

# Total absolute curvature as a tool for modelling curves and surfaces

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## Abstract

Total absolute curvature (TAC) is proposed as a tool for modelling curves and surfaces from discrete two- or three-dimensional data.

## 1 Introduction

The work presented in this communication is dedicated to the analysis of properties of discrete curves and surfaces with respect to their *total absolute curvature*. Under *discrete curves and surfaces* we understand *polygonal curves and polyhedral surfaces* that are reconstructed on the basis of given discrete data. The problem of reconstructing curves and surfaces arises in various domains such as Computer Vision, Computer Graphics, Reverse Engineering, Image Processing, Mathematics and Chemistry.

This work is a continuation of the research concerning discrete curvatures and optimality criteria based on curvatures, presented by the first author in a number of papers (see, for example, [2, 3, 4]).

We are interested in the following problem:

- **Main problem.** To what extent a discrete curve or surface is characterised by its total absolute curvature?

Motivation to study discrete curves and surfaces with respect to their total absolute curvature is due to a couple of reasons. First of all, curvatures

truly describe the shape of objects. For example, the curvature of a plane curve determines the smooth curve up to isometries. The *total absolute curvature of a curve is one of its global properties*, an important fact is that the total absolute curvature reaches its minimum value  $2\pi$  on convex curves. Discrete curves and surfaces belong to the class of non-regular curves and surfaces, for which the concept of curvature is also determined [5]. Research on discrete curvatures (*i.e.*, curvatures that can be computed on a set of points) is of growing interest in geometric modelling (some overview is given in [2]).

We are especially interested in the following subproblem of the Main problem:

- **Problem 1.** Given discrete two-dimensional (correspondingly, three-dimensional) data. Among all closed polygonal curves (correspondingly, closed polyhedral surfaces) that span these data find a curve or curves (correspondingly, a surface or surfaces) with the minimum total absolute curvature.

Motivation to study the above-mentioned problems is concerned with the concept of Tight submanifolds [6]. One of the main properties of one- and two-dimensional tight submanifolds in  $\mathbb{R}^3$  is that they possess the minimum total absolute curvature. Surface triangulations of minimum total absolute (extrinsic) curvature (MTAEC), reconstructed from discrete data, were introduced as well. The MTAEC triangulation coincides with the convex triangulation of the data if the data are in convex position, but the properties of triangulations with MTAEC for non-convex data are still mostly unknown (see [?] for a review).

The problem of finding a MTAEC triangulation is directly related to the *optimisation problem*. Indeed, in many applications the first step in reconstructing an object, consists of putting an initial structure on the data obtained by some measurements from this object. In surface reconstruction, this initial structure represents often a triangulation, which then is subjected to an optimisation process. A great majority of optimisation criteria consist of minimising a certain energy functional, such as minimisation of potential energy of tension of thin membrane, or potential energy of thin plate. The energy functional has to be presented in a discretised form, because the data are discrete and finite. Such discretisation often yields high computational cost and numerical difficulties. In order to avoid this, new methods are being developed. Instead of using an energy functional defined for a smooth surface in a discretised form, an analogue of this functional is used, which is defined directly on a discrete, *i.e.*, polyhedral surface. The

method of optimising an initial triangulation by minimising total absolute extrinsic curvature as far as we know, was the starting point in developing these new methods. Since its introduction, several other cost functions that measure various types of discrete curvatures have been proposed ([4, 7]). The idea behind minimisation of a certain energy is to obtain a fair surface, *i.e.*, a good representation of the underlying real continuous surface. Here the following question naturally arises:

- Does the minimisation of a certain energy functional or cost function yield a correct surface?

First of all, the ‘correct’ surface is often not defined. For example, in medical applications, one may have a cloud of points that is a discrete representation of an organ, but how to evaluate a surface reconstructed from the data? Another problem is that energy functionals can have multiple local minima. In general, any local minimum of energy is considered as a solution to the optimisation problem, mostly because the global minimum is hard to reach. However, the global minimum might also be not unique, moreover, it might be far away from the input configuration ([8]). A local minimum, which is as close as possible to the initial configuration, might be more meaningful. Indeed, by minimising energy functionals, one uses methods of continuous mathematics, but to what extent is it legitimate to apply continuous methods to modelling objects from discrete finite data sets? Even, if we use an analogue of such a functional, defined for discrete (polyhedral) surfaces, one cannot guarantee that a solution to the optimisation problem will yield a desirable result. Given the data, one can construct many polyhedral surfaces that span the data, and minimisation of some discrete energy of the triangulated surface (control mesh) does not necessarily lead to the minimisation of the corresponding energy of the associated continuous surface.

