be given by $\pi(\omega, x) = x$. Then

$$\pi\big(K(\omega, x)\big) = \begin{cases} T_0 x - \omega_1, & \text{if } x > 0, \\ T_1(x+1) - \omega_1, & \text{if } x < 0. \end{cases}$$

Set $\omega_0 = 0$ if $x \geq 0$ and 1 otherwise. This gives

$$(3) \qquad\qquad \pi\big(K(\omega, x)\big) = T_{\omega_0}(x + \omega_0) - \omega_1.$$

Note that $\pi\big(K(\omega, x)\big) \in [0, 1]$ if $\omega_1 = 0$ and $\pi\big(K(\omega, x)\big) \in [-1, 0]$ if $\omega_1 = 1$. By iterating we see that for each $n \geq 1$ and all $(\omega, x) \in \Omega \times [-1, 1]$, such that $\pi\big(K^m(\omega, x)\big) \neq 0$ for $0 \leq m < n$, we have

$$\pi\big(K^n(\omega, x)\big) = (T_{\omega_{n-1}} \circ \cdots \circ T_{\omega_0})(x + \omega_0) - \omega_n.$$

2.2. **Random continued fraction expansions.** For $n = 1$, set

$$d_1 = d_1(\omega, x) = \begin{cases} k + \omega_1, & \text{if } |x| \in \left(\dfrac{1}{k+1}, \dfrac{1}{k} \right], \\ \infty, & \text{if } x = 0, \end{cases}$$

and for $n \geq 2$, define $d_n(\omega, x) = d_1\big(K^{n-1}(\omega, x)\big)$. We have

$$(4) \qquad \pi\big(K(\omega, x)\big) = (-1)^{\omega_0} \frac{1}{x} - d_1.$$

and for each $n \geq 1$ such that $\pi(K^m(\omega, x)) \neq 0$ for all $0 \leq m \leq n$,

$$x = \cfrac{(-1)^{\omega_0}}{d_1 + \pi(K(\omega, x))} = \cfrac{(-1)^{\omega_0}}{d_1 + \cfrac{(-1)^{\omega_1}}{d_2 + \pi(K(\omega, x))}} = \cdots = \cfrac{(-1)^{\omega_0}}{d_1 + \cfrac{(-1)^{\omega_1}}{d_2 + \cfrac{\ddots}{} + \cfrac{(-1)^{\omega_{n-1}}}{d_n + \pi(K^n(\omega, x))}}}.$$

The digit sequence $\big(d_n(\omega, x)\big)_{n \geq 1}$ represents the continued fraction representation of the pair (ω, x) as given by K. If there is a smallest integer n such that $d_n(\omega, x) = \infty$, then (ω, x) has **finite random continued fraction expansion**

$$x = \cfrac{(-1)^{\omega_0}}{d_1 + \cfrac{(-1)^{\omega_1}}{d_2 + \cfrac{\ddots}{} + \cfrac{(-1)^{\omega_{n-2}}}{d_{n-1}}}}.$$

Now suppose that $d_n(\omega, x)$ is finite for all $n \geq 1$. We want to show that (ω, x) has **infinite random continued fraction expansion**

$$x = \cfrac{(-1)^{\omega_0}}{d_1 + \cfrac{(-1)^{\omega_1}}{d_2 + \cfrac{\ddots}{} + \cfrac{(-1)^{\omega_{n-1}}}{d_n + \ddots}}}.$$

For each $n \geq 1$, let $\frac{p_n}{q_n}$ denote the convergents of the random continued fraction of (ω, x), i.e., write

$$\frac{p_n}{q_n} = \cfrac{(-1)^{\omega_0}}{d_1 + \cfrac{(-1)^{\omega_1}}{d_2 + \cfrac{\ddots}{} + \cfrac{(-1)^{\omega_{n-1}}}{d_n}}}.$$

2.3. **Convergence of Partial Sums.** To show convergence, we follow the approach for regular continued fractions with the important difference that in our case the numbers q_n do not necessarily increase. As for regular continued fractions (see for example [DK02]), one can obtain the following relations:

$$p_{-1} = 1, \quad p_0 = 0, \quad p_n = d_n p_{n-1} + (-1)^{\omega_{n-1}} p_{n-2},$$
$$q_{-1} = 0, \quad q_0 = 1, \quad q_n = d_n q_{n-1} + (-1)^{\omega_{n-1}} q_{n-2}.$$

Using these recurrences, induction easily gives that

$$(5) \qquad x = \frac{p_n + p_{n-1}\pi\big(K^n(\omega, x)\big)}{q_n + q_{n-1}\pi\big(K^n(\omega, x)\big)}$$

and that

$$(6) \qquad p_{n-1}q_n - p_n q_{n-1} = (-1)^n (-1)^{\omega_0 + \cdots + \omega_{n-1}}.$$

The next lemmas are needed to show that although the sequence $\{q_n\}_{n\geq 1}$ is not necessarily increasing, we still have $\lim_{n\to\infty} \frac{1}{q_n} = 0$.

Lemma 2.1. *For each $n \geq 2$, $q_n > 0$ and if $q_n \leq q_{n-1}$, then $\omega_{n-1} = 1$ and $d_n = 1$, so $\omega_n = 0$.*

Proof. We prove this by induction. For $n = 1$ we have $q_1 = d_1 \geq 1 = q_0$. For $n = 2$, we have $q_2 = d_2 d_1 + (-1)^{\omega_1}$ and hence

$$q_2 \leq q_1 \quad \Leftrightarrow \quad d_2 \leq 1 - \frac{(-1)^{\omega_1}}{d_1}.$$

This can happen only if $\omega_1 = 1$, thus $d_1 > 1$, and $d_2 = 1$. This gives $q_2 = d_1 - 1 > 0$ and the lemma. Now suppose the statements hold for all $k < n$, i.e., $q_k > 0$ and if $q_k \leq q_{k-1}$, then $\omega_{k-1} = 1$ and $d_k = 1$, so $\omega_k = 0$. If $q_n > q_{n-1}$, then automatically $q_n > 0$. So, suppose $q_n \leq q_{n-1}$. We have

$$q_n = d_n q_{n-1} + (-1)^{\omega_{n-1}} q_{n-2} \leq q_{n-1} \quad \Leftrightarrow \quad 1 \leq d_n \leq 1 - (-1)^{\omega_{n-1}} \frac{q_{n-2}}{q_{n-1}}.$$

Since $q_{n-2}, q_{n-1} > 0$, $q_n \leq q_{n-1}$ implies that $\omega_{n-1} = 1$ and hence $d_{n-1} > 1$. The induction hypothesis then gives that $\frac{q_{n-2}}{q_{n-1}} < 1$ and hence $1 \leq d_n < 2$. This gives the lemma. $\qquad \square$

Lemma 2.2. *If $q_n \leq q_{n-1}$, then $q_{n-2} < q_{n-1} < q_{n+1}$. This implies that $q_n \neq q_{n-1}$.*

Proof. By the previous lemma we have $\omega_{n-1} = 1$, so $q_{n-1} > q_{n-2}$. We also have $d_n = 1$ and thus $\omega_n = 0$. Since $q_n > 0$, this implies that $q_{n+1} = d_{n+1}q_n + q_{n-1} > q_{n-1}$. For the second part, note that if $q_n = q_{n-1}$, then $d_n = 1 - (-1)^{\omega_{n-1}} \frac{q_{n-2}}{q_{n-1}}$ and by the previous lemma $q_{n-1} > q_{n-2}$. Since d_n is an integer, this is not possible. $\qquad \square$

Before we can prove that the process converges, we need a lower bound on the q_n's in case $q_n < q_{n-1}$. This is done in the next lemma.

