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Abstract

We determine all 120 nonisomorphic systems obtainable from the projective Steiner triple system of order 31 by at most three Pasch trades. Exactly three of these, each corresponding to three Pasch trades, are rigid. Thus three Pasch trades suffice, and are required, in order to convert the projective system of order 31 to a rigid system. This contrasts with the projective system of order 15 where four Pasch trades are required. We also show that four Pasch trades are required in order to convert the projective system of order 63 to a rigid system.

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1 Introduction

A *Steiner triple system* of order v, STS(v), is an ordered pair $(V, \mathcal{B})$ where V is a v-element set (the *points*) and $\mathcal{B}$ is a set of triples from V (the *blocks*), such that each pair from V appears in precisely one block. The necessary and sufficient condition for the existence of an STS(v) is that $v \equiv 1$ or 3 (mod 6) [4]; such values of v are called *admissible.* As is usual, we often omit set brackets and commas from triples of points, so that $\{x, y, z\}$ may be written as xyz when no confusion is likely. An *automorphism* of an STS(v) $= (V, \mathcal{B})$ is a permutation on the points of V that preserves the set of blocks $\mathcal{B}$. An STS(v) is *rigid* (*automorphism-free* or *asymmetric*) if its only automorphism is the identity permutation.

If T_1 and T_2 are disjoint sets of triples from a common point set V that cover the same pairs of points, then the pair $\mathcal{T} = \{T_1, T_2\}$ is called a *trade pair* and T_1 and T_2 are *tradeable configurations.* If an STS(v) contains a copy of T_1, then that copy may be replaced by the corresponding copy of T_2 to give another STS(v). This operation is called a $\mathcal{T}$-trade. The set of points covered by T_1 and T_2 is called the *foundation* of the trade, and the number of blocks in each T_i $(i = 1, 2)$ is called the *volume* of the trade. A *Pasch configuration* or *quadrilateral* or 4-*cycle* $P(a, b, c, d, e, f)$ is a set of four triples abc, ade, bdf, cef on six distinct points $\{a, b, c, d, e, f\}$. The *opposite* Pasch configuration is $\overline{P}(a, b, c, d, e, f) = P(f, b, c, d, e, a)$, and this covers the same pairs with a disjoint set of triples. If P_1 and P_2 are opposite Pasch configurations then $\mathcal{P} = \{P_1, P_2\}$ is a trade pair and the corresponding replacement operation is called a *Pasch trade.* This is the smallest possible trade in an STS(v), both by foundation and by volume.

The *projective triple system* S_n of order $v = 2^n - 1$ is the point-line design of the projective space PG$(n-1, 2)$. It may be realized as an STS(v) $= (V, \mathcal{B})$ with point set $V = \mathbb{Z}_2^n \setminus \{\mathbf{0}\}$ and whose blocks comprise all triples of points $\mathbf{xyz}$ such that $\mathbf{x} \oplus \mathbf{y} \oplus \mathbf{z} = \mathbf{0}$, where $\oplus$ denotes vector addition in $\mathbb{Z}_2^n$. Most of our results below relate to small values of n and it is then convenient to represent $\mathbf{x} = (x_{n-1}, x_{n-2}, \cdots, x_0) \in \mathbb{Z}_2^n$ as the number $\sum_{i=0}^{n-1} x_i 2^i$; for example $(1, 0, 1, 1)$ is represented as $8 + 2 + 1 = 11$.

It is known that, amongst Steiner triple systems on $v = 2^n - 1$ points, S_n has the largest group of automorphisms and the largest number of Pasch configurations. In fact $|\mathrm{Aut}(S_n)| = 2^{\frac{n(n-1)}{2}} \Pi_{i=2}^{n} (2^i - 1)$, see [3, page 41], and the number of Pasch configurations is $v(v-1)(v-3)/24$, [5]. Although the automorphism group of S_n is large, we showed in [2] that, if $n \geq 4$, then by applying n specific Pasch trades to S_n it is possible to obtain a rigid STS$(2^n - 1)$. This result is the best possible for $n = 4$, since in [1] there is an analysis of all 80 nonisomorphic STS(15)s, and it is shown that four Pasch trades are necessary in order to convert S_4 to a rigid system.