We are interested in determining the general properties of discrete curves and surfaces of minimum total absolute curvature, in order to better understand the optimisation problem. In this paper we present several results concerning curves, *i.e.*, concentrate on the  $2D$ -version of Problem 1. As surfaces are much more complex objects than curves, the study of the one-dimensional version of the MTAEC triangulation provides useful insights for research on triangulations. Problem 1 concerning curves, was partially studied in [1]. Note also, that the study of polygonal curves with respect to their total absolute curvature is also interesting for its own sake, as the total absolute curvature is related to the global properties of the curve and might be used to characterise diverse curve profiles [9].

## 2 Polygonal Curves. Notions related to the concept of curvature

### 2.1 Formulae and definitions

In this paper we discuss only curves, which are continuous images of the circle  $S^1$  into the plane, *i.e.*, *closed polygons*. Let us denote with  $\alpha(v_i)$  the exterior angle at the vertex  $v_i$  of a curve. Then expression

$$\omega = \sum_{i=1}^n \alpha(v_i) \quad (1)$$

represents the total curvature of a closed polygonal curve. If this curve is the boundary of a closed simple polygon,  $\omega$  is always equal to  $2\pi$  (an elementary case of the Gauss-Bonnet theorem). The following expression:

$$\hat{\omega} = \sum_{i=1}^n |\alpha(v_i)| \quad (2)$$

is called the *absolute total curvature* of a curve. An important fact is that  $\hat{\omega}$  reaches its minimum value  $2\pi$  on convex curves (polygons).

Therefore for a non-convex curve the excess in the total absolute curvature with respect to the curvature of the convex curve can be used as a measure of deviation from the convex curve.

Given the data, we take as a *convex curve of reference* the boundary of the convex hull of the data, (referred to as the *CB-curve*). The statement of Problem 1 allows self-intersecting curves, but in this communication we presume that polygonal curves represent boundaries of simple polygons, *i.e.*, are not self-intersecting.

We want to determine the measure of deviation of the curve  $L$  with respect to the *CB-curve*. Intuitively, it is clear that we should minimise the amount of curvature contributed by *reflex vertices*. Let us make this assumption more precise.

A simple polygonal curve  $L$ , that spans the data, is represented by its vertices  $V_1, V_2, \dots, V_{n-1}, V_n$ . Then  $V_i V_j$ , where  $j = i + 1$ , is a line segment, and its length is denoted with  $l_{ij}$ ;  $l_{ij} = l_{ji}$ . We consider only closed curves, therefore the line segment  $V_n V_1$  belongs to the curve  $L$ . The *star* of a vertex  $V_i$  is the union of the vertex and its two adjacent line segments  $V_{i-1} V_i$  and  $V_i V_{i+1}$ .

We designate the vertices that lie on *CB-curve* with  $V_{CB}^i$ , and with  $\gamma_i, i = 1, \dots, l$  the exterior angles at these vertices with respect to the *CB-curve*. It is clear that  $\sum \gamma_i = 2\pi$ .

Let us denote with  $V_{conv}^j$  convex vertices of  $L$  and with  $\alpha_j, j = 1, \dots, m$  the exterior angles at these vertices with respect to  $L$ , and with  $V_{reflex}^k$  - reflex vertices of  $L$  and with  $\beta_k, k = 1, \dots, p$ , the corresponding exterior angles at these vertices with respect to  $L$ ; where  $m + p = n$ .

For a simple closed polygon the following equality holds:

$$\sum \alpha_j - \sum \beta_k = 2\pi \quad (3)$$

As  $\hat{\omega} = \sum \alpha_j + \sum \beta_k$ , the total absolute curvature of a polygonal closed simple curve can be represented as

$$\hat{\omega} = 2\pi + 2 \sum_k \beta_k \quad (4)$$

The set of convex vertices  $V_{conv}^j$  can be further split into three disjoint subsets, namely, the subset  $V_{conv-CB}^{j_1}$  of vertices that lie on the *CB-curve* and such that their stars belong to the *CB-curve*; the subset  $V_{conv-int}^{j_2}$  of vertices that are convex but do not belong to the *CB-curve*; and the subset  $V_{conv-corn}^{j_3}$  of vertices that lie on the *CB-curve* but their stars do not belong to the *CB-curve*. Vertices of the last type are called *corner vertices* as at these vertices the curve is deviated from the *CB-curve*. The corresponding exterior angles are denoted as  $\alpha_{j_1}, \alpha_{j_2}$  and  $\alpha_{j_3}$ . Obviously, any part of the curve that is deviated from the *CB-curve* starts and ends at the neighbouring vertices of  $V_{conv-corn}^{j_3}$ , since our curve has no self-intersections. Segments that have a corner vertex as an end-vertex, but do not belong to the *CB-curve*, are called *deviating segments*.