Lemma 2.3. *Suppose $q_n < q_{n-1}$, so $d_{n-1} > 1$.*

  (i) *If $d_{n-1} > 2$, then $q_n > q_{n-2}$.*
  (ii) *If $d_{n-1} = 2$ and $(\omega_k, d_k) = (1, 2)$ for all $1 \leq k \leq n-1$, then $q_n = 1$ and $q_{n-1} = n$.*
  (iii) *Suppose $d_{n-1} = 2$ and there is a $1 \leq k < n-1$ such that $(\omega_k, d_k) \neq (1, 2)$. Let k be the largest such index. Then $q_n > q_{k-1}$.*

Proof. Recall that $q_n < q_{n-1}$ implies that $d_n = 1$, $\omega_n = 0$ and $\omega_{n-1} = 1$, so $d_{n-1} > 1$. Hence, $q_n = q_{n-1} - q_{n-2}$.

(i) If $d_{n-1} > 2$, then

$$
\begin{aligned}
q_n &= d_{n-1}q_{n-2} + (-1)^{\omega_{n-2}}q_{n-3} - q_{n-2} = (d_{n-1} - 1)q_{n-2} + (-1)^{\omega_{n-2}}q_{n-3} \\
&\geq 2q_{n-2} + (-1)^{\omega_{n-2}}q_{n-3}.
\end{aligned}
$$

If $q_{n-2} > q_{n-3}$, then $q_n \geq 2q_{n-2} - q_{n-3} > q_{n-2}$. If $q_{n-2} < q_{n-3}$, then Lemma 2.1 gives that $\omega_{n-2} = 0$ and hence $q_n \geq 2q_{n-2} + q_{n-3} > q_{n-3} > q_{n-2}$.

For both (ii) and (iii), note that if $d_k = 2$ and $\omega_k = 1$ for some $1 \leq k \leq n - 1$, then $q_k = 2q_{k-1} - q_{k-2}$, so

(7)
$$
q_k - q_{k-1} = q_{k-1} - q_{k-2}.
$$

(ii) From (7) it follows that

$$
q_n = q_{n-1} - q_{n-2} = q_2 - q_1 = 2 \cdot 2 + (-1)^1 - 2 = 1.
$$

Moreover, for each $1 \leq k \leq n - 1$ it holds that

$$
q_k = 2q_{k-1} - q_{k-2} = q_{k-1} + (q_{k-1} - q_{k-2}) = q_{k-1} + 1.
$$

Hence, $q_{n-1} = n - 2 + q_1 = n$.

(iii) Let k be as given in the lemma, so $(\omega_k, d_k) \neq (1, 2)$ and $(\omega_j, d_j) = (1, 2)$ for $k + 1 \leq j \leq n - 1$. Then,

$$
q_n = q_{n-1} - q_{n-2} = q_{k+1} - q_k = d_{k+1}q_k + (-1)^{\omega_k}q_{k-1} - q_k = q_k + (-1)^{\omega_k}q_{k-1}.
$$

If $\omega_k = 0$, then $q_n = q_k + q_{k-1} > q_{k-1}$. If $\omega_k = 1$, then $d_k \geq 3$ and $q_k > q_{k-1}$ by Lemma 2.1. This gives

$$
q_n = q_k - q_{k-1} = (d_k - 1)q_{k-1} + (-1)^{\omega_{k-1}}q_{k-2} \geq 2q_{k-1} + (-1)^{\omega_{k-1}}q_{k-2}.
$$

As in the proof of part (i) we now get that if $q_{k-1} > q_{k-2}$, then $q_n > q_{k-1}$ and if $q_{k-1} < q_{k-2}$, then $\omega_{k-1} = 0$ and we also get $q_n > q_{k-1}$. $\qquad\square$

Proposition 2.1. *Let $x \in [-1, 1] \backslash \mathbb{Q}$. For each ω, the digits $d_n(\omega, x)$ give a continued fraction expansion of x.*

Proof. For all $n \geq 1$ we have using (5) and (6)

$$
\begin{aligned}
\left| x - \frac{p_n}{q_n} \right| &= \left| \frac{q_n\big(p_n + p_{n-1}\pi\big(K^n(\omega, x)\big)\big) - p_n\big(q_n + q_{n-1}\pi\big(K^n(\omega, x)\big)\big)}{q_n\big(q_n + q_{n-1}\pi\big(K^n(\omega, x)\big)\big)} \right| \\
&= \left| \frac{\pi\big(K^n(\omega, x)\big)\big(q_n p_{n-1} - p_n q_{n-1}\big)}{q_n\big(q_n + q_{n-1}\pi\big(K^n(\omega, x)\big)\big)} \right| \\
&\leq \frac{1}{\left| q_n\big(q_n + q_{n-1}\pi\big(K^n(\omega, x)\big)\big)\right|} \leq \frac{1}{q_n|q_n - q_{n-1}|} \leq \frac{1}{q_n}.
\end{aligned}
$$

We now show that $\lim_{n\to\infty} \frac{1}{q_n} = 0$. Let $\varepsilon > 0$. By Lemma 2.2 there exists a subsequence $(q_{n_k})_{k\geq 0}$ such that $\lim_{k\to\infty} \frac{1}{q_{n_k}} = 0$ and hence there exists an N_1 such that $\frac{1}{q_{N_1}} < \varepsilon$. If there

is no $n > N_1$, such that $q_n < q_{n-1}$, then we are done. If there is, let k be the smallest such index. If $(\omega_{N_1+1}, d_{N_1+1}) \neq (1,2)$, by Lemma 2.3 $q_k \geq q_{N_1}$ and then same holds for all other $n > N_1$. If $(\omega_{N_1+1}, d_{N_1+1}) = (1,2)$, then set $N = k+1$. We then have $\frac{1}{q_N} < \frac{1}{q_{k-1}} < \frac{1}{q_{N_1}} < \varepsilon$. Moreover, for all $n > N$ we have $q_n > q_{k-1}$, since $(\omega_k, d_k) = (0,1)$. This shows that the limit exists and is equal to 0. Hence we get

$$x = \cfrac{(-1)^{\omega_0}}{d_1 + \cfrac{(-1)^{\omega_1}}{d_2 + \cfrac{(-1)^{\omega_2}}{d_3 + \cdots}}}$$

for each $x \in [-1, 1] \backslash \mathbb{Q}$. $\square$

3. Invariant densities

Recall that $T_0, T_1 : [0,1] \to [0,1]$ are the Gauss and Rényi map respectively. To study the dynamical properties of random continued fractions we replace the map K by the transformation $R : \Omega \times [0,1] \to \Omega \times [0,1]$ given by

$$R(\omega, x) = (\sigma\omega, T_{\omega_1}x).$$

The ergodic properties of R are easier to study and (3) gives the following simple relation between K and R. Recall that $\omega_0 = 0$ if $x \geq 0$ and 1 otherwise. Then,

$$\pi\big(R^{n-1}(\omega, T_0x)\big) = \pi\big(K^n(\omega, x)\big) + \omega_n,$$

and hence,

$$\left|\pi\big(K^n(\omega, x)\big)\right| = \omega_n + (-1)^{\omega_n}\pi\big(R^{n-1}(\omega, T_0x)\big).$$