Denote by $\mu(n)$ the minimum number of Pasch trades needed to convert S_n to a rigid system, so that $\mu(4) = 4$. In [2] we derived the bounds

$$n/3 \leq \mu(n) \leq n, \tag{1}$$

but the precise value of $\mu(n)$ is generally unknown. An obvious question to ask is whether or not $\mu(n)$ is monotonically increasing. This is not the case since, rather surprisingly, $\mu(5) = 3$. In this note we determine all 120 nonisomorphic systems obtainable from S_5 by at most three Pasch trades and show that exactly three of these, each corresponding to three Pasch trades, are rigid. We also show that $\mu(6) = 4$.

2 Results

Our results for $n = 5$ are presented in Tables 1–3. Table i, $1 \leq i \leq 3$, contains all the non-isomorphic STS(31)s which are obtained from S_5 by i Pasch trades. For comparison, Table 1 also includes the original system S_5. Of course, in Table i we present only systems which do not appear in Table j for $j < i$. For every pair of Pasch configurations in S_n there is an automorphism mapping one of these configurations to the other. Hence the first Pasch trade in every case may be taken as $\{P_1, \overline{P_1}\}$, where $P_1 = P(1, 2, 3, 4, 5, 6)$ and therefore this trade is not listed in any of the tables. In Tables 2 and 3 we give the Pasch configuration(s) P, such that $\{P, \overline{P}\}$ is the corresponding trade. These trades are followed by the order of the automorphism group (|Aut|) and by the number of Pasch configurations of the system ($|P|$). The tables also give an interesting invariant (Pasch-point incidence), namely the number of Pasch configurations incident with each of the 31 points of the system. An entry such as 198^{24} indicates that 24 points each have 198 incident Pasch configurations. With the exception of one pair of systems appearing in Table 3 (#89 and #90), each having 48 automorphisms and 821 Pasch configurations, this invariant together with the number of automorphisms and the number of Pasch configurations uniquely distinguishes the systems. Note that after making a trade, new blocks appear in the system and these may lie in Pasch configurations that can subsequently be traded. Thus Tables 2 and 3 include the results of trading Pasch configurations not present in the original system S_5.

#	Number of trades	|Aut|	$\|P\|$	Pasch-point incidence
1.	0	9999360	1085	210^{31}
2.	1	9216	989	$210^{1}198^{24}162^{6}$

Table 1. S_5 and a single Pasch trade.

#	Trade 2	|Aut|	|P|	Pasch-point incidence
3.	$P(1,8,9,16,17,24)$	32	901	$198^2 187^8 186^{10} 154^8 150^2 122^1$
4.	$P(7,8,15,16,23,24)$	96	893	$198^1 186^{18} 162^1 150^{11}$
5.	$P(1,6,7,8,9,14)$	128	901	$198^1 188^4 186^{16} 162^1 158^1 154^6 118^2$
6.	$P(1,8,9,15,14,7)$	512	901	$188^4 186^{16} 162^2 154^8 122^1$
7.	$P(1,8,9,14,15,6)$	1024	925	$210^1 194^4 186^{16} 166^8 130^2$
8.	$P(8,16,24,31,23,15)$	1152	893	$210^1 186^{18} 150^{12}$
9.	$P(1,2,4,5,3,7)$	4608	941	$192^{24} 162^3 138^4$

Table 2. Two Pasch trades.

There are several interesting observations related to the 120 STS(31)s presented in Tables 1 to 3. First, observe that although $|\mathrm{Aut}(S_5)|$ is not a power of 2, as many as 96 out of these 120 systems have an automorphism group order which is a power of 2. Second, the least number of Pasch configurations, namely 797, does not occur with a rigid system, but with three systems having 24, 32 and 576 automorphisms (#72, #73 and #115 respectively). Third, only 11 of the systems are obtained by Pasch trades which use a block not present in S_5 (i.e., at least one of the blocks appeared after a previous trade). Perhaps surprisingly, such systems have relatively many automorphisms. One of these systems is the last in Table 2 (#9) and the others are the systems #57, #71, #98, #99, #100, #101, #109, #110, #112 and #120.

