A TREATISE

ON THE

THEORY OF DETERMINANTS.

Cambridge:
PRINTED BY C. J. CLAY, M.A.,
AT THE UNIVERSITY PRESS.

A TREATISE

ON THE

THEORY OF DETERMINANTS

AND

THEIR APPLICATIONS IN ANALYSIS AND GEOMETRY,

BY

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Cambridge:
AT THE UNIVERSITY PRESS.

London: CAMBRIDGE WAREHOUSE, 17, PATERNOSTER ROW.
Cambridge: DEIGHTON, BELL, AND CO.
Leipzig: F. A. BROCKHAUS.
1880

PREFACE.

In the present treatise I have attempted to give an exposition of the Theory of Determinants and their more important applications. In every case where it was possible I have consulted the original works and memoirs on the subject; a list of those I have been able to see is appended as it may be useful to others pursuing the same line of study. At one time I hoped to make this list exhaustive, supplementing my own researches from the literary notices in foreign mathematical journals, but even with this aid I found that it would be necessarily incomplete. In consequence of this the list has been restricted to those memoirs which I have seen, the leading results of which are incorporated either in the body of the text or in the examples.

The principal novelty of the treatise lies in the systematic use of Grassmann's alternate units, by means of which the study of determinants is, I believe, much simplified.

I have to thank my friend Mr Jas. Barnard, M.A. of St John's College and Mathematical Master at the Proprietary School, Blackheath, for the care he has bestowed on correcting the proofs and for many valuable suggestions.

R. F. SCOTT.

Feb. 1880.

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THEORY OF DETERMINANTS.

CHAPTER I.

Introduction.

1. THE object of the theory of Determinants is to obtain compendious and simple methods of dealing with large numbers of quantities. In the words of Professor Sylvester, "It is an algebra upon an algebra; a calculus which enables us to combine and foretell the results of algebraical operations in the same way as algebra itself enables us to dispense with the performance of the special operations of arithmetic."

It will be found that the advantages and success of the method depend in great measure upon the notations which have been employed.

2. To indicate concisely the quantities discussed different notations have been used. The numbers belonging to the same class being denoted by the same letter, the different numbers of that class are distinguished by affixing numbers or letters, e.g.

We have frequently to deal with a series of such classes, each containing the same number of elements; these when written one under the other in rows form a rectangular array, the class being denoted by the letter while the suffix indicates the position of the element in the class.

E.g.

$$\begin{array}{l} a_1,\ a_2,\ a_3 \ldots\ldots \\ b_1,\ b_2,\ b_3 \ldots\ldots \\ c_1,\ c_2,\ c_3 \ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

3. In the theory of determinants we have frequently to deal with several such arrays, and it will be found that the most convenient notation is the following:

$$\begin{array}{l} a_{11},\ a_{12},\ a_{13},\ \ldots a_{1p}, \\ a_{21},\ a_{22},\ a_{23},\ \ldots a_{2p}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1},\ a_{m2},\ a_{m3} \ldots a_{mp}, \end{array}$$

where there are m horizontal and p vertical rows of elements.

Then a_{ks} is that element in the array of a's which is situated at the intersection of the k^{th} horizontal and s^{th} vertical rows.

The first suffix tells us the horizontal and the second suffix the vertical row in which the element stands.

In the present work these horizontal and vertical rows will be called rows and columns; a_{ks} therefore stands in the k^{th} row and s^{th} column.

Occasionally when we are dealing with a single array the letter is omitted, and instead of a_{ks} we write (ks) only. Such a notation is called an *umbral* notation, (ks) being not a quantity, [illegible] were, the shadow of one.

(ks) simply, and the whole set of lines joining the points of the two groups would be denoted by the array in § 3. At the same time the meaning of any selected element d_{ij} is perceived at once.

5. If we have any n elements $a_1, a_2, \ldots a_n$, we may call

$$a_1, \ a_2, \ldots a_n,$$

where the elements are arranged according to the magnitude of the numbers forming the suffixes, the natural or original order of the letters. Any other order is called a permutation of the elements. One element is said to be higher than another when it has the greater suffix. When in any permutation an element with a higher suffix precedes another with a lower we have an *inversion.*

Thus the permutation a_4, a_2, a_1, a_3, of four letters, contains the following four inversions,

$$a_4a_2, \ a_4a_1, \ a_4a_3, \ a_2a_1,$$

where we compare each element with all that follow it.

Following Cramer it is usual to divide the permutations of a given set of elements into two classes; the first class contains those permutations which have an even number of inversions, the second those which have an odd number.

6. By permutating the elements $a_1, a_2, \ldots a_n$ we obtain all possible ways in which they can be written. The same result is arrived at by writing down all the permutations of the suffixes $1, 2, \ldots n$ and then putting a's above them.

By repeated interchange of two suffixes we can get every permutation of the given elements from their original order.

For if we start with two suffixes 1, 2, they have but two arrangements,

$$1, 2, \qquad 2, 1,$$

of which the second is got from the first by a simple interchange. Taking three elements 1, 2, 3 out of these we can select the duad 2, 3, whose permutations are 2, 3; 3, 2. Prefixing 1 to each of these we get 1, 2, 3; 1, 3, 2, which are two permutations of the given elements. Proceeding in like manner with the other duads 1, 3; 1, 2, we get the six arrangements of three figures

1 2 3,	1 3 2,	2 3 1
2 1 3,	3 1 2,	3 2 1.

Next take four numbers 1, 2, 3, 4. We get four triplets by leaving out one number, viz.

1 2 3, 1 2 4, 1 3 4, 2 3 4.

For each triplet we can write down six arrangements by the rule just given for three numbers, then adding on the missing number we get twenty-four arrangements of four numbers, viz.

1 2 3 4	1 2 4 3	1 3 4 2	2 3 4 1
2 1 3 4	2 1 4 3	3 1 4 2	3 2 4 1
1 3 2 4	1 4 2 3	1 4 3 2	2 4 3 1
3 1 2 4	4 1 2 3	4 1 3 2	4 2 3 1
2 3 1 4	2 4 1 3	3 4 1 2	3 4 2 1
3 2 1 4	4 2 1 3	4 3 1 2	4 3 2 1.

And so we could go on to write down the arrangements of any set of elements.

The number of arrangements of n letters is $1.2.3...n$ or $n!$ an even number.

7. If in a given permutation two elements be interchanged while all the others remain unaltered in position, the two resulting permutations belong to different classes. This will be proved if we can shew that the difference between the number of inversions in the two permutations is an odd number.

We can represent any permutation of a group of elements by

$$A \ d \ B \ e \ C \text{..........................} (1),$$

where d and e are the two elements to be presently interchanged, A the group of elements which precede d, B the group between d and e, and C the group which follows e. The permutation we obtain is

$$A \ e \ B \ d \ C \text{..........................} (2).$$

The number of inversions in the two permutations (1) and (2) due to the elements contained in the groups A, B and C is in each case the same. And since the elements of A precede d and e in both permutations we get no new inversions in (2) from these; the elements of C follow both d and e, and therefore give rise to no new inversions. We have therefore only to consider the changes in the two permutations

$$d\ B\ e \text{ and } e\ B\ d \dots\dots\dots\dots\dots\dots (3).$$

Suppose that e is higher than d; let B contain b elements of which b_1 are higher than d and b_2 higher than e. Then in the permutation $d\ B\ e$ we have, independently of the inversions contained in B itself, $b - b_1 + b_2$ inversions, because there are $b - b_1$ elements lower than d and b_2 higher than e.

In $e\,B\,d$ we have $b - b_2$ inversions on account of e, b_1 on account of d, and one because e is higher than d; thus, without counting the inversions in B, we have $b - b_2 + b_1 + 1$. The difference between the number of inversions in the permutations (3), and therefore in (1) and (2), is thus

$$b - b_2 + b_1 + 1 - (b - b_1 + b_2) = 2(b_1 - b_2) + 1,$$

which is an odd number, shewing that the permutations belong to different classes.

8. The same result may be arrived at as follows: If there be n quantities whose natural order is

$$a_1, a_2, \dots a_n,$$

and if in any arrangement we subtract each suffix from all that follow it and multiply these differences together, we shall have a product whose sign will depend on the number of inversions in the given arrangement, the sign being positive if the number of inversions is even and negative if the number of inversions is odd. If then i, k be any two suffixes chosen arbitrarily which are to be interchanged, i preceding k in the given arrangement, the product of the differences will consist of four parts.

(i) The factor $k - i$.

(ii) and (iii) A set of factors such as $r - k$, and $r - i$, where r is some number of the series $1 \dots n$ excluding i and k.

(iv) A set of factors such as $r-s$, where r, s are any two numbers of the series $1, 2 \ldots n$ excluding i and k.

Then for the given arrangement the product of the differences will be

$$\pm (k-i)\,\Pi\,(r-i)(r-k)\,\Pi\,(r-s),$$

where the symbol Π stands for "the product of all such factors." If now we interchange i and k, the signs of all factors such as $(r-k)(r-i)$, $(r-s)$ remain unchanged, while $k-i$ changes sign.

Thus on interchanging two elements the product of the differences changes sign, i.e. by interchanging two suffixes we have introduced an odd number of negative factors and therefore of inversions, hence the two arrangements considered belong to different classes.

9. If in a series of elements each is replaced by the one which follows it, and the last by the first, we are said to have got a cyclical permutation of the given arrangement. If the system of elements

$$a_1, a_2 \ldots\ldots a_n$$

be considered as forming an endless band, if we cut this band between a_1 and a_n we have the natural order, cutting it between a_1 and a_2 we have a cyclical permutation of the first order, and so on.

Such a cyclical permutation is equivalent to $n-1$ simple interchanges, viz. we move a_1 from the first to the last place by interchanging the first and second elements, then the second and third, and so on, in all $n-1$ simple interchanges. Thus a cyclical permutation of a given arrangement belongs to the same or opposite class as the given one according as the number of elements is odd or even.

10. Every permutation of a given set of elements may be considered as derived from a fixed permutation by means of cyclical permutations of groups of the elements.

This is best illustrated by an example. Let the suffixes of two permutations of nine elements be

$$7,\ 6,\ 3,\ 2,\ 1,\ 4,\ 8,\ 5,\ 9$$
$$8,\ 7,\ 9,\ 5,\ 1,\ 6,\ 4,\ 3,\ 2$$

Here the second permutation is obtained by replacing in the first 7 by 8, 8 by 4, 4 by 6 and 6 by 7, which completes a cycle. Then 3 is replaced by 9, 9 by 2, 2 by 5 and 5 by 3, which completes the second cycle. Lastly, 1 forms a cycle by itself.

11. If elements which remain unchanged like 1 in the preceding example be considered as forming a cycle of one letter, we may state the following theorem: Two permutations belong to the same or different classes, according as the difference between the number of elements and the number of groups by whose cyclical interchange one permutation is got from the other, is even or odd.

For if there be n elements altogether, and p cycles of n_1, $n_2 \dots n_p$ letters, the cyclical interchanges are equivalent to

$$\begin{aligned}(n_1-1)+(n_2-1)+\dots+(n_p-1) &= n_1+n_2\dots+n_p-p \\ &= n-p\end{aligned}$$

simple interchanges, which proves the theorem.

In the example in Art. 10, $n=9$, $p=3$, and thus they belong to the same class.

12. If the number of rows and columns in an array be the same, we have a square array. Let such an array, containing n^2 elements, be

$$\begin{array}{c} a_{11},\ a_{12}\ \dots\dots\ a_{1n}, \\ a_{21},\ a_{22}\ \dots\dots\ a_{2n}, \\ \dots\dots\dots\dots\dots\dots \\ a_{n1},\ a_{n2}\ \dots\dots\ a_{nn}. \end{array}$$

The diagonal of elements $a_{11}, a_{22} \dots a_{nn}$ will be called the leading or principal diagonal.

A certain function, which is called a determinant, can be formed with the elements of this array as follows: From the array choose n different elements such that there is one and only one element from each row and column, multiply these elements together, the product will be a term of the determinant of n letters. For example, the set of elements

$$a_{11},\ a_{22}\ \dots\dots\ a_{nn},$$

situated in the principal diagonal of the square array, form a term of the determinant; this will be called the leading term, and to it we assign the positive sign.

The sign of any other term

$$a_{fg} \, . \, a_{hk} \, \ldots\ldots \, a_{st}$$

is determined as follows. From the mode in which the elements were selected, it follows that

$$f, \; h \ldots s, \quad \text{and} \quad g, \; k \ldots t$$

are each of them permutations of $1, 2 \ldots n$. Let them contain p and q inversions respectively, then the sign of the term

$$a_{fg} \, . \, a_{hk} \ldots a_{st}$$

is $(-1)^{p+q}$. The sum of all the possible terms with their proper signs is the determinant of the array.

More simple rules may be given for determining the sign of any term. If we interchange any two elements a_{hk} and a_{ij} the term does not change its sign. For this interchange is equivalent to the interchange of i with h and j with k. By these two interchanges we increase both p and q by an odd number, and hence the sign of the term is unaltered. It is therefore usual to give to one series of suffixes their natural order, when one of the two numbers p or q is zero, and the sign of the term of the determinant depends solely on the number of inversions in the other series, and is the same whether the first or second series of suffixes retains its natural order.

It is thus clear that all the terms of the determinant will be obtained from the leading term

$$a_{11}a_{22} \ldots\ldots a_{nn}$$

by keeping the first suffixes fixed in their natural order, and writing for the second suffixes in succession all possible permutations of the elements $1, 2 \ldots n$, giving to the product of the elements the positive or negative sign according as the number of inversions is even or odd.

Such a determinant is said to be of the n^{th} degree, since each term is the product of n elements. It has $n!$ terms in all, since this is the number of permutations of the second suffixes, each of which gives a term of the determinant. One half of these terms have the positive, the other half the negative sign.

13. Various notations are employed for the determinant of a system of n^2 elements. Cauchy and Jacobi denoted it by drawing two vertical lines at the sides of the array, or by writing $\pm$ before the leading term and prefixing a summation sign,

$$\begin{vmatrix} a_{11}, & a_{12} \ldots\ldots & a_{1n} \\ a_{21}, & a_{22} \ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots \\ a_{n1}, & a_{n2} \ldots\ldots & a_{nn} \end{vmatrix} = \Sigma \pm a_{11} a_{22} \ldots a_{nn}.$$

Sylvester uses the umbral notation

$$\begin{vmatrix} 1, & 2 \ldots\ldots & n, \\ 1, & 2 \ldots\ldots & n. \end{vmatrix}$$

If the determinant be written in the form

$$\begin{vmatrix} x_1, & y_1, & z_1 & \ldots \\ x_2, & y_2, & z_2 & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ x_n, & y_n, & z_n & \ldots \end{vmatrix}$$

we may denote it by

$$| x_i, y_i, z_i \ldots | \ (i = 1, 2 \ldots n),$$

meaning by this that i is to take the different values $1, 2 \ldots n$ in succession. Lastly, the determinant with double suffixes may be denoted by

$$| a_{ik} | \ (i, k = 1, 2 \ldots n),$$

the bracket at the side telling us what values the suffixes i and k take.

This ~~bracket~~ is frequently omitted in practice.

This notation is, I believe, due to Prof. H. J. S. Smith, who employs it in his report on the theory of numbers, *Brit. Ass. Rep.*, 1861, p. 504.

14. From Art. 6 we know the permutations of a system of two, three, or four elements. These give us the determinants of degree two, three, and four, viz.

$$\begin{vmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21},$$

$$\begin{vmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{21}, & a_{22}, & a_{23} \\ a_{31}, & a_{32}, & a_{33} \end{vmatrix} = \begin{array}{l} a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} \\ - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31}, \end{array}$$

$$\begin{vmatrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \\ a_4, & b_4, & c_4, & d_4 \end{vmatrix} = \begin{array}{l} a_1b_2c_3d_4 - a_2b_1c_3d_4 - a_1b_3c_2d_4 + a_3b_1c_2d_4 \\ + a_2b_3c_1d_4 - a_3b_2c_1d_4 - a_1b_2c_4d_3 + a_2b_1c_4d_3 \\ + a_1b_4c_2d_3 - a_4b_1c_2d_3 - a_2b_4c_1d_3 + a_4b_2c_1d_3 \\ + a_1b_3c_4d_2 - a_3b_1c_4d_2 - a_1b_4c_3d_2 + a_4b_1c_3d_2 \\ + a_3b_4c_1d_2 - a_4b_3c_1d_2 - a_2b_3c_4d_1 + a_3b_2c_4d_1 \\ + a_2b_4c_3d_1 - a_4b_2c_3d_1 - a_3b_4c_2d_1 + a_4b_3c_2d_1. \end{array}$$

A useful mnemonical rule for writing down the expansion of any determinant of the third order is the following, due to Sarrus.

Let the determinant be

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}.$$

Alongside of this repeat the first and second columns in order

$$\begin{array}{ccccccccc} a_1 & & b_1 & & c_1 & & a_1 & & b_1 \\ & \diagdown & & \times & & \times & & \diagup & \\ a_2 & & b_2 & & c_2 & & a_2 & & b_2, \\ & \diagup & & \times & & \times & & \diagdown & \\ a_3 & & b_3 & & c_3 & & a_3 & & b_3 \end{array}$$

and form the product of each set of three elements lying in lines parallel to the diagonals of the original square. Those which lie in lines descending from left to right have the positive, the others the negative sign.

Thus the determinant is

$$a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3$$
$$- c_1b_2a_3 - a_1c_2b_3 - b_1a_2c_3.$$

In practice it is not necessary actually to repeat the columns, but only to imagine them repeated.

It is not difficult to devise similar rules for determinants of higher order than the third, but we shall obtain methods for reducing the expansion of a determinant to that of several determinants of lower order, and for reducing the order of a determinant, so that they are unnecessary.

15. If we interchange rows and columns in the determinant of Art. 13, we get

$$\begin{vmatrix} a_{11}, & a_{21}, & \dots & a_{n1} \\ a_{12}, & a_{22}, & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1n}, & a_{2n}, & \dots & a_{nn} \end{vmatrix}.$$

This is the same as the original determinant with the suffixes of each element interchanged. Its expansion is then obtained from that of the original determinant by interchanging in each term the suffixes of each element. That is to say, in the term $a_{11}, a_{22} \dots a_{nn}$ we keep the second suffixes fixed in their natural order and write for the first suffixes all possible permutations of $1, 2 \dots n$. But the reasoning of Art. 12 shews that each term in the new determinant has the same sign as the corresponding one in the original determinant.

Thus a determinant remains unchanged in value when its rows and columns are interchanged.

Alternate Numbers.

16. The magnitudes with which we deal in ordinary or arithmetical algebra are subject, as regards their addition and multiplication, to the following principal laws:

(i) The associative law, which states that

$$(a + b) + c = a + (b + c) = a + b + c,$$

or that $$ab \,.\, c = a \,.\, bc = abc.$$

(ii) The commutative law, which states that

$$a + b = b + a,$$

$$ab = ba.$$

(iii) The distributive law, which states that

$$(b+c)\,a = ba + ca,$$
$$a\,(b+c) = ab + ac.$$

The researches of modern algebraists have led them to consider quantities for which one or more of these laws ceases to hold, or for which one or more of these laws assumes a different form.

Numbers, whether real or ideal, which follow the laws of arithmetical algebra will be called *scalar* quantities.

We shall find it useful to consider a class of numbers which have received the name of alternate numbers. These are determined by means of a system of independent units given in sets like the co-ordinates of a point in space; such a set will be denoted by $e_1, e_2, \ldots e_n$. A number such as

$$A = a_1e_1 + a_2e_2 + \ldots + a_ne_n,$$

formed by adding the units together, each multiplied by a scalar, will be called an alternate number of the n^{th} order.

In combination with scalar quantities and with units of other sets these units follow the laws of ordinary algebra. In combination with each other the units of a system follow the associative law and the commutative law as regards addition, but for multiplication we have the new equation

$$e_ie_j = -e_je_i \qquad \text{(1).}$$

As a consequence of which it follows at once that

$$e_i^2 = 0 \qquad \text{(2)}$$

for all values of i.

17. If
$$A = a_1e_1 + a_2e_2 + \ldots + a_ne_n,$$
$$B = b_1e_1 + b_2e_2 + \ldots + b_ne_n$$

be two alternate numbers of the n^{th} order, we define their product as follows:

$$AB = \sum_i a_ie_i \sum_j b_je_j$$
$$= \sum_{i,j} a_ie_i \,.\, b_je_j$$
$$= \sum_{i,j} a_ib_je_ie_j.$$

Hence, by equations (1) and (2) of Art. 16,

$$AB = (a_1b_2 - a_2b_1)\,e_1e_2 + (a_1b_3 - a_3b_1)\,e_1e_3 + \dots$$
$$+ (a_{n-1}b_n - a_nb_{n-1})\,e_{n-1}e_n.$$

Thus clearly $AB = -BA$ and $A^2 = 0$, proving that alternate numbers have the same commutative law of multiplication as the units.

This kind of multiplication, where $AB = -BA$, is called polar because the product AB has opposite properties at its two ends.

18. If k be any scalar

$$(A + kB)\,B = AB + kB^2 = AB,$$

so that the product of two alternate numbers is not altered if one be increased by a multiple of the other.

If we have a product of more than two numbers

$$ABC \dots\dots L,$$

it follows that for one of them, say C, we can write

$$C + k_1A + k_2B + \dots + k_rL,$$

and the product will still remain unaltered.

The alternate numbers belong to that class of algebraical magnitudes for which multiplication is a determinate, but division an indeterminate process. Viz.

$$\frac{AB}{B} = A + kB,$$

where k is an arbitrary scalar.

The continued product $e_1e_2 \dots e_n$ of all the units of a set will in future be assumed to be unity. An explanation of this assumption will be given later on.

19. If now we take a square array of elements such as that in Art. 12, we can form a system of n alternate numbers of the n^{th} order by taking the elements of each row to form the coefficients of the units in the numbers. Let P be the product of all these numbers, so that

$$P = (a_{11}e_1 + a_{12}e_2 + \dots + a_{1n}e_n)\,(a_{21}e_1 + a_{22}e_2 + \dots + a_{2n}e_n)\dots$$
$$(a_{n1}e_1 + a_{n2}e_2 + \dots + a_{nn}e_n).$$

On multiplying out the factors on the right,

$$P = \Sigma a_{1p} a_{2q} \dots a_{ns} e_p e_q \dots e_s.$$

If $e_1, e_2 \dots e_n$ were ordinary scalars the product $e_p e_q \dots e_s$ would be formed by taking n numbers from $e_1, e_2 \dots e_n$, and any number might be repeated $1, 2 \dots n$ times; but since $e_p e_q \dots e_s = 0$ if any two units are alike, it follows that $p, q \dots s$ is to be a permutation of $1, 2 \dots n$. It follows at once from the law of multiplication (equation 1, Art. 16) that

$$e_p e_q \dots e_s = (-1)^\nu e_1 e_2 \dots e_n,$$

where ν is the number of inversions in the series $e_p e_q \dots e_s$.

Thus $$P = \cancel{e_1 e_2 \dots e_n} \Sigma (-1)^\nu a_{1p} a_{2q} \dots a_{ns},$$

but the term under the summation sign is a term of the determinant of the system of elements, with its proper sign. Thus

$$\begin{aligned} P &= | a_{ik} | e_1 e_2 \dots e_n \\ &= | a_{ik} |. \end{aligned}$$

Hence the determinant of a system of n^2 elements is expressed as a product of n alternate numbers linear in these elements. From this it immediately follows that if all the elements of a row are multiplied by the same number the determinant is multiplied by that number, and if all the elements of a row vanish the determinant vanishes.

In future we shall write for a determinant of the n^{th} order whichever of the forms

$$|a_{ik}|, \quad \Pi A_j, \quad \Sigma \pm a_{11} a_{22} \dots a_{nn},$$

$(A_j = a_{j1} e_1 + a_{j2} e_2 + \dots + a_{jn} e_n)$ is most convenient. The letters i, k, j taking all the values $1, 2 \dots n$.

20. If the determinant is so constituted that the different factors of which it is composed do not contain all the units, its evaluation is frequently readily effected.

For example, the determinant

$$\begin{vmatrix} a_{11}, & 0, & 0 & \dots\dots 0 \\ a_{21}, & a_{22}, & 0 & \dots\dots 0 \\ a_{31}, & a_{32}, & a_{33} & \dots\dots 0 \\ \dots & \dots & \dots & \dots \\ a_{n1}, & a_{n2}, & a_{n3} & \dots\dots a_{nn} \end{vmatrix}$$

in which all the elements above the leading diagonal vanish reduces to the product $a_{11}a_{22} \ldots a_{nn}$.

For it is equal to the product of the alternate numbers

$$\begin{array}{l} a_{11}e_1 \\ a_{21}e_1 + a_{22}e_2 \\ a_{31}e_1 + a_{32}e_2 + a_{33}e_3 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}e_1 + a_{n2}e_2 + a_{n3}e_3 + \ldots + a_{nn}e_n. \end{array}$$

Since the first number contains e_1, and e_1 only, all terms in the product of the remaining factors which contain e_1 disappear when multiplied by this factor, so that as far as we are concerned we may suppose a_{21}, $a_{31} \ldots a_{n1}$ to vanish. The second number reduces to $a_{22}e_2$, and the product of the first two to $a_{11}e_1a_{22}e_2$. We may shew in like manner that a_{32}, $a_{42} \ldots$ may vanish, and so on. Finally the product reduces to

$$a_{11}e_1a_{22}e_2 \ldots a_{nn}e_n = a_{11}a_{22} \ldots a_{nn}.$$

By an interchange of rows and columns it follows that the determinant for which all the elements below the leading diagonal vanish also reduces to its leading term.

21. As another example let us consider the determinant

$$D = \begin{vmatrix} 0, & \cos(a_1 + a_2), & \cos(a_1 + a_3) & \ldots\ldots \\ \cos(a_2 + a_1), & 0, & \cos(a_2 + a_3) & \ldots\ldots \\ \cos(a_3 + a_1), & \cos(a_3 + a_2), & 0 & \ldots\ldots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \end{vmatrix}$$

of order n: the element in the i^{th} row and j^{th} column is $\cos(a_i + a_j)$ unless $i = j$, when it vanishes.