Using this we can recover the digit sequence $\big(d_n(\omega, x)\big)_{n \geq 1}$ generated by K as follows. For $(\omega, x) \in \Omega \times [0,1]$ define

$$(8) \qquad\qquad b(\omega, x) = \begin{cases} k + \omega_2, & \text{if } \omega_1 + (-1)^{\omega_1}x \in \big(\frac{1}{k+1}, \frac{1}{k}\big], \\ \infty, & \text{if } \omega_1 + (-1)^{\omega_1}x = 0. \end{cases}$$

Write 0ω for the sequence $\omega' \in \Omega$ satisfying $\omega'_1 = 0$ and $\omega'_{n+1} = \omega_n$ for all $n \geq 1$. Then for $n \geq 1$,

$$(9) \qquad\qquad d_n(\omega, x) = b\big(R^{n-1}(0\omega, x)\big).$$

Remark 3.1. The Gauss map T_0 is intimately related to the well known Farey map, which is defined by

$$Fx = \begin{cases} \dfrac{x}{1-x}, & \text{if } 0 \leq x \leq \frac{1}{2}, \\[2mm] \dfrac{1-x}{x}, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

The Gauss map can be obtained from F by inducing on the first passage time to the interval $\left[\frac{1}{2}, 1\right]$. For $x \in [0, 1]$ and $n \geq 1$, set $\omega_n = 1$ if $\pi\left(R^{n-1}(\omega, x)\right) < \frac{1}{2}$ and $\omega_n = 0$ if $\pi\left(R^{n-1}(\omega, x)\right) > \frac{1}{2}$. If $\pi\left(R^{n-1}(\omega, x)\right) = \frac{1}{2}$, you can choose. Then $\pi\left(R^n(\omega, x)\right) = F^n x$ for each $n \geq 1$.

3.1. **Existence.** Let λ denote the Lebesgue measure on $[0, 1]$. We are going to show that R has invariant measures of type $m_p \otimes \mu_p$, where m_p is the $(p, 1-p)$-Bernoulli measure on Ω and μ_p is a probability measure on $[0, 1]$ absolutely continuous with respect to λ. For this we use [Ino12]. Recall that a function $g : [0, 1] \to [0, 1]$ is said to have bounded variation if

$$\bigvee_{[0,1]} g := \sup_{0=x_0<x_1<x_2<\cdots<x_n=1} \sum_{i=1}^n |g(x_i) - g(x_{i-1})| < \infty.$$

The following is a very simplified, discrete version of the main theorem of [Ino12]:

Theorem 3.1 (Inoue, [Ino12]). *Given two non-singular maps $T_0, T_1 : [0, 1] \to [0, 1]$, let $R : \{0, 1\}^{\mathbb{N}} \times [0, 1] \to \{0, 1\}^{\mathbb{N}} \times [0, 1]$ be given by*

$$R(\omega, x) = (\sigma(\omega), T_{\omega_1}(x)).$$

Let $p \in [0, 1]$ and set $p_0 = p$ and $p_1 = 1-p$. For $i \in \{0, 1\}$ let $\{I_{i,k}\}$ be a countable partition of $[0, 1]$ into intervals and use $int(I_{i,k})$ to denote the interior of these intervals. Let $g(i, x)$ be functions satisfying

$$(10) \qquad\qquad g(i, x) = \frac{p_i}{|T_i'(x)|}$$

on $\bigcup_k int(I_{i,k})$. Assume that the following conditions are satisfied:

(I1) *The restrictions of T_i to each interval $int(I_{i,k})$ are C^1 and monotone.*

(I2) *The weighted average expansion of T_i is uniformly positive for all x, i.e.,*

$$\sup_{x \in [0,1]} \big(g(0, x) + g(1, x)\big) < 1.$$

(I3) *For each $i \in \{0, 1\}$ the functions $g(i, x) : [0, 1] \to \mathbb{R}$ are of bounded variation.*

Then there exists a probability measure μ_p on $[0, 1]$ absolutely continuous with respect to the Lebesgue measure λ with density function h_p that is of bounded variation. Moreover, μ_p has the property that

$$\mu_p(A) = p\mu_p(T_0^{-1}A) + (1-p)\mu_p(T_1^{-1}A)$$

for each Borel measurable set $A \subseteq [0, 1]$.

We have not included condition $(A2)$ of Inoue since this automatically holds when one only has a finite number of maps T_i. We apply Theorem 3.1 to our setting.

Proposition 3.1. *Suppose $p \in (0, 1)$. Then the maps T_0 and T_1 satisfy the conditions (I1), (I2) and (I3) of Theorem 3.1.*

Proof. For $k \geq 1$, let $I_{0,k} = \left(\frac{1}{k+1}, \frac{1}{k}\right]$ and $I_{1,k} = \left[\frac{k-1}{k}, \frac{k}{k+1}\right)$. Consider the following partitions of $[0,1]$:

(11) $$\mathcal{I}_0 = \{I_{0,k}\}_{k \geq 1} \quad \text{and} \quad \mathcal{I}_1 = \{I_{1,k}\}_{k \geq 1}.$$

Then T_0 and T_1 are both C^1 and monotone on the interiors of the intervals of their respective partitions, so condition (I1) is satisfied. Note that the functions $g(i, \cdot)$ from (10) become

$$g(0, x) = px^2 \quad \text{and} \quad g(1, x) = (1-p)(1-x)^2.$$

almost everywhere. We see that

$$\sup_{x \in [0,1]} \big(g(0,x) + g(1,x)\big) = \sup_{x \in [0,1]} \big(x^2 - 2(1-p)x + (1-p)\big) = \max\{1-p, p\}.$$

So condition (I2) is satisfied for all $p \in (0,1)$. Since both $g(0, \cdot)$ and $g(1, \cdot)$ are monotone functions on the interval $[0,1]$, we have

$$\bigvee_{[0,1]} g(0, \cdot) = p \quad \text{and} \quad \bigvee_{[0,1]} g(1, \cdot) = 1 - p.$$

This gives (I3), and the proof $\qquad\qquad\square$

The conclusions of Theorem 3.1 yield the following result, the proof of which is standard, but included for the convenience of the reader.

Theorem 3.2. *For any choice of parameter $0 < p < 1$, there is an absolutely continuous probability measure $\mu_p \ll \lambda$ such that the product measure $m_p \otimes \mu_p$ is invariant for R. The probability density function h_p of the measure μ_p is of bounded variation.*

Proof. Theorem 3.1 gives an absolutely continuous measure μ_p with the property that for each Borel set $A \subset [0,1]$, we have

(12) $$\mu_p(A) = p\mu_p(T_0^{-1}A) + (1-p)\mu_p(T_1^{-1}A).$$

Take a cylinder $[j_1 \cdots j_n] \in \{0,1\}^{\mathbb{N}}$ and an interval $(a,b) \subset [0,1]$. Then

$$R^{-1}\big([j_1 \cdots j_n] \times (a,b)\big) = [0j_1 \cdots j_n] \times T_0^{-1}\big((a,b)\big) \cup [1j_1 \cdots j_n] \times T_1^{-1}\big((a,b)\big).$$

Hence,

$$(m_p \otimes \mu_p)\Big(R^{-1}\big([j_1 \cdots j_n] \times (a,b)\big)\Big)$$

$$= p \, m_p([j_1 \cdots j_n])\mu_p\Big(T_0^{-1}\big((a,b)\big)\Big) + (1-p) \, m_p([j_1 \cdots j_n])\mu_p\Big(T_1^{-1}\big((a,b)\big)\Big)$$

$$= m_p([j_1 \cdots j_n])\mu_p\big((a,b)\big) = (m_p \otimes \mu_p)\big([j_1 \cdots j_n] \times (a,b)\big).$$

This gives the first part of the result. The density h_p is given by Theorem 3.1. $\qquad\square$

If $p = 1$, then R reduces to the Gauss map and $h_1(x) = \frac{1}{\log 2}\frac{1}{x+1}$. If, on the other hand, we take $p = 0$, then R reduces to the map T_1, which has no absolutely continuous invariant probability measure, but does have an infinite and σ-finite absolutely continuous invariant

measure with density $h_0(x) = \frac{1}{x}$. This was proved by Rényi in [Rén57]. The fact that R is expanding on average causes that h_p is a probability density for all $0 < p < 1$. It would be interesting to analyse the behaviour of h_p as $p \to 0$.