Our final remark is related to the analogous problem for S_6. Using the automorphisms of the STS(63) obtained from S_6 by the Pasch trade $\{P_1, \overline{P_1}\}$, where $P_1 = P(1,2,3,4,5,6)$, we found that three Pasch trades do not suffice to convert S_6 to a rigid system. However, four Pasch trades do suffice. One such quadruple of trades is determined by the Pasch configurations $P(1,2,3,4,5,6)$, $P(1,6,7,8,9,14)$, $P(1,10,11,16,17,26)$ and $P(2,5,7,32,34,37)$, but there are many others. Listing all quadruples of Pasch trades that yield nonisomorphic rigid STS(63)s is currently beyond our computational resources.

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#	Trade 2	Trade 3	|Aut|	$\|P\|$	Pasch-point incidence
10.	$P(1,8,9,16,17,24)$	$P(2,12,14,25,27,21)$	1	813	$186^2 176^2 175^8 174^3 154^1 146^2 143^2 142^7 139^2 114^2$
11.	$P(1,8,9,16,17,24)$	$P(2,12,14,27,25,23)$	1	813	$187^1 186^1 176^2 175^8 174^3 154^1 146^2 143^2 142^7 139^1 138^1 114^2$
12.	$P(1,6,7,8,9,14)$	$P(2,9,11,16,18,25)$	1	820	$187^1 186^1 178^1 177^2 176^1 175^6 174^5 154^1 150^1 147^1 146^4 143^1 142^2 117^2 113^1 110^1$
13.	$P(1,8,9,16,17,24)$	$P(7,10,13,28,27,22)$	2	805	$186^1 175^6 174^7 154^1 150^1 142^7 139^2 138^5 110^1$
14.	$P(1,6,7,8,9,14)$	$P(2,13,15,16,18,29)$	2	813	$186^1 177^2 176^1 175^4 174^8 154^1 150^1 146^3 144^1 143^2 142^3 138^1 114^1 110^2$
15.	$P(1,8,9,16,17,24)$	$P(2,12,14,21,23,25)$	2	813	$186^2 176^4 175^4 174^5 150^1 146^2 143^4 142^6 138^1 114^2$
16.	$P(1,8,9,16,17,24)$	$P(6,10,12,25,31,19)$	2	813	$187^1 186^1 175^{10} 174^3 154^1 146^4 143^4 142^3 139^1 138^1 110^2$
17.	$P(1,6,7,8,9,14)$	$P(2,8,10,16,18,24)$	2	819	$186^2 178^2 176^3 175^4 174^6 150^1 146^6 142^3 120^1 116^1 113^2$
18.	$P(1,8,9,15,14,7)$	$P(2,8,10,16,18,24)$	2	820	$186^1 178^1 177^2 176^1 175^6 174^5 154^2 147^1 146^5 143^2 142^2 117^3$
19.	$P(1,8,9,16,17,24)$	$P(2,8,10,21,23,29)$	2	820	$186^3 177^2 176^4 175^4 174^3 147^1 146^5 143^2 142^4 117^3$
20.	$P(1,8,9,16,17,24)$	$P(2,8,10,23,21,31)$	2	820	$187^1 186^2 177^1 176^5 175^5 174^2 146^6 143^2 142^4 117^3$