Each  $\alpha_{j_3}$  is equal to  $\gamma_{j_3} + \alpha(CB)_{j_3}$ ; where by  $\alpha(CB)_{j_3}$  we denote the angle at a corner vertex of the curve  $L$  with respect to the *CB-curve*, or in other words, the angle between the deviating segment of  $L$  that has this corner vertex as one of the end-vertices and the (imaginary) segment of the *CB-curve*, that would have the same corner vertex as one of the end-vertices. We call such an angle a *deviating angle* (see Fig. 1)

After some calculation, equation 3 is transformed to the following one:

$$\sum \alpha_{j_2} + \sum \alpha(CB)_{j_3} = \sum \beta_k \quad (5)$$

Therefore, to find the curve (or curves) of minimum total absolute curvature among all curves that span the given data, it is sufficient to minimise either

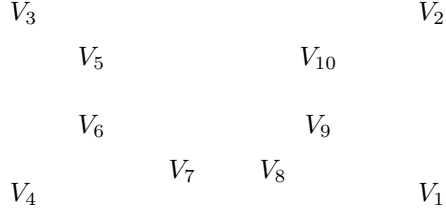


Figure 1: A non-convex curve. The deviating angle at vertex  $V_4$  is formed by the deviating segment  $V_4V_5$  and the (imaginary) segment  $V_4V_1$  of the  $CB$ -curve

the sum of exterior angles at reflex vertices or the sum of exterior angles at the internal convex vertices (if they exist) and the deviating angles. Let us remind that deviating angles always appear in pairs and each pair indicates a deviation of the curve from the reference convex curve. Each deviated part  $L_d$ ,  $d = 1, \dots, f$  contains in its turn some convex internal vertices  $V_{conv_d}^{j_d}$  (its number may be equal to zero) and reflex vertices  $V_{reflex_d}^{k_d}$ . Equality 5 holds for each deviated part:

$$\sum (\alpha_{j_2^d} + \alpha(CB)_{j_3^d}) = \sum \beta_{k^d} \quad (6)$$

Let us define a convex region  $R_{conv}$  as a part  $V_i, V_{i+1}, \dots, V_{i+j}$  of a curve  $L$  that satisfies the condition that each vertex that belongs to  $R_{conv}$  is convex, but vertices  $V_{i-1}$  and  $V_{i+j+1}$  are reflex. A concave region  $R_{concave}$  is defined analogously. From the above mentioned we get the following statement:

**Lemma 2.1** *Suppose we construct a curve  $L_1$  which has  $g$  convex regions and  $h$  concave ones. If we add new vertices in such a way that the obtained regions are preserved, then a new curve  $L_2$  will possess the same total absolute curvature as  $L_1$ .*

Those segments of a curve whose vertices are of different types (i.e., one is reflex, and another - convex) are called *separating segments*.

## 2.2 Spherical image of a curve. Curvature identical curves

The *spherical image* of a curve illustrates graphically the concept of *total absolute curvature*.

The spherical image for a polygonal curve is constructed by means of outwards unit normals to the line segments of the curve, all of them are ‘translated’ to the same origin. The ends of the unit normals of a planar curve will lie on the unit circle. Let us suppose that we walk around the boundary of a polygon, for example, in anticlockwise direction starting from vertex  $V_1$ , passing through all the vertices according to their order until we arrive again at the vertex  $V_1$ . To this walk a corresponding walk on the circle is generated, some of its parts are walked several time for a non-convex curve. The length of this walk is equal to the total absolute curvature of a curve. The spherical image provides a vertex classification of the curve as for a reflex vertex the direction will be opposite to the chosen one. The spherical images can be put in one-to-one correspondence for two curves of the same data set if the numbers of concavities/convexities and corresponding ‘incorporated’ curvatures for the both curves are the same. We say in this case that two curves are *curvature identical*. If the total absolute curvatures are equal for two curves, but they are not curvature identical, their spherical images will be different; because in this case either the numbers of concavities (*i.e.*, concave regions of a curve) will be different or the ‘incorporated’ curvatures will be different. Therefore, the spherical image can be used as a representative of any subset of curvature identical curves that span the same data set.