3.2. **Properties of the invariant measure.** In this section we list some of the properties of the invariant measure $m_p \otimes \mu_p$. In [Ino12] Inoue proved the existence of the density h_p by analysing a random version of the Perron-Frobenius operator. In our case, the corresponding operator $\mathcal{L}_p : L^1(\lambda) \to L^1(\lambda)$ is given by

$$(13) \qquad (\mathcal{L}_p f)(x) = \sum_{k \geq 1} \left[\frac{p}{(k+x)^2} f\left(\frac{1}{k+x}\right) + \frac{1-p}{(k+x)^2} f\left(1 - \frac{1}{k+x}\right) \right].$$

Theorem 3.1 is proved in [Ino12] by showing that this operator has a fixed point in the space of functions of bounded variation, which is our function h_p. Recall that functions of bounded variation can be modified on a countable number of points to obtain a lower semi-continuous function. From now on we assume that h_p is lower semi-continuous. Under the conditions of Theorem 3.1 the operator $\mathcal{L}_p$ is quasi-compact and constrictive. In this section we use these results to derive some dynamical properties of R. We first prove that R satisfies a strong Random Covering Property.

Proposition 3.2. *Let $I \subseteq [0,1]$ be a non-trivial interval. Then for every $\omega \in \Omega$, there is an $n \geq 1$ such that*

$$\left(T_{\omega_n} \circ \cdots \circ T_{\omega_1} \right) I = [0,1).$$

Proof. Recall the definition of the partitions $\mathcal{I}_0$ and $\mathcal{I}_1$ in the proof of Proposition 3.1. If a non-trivial interval J is contained in one of the intervals in $\mathcal{I}_i$, then $T_i J$ is again an interval and $\lambda(T_i J) > \lambda(J)$. Hence, there is an $m \geq 1$, such that $\left(T_{\omega_m} \circ \cdots \circ T_{\omega_1} \right) I$ contains an endpoint of one of the intervals in $\mathcal{I}_{\omega_{m+1}}$. This means that $\left(T_{\omega_{m+1}} \circ T_{\omega_m} \circ \cdots \circ T_{\omega_1} \right) I$ contains an interval of the form $[0, c)$ and an interval of the form $(1 - c, 1)$ for some $c > 0$. Thus $\left(T_{\omega_{m+2}} \circ \cdots \circ T_{\omega_1} \right) I = [0,1)$. $\square$

It now follows that the measure μ_p is in fact equivalent to λ. The proof below is essentially the one from [ANV].

Proposition 3.3. *Let h_p be the probability density function from Theorem 3.2. Then $h_p(x) > 0$ for all $x \in [0,1)$.*

Proof. We know that h_p is a function of bounded variation satisfying $h_p \geq 0$, $\int_{[0,1]} h_p d\mu_p = 1$ and $\mathcal{L}_p h_p = h_p$. Therefore there exists a non-trivial interval $I \subseteq [0,1]$ and an $\alpha > 0$, such

that $h_p \geq \alpha 1_I$. Fix a sequence $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \dots) \in \Omega$. Then for any $n \geq 1$,

$$
\begin{aligned}
h_p(x) &= \mathcal{L}_p^n h_p(x) \geq \alpha \mathcal{L}_p^n 1_I(x) \\
&= \sum_{(\omega_1, \dots, \omega_n) \in \Omega^n} \sum_{y \in (T_{\omega_n} \circ \cdots \circ T_{\omega_1})^{-1}\{x\}} \frac{p_{\omega_1} \cdots p_{\omega_n}}{|(T_{\omega_n} \circ \cdots \circ T_{\omega_1})'y|} 1_I(y) \\
&\geq \sum_{y \in (T_{\bar{\omega}_n} \circ \cdots \circ T_{\bar{\omega}_1})^{-1}\{x\}} \frac{p_{\bar{\omega}_1} \cdots p_{\bar{\omega}_n}}{|(T_{\bar{\omega}_n} \circ \cdots \circ T_{\bar{\omega}_1})'y|} 1_I(y).
\end{aligned}
$$

By Proposition 3.2 there is an $n \geq 1$ such that for all $x \in [0, 1)$,

$$
(T_{\bar{\omega}_n} \circ \cdots \circ T_{\bar{\omega}_n})^{-1}\{x\} \cap I \neq \emptyset.
$$

Hence, for all $0 < p < 1$ and all $x \in [0, 1)$, $h_p(x) > 0$.   □

This proposition leads to the following result, the proof of which uses a standard technique that can be found for example in [BG97].

Proposition 3.4. *The probability measure h_p is bounded from above and away from 0.*

Proof. The fact that h_p is bounded from above follows since h_p is of bounded variation. In the previous proposition, we have established that $h_p(x) > 0$ for all $x \in [0, 1)$. Since h_p is lower semi-continuous, it takes its minimum on [0,1]. Therefore, it is enough to show that $h_p(1) > 0$. Let $\varepsilon > 0$ be small and let $k \geq 1$. We consider part of the inverse image of the interval $(1 - \varepsilon, 1)$ under T_0 (we could just as well use T_1). Note that

$$
\left(\frac{1}{k+1}, \frac{1}{k+1-\varepsilon} \right) \subseteq T_0^{-1}(1 - \varepsilon, 1)
$$

and that

$$
\lambda \left(\left(\frac{1}{k+1}, \frac{1}{k+1-\varepsilon} \right) \right) = \frac{\varepsilon}{(k+1)(k+1-\varepsilon)}.
$$

Hence,

$$
k^2 \lambda \left(\left(\frac{1}{k+1}, \frac{1}{k+1-\varepsilon} \right) \right) < \lambda((1 - \varepsilon, 1)) < (k+1)^2 \lambda \left(\left(\frac{1}{k+1}, \frac{1}{k+1-\varepsilon} \right) \right).
$$

It then follows that

$$
\begin{aligned}
\lim_{x \uparrow 1} h_p(x) &= \lim_{\varepsilon \to 0} \frac{1}{\lambda((1 - \varepsilon, 1))} \int_{1-\varepsilon}^1 h_p(x) dx = \lim_{\varepsilon \to 0} \frac{\mu_p((1 - \varepsilon, 1))}{\lambda((1 - \varepsilon, 1))} \\
&\geq \lim_{\varepsilon \to 0} \frac{p \mu_p(T_0^{-1}(1 - \varepsilon, 1)) + (1 - p)\mu_p(T_1^{-1}(1 - \varepsilon, 1))}{(k+1)^2 \lambda((\frac{1}{k+1}, \frac{1}{k+1-\varepsilon}))} \\
&\geq \lim_{\varepsilon \to 0} \frac{p \mu_p((\frac{1}{k+1}, \frac{1}{k+1-\varepsilon}))}{(k+1)^2 \lambda((\frac{1}{k+1}, \frac{1}{k+1-\varepsilon}))} = \frac{p}{(k+1)^2} h_p\left(\frac{1}{k+1} \right) > 0.
\end{aligned}
$$