21.	$P(1,6,7,8,9,14)$	$P(1,10,11,16,17,26)$	2	821	$187^1 186^1 177^2 175^8 174^4 154^1 151^1 148^2 147^3 146^4 143^1 138^1 110^1 86^1$
22.	$P(1,8,9,16,17,24)$	$P(2,12,14,24,26,20)$	2	821	$187^1 186^2 177^1 176^5 175^5 174^2 147^3 146^5 143^2 142^2 118^1 114^2$
23.	$P(1,8,9,16,17,24)$	$P(2,12,14,26,24,22)$	2	821	$187^3 177^1 176^5 175^5 174^2 147^1 146^7 143^2 142^2 118^1 114^2$
24.	$P(1,8,9,16,17,24)$	$P(7,10,13,25,30,19)$	4	805	$187^1 175^4 174^9 150^2 142^8 139^3 138^3 110^1$
25.	$P(1,6,7,8,9,14)$	$P(2,16,18,27,25,11)$	4	813	$186^1 176^3 175^8 174^4 154^1 150^1 146^3 142^6 140^1 114^1 110^2$
26.	$P(1,6,7,8,9,14)$	$P(7,10,13,16,23,26)$	4	813	$186^2 177^2 175^4 174^8 146^3 144^2 143^2 142^4 138^1 122^1 110^2$
27.	$P(1,6,7,8,9,14)$	$P(9,16,25,26,19,10)$	4	813	$186^1 176^3 175^8 174^4 154^2 150^1 146^1 142^7 140^1 114^1 110^1 106^1$
28.	$P(1,8,9,15,14,7)$	$P(2,16,18,26,24,10)$	4	813	$176^3 175^8 174^4 154^2 150^1 146^2 142^8 140^1 114^2$
29.	$P(1,8,9,16,17,24)$	$P(2,12,14,23,21,27)$	4	813	$198^1 186^1 176^4 175^4 174^5 146^2 143^4 142^6 138^2 114^2$
30.	$P(1,6,7,8,9,14)$	$P(1,14,15,16,17,30)$	4	821	$186^1 177^4 176^4 175^4 174^4 154^2 150^2 146^6 142^1 114^1 110^1 82^1$
31.	$P(1,6,7,8,9,14)$	$P(1,16,17,26,27,10)$	4	821	$188^1 186^1 176^6 175^8 154^1 150^1 146^9 142^1 140^1 110^1 86^1$
32.	$P(1,6,7,8,9,14)$	$P(2,16,18,25,27,9)$	4	821	$188^1 186^1 176^7 175^8 154^1 150^1 146^7 142^1 114^3 110^1$
33.	$P(1,8,9,16,17,24)$	$P(1,10,11,22,23,28)$	4	821	$187^1 186^2 176^4 175^8 147^6 146^6 139^1 138^2 90^1$
34.	$P(1,8,9,16,17,24)$	$P(2,12,14,20,22,24)$	4	821	$187^1 186^2 176^4 175^8 174^1 147^5 146^5 143^2 114^2 110^1$
35.	$P(1,8,9,14,15,6)$	$P(2,8,10,16,18,24)$	4	841	$198^1 186^1 184^2 182^1 176^2 175^4 174^6 157^4 154^2 150^1 146^1 142^2 126^2 124^2$
36.	$P(1,8,9,16,17,24)$	$P(2,8,10,31,29,23)$	6	820	$187^3 177^1 176^3 175^9 146^6 142^6 117^3$
37.	$P(1,8,9,16,17,24)$	$P(2,8,10,20,22,28)$	6	821	$186^3 177^4 175^6 174^3 148^3 146^3 142^6 118^3$