Examples of two *curvature identical* curves and their corresponding spherical images (schematically depicted), are given in Fig. 2 and Fig. 3.

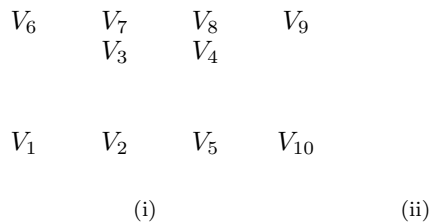


Figure 2: Ten-points data set. Curve 1 and its spherical image

If we change the starting point of our walk in one of the polygons, the spherical images of both polygons can be put in one-to-one correspondence. From the definition of the spherical image it follows that in order to determine the excess of total absolute curvature of a given curve with respect to the convex curve, it is sufficient to reconstruct the outwards normals only to

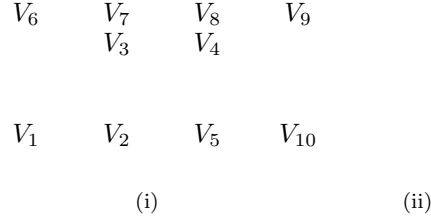


Figure 3: Ten-points data set. Curve 2 and its spherical image

separating and deviating segments. Then the length of the walk on a circle that the end-points of these normals generate, by observing the proper ordering, will be equal to this excess.

As we saw in the previous subsection, a curve or a surface of minimum total absolute curvature, that span the data may be not unique. An open question is how to determine all curves or surfaces of minimum *TAC* for the given data. This problem can be reformulated in a more general form:

- **Problem 2.** Given a two- or three- dimensional data set and a certain positive number  $v_\alpha$ . Determine the subset of curves or, correspondingly, surfaces, that span the data and whose total absolute curvature  $\hat{\omega}$  is equal to  $v_\alpha$ .

Of course, such a subset may be empty. In general, at least one curve or surface with total absolute curvature equal to a certain value  $v_\alpha$  is to be reconstructed and then other curves or surfaces with the same value of total absolute curvature might be determined.

In general, we can assume, that the set  $L_n$  of all admissible curves that span the same finite discrete data, is divided into a finite number of disjoint subsets, each subset  $L_n^i$  containing curves with the same value of total absolute curvature  $v_{\alpha_i}$ .

Curves with the total absolute curvature equal to  $v_{\alpha_i}$  might be not curvature identical. It might be the case that the subset (possibly empty) of curves with the same  $\hat{\omega}$  are further divided in several disjoint subsets of curvature identical curves. The curve that has the minimum total absolute curvature with respect to all admissible curves, is denoted with  $L_{min}$ .



### 3 Approach and examples

From the previous section, we can conclude that in order to obtain a curve of minimum total absolute curvature that spans a given data set  $S$ , three parameters may be minimised: the *number of deviations* of a curve from the  $CB$ -curve, the *number of convex regions in a deviation*, and the *amount of curvature contributed by reflex vertices in each deviation*. The dependence among these parameters is not straightforward. What we really need to do, is to decrease the amount of curvature contributed by all the reflex vertices in a curve, and a curve with only one deviation does not guarantee to be a curve of minimum  $TAC$ . However, the above remarks give us an indication on how to approach the study of properties of discrete curves with respect to their total absolute curvature and how to design appropriate algorithms. If we have the data set  $S$  in a non-convex position, then we can determine the *layers* of this set by repeatedly removing all convex hull elements and considering the convex hull of the remaining set. The set  $S$  has  $k$  *layers* if this process terminates after precisely  $k$  steps. So the first layer is the convex hull of data points, and other layers are *nested convex hulls*. A segment of the curve whose end-points belong to two different layers is called *bridgeable*. Two segments form a *bridge* if they belong to the same deviation and their end-points belong to the same not necessarily successive layers. A deviation is called *simple* if in this deviation no two bridges have end-points that belong to the same layers. The following statements are valid:

**Lemma 3.1** *If the the data has  $k$  layers then the number of bridgeable segments in any admissible polygonal curve is greater or equal to  $k - 1$ . It reaches the minimum value only if the curve is convex. For non-convex curves the minimum number of bridgeable segments is equal to  $k$ .*