S· bstitute for the cosines their exponential values and write
erm

$$e^{a\sqrt{-1}} = \alpha.$$; $e^{i\theta} = \cos\theta + i\sin\theta$; $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$; $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

que
like tD is the product of such factors as

$$\left[\left(\alpha_1\alpha_2 + \frac{1}{\alpha_1\alpha_2}\right)e_2 + \left(\alpha_1\alpha_3 + \frac{1}{\alpha_1\alpha_3}\right)e_3 + \ldots + \left(\alpha_1\alpha_n + \frac{1}{\alpha_1\alpha_n}\right)e_n\right]$$

3.

more n $\left[\alpha_1 E + \frac{F}{\alpha_1} - \left(\alpha_1^2 + \frac{1}{\alpha_1^2}\right)e_1\right]$,

s.

where $$E = a_1 e_1 + a_2 e_2 + \ldots + a_n e_n,$$

$$F = \frac{e_1}{a_1} + \frac{e_2}{a_2} + \ldots + \frac{e_n}{a_n}.$$

Thus if $$a_i E + \frac{F}{a_i} = A_i,$$

we see that $$(-2)^n D = \Pi\,(2\cos 2a_i \,.\, e_i - A_i).$$

Now observe that since the quantities A_i depend only on the two alternate numbers E and F, the product of more than two of them must vanish. Hence expanding

$$(-2)^n D = 2^n \cos 2a_1 \cos 2a_2 \ldots \cos 2a_n - 2^n \cos 2a_1 \ldots \cos 2a_n \Sigma \frac{e_1 e_2 \ldots A_n}{2\cos 2a_n}$$

$$+ 2^n \cos 2a_1 \ldots \cos 2a_n \Sigma \frac{e_1 e_2 \ldots e_{n-2} A_{n-1} A_n}{4\cos 2a_{n-1}\cos 2a_n}.$$

Now $$e_1 e_2 \ldots e_{n-1} A_n = e_1 \ldots e_{n-1}\left(a_n E + \frac{F}{a_n}\right)$$

$$= \frac{1}{a_n^2} + a_n^2 = 2\cos 2\,a_n.$$

$$e_1 e_2 \ldots A_{n-1} A_n = e_1 \ldots e_{n-2}\left(\frac{a_{n-1}}{a_n} - \frac{a_n}{a_{n-1}}\right) EF$$

$$= \left(\frac{a_{n-1}}{a_n} - \frac{a_n}{a_{n-1}}\right)^2 = -4\sin^2(a_{n-1} - a_n).$$

Thus $$\frac{(-1)^n D}{\cos 2a_1 \cos 2a_2 \ldots \cos 2a_n} = 1 - n - \Sigma \frac{\sin^2(a_i - a_k)}{\cos 2a_i \cos 2a_k};$$

or $$\frac{(-1)^{n-1} D}{\cos 2a_1 \ldots \cos 2a_n} = (n-1) + \Sigma \frac{\sin^2(a_i - a_k)}{\cos 2a_i \cos 2a_k},$$

where (i, k) are all duads derived from $1, 2 \ldots n$.

CHAPTER II.

GENERAL PROPERTIES OF DETERMINANTS.

1. If two columns or rows of a determinant be interchanged the resulting determinant is equal in value to the original, but of opposite sign.

Let $\quad D = \Pi\,(a_{i1} e_1 + \ldots + a_{ij} e_j + \ldots + a_{ik} e_k + \ldots + a_{in} e_n)$, then, if D' is the determinant got by interchanging the j^{th} and k^{th} columns,

$$D' = \Pi\,(a_{i1} e_1 + \ldots + a_{ik} e_j + \ldots + a_{ij} e_k + \ldots + a_{in} e_n);$$

but since in addition we follow the ordinary commutative law, D' is got from D by interchanging e_j and e_k in the product on the right. This leaves the scalar factor unaltered but changes the sign of the product of the units, thus

$$D' = -D.$$

Interchanging two rows of a determinant, say the j^{th} and k^{th}, is the same as interchanging the two factors A_j and A_k on the right: this is equivalent to an odd number of inversions, and hence by the rule of multiplication changes the sign of the product.

2. If two rows or columns of a determinant be identical the determinant vanishes. For by the interchange of the two columns in question the determinant changes sign, but both columns being alike the determinant remains the same, thus

$$D = -D \quad \text{or} \quad D = 0.$$

3. If each element of the i^{th} row consist of the sum of two or more numbers the determinant splits up into the sum of two or

more determinants having for elements of the i^{th} row the separate terms of the elements of the i^{th} row of the given determinant.

For if $$D = \Pi A_k,$$

and
$$\begin{aligned} A_i &= (a_{i1} + b_{i1})\, e_1 + (a_{i2} + b_{i2})\, e_2 + \ldots + (a_{in} + b_{in})\, e_n \\ &= (a_{i1} e_1 + \ldots + a_{in} e_n) + (b_{i1} e_1 + \ldots + b_{in} e_n) \\ &= A'_i + B_i\,; \end{aligned}$$

since
$$\begin{aligned} A_1 \ldots A_i \ldots A_n &= A_1 \ldots (A'_i + B_i) \ldots A_n \\ &= A_1 \ldots A'_i \ldots A_n + A_1 \ldots B_i \ldots A_n, \end{aligned}$$

we have $$D = D_1 + D_2,$$

where D_1 and D_2 are determinants having for elements of the i^{th} row in the k^{th} place a_{ik} and b_{ik} respectively.

Repeated applications of this reasoning shew that if the elements of the i^{th} row consist each of the sum of p elements, then the original determinant can be resolved into the sum of p determinants having for their i^{th} rows the terms of the elements of the i^{th} row of the given determinant.

The same theorem would apply if the elements of a column consisted of the sum of elements. In fact whenever a theorem applies to rows it applies equally to columns, as these can be interchanged (I. 15).

In future, when a theorem is stated with regard either to rows or columns, it is to be understood as applying also to the other.

4. The value of a determinant is not altered if we add to the elements of any row the corresponding elements of another row, each multiplied by the same constant factor.

For if we add to the elements of the i^{th} row those of the k^{th} row, each multiplied by p, the resulting determinant is

$$\begin{aligned} A_1 \ldots (A_i + pA_k) \ldots A_k \ldots A_n &= A_1 \ldots A_i \ldots A_k \ldots A_n + pA_1 \ldots A_k \ldots A_k \ldots A_n \\ &= A_1 \ldots A_i \ldots A_k \ldots A_n, \end{aligned}$$

the latter product vanishing, since it contains two identical factors.

For brevity the operation of adding corresponding elements of two rows is usually spoken of as adding the rows.

5. The theorem of the last article is of great importance in the reduction of determinants. The following are examples of its application:

(i) If corresponding elements of two rows of a determinant have a constant ratio the determinant vanishes. For we have only to multiply the elements of one row by a proper factor and subtract them from the elements of the other when all the elements in that row will vanish, and consequently the determinant vanishes.

Of a similar nature are the two following theorems, whose proof presents no difficulty:

(ii) If the ratio of the differences of corresponding elements in the p^{th} and q^{th} rows to the difference of corresponding elements in the r^{th} and s^{th} rows be constant, then the determinant vanishes.

(iii) If from the corresponding elements of $i+1$ rows we form the i^{th} differences and from the corresponding elements of $m+1$ rows the m^{th} differences (the second set of rows being at least partially different from the first set); then, if the ratio of corresponding differences is constant, the determinant vanishes.

(iv) Let
$$D = \begin{vmatrix} u_1, & v_1 & \dots & t_1 \\ u_2, & v_2 & \dots & t_2 \\ \dots & \dots & \dots & \dots \\ u_n, & v_n & \dots & t_n \end{vmatrix}.$$

Subtract each row from the one which follows it, beginning with the last but one. Then, if
$$\Delta u_i = u_{i+1} - u_i,$$

we have
$$D = \begin{vmatrix} u_1, & v_1 & \dots & t_1 \\ \Delta u_1, & \Delta v_1 & \dots & \Delta t_1 \\ \Delta u_2, & \Delta v_2 & \dots & \Delta t_2 \\ \dots & \dots & \dots & \dots \\ \Delta u_{n-1}, & \Delta v_{n-1} & \dots & \Delta t_{n-1} \end{vmatrix}.$$

Repeat the same operation, stopping short at the second row.

Then, if
$$\Delta^2 u_i = \Delta u_{i+1} - \Delta u_i,$$
$$D = \begin{vmatrix} u_1, & v_1 & \dots & t_1 \\ \Delta u_1, & \Delta v_1 & \dots & \Delta t_1 \\ \Delta^2 u_1, & \Delta^2 v_1 & \dots & \Delta^2 t_1 \\ \dots & \dots & \dots & \dots \\ \Delta^2 u_{n-2}, & \Delta^2 v_{n-2} & \dots & \Delta^2 t_{n-2} \end{vmatrix}.$$

Proceed in this way, leaving out a row each time, and we see that
$$D = \begin{vmatrix} u_1, & v_1 & \dots & t_1 \\ \Delta u_1, & \Delta v_1 & \dots & \Delta t_1 \\ \Delta^2 u_1, & \Delta^2 v_1 & \dots & \Delta^2 t_1 \\ \dots & \dots & \dots & \dots \\ \Delta^{n-1} u_1, & \Delta^{n-1} v_1 & \dots & \Delta^{n-1} t_1 \end{vmatrix};$$
where generally:
$$\Delta^r u_i = \Delta^{r-1} u_{i+1} - \Delta^{r-1} u_i.$$
Suppose now that u_1 is a function of degree 0, v_1 of degree 1, and so on, then all the elements below the leading diagonal of D vanish, and
$$D = u_1 . \Delta v_1 . \Delta^2 w_1 \dots \Delta^{n-1} t_1.$$
For example, if
$$m_p = \frac{m(m-1)\dots(m-p+1)}{1.2\dots p}, \quad m_0 = 1,$$
$$\begin{vmatrix} m_0, & m_1 & \dots & m_r \\ (m+d)_0, & (m+d)_1 & \dots & (m+d)_r \\ \dots & \dots & \dots & \dots \\ (m+rd)_0, & (m+rd)_1 & \dots & (m+rd)_r \end{vmatrix} = 1 . d . d^2 \dots d^r = d^{\frac{r(r+1)}{2}}.$$
For here
$$\Delta^t (m+td)_t = d^t.$$

6. In a determinant of the form
$$\begin{vmatrix} 0, & 1, & 1, & 1 & \dots \\ 1, & a_{11}, & a_{12}, & a_{13} & \dots \\ 1, & a_{21}, & a_{22}, & a_{23} & \dots \\ 1, & a_{31}, & a_{32}, & a_{33} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix},$$
every element of which a_{rs} is a type can be replaced by
$$A_{rs} = a_{rs} + h_r + k_s,$$
where h_r and k_s are arbitrary quantities, without altering the value of the determinant.

For multiply the first row by h_r and add it to the r^{th} row, then in this row the first element is still 1, while in place of a_{rs} we have $a_{rs}+h_r$. Now multiply the first column by k_s and add it to the s^{th} column; the element in the first row is still unchanged, while the element under discussion has become $a_{rs}+h_r+k_s$.

These transformations have left the value of the determinant unaltered.

7. We are now in a position to solve the system of linear equations

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = u_1, \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = u_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = u_n \end{array}$$

[or, as they may be more briefly written,

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = u_i \; (i = 1, 2 \dots n)].$$

We have

$$\begin{vmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n - u_1, & a_{12}, & a_{13} \dots a_{1n} \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n - u_2, & a_{22}, & a_{23} \dots a_{2n} \\ \dots\dots\dots\dots\dots\dots & & \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n - u_n, & a_{n2}, & a_{n3} \dots a_{nn} \end{vmatrix} = 0,$$

for each element in the first column vanishes (I. 12).

Since the elements of the first column of this determinant consist of $n+1$ elements, it can be resolved into the sum of $n+1$ determinants.

The first of these is

$$\begin{vmatrix} a_{11}x_1, & a_{12}, & a_{13} \dots a_{1n} \\ a_{21}x_1, & a_{22}, & a_{23} \dots a_{2n} \\ \dots\dots\dots & & \\ a_{n1}x_1, & a_{n2}, & a_{n3} \dots a_{nn} \end{vmatrix} = x_1 |a_{ik}|.$$

The last is

$$\begin{vmatrix} -u_1, & a_{12}, & a_{13} \dots a_{1n} \\ -u_2, & a_{22}, & a_{23} \dots a_{2n} \\ \dots\dots\dots & & \\ -u_n, & a_{n2}, & a_{n3} \dots a_{nn} \end{vmatrix} = - \begin{vmatrix} u_1, & a_{12} \dots a_{1n} \\ u_2, & a_{22} \dots a_{2n} \\ \dots\dots\dots & \\ u_n, & a_{n2} \dots a_{nn} \end{vmatrix}.$$

While any of the others, such as

$$\begin{vmatrix} a_{1i}x_i, & a_{12}\dots a_{1n} \\ a_{2i}x_i, & a_{22}\dots a_{2n} \\ \dots & \dots \\ a_{ni}x_i, & a_{n2}\dots a_{nn} \end{vmatrix}$$

vanishes, because the elements of the first column are proportional to those of the i^{th} column.

Thus $$x_1|a_{ik}| = \begin{vmatrix} u_1, & a_{12}\dots a_{1n} \\ u_2, & a_{22}\dots a_{2n} \\ \dots & \dots \\ u_n, & a_{n2}\dots a_{nn} \end{vmatrix}.$$

And in general x_i is obtained by substituting in the determinant $|a_{ik}|$ for the elements of the i^{th} column the quantities $u_1\dots u_n$, and dividing the resulting determinant by $|a_{ik}|$.

8. If p rows of a determinant whose elements are functions of x become identical when $x=a$, then the determinant is divisible by $(x-a)^{p-1}$. For, subtract any one of these rows from the remaining $p-1$ rows; the determinant remains unchanged, but now when $x=a$ all the elements of these $p-1$ new rows vanish, hence each element divides by $x-a$, and thus dividing each of the $p-1$ rows by this factor we see that the determinant divides by $(x-a)^{p-1}$.

If when $x=a$ the rows are not equal, but only proportional, the theorem is still true.

Ex. The value of the determinant

$$\begin{vmatrix} x, & a\dots\dots a \\ a, & x\dots\dots a \\ \dots & \dots \\ a, & a\dots\dots x \end{vmatrix} \quad (n \text{ rows})$$

is $$\{x+(n-1)a\}(x-a)^{n-1}.$$

For if $x=a$ the n rows all become identical, thus the determinant divides by $(x-a)^{n-1}$.

Adding all the rows to the first, each element in that row becomes $x + (n-1)a$, this is therefore a factor in the determinant. Thus the determinant divides by

$$\{x + (n-1)a\}(x-a)^{n-1}.$$

This is of the same degree as the determinant, and as the coefficient of x^n in the determinant and in the product is unity the determinant must be equal to the product.

CHAPTER III.

ON THE MINORS AND ON THE EXPANSION OF A DETERMINANT.

1. If from the n rows of the array

$$\begin{array}{l} a_{11},\ a_{12}\ \dots\ a_{1n} \\ a_{21},\ a_{22}\ \dots\ a_{2n} \\ \dots\dots\dots\dots\dots\dots \\ a_{n1},\ a_{n2}\ \dots\ a_{nn} \end{array}$$

we select any p rows, and then from the new array which these form select p columns, these when written in the form of a determinant constitute a minor of the given system. Such a minor is said to be of the p^{th} order.

Since we can select p rows from n in

$$\frac{n(n-1)\ \dots\ (n-p+1)}{1\ .\ 2\ \dots\ p} = n_p$$

ways, and p columns from n columns in a like number of ways, it follows that the given system of order n has $(n_p)^2$ minors of order p.

2. If out of the $n-p$ rows which remain after the above p have been selected we take those $n-p$ columns whose column suffixes are different from those selected in the minor of order p, we have another determinant of order $n-p$ said to be complementary to that of order p.

For example, in the determinant

$$\begin{vmatrix} a_{11}, & a_{12}, & a_{13}, & a_{14}, & a_{15} \\ a_{21}, & a_{22}, & a_{23}, & \dots & a_{25} \\ \dots & \dots & \dots & \dots & \dots \\ a_{51}, & a_{52}, & \dots & \dots & a_{55} \end{vmatrix},$$

$$\begin{vmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{vmatrix} \text{ and } \begin{vmatrix} a_{33}, & a_{34}, & a_{35} \\ a_{43}, & a_{44}, & a_{45} \\ a_{53}, & a_{54}, & a_{55} \end{vmatrix}$$

are cómplementary minors.

3. If $p = 1$, i.e. if we take a single element, the complementary minor is a determinant of order $n-1$, which is called the complement of the element. This complement is obtained from the original determinant by omitting the row and column in which the selected element stands. For example, the complement of the element a_{ik}, which we denote by A_{ik}, is

$$\begin{vmatrix} a_{11} & \dots & a_{1k-1}, & a_{1k+1} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i-11} & \dots & a_{i-1k-1}, & a_{i-1k+1} & \dots & a_{i-1n} \\ a_{i+11} & \dots & a_{i+1k-1}, & a_{i+1k+1} & \dots & a_{i+1n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{nk-1}, & a_{nk+1} & \dots & a_{nn} \end{vmatrix}.$$

This is sometimes spoken of as a first minor of the given determinant. In like manner the determinant formed by omitting p rows and p columns would be called a p^{th} minor; it is to be observed that a p^{th} minor is a determinant of order $n-p$.

4. We may extend the meaning of complementary minors as follows: From the array in Art. 1 select p rows and p columns, then from those that remain q rows and q columns, from those that remain r rows and r columns, and so on. With the elements in these selected rows and columns form determinants; these will form a complementary system of minors if

$$p + q + r + \dots = n.$$

The number of ways in which we can form such a system is

$$\left\{\frac{n!}{p!\,q!\,r!\dots}\right\}^2.$$

It is of course permissible that one or more of the numbers $p, q, r \ldots$ should be unity; the corresponding minor is then a single element. For the determinants

$$\begin{vmatrix} a_{11} & \cdots & a_{16} \\ \cdots & \cdots & \cdots \\ a_{61} & \cdots & a_{66} \end{vmatrix};$$

$$\begin{vmatrix} a_{24}, & a_{25} \\ a_{34}, & a_{35} \end{vmatrix}; \qquad \begin{vmatrix} a_{12}, & a_{13}, & a_{16} \\ a_{42}, & a_{43}, & a_{46} \\ a_{62}, & a_{63}, & a_{66} \end{vmatrix}; \qquad a_{51}$$

form such a complementary system, and there are 3600 such systems.

5. We have hitherto only considered the product of a set of alternate numbers equal in number to the number of units. Let us now consider the product

$$(a_{11}e_1 + a_{12}e_2 + \ldots + a_{1n}e_n) \ldots (a_{m1}e_1 + a_{m2}e_2 + \ldots + a_{mn}e_n);$$

this is equal to

$$\Sigma a_{1p}a_{2q} \ldots a_{mr}e_pe_q \ldots e_r,$$

where $p, q \ldots r$ consist of all combinations m at a time from $1, 2 \ldots n$, repetitions being allowed.

First, if $m > n$, we must have repetitions in every term of the sum, and hence (I. 16, Equation 2) the whole vanishes.

If $m = n$, we have the case of I. 19, and the sum is the determinant $| a_{ik} |$.

But if $m < n$, the sum is formed by taking for $p, q \ldots r$ all m-ads from $1, 2 \ldots n$ and permutating the elements of each m-ad in all possible ways.

Namely, the term

$$a_{1p}a_{2q} \ldots a_{mr}e_pe_q \ldots e_r$$

is got by taking $a_{1p}e_p$ from the first factor of the product, $a_{2q}e_q$ from the second ..., and $a_{mr}e_r$ from the last factor. But we should still get the product of the units $e_pe_q \ldots e_r$, though in a different order, if we take the p^{th} term of some other factor than the first, the q^{th} of some other than the second, and so on. The term of the product which multiplies $e_p e_q \dots e_r$ is thus got from

$$a_{1p}a_{2q} \dots a_{mr}$$

by permutating $p, q \dots r$ in all possible ways, and giving to each term the sign corresponding to the number of inversions in its second suffixes, $p, q \dots r$ being considered the original order. The sum of these products is

$$\begin{vmatrix} a_{1p}, & a_{1q} & \dots & a_{1r} \\ a_{2p}, & a_{2q} & \dots & a_{2r} \\ \dots & \dots & \dots & \dots \\ a_{mp}, & a_{mq} & \dots & a_{mr} \end{vmatrix}.$$

Hence the product of the m factors is equal to

$$\Sigma \begin{vmatrix} a_{1p}, & a_{1q} & \dots & a_{1r} \\ a_{2p}, & a_{2q} & \dots & a_{2r} \\ \dots & \dots & \dots & \dots \\ a_{mp}, & a_{mq} & \dots & a_{mr} \end{vmatrix} e_p e_q \dots e_r \dots\dots\dots\dots (1).$$

In like manner, if we take the remaining factors necessary to form the determinant $|a_{ik}|$, we have

$$(a_{m+1\,1}e_1 + \dots + a_{m+1n}e_n) \dots (a_{n1}e_1 + \dots + a_{nn}e_n)$$

$$= \Sigma \begin{vmatrix} a_{m+1u}, & a_{m+1v} & \dots & a_{m+1w} \\ a_{m+2u}, & a_{m+2v} & \dots & a_{m+2w} \\ \dots & \dots & \dots & \dots \\ a_{nu}, & a_{nv} & \dots & a_{nw} \end{vmatrix} e_u e_v \dots e_w \dots\dots\dots\dots (2),$$

where $u, v \dots w$ is a combination of $n-m$ numbers selected from $1, 2 \dots n$.

Now multiply the equation (2) by the equation (1) and we obtain

$$|a_{ik}| = \Sigma \left\{(-1)^\nu \begin{vmatrix} a_{1p}, & a_{1q} & \dots & a_{1r} \\ \dots & \dots & \dots & \dots \\ a_{mp}, & a_{mq} & \dots & a_{mr} \end{vmatrix} \begin{vmatrix} a_{m+1u} & \dots & a_{m+1w} \\ \dots & \dots & \dots \\ a_{nu} & \dots & a_{nw} \end{vmatrix} \right\};$$

where from the nature of the alternate numbers e it follows that the two determinant factors under the summation sign are complementary minors, and ν is the number of inversions in

$$e_p e_q \dots e_r e_u e_v \dots e_w \text{ or in } p,\ q \dots r,\ u,\ v \dots w.$$

This theorem, usually called Laplace's theorem, gives the expansion of a determinant in the form of a sum of products of complementary minors.

It is assumed in the above that the complementary minors are formed from the first m and last $n-m$ rows. Since by a suitable change of the order of the rows and sign of the determinant any m rows can be brought into the first m places, this is no real restriction.

6. For examples we have

$$\begin{vmatrix} a_1, & a_2, & a_3, & a_4 \\ b_1, & b_2, & b_3, & b_4 \\ \hline c_1, & c_2, & c_3, & c_4 \\ d_1, & d_2, & d_3, & d_4 \end{vmatrix} = \begin{aligned} &(12)\,(34)+(23)\,(14)+(31)\,(24) \\ &+(34)\,(12)+(14)\,(23)+(24)\,(31), \end{aligned}$$

where for brevity

$$(12)\,(34) = \begin{vmatrix} a_1, & a_2 \\ b_1, & b_2 \end{vmatrix} \cdot \begin{vmatrix} c_3, & c_4 \\ d_3, & d_4 \end{vmatrix} \text{ \&c.}$$

In like manner

$$\begin{vmatrix} a_1, & a_2, & a_3, & a_4, & a_5 \\ b_1, & b_2, & b_3, & b_4, & b_5 \\ c_1, & c_2 \ldots\ldots & & & \\ \hline d_1, & d_2 \ldots\ldots & & & \\ e_1, & e_2 \ldots\ldots & & & \end{vmatrix} = \begin{aligned} &(123)(45)+(142)(35)+(134)(25)+(243)(15) \\ &+(125)\,(34)+(315)\,(24)+(235)\,(14) \\ &+(145)\,(23)+(425)\,(13) \\ &+(345)\,(12), \end{aligned}$$

where

$$(123)\,(45) = \begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix} \cdot \begin{vmatrix} d_4, & d_5 \\ e_4, & e_5 \end{vmatrix} \text{ \&c.}$$

7. If when the determinant is divided into two sets of m and $n-m$ rows there are $n-m$ columns of zeros in the set of m rows, the determinant reduces to the product of the minor of the remaining m columns and its complementary minor.

This is clear, for with the exception of this single minor of order m all the others vanish because they contain at least one column of zero elements.

If the set of m rows contains more than $n-m$ columns of zeros the determinant vanishes.

Thus, for example:

$$\begin{vmatrix} a_1, & a_2, & 0, & 0 \\ b_1, & b_2, & 0, & 0 \\ c_1, & c_2, & c_3, & c_4 \\ d_1, & d_2, & d_3, & d_4 \end{vmatrix} = \begin{vmatrix} a_1, & a_2 \\ b_1, & b_2 \end{vmatrix} \begin{vmatrix} c_3, & c_4 \\ d_3, & d_4 \end{vmatrix},$$

while

$$\begin{vmatrix} a_1, & a_2, & 0, & 0, & 0 \\ b_1, & b_2, & 0, & 0, & 0 \\ c_1, & c_2, & 0, & 0, & 0 \\ d_1, & d_2, & d_3, & d_4, & d_5 \\ e_1, & e_2, & e_3, & e_4, & e_5 \end{vmatrix} = 0.$$

8. In Art. 5 we resolved a determinant into the sum of products of pairs of complementary minors. We can however resolve it into a sum of products of as many complementary minors as we please.