Table 3. Three Pasch trades (page 1 of 4).

#	Trade 2	Trade 3	|Aut|	|P|	Pasch-point incidence
38.	$P(1,6,7,8,9,14)$	$P(10,16,26,27,17,11)$	8	805	$186^1 176^2 174^{12} 150^1 146^1 142^6 140^2 138^4 118^1 106^1$
39.	$P(1,6,7,8,9,14)$	$P(10,16,26,31,21,15)$	8	805	$176^3 174^{12} 154^1 150^2 146^1 142^5 140^1 138^4 106^2$
40.	$P(1,8,9,16,17,24)$	$P(7,10,13,20,19,30)$	8	805	$198^1 175^4 174^9 150^1 142^8 139^4 138^3 110^1$
41.	$P(1,8,9,16,17,24)$	$P(7,10,13,21,18,31)$	8	805	$186^1 175^4 174^9 150^2 142^8 139^4 138^2 110^1$
42.	$P(1,8,9,16,17,24)$	$P(10,22,28,29,23,11)$	8	805	$186^2 175^6 174^6 142^8 139^2 138^6 122^1$
43.	$P(1,6,7,8,9,14)$	$P(2,16,18,19,17,3)$	8	813	$186^1 176^8 174^8 150^1 146^3 142^6 118^1 114^1 110^2$
44.	$P(1,6,7,8,9,14)$	$P(2,16,18,26,24,10)$	8	813	$186^1 176^3 175^8 174^4 158^1 150^1 146^2 142^7 140^1 114^1 110^2$
45.	$P(1,6,7,8,9,14)$	$P(2,16,18,29,31,13)$	8	813	$198^1 176^3 175^8 174^4 150^1 146^3 142^7 140^1 114^1 110^2$
46.	$P(1,6,7,8,9,14)$	$P(2,16,18,31,29,15)$	8	813	$188^1 176^3 175^8 174^4 150^2 146^3 142^7 114^1 110^2$
47.	$P(1,6,7,8,9,14)$	$P(7,16,23,26,29,10)$	8	813	$188^1 186^1 176^2 175^8 174^4 146^3 142^8 140^1 122^1 110^2$
48.	$P(1,6,7,8,9,14)$	$P(8,16,24,26,18,10)$	8	813	$186^1 176^3 175^8 174^4 154^1 150^1 146^2 142^7 140^1 118^1 110^2$
49.	$P(1,8,9,15,14,7)$	$P(6,10,12,16,22,26)$	8	813	$186^1 176^2 175^4 174^8 150^1 146^4 144^2 143^4 142^2 138^1 122^1 110^1$
50.	$P(1,8,9,16,17,24)$	$P(6,10,12,19,21,25)$	8	813	$186^2 175^8 174^5 150^1 146^4 143^8 138^1 110^2$
51.	$P(1,6,7,8,9,14)$	$P(2,5,7,16,18,21)$	8	821	$186^2 177^4 176^4 175^2 174^6 147^2 146^6 122^1 114^4$
52.	$P(1,6,7,8,9,14)$	$P(2,16,18,23,21,7)$	8	821	$186^1 177^4 176^4 175^4 174^4 154^1 146^9 122^1 114^3$
53.	$P(1,6,7,8,9,14)$	$P(2,16,18,24,26,8)$	8	821	$188^1 186^1 176^{11} 174^4 150^1 146^8 142^1 118^1 114^3$
54.	$P(1,8,9,15,14,7)$	$P(2,16,18,25,27,9)$	8	821	$188^1 176^7 175^8 154^2 146^8 142^2 118^1 114^2$
55.	$P(1,8,9,14,15,6)$	$P(2,16,18,26,24,10)$	8	837	$198^1 182^3 175^8 174^4 166^1 158^2 154^4 146^1 142^4 126^1 122^2$
56.	$P(1,8,9,16,17,24)$	$P(1,14,15,26,27,20)$	12	821	$187^3 177^2 176^6 174^4 146^{12} 139^3 90^1$
57.	$P(1,2,4,5,3,7)$	$P(2,8,10,16,18,24)$	12	857	$192^1 181^{12} 180^6 154^3 150^4 144^1 130^3 102^1$
58.	$P(1,6,7,8,9,14)$	$P(10,16,26,29,23,13)$	16	805	$186^1 176^2 174^{12} 162^1 146^1 142^6 140^2 138^4 106^2$
59.	$P(1,8,9,16,17,24)$	$P(7,10,13,19,20,25)$	16	805	$186^1 175^4 174^9 150^2 142^8 139^4 138^2 110^1$
60.	$P(1,8,9,16,17,24)$	$P(10,20,30,31,21,11)$	16	805	$186^2 175^4 174^8 142^8 139^4 138^4 122^1$
61.	$P(1,6,7,8,9,14)$	$P(7,16,23,25,30,9)$	16	813	$186^1 176^8 174^8 154^1 150^1 142^8 118^1 114^1 110^2$
62.	$P(1,6,7,8,9,14)$	$P(9,16,25,31,22,15)$	16	813	$176^8 174^8 154^1 150^2 146^1 142^8 118^1 114^1 110^1$
63.	$P(1,8,9,15,14,7)$	$P(6,16,22,26,28,10)$	16	813	$188^1 176^2 175^8 174^4 150^1 146^4 142^8 140^1 122^1 110^1$
64.	$P(1,8,9,16,17,24)$	$P(6,10,12,21,19,31)$	16	813	$198^1 186^1 175^8 174^5 146^4 143^8 138^2 110^2$
65.	$P(1,6,7,8,9,14)$	$P(1,16,17,31,30,15)$	16	821	$176^8 175^8 154^2 150^2 146^9 110^1 86^1$