**Lemma 3.2** *If the number of bridgeable segments is equal to the number of layers then a curve has one simple deviation. The number of bridges in a curve with one simple deviation does not exceed  $k - 1$  where  $k$  is the number of layers.*

**Corollary 3.1** *The maximal number of bridgeable segments in a curve with one simple deviation is equal to  $2(k - 1)$  where  $k$  is the number of layers.*

**Lemma 3.3** *The total curvature of a curve with one simple deviation does not exceed  $2\pi + 4\pi(k - 1)$ , where  $k$  is the number of layers. This bound is exact.*

Because of the lack of space we omit the proofs here. Now the question is: *how do multiple deviations influence the TAC?*

This difficult problem is still under study, but we can make several observations. First is that any new deviation may contribute to *TAC* up to  $2\pi + 4\pi(k - 1)$ . Adding new bridges may also contribute to curvature. So, a heuristic idea is to keep deviations as simple as possible, and deviating angles as small as possible. Below we illustrate how this idea works in practice.

We designed several algorithms to minimise *TAC*. They can be divided in two groups, algorithms of the first group search for a curve with minimum *TAC* among curves with only one deviation, and in the second group – among curves with multiple deviations. In the latter case the output curve with only one deviation is also possible. We illustrate the algorithms by two examples.

In both examples the data consist of two layers. For simplicity, we assume that the data lie on two concircular circumferences, of radii  $r_{outer}$  and  $r_{inner}$  correspondingly. We denote a curve of minimum *TAC* with  $L_{min}$ , and with  $L_{min}^0$  - the curve with one deviation that has the minimum *TAC* with respect to all curves with one deviation that span the given data.

**Example 1.** In this example the data are situated so that if we connect the centre  $O$  with a point  $v$  on the outer circumference, there is a point  $u$  that lies on the inner circumference and that belongs to the segment  $Ov$ . On each circumference lie 12 points. This example clearly shows that the concept of ‘nearness’ between two sample points is not essential for changing the total absolute curvature of a curve. We can show that the curvature  $v_\alpha$  of a curve with multiple deviations is always larger than the  $L_{min}^0$ -curve for these data. Moreover,  $v_\alpha$  keeps the same value no matter how small is the value  $\delta r = r_{outer} - r_{inner}$  (see Fig 4).

This example also shows that a global minimum to a discrete optimisation problem might be very far indeed from the input configuration. For example, we can assume that sample points are taken from the curve  $L_{saw}$  presented in Fig. 4, (iii). In these points high values of curvature are ‘incorporated’, and therefore they are considered as significant for shape description. The curve (ii) in Fig. 4 might be served as an initial approximation of  $L_{saw}$ , but not the curve (i).

**Example 2.** In this example on each circumference lie 10 points. The points on the inner circumference are slightly shifted at the same distance  $\epsilon$  along the circumference, so that no point on the inner conference belongs to the segment  $Ov$  anymore. In this case we can show that by decreasing the distance  $\delta r = r_{outer} - r_{inner}$  at a certain moment the curve of minimum

(i) (ii) (iii)

Figure 4: Example 1. (i) - curve  $L_{min}^o$  that is also  $L_{min}$ ; (ii) - curve of minimum  $TAC$  among curves with multiple deviations; (iii) -  $L_{saw}$ -curve

$TAC$  among curves with multiple deviations becomes  $L_{min}$  and its  $TAC$  tends to  $2\pi$  (see Fig 5).

## 4 Conclusion

We have introduced and discussed ‘discrete versions’ of minimum total absolute curvature curves. We have showed that given the data a curve of minimum  $TAC$  may be not unique, and that a solution to a discrete optimisation problem may lead to an unexpected curve and might be very far from the input configuration. Nevertheless, by imposing additional constraints, the criterion of minimising  $TAC$  may be very useful for a variety of applications. Other types of discrete energy should also be considered. Some of them were discussed in [1]. The research on the discrete optimisation problem provides new insights for geometric modelling.

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(i)

(ii)

(iii)

(iv)

Figure 5: Example 2. (i) - curve  $L_{min}^o$  that is also  $L_{min}$ ; (ii) - curve of minimum  $TAC$  among curves with multiple deviations; (iii) - curve  $L_{min}^o$  that is no longer  $L_{min}$ ; (iv) - curve with multiple deviations that is also  $L_{min}$

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