For we can divide up the n factors whose product is $|a_{ik}|$ as follows: Take the first u, the second v ..., the last w. The product of the first u factors would be of the form

$$\Sigma \begin{vmatrix} a_{1p}, & a_{1q} & \ldots & a_{1r} \\ a_{2p}, & a_{2q} & \ldots & a_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ a_{up}, & a_{uq} & \ldots & a_{ur} \end{vmatrix} e_p e_q \ldots e_r,$$

or

$$\Sigma D_u e_p e_q \ldots e_r,$$

$p, q \ldots r$ being u numbers taken from $1, 2 \ldots n$ without repetition and D_u a minor of order u from the first u rows.

In like manner the product of the next v factors would be

$$\Sigma D_v e_f e_g \ldots e_h,$$

D_v being a minor of order v chosen from the v rows.

Lastly, the product of the w factors would be

$$\Sigma D_w e_r e_s \ldots e_t,$$

with a similar meaning for the quantities involved.

Now form the product of all the factors, taking care to keep them in their proper order, and

$$|a_{ik}| = \Sigma D_u D_v \ldots D_w,$$

where $D_u, D_v, \ldots D_w$ form a system of complementary minors of the determinant $|a_{ik}|$.

The sign of the term is determined from the number of inversions in

$$p, q \ldots r, \ f, g \ldots h, \ r, s \ldots t.$$

9. If in Art. 5 we restrict the first product to the single factor

a

$$a_{i1}e_1 + a_{i2}e_2 + \ldots + a_{in}e_n \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

the second product becomes

$$A_{i1}E_1 + A_{i2}E_2 + \ldots + A_{in}E_n \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where A_{ij} is the complement of a_{ij} (Art. 3) and

$$E_j = e_1 e_2 \ldots e_{j-1}e_{j+1} \ldots e_n.$$

For we get a term of the product by leaving out each unit such as e_j in turn, i.e. by forming a determinant with the remaining $n-1$ columns; and since we previously omitted the i^{th} row of the given determinant, this determinant is A_{ij}.

Now multiply the $n-1$ factors which form (2) by the remaining factor (1); we obtain

$$(-1)^{i-1}|a_{ik}| = a_{i1}A_{i1} - a_{i2}A_{i2} + \ldots + (-1)^{j-1} a_{ij}A_{ij} + \ldots$$

For
$$\begin{aligned} e_jE_j &= e_j \,.\, e_1 \ldots e_{j-1}e_{j+1} \ldots e_n \\ &= (-1)^{j-1}e_1 \ldots e_n = (-1)^{j-1}, \end{aligned}$$

$$e_jE_k = 0 \text{ if } j \text{ is not equal to } k.$$

The factor $(-1)^{i-1}$ on the left is accounted for in the same way.

Thus
$$|a_{ik}| = \Sigma (-1)^{i+j} a_{ij}A_{ij}.$$

For example,

$$\begin{vmatrix} a_1, & a_2, & a_3, & a_4 \\ b_1, & b_2, & b_3, & b_4 \\ c_1, & c_2, & c_3, & c_4 \\ d_1, & d_2, & d_3, & d_4 \end{vmatrix} = c_1 \begin{vmatrix} a_2, & a_3, & a_4 \\ b_2, & b_3, & b_4 \\ d_2, & d_3, & d_4 \end{vmatrix} - c_2 \begin{vmatrix} a_1, & a_3, & a_4 \\ b_1, & b_3, & b_4 \\ d_1, & d_3, & d_4 \end{vmatrix}$$

$$+ c_3 \begin{vmatrix} a_1, & a_2, & a_4 \\ b_1, & b_2, & b_4 \\ d_1, & d_2, & d_4 \end{vmatrix} - c_4 \begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ d_1, & d_2, & d_3 \end{vmatrix}.$$

10. In the final equation of Art. 9 A_{ij} is got from $|a_{ik}|$ by erasing the i^{th} row and j^{th} column and writing the remainder as a determinant. It is however more symmetrical, and sometimes convenient, to give to A_{ij} a different form obtained by a series of cyclical permutations of rows and columns.

In A_{ij} remove the first row by a series of interchanges to the last place, then move what is now the first row to the last place, and so on, until we arrive at what was the $(i-1)^{\text{th}}$ row, which we remove to the last place. This introduces $(i-1)(n-2)$ changes of sign.

Now remove the first column to the last place, and so on, $j-1$ times, necessitating $(j-1)(n-2)$ changes of sign. In all we have introduced

$$(i-1)(n-2)+(j-1)(n-2), \text{ or } (i+j)n$$

changes of sign (an even number of changes being neglected). So that, if the new determinant is called A'_{ij}, we have

$$A'_{ij} = (-1)^{n(i+j)} A_{ij},$$

and

$$|a_{ik}| = \Sigma (-1)^{(n+1)(i+j)} a_{ij} A'_{ij},$$

where

$$A'_{ij} = \begin{vmatrix} a_{i+1\,j+1}, & a_{i+1\,j+2} \dots a_{i+1\,n} a_{i+1\,1} \dots a_{i+1\,j-1} \\ \dots\dots\dots & \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n\,j+1}, & a_{n\,j+2} \dots\dots a_{nn}, \; a_{n1} \dots\dots a_{n\,j-1} \\ a_{1\,j+1}, & a_{1\,j+2} \dots\dots a_{1n}, \; a_{11} \dots\dots a_{1\,j-1} \\ \dots\dots\dots & \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{i-1\,j+1}, & a_{i-1\,j+2} \dots a_{i-1\,n} a_{i-1\,1} \dots a_{i-1\,j-1} \end{vmatrix}.$$

For example,

$$\begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix} = b_1 \begin{vmatrix} c_2, & c_3 \\ a_2, & a_3 \end{vmatrix} + b_2 \begin{vmatrix} c_3, & c_1 \\ a_3, & a_1 \end{vmatrix} + b_3 \begin{vmatrix} c_1, & c_2 \\ a_1, & a_2 \end{vmatrix}.$$

In future we shall always write

$$|a_{ik}| = \Sigma a_{ij} A_{ij},$$

and suppose that A_{ij} has its proper sign.

11. We may arrange the complements of the elements of a determinant in another square array, and then the two arrays

$$\left.\begin{matrix} a_{11} \ldots\ldots a_{1n} \\ \ldots\ldots\ldots\ldots \\ a_{n1} \ldots\ldots a_{nn} \end{matrix}\right\} \ldots\ldots\ldots (1), \qquad \left.\begin{matrix} A_{11} \ldots\ldots A_{1n} \\ \ldots\ldots\ldots\ldots \\ A_{n1} \ldots\ldots A_{nn} \end{matrix}\right\} \ldots\ldots\ldots (2),$$

are said to be reciprocal.

If now a sum be formed by multiplying each element of a row of (1) by the corresponding element of a row of (2), and adding these products together, the sum is equal to the original determinant or zero, according as the two rows have the same suffix or not. Namely,

$$a_{i1} A_{j1} + a_{i2} A_{j2} + \ldots + a_{in} A_{jn} = |a_{ik}| \text{ or } 0,$$

according as i is or is not equal to j.

For if i is equal to j the sum on the left is the expansion of the determinant according to the elements of the i^{th} row, but if i is not equal to j the sum on the left is what the expansion of the determinant would be, if its i^{th} and j^{th} rows were identical, but if the elements of two rows are identical the determinant vanishes. In like manner, if we multiply the elements of a column of (1) by the corresponding elements of a column of (2), we get

$$a_{1i} A_{1j} + a_{2i} A_{2j} + \ldots + a_{ni} A_{nj},$$

and this sum is equal to $|a_{ik}|$ or 0, according as i is or is not equal to j.

12. If all the elements of a row vanish the determinant vanishes, as we see at once by expanding the determinant according to the elements of that row. If all but one vanish the

determinant reduces to the product of that element and its complement; viz. if all the elements of the i^{th} row vanish except a_{ik}, then the determinant reduces to $a_{ik} A_{ik}$.

Thus for example,

$$\begin{vmatrix} a_{11}, & a_{12} \dots a_{1n} \\ 0, & a_{22} \dots a_{2n} \\ 0, & a_{32} \dots a_{3n} \\ \multicolumn{2}{c}{\dots\dots\dots\dots} \\ 0, & a_{n2} \dots a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} \dots a_{2n} \\ \dots\dots\dots \\ a_{n2} \dots a_{nn} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \dots a_{1n} \\ 0, & a_{22} & a_{23} \dots a_{2n} \\ 0, & 0, & a_{33} \dots a_{3n} \\ \multicolumn{3}{c}{\dots\dots\dots\dots\dots} \\ 0, & 0, & 0 \dots\dots a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22}, & a_{23} \dots a_{2n} \\ 0, & a_{33} \dots a_{3n} \\ \multicolumn{2}{c}{\dots\dots\dots\dots} \\ 0, & 0 \dots\dots a_{nn} \end{vmatrix}$$

$$= a_{11} a_{22} \begin{vmatrix} a_{33} \dots a_{3n} \\ \dots\dots\dots \\ 0 \dots\dots a_{nn} \end{vmatrix}$$

$$= a_{11} a_{22} a_{33} \dots a_{nn}.$$

13. The theorem of the preceding article is of use in evaluating a determinant by reducing it to one of lower order. If the determinant is not of the required form to begin with, it can sometimes be reduced to it. We may exemplify this by finding the value of the determinant

$$D_r = \begin{vmatrix} 0, & a, & a \dots a \\ b, & 0, & a \dots a \\ b, & b, & 0 \dots a \\ \multicolumn{3}{c}{\dots\dots\dots\dots} \\ b, & b, & b \dots 0 \end{vmatrix} (r),$$

the suffixes denoting the order of the determinant. The elements of the leading diagonal are zero, those to the right of it all equal to a, and those to the left all equal to b.

If we subtract each row from the one which follows it, beginning with the last but one,

$$D_r = \begin{vmatrix} 0, & a, & a, & a \ldots\ldots a \\ b, & -a, & 0, & 0 \ldots\ldots 0 \\ 0, & b, & -a, & 0 \ldots\ldots 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0, & 0, & 0, & 0 \ldots -a \end{vmatrix} (r).$$

The first column contains only one element, hence

$$D_r = -b \begin{vmatrix} a, & a, & a, & a \ldots \\ b, & -a, & 0, & 0 \ldots \\ 0, & b, & -a, & 0 \ldots \\ 0, & 0, & b, & -a \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} (r-1).$$

Regard the elements in the first row as

$$a+0, \qquad 0+a, \qquad 0+a \ldots$$

then (II. 3) we can resolve the determinant into the sum of two.

$$D_r = -b \begin{vmatrix} a, & 0, & 0, & 0 \ldots \\ b, & -a, & 0, & 0 \ldots \\ 0, & b, & -a, & 0 \ldots \\ 0, & 0, & b, & -a \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} (r-1) \quad -b \begin{vmatrix} 0, & a, & a, & a \ldots \\ b, & -a, & 0, & 0 \ldots \\ 0, & b, & -a, & 0 \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} (r-1).$$

In the first of these two determinants all the elements above the leading diagonal vanish, hence its value is $(-1)^{r-2} a^{r-1}$. The second determinant is of the same form as that to which we first reduced D_r, hence

$$D_r = -bD_{r-1} + b(-a)^{r-1}.$$

This is an equation of differences with constant coefficients for D_r, its solution is

$$D_r = \frac{(-1)^{r-1}ab}{a-b}(a^{r-1} - b^{r-1}).$$

14. In Art. 11 we saw how under certain circumstances the order of a determinant might be reduced. Conversely we are enabled to increase the order of a determinant without altering its value, namely, by bordering it with a new row and column in one of which all the elements vanish except that common to the other. Thus

$$|a_{ik}| = \begin{vmatrix} 1, & 0, & 0, & 0 & \dots \\ x, & a_{11}, & a_{12}, & a_{13} & \dots \\ y, & a_{21}, & a_{22}, & a_{23} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

$$= (-1)^n \begin{vmatrix} 0, & 0, & 0 & \dots & 0, & 1 \\ a_{11}, & a_{12}, & a_{13} & \dots & a_{1n}, & x \\ a_{21}, & a_{22}, & a_{23} & \dots & a_{2n}, & y \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

where the quantities $x, y \dots$ are any whatever. By adding on to these a new row and column we can raise the order of the determinant to $n+2$ and so on.

15. In the determinant $D = |a_{ik}|$, if we suppose only the element a_{ik} to vary, since on expanding according to the elements of the i^{th} row

$$D = a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + a_{ik}A_{ik} + \dots$$

the only variable term on the right is the product $a_{ik}A_{ik}$, we see at once that

$$\frac{dD}{da_{ik}} = A_{ik}.$$

If among the elements of A_{ik} only a_{rs} is variable, we see that

$$\frac{dA_{ik}}{da_{rs}} = \frac{d^2D}{da_{rs}da_{ik}}.$$

Thus $\dfrac{d^2D}{da_{rs}da_{ik}}a_{rs}a_{ik}$ is the sum of all terms in D which contain the product $a_{ik}a_{rs}$.

The differential coefficient

$$\frac{d^2D}{da_{rs}da_{ik}}$$

is the determinant obtained by erasing in D the i^{th} and r^{th} rows and the k^{th} and s^{th} columns, it is complementary to

$$\begin{vmatrix} a_{ik}, & a_{is} \\ a_{rk}, & a_{rs} \end{vmatrix}.$$

In like manner it is plain that

$$\frac{d^{n-m}D}{da_{fr}da_{gs}\dots} \text{ and } \frac{d^m D}{da_{pu}da_{qv}\dots}$$

are complementary determinants if

$$f,\ g\ \dots\ p,\ q\ \dots$$
$$r,\ s\ \dots\ u,\ v\ \dots$$

are each of them permutations of 1, 2 ... n, i.e. if the product

$$a_{fr}a_{gs}\ \dots\ a_{pu}a_{qv}\ \dots$$

is a term of the determinant D.

16. If all the elements of a determinant are functions of a variable t we see that

$$\frac{dD}{dt} = \Sigma \frac{dD}{da_{ik}} \cdot \frac{da_{ik}}{dt} \quad (i,\ k = 1,\ 2 \dots n).$$

If we denote differential coefficients with respect to t by accents we have

$$D' = \Sigma A_{i1} a'_{i1} + \Sigma A_{i2} a'_{i2} + \dots$$

$$= \begin{vmatrix} a'_{11}, & a_{12} & \dots & a_{1n} \\ a'_{21}, & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \end{vmatrix} + \begin{vmatrix} a_{11}, & a'_{12}, & \dots & a_{1n} \\ a_{21}, & a'_{22}, & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \end{vmatrix} + \dots$$

So that D' is the sum of n determinants obtained by substituting for the elements of each column of D in succession their differential coefficients with respect to t.

An interesting example of this is to consider the differential coefficient of

$$D = \begin{vmatrix} u, & u', & u'', & \dots & u^{(n-1)} \\ v, & v', & v'', & \dots & v^{(n-1)} \\ w, & w', & w'', & \dots & w^{(n-1)} \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix},$$

accents denoting differential coefficients with respect to t.

Each of the first $n-1$ determinants obtained by the preceding rule vanishes because it has two columns alike, the last alone does not vanish, so that

$$\frac{dD}{dt} = \begin{vmatrix} u, & u' & \dots & u^{(n-2)}, & u^{(n)} \\ v, & v' & \dots & v^{(n-2)}, & v^{(n)} \\ w, & w' & \dots & w^{(n-2)}, & w^{(n)} \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}.$$

As another example take the determinant

$$D_n = \begin{vmatrix} 1, & 1 & \dots & 1 \\ t_1, & t_2 & \dots & t_n \\ t_1^2, & t_2^2 & \dots & t_n^2 \\ \dots & \dots & \dots & \dots \\ t_1^{n-1}, & t_2^{n-1} & \dots & t_n^{n-1} \end{vmatrix}.$$

Then $\dfrac{dD_n}{dt_r}$ is got from D_n by substituting for the elements of the r^{th} column

$$0,\ 1,\ 2t_r,\ 3t_r^2\ \dots\ (n-1)\,t_r^{n-2}.$$

Hence

$$\frac{d^{n-1}D_n}{dt_1 dt_2 \dots dt_{n-1}} = \begin{vmatrix} 0, & 0 & \dots & 0, & 1 \\ 1, & 1 & \dots & 1, & t_n \\ 2t_1, & 2t_2 & \dots & 2t_{n-1}, & t_n^2 \\ \dots & \dots & \dots & \dots & \dots \\ (n-1)\,t_1^{n-2}, & (n-1)\,t_2^{n-2} & \dots & (n-1)\,t_{n-1}^{n-2}, & t_n^{n-1} \end{vmatrix}$$

$$= (-1)^{n-1}(n-1)!\ D_{n-1}.$$

17. We may use the theorems of Art. 11 of the present chapter to prove those of Arts. 3 and 4 of Chap. II.

If each element of a row of a determinant is the sum of p terms, the determinant is equal to the sum of p determinants having for their elements the separate terms of the sum in question.

For if $\qquad a_{ik} = p_k + q_k + \dots + t_k.$

Then

$$\begin{aligned} |\,a_{ik}\,| &= \Sigma a_{ik} A_{ik} \\ &= \Sigma p_k A_{ik} + \Sigma q_k A_{ik} + \dots + \Sigma t_k A_{ik} \\ &= P + Q + \dots + T, \end{aligned}$$

where P is the determinant obtained from the given one by writir $p_1, p_2 \dots p_n$ for the elements of the i^{th} row and $Q \dots T$ have simil meanings.

The value of a determinant is not altered by adding to tl elements of any row those of another row multiplied by a consta factor. For if to the elements of the i^{th} row we add those the j^{th} row, each multiplied by p, the resulting determinant equal to

$$\Sigma\,(a_{ik} + pa_{jk})\,A_{ik} = \Sigma a_{ik} A_{ik} + p\Sigma a_{jk} A_{ik}$$
$$= |\, a_{ik} \,|.$$

The last sum vanishing by Art. 11.

18. If each element of a determinant consists of the sum p terms, we could by continued application of the first theorem Art. 17 reduce this determinant to a sum of determinants who elements are all single terms. But a formula of expansion h been given by Albeggiani which presents the result in a mo suitable form for applications.

Let
$$a_{ik} = a_{ik1} + a_{ik2} + \dots + a_{ikp},$$
so that each element in the determinant is the sum of p tern Then each column of the determinant when written at full leng would consist of p partial columns whose suffixes are the thi suffixes of the above elements. With these partial columns can form p determinants, taking all the partial columns with tl third suffix 1 to form the first, those with the third suffix to form the second, and so on. We shall denote these dete minants by

$$D_1^{(n)},\ D_2^{(n)} \dots D_p^{(n)},$$

so that

$$D_u^{(n)} = \begin{vmatrix} a_{11u}, & a_{12u} & \dots & a_{1nu} \\ a_{21u}, & a_{22u} & \dots & a_{2nu} \\ \dots & \dots & \dots & \dots \\ a_{n1u}, & a_{n2u} & \dots & a_{nnu} \end{vmatrix}.$$

The first two suffixes tell us the row and column in which t element stands, the third the determinant to which it belongs. T original determinant is denoted by $D^{(n)}$. The index in brack tells us the order of the determinant.

19. We shall find it necessary to employ the term complementary minors in the following sense. From the elements of $D_1^{(n)}$, form a minor $D_1^{(\alpha)}$ of order α by selecting α rows and columns. Then in $D_2^{(n)}$ select β rows and columns, whose suffixes are different from those selected to form $D_1^{(\alpha)}$, these form a determinant $D_2^{(\beta)}$, and so on until we take π rows and columns from $D_p^{(n)}$, to form a determinant $D_p^{(\pi)}$ none of which have the same suffix as any of the preceding. Then if

$$\alpha + \beta + \gamma + \ldots + \pi = n \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$D_1^{(\alpha)},\ D_2^{(\beta)},\ D_3^{(\gamma)} \ldots D_p^{(\pi)}$$

shall be called a series of complementary minors. Any one or more of the numbers $\alpha, \beta \ldots \pi$ can be unity or zero.

20. We shall now prove that

$$D^{(n)} = S\Sigma D_1^{(\alpha)} D_2^{(\beta)} \ldots D_p^{(\pi)},$$

where the meanings of the summation signs will be explained presently. For we have

$$D^{(n)} = \Pi\ (a_{i1}e_1 + a_{i2}e_2 + \ldots + a_{in}e_n),$$

and if

$$u_{ij} = a_{i1j}e_1 + a_{i2j}e_2 + \ldots + a_{inj}e_n,$$

$$D^{(n)} = \Pi\ (u_{i1} + u_{i2} + \ldots + u_{ip}) \ldots\ldots\ldots\ldots\ldots (2),$$

the product containing n factors.

We shall obtain a term of the product on the right if we take α factors such as u_{i1}, β factors such as $u_{i2} \ldots \pi$ factors such as u_{ip}, provided the equation (1) is satisfied.

But from the definition of a determinant this product of factors is equal to a determinant of order n the first α of whose rows come from $D_1^{(n)}$, the next β from $D_2^{(n)} \ldots$ the last π from $D_p^{(n)}$. Expand this determinant in the sum of products of complementary minors of order $\alpha, \beta \ldots \pi$ selecting the rows of the minors from the first α, the next $\beta \ldots$ the last π, its value is then (Art. 8)

$$\Sigma D_1^{(\alpha)} D_2^{(\beta)} \ldots D_p^{(\pi)},$$

with the notation of Art. 19, and the summation sign means that we are to take all the possible complementary minors.

This is only a single term in the expansion of the product (2), the whole product is obtained by summing this for all values of $\alpha, \beta \ldots \pi$ which satisfy the equation (1).

Thus $$D^{(n)} = S\Sigma D_1^{(\alpha)} D_2^{(\beta)} \ldots D_p^{(\pi)} \quad\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

21. The number of terms in the sum Σ is

$$\frac{n!}{\alpha!\beta!\ldots\pi!}.$$

Let us compare the expansion (3) with the expansion of the multinomial

$$(D_1 + D_2 + \ldots + D_p)^n.$$

The general term is

$$CD_1^{\alpha} D_2^{\beta} \ldots D_p^{\pi} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

where $\alpha, \beta \ldots \pi$ satisfy (1) and

$$C = \frac{n!}{\alpha!\beta!\ldots\pi!}.$$

Comparing (3) and (4) we see that in expanding the determinant we replace C by Σ, and $\alpha, \beta \ldots \pi$ are no longer exponents, but merely indicate the order of the determinant.

Hence we may write symbolically for the expansion of our determinant

$$(D_1 + D_2 + \ldots + D_p)^n,$$

where in every term of the multinomial expansion we replace the coefficient by a summation sign, the number of terms in the sum being given by the multinomial coefficient and the exponents $\alpha, \beta \ldots \pi$ now indicating the orders of the complementary minors. Thus finally we have the symbolical equation

$$D^{(n)} = (D_1 + D_2 + \ldots + D_p)^n.$$

22. Let us make use of this theorem to expand the determinant

$$D = \begin{vmatrix} a_{11} + z_1, & a_{12}, & a_{13} & \ldots & a_{1n} \\ a_{21}, & a_{22} + z_2, & a_{23} & \ldots & a_{2n} \\ a_{31}, & a_{32}, & a_{33} + z_3 & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1}, & a_{n2}, & a_{n3} & \ldots & a_{nn} + z_n \end{vmatrix}$$

according to products of the quantities $z_1, z_2 \ldots z_n$.

Here we must write

$$D_1^{(n)} = \begin{vmatrix} a_{11} \dots a_{1n} \\ \dots\dots\dots \\ a_{n1} \dots a_{nn} \end{vmatrix}, \qquad D_2^{(n)} = \begin{vmatrix} z_1, & 0 \dots 0 \\ 0, & z_2 \dots 0 \\ \dots & \dots\dots \\ 0, & 0 \dots z_n \end{vmatrix}.$$

Then by the above theorem

$$\begin{aligned} D &= (D_1 + D_2)^n \\ &= D_1^{(n)} + \Sigma D_1^{(n-1)} D_2^{(1)} + \Sigma D_1^{(n-2)} D_2^{(2)} + \dots + D_2^n. \end{aligned}$$

Now clearly all minors of $D_2^{(n)}$ vanish except those whose leading diagonal is part of the leading diagonal of $D_2^{(n)}$.

Thus

$$D_2^{(1)} = z_i, \; D_2^{(2)} = z_i z_k \dots D_2^{(n)} = z_1 z_2 \dots z_n.$$

The corresponding minors $D_1^{(n-1)}$, $D_1^{(n-2)}$ ... are got by erasing in $D_1^{(n)}$ the i^{th} row and column, the i^{th} and k^{th} rows and columns, &c.

Thus

$$D = D_1^{(n)} + \Sigma z_i D_1^{(n-1)} + \Sigma z_i z_k D_1^{(n-2)} + \dots + z_1 z_2 \dots z_n.$$

Or if we simply denote $D_1^{(n)}$ by D_1,

$$D = D_1 + \Sigma z_i \frac{dD_1}{da_{ii}} + \Sigma z_i z_k \frac{d^2 D_1}{da_{ii} da_{kk}} + \dots + z_1 z_2 \dots z_n.$$

If $z_1 = z_2 \dots = z_n$ we get

$$D = D_1 + z\Sigma \frac{dD_1}{da_{ii}} + z^2 \Sigma \frac{d^2 D_1}{da_{ii} da_{kk}} + \dots + z^n.$$

23. Any determinant can be written in the form

$$D = \begin{vmatrix} 0 + a_{11}, & a_{12} & \dots & a_{1n} \\ a_{21} \;, & 0 + a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} \;, & a_{n2} & \dots & 0 + a_{nn} \end{vmatrix}.$$

We may now apply the theorem of Art. 22 by supposing

$$D_1 = \begin{vmatrix} 0, & a_{12} \dots a_{1n} \\ a_{21}, & 0 \;\; \dots a_{2n} \\ \dots & \dots\dots \\ a_{n1}, & a_{n2} \dots 0 \end{vmatrix}$$

and

$$z_i = a_{ii}.$$

Then

$$D = D_1 + \Sigma a_{ii}\frac{dD_1}{da_{ii}} + \Sigma a_{ii}a_{kk}\frac{d^2D_1}{da_{ii}da_{kk}} + \ldots + a_{11}a_{22}\ldots a_{nn}.$$

The general term being

$$\Sigma a_{ii}a_{kk}\ldots a_{rr}D_1^{(n-m)}.$$

Where $D_1^{(n-m)}$ is the minor obtained from D_1 by suppressing the i^{th}, k^{th} ... r^{th} rows and columns, m in number.