Table 3. Three Pasch trades (page 2 of 4).

#	Trade 2	Trade 3	$\lvert\mathrm{Aut}\rvert$	$\lvert P\rvert$	Pasch-point incidence
66.	$P(1,8,9,15,14,7)$	$P(1,10,11,16,17,26)$	16	821	$186^{1}176^{2}175^{8}174^{4}150^{2}148^{2}147^{8}146^{2}138^{1}90^{1}$
67.	$P(1,6,7,8,9,14)$	$P(2,16,18,21,23,5)$	16	837	$186^{1}182^{4}176^{4}174^{8}162^{1}158^{2}154^{4}146^{1}142^{2}122^{4}$
68.	$P(1,6,7,8,9,14)$	$P(1,16,17,30,31,14)$	16	845	$198^{1}182^{4}176^{4}175^{8}166^{1}162^{1}158^{5}146^{4}122^{2}94^{1}$
69.	$P(1,8,9,14,15,6)$	$P(1,10,11,16,17,26)$	16	845	$198^{1}186^{1}182^{2}175^{8}174^{4}159^{8}154^{2}146^{2}138^{1}118^{1}98^{1}$
70.	$P(1,8,9,14,15,6)$	$P(2,16,18,24,26,8)$	16	845	$198^{1}194^{1}182^{3}176^{8}174^{4}158^{4}154^{2}146^{4}126^{4}$
71.	$P(1,2,4,5,3,7)$	$P(1,8,9,16,17,24)$	16	853	$192^{1}181^{8}180^{10}150^{2}148^{4}144^{1}130^{4}122^{1}$
72.	$P(7,8,15,16,23,24)$	$P(9,18,27,31,22,13)$	24	797	$174^{13}150^{3}138^{15}$
73.	$P(7,8,15,16,23,24)$	$P(9,18,27,28,21,14)$	32	797	$186^{1}174^{12}162^{1}138^{17}$
74.	$P(1,8,9,15,14,7)$	$P(10,16,26,28,22,12)$	32	805	$176^{2}174^{12}162^{1}150^{1}142^{8}140^{2}138^{4}110^{1}$
75.	$P(1,6,7,8,9,14)$	$P(7,16,23,31,24,15)$	32	813	$176^{8}174^{8}158^{1}150^{1}146^{2}142^{8}122^{1}110^{2}$
76.	$P(1,6,7,8,9,14)$	$P(8,16,24,31,23,15)$	32	813	$176^{8}174^{8}162^{1}150^{1}146^{2}142^{8}118^{1}110^{2}$
77.	$P(1,6,7,8,9,14)$	$P(2,5,7,9,11,12)$	32	819	$188^{1}180^{2}174^{16}154^{1}150^{1}147^{4}118^{1}116^{2}114^{1}113^{2}$
78.	$P(1,6,7,16,17,22)$	$P(2,16,18,23,21,7)$	32	819	$180^{2}174^{16}154^{2}150^{1}148^{1}147^{4}118^{1}116^{2}113^{2}$