The fact that h_p is bounded from above follows since h_p is a function of bounded variation on a closed and bounded interval.   □

The fact that the operator $\mathcal{L}_p$ is quasi-compact on the set of functions of bounded variation allows us to obtain a number of consequences from the Ionescu-Tulcea and Marinescu Theorem, as is done in many similar situations. The reader is referred to [Bal00] for example for an outline of this approach for deterministic maps. The spectral decomposition (which is already given in [Ino12]) together with Proposition 3.4 gives that 1 is a simple eigenvalue for $\mathcal{L}_p$, that there are no other eigenvalues on the unit circle and that $\mathcal{L}_p^n f \to \int f d\lambda h_p$ in $L^1(\lambda)$. This means that the system R satisfies the conditions of [ANV]. (Note that R is not contained in the class of random Lasota-Yorke systems discussed in Example 2.1 of [ANV], since both T_0 and T_1 have infinitely many branches.) From [ANV] we then immediately get exponential decay of correlations:

Proposition 3.5 (see [ANV] Proposition 3.1). *There exist constants $C \geq 0$ and $\rho < 1$, such that for all functions f of bounded variation and all $g \in L^\infty(\lambda)$,*

$$\left| \int_{[0,1]} \mathcal{L}_p^n f \cdot g d\mu_p - \int_{[0,1]} f d\mu_p \int_{[0,1]} g d\mu_p \right| \leq C\rho^n \|f\|_{BV} \|g\|_\infty.$$

In particular, R is mixing. A proof can be found in [Mor85] for example.

Theorem 3.3. *The random transformation R is mixing with respect to $m_p \otimes \mu_p$.*

Proof. Define the function $\phi : \Omega \times [0,1] \to [0,1]$ by $\phi(\omega, x) = \omega_1 + (-1)^{\omega_1} x$. Define the cylinder sets of order 1 by

$$[d] \times \Delta(a)_j = [d] \times \phi\left([d] \times \left(\frac{1}{a+1}, \frac{1}{a}\right]\right).$$

In general the cylinders of order k are given by $[d_1 \cdots d_k] \times \Delta(a_1, \ldots, a_k)_{d_1 \cdots d_k}$, where

$$\Delta(a_1, \ldots, a_k)_{d_1 \cdots d_k} = \bigcap_{j=1}^k \left(T_{d_{j-1}} \circ \cdots \circ T_{d_1} \right)^{-1} \phi\left([d_j] \times \left(\frac{1}{a_j+1}, \frac{1}{a_j}\right]\right).$$

Write $[\bar{d}] \times \Delta(\bar{a})_{\bar{d}}^k$ for $[d_1 \cdots d_k] \times \Delta(a_1, \ldots, a_k)_{d_1 \cdots d_k}$. These sets form a generating semi-algebra for the σ-algebra on $\Omega \times [0,1]$. A straightforward computation using the previous proposition now gives that for any two such cylinders $[\bar{d}] \times \Delta(\bar{a})_{\bar{d}}^k$ and $[\bar{c}] \times \Delta(\bar{b})_{\bar{c}}^\ell$,

$$\lim_{n\to\infty} (m_p \otimes \mu_p)\left(R^{-n}\left([\bar{c}] \times \Delta(\bar{b})_{\bar{c}}^\ell\right) \cap \left([\bar{d}] \times \Delta(\bar{a})_{\bar{d}}^k\right)\right)$$

$$= (m_p \otimes \mu_p)\left([\bar{c}] \times \Delta(\bar{b})_{\bar{c}}^\ell\right)(m_p \otimes \mu_p)\left([\bar{d}] \times \Delta(\bar{a})_{\bar{d}}^k\right),$$

where the convergence is in L^1. $\qquad\square$

We also immediately obtain the Central Limit Theorem and the Large Deviation Principle. Let ϕ be an observable of bounded variation with $\int_{[0,1]} \phi d\mu_p = 0$. Define $X_k : \Omega \times [0,1] \to \mathbb{R}$

by $X_k(\omega, x) = \phi\big((T_{\omega_k} \circ \cdots \circ T_{\omega_1})x\big)$ and $S_n = \sum_{k=0}^{n-1} X_k$. The asymptotic variance σ^2 is defined by

$$\sigma^2 = \lim_{n \to \infty} \frac{1}{n} \int_{\Omega \times [0,1]} S_n^2 \, d(m_p \otimes \mu_p).$$

In [ANV] it is proved that this limit exists in Proposition 3.2. Let $\mathcal{M}$ be the set of all Borel probability measures on $[0, 1]$ that are absolutely continuous w.r.t. λ with a density of bounded variation.

Theorem 3.4 (Central Limit Theorem, see [ANV] Theorem 3.5). *Let $\nu \in \mathcal{M}$. Then the process $\big(\frac{S_n}{\sqrt{n}}\big)_{n \geq 1}$ converges in law to $\mathcal{N}(0, \sigma^2)$ under the probability measure $m_p \otimes \nu$.*

Theorem 3.5 (Large Deviation Principle, see [ANV] Theorem 3.6). *Suppose $\sigma^2 > 0$. Then there exists a non-negative rate function c, continuous, strictly convex, vanishing only at 0, such that for every $\nu \in \mathcal{M}$ and every sufficiently small $\varepsilon > 0$,*

$$\lim_{n \to \infty} \frac{1}{n} \log \big((m_p \otimes \nu)(S_n > n\varepsilon)\big) = -c(\varepsilon).$$

In particular these results hold for λ and μ_p.

4. Further Properties of the continued fractions

Here we make a few observations about the continued fractions produced by K.

4.1. **Counting Expansions.** It is clear that K produces all semi-regular continued fractions of points $x \in [-1, 1]$. Since typically each sequence $\omega \in \Omega$ characterises a unique continued fraction expansion, K will generate uncountably many different continued fraction expansions for most x.

Proposition 4.1. *Each $x \in [-1, 1]\backslash\mathbb{Q}$ has uncountably many different continued fraction expansions given by K. Each $x \in ([-1, 1] \cap \mathbb{Q})\backslash\{0\}$ has countably infinitely many.*

Proof. To prove this we first show that $x \in ([-1, 1] \cap \mathbb{Q})\backslash\{0\}$ if and only if for each $\omega \in \Omega$ there exists an $n \geq 0$ such that $\pi\big(K^n(\omega, x)\big) \in \{\frac{1}{k}, -\frac{1}{k} : k \in \mathbb{N}\}$. One direction is clear, since if x is irrational then $\pi(K(\omega, x))$ is irrational for any $\omega \in \Omega$. Now assume that $x \in [-1, 1] \cap \mathbb{Q}\backslash\{0\}$. First note that $\pi\big(K(\omega, x)\big) \in \{-1, 0, 1\}$ if and only if $x \in \{\frac{1}{k}, -\frac{1}{k} : k \geq 1\}$. Write $x = \frac{r_1}{r_0}$ with $r_0, r_1 \in \mathbb{Z}$ and $|r_1| \leq |r_0|$. If $|r_0| = |r_1|$, then the proposition is obtained with $n = 0$. If not, then

$$\pi\big(K(\omega, x)\big) = \frac{(-1)^{\omega_0}}{x} - d_1 = \frac{(-1)^{\omega_0} r_0 - r_1 d_1}{r_1} = \frac{r_2}{r_1} \in [-1, 1].$$

Here $r_2 \in \mathbb{Z}$ and $|r_2| \leq |r_1|$. If $|r_2| = |r_1|$, then we have the proposition with $n = 0$. If not, then by continuing in the same manner we obtain a sequence of integers $r_0, r_1, \ldots, r_n$ with $|r_n| < \cdots < |r_1| < |r_0|$. This shows that the process must terminate after at most r_0 steps.