It is clear that $D_1^{(1)}$ is zero, for a term of D cannot contain $n-1$ terms from the leading diagonal only, if it does it must contain n.

Ex. If

$$\begin{vmatrix} 0, & a_{12} \\ a_{21}, & 0 \end{vmatrix} = (12), \text{ \&c.}$$

we have

$$\begin{vmatrix} a_{11} & \cdots & a_{14} \\ \cdots & \cdots & \cdots \\ a_{41} & \cdots & a_{44} \end{vmatrix} = \begin{aligned}[t] & a_{11}a_{22}a_{33}a_{44} + a_{11}a_{22}(34) + a_{11}a_{33}(24) + a_{11}a_{44}(23) \\ & + a_{22}a_{33}(14) + a_{22}a_{44}(13) + a_{33}a_{44}(12) \\ & + a_{11}(234) + a_{22}(134) + a_{33}(124) + a_{44}(123) \\ & + (1234). \end{aligned}$$

As another example we may find the value of the determinant

$$D = \begin{vmatrix} c_1, & a, & a, & a \ldots a \\ b, & c_2, & a, & a \ldots a \\ b, & b, & c_3, & a \ldots a \\ \cdots & \cdots & \cdots & \cdots \\ b, & b, & b, & b \ldots c_n \end{vmatrix}.$$

The general term in the expansion of this determinant is

$$\Sigma c_i c_k \ldots c_r D_1^{(n-m)},$$

when c_i, c_k ... c_r are any m elements of the leading diagonal. But by Art. 13

$$D_1^{(n-m)} = (-1)^{n-m-1}\frac{ab}{a-b}(a^{n-m-1} - b^{n-m-1}).$$

Whence if $f(x) = (c_1 - x)(c_2 - x)\ldots(c_n - x)$,

it is clear that

$$D = \frac{af(b) - bf(a)}{a-b}.$$

If we write down the similar determinant of order $n+1$, for which $c_{n+1}=0$, after dividing both sides by ab, we get

$$\begin{vmatrix} c_1, & a & \dots & a, & 1 \\ b, & c_2 & \dots & a, & 1 \\ \dots & \dots & \dots & \dots & \dots \\ b, & b & \dots & c_n, & 1 \\ 1, & 1 & \dots & 1, & 0 \end{vmatrix} = \frac{f(a)-f(b)}{a-b}.$$

If we suppose $a=b$, we get on evaluating the vanishing fraction in this latter determinant a determinant expression for $f'(a)$.

24. We have seen how to expand a determinant according to the elements of a row or column. It is frequently useful to be able to expand a determinant according to the elements of a row and column. This is effected by means of the following theorem due to Cauchy,

$$|\,a_{ik}\,| = a_{rs}A_{rs} - \Sigma a_{rk}a_{is}B_{ik},$$

which expands a determinant according to the products of elements standing in the r^{th} row and s^{th} column.

A_{rs} is the complement of a_{rs} and B_{ik} is the complement of a_{ik} in A_{rs}, and is therefore a second minor of the original determinant.

For every term which does not contain a_{rs} must contain some other element from the r^{th} row and some other element from the s^{th} column, and hence contains such a product as $a_{rk}a_{is}$, where i and k are different from r and s respectively. The aggregate of all terms which multiply a_{rs} is A_{rs}; now $a_{rk}a_{is}$ differs from $a_{rs}a_{ik}$ by the interchange of the suffixes k and s, thus the aggregate of terms which multiplies $a_{rk}a_{is}$ differs in sign only from that which multiplies $a_{rs}a_{ik}$, that is to say, differs in sign only from the coefficient of a_{ik} in A_{rs}. Hence $-B_{ik}$ is the coefficient in question.

25. This theorem is useful for expanding a determinant which has been bordered. For example by this theorem

$$D = \begin{vmatrix} b_{pq}, & b_{p1}, & b_{p2} & \dots \\ b_{1q}, & a_{11}, & a_{12} & \dots \\ b_{2q}, & a_{21}, & a_{22} & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

$$= b_{pq}\,|\,a_{ik}\,| - \Sigma b_{pk}b_{iq}A_{ik},$$

where A_{ik} is the complement of a_{ik} in $|\,a_{ik}\,|$.

By the selection of a suitable bordering we are often able t evaluate a determinant by means of this theorem.

For example, let

$$D = \begin{vmatrix} x_1, & a_2, & a_3 & \dots & a_n \\ a_1, & x_2, & a_3 & \dots & a_n \\ a_1, & a_2, & x_3 & \dots & a_n \\ \dots & \dots & \dots & \dots & \dots \\ a_1, & a_2, & a_3 & \dots & x_n \end{vmatrix}$$

all the elements in the i^{th} column being a_i except that in the i row which is x_i.

Then by Art. 14

$$D = \begin{vmatrix} 1, & 0, & 0, & 0 & \dots \\ 1, & x_1, & a_2, & a_3 & \dots \\ 1, & a_1, & x_2, & a_3 & \dots \\ 1, & a_1, & a_2, & x_3 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}.$$

Multiply the first column by a_i, and subtract it from the i column; do this for each column, the value of the determinant i unaltered, and

$$D = \begin{vmatrix} 1, & -a_1, & -a_2, & -a_3, & \dots \\ 1, & x_1 - a_1, & 0, & 0, & \dots \\ 1, & 0, & x_2 - a_2, & 0, & \dots \\ 1, & 0, & 0, & x_3 - a_3, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}.$$

Here the bordered determinant is

$$\begin{matrix} x_1 - a_1, & 0, & 0 & \dots \\ 0, & x_2 - a_2, & 0 & \dots \\ 0, & 0, & x_3 - a_3 & \dots \\ \dots & \dots & \dots & \dots \end{matrix}$$

for which all first minors vanish except those of diagonal element

Hence, in the theorem of this article, we must suppose $i = k$;

$$f = (x_1 - a_1)(x_2 - a_2) \dots (x_n - a_n),$$

$$f'(x_r) = \frac{df}{dx_r},$$

it follows that

$$D = f + \Sigma a_r f'(x_r),$$

a theorem due to Sardi.

CHAPTER IV.

ON THE MULTIPLICATION OF DETERMINANTS.

1. IF we have two arrays

$$\begin{array}{l} a_{11},\ a_{12}\ \ldots\ a_{1n} \\ a_{21},\ a_{22}\ \ldots\ a_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1},\ a_{m2}\ \ldots\ a_{mn} \end{array} \quad (1), \qquad \begin{array}{l} b_{11},\ b_{12}\ \ldots\ b_{1n} \\ b_{21},\ b_{22}\ \ldots\ b_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ b_{m1},\ b_{m2}\ \ldots\ b_{mn} \end{array} \quad (2),$$

and form a new set of elements c_{ik} by multiplying each element in the i^{th} row of (1) by the corresponding element in the k^{th} row of (2) and adding the products, these elements form a new square array of m^2 elements where

$$c_{ik} = a_{i1}b_{k1} + a_{i2}b_{k2} + \ldots + a_{in}b_{kn}.$$

This array is said to be compounded of the arrays (1) and (2).

2. We shall now shew that the determinant $|c_{ik}|$ is equal to zero if the two arrays (1) and (2) are redundant $(m > n)$; is equal to the product of the two determinants $|a_{ik}|$, $|b_{ik}|$ if $m = n$; and if the arrays are defective $(m < n)$ is equal to the sum of the n_m products of determinants got by taking any m columns from (1) to form a determinant and multiplying it by the determinant of the corresponding m columns of (2).

Let $\qquad C_i = c_{i1}e_1 + c_{i2}e_2 + \ldots + c_{im}e_m,$

then $\qquad |c_{ik}| = \Pi C_i. \qquad i = 1, - - m$

Now

$$\begin{aligned} C_i = &(a_{i1}b_{11} + a_{i2}b_{12} + \ldots + a_{in}b_{1n})\,e_1 \\ &+ (a_{i1}b_{21} + a_{i2}b_{22} + \ldots + a_{in}b_{2n})\,e_2 \\ &+ \ldots \\ &+ (a_{i1}b_{m1} + a_{i2}b_{m2} + \ldots + a_{in}b_{mn})\,e_m \\ = &\,a_{i1}B_1 + a_{i2}B_2 + \ldots + a_{in}B_n, \end{aligned}$$

where $$B_k = b_{1k}e_1 + b_{2k}e_2 + \dots + b_{mk}e_m$$

form a system of alternate numbers of the m^{th} order.

Thus $$|c_{ik}| = \Pi\,(a_{i1}B_1 + a_{i2}B_2 + \dots + a_{in}B_n).$$ $i = 1, \dots m$

(i) If $m > n$ the product on the right vanishes, for on multiplying it out, in each term some one of the B's is repeated and the product vanishes.

(ii) If $m = n$ since by I. 17 the B's follow the same law as the units e,

$$\begin{aligned}|c_{ik}| &= |a_{ik}|\,.\,\Pi B_i \qquad (i = 1, 2 \dots n)\\ &= |a_{ik}|\,.\,|b_{ik}|.\end{aligned}$$

(iii) If $m < n$ the product on the right is the sum of such terms as

$$\begin{vmatrix} a_{1p}, & a_{1q}, & a_{1r} & \dots \\ a_{2p}, & a_{2q}, & a_{2r} & \dots \\ \dots & \dots & \dots & \dots \\ a_{mp}, & a_{mq}, & a_{mr} & \dots \end{vmatrix} B_p B_q B_r \dots$$

when $p, q, r \dots$ are m numbers taken from $1, 2 \dots n$ (III. 5).

But

$$B_p B_q B_r \dots = \begin{vmatrix} b_{1p}, & b_{1q}, & b_{1r} & \dots \\ b_{2p}, & b_{2q}, & b_{2r} & \dots \\ \dots & \dots & \dots & \dots \\ b_{mp}, & b_{mq}, & b_{mr} & \dots \end{vmatrix} e_1 e_2 e_3 \dots e_m.$$

Thus

$$|c_{ik}| = \Sigma \begin{vmatrix} a_{1p}, & a_{1q}, & a_{1r} & \dots \\ a_{2p}, & a_{2q}, & a_{2r} & \dots \\ \dots & \dots & \dots & \dots \\ a_{mp}, & a_{mq}, & a_{mr} & \dots \end{vmatrix} . \begin{vmatrix} b_{1p}, & b_{1q}, & b_{1r} & \dots \\ b_{2p}, & b_{2q}, & b_{2r} & \dots \\ \dots & \dots & \dots & \dots \\ b_{mp}, & b_{mq}, & b_{mr} & \dots \end{vmatrix}$$

where for $p, q, r \dots$ we are to write all possible m-*ads* from the n numbers $1, 2 \dots n$.

3. The second case of Art. 2 gives us the rule for multiplying two determinants. We see also that the product of two determinants of the n^{th} order is also a determinant of the n^{th} order. Thus

$$\begin{vmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} . \begin{vmatrix} b_{11} & \dots & b_{1n} \\ \dots & \dots & \dots \\ b_{n1} & \dots & b_{nn} \end{vmatrix} = \begin{vmatrix} c_{11} & \dots & c_{1n} \\ \dots & \dots & \dots \\ c_{n1} & \dots & c_{nn} \end{vmatrix}$$

where the quantities c_{ik} are given by

$$c_{ik} = a_{i1}\,b_{k1} + a_{i2}\,b_{k2} + \dots + a_{in}b_{kn}.$$

But since in either, or both of the determinants $|a_{ik}|$, $|b_{ik}|$ we may interchange rows and columns without affecting their value, we see that the product of two determinants can be obtained in the form of a determinant in four different ways, viz. the element c_{ik} has one of the four forms :

$$a_{i1}b_{k1} + a_{i2}b_{k2} + \dots + a_{in}b_{kn},$$
$$a_{i1}b_{1k} + a_{i2}b_{2k} + \dots + a_{in}b_{nk},$$
$$a_{1i}b_{1k} + a_{2i}b_{2k} + \dots + a_{ni}b_{nk},$$
$$a_{1i}b_{k1} + a_{2i}b_{k2} + \dots + a_{ni}b_{kn},$$

where we multiply the elements of a row of $|a_{ik}|$ by the corresponding elements of a row or column of $|b_{ik}|$; or the elements of a column of $|a_{ik}|$ by the corresponding elements of a column or row of $|b_{ik}|$. There are really only two essentially distinct cases: multiplying by rows, when we multiply corresponding elements of two rows together; and multiplying by rows and columns, where we multiply the elements of a row by the corresponding elements of a column.

4. We can only compound two arrays when they have the same number of rows and columns, but we can always form the product of two determinants, for by III. 14 the order of one of them can be increased until it is equal to that of the other without altering the value of the determinant. So that the product of two determinants of orders n and m $(n > m)$ is a determinant of order n.

5. *Examples.* Compounding the two systems

$$\begin{matrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \end{matrix} \qquad \begin{matrix} p_1, & q_1, & r_1 \\ p_2, & q_2, & r_2 \end{matrix}$$

we get the theorem

$$\begin{vmatrix} a_1p_1 + b_1q_1 + c_1r_1, & a_2p_1 + b_2q_1 + c_2r_1 \\ a_1p_2 + b_1q_2 + c_1r_2, & a_2p_2 + b_2q_2 + c_2r_2 \end{vmatrix}$$
$$= \begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix} \cdot \begin{vmatrix} p_1, & q_1 \\ p_2, & q_2 \end{vmatrix} + \begin{vmatrix} a_1, & c_1 \\ a_2, & c_2 \end{vmatrix} \cdot \begin{vmatrix} p_1, & r_1 \\ p_2, & r_2 \end{vmatrix} + \begin{vmatrix} b_1, & c_1 \\ b_2, & c_2 \end{vmatrix} \cdot \begin{vmatrix} q_1, & r_1 \\ q_2, & r_2 \end{vmatrix},$$

while if we compound the systems

$$\begin{matrix} a_1, & a_2 \\ b_1, & b_2 \\ c_1, & c_2 \end{matrix} \qquad \begin{matrix} p_1, & p_2 \\ q_1, & q_2 \\ r_1, & r_2 \end{matrix}$$

we get

$$\begin{vmatrix} a_1p_1 + a_2p_2, & a_1q_1 + a_2q_2, & a_1r_1 + a_2r_2 \\ b_1p_1 + b_2p_2, & b_1q_1 + b_2q_2, & b_1r_1 + b_2r_2 \\ c_1p_1 + c_2p_2, & c_1q_1 + c_2q_2, & c_1r_1 + c_2r_2 \end{vmatrix} = 0.$$

Again, the product of the two determinants

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} \qquad \begin{vmatrix} p_1, & q_1, & r_1 \\ p_2, & q_2, & r_2 \\ p_3, & q_3, & r_3 \end{vmatrix}$$

is the determinant

$$\begin{vmatrix} a_1p_1 + b_1q_1 + c_1r_1, & a_1p_2 + b_1q_2 + c_1r_2, & a_1p_3 + b_1q_3 + c_1r_3 \\ a_2p_1 + b_2q_1 + c_2r_1, & a_2p_2 + b_2q_2 + c_2r_2, & a_2p_3 + b_2q_3 + c_2r_3 \\ a_3p_1 + b_3q_1 + c_3r_1, & a_3p_2 + b_3q_2 + c_3r_2, & a_3p_3 + b_3q_3 + c_3r_3 \end{vmatrix}.$$

While

$$\begin{vmatrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \\ a_4, & b_4, & c_4, & d_4 \end{vmatrix} \cdot \begin{vmatrix} p_1, & q_1 \\ p_2, & q_2 \end{vmatrix} = \begin{vmatrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \\ a_4, & b_4, & c_4, & d_4 \end{vmatrix} \begin{vmatrix} p_1, & q_1, & 0, & 0 \\ p_2, & q_2, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \end{vmatrix}$$

(forming the product by rows and columns)

$$= \begin{vmatrix} a_1p_1 + b_1p_2, & a_1q_1 + b_1q_2, & c_1, & d_1 \\ a_2p_1 + b_2p_2, & a_2q_1 + b_2q_2, & c_2, & d_2 \\ a_3p_1 + b_3p_2, & a_3q_1 + b_3q_2, & c_3, & d_3 \\ a_4p_1 + b_4p_2, & a_4q_1 + b_4q_2, & c_4, & d_4 \end{vmatrix}.$$

Multiplying by rows we have

$$\begin{vmatrix} a, & b \\ -b', & a' \end{vmatrix} \begin{vmatrix} c, & d \\ -d', & c' \end{vmatrix} = \begin{vmatrix} ac + bd, & -ad' + bc' \\ -b'c + a'd, & b'd' + a'c' \end{vmatrix}.$$

Now if a, b, c, d are the complex numbers

$$\begin{matrix} a = x + iy & b = u + iv \\ c = p + iq & d = r + is \end{matrix} \qquad i = \sqrt{-1},$$

and a', b', c', d' their conjugates, $a' = x - iy$, &c. On multiplying out the three determinants we have Euler's theorem concerning the product of two numbers each the sum of four squares, viz.

$$\begin{aligned}&(x^2+y^2+u^2+v^2)(p^2+q^2+r^2+s^2),\\ &= (px-qy+ru-sv)^2+(py+qx+rv+su)^2\\ &+(pu+qv-rx-sy)^2+(pv-qu-ry+sx)^2.\end{aligned}$$

6. We may compound an array with itself, thus if we compound the first array in Art. 1 with itself, the resulting determinant has for elements

$$c_{ik} = a_{i1}a_{k1} + a_{i2}a_{k2} + \dots + a_{in}a_{kn} = c_{ki},$$

and

$$|c_{ik}| = \Sigma \begin{vmatrix} a_{1p}, & a_{1q}, & a_{1r} & \dots \\ a_{2p}, & a_{2q}, & a_{2r} & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}^2,$$

or the determinant is the sum of n_m squares. If then the elements a_{ik} are all real the determinant $|c_{ik}|$ can only vanish when the determinant under the summation sign on the right vanishes for all values of $p, q, r\dots$

Thus compounding

$$\begin{matrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \end{matrix}$$

with itself we see that

$$\begin{vmatrix} a_1^2+b_1^2+c_1^2, & a_1a_2+b_1b_2+c_1c_2 \\ a_1a_2+b_1b_2+c_1c_2, & a_2^2+b_2^2+c_2^2 \end{vmatrix} = \begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1, & c_1 \\ b_2, & c_2 \end{vmatrix}^2 + \begin{vmatrix} a_1, & c_1 \\ a_2, & c_2 \end{vmatrix}^2,$$

or

$$\begin{aligned}&(a_1^2+b_1^2+c_1^2)(a_2^2+b_2^2+c_2^2)-(a_1a_2+b_1b_2+c_1c_2)^2\\ &= (a_1b_2-a_2b_1)^2+(b_1c_2-b_2c_1)^2+(a_1c_2-a_2c_1)^2.\end{aligned}$$

Again

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}^2 = \begin{vmatrix} a_1^2+b_1^2+c_1^2, & a_1a_2+b_1b_2+c_1c_2, & a_1a_3+b_1b_3+c_1c_3 \\ a_1a_2+b_1b_2+c_1c_2, & a_2^2+b_2^2+c_2^2, & a_2a_3+b_2b_3+c_2c_3 \\ a_1a_3+b_1b_3+c_1c_3, & a_2a_3+b_2b_3+c_2c_3, & a_3^2+b_3^2+c_3^2 \end{vmatrix}.$$

7. Prof. Sylvester has shewn how, by the artifice of bordering the determinants as in III. 14, the product of two determinants of order n can be represented in $n+1$ distinct forms. We shall illustrate this for the case $n = 3$.

The product of the two determinants

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}, \qquad \begin{vmatrix} p_1, & q_1, & r_1 \\ p_2, & q_2, & r_2 \\ p_3, & q_3, & r_3 \end{vmatrix},$$

is the determinant of order 3:

$$\begin{vmatrix} a_1p_1+b_1q_1+c_1r_1, & a_1p_2+b_1q_2+c_1r_2, & a_1p_3+b_1q_3+c_1r_3 \\ a_2p_1+b_2q_1+c_2r_1, & a_2p_2+b_2q_2+c_2r_2, & a_2p_3+b_2q_3+c_2r_3 \\ a_3p_1+b_3q_1+c_3r_1, & a_3p_2+b_3q_2+c_3r_2, & a_3p_3+b_3q_3+c_3r_3 \end{vmatrix}.$$

But if before forming their product we write the determinants in the respective forms

$$\begin{vmatrix} a_1, & b_1, & c_1, & 0 \\ a_2, & b_2, & c_2, & 0 \\ a_3, & b_3, & c_3, & 0 \\ 0, & 0, & 0, & 1 \end{vmatrix} \qquad -\begin{vmatrix} p_1, & q_1, & 0, & r_1 \\ p_2, & q_2, & 0, & r_2 \\ p_3, & q_3, & 0, & r_3 \\ 0, & 0, & 1, & 0 \end{vmatrix},$$

their product by rows is the determinant of order 4:

$$-\begin{vmatrix} a_1p_1+b_1q_1, & a_1p_2+b_1q_2, & a_1p_3+b_1q_3, & c_1 \\ a_2p_1+b_2q_1, & a_2p_2+b_2q_2, & a_2p_3+b_2q_3, & c_2 \\ a_3p_1+b_3q_1, & a_3p_2+b_3q_2, & a_3p_3+b_3q_3, & c_3 \\ r_1, & r_2, & r_3, & 0 \end{vmatrix}.$$

Again writing the original determinants in the forms

$$\begin{vmatrix} a_1, & b_1, & c_1, & 0, & 0 \\ a_2, & b_2, & c_2, & 0, & 0 \\ a_3, & b_3, & c_3, & 0, & 0 \\ 0, & 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 0, & 1 \end{vmatrix}, \qquad \begin{vmatrix} p_1, & 0, & 0, & q_1, & r_1 \\ p_2, & 0, & 0, & q_2, & r_2 \\ p_3, & 0, & 0, & q_3, & r_3 \\ 0, & 1, & 0, & 0, & 0 \\ 0, & 0, & 1, & 0, & 0 \end{vmatrix},$$

their product is now the determinant of order 5:

$$\begin{vmatrix} a_1p_1, & a_1p_2, & a_1p_3, & b_1, & c_1 \\ a_2p_1, & a_2p_2, & a_2p_3, & b_2, & c_2 \\ a_3p_1, & a_3p_2, & a_3p_3, & b_3, & c_3 \\ q_1, & q_2, & q_3, & 0, & 0 \\ r_1, & r_2, & r_3, & 0, & 0 \end{vmatrix}.$$

While writing the determinants in the forms

$$\begin{vmatrix} a_1, & b_1, & c_1, & 0, & 0, & 0 \\ a_2, & b_2, & c_2, & 0, & 0, & 0 \\ a_3, & b_3, & c_3, & 0, & 0, & 0 \\ 0, & 0, & 0, & 1, & 0, & 0 \\ 0, & 0, & 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 0, & 0, & 1 \end{vmatrix}, \qquad \begin{vmatrix} 1, & 0, & 0, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0, & 0, & 0 \\ 0, & 0, & 1, & 0, & 0, & 0 \\ 0, & 0, & 0, & p_1, & q_1, & r_1 \\ 0, & 0, & 0, & p_2, & q_2, & r_2 \\ 0, & 0, & 0, & p_3, & q_3, & r_3 \end{vmatrix},$$

their product is the determinant of the sixth order

$$\begin{vmatrix} a_1, & b_1, & c_1, & 0, & 0, & 0 \\ a_2, & b_2, & c_2, & 0, & 0, & 0 \\ a_3, & b_3, & c_3, & 0, & 0, & 0 \\ 0, & 0, & 0, & p_1, & q_1, & r_1 \\ 0, & 0, & 0, & p_2, & q_2, & r_2 \\ 0, & 0, & 0, & p_3, & q_3, & r_3 \end{vmatrix}.$$

This rule is interesting as giving us a complete scale whereby we may represent the product of two determinants of order n by a determinant of any order from n to $2n$ inclusive; it is also frequently useful in applications of the theory.

8. The fundamental theorem of Art. 2 regarding the determinant formed by compounding two arrays can be deduced as follows from Laplace's theorem, III. 5.

We can write the determinant $|c_{ik}|$ in the form of the determinant of order $(n+m)$, III. 14.

$$\begin{vmatrix} c_{11} \dots c_{1m}, & b_{11} \dots b_{1n} \\ \dots\dots\dots & \dots\dots\dots \\ c_{m1} \dots c_{mm}, & b_{m1} \dots b_{mn} \\ 0 \ \dots \ 0, & 1 \dots 0 \\ \dots\dots\dots & \dots\dots\dots \\ 0 \ \dots \ 0, & 0 \dots 1 \end{vmatrix},$$

where c_{ik} has the value ascribed to it in Art. 1.