79.	$P(1,6,7,8,9,14)$	$P(1,16,17,23,22,7)$	32	821	$186^{1}176^{12}174^{4}150^{2}146^{9}118^{2}82^{1}$
80.	$P(1,6,7,8,9,14)$	$P(1,16,17,24,25,8)$	32	821	$186^{1}176^{8}175^{8}154^{2}146^{9}114^{2}82^{1}$
81.	$P(1,6,7,8,9,14)$	$P(1,16,17,25,24,9)$	32	821	$186^{1}176^{8}175^{8}158^{2}146^{9}110^{2}82^{1}$
82.	$P(1,8,9,15,14,7)$	$P(1,16,17,26,27,10)$	32	821	$188^{1}176^{10}174^{4}150^{2}146^{12}140^{1}90^{1}$
83.	$P(1,8,9,15,14,7)$	$P(2,16,18,23,21,7)$	32	821	$176^{8}175^{8}154^{2}146^{10}122^{1}114^{2}$
84.	$P(1,8,9,14,15,6)$	$P(7,10,13,16,23,26)$	32	829	$186^{1}182^{2}174^{12}162^{1}154^{8}146^{2}138^{3}118^{2}$
85.	$P(1,6,7,8,9,14)$	$P(7,16,23,24,31,8)$	32	837	$198^{1}182^{4}176^{4}174^{8}158^{2}154^{4}142^{4}130^{1}126^{1}122^{2}$
86.	$P(1,8,9,14,15,6)$	$P(2,16,18,23,21,7)$	32	837	$182^{4}176^{4}174^{8}166^{1}162^{1}158^{2}154^{4}142^{4}126^{1}122^{2}$
87.	$P(1,8,9,14,15,6)$	$P(1,16,17,26,27,10)$	32	845	$198^{1}194^{1}182^{2}176^{8}174^{4}158^{8}146^{5}118^{1}98^{1}$
88.	$P(1,6,7,8,9,14)$	$P(1,10,11,14,15,4)$	48	820	$180^{2}174^{16}154^{3}151^{3}148^{3}113^{3}78^{1}$
89.	$P(1,8,9,16,17,24)$	$P(1,10,11,20,21,30)$	48	821	$186^{3}175^{12}147^{12}138^{3}90^{1}$
90.	$P(1,8,9,16,17,24)$	$P(1,10,11,21,20,31)$	48	821	$186^{3}175^{12}147^{12}138^{3}90^{1}$
91.	$P(1,8,9,16,17,24)$	$P(6,10,12,18,20,24)$	48	821	$186^{3}175^{12}174^{1}147^{12}110^{3}$
92.	$P(1,8,9,15,14,7)$	$P(10,16,26,27,17,11)$	64	805	$176^{2}174^{12}150^{2}142^{8}140^{2}138^{4}122^{1}$
93.	$P(1,6,7,8,9,14)$	$P(9,16,25,30,23,14)$	64	813	$186^{1}176^{8}174^{8}162^{1}154^{1}142^{8}110^{4}$