Now suppose $x \in [-1, 1]\backslash\mathbb{Q}$. Then there is no $\omega \in \Omega$ and no $n \geq 0$, such that $\pi\left(K^n(\omega, x)\right) \in \left\{\pm\frac{1}{k} : k \geq 1\right\}$. Hence, each ω produces a unique continued fraction expansion for x. This gives the first part of the proposition.

For the second part of the statement, let $\frac{p}{q}$ be an arbitrary element of $([-1, 1] \cap \mathbb{Q})\backslash\{0\}$ with $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and $0 < |p| < q$. Then for each $\omega \in \Omega$ there is an $N \leq q$, such that $\pi\left(K^N(\omega, \frac{p}{q})\right) \in \{\frac{1}{k}, -\frac{1}{k}\}$. If $\omega_n = 1$ for all $n > N$, then there is some $k \geq 1$ such that

$$\left(d_n\left(\omega, \frac{p}{q}\right)\right)_{n \geq 1} = (d_1, \ldots, d_N, k + 1, 2, 2, 2, \ldots).$$

If there is a smallest integer $M \geq 2$ such that $\omega_{N+M} = 0$, then

$$\left(d_n\left(\omega, \frac{p}{q}\right)\right)_{n \geq 1} = (d_1, \ldots, d_N, k + 1, \underbrace{2, \ldots, 2}_{M-1 \text{ times}}, 1).$$

Finally, if $\omega_{N+1} = 0$, then

$$\left(d_n\left(\omega, \frac{p}{q}\right)\right)_{n \geq 1} = (d_1, \ldots, d_N, k).$$

Hence, K generates only countably many different expansions for x. $\qquad\square$

Proposition 4.1 gives all the possible endings of digit sequences of rational points $\frac{p}{q}$ generated by K. From Figure 1 it is clear however that $\frac{p}{q}$ also has expansions that are not generated by K. The corresponding digit sequences are

$$(d_1, \ldots, d_N, k - 1, 1), \quad (d_1, \ldots, d_N, k - 1, 2, 2, 2, \ldots),$$

and for each $M \geq 2$,

$$(d_1, \ldots, d_N, k - 1, \underbrace{2, \ldots, 2}_{M-1 \text{ times}}, 1).$$

The fact that K does not generate these expansions is a consequence of having defined K by taking the intervals $\left(\frac{1}{k+1}, \frac{1}{k}\right]$ left-open and right-closed. So all points in $[-1, 1] \cap \mathbb{Q}$ have expansions of the form (1) that are not generated by K, but we know exactly what the missing expansions are.

α-continued fractions. The α-continued fraction transformation was first introduced by Nakada in [Nak81]. Given $\alpha \in [0, 1]$, define the map $T_\alpha : [\alpha - 1, \alpha) \to [\alpha - 1, \alpha)$ by $T_\alpha 0 = 0$ and

$$T_\alpha x = \left|\left|\frac{1}{x}\right| - \left\lfloor\left|\frac{1}{x}\right| - (\alpha - 1)\right\rfloor\right|$$

for $x \neq 0$. First note that for $x < 0$, $T_1(1 + x) = \left|\frac{1}{x}\right| - \left\lfloor\left|\frac{1}{x}\right|\right\rfloor$. Also note that

$$\left|\frac{1}{x}\right| - \left\lfloor\left|\frac{1}{x}\right|\right\rfloor < \alpha \quad \Leftrightarrow \quad \left\lfloor\left|\frac{1}{x}\right| - \alpha + 1\right\rfloor = \left\lfloor\left|\frac{1}{x}\right|\right\rfloor$$

and

$$\left|\frac{1}{x}\right| - \left\lfloor\left|\frac{1}{x}\right|\right\rfloor \geq \alpha \quad \Leftrightarrow \quad \left\lfloor\left|\frac{1}{x}\right| - \alpha + 1\right\rfloor = \left\lfloor\left|\frac{1}{x}\right|\right\rfloor + 1.$$

We can define a sequence $\omega = (\omega_n)_{n \geq 1} \in \Omega$ such that for each $n \geq 0$, $\pi\big(K^n(\omega, x)\big) = T_\alpha^n x$ as follows. Note that if $0 < x < \alpha$, then either $T_0 x \in [\alpha - 1, \alpha)$ in which case we set $\omega_1 = 0$ or $T_0 x - 1 \in [\alpha - 1, \alpha)$, in which case we set $\omega_1 = 1$. Similarly for $\alpha - 1 \leq x < 0$ and the map $T_1(x + 1)$. Then

$$T_{\omega_0}(x + \omega_0) - \omega_1 = T_\alpha x.$$

We can apply the same procedure for every $n \geq 2$ with the point $(T_{\omega_{n-1}} \circ \cdots \circ T_{\omega_0})(x + \omega_0) - \omega_n$. One can easily show, using (3) and induction, that for each $x \in [\alpha - 1, \alpha)$ and $n \geq 1$, one gets $\pi\big(K^n(\omega, x)\big) = T_\alpha^n x$.

4.2. **Restrictions on digits.** For regular continued fractions, expansions with restrictions on the digits have been thoroughly investigated. Continued fractions with bounded digits have applications in many fields, ranging from formal language theory to diophantine approximation to pseudo-random number generators, see [Sha92] for a survey on this topic. For random continued fractions the following is easily observed.

Proposition 4.2. *Let $\Lambda \subset \mathbb{N}$ be such that it does not miss two or more consecutive integers, i.e., if $k \notin \Lambda$, then $k - 1, k + 1 \in \Lambda$. Then all numbers in $[-1, 1]$ have a continued fraction expansion of the form (1) with $d_n \in \Lambda$ for each n.*

Proof. From Figure 1 it is clear that if we delete the branches $\frac{1}{x} - k$ and $-\frac{1}{x} - k$ for $k \in \Lambda$, then all the other branches still cover all of the interval $[-1, 1]$. This means that for each $x \in [-1, 1]$ we can choose an $\omega \in \Omega$ such that $d_n(\omega, x) \neq k$ for all $n \geq 1$. □

From this proposition it follows immediately that each $x \in [-1, 1]$ has a continued fraction expansion with only even or only odd digits. Moreover, note that if we remove all branches corresponding to the odd (or even) digits, then there is no overlap in the system except on the points $\pm\frac{1}{k}$, implying that each irrational $x \in [-1, 1]$ has a unique expansion with only odd (or even) digits.