Now from the i^{th} column subtract the last n columns multiplied respectively by $a_{i1}, a_{i2} \ldots$ then from the value of c_{ik} it follows that

$$|c_{ik}| = \begin{vmatrix} 0 & \ldots & 0, & b_{11} & \ldots & b_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \ldots & 0, & b_{m1} & \ldots & b_{mn} \\ -a_{11} & \ldots & -a_{m1}, & 1 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_{1n} & \ldots & -a_{mn}, & 0 & \ldots & 1 \end{vmatrix}.$$

In the determinant on the right multiply the first m columns by -1 and then move the second m rows to the beginning, then (after $m + m^2$ changes of sign) our determinant is equal to

$$\begin{vmatrix} a_{11} & \ldots & a_{m1}, & 1 & \ldots & 0, & 0 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1m} & \ldots & a_{mm}, & 0 & \ldots & 1, & 0 & \ldots & 0 \\ 0 & \ldots & 0, & b_{11} & \ldots & b_{1m}, & b_{1m+1} & \ldots & b_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \ldots & 0, & b_{m1} & \ldots & b_{mm}, & b_{mm+1} & \ldots & b_{mn} \\ a_{1m+1} & \ldots & a_{mm+1}, & 0 & \ldots & 0, & 1 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{nn} & \ldots & a_{mn}, & 0 & \ldots & 0, & 0 & \ldots & 1 \end{vmatrix}.$$

Now expand this by Laplace's theorem according to minors of the first m columns. Let us find the complement of the minor

$$\begin{vmatrix} a_{f1}, & a_{f2} \ldots \\ a_{g1}, & a_{g2} \ldots \\ \cdots & \cdots \end{vmatrix}.$$

For this purpose we move the rows of a's having the suffixes $f, g \ldots$ up to the beginning; then move those columns of b's which have the suffixes $f, g \ldots$ into the $(m+1)^{\text{st}}$, $(m+2)^{\text{nd}} \ldots$ places. This does not alter the value or sign of the determinant, and in every place where a 1 stood before, will now again stand 1. Hence the required complement is

$$\begin{vmatrix} b_{1f}, & b_{1g} \ldots 0 & 0 \\ b_{2f}, & b_{2g} \ldots 0 & 0 \\ \cdots & \cdots & \cdots \\ 0 \quad 0 & 1 & 0 \\ 0 \ldots 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} b_{1f}, & b_{1g} \ldots \\ b_{2f}, & b_{2g} \ldots \\ \cdots & \cdots \end{vmatrix}.$$

Hence

$$|c_{ik}| = \Sigma \begin{vmatrix} a_{f1}, & a_{f2} \dots \\ a_{g1}, & a_{g2} \dots \\ \dots & \dots \end{vmatrix} \cdot \begin{vmatrix} b_{1f}, & b_{1g} \dots \\ b_{2f}, & b_{2g} \dots \\ \dots & \dots \end{vmatrix}$$

when $f, g \dots$ is an m-*ad* from $1, 2 \dots n$. This agrees with our former result.

9. The value of any minor of order μ of the determinant $|c_{ik}|$, the product of two determinants $|a_{ik}|$ and $|b_{ik}|$, say,

$$C_\mu = \begin{vmatrix} c_{fp}, & c_{fq} \dots c_{fs} \\ c_{gp}, & c_{gq} \dots c_{gs} \\ \dots & \dots \\ c_{kp}, & c_{kq} \dots c_{ks} \end{vmatrix},$$

can be expressed as the sum of products of corresponding minors of order μ of the determinants $|a_{ik}|$ and $|b_{ik}|$.

For the elements of C_μ are got by compounding the two arrays

$$\begin{matrix} a_{f1}, & a_{f2} \dots a_{fn} \\ a_{g1}, & a_{g2} \dots a_{gn} \\ \dots & \dots \\ a_{k1}, & a_{k2} \dots a_{kn} \end{matrix} \qquad \begin{matrix} b_{p1}, & b_{p2} \dots b_{pn} \\ b_{q1}, & b_{q2} \dots b_{qn} \\ \dots & \dots \\ b_{s1}, & b_{s2} \dots b_{sn}. \end{matrix}$$

And since these arrays have more columns than rows, it follows that C_μ is the sum of n_μ products of determinants of order μ, formed by selecting μ columns from the two arrays. Thus

$$C_\mu = \Sigma \begin{vmatrix} a_{fi}, & a_{fj} \dots a_{fr} \\ a_{gi}, & a_{gj} \dots a_{gr} \\ \dots & \dots \end{vmatrix} \begin{vmatrix} b_{pi}, & b_{pj} \dots b_{pr} \\ b_{qi}, & b_{qj} \dots b_{qr} \\ \dots & \dots \end{vmatrix}$$

when $i, j \dots r$ is any μ-*ad* from $1, 2 \dots n$.

One particular case of this we shall find presently of importance; namely, when the two systems a and b are identical, and when moreover $f = p$, $g = q \dots k = s$, so that the leading diagonal of C_μ consists of elements from the leading diagonal of $|c_{ik}|$.

Then we see that

$$C_\mu = \Sigma \begin{vmatrix} a_{fi}, & a_{fj} \dots a_{fr} \\ a_{gi}, & a_{gj} \dots a_{gr} \\ \dots & \dots \end{vmatrix}^2$$

is a sum of n_μ squares.

10. The differential coefficients of a determinant C, elements c_{ik}, which is the product of two determinants A, B, elements a_{ik}, b_{ik}, can be represented as the sum of products of differential coefficients of these determinants.

We have $$AB = C \qquad \text{(1)},$$
and $$c_{ik} = a_{i1}b_{k1} + a_{i2}b_{k2} + \ldots + a_{in}b_{kn}.$$
Differentiate (1) with regard to a_{ip}; remembering that $c_{i1}, c_{i2} \ldots c_{in}$ are functions of this, we get

$$B\frac{dA}{da_{ip}} = \frac{dC}{dc_{i1}}b_{1p} + \frac{dC}{dc_{i2}}b_{2p} + \ldots + \frac{dC}{dc_{in}}b_{np}.$$

Multiply this equation by

$$\frac{dB}{db_{kp}} = B_{kp}$$

and add together all the equations which can be obtained from it by writing for p the values $1, 2 \ldots n$. Thus we get

$$B\Sigma\frac{dA}{da_{ip}}\cdot\frac{dB}{db_{kp}} = \frac{dC}{dc_{i1}}\Sigma B_{kp}b_{1p} + \ldots + \frac{dC}{dc_{in}}\Sigma B_{kp}b_{np}.$$

But by III. 11 all the sums on the right vanish except $\Sigma B_{kp}b_{kp}$, which is equal to B, hence

$$\frac{dC}{dc_{ik}} = \Sigma\frac{dA}{da_{ip}}\cdot\frac{dB}{db_{kp}}\ (p = 1, 2 \ldots n).$$

Similarly we can prove the equations,

$$\frac{d^2C}{dc_{ik}dc_{rs}} = \frac{1}{1.2}\Sigma\frac{d^2A}{da_{ip}da_{rq}}\cdot\frac{d^2A}{db_{kp}db_{sq}}\ (p, q = 1, 2 \ldots n),$$

$$\frac{d^3C}{dc_{ik}dc_{pq}dc_{rs}} = \frac{1}{1.2.3}\Sigma\frac{d^3A}{da_{iu}da_{pv}da_{rw}}\cdot\frac{d^3B}{db_{ku}db_{qv}db_{sw}}$$

$$(u, v, w = 1, 2 \ldots n),$$

whence the general law is obvious.

CHAPTER V.

ON DETERMINANTS OF COMPOUND SYSTEMS.

1. If the elements of a determinant are not simple quantities but themselves determinants, the determinant is called a compound determinant.

Compound determinants are usually formed from the minors of one or more determinants.

2. The number of all possible minors of order m of a given determinant is $\{n_m\}^2$ (III. 1). We can form a square array with these minors, writing in the same row all those which proceed from the same selection of rows of the given determinant, and similarly for the columns.

If $n_m = \mu$ and we give to the combinations of rows and columns taken to form minors the suffixes $1, 2 \ldots \mu$, we may denote that minor whose elements belong to the i^{th} combination of rows and j^{th} combination of columns, by p_{ij}, and the whole system of minors will be

$$\left.\begin{matrix} p_{11} \cdots p_{1\mu} \\ \cdots\cdots\cdots \\ p_{\mu 1} \cdots p_{\mu\mu} \end{matrix}\right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (1).$$

Corresponding to each element in this array, which is a minor of the original determinant, we have a complementary minor of order $n - m$. We shall denote the complement of p_{ij} by q_{ij}, then these form a new array,

$$\left.\begin{matrix} q_{11} \cdots q_{1\mu} \\ \cdots\cdots\cdots \\ q_{\mu 1} \cdots q_{\mu\mu} \end{matrix}\right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (2).$$

The arrays (1) and (2) are called reciprocal arrays of the m^{th} order. Minors of these arrays formed from the same selection of rows and columns in each are called conjugate minors. The simplest instance of two such arrays is the original system and its system of first minors, viz.

$$\begin{array}{ccc} a_{11} \dots a_{1n} & \qquad & A_{11} \dots A_{1n} \\ \dots\dots\dots & & \dots\dots\dots \\ a_{n1} \dots a_{nn} & & A_{n1} \dots A_{nn}. \end{array}$$

3. If we multiply the elements of the i^{th} row of the array (1) by the corresponding elements of the k^{th} row of (2) the sum of the products is equal to A or zero according as i is or is not equal to k, viz.

$$p_{i1}q_{k1} + p_{i2}q_{k2} + \dots + p_{i\mu}q_{k\mu} = A \text{ or } 0.$$

For if i is equal to k this is nothing else than the expansion of the given determinant A according to products of minors of order m and $n-m$ by Laplace's theorem. If i is not equal to k the sum represents the expansion of the determinant when the i^{th} selection of rows is replaced by the k^{th}; the rows of this determinant are not all different, hence it vanishes. The particular case

$$a_{i1}A_{k1} + a_{i2}A_{k2} + \dots + a_{in}A_{kn} = A \text{ or } 0$$

according as i is or is not equal to k is considered in III. 11.

4. Let $$A = | a_{ik} |, \qquad B = | b_{ik} |$$
be two determinants each of order n for which we have formed the systems of μ^2 elements discussed in Art. 2; the systems for the determinant A being denoted by p_{ik}, q_{ik}, those for the determinant B by p'_{ik}, q'_{ik}.

We can form two new systems each of μ^2 elements as follows. In the determinant A replace each combination of the rows m at a time by the fixed selection of rows marked i in the determinant B, this will give us μ determinants which we shall denote by t_{i1}, $t_{i2} \dots t_{i\mu}$. In the determinant B replace the fixed selection of rows marked k by each combination from A in turn; these determinants are called u_{k1}, $u_{k2} \dots u_{k\mu}$. We have then two new systems

$$\begin{array}{ccc} t_{11} \dots t_{1\mu} & \qquad & u_{11} \dots u_{1\mu} \\ \dots\dots\dots & & \dots\dots\dots \\ t_{\mu 1} \dots t_{\mu\mu} & & u_{\mu 1} \dots u_{\mu\mu}. \end{array}$$

Then by Laplace's theorem we have the two sets of equations:

$$A = p_{h1}q_{h1} + p_{h2}q_{h2} + \ldots \qquad B = p'_{h1}q'_{h1} + p'_{h2}q'_{h2} + \ldots$$
$$t_{i1} = p'_{i1}q_{11} + p'_{i2}q_{12} + \ldots \qquad u_{k1} = p_{11}q'_{k1} + p_{12}q'_{k2} + \ldots$$
$$t_{i2} = p'_{i1}q_{21} + p'_{i2}q_{22} + \ldots \qquad u_{k2} = p_{21}q'_{k1} + p_{22}q'_{k2} + \ldots$$

Whence by Art. 3,

$$t_{i1}p_{11} + t_{i2}p_{21} + \ldots = p'_{i1}A,$$
$$t_{i1}p_{12} + t_{i2}p_{22} + \ldots = p'_{i2}A.$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

And hence

$$t_{i1}(p_{11}q'_{k1} + p_{12}q'_{k2} + \ldots) + t_{i2}(p_{21}q'_{k1} + p_{22}q'_{k2} + \ldots) + \ldots$$

or
$$t_{i1}u_{k1} + t_{i2}u_{k2} + \ldots = A\,(p'_{i1}q'_{k1} + p'_{i2}q'_{k2} + \ldots).$$

That is to say by compounding the i^{th} and k^{th} rows of the new arrays the sum is AB or 0 according as i is or is not equal to k.

5. We now proceed to investigate properties of determinants of the elements of reciprocal systems, and first we shall examine the system of the first order.

Let
$$A = |\,a_{ik}\,|, \qquad D = |\,A_{ik}\,|.$$

Forming the product of these two,

$$AD = |\,C_{ik}\,|,$$

where
$$C_{ik} = a_{i1}A_{k1} + a_{i2}A_{k2} + \ldots + a_{in}A_{kn};$$

and hence $C_{ik} = A$ or 0 according as i is or is not equal to k. Thus

$$AD = \begin{vmatrix} A, & 0, & 0 \ldots \\ 0, & A, & 0 \ldots \\ 0, & 0, & A \ldots \\ \ldots & \ldots & \ldots \end{vmatrix} = A^n;$$

$$\therefore\ D = A^{n-1}.$$

6. Any minor of order p in the system A_{ik} is equal to the complementary minor of its conjugate in A multiplied by A^{p-1}.

Let

$$\Sigma \pm a_{fi}a_{gk} \ldots = \begin{vmatrix} a_{fi}, & a_{fk} \ldots \\ a_{gi}, & a_{gk} \ldots \\ \ldots & \ldots \end{vmatrix}$$

and $\Sigma \pm A_{fi}A_{gk} \dots$ be two conjugate minors in the two systems each of order p, and let $\Sigma \pm a_{ru}a_{sv} \dots$ be the complement of $\Sigma \pm a_{fi}a_{gk} \dots$. So that

$$A = \begin{vmatrix} a_{fi}, & a_{fk} & \dots & a_{fu}, & a_{fv} & \dots \\ a_{gi}, & a_{gk} & \dots & a_{gu}, & a_{gv} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{ri}, & a_{rk} & \dots & a_{ru}, & a_{rv} & \dots \\ a_{si}, & a_{sk} & \dots & a_{su}, & a_{sv} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} = \Sigma \pm a_{fi}a_{gk} \dots a_{ru}a_{sv} \dots \quad (1).$$

We may write $\Sigma \pm a_{ru}a_{sv} \dots = co\, \Sigma \pm a_{fi}a_{gk} \dots$

Now we may write $\Sigma \pm A_{fi}A_{gk} \dots$ as the determinant of order n,

$$\begin{vmatrix} A_{fi}, & A_{fk} & \dots & A_{fu}, & A_{fv} & \dots \\ A_{gi}, & A_{gk} & \dots & A_{gu}, & A_{gv} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0, & 0 & \dots & 1, & 0 & \dots \\ 0, & 0 & \dots & 0, & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix},$$

which consists of four parts. The first square consists of the elements of $\Sigma \pm A_{fi}A_{gk} \dots$; to the right of this is a rectangle of $n-p$ columns and p rows containing the remaining elements of the f^{th}, $g^{\text{th}} \dots$ rows. The rectangle on the left below of p columns and $n-p$ rows consists solely of zeros, and the square on the right of $n-p$ rows and columns contains 1's in the leading diagonal and zeros elsewhere. Multiply this by the determinant A written in the form (1) above. Then (III. 11) we have

$$A \Sigma \pm A_{fi}A_{gk} \dots = \begin{vmatrix} A, & 0 & \dots & a_{fu}, & a_{fv} & \dots \\ 0, & A & \dots & a_{gu}, & a_{gv} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0, & 0 & \dots & a_{ru}, & a_{rv} & \dots \\ 0, & 0 & \dots & a_{su}, & a_{sv} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

$$= A^p \Sigma \pm a_{ru}a_{sv} \dots$$

If we resolve the determinant on the right into products of minors of the first p and last $n-p$ columns,

$$\therefore \Sigma \pm A_{fi}A_{gk} \dots = A^{p-1}\, co \Sigma \pm a_{fi}a_{gk} \dots$$

From this it follows that the ratio of two minors of the same order of the system A_{ik} is the same as the ratio of the complementary minors of their conjugates.

$$\frac{\Sigma \pm A_{fi} A_{gh} \dots}{\Sigma \pm A_{kl} A_{pq} \dots} = \frac{co\Sigma \pm a_{fi} a_{gh} \dots}{co\Sigma \pm a_{kl} a_{pq} \dots}$$

7. As examples of the theorem in Art. 6, we have

$$\begin{vmatrix} A_{11} & \dots & A_{1p} \\ \dots & \dots & \dots \\ A_{p1} & \dots & A_{pp} \end{vmatrix} = A^{p-1} \begin{vmatrix} a_{p+1,\,p+1} & \dots & a_{p+1,\,n} \\ \dots & \dots & \dots \\ a_{np+1} & \dots & a_{nn} \end{vmatrix},$$

$$\begin{vmatrix} A_{p+1,\,p+1} & \dots & A_{p+1n} \\ \dots & \dots & \dots \\ A_{np+1} & \dots & A_{nn} \end{vmatrix} = A^{n-p-1} \begin{vmatrix} a_{11} & \dots & a_{1p} \\ \dots & \dots & \dots \\ a_{p1} & \dots & a_{pp} \end{vmatrix}.$$

$AC' = A^{2}\Gamma$

$C = d$ since $\neq 0$.

The relation

$$\begin{vmatrix} A_{ik}, & A_{is} \\ A_{rk}, & A_{rs} \end{vmatrix} = A\, co \begin{vmatrix} a_{ik}, & a_{is} \\ a_{rk}, & a_{rs} \end{vmatrix}$$

may also be written

$$\frac{dA}{da_{ik}} \cdot \frac{dA}{da_{rs}} - \frac{dA}{da_{is}} \frac{dA}{da_{rk}} = A \frac{d^2 A}{da_{ik} da_{rs}},$$

in particular

$$\frac{dA}{da_{n-1,\,n-1}} \cdot \frac{dA}{da_{nn}} - \frac{dA}{da_{n-1n}} \cdot \frac{dA}{da_{nn-1}} = A \frac{d^2 A}{da_{n-1,\,n-1}\, da_{nn}}.$$

If $A = 0$, we see that

$$\begin{vmatrix} A_{ik}, & A_{is} \\ A_{rk}, & A_{rs} \end{vmatrix} = 0,$$

or

$$\frac{A_{ik}}{A_{rk}} = \frac{A_{is}}{A_{rs}}.$$

That is to say, if the determinant vanishes, the minors of the elements of any row are proportional to the corresponding minors of the elements of any other row.

8. As an example of the use of the method of Arts. 20 and 21 of Chap. III., let us discuss the value of the determinant

$$P = | \lambda a_{ik} + \mu b_{ik} |,$$

a_{ik} and b_{ik} being elements of two determinants of the n^{th} order

$$A^{(n)} = | a_{ik} |, \qquad B^{(n)} = | b_{ik} |.$$

Symbolically we can write

$$P = (\lambda A + \mu B)^n$$
$$= A^n B^n \left(\frac{\lambda}{B} + \frac{\mu}{A}\right)^n.$$

Now let $A_1^{(n)}$, $B_1^{(n)}$ be two determinants of order n, whose elements are

$$\alpha_{ik} = \frac{1}{A^{(n)}} \frac{dA^{(n)}}{da_{ik}}, \quad \beta_{ik} = \frac{1}{B^{(n)}} \cdot \frac{dB^{(n)}}{db_{ik}},$$

then by Art. 5

$$A_1^{(n)} = \frac{| A_{ik}^{(n)} |}{(A^{(n)})^n} = \frac{1}{A^{(n)}},$$

so
$$B_1^{(n)} = \frac{1}{B^{(n)}}.$$

Or, symbolically,

$$A_1 = \frac{1}{A}, \quad B_1 = \frac{1}{B}.$$

Thus
$$P = A^n B^n (\lambda B_1 + \mu A_1)^n.$$

But $(\lambda B_1 + \mu A_1)^n$ is the symbolical expression for a determinant of order n with binomial elements of the form

$$\lambda\beta_{ik} + \mu\alpha_{ik}.$$

Hence, passing from symbolic to real expressions, we have the determinant equation:

$$| \lambda a_{ik} + \mu b_{ik} | = | a_{ik} | \cdot | b_{ik} | \cdot | \lambda\beta_{ik} + \mu\alpha_{ik} | .$$

Numerous other transformations of the determinant on the left can be effected.

9. Next let us consider reciprocal arrays of order m. (Art. 2.)

Let
$$\Delta = | p_{ik} |, \quad \Delta' = | q_{ik} | .$$

The product $\Delta\Delta'$ is a determinant of order μ whose general element is

$$p_{i1}q_{k1} + p_{i1}q_{k2} + \dots + p_{i\mu}q_{k\mu},$$

which is equal to A or 0 according as i is or is not equal to k. (Art. 3.) Hence in the product determinant all the elements vanish except those in the principal diagonal.

Thus
$$\Delta\Delta' = A^{\mu}.$$

It follows therefore that Δ is a divisor of A^μ. Now A is a linear function of one of its elements, say a_{11}, hence Δ can only differ from a power of A by a coefficient independent of the elements of A. Among the combinations m at a time of the numbers 1, 2 ... n there are

$$\lambda = (n-1)_{m-1},$$

which contain 1. Hence there are λ elements of Δ, which contain a_{11}, such for example as $p_{11}, p_{22} \dots p_{\lambda\lambda}$.

Hence $$\Delta = xA^\lambda,$$

where x does not depend on the elements of A.

To determine the value of x, let $a_{ik}=0$ except when $i=k$, and let $a_{ii}=1$. The same will be the case with the elements p_{ik};

$$\therefore\ A=1, \quad \Delta=1, \quad \text{and} \therefore\ x=1.$$

Thus $$\Delta = A^{(n-1)_{m-1}},$$

and $$\Delta' = A^{(n-1)_m}$$

for $$n_m-(n-1)_{m-1}=(n-1)_m.$$

10. A minor of order r of the system q_{ik} is equal to the complement of its conjugate multiplied by $A^{r-\lambda}$.

For if we multiply the determinant $\Sigma \pm q_{fi}q_{gk} \dots$ by the determinant Δ in the same manner as we did in Art. 6 for systems of the first order, we get:

$$\Delta\Sigma \pm q_{fi}q_{gk} \dots = A^r co\Sigma \pm p_{fi}p_{gk} \dots;$$

$$\therefore\ \Sigma \pm q_{fi}q_{gk} \dots = A^{r-\lambda} co\Sigma \pm p_{fi}p_{gk} \dots$$

And in like manner

$$\Sigma \pm p_{fi}p_{gk} \dots = A^{r-(n-1)_m} co\Sigma \pm q_{fi}q_{gk} \dots$$

11. Let A_h be a minor of A, with h rows and columns. From this let us form the determinant whose elements are all the minors of order m of A_h. These last are minors of order m of A, and are hence elements of Δ. On the other hand, those among them which arise from the same rows or columns of A, and are hence in the same row or column of Δ, also arise from elements belonging to the same row or column of A_h, which is a minor of A; altogether they form a minor M of Δ, which has h_m rows and columns. While by Art. 9 we have

$$M = A_h^{(h-1)_{m-1}},$$

which gives a representation of the minors of Δ by means of powers of minors of A.

12. If in the determinant A we select a minor A_h of order h, and form all the minors of order m in A $(m > h)$, which contain neither all the h rows nor all the h columns of A_h, we shall form a minor of Δ with $n_m - (n-h)_{m-h}$ rows and columns, which is equal to

$$A_{n-h}^{(n-h-1)_{m-h}} \cdot A^{\{(n-1)_{m-1}-(n-h)_{m-h}\}},$$

where A_{n-h} is the complement of A_h in A.

Let us suppose that, as in Art. 11, we have formed the minor M in Δ' with $(n-h)_{m-h}$ rows and columns, which is equal to

$$A_{n-h}^{(n-h-1)_{m-h}},$$

and let us consider the conjugate minor α_1 in Δ, i.e. that determinant whose elements are the complementary minors in A of the elements of M.

From the law of formation of M this minor has for elements all the minors of A of order m, which have A_h as a minor.

If α is the complement of α_1 in Δ, it follows from Art. 10 that

$$\alpha = M \cdot A^{(n-1)_{m-1}-(n-h)_{m-h}}.$$

Substituting for M its value we have

$$\alpha = A_{n-h}^{(n-h-1)_{m-h}} \cdot A^{(n-1)_{m-1}-(n-h)_{m-h}}.$$

The theorem is therefore proved, if we can shew that α is formed as prescribed. For this purpose we must remember that α_1 has for elements all minors of A which have A_h for one of their minors; to get α we have then to suppress among the combinations m at a time of the rows and columns of A all those which contain all the rows or columns of A_h; thus α has for its elements all the minors of A with m rows and columns, such that they do not contain all the h rows or columns of A_h.

13. Next let us consider the determinant of the system of elements t_{ik} in Art. 4, calling this determinant T, so that

$$T = |t_{ik}|.$$

Since $$t_{ik} = p'_{i1}q_{k1} + p'_{i2}q_{k2} + \dots,$$

it follows that T is the product of the two determinants

$$|p'_{ik}| \text{ and } |q_{ik}|,$$

that is, by Art. 9,

$$T = A^{(n-1)_m} \,.\, B^{(n-1)_{m-1}}.$$

The value of the determinant of the elements u_{ik} is obtained by interchanging A and B, and at the same time writing $n-m$ for m. Thus

$$U = A^{(n-1)_{m-1}} \,.\, B^{(n-1)_m}.$$

14. The ratio of complementary minors of T and U is a power of A multiplied by a power of B.

For if

$$T_h = \begin{vmatrix} t_{11} \dots t_{1h} \\ \dots\dots\dots \\ t_{h1} \dots t_{hh} \end{vmatrix}, \quad U_{\mu-h} = \begin{vmatrix} u_{h+1h+1} \dots u_{h+1\mu} \\ \dots\dots\dots\dots\dots \\ u_{\mu h+1} \dots u_{\mu\mu} \end{vmatrix}.$$

Since

$$T_h = \begin{vmatrix} t_{11} \dots t_{1h}, & t_{1h+1} \dots t_{1\mu} \\ \dots\dots\dots & \dots\dots\dots \\ t_{h1} \dots t_{hh}, & t_{hh+1} \dots t_{h\mu} \\ 0 \dots 0, & 1 \dots 0 \\ \dots\dots\dots & \dots\dots\dots \\ 0 \dots 0, & 0 \dots 1 \end{vmatrix}$$

we have by the theorem of Art. 4

$$UT_h = \begin{vmatrix} AB, & 0 & \dots 0, & u_{1h+1} & \dots u_{1\mu} \\ 0, & AB & \dots 0, & u_{2h+1} & \dots u_{2\mu} \\ \dots & \dots & \dots & \dots & \dots \\ 0, & 0 & \dots AB, & u_{hh+1} & \dots u_{h\mu} \\ 0, & 0 & \dots 0, & u_{h+1h+1} & \dots u_{h+1\mu} \\ \dots & \dots & \dots & \dots & \dots \\ 0, & 0 & \dots 0, & u_{\mu h+1} & \dots u_{\mu\mu} \end{vmatrix}$$

$$= (AB)^h \,.\, U_{\mu-h},$$

which gives when we substitute for U

$$\frac{U_{\mu-h}}{T_h} = A^{\lambda_1} \,.\, B^{\lambda_2}$$

where $\lambda_1 = (n-1)_{m-1} - h, \quad \lambda_2 = (n-1)_m - h.$

15. If the determinants A and B of Art. 4 had not been of the same order we must have increased the order of one of them, as in III. 14, until they were both of order n. We shall make use of this to investigate some further properties of the minors of A and compound determinants formed with them.