Table 3. Three Pasch trades (page 3 of 4).

#	Trade 2	Trade 3	$\lvert\mathrm{Aut}\rvert$	$\lvert P\rvert$	Pasch-point incidence
94.	$P(1,8,9,14,15,6)$	$P(7,16,23,26,29,10)$	64	829	$194^{1}182^{2}174^{12}162^{1}154^{8}146^{1}138^{4}118^{2}$
95.	$P(1,8,9,14,15,6)$	$P(10,16,26,27,17,11)$	64	829	$198^{1}182^{2}174^{12}154^{8}146^{2}138^{4}130^{1}118^{1}$
96.	$P(1,6,7,8,9,14)$	$P(2,8,10,13,15,5)$	64	837	$186^{2}174^{16}166^{1}162^{1}156^{4}150^{2}126^{1}122^{4}$
97.	$P(1,6,7,8,9,14)$	$P(1,16,17,22,23,6)$	64	845	$186^{1}182^{4}176^{8}174^{4}162^{1}158^{8}154^{1}146^{2}90^{2}$
98.	$P(1,2,4,5,3,7)$	$P(1,8,9,10,11,2)$	64	857	$183^{4}180^{16}154^{2}150^{4}138^{1}134^{2}118^{1}98^{1}$
99.	$P(1,8,9,14,15,6)$	$P(1,2,4,8,14,10)$	64	869	$198^{1}194^{1}190^{1}186^{1}180^{16}156^{4}150^{1}138^{4}122^{1}118^{1}$
100.	$P(1,2,4,5,3,7)$	$P(8,16,24,26,18,10)$	72	845	$180^{18}150^{3}144^{6}138^{1}126^{3}$
101.	$P(1,2,4,5,3,7)$	$P(8,16,24,25,17,9)$	96	845	$180^{18}162^{1}150^{2}144^{6}126^{4}$
102.	$P(1,6,7,8,9,14)$	$P(1,10,11,13,12,7)$	128	821	$188^{1}174^{16}152^{2}148^{9}118^{2}82^{1}$
103.	$P(1,6,7,8,9,14)$	$P(2,5,7,15,13,10)$	128	825	$180^{2}174^{16}158^{2}150^{6}126^{1}116^{4}$
104.	$P(1,8,9,14,15,6)$	$P(10,16,26,29,23,13)$	128	829	$210^{1}182^{2}174^{12}154^{8}146^{2}138^{4}118^{2}$
105.	$P(1,6,7,8,9,14)$	$P(2,5,7,8,10,13)$	128	837	$198^{1}186^{2}174^{16}156^{4}150^{2}130^{1}126^{1}122^{4}$
106.	$P(1,6,7,16,17,22)$	$P(2,17,19,20,22,5)$	128	837	$194^{1}186^{2}174^{16}162^{1}156^{4}154^{1}122^{6}$
107.	$P(1,8,9,14,15,6)$	$P(1,16,17,23,22,7)$	128	845	$182^{4}176^{8}174^{4}162^{1}158^{8}146^{4}130^{1}98^{1}$
108.	$P(1,8,9,14,15,6)$	$P(2,16,18,21,23,5)$	128	861	$210^{1}182^{8}174^{8}170^{2}154^{8}134^{4}$
109.	$P(1,2,4,5,3,7)$	$P(2,8,10,11,9,3)$	128	869	$186^{4}180^{16}166^{1}162^{1}158^{1}156^{4}138^{2}102^{2}$
110.	$P(1,2,4,5,3,7)$	$P(1,6,7,8,9,14)$	192	857	$192^{1}183^{4}180^{16}152^{3}132^{4}118^{3}$
111.	$P(1,6,7,8,9,14)$	$P(1,10,11,12,13,6)$	256	845	$194^{1}174^{16}162^{2}160^{8}154^{2}90^{2}$
112.	$P(1,2,4,5,3,7)$	$P(1,8,9,14,15,6)$	256	869	$186^{4}180^{16}162^{1}156^{4}138^{4}130^{1}122^{1}$
113.	$P(1,8,9,15,14,7)$	$P(6,16,22,23,17,7)$	384	821	$176^{12}174^{4}146^{12}122^{3}$
114.	$P(1,8,9,14,15,6)$	$P(1,10,11,13,12,7)$	512	845	$174^{16}162^{1}160^{8}154^{4}130^{1}98^{1}$
115.	$P(8,16,24,31,23,15)$	$P(9,18,27,28,21,14)$	576	797	$210^{1}174^{12}138^{18}$
116.	$P(1,8,9,14,15,6)$	$P(2,8,10,13,15,5)$	768	861	$210^{1}198^{2}174^{16}162^{6}134^{6}$
117.	$P(1,8,9,14,15,6)$	$P(1,16,17,22,23,6)$	768	893	$210^{1}182^{12}174^{4}170^{12}114^{2}$
118.	$P(1,8,9,15,14,7)$	$P(6,10,12,13,11,7)$	1536	821	$174^{16}148^{12}122^{3}$
119.	$P(1,8,9,14,15,6)$	$P(1,10,11,12,13,6)$	3072	893	$210^{1}178^{12}174^{16}114^{2}$
120.	$P(1,2,4,5,3,7)$	$P(1,2,5,6,7,3)$	8064	917	$189^{24}138^{7}$

Table 3. Three Pasch trades (page 4 of 4).

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