Regular continued fractions with bounded digits are very well studied. It has been known since the works of Jarnik ([Jar32]) and Good ([Goo41]) that for each $N \geq 2$ the set of points $x \in [0, 1]$ that has a regular continued fraction expansion with digits not exceeding N has Hausdorff dimension strictly between 0 and 1. Many people have given estimates for the Hausdorff dimension of the set $E_{1,2}$ of points with a regular continued fraction expansion using only digits 1 and 2, with the most recent result by Pollicott and Jenkinson ([JP01]) calculated the first 25 digits of the Hausdorff dimension of this set:

$$\dim_H(E_{1,2}) = 0.531280506277205141624686\ldots.$$

In the random case we have the following.

Proposition 4.3. *The set of points $x \in [-1, 1]$ that have a continued fraction expansion of the form (1) with $d_n \in \{1, 2\}$ for all $n \geq 1$, has positive Lebesgue measure. In fact, it is a countable union of intervals, containing the interval $\left(\frac{1}{2}, 1\right]$.*

Proof. Let $A_{1,2}$ denote the set of points that have an expansion of the form (1) with only digits 1 and 2. Then $A_{1,2} \cap \left(-\frac{1}{3}, \frac{1}{3} \right) = \emptyset$. First note that the points $\pm\frac{1}{k}$, $k = 2, 3$, have countably many expansions using only 1 and 2 as discussed after the proof of Proposition 4.1. These expansions are not generated by K. For all other points in $A_{1,2}$ such expansions are generated by K. We consider these expansions, see Figure 2 for an illustration.

The first digit of any $x \in A_{1,2} \cap \left(\left(\frac{1}{3}, \frac{1}{2} \right] \cup \left[-\frac{1}{2}, -\frac{1}{3} \right) \right)$ must be 2, coming from $\omega_1 = 0$. For $n \geq 1$, let S_n denote the map $S_n x = \left| \frac{1}{x} \right| - n$. Then for points in the set

$$E = \bigcup_{n \geq 1} S_2^{-n} \left(0, \frac{1}{3} \right) \cap \left(\left(\frac{1}{3}, \frac{1}{2} \right] \cup \left[-\frac{1}{2}, -\frac{1}{3} \right) \right)$$

K does not generate an expansion with only digits 1 and 2. Note that S_2 has fixed point $\sqrt{2} - 1$ and $|S_2' x| > 1$ for all $x \in \left(\frac{1}{3}, \frac{1}{2} \right] \cup \left[-\frac{1}{2}, -\frac{1}{3} \right)$. Since

$$\sqrt{2} - 1 \notin S_2^{-1} \left[0, \frac{1}{3} \right) = \left(\frac{3}{7}, \frac{1}{2} \right] \cup \left[-\frac{1}{2}, -\frac{3}{7} \right)$$

and

$$S_2^{-2} \left[0, \frac{1}{3} \right) = \left[\frac{2}{5}, \frac{7}{17} \right) \cup \left(-\frac{7}{17}, \frac{2}{5} \right],$$

it follows that $E \subseteq \left[\frac{2}{5}, \frac{1}{2} \right] \cup \left[-\frac{1}{2}, -\frac{2}{5} \right]$ and that E is a countable union of intervals. For the set $\left(\frac{1}{2}, 1 \right] \cap \left[-1, -\frac{1}{2} \right)$ we have the following. Since

$$S_1^{-1} \left[0, \frac{1}{3} \right) = \left(\frac{3}{4}, 1 \right] \cup \left[-1, -\frac{3}{4} \right) \quad \text{and} \quad S_2^{-1} \left(-\frac{1}{3}, 0 \right] = \left[\frac{1}{2}, \frac{3}{5} \right) \cup \left(-\frac{3}{5}, -\frac{1}{2} \right]$$

we see that on $\left(\frac{3}{4}, 1 \right]$ and $\left[-1, -\frac{3}{4} \right)$ we will have to use S_2 and on $\left(\frac{1}{2}, \frac{3}{5} \right)$ and $\left(-\frac{3}{5}, -\frac{1}{2} \right)$ we will have to use S_1. Note that

$$S_1^{-1} E \cap \left(\frac{1}{2}, 1 \right] \subseteq \left[\frac{2}{3}, \frac{5}{7} \right].$$

Since $\frac{3}{5} < \frac{2}{3} < \frac{5}{7} < \frac{3}{4}$, on all of the interval $\left[\frac{1}{2}, 1 \right]$ we can avoid being mapped into E by making appropriate choices between S_1 and S_2. Hence

$$A_{1,2} = \left(\left[\frac{1}{3}, 1 \right] \cup \left[-1, -\frac{1}{3} \right] \right) \setminus E. \qquad \square$$

In 1994 Lehner ([Leh94]) proved that every irrational number x in the interval $[1, 2)$ has a unique continued fraction expansion of the form

$$x = b_0 + \cfrac{\epsilon_0}{b_1 + \cfrac{\epsilon_1}{b_2 + \ddots}},$$

where (ϵ_n, b_n) equals either $(1, 1)$ or $(-1, 2)$. In [DK00] Dajani and Kraaikamp established a relation between these Lehner expansions and the regular continued fraction expansions, produced by the Gauss map. Here we see that all points in $A_{1,2}$ have a Lehner-like expansion
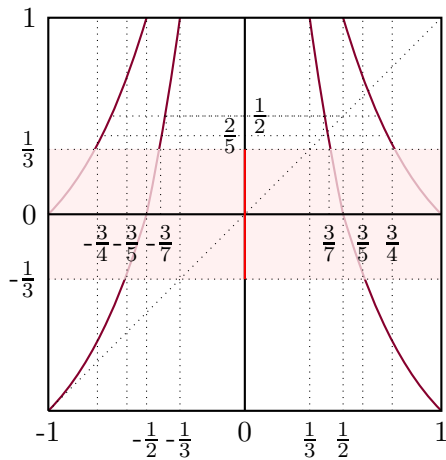
FIGURE 2. Points that have a continued fraction expansion generated by K with only digits 1 and 2 only use the maps $x \mapsto \left|\frac{1}{x}\right| - 1$ and $x \mapsto \left|\frac{1}{x}\right| - 2$.

with $b_0 = 0$. Moreover, there are points that have more than 1 such expansions. This holds for example for all points in the interval $\left(\frac{3}{5}, \frac{2}{3}\right)$ or in the interval $\left(\frac{5}{7}, \frac{3}{4}\right)$.

In spite of the result from Proposition 4.3, we have the following results on the asymptotic geometric and arithmetic mean of the digits, which are similar for the regular continued fractions.

Proposition 4.4. *For $(m_p \otimes \mu_p)$-a.e. $(\omega, x) \in \Omega \times [0,1]$, we have*

$$1 < \lim_{n \to \infty} \left(d_1(\omega, x) \cdots d_n(\omega, x)\right)^{1/n} < \infty,$$

and

$$\lim_{n \to \infty} \frac{d_1(\omega, x) + \cdots + d_n(\omega, x)}{n} = \infty.$$

Proof. We use the ergodicity of the map R. Recall the definition of the function $b : \Omega \times [0,1] \to \mathbb{N} \cup \{\infty\}$ from (8). For the first statement, we first show that $\log b \in L^1$. Suppose $h_p(x) \leq M$ for all $x \in [0,1]$. Then,

$$
\begin{aligned}
\int_{\Omega \times [0,1]} \log b \, d(m_p \otimes \mu_p) \ &\leq \ M \sum_{n \geq 1} \left[p^2 \int_{\frac{1}{n+1}}^{\frac{1}{n}} \log n \, dx + p(1-p) \int_{\frac{1}{n+1}}^{\frac{1}{n}} \log(n+1) \, dx \right. \\
&\qquad \left. + p(1-p) \int_{\frac{n-1}{n}}^{\frac{n}{n+1}} \log n \, dx + (1-p)^2 \int_{\frac{n-1}{n}}^{\frac{n}{n+1}} \log(n+1) \, dx \right] \\
&= \ M \sum_{n \geq 1} \left[\frac{p \log n}{n(n+1)} + \frac{(1-p) \log(n+1)}{n(n+1)} \right] < \infty.
\end{aligned}
$$