16. If A_h is a minor of order h of A, and if we border it in all possible ways with m of the remaining rows and columns of A, we get the elements of a new determinant M_m of order $(n-h)_m$, whose value is

$$A_h^{(n-h-1)_m} \cdot A^{(n-h-1)_{m-1}}.$$

For we have

$$A_h = \begin{vmatrix} a_{11} & \dots & a_{1h}, & 0, & 0 & \dots & 0 \\ a_{21} & \dots & a_{2h}, & 0, & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{h1} & \dots & a_{hh}, & 0, & 0 & \dots & 0 \\ a_{h+11} & \dots & a_{h+1h}, & 1, & 0 & \dots & 0 \\ a_{h+21} & \dots & a_{h+2h}, & 0, & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{nh}, & 0, & 0 & \dots & 1 \end{vmatrix}.$$

Now let us write A_h and A for A and B in the theorem of Art. 13 and combine columns instead of rows (m is supposed less than h).

Each combination m at a time of the first h columns of A will give a row of T, of which only a single element does not vanish; the value of that element is A_h, and it will lie in the leading diagonal. The number of such rows is h_m. Each combination m at a time of the columns of A taken from $h-1$ of the first h columns, the last being replaced by one of the other columns, will give a row of T, in which, besides h_m elements of order h which have no influence, there will be $n-h$ elements of order $h+1$ which will be the minor A_h, bordered with a row and column of A.

The first $h-1$ columns of this combination remaining fixed while the last varies among the last $n-h$ columns of A, we shall get $n-h$ analogous rows in T, which will give in the diagonal of T a square of elements consisting of A_h with the simple border. The same will be the case for each combination $h-1$ at a time of the first h columns of A, and the determinant of elements with simple border will appear h_{m-1} times. Similarly we should have the determinant of elements with a border of k rows and columns repeated $h_{m-\kappa}$ times, and hence

$$T = A_h^{\,h_m} . M_1^{\,h_{m-1}} . M_2^{\,h_{m-2}} \ldots M_m^{\,h_0} \qquad (1),$$

while, by Art. 13,

$$T = A_h^{\,(n-1)_m} . A^{(n-1)_{m-1}} \qquad (2).$$

Hence, if we admit the law,

$$M_{k-1} = A_h^{\,(n-h-1)_{k-1}} . A^{(n-h-1)_{k-2}}$$

(which is true for $k=1$, for then $M_0 = A_h$). Substituting for $M_1, M_2 \ldots M_{m-1}$, the exponent of A_h is

$$h_m + (n-h-1)_1 . h_{m-1} + (n-h-1)_2 . h_{m-2} + \ldots + (n-h-1)_{m-1} . h_1 ;$$

if we add $(n-h-1)_m$ to this, by a known property of binomial coefficients it becomes $(n-1)_m$.

Similarly the exponent of A is

$$h_{m-1} + (n-h-1)_1 . h_{m-2} + (n-h-1)_2 . h_{m-3} + \ldots + (n-h-1)_{m-2} . h_1$$
$$= (n-1)_{m-1} - (n-h-1)_{m-1}.$$

Thus from (1) and (2)

$$M_m = A_h^{\,(n-h-1)_m} . A^{(n-h-1)_{m-1}}.$$

17. Another way of stating the theorem of Art. 16 is the following: If A_h is a minor of order h of A, and we form all the minors of A with m rows and columns which have it as a minor, we get the elements of a new determinant of order $(n-h)_{m-h}$, whose value is

$$A_h^{\,(n-h-1)_{m-h}} . A^{(n-h-1)_{m-h-1}}.$$

18. The particular case of $m=1$ is so easily stated that it is of advantage to give it here.

The elements of the new determinant are of the form

$$c_{ik} = \begin{vmatrix} a_{11} & \ldots & a_{1h}, & a_{1h+i} \\ \cdots & \cdots & \cdots & \cdots \\ a_{h1} & \ldots & a_{hh}, & a_{hh+i} \\ a_{h+k1} & \ldots & a_{h+kh}, & a_{h+kh+i} \end{vmatrix} \quad (i,\ k = 1,\ 2 \ldots n-h),$$

and $\qquad | c_{ik} | = A_h^{\,n-h-1} . A.$

This theorem and the theorem of Art. 16 are due to Prof. Sylvester, the proofs here given are due to M. Picquet.

19. Another modification of the theorem of Art. 16 can be obtained as follows: Let us return to the determinants Δ, Δ' of Art. 9, and form a determinant M_1 with the minors of A_{n-h} of order $n-m$; this is a minor of Δ' of order $(n-h)_{m-h}$. The conjugate minor in Δ has for elements those minors of A of order m complementary to those of M_1, and hence all those which have A_h as a minor. This is precisely the determinant of Art. 17. Whence the theorem can be stated as follows: If A_{n-h} is a minor of A of order $n-h$, and if we form a determinant M_1 with all the minors of order $n-m$ of A_{n-h}, and then replace each element by its complement in A, we get a new determinant, whose value is

$$M = A_h^{(n-h-1)_{m-h}} \cdot A^{(n-h-1)_{m-h-1}}.$$

20. If now we form all minors of A of order $n-m$ $(m>h)$ such that neither all their rows nor all their columns belong to A_{n-h}, which in A therefore overlap A_{n-h} or belong altogether to A_h, these form a determinant N of order $n_m-(n-h)_{m-h}$ which is equal to

$$A_h^{(n-h-1)_{m-h}} \cdot A^{(n-1)_m-(n-h-1)_{m-h}}.$$

First notice that this is essentially different from the theorem of Art. 12, applied to A_h. There the determinant is formed with all the minors of the same order of A with more elements than A_h, and which do not admit all the rows and columns of A_h. Here the determinant is formed with minors of the same order of A with fewer elements than A_{n-h}, and which do not admit all the rows and columns A_{n-h}.

To prove the theorem it is sufficient to consider in Δ' the minor N complementary to α_1 in Δ or to M in Δ'. For N is exactly formed with regard to A_{n-h} as the enunciation prescribes; it has $n_m-(n-h)_{m-h}$ rows, apply to it the theorem of Art. 10,

$$\frac{\alpha_1}{N} = A^{(n-1)_{m-1}-n_m+(n-h)_{m-h}},$$

$$\therefore \quad N = \alpha_1 A^{(n-1)_m-(n-h)_{m-h}};$$

or, replacing α_1 by its value, from Art. 17,

$$N = A_h^{(n-h-1)_{m-h}} \cdot A^{(n-1)_m-(n-h-1)_{m-h}}.$$

CHAPTER VI.

DETERMINANTS OF SPECIAL FORMS.

1. WHEN a square array is written down, it is natural to inquire what simplifications arise in the determinant of the array when special relations are supposed to exist between the elements. And looking at the figure the relations which naturally suggest themselves are those which depend on the geometrical form which the array assumes. Hence we have various forms of determinants obtained by supposing relationships, of equality or otherwise, to exist between elements situated symmetrically in the figure; this shews how the notation employed has influenced the development of the theory.

The most important of these special forms are symmetrical and skew symmetrical determinants. Here the special form of geometrical symmetry considered is with regard to the diagonal. Elements which are situated in regard to the diagonal in the position of a point and its image with respect to a mirror coinciding with the diagonal, have been called conjugate: two such elements are denoted by a_{ik} and a_{ki}.

2. If $a_{ik} = a_{ki}$, the determinant is called symmetrical.

The square of any determinant is a symmetrical determinant.

For
$$| a_{ik} |^2 = | c_{ik} |$$
where
$$\begin{aligned} c_{ik} &= a_{i1}a_{k1} + a_{i2}a_{k2} + \dots \\ &= c_{ki}. \end{aligned}$$

It follows from this that every even power of a determinant is a symmetrical determinant.

3. We may also suppose the determinant to be symmetrical with respect to the centre of the square formed by the elements of the determinant.

Two cases arise, according as the determinant is of even or odd order.

First, if the order of the determinant is $2r$, we may write it in the form:

$$D = \begin{vmatrix} a_1, & b_1, & c_1 & \dots & m_1, & n_1, & \nu_1, & \mu_1 & \dots & \gamma_1, & \beta_1, & \alpha_1 \\ a_2, & b_2, & c_2 & \dots & m_2, & n_2, & \nu_2, & \mu_2 & \dots & \gamma_2, & \beta_2, & \alpha_2 \\ \dots & & & & & & & & & & & \dots \\ a_r, & b_r, & c_r & \dots & m_r, & n_r, & \nu_r, & \mu_r & \dots & \gamma_r, & \beta_r, & \alpha_r \\ \alpha_r, & \beta_r, & \gamma_r & \dots & \mu_r, & \nu_r, & n_r, & m_r & \dots & c_r, & b_r, & a_r \\ \dots & & & & & & & & & & & \dots \\ \alpha_2, & \beta_2, & \gamma_2 & \dots & \mu_2, & \nu_2, & n_2, & m_2 & \dots & c_2, & b_2, & a_2 \\ \alpha_1, & \beta_1, & \gamma_1 & \dots & \mu_1, & \nu_1, & n_1, & m_1 & \dots & c_1, & b_1, & a_1 \end{vmatrix}$$

In this determinant add the last column to the first, the last but one to the second, the $(r+1)^{\text{st}}$ to the r^{th}, then it becomes

$$D = \begin{vmatrix} a_1+\alpha_1, & b_1+\beta_1 & \dots & n_1+\nu_1, & \nu_1, & \mu_1 & \dots & \beta_1, & \alpha_1 \\ a_2+\alpha_2, & b_2+\beta_2 & \dots & n_2+\nu_2, & \nu_2, & \mu_2 & \dots & \beta_2, & \alpha_2 \\ \dots & & & & & & & & \dots \\ a_r+\alpha_r, & b_r+\beta_r & \dots & n_r+\nu_r, & \nu_r, & \mu_r & \dots & \beta_r, & \alpha_r \\ a_r+\alpha_r, & b_r+\beta_r & \dots & n_r+\nu_r, & n_r, & m_r & \dots & b_r, & a_r \\ \dots & & & & & & & & \dots \\ a_2+\alpha_2, & b_2+\beta_2 & \dots & n_2+\nu_2, & n_2, & m_2 & \dots & b_2, & a_2 \\ a_1+\alpha_1, & b_1+\beta_1 & \dots & n_1+\nu_1, & n_1, & m_1 & \dots & b_1, & a_1 \end{vmatrix}.$$

Now subtract the first row from the last, the second from the last but one, the r^{th} from the $(r+1)^{\text{st}}$, then

$$D = \begin{vmatrix} a_1+\alpha_1, & b_1+\beta_1 & \dots & n_1+\nu_1, & \nu_1\;, & \mu_1 & \dots & \beta_1\;, & \alpha_1 \\ a_2+\alpha_2, & b_2+\beta_2 & \dots & n_2+\nu_2, & \nu_2\;, & \mu_2 & \dots & \beta_2\;, & \alpha_2 \\ \dots & & & & & & & & \dots \\ a_r+\alpha_r, & b_r+\beta_r & \dots & n_r+\nu_r, & \nu_r\;, & \mu_r & \dots & \beta_r\;, & \alpha_r \\ 0, & 0 & \dots & 0, & n_r-\nu_r, & m_r-\mu_r & \dots & b_r-\beta_r, & a_r-\alpha_r \\ \dots & & & & & & & & \dots \\ 0, & 0 & \dots & 0, & n_2-\nu_2, & m_2-\mu_2 & \dots & b_2-\beta_2, & a_2-\alpha_2 \\ 0, & 0 & \dots & 0, & n_1-\nu_1, & m_1-\mu_1 & \dots & b_1-\beta_1, & a_1-\alpha_1 \end{vmatrix}.$$

Hence (III. 7),

$$D = \begin{vmatrix} a_1 + \alpha_1 & \dots & n_1 + \nu_1 \\ \dots & \dots & \dots \\ a_r + \alpha_r & \dots & n_r + \nu_r \end{vmatrix} \cdot \begin{vmatrix} n_r - \nu_r & \dots & a_r - \alpha_r \\ \dots & \dots & \dots \\ n_1 - \nu_1 & \dots & a_1 - \alpha_1 \end{vmatrix}.$$

But if the order of the determinant is $2r+1$, it may be written in the form

$$D = \begin{vmatrix} a_1, & b_1 & \dots & n_1, & u_1, & \nu_1 & \dots & \beta_1, & \alpha_1 \\ a_2, & b_2 & \dots & n_2, & u_2, & \nu_2 & \dots & \beta_2, & \alpha_2 \\ \dots & & & & & & & & \dots \\ a_r, & b_r & \dots & n_r, & u_r, & \nu_r & \dots & \beta_r, & \alpha_r \\ v_1, & v_2 & \dots & v_r, & p, & v_r & \dots & v_2, & v_1 \\ \alpha_r, & \beta_r & \dots & \nu_r, & u_r, & n_r & \dots & b_r, & a_r \\ \dots & & & & & & & & \dots \\ \alpha_1, & \beta_1 & \dots & \nu_1, & u_1, & n_1 & \dots & b_1, & a_1 \end{vmatrix}.$$

By proceeding exactly as in the former case, we can shew that

$$D = \begin{vmatrix} a_1 + \alpha_1 & \dots & n_1 + \nu_1, & u_1 \\ \dots & \dots & \dots & \dots \\ a_r + \alpha_r & \dots & n_r + \nu_r, & u_r \\ 2v_1, & \dots & 2v_r, & p \end{vmatrix} \cdot \begin{vmatrix} n_r - \nu_r & \dots & a_r - \alpha_r \\ \dots & \dots & \dots \\ n_1 - \nu_1 & \dots & a_1 - \alpha_1 \end{vmatrix}.$$

So that when a determinant is symmetrical with respect to the centre of the square formed by its elements, it reduces to the product of two other determinants.

4. If in a determinant the conjugate elements are equal in magnitude but opposite in sign, i.e. if

$$a_{ik} = -a_{ki},$$

the determinant is called a skew determinant. If, moreover,

$$a_{ii} = 0,$$

the determinant is called a skew symmetrical determinant.

5. It will be useful to notice the connexion between two minors of these systems, such that the rows and columns suppressed to obtain the one minor correspond to the columns and rows suppressed to obtain the other. Two such minors may be denoted by

$$P = \begin{vmatrix} a_{pf}, & a_{pg} & \dots \\ a_{qf}, & a_{qg} & \dots \\ \dots & \dots & \dots \end{vmatrix}, \quad Q = \begin{vmatrix} a_{fp}, & a_{fq} & \dots \\ a_{gp}, & a_{gq} & \dots \\ \dots & \dots & \dots \end{vmatrix}.$$

6. If the determinant is symmetrical,

i.e. if $$a_{ik} = a_{ki},$$

clearly $$P = Q.$$

A special case of this is, that in a symmetrical determinant

$$A_{ik} = A_{ki},$$

for A_{ik} is got by suppressing the i^{th} row and k^{th} column, while A_{ki} is got by suppressing the k^{th} row and i^{th} column, thus these determinants are of the same nature as P and Q, and are therefore equal. Thus the determinant of the reciprocal system is also symmetrical. If A is the determinant of the system

$$\frac{dA}{da_{ik}} = A_{ik} + A_{ki}\frac{da_{ki}}{da_{ik}}$$

$$= 2A_{ik}.$$

But $$\frac{dA}{da_{ii}} = A_{ii}.$$

In a symmetrical determinant A_{ii} and the like are still symmetrical determinants.

7. If in Art. 5 $$a_{ik} = -a_{ki},$$
we see that

$$P = \begin{vmatrix} a_{pf}, & a_{pg} & \dots \\ a_{qf}, & a_{qg} & \dots \\ \dots & \dots & \dots \end{vmatrix} = \begin{vmatrix} -a_{fp}, & -a_{gp} & \dots \\ -a_{fq}, & -a_{gq} & \dots \\ \dots & \dots & \dots \end{vmatrix} = (-1)^m Q,$$

m being the order of the minors. Thus if m is even

$$P = Q,$$

but if m is odd $$P = -Q.$$

8. The calculation of skew determinants reduces to that of skew symmetrical determinants, which we shall therefore now consider. A skew symmetrical determinant of odd order vanishes, for if we multiply each row by -1, since $a_{ik} = -a_{ki}$, this changes the rows into columns, which does not alter the value of the determinant.

Hence, if n be its order,

$$A = (-1)^n A;$$

and hence $A = 0$ if n is odd.

The minor A_{ik} differs from A_{ki} by the sign of every element; hence

$$A_{ik} = (-1)^{n-1} A_{ki}.$$

Thus $A_{ki} = A_{ik}$ if n is odd, but $= -A_{ik}$ if n is even.

Thus the reciprocal system is skew if n is even, but symmetrical if n is odd.

A_{ii} is a skew symmetrical determinant of order $n-1$, and hence vanishes if n is even.

We have

$$\begin{aligned} \frac{dA}{da_{ik}} &= A_{ik} + A_{ki}\frac{da_{ki}}{da_{ik}} \\ &= A_{ik} - A_{ki} \\ &= 2A_{ik} \text{ if } n \text{ is even} \\ &= 0 \text{ if } n \text{ is odd.} \end{aligned}$$

9. A skew symmetrical determinant of even order is a complete square.

For if $$A = |\, a_{ik} \,|$$

is the determinant, since A_{11} is a skew symmetrical determinant of odd order it vanishes. Hence (v. 7), if α_{ik} is the complement of a_{ik} in A_{11},

$$\begin{vmatrix} \alpha_{ii}, & \alpha_{ik} \\ \alpha_{ki}, & \alpha_{kk} \end{vmatrix} = 0, \text{ or } \alpha_{ii}\alpha_{kk} = \alpha_{ik}^{\,2},$$

since $\alpha_{ik} = \alpha_{ki}$ (Art. 8).

Now by (III. 24) if we expand according to products of elements in the first row and first column, since $A_{11} = 0$

$$A = -\Sigma a_{1i} a_{k1} \alpha_{ik},$$

where i, k take the values 2, 3 ... n;

or $$\begin{aligned} A &= \Sigma a_{1i}\, a_{1k} \sqrt{\alpha_{ii}\, \alpha_{kk}} \\ &= \{\Sigma a_{1i} \sqrt{\alpha_{ii}}\}^2. \end{aligned}$$

Thus A is the square of a linear function of the elements of a row. Now α_{ii} is a determinant of order $n-2$, which is even if n is even. Thus a skew symmetrical determinant of order n will be the square of a rational function of its elements if one of order $n-2$ is so. But when $n=2$,

$$\begin{vmatrix} 0, & a_{12} \\ a_{21}, & 0 \end{vmatrix} = a_{12}^2.$$

Thus skew symmetrical determinants of orders $4, 6 \dots 2r$ are squares of rational functions of their elements.

10. Since if $n=2$ the square root contains one term, when $n=4$ the square root will contain 3, when $n=6$ it will contain 5.3 terms, and so on. Hence a skew symmetrical determinant of even order n is the square of an aggregate of

$$1.3.5 \dots n-1$$

terms, each consisting of the product of $\frac{1}{2}n$ terms of A.

In particular $a_{12}\,a_{34} \dots a_{n-1n}$ is a term of $\sqrt{A}$, for

$$(a_{12}a_{34} \dots a_{n-1n})^2 = (-1)^{\frac{n}{2}} a_{12}a_{34} \dots a_{n-1n}a_{21}a_{43} \dots a_{nn-1}.$$

11. This function $\sqrt{A}$ is of importance in analysis, and has been called a Pfaffian by Prof. Cayley on account of the use made of it by Jacobi in his discussion of Pfaff's problem.

That value of $\sqrt{A}$ which contains $a_{12}a_{34} \dots a_{n-1n}$ as first term with positive sign will be denoted by

$$P = [1, 2 \dots n].$$

The remaining terms of P are got from the first term,

$$a_{12}a_{34} \dots a_{n-1n},$$

by interchanging all the suffixes $2, 3 \dots n$ in all possible ways, and giving a sign corresponding to the number of inversions. Since $a_{ik} = -a_{ki}$ it is possible to effect the interchange in such a way that all the terms are positive.

The Pfaffian changes sign on interchanging only two suffixes i and k. For if we interchange i and k in the determinant, this interchanges the i^{th} and k^{th} rows as well as the i^{th} and k^{th} columns, thus the value of the determinant remains unchanged. If P_1 is the new value of P,

$$P_1^2 = P^2.$$

Hence $$P_1 = \pm P.$$

To determine which sign we are to take, let us consider the aggregate of terms $a_{ik}p_{ik}$ which contain a_{ik}. Then p_{ik} only contains terms whose suffixes are independent of i and k. The corresponding aggregate for P_1 is

$$a_{ki}\,p_{ik},$$

which, in consequence of the relation $a_{ki} = -a_{ik}$, proves that

$$P_1 = -P.$$

12. The minor α_{ii} is also a skew symmetrical determinant. We shall shew that

$$\sqrt{\alpha_{ii}} = (-1)^i\,[2, \ldots i-1, i+1, \ldots n],$$

or with $i-2$ cyclical interchanges

$$\sqrt{\alpha_{ii}} = [i+1, \ldots n, 2 \ldots i-1].$$

Since

$$\alpha_{ik}{}^2 = \alpha_{ii}\alpha_{kk},$$

it follows that the terms of the product $\sqrt{\alpha_{ii}}\sqrt{\alpha_{kk}}$ are either equal to those of α_{ik}, or equal with opposite signs.

Now the product

$$(-1)^{i+k}[2 \ldots i-1, i+1 \ldots n]\,[2 \ldots k-1, k+1 \ldots n]$$

and the determinant

$$\alpha_{ik} = \begin{vmatrix} a_{22} \ldots\ldots a_{2k-1}, & a_{2k+1} \ldots\ldots \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ a_{i-12} \ldots a_{i-1\,k-1}, & a_{i-1\,k+1} \ldots \\ a_{i+12} \ldots a_{i+1\,k-1}, & a_{i+1\,k+1} \ldots \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \end{vmatrix} (-1)^{i+k},$$

by the same number of interchanges of two suffixes, become respectively

$$[k, p.\, q, r, s \ldots u, v]\,[p, q, r, s \ldots v, i]$$

and

$$\begin{vmatrix} a_{kp}, & a_{kq}, & a_{kr} \ldots a_{ki} \\ a_{pp}, & a_{pq}, & a_{pr} \ldots a_{pi} \\ \ldots & \ldots & \ldots\ldots \\ a_{vp}, & a_{vq}, & a_{vr} \ldots a_{vi} \end{vmatrix}.$$

And the term

$$a_{kp}a_{qr} \ldots a_{uv} \,.\, a_{pq}a_{rs} \ldots a_{vi}$$

of the product agrees in sign with the first term of the determinant

$$a_{kp}a_{pq}a_{qr} \ldots a_{vi},$$

whence the theorem follows.

13. Since we have shewn in Art. 9 that

$$\sqrt{A} = a_{12}\sqrt{\alpha_{22}} + a_{13}\sqrt{\alpha_{33}} + \ldots + a_{1n}\sqrt{\alpha_{nn}},$$

it follows that

$$[1, 2 \ldots n] = a_{12}[3 \ldots n] + a_{13}[4 \ldots n, 2] + \ldots + a_{1n}[2 \ldots n-1];$$

a relation which enables us to determine Pfaffians of order n from those of order $n-2$.

Observe that after we have selected the suffix 1, the others are written cyclically. Hence

$$\begin{aligned}
[1, 2] &= a_{12} \\
[1, 2, 3, 4] &= a_{12}a_{34} + a_{13}a_{42} + a_{14}a_{23} \\
[1, 2, 3, 4, 5, 6] &= a_{12}[3, 4, 5, 6] + a_{13}[4, 5, 6, 2] + a_{14}[5, 6, 2, 3] \\
&\quad + a_{15}[6, 2, 3, 4] + a_{16}[2, 3, 4, 5] \\
&= a_{12}a_{34}a_{56} + a_{12}a_{35}a_{64} + a_{12}a_{36}a_{45} \\
&\quad + a_{13}a_{45}a_{62} + a_{13}a_{46}a_{25} + a_{13}a_{42}a_{56} \\
&\quad + a_{14}a_{56}a_{23} + a_{14}a_{52}a_{36} + a_{14}a_{53}a_{62} \\
&\quad + a_{15}a_{62}a_{34} + a_{15}a_{63}a_{42} + a_{15}a_{64}a_{23} \\
&\quad + a_{16}a_{23}a_{45} + a_{16}a_{24}a_{53} + a_{16}a_{25}a_{34}.
\end{aligned}$$

In particular

$$\begin{vmatrix} 0, & a, & -b, & c \\ -a, & 0, & f, & e \\ b, & -f, & 0, & d \\ -c, & -e, & -d, & 0 \end{vmatrix} = (ad + be + cf)^2.$$

14. In a skew symmetrical determinant of even order, A_{ii} vanishes, being a skew symmetrical determinant of odd order.

But (Art. 8),

$$\begin{aligned}
A_{ik} &= \tfrac{1}{2}\frac{dA}{da_{ik}} \\
&= \tfrac{1}{2}\frac{d}{da_{ik}}[1, 2 \ldots n]^2 \\
&= [1, 2 \ldots n]\frac{d}{da_{ik}}[1, 2 \ldots n].
\end{aligned}$$

Now

$$\begin{aligned}
P = [1, 2 \ldots n] &= (-1)^{i-1}[i, 1 \ldots i-1, i+1 \ldots n] \\
&= (-1)^{i-1}\{a_{i1}[2 \ldots i-1, i+1 \ldots n] + \ldots \\
&\quad + a_{ik}(-1)^{k-1}[1, 2 \ldots i-1, i+1 \ldots k-1\; k+1 \ldots n] + \ldots\};
\end{aligned}$$

hence $$A_{ik} = (-1)^{i+k} [1, 2 \dots n] \{ik\},$$

where $\{ik\}$ is the Pfaffian got by omitting i and k in $[1, 2 \dots n]$.

15. In a skew symmetrical determinant of odd order A_{ii} is a skew symmetrical determinant of even order, and is hence the square of a Pfaffian;

viz. $$A_{ii} = [1 \dots i-1,\ i+1 \dots n]^2,$$

$$\sqrt{A_{ii}} = (-1)^{i-1} [1 \dots i-1,\ i+1 \dots n]$$

$$= [i+1 \dots n,\ 1 \dots i-1].$$

Also, since $$A = 0,$$

$$A_{ik}^2 = A_{ii} A_{kk}.$$

Hence $$A_{ik} = [i+1 \dots n,\ 1 \dots i-1] [k+1 \dots n,\ 1 \dots k-1].$$

16. The result of bordering a skew symmetrical determinant is also of interest. The result assumes different forms according as the determinant which we border is of odd or even order.

Let the original skew symmetrical determinant be

$$A = | a_{ik} |,$$

and let the bordered determinant be

$$D = \begin{vmatrix} a_{\alpha\beta}, & a_{\alpha 1}, & a_{\alpha 2}, & a_{\alpha 3} & \dots \\ a_{1\beta}, & a_{11}, & a_{12}, & a_{13} & \dots \\ a_{2\beta}, & a_{21}, & a_{22}, & a_{23} & \dots \\ a_{3\beta}, & a_{31}, & a_{32}, & a_{33} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}.$$

By Cauchy's theorem (III. 24)

$$D = a_{\alpha\beta} A - \Sigma a_{\alpha i} a_{k\beta} A_{ik}.$$

Now, if A is of odd order it vanishes, and

$$A_{ik} = [i+1 \dots n,\ 1 \dots i-1] [k+1 \dots n,\ 1 \dots k-1];$$

hence, if we suppose that $a_{\beta k} = -a_{k\beta}$,

$$A = \Sigma a_{\alpha i} a_{\beta k} [i+1 \dots n,\ 1 \dots i-1] [k+1 \dots n,\ 1 \dots k-1]$$ D

$$= (a_{\alpha 1} [2, 3 \dots n] + \dots)(a_{\beta 1} [2, 3 \dots n] + \dots)$$

$$= [\alpha, 1, 2 \dots n] [\beta, 1, 2 \dots n],$$

where in the Pfaffians such expressions as $a_{i\alpha}$, $a_{\beta k}$ which do not occur in the determinant are supposed to mean $-a_{\alpha i}$, $-a_{k\beta}$.

But if A is of even order,

$$D = a_{\alpha\beta}[1, 2 \dots n]^2 + \Sigma a_{\alpha i}\, a_{\beta k}\, (-1)^{i+k} \{ik\} [1, 2 \dots n] \quad \text{(Art. 14)}$$
$$= [1, 2 \dots n] [\alpha, \beta, 1, 2 \dots n].$$

17. We have hitherto treated of skew symmetrical determinants: it is easy to reduce to these the calculation of skew determinants. Namely, by III. 23,

$$D' = D + \Sigma a_{ii} D_i + \Sigma a_{ii} a_{kk} D_{ik} + \dots + a_{11} a_{22} \dots a_{nn},$$

where D is what D' becomes when all the diagonal elements vanish. D_i is what the coefficient of a_{ii} in D' becomes when the diagonal elements vanish; D_{ik} the coefficient of $a_{ii}\, a_{kk}$ in D' with the elements in the leading diagonal zeros, and so on.

If all the elements in the leading diagonal are equal to x we can write this

$$D' = x^n + x^{n-2} \Sigma D_2 + x^{n-4} \Sigma D_4 + \dots + x^{n-m} \Sigma D_m + \dots$$

Where D_m is a minor of order m got by suppressing $n-m$ rows and columns which meet in a diagonal element, the other diagonal elements being put zero, the summation extends to all m-ads in n.

If m is odd, D_m vanishes, and if m is even it is a complete square.

Thus, the elements being skew,

$$\begin{vmatrix} x, & a_{12}, & a_{13} \\ a_{21}, & x, & a_{23} \\ a_{31}, & a_{32}, & x \end{vmatrix} = x^3 + x\,(a_{12}^2 + a_{13}^2 + a_{23}^2)$$

$$\begin{vmatrix} x, & a_{12}, & a_{13}, & a_{14} \\ a_{21}, & x, & a_{23}, & a_{24} \\ a_{31}, & a_{32}, & x, & a_{34} \\ a_{41}, & a_{42}, & a_{43}, & x \end{vmatrix} = x^4 + x^2\,(a_{12}^2 + a_{13}^2 + a_{14}^2 + a_{23}^2 + a_{24}^2 + a_{34}^2) + (a_{12}a_{34} + a_{13}\, a_{42} + a_{14}a_{23})^2.$$

18. We can apply this last theorem to prove Euler's theorem concerning the product of two numbers, each of which is the sum

of four squares. Namely, we have

$$\begin{vmatrix} a, & b, & c, & d \\ -b, & a, & -d, & c \\ -c, & d, & a, & -b \\ -d, & -c, & b, & a \end{vmatrix} = (a^2+b^2+c^2+d^2)^2,$$

$$\begin{vmatrix} p, & q, & r, & s \\ -q, & p, & -s, & r \\ -r, & s, & p, & -q \\ -s, & -r, & q, & p \end{vmatrix} = (p^2+q^2+r^2+s^2)^2.$$

Now multiply these two determinants by rows, then if we write

$$A = ap + bq + cr + ds, \qquad B = -aq + bp - cs + dr,$$
$$C = -ar + bs + cp - dq, \quad D = -as - br + cq + dp,$$

we get a skew determinant of the same form as the other two, whose value is

$$(A^2 + B^2 + C^2 + D^2)^2,$$

whence

$$(a^2 + b^2 + c^2 + d^2)(p^2 + q^2 + r^2 + s^2) = A^2 + B^2 + C^2 + D^2.$$

If we were to effect the multiplication by rows and columns we should get another form of the same theorem; by permutating the rows and columns we get still further representations of the way in which the product of two numbers, each of which is the sum of four squares, can be represented as the sum of four squares. The total number of different ways is 48. The product of n numbers, each of which consists of the sum of four squares, can be represented as the sum of four squares in 48^{n-1} different ways.

19. We have seen that the square of any determinant is a symmetrical determinant (Art. 2). Cayley and Brioschi have shewn independently that the square of a determinant of even order can be represented by a skew symmetrical determinant of even order.

The process of the latter is as follows: We have

$$A = \begin{vmatrix} a_{11}, & a_{12} \dots a_{1n-1}, & a_{1n} \\ a_{21}, & a_{22} \dots a_{2n-1}, & a_{2n} \\ \dots & \dots & \dots \\ a_{n1}, & a_{n2} \dots a_{nn-1}, & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{12}, & -a_{11} \dots a_{1n}, & -a_{1n-1} \\ a_{22}, & -a_{21} \dots a_{2n}, & -a_{2n-1} \\ \dots & \dots & \dots \\ a_{n2}, & -a_{n1} \dots a_{nn}, & -a_{nn-1} \end{vmatrix}.$$

Multiply these two equal determinants together by rows, and we obtain:

$$A^2 = \begin{vmatrix} 0, & l_{12}, & l_{13} \dots l_{1n} \\ l_{21}, & 0, & l_{23} \dots l_{2n} \\ \dots & \dots & \dots \\ l_{n1}, & l_{n2}, & l_{n3} \dots 0 \end{vmatrix},$$

where

$$l_{rs} = a_{r1}a_{s2} - a_{r2}a_{s1} + a_{r3}a_{s4} - a_{r4}a_{s3} + \dots + a_{rn-1}a_{sn} - a_{rn}a_{sn-1},$$

then $$l_{ss} = 0, \quad l_{rs} + l_{sr} = 0.$$

Thus A^2 is represented as a skew symmetrical determinant. It follows that A can be represented as a Pfaffian of the functions l. If $n = 4$, for example,

$$\begin{vmatrix} a_{11} \dots a_{14} \\ \dots\dots \\ a_{41} \dots a_{44} \end{vmatrix} = l_{12}l_{34} + l_{13}l_{42} + l_{14}l_{23}.$$

The sign is determined by making the sign of a single term in the determinant and Pfaffian agree.

If instead of interchanging columns, we interchanged rows, we should get another independent representation of the determinant as a Pfaffian.

20. A third class of determinants are those of the form

$$D = \begin{vmatrix} a_1, & a_2, & a_3 & \dots a_n \\ a_2, & a_3, & a_4 & \dots a_{n+1} \\ a_3, & a_4, & a_5 & \dots a_{n+2} \\ \dots & \dots & \dots & \dots \\ a_n, & a_{n+1}, & a_{n+2} & \dots a_{2n-1} \end{vmatrix},$$

where all the elements in a line at right angles to the leading diagonal are the same. If the elements had been written with double suffixes we should have had the relation

$$a_{pq} = a_{p\pm k q\pm k}.$$

Such determinants have been called orthosymmetrical. Their most important property is that we can replace the elements by differences of a_1.

For if we operate on the rows as we did in Chap. II. 5 (iv), if

$$\Delta a_x = a_{x+1} - a_x, \text{ \&c.}$$

$$D = \begin{vmatrix} a_1, & a_2, & \dots & a_n \\ \Delta a_1, & \Delta a_2, & \dots & \Delta a_n \\ \Delta^2 a_1, & \Delta^2 a_2, & \dots & \Delta^2 a_n \\ \dots & \dots & \dots & \dots \\ \Delta^{n-1} a_1, & \Delta^{n-1} a_2, & \dots & \Delta^{n-1} a_n \end{vmatrix}.$$

Now repeat the same series of operations on the columns, beginning at the last, then

$$D = \begin{vmatrix} a_1, & \Delta a_1, & \dots & \Delta^{n-1} a_1 \\ \Delta a_1, & \Delta^2 a_1, & \dots & \Delta^n a_1 \\ \Delta^2 a_1, & \Delta^3 a_1, & \dots & \\ \dots & \dots & \dots & \dots \\ \Delta^{n-1} a_1, & \Delta^n a_1, & \dots & \Delta^{2n-2} a_1 \end{vmatrix}.$$

An important example of this class of determinants is that where a_k is a function of k of the m^{th} degree in k, whose highest term has coefficient unity, the quantities $a_1, a_2 \dots$ form an arithmetic series of the m^{th} order. If $m = n - 1$ all the elements below the second diagonal vanish, while all those in it are equal to $(n-1)!$, whence the value of the determinant is

$$(-1)^{\frac{n(n-1)}{2}} \{(n-1)!\}^n.$$

If m is less than $n - 1$ the determinant vanishes.

21. The determinant of order $r + 1$,

$$\begin{vmatrix} m_p, & m_{p+1}, & m_{p+2} & \dots & m_{p+r} \\ (m+1)_p, & (m+1)_{p+1}, & (m+1)_{p+2} & \dots & (m+1)_{p+r} \\ (m+2)_p, & (m+2)_{p+1}, & (m+2)_{p+2} & \dots & (m+2)_{p+r} \\ \dots & \dots & \dots & \dots & \dots \\ (m+r)_p, & (m+r)_{p+1}, & (m+r)_{p+2} & \dots & (m+r)_{p+r} \end{vmatrix},$$

where $$m_p = \frac{m(m-1)\dots(m-p+1)}{1 \,.\, 2 \dots p},$$

though not orthosymmetrical, is of a similar nature; let us call it $V_{m,\,p}$.

Divide its first row by m, the second by $m+1, \ldots$ its $(r+1)^{\text{th}}$ by $m+r$. Then multiply the first column by p, the second by $p+1, \ldots$ the last by $p+r$. Then

$$V_{m,\,p} = \frac{m(m+1)\ldots(m+r)}{p(p+1)\ldots(p+r)} \times \begin{vmatrix} (m-1)_{p-1}, & (m-1)_p & \ldots (m-1)_{p+r-1} \\ m_{p-1}, & m_p & \ldots m_{p+r-1} \\ \cdots & \cdots & \cdots \\ (m+r-1)_{p-1}, & (m+r-1)_p & \ldots (m+r-1)_{p+r-1} \end{vmatrix},$$

or, if we multiply numerator and denominator of the fraction by

$$(r+1)!,$$

$$V_{m,\,p} = \frac{(m+r)_{r+1}}{(p+r)_{r+1}} V_{m-1,\,p-1}.$$

Thus we obtain a series of equations by giving to m and p different values in this

$$V_{m-1,\,p-1} = \frac{(m+r-1)_{r+1}}{(p+r-1)_{r+1}} V_{m-2,\,p-2}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$V_{m-p+1,\,1} = \frac{(m+r-p+1)_{r+1}}{(r+1)_{r+1}} V_{m-p,\,0}.$$

Now $V_{m-p,\,0}$ is the value of the last determinant in II. 5, when we write $m-p$ for m and 1 for d. Hence its value is unity, which gives, when we multiply the above equations together and cancel like factors,

$$V_{m,\,p} = \frac{(m+r)_{r+1}(m+r-1)_{r+1}\ldots(m+r-p+1)_{r+1}}{(p+r)_{r+1}(p+r-1)_{r+1}\ldots(r+1)_{r+1}}.$$

Another expression can be obtained for the determinant by dividing the first row by m_p, the second by $(m+1)_p, \ldots$ the last by $(m+r)_p$. Then multiply the first column by p_0, the second by $(p+1)_1$, the last by $(p+r)_r$; the transformation gives

$$V_{m,\,p} = \frac{m_p\,(m+1)_p\,(m+2)_p\ldots(m+r)_p}{p_p\,(p+1)_p\,(p+2)_p\ldots(p+r)_p}.$$

A remarkable special case of the first form is when $p=1$, the value of the determinant being $(m+r)_{r+1}$, i.e. the last element in its leading diagonal.

22. If in the determinant of Art. 20

$$a_{k+1} = (c+k+m)_m = \frac{(c+k+m)(c+k+m-1)\dots(c+k+1)}{1.2\dots m},$$

then if $m = n-1$, $\Delta^{n-1}a_1 = 1$, and we have

$$\begin{vmatrix} (c+n-1)_{n-1}, & (c+n)_{n-1} & \dots (c+2n-2)_{n-1} \\ (c+n)_{n-1}, & (c+n+1)_{n-1} & \dots (c+2n-1)_{n-1} \\ \dots\dots & \dots\dots & \dots\dots \\ (c+2n-2)_{n-1}, & (c+2n-1)_{n-1} & \dots (c+3n-3)_{n-1} \end{vmatrix} = (-1)^{\frac{n(n-1)}{2}}.$$

23. Another class of determinants are those of the form

$$D = \begin{vmatrix} a_1, & a_2 & \dots & a_n \\ a_n, & a_1 & \dots & a_{n-1} \\ a_{n-1}, & a_n & \dots & a_{n-2} \\ \dots & \dots & \dots & \dots \\ a_2, & a_3 & \dots & a_1 \end{vmatrix};$$

where the element in the leading diagonal is always a_1, and the rest of the row is filled up with $a_2 \dots a_n$ in cyclical order.

The peculiar property of this determinant is that it divides by

$$a_1 + a_2\omega + a_3\omega^2 + \dots + a_n\omega^{n-1},$$

where ω is a root of the equation $x^n = 1$.

For if $A_1, A_2 \dots A_n$ are the complements of the elements of the first row of this determinant we have (III. 11)

$$\left.\begin{aligned} a_1A_1 + a_2A_2 + \dots + a_nA_n &= D \\ a_1A_2 + a_2A_3 + \dots + a_nA_1 &= 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots & \\ a_1A_n + a_2A_1 + \dots + a_nA_{n-1} &= 0 \end{aligned}\right\} \quad (1).$$

Now consider the product

$$(a_1 + a_2\omega + a_3\omega^2 + \dots + a_n\omega^{n-1})(A_1 + A_2\omega^{-1} + A_3\omega^{-2} + \dots + A_n\omega^{-n+1}).$$

The coefficient of ω^{k-1} is

$$A_1a_k + A_2a_{k+1} + \dots + A_na_{k-1}.$$

If k is equal to unity this is equal to D, by the first of equations (1), but if k is not unity it vanishes by one of the other equations. Thus D divides by

$$a_1 + a_2\omega + \dots + a_n\omega^{n-1}.$$

Hence

$$D = (a_1 + a_2 \dots + a_n)\, \Pi\, (a_1 + a_2\omega + a_3\omega^2 + \dots + a_n\omega^{n-1}),$$

where ω is one of the roots of the equation $x^n - 1 = 0$, unity excepted.

24. Another elegant demonstration of the theorem of the preceding article is the following. If $\omega_1, \omega_2 \dots \omega_n$ are the n roots of unity let

$$P = \begin{vmatrix} 1, & \omega_1, & \omega_1^2 \dots \omega_1^{n-1} \\ 1, & \omega_2, & \omega_2^2 \dots \omega_2^{n-1} \\ \dots & \dots & \dots \\ 1, & \omega_n, & \omega_n^2 \dots \omega_n^{n-1} \end{vmatrix}.$$

Then if we write

$$a_1 + a_2\omega + a_3\omega^2 + \dots + a_n\omega^{n-1} = \phi(\omega),$$

and remember that $\omega^n = 1$,

$$DP = \begin{vmatrix} \phi(\omega_1), & \phi(\omega_2) & \dots \\ \omega_1\phi(\omega_1), & \omega_2\phi(\omega_2) & \dots \\ \omega_1^2\phi(\omega_1), & \omega_2^2\phi(\omega_2) & \dots \\ \dots & \dots & \dots \\ \omega_1^{n-1}\phi(\omega_1), & \omega_2^{n-1}\phi(\omega_2) & \dots \end{vmatrix}$$

$$= P\phi(\omega_1)\,\phi(\omega_2) \dots \phi(\omega_n),$$

whence $$D = \phi(\omega_1)\,\phi(\omega_2) \dots \phi(\omega_n).$$

25. Mr Glaisher has shewn that a determinant, such as that in Art. 23, of order $2n$, can be expressed as a similar determinant of order n. Namely

$$\begin{vmatrix} a_1, & a_2 & \dots a_{2n} \\ a_{2n}, & a_1 & \dots a_{2n-1} \\ a_{2n-1}, & a_{2n} & \dots a_{2n-2} \\ \dots & \dots & \dots \\ a_2, & a_3 & \dots a_1 \end{vmatrix} = \begin{vmatrix} A_1, & A_2 \dots A_n \\ A_n, & A_1 \dots A_{n-1} \\ \dots & \dots \\ A_2, & A_3 \dots A_1 \end{vmatrix},$$

where

$$A_1 = a_1\,a_1 - a_{2n}\,a_2 + a_{2n-1}\,a_3 \dots - a_2\,a_{2n}$$
$$A_2 = a_3\,a_1 - a_2\,a_2 + a_1\,a_3 \dots - a_4\,a_{2n}$$
$$\dots\dots\dots$$
$$A_r = a_{2r-1}\,a_1 - a_{2r-2}\,a_2 + a_{2r-3}\,a_3 \dots - a_{2r}\,a_{2n}$$
$$\dots\dots\dots$$

For the first determinant

$$= \Pi\,(a_1 + a_2\omega + a_3\omega^2 + \ldots + a_{2n}\omega^{2n-1}),$$

ω being a $2n^{\text{th}}$ root of unity; and since for every root ω there is a root $-\omega$, this

$$= \Pi\,(A_1 + A_2\omega^2 + A_3\omega^4 + \ldots + A_n\omega^{2n-2}),$$

which product is equal to the second determinant. For the $2n^{\text{th}}$ roots of unity being denoted by $\pm 1, \pm\omega_1, \pm\omega_2 \ldots \pm\omega_{n-1}$, the n^{th} roots of unity are $1, \omega_1^2, \omega_2^2 \ldots \omega_{n-1}^2$.

For example if $n = 2$

$$\begin{vmatrix} a, & b, & c, & d \\ d, & a, & b, & c \\ c, & d, & a, & b \\ b, & c, & d, & a \end{vmatrix} = \begin{vmatrix} A, & B \\ B, & A \end{vmatrix},$$

where $$A = a^2 + c^2 - 2bd,$$

$$B = -b^2 - d^2 + 2ac,$$

and the value of the determinant is

$$a^4 - b^4 + c^4 - d^4 - 2a^2c^2 + 2b^2d^2 - 4a^2bd + 4b^2ac - 4c^2bd + 4d^2ac.$$

26. If in the determinant of Art. 23 we suppose

$$a_r = \frac{x^{r-1}}{(r-1)!} + \frac{x^{n+r-1}}{(n+r-1)!} + \frac{x^{2n+r-1}}{(2n+r-1)!} + \ldots$$

$$\begin{aligned} D &= \epsilon^x \Pi\,(a_1 + a_2\omega + a_3\omega^2 + \ldots + a_n\omega^{n-1}) \\ &= \epsilon^x \Pi\, \epsilon^{\omega x} \\ &= \epsilon^{x(1+\omega_1+\omega_2+\ldots+\omega_{n-1})} \\ &= 1. \end{aligned}$$

27. Determinants whose elements are binomial coefficients have been discussed with great minuteness by v. Zeipel, who has given an immense number of theorems relating to this class of determinants. One or two of these we shall now consider.

The value of the determinant

$$\begin{vmatrix} m_k, & n, & p\,m_1, & q\,m_2 & \cdots & t\,m_{k-1} \\ (m+1)_k, & n+1, & (p+1)(m+1)_1, & (q+1)(m+1)_2 & \cdots & (t+1)(m+1)_{k-1} \\ (m+2)_k, & n+2, & (p+2)(m+2)_1, & (q+2)(m+2)_2 & \cdots & (t+2)(m+2)_{k-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ (m+k)_k, & n+k, & (p+k)(m+k)_1, & (q+k)(m+k)_2 & \cdots & (t+k)(m+k)_{k-1} \end{vmatrix}$$

is $\qquad (m-n)(m-p-1)(m-q-2)\ldots(m-t-k+1).$

We must first shew that the determinant vanishes when m is equal to any one of the quantities

$$n,\ p+1,\ q+2\ \ldots\ t+k-1.$$

First let $m=n$, then the determinant is

$$\begin{vmatrix} m_k, & m, & p\,m_1, & q\,m_2 & \cdots \\ (m+1)_k, & m+1, & (p+1)(m+1)_1, & (q+1)(m+1)_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (m+k)_k, & m+k, & (p+k)(m+k)_1, & (q+k)(m+k)_2 & \cdots \end{vmatrix}.$$

If we subtract the second column, multiplied by p, from the third we see that the determinant is independent of p. Do this, and divide the first row by m, the second by $m+1$, the third by $m+2\ldots$, then multiply the first column by k, the fourth by 2, the fifth by $3\ldots$, then the determinant reduces to the product of

$$\frac{m(m+1)(m+2)\ldots(m+k)}{1.2\ldots k}$$

and the determinant

$$\begin{vmatrix} (m-1)_{k-1}, & 1, & 0, & q(m-1)_1, & r(m-1)_2 & \cdots \\ m_{k-1}, & 1, & 1, & (q+1)\,m_1, & (r+1)\,m_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ (m+k-1)_{k-1}, & 1, & k, & (q+k)(m+k-1)_1, & (r+k)(m+k-1)_2 & \cdots \end{vmatrix}$$

Multiply the second column by $q(m-1)_1$, the third by

$$q(m-1)_0+1\,.\,m_1,$$

and subtract their sum from the fourth column, and we get the new determinant

$$2\begin{vmatrix} (m-1)_{k-1}, & 1, & 0, & 0, & r(m-1)_2 & \dots \\ m_{k-1}, & 1, & 1, & 0, & (r+1)\,m_2 & \dots \\ (m+1)_{k-1}, & 1, & 2, & 1, & (r+2)(m+1)_2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ (m+k-1)_{k-1}, & 1, & k_1, & k_2, & (r+k)(m+k-1)_2 & \dots \end{vmatrix}.$$

In this determinant multiply the second column by $r(m-1)_2$, the third by $r(m-1)_1+1\,.\,m_2$, the fourth by $r(m-1)_0+2\,.\,m_1$, and subtract the sum of their elements so multiplied from the elements of the fifth column, and proceed in a similar way with the altered determinant. Finally we reduce the determinant to the product of a finite number of factors and

$$\begin{vmatrix} (m-1)_{k-1}, & 1, & 0, & 0 \dots 0, & 0 \\ m_{k-1}, & 1, & 1, & 0 \dots 0, & 0 \\ (m+1)_{k-1}, & 1, & 2, & 1 \dots 0, & 0 \\ \dots & \dots & \dots & \dots & \dots \\ (m+k-1)_{k-1}, & 1, & k_1, & k_2 \dots k_{k-2}, & k_{k-1} \end{vmatrix}.$$

In this determinant multiply the second column by $(m-1)_{k-1}$, the third by $(m-1)_{k-2}$, the fourth by $(m-1)_{k-3}$, &c., and subtract their sum from the elements of the first column, then each element of the first column, and consequently the determinant vanishes. Hence our determinant divides by $m-n$. Similarly we can shew that it divides by each of the other factors, hence it is equal to

$$C(m-n)(m-p-1)(m-q-2)\dots(m-t-k+1).$$

To find the value of C put

$$n=p=q=\dots=t=0;$$

then we get

$$m_k\begin{vmatrix} 1, & (m+1)_1, & (m+1)_2 & \dots \\ 2, & 2(m+2)_1, & 2(m+2)_2 & \dots \\ 3, & 3(m+3)_1, & 3(m+3)_2 & \dots \\ \dots & \dots & \dots & \dots \\ k, & k(m+k)_1, & k(m+k)_2 & \dots \end{vmatrix}$$

$$= Cm(m-1)\dots(m-k+1).$$

But the determinant $= k\,!$ as we see by putting $d = 1$ in the last determinant of II. 5. Hence

$$C = 1;$$

thus the theorem is proved.