There also is an $m > 0$, such that $h_p(x) \geq m$ for all $x \in [0,1]$. Similarly as above, we obtain

$$\int_{\Omega \times [0,1]} \log b \, d(m_p \otimes \mu_p) \geq m \sum_{n \geq 1} \left[\frac{p \log n}{n(n+1)} + \frac{(1-p) \log(n+1)}{n(n+1)} \right] > 0.$$

Then by the Birkhoff Ergodic Theorem, for $(m_p \otimes \mu_p)$-a.e. (ω, x),

$$0 < \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log b\big(R^i(\omega, x)\big) = \int_{\Omega \times [0,1]} \log b \, d(m_p \otimes \mu_p) < \infty.$$

The result now follows from (9).

For the other statement, one first notices that $b(\omega, x) \geq \frac{1}{x} - 1$ if $\omega_1 = 0$ and $b(\omega, x) \geq \frac{1}{1-x} - 1$ if $\omega_1 = 1$. Hence, $b \notin L^1$. Therefore, consider the functions

$$b_N(\omega, x) = \begin{cases} b(\omega, x) 1_{(\frac{1}{N+1}, 1]}(x), & \text{if } \omega_1 = 0, \\ b(\omega, x) 1_{[0, \frac{N}{N+1})}(x), & \text{if } \omega_1 = 1. \end{cases}$$

Then each b_N is bounded and the sequence $\{b_N\}$ is increasing, so by Beppo-Levi's Theorem,

$$(14) \qquad \lim_{N \to \infty} \int_{\Omega \times [0,1]} b_N \, d(m_p \otimes \mu_p) = \int_{\Omega \times [0,1]} b \, d(m_p \otimes \mu_p) = \infty.$$

Moreover, the Birkhoff Ergodic Theorem gives the existence of a $(m_p \otimes \mu_p)$-measure 1 set of (ω, x), such that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} b_N\big(R^i(\omega, x)\big) = \int_{\Omega \times [0,1]} b_N \, d(m_p \otimes \mu_p)$$

for all $N \geq 1$. Then by (14) we get that for $(m_p \otimes \mu_p)$-a.e. (ω, x),

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} d_i(\omega, x) = \liminf_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} b\big(R^i(0\omega, x)\big)$$

$$\geq \lim_{N \to \infty} \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} b_N\big(R^i(0\omega, x)\big) = \int_{\Omega \times [0,1]} b_N \, d(m_p \otimes \mu_p) = \infty.$$

This gives the result. $\qquad \square$

5. Further Questions and Comments

So far we have established that the density h_p is of bounded variation and is bounded away from zero. It satisfies the equation

$$h_p(x) = \mathcal{L}_p h_p(x) = \sum_{k=1}^{\infty} \left[\frac{p}{(k+x)^2} h_p\left(\frac{1}{k+x}\right) + \frac{1-p}{(k+x)^2} h_p\left(1 - \frac{1}{x+k}\right) \right]$$

$$= p \mathcal{L}_0 h_p(x) + (1-p) \mathcal{L}_1 h_p(x),$$

where $\mathcal{L}_0$, $\mathcal{L}_1$ are transfer operators for T_0 and T_1 respectively. Both operators $\mathcal{L}_0$ and $\mathcal{L}_1$ preserve cones of positive smooth (analytic) functions on $[0, 1]$. Considerable effort has been put in the analysis of the spectral properties of $\mathcal{L}_0$, which are now well understood, see [Ios14] for a recent overview. It would be interesting to study the spectral properties of $\mathcal{L}_p$ in more detail. Based on simulations, we suspect the following.

Conjecture 5.1. *For each $0 < p < 1$ the function h_p is a* smooth *function on $[0, 1]$.*

The exact smoothness condition is to be determined. We presume that applying relatively standard techniques one could strengthen the results of the present paper by showing that $h_p \in C^k([0, 1])$ for all $k \in \mathbb{N}$. In fact, we believe that the density is C^∞; however, most probably it is not real-analytic (which is the case for the Gauss map).

This conjecture is motivated as follows. In [MR87, JGU03] the authors investigate the relation between $\mathcal{L}_0$ and the the integral operator $\mathcal{K}_0$ acting on the Hilbert space $L^2(\mathbb{R}_+, \mu)$, given by

$$\mathcal{K}_0\phi(s) = \int_0^\infty \frac{J_1(2\sqrt{st})}{\sqrt{st}}\phi(t)\,d\mu(t),$$

where J_1 is the Bessel function of the first kind, and μ is the measure on $\mathbb{R}_+$ with the density

$$d\mu = \frac{t}{e^t - 1}dt.$$

The operator $\mathcal{K}_0$ has a symmetric kernel $K_0(s, t) = \frac{J_1(2\sqrt{st})}{\sqrt{st}}$, and has several nice properties, e.g., is nuclear. In a similar fashion, existence of a positive smooth fixed point of $\mathcal{L}_p$ will follow from the existence of a positive fixed point of $\mathcal{K}_p$

$$\begin{aligned}(15) \quad \mathcal{K}_p\phi(s) &= p\mathcal{K}_0\phi(s) + (1-p)\mathcal{K}_1\phi(s) \\ &= p\int_0^\infty \frac{J_1(2\sqrt{st})}{\sqrt{st}}\phi(t)\,d\mu(t) + (1-p)\int_0^\infty \frac{I_1(2\sqrt{st})}{\sqrt{st}}\phi(t)e^{-t}\,d\mu(t),\end{aligned}$$

where I_1 is the modified Bessel function of the first kind. Technical difficulties arise from the fact that the kernel $K_1 = \frac{I_1(2\sqrt{st})}{\sqrt{st}}$ albeit monotonic and positive (c.f., K_0 is oscillating), is not integrable.

An exact formula for the density h_p would of course settle the conjecture. The most successful approach to constructing invariant densities for continued fraction transformations has been to build natural extensions of the transformations, see for example [Nak81, Haa02, IS06, DKS09, KSS10, KSS12] and the references therein. This approach was also effective in determining the invariant density of the random β-transformation (see [Kem14]), in which case the invariant density was not just a linear combination of the invariant densities of the two maps making up the random transformation. We have so far not been able to build a natural extension of the random continued fraction map.

With an expression for the density one could study frequencies of digits for the continued fraction expansions. It would also be interesting to consider the size of subsets of $[-1, 1]$

obtained like those in Propositions 4.2 and 4.3. For example, for any two consecutive digits n and $n + 1$ is there set with positive Lebesgue measure such that all points in this set have an expansion using only these digits?

Even without a formula for the density, one could study the behaviour of h_p as a function of p. Of particular interest is the case when $p \to 0$, since h_0 is unbounded and h_p is of bounded variation for each $p > 0$. A similar question can be asked for the metric entropy of R. Can one calculate this? The Shannon-McMillan-Breiman Theorem with the cylinder sets from the proof of Theorem 3.3 can be of help here. How does it behave as $p \to 0$? This last question could be seen as an analogue of the question of studying the entropy of α continued fractions as was done in [CT13].

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