28. The determinant

$$\begin{vmatrix} m_k, & n, & p\,m_1 & \dots\; q\,m_{k-1},\; s\,m_k, \;\dots & u\,m_{r-1} \\ (m+1)_k, & n+1, & (p+1)(m+1)_1 & \dots\dots\dots\dots & (u+1)(m+1)_{r-1} \\ \dots & \dots & \dots & \dots & \dots \\ (m+k)_k, & n+k, & (p+k)(m+k)_1 & \dots\dots\dots\dots & (u+k)(m+k)_{r-1} \\ \dots & \dots & \dots & \dots & \dots \\ (m+r)_k, & n+r, & (p+r)(m+r)_1 & \dots\dots\dots\dots & (u+r)(m+r)_{r-1} \end{vmatrix}$$

is equal to the product of

$$(k+1)(k+2)\dots r$$

and

$$\begin{vmatrix} m_k, & n, & p\,m_1, & \dots & q\,m_{k-1} \\ (m+1)_k, & n+1, & (p+1)(m+1)_1 & \dots & (q+1)(m+1)_{k-1} \\ \dots & \dots & \dots & \dots & \dots \\ (m+k)_k, & n+k, & (p+k)(m+k)_1 & \dots & (q+k)(m+k)_{k-1} \end{vmatrix} \quad (1).$$

That is to say, is independent of the $r-k$ quantities $s, \dots u$.

To this determinant apply the operations of II. 5. iv. Then in place of any element P in the j^{th} row we must write

$$\Delta^{j-1}P.$$

Then in the first column every element after the $(k+1)^{\text{st}}$ vanishes, while in each of the others every element below the leading diagonal vanishes, the element in the leading diagonal of the i^{th} column being $(i-1)$.

Hence if we expand the determinant by Laplace's theorem, according to minors of the first k columns it reduces to

$$(k+1)(k+2)\dots r \begin{vmatrix} m_k, & n, & p\,m_1 & \dots & qm_{k-1} \\ m_{k-1}, & 1, & [p\,m_0 + (m+1)_0] & \dots & \\ m_{k-2}, & 0, & 2\,(m+1)_0 & \dots & \\ \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & 0, & \dots & \end{vmatrix},$$

which proves the theorem. For the last determinant is the result of operating, as in II. 5. iv., on the determinant (1). The determinant (1) is known by Art. 27, and hence we know the value of the new determinant.

29. Next let us consider

$$\begin{vmatrix} m_k, & n\, m_d, & p\, m_{d+1} & \dots & u\, m_{d+r-1} \\ (m+1)_k, & (n+1)(m+1)_d, & (p+1)(m+1)_{d+1} & \dots & \dots \\ \dots\dots & \dots\dots & \dots\dots & \dots & \dots \\ (m+r)_k, & (n+r)(m+r)_d, & (p+r)(m+r)_{d+1} & \dots & \dots \end{vmatrix};$$

where k has any value from d to $d+r-1$ inclusive.

Divide the rows by

$$m_d,\ (m+1)_d \dots (m+r)_d$$

respectively, and multiply the columns by

$$k_{k-d},\ 1,\ (d+1)_1,\ (d+2)_2 \dots$$

Then our determinant is equal to

$$\frac{m_d\,(m+1)_d\,(m+2)_d \dots (m+r)_d}{k_{k-d}\,(d+1)_1\,(d+2)_2 \dots (d+r-1)_{r-1}} \quad \dots\dots\dots\dots(1)$$

multiplied by the determinant

$$\begin{vmatrix} (m-d)_{k-d}, & n, & p\,(m-d)_1 & \dots \\ (m-d+1)_{k-d}, & n+1, & (p+1)(m-d+1)_1 & \dots \\ \dots\dots & \dots & \dots\dots & \dots \\ (m-d+r)_{k-d}, & n+r, & (p+r)(m-d+r)_1 & \dots \end{vmatrix},$$

which by the preceding articles is equal to

$$(k-d+1)(k-d+2)\dots r\,(m-d-n)(m-d-p-1)$$
$$(m-d-q-2)\dots \qquad (2),$$

being independent of the last $r-k+d$, of the quantities $n, p \dots u$.

The determinant we started with is equal to the product of (1) and (2).

30. In the determinant of the last article let

$$n = p = \dots = u = \tfrac{1}{2},\quad k = d = 1\,;$$

then if we multiply both sides by 2^r

$$\begin{vmatrix} m_1, & m_1, & m_2 & \dots & m_r \\ (m+1)_1, & 3\,(m+1)_1, & 3\,(m+1)_2 & \dots & 3\,(m+1)_r \\ (m+2)_1, & 5\,(m+2)_1, & 5\,(m+2)_2 & \dots & 5\,(m+2)_r \\ \dots\dots & \dots\dots & \dots\dots & \dots & \dots\dots \\ (m+r)_1, & (2r+1)\,(m+r)_1, & (2r+1)\,(m+r)_2 & \dots & (2r+1)\,(m+r)_r \end{vmatrix}$$

$$= 2^r\, m\,(m+1)\dots(m+r).$$

Divide both sides by $m\,(m+1)\dots(m+r)$, and then multiply both sides by $r\,!$, thus

$$\begin{vmatrix} 1, & 1, & (m-1)_1, & \dots & (m-1)_{r-1} \\ 1, & 3, & 3\,m_1, & \dots & 3\,(m)_{r-1} \\ 1, & 5, & 5\,(m+1)_1 & \dots & 5\,(m+1)_{r-1} \\ \dots & \dots & \dots\dots & \dots & \dots\dots \end{vmatrix} = 2\,.\,4\,.\,6\dots 2r.$$

Hence, changing $m-1$ into m, if we write

$$u_r = \begin{vmatrix} 1, & 1, & m_1, & m_2 & \dots & m_r \\ \tfrac{1}{3}, & 1, & (m+1)_1, & (m+1)_2 & \dots & (m+1)_r \\ \tfrac{1}{5}, & 1, & (m+2)_1, & (m+2)_2 & \dots & (m+2)_r \\ \dots & \dots & \dots\dots & \dots\dots & \dots & \dots\dots \end{vmatrix},$$

we have by Wallis' theorem

$$\text{Lim.}\quad (2r+1)\,u^2_{r-1} = \frac{\pi}{2},$$

when r, and therefore the order of the determinant, is infinite.

CHAPTER VII.

ON CUBIC DETERMINANTS AND DETERMINANTS WITH MULTIPLE SUFFIXES.

1. JUST as when n^2 elements are given we can arrange them in the form of a square, so when n^3 elements are given we can arrange them in the form of a cube. Then we can indicate the position of the elements by means of three suffixes. The elements will lie in three sets of parallel planes; supposing the cube containing the elements to stand on a table with one face towards us, we may for convenience call those planes parallel to the face on which the cube rests strata, those parallel to the face in front of us planes, and the perpendicular planes sections.

2. An element of such an array will be denoted by a_{ijk}, where the suffixes mean that it stands in the i^{th} stratum, j^{th} plane, and k^{th} section.

The set of elements in the leading diagonal will be

$$a_{111}a_{222} \dots a_{nnn}.$$

From this we can form a function analogous to a determinant, and hence called a cubic determinant, by the following process.

From the leading term $a_{111}a_{222} \dots a_{nnn}$ we form $n!$ terms by writing for the series of third suffixes all possible permutations of $1, 2 \dots n$, giving to each of these terms a sign corresponding to the class of the permutation. Then from each of the terms so obtained we derive $n!$ new terms by writing for the series of second suffixes all possible permutations of $1, 2 \dots n$, giving to each new term, relatively to the term from which it is derived, the sign corresponding to the class of the permutation. The sum of all these $\{n\,!\}^2$ terms is called a cubic determinant, and is denoted by

$$\Sigma \pm a_{111} a_{222} \ldots a_{nnn},$$

or by

$$|a_{ijk}| \ (i,\ j,\ k = 1,\ 2 \ldots n).$$

3. Just as an ordinary determinant can be represented as the product of n alternate numbers, so a cubic determinant can be represented as the product of n factors lineo-linear in two sets of alternate units.

If $e_1, e_2 \ldots e_n$; $\epsilon_1, \epsilon_2 \ldots \epsilon_n$ are two independent sets of alternate units, then the determinant of Art. 2 is equal to the product

$$\begin{aligned}\Pi\ \{&a_{i11}\epsilon_1 e_1 + a_{i12}\epsilon_1 e_2 + \ldots + a_{i1n}\epsilon_1 e_n \\ &+ a_{i21}\epsilon_2 e_1 + a_{i22}\epsilon_2 e_2 + \ldots + a_{i2n}\epsilon_2 e_n \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ a_{in1}\epsilon_n e_1 + a_{in2}\epsilon_n e_2 + \ldots + a_{inn}\epsilon_n e_n\}\end{aligned} \qquad (i = 1,\ 2 \ldots n).$$

For if we consider any term of the product, it will vanish if it contains two ϵ's or two e's with the same suffix, i.e. if two a's with like second or third suffix occur in the term, which ensures that all terms which do not belong to the determinant vanish. Thus every term which does not vanish contains some permutation of the units $\epsilon_1, \epsilon_2 \ldots \epsilon_n$ and $e_1, e_2 \ldots e_n$ as a factor, and if the units be brought to this order the sign of the term will be $(-1)^{\mu+\nu}$; where μ is the number of inversions in the ϵ's, i.e. in the second suffixes of the term, and ν the like number for the e's or third suffixes. That is to say each term of the product is a term of the determinant with its proper sign. Thus the determinant is correctly represented by the product.

Just as an ordinary determinant is the product of linear functions of the elements of a row, a cubic determinant is the product of linear factors of the elements of a stratum.

By means of this representation we can deduce the properties of cubic determinants.

4. The sign of the determinant is changed if we interchange two planes or sections.

For interchanging two planes is the same thing as interchanging two ϵ's, and interchanging two sections the same as interchanging two e's. Either of these changes alters the sign of every term, and therefore of the whole determinant.

5. Interchanging two strata does not alter the sign of the determinant.

For we can represent the determinant by either of the two products

$$\begin{matrix} \Pi\,(b_{i1}e_1 + b_{i2}e_2 + \ldots + b_{in}e_n) \\ \Pi\,(c_{i1}\epsilon_1 + c_{i2}\epsilon_2 + \ldots + c_{in}\epsilon_n) \end{matrix} \quad (i = 1,\ 2 \ldots n)$$

where
$$b_{ik} = a_{i1k}\epsilon_1 + a_{i2k}\epsilon_2 + \ldots + a_{ink}\epsilon_n$$

$$c_{ik} = a_{ik1}e_1 + a_{ik2}e_2 + \ldots + a_{ikn}e_n.$$

From the first form we see that the determinant, on interchanging two strata, suffers a change of sign as being the product of alternate numbers belonging to the system e; from the second we see that it also suffers a change of sign as being the product of alternate numbers belonging to the system ϵ. Thus on interchanging two strata the determinant undergoes two changes of sign, and hence remains unaltered.

6. A cubic determinant of order n is the sum of $n\,!$ ordinary determinants, each of order n.

For as in Art. 5

$$A = \Pi\,(c_{i1}\epsilon_1 + c_{i2}\epsilon_2 + \ldots + c_{in}\epsilon_n)$$

where c_{ik} has the same meaning as in Art. 5. Hence, by I. 19,

$$A = |\,c_{ik}\,|.$$

Thus the cubic determinant is equal to an ordinary determinant of the same order, whose elements are alternate numbers. To split up this determinant into others with simple elements we must take a partial column from each column of the determinant, but if we take a partial column in the p^{th} place from one column we cannot take a partial column in the p^{th} place from any other column, for then e_p would occur twice, and the corresponding determinant must vanish. Hence each selection of partial columns must be a permutation of $1, 2 \ldots n$, there are $n\,!$ such selections, and as many determinants with simple elements.

Thus $$A = \Sigma \mid a_{ij(k)} \mid,$$
where the determinant on the right is an ordinary determinant; k is put in brackets to remind us that though it varies from one column to another, in the same determinant it remains fixed. This theorem is also an obvious consequence of Art. 2.

7. If in the preceding article we suppose all the first suffixes to be the same, all the determinants on the right would become alike, only their columns being permutated, and each determinant would have the sign corresponding to that permutation, hence suppressing the first suffixes altogether, the cubic determinant is now equal to

$$(n\,!) \mid a_{jk} \mid (j, k = 1, 2 \dots n).$$

This then is the value of the cubic determinant whose strata consist of the determinant

$$\begin{vmatrix} a_{11} \cdots a_{1n} \\ \cdots\cdots\cdots\cdots \\ a_{n1} \cdots a_{nn} \end{vmatrix}$$

repeated n times.

8. The product of two ordinary determinants, each of order n, is a cubic determinant of order n.

Let
$$A = \mid a_{ik} \mid = A_1 A_2 \dots A_n,$$
$$B = \mid b_{ik} \mid = B_1 B_2 \dots B_n,$$
where
$$A_i = a_{i1}\epsilon_1 + a_{i2}\epsilon_2 + \dots + a_{in}\epsilon_n,$$
$$B_i = b_{i1}e_1 + b_{i2}e_2 + \dots + b_{in}e_n,$$
the systems of units e and ϵ being independent.

Then
$$\begin{aligned} AB &= \Pi A_i B_i \\ &= \Pi\,(a_{i1}b_{i1}\epsilon_1 e_1 + a_{i1}b_{i2}\epsilon_1 e_2 + \dots + a_{i1}b_{in}\epsilon_1 e_n \\ &\quad + a_{i2}b_{i1}\epsilon_2 e_1 + a_{i2}b_{i2}\epsilon_2 e_2 + \dots + a_{i2}b_{in}\epsilon_2 e_n \\ &\quad + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &\quad + a_{in}b_{i1}\epsilon_n e_1 + a_{in}b_{i2}\epsilon_n e_2 + \dots + a_{in}b_{in}\epsilon_n e_n). \end{aligned}$$

Now if
$$c_{ijk} = a_{ij}b_{ik},$$
the product on the right is the cubic determinant of the elements c_{ijk}. Thus the theorem is proved.

By multiplying A_i and B_i together we avoided any inversion of the A's and B's among themselves. If we allow for the consequent changes of sign we can have as many such inversions as we please, and so vary the form of the cubic determinant which represents the product.

9. The product of a cubic determinant A, whose elements are a_{ijk}, and of an ordinary determinant B, whose elements are b_{ik}, is a cubic determinant C, whose elements are c_{ijk}, where

$$c_{ijk} = b_{j1} a_{ik1} + b_{j2} a_{ik2} + \dots + b_{jn} a_{ikn}.$$

Or we treat each stratum of A as if it were an ordinary determinant to be multiplied by B, the resulting strata give C.

For
$$\begin{aligned} C = \Pi\ (&c_{i11}\epsilon_1 e_1 + c_{i12}\epsilon_1 e_2 + \dots + c_{i1n}\epsilon_1 e_n \\ &+ c_{i21}\epsilon_2 e_1 + c_{i22}\epsilon_2 e_2 + \dots + c_{i2n}\epsilon_2 e_n \\ &+ \dots \\ &+ c_{in1}\epsilon_n e_1 + c_{in2}\epsilon_n e_2 + \dots + c_{inn}\epsilon_n e_n) \\ = \Pi\ (&a_{i11} B_1 e_1 + a_{i12} B_1 e_2 + \dots a_{i1n} B_1 e_n \\ &+ a_{i21} B_2 e_1 + a_{i22} B_2 e_2 + \dots \\ &+ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots); \end{aligned}$$

where
$$B_j = b_{1j}\epsilon_1 + b_{2j}\epsilon_2 + \dots + b_{nj}\epsilon_n.$$

Since the alternate numbers B_j follow the same laws as units, this last product is a representation of the cubic determinant A by means of the units e and B. Thus

$$\begin{aligned} C &= A \,.\, e_1 \dots e_n \,.\, B_1 \dots B_n \\ &= AB. \end{aligned}$$

10. It is now an obvious step to consider those functions formed of letters with more than three suffixes analogously to determinants, though when we take elements with more than three suffixes we cease to be able to picture to ourselves their arrangement topographically as we can in the case of elements with one, two or three suffixes. We can, however, conceive a set of elements with p suffixes such as

$$a_{ijk\dots l},$$

n^p in number, to be arranged in p sets of rectangular planes in a space of p dimensions, and forming a rectangular paralleloschemon of p dimensions. (Cf. Schläfli, *Quarterly Jour.* II. p. 278.) The elements which have all suffixes the same, except i, lie in the same line, those which have all suffixes the same, with the exception of i and j, lie in the same plane, ... those which have only l in common lie in a rectangular paralleloschemon of $p-1$ dimensions.

The product of the elements

$$a_{11\cdots 1}\, a_{22\cdots 2} \cdots a_{nn\cdots n}$$

is called the leading term of the determinant of the p^{th} class, which is formed by keeping the first suffixes unaltered, and writing for each set of the other suffixes all possible permutations of $1, 2 \ldots n$. To each term so obtained we give the sign corresponding to the sum of the number of inversions in the $p-1$ sets of variable suffixes.

The whole number of terms is $\{n!\}^{p-1}$.

11. The determinant of the p^{th} class can be represented as a product of linear factors of the elements which lie in the same paralleloschemon of $p-1$ dimensions.

If

$$\begin{array}{c} e_1, \; e_2 \;\ldots\; e_n \\ \epsilon_1, \; \epsilon_2 \;\ldots\; \epsilon_n \\ \ldots\ldots\ldots\ldots\ldots \\ \eta_1, \; \eta_2 \cdots \eta_n \end{array}$$

be $p-1$ sets of alternate units; it is plain from reasoning similar to that in Art. 3, that the function

$$A = \Pi \Sigma a_{ijk\ldots l}\, e_j \epsilon_k \ldots \eta_l$$

(where the sum is formed by giving to each of the suffixes $j, k \ldots l$ all values from 1 to n, and then forming the product of such sums for the values $1, 2 \ldots n$ of i) is a determinant of the p^{th} class and n^{th} order, such as we have defined in Art. 10.

12. This definition is strictly analogous to those for determinants of the second and third class. A determinant of the second class is the product of linear functions of the elements of a row, one of the third class the product of n factors linear in the elements of a stratum. Here the determinant of the p^{th} class is the product of n factors linear in the elements of a paralleloschemon of $p-1$ dimensions.

13. It is clear that by the interchange of any two suffixes, except the first, the determinant changes sign. Also since the factors of the determinant can be written as linear expressions of each of the $p-1$ sets of alternate units, it follows by the interchange of two first suffixes the determinant undergoes $p-1$ changes of sign. Thus the determinant remains unaltered or changes sign according as its class is odd or even.

14. We have kept the first suffixes in their natural order. It is however indifferent which set of suffixes is retained fixed. If the class of the determinant is odd, it is perhaps more symmetrical to keep the middle suffix unaltered; the determinant is however not the same as before.

15. The product of a cubic determinant A, whose elements are a_{ijk}, and of an ordinary determinant B, whose elements are b_{ik}, can be represented as a determinant of the fourth class C, whose elements c_{ijkl} are given by

$$c_{ijkl} = a_{ijk} b_{il}.$$

For
$$A = \Pi\,(a_{i11}\epsilon_1 e_1 + a_{i12}\epsilon_1 e_2 + \dots + a_{i1n}\epsilon_1 e_n + a_{i21}\epsilon_2 e_1 + \dots\dots\dots\dots),$$

$$B = \Pi\,(b_{i1}\eta_1 + b_{i2}\eta_2 + \dots + b_{in}\eta_n).$$

Thus clearly

$$AB = \Pi\,(\Sigma c_{ijkl}\epsilon_j e_k \eta_l) \qquad \begin{array}{l}(\text{In } \Sigma\ j,\, k,\, l = 1,\, 2 \dots n)\\ (\text{In } \Pi\ \ i = 1,\, 2 \dots n)\end{array}$$

which proves the theorem.

16. The product of two cubic determinants A and B, whose elements are a_{ijk} and b_{ijk}, both of order n, can be represented either as a determinant of the fifth class, whose elements are

$$c_{ipqrs} = a_{ipq} b_{irs},$$

or as a determinant of the fourth class, whose elements are given by

$$c_{ijkl} = \Sigma a_{pij}\, b_{pkl} \qquad (p = 1,\, 2 \dots n);$$

the order of both determinants being n.

The first part of the theorem is proved as follows:

$$A = \Pi \Sigma a_{ipq} \epsilon_p e_q.$$

$$(\text{In } \Sigma\ \ p,\, q = 1,\, 2 \dots n;\ \text{ in } \Pi\ \ i = 1,\, 2 \dots n.)$$

$$B = \Pi\Sigma b_{irs} j_r k_s.$$

$$(\text{In } \Sigma \quad r, s = 1, 2 \ldots n; \text{ in } \Pi \quad i = 1, 2 \ldots n.)$$

Thus
$$AB = \Pi\Sigma\, a_{ipq} b_{irs}\, \epsilon_p\, e_q j_r k_s$$
$$= \Pi\Sigma\, c_{ipqrs}\, \epsilon_p e_q j_r k_s.$$

$$(\text{In } \Sigma \quad p, q, r, s = 1, 2 \ldots n; \text{ in } \Pi \quad i = 1, 2 \ldots n.)$$

Which by definition proves the theorem.

For the second part of the theorem we have

$$C = \Pi\Sigma c_{ijkl} e_j\, \epsilon_k\, \eta_l.$$

Now the sum under the product sign

$$= \Sigma e_j \{a_{ij1} B_1 + a_{ij2} B_2 + \ldots + a_{ijn} B_n\} \qquad (j = 1, 2 \ldots n),$$

where
$$B_p = b_{p11}\, \epsilon_1\, \eta_1 + b_{p12}\, \epsilon_1\, \eta_2 + \ldots + b_{p1n}\, \epsilon_1\, \eta_n$$
$$+ b_{p21}\, \epsilon_2\, \eta_1 + b_{p22}\, \epsilon_2\, \eta_2 + \ldots + b_{p2n}\, \epsilon_2\, \eta_n$$
$$+ \ldots$$

and if we write

$$A_{iq} = a_{i1q}\, e_1 + a_{i2q}\, e_2 + \ldots + a_{inq} e_n$$

the sum becomes

$$B_1 A_{i1} + B_2 A_{i2} + \ldots + B_n A_{in}.$$

The product of this has to be taken for all values of i. It must always be taken so that in each term we have the product $B_1 B_2 \ldots B_n$; for if two B's are repeated the term vanishes. The value of this product is B.

The remaining factors in the term are

$$A_{1p} A_{2q} \ldots A_{nr},$$

where $p, q \ldots r$ is a permutation of $1, 2 \ldots n$. This is an ordinary determinant of class 2. Comparing this with Art. 6, we see that it is a term in the expansion of the cubic determinant A in a sum of determinants of class 2. All these terms occur in our product. Thus

$$C = A \,.\, B.$$

17. The following theorem regarding the product of two determinants of any class can be proved by the preceding methods.

The product of two determinants of classes p and q, whose elements are $a_{ij\ldots l}$ and $b_{ij\ldots k}$ respectively, can be represented either as a determinant of class $p+q-1$, whose elements are

$$c_{ij\ldots luv\ldots s} = a_{ij\ldots l}\, b_{iu\ldots s},$$

or as a determinant of class $p+q-2$, whose elements are

$$c_{j\ldots luv\ldots s} = \Sigma a_{ij\ldots l}\, b_{iu\ldots s}\ (i = 1, 2 \ldots n),$$

all the determinants being of order n.

18. It is not difficult to see how the theorems with regard to determinants of the second class (i.e. ordinary determinants) can be extended to determinants of any other class. It is probable that determinants of higher class possess many properties peculiar to themselves, though as yet not many of these have been investigated. The complement of any element of a determinant is a determinant of the same class and next lower order. The extension of Laplace's theorem would shew how a determinant of class p and order n could be expanded in a series of products of pairs of determinants of class p and orders m and $n-m$.

19. There is no difficulty in writing down the expansions of determinants of any required class or order. The number of terms however increases very rapidly.

The following are the expansions of determinants of the second order, and classes 3 and 4 respectively :

$$\Sigma \pm (111)(222) = (111)(222) - (121)(212) + (122)(211) - (112)(221)$$

$$\begin{aligned}\Sigma \pm (1111)(2222) = (1111)(2222) &- (1112)(2221) + (1212)(2121)\\ &- (1211)(2122) + (1122)(2211) - (1121)(2212)\\ &+ (1221)(2112) - (1222)(2111),\end{aligned}$$

while for the determinant of class 3 and order 3,

$$\begin{aligned}\Sigma \pm (111)(222)(333) = (111)(222)(333) &- (121)(212)(333)\\ - (111)(232)(323) &+ (131)(212)(323)\\ + (121)(232)(313) &- (131)(222)(313)\\ - (112)(221)(333) &+ (122)(211)(333)\\ + (112)(231)(323) &- (132)(211)(323)\\ - (122)(231)(313) &+ (132)(221)(313)\\ - (111)(223)(332) &+ (121)(213)(332)\\ + (111)(233)(322) &- (131)(213)(322)\\ - (121)(233)(312) &+ (131)(223)(312)\\ + (113)(221)(332) &- (123)(211)(332)\end{aligned}$$

$$
\begin{aligned}
&-(113)(231)(322)+(133)(211)(322)\\
&+(123)(231)(312)-(133)(221)(312)\\
&+(112)(223)(331)-(122)(213)(331)\\
&-(112)(233)(321)+(132)(213)(321)\\
&+(122)(233)(311)-(132)(223)(311)\\
&-(113)(222)(331)+(123)(212)(331)\\
&+(113)(232)(321)-(133)(212)(321)\\
&-(123)(232)(311)+(133)(222)(311).
\end{aligned}
$$

20. We shall conclude this chapter with the following general theorems.

A determinant of any class, all of whose elements are equal to a, except those in the leading diagonal which are equal to x, is equal to

$$\{x+(n-1)a\}(x-a)^{n-1},$$

n being the order of the determinant.

We shall prove this for a cubic determinant, but the method is perfectly general.

$$
\begin{aligned}
D = \Pi\,(&ae_1\epsilon_1 + ae_1\epsilon_2 + \dots\\
&+ ae_2\epsilon_1 + ae_2\epsilon_2 + \dots\\
&+ \dots \quad + xe_i\epsilon_i + \dots)\\
= \Pi\{&aEE' + (x-a)\,e_i\epsilon_i\},
\end{aligned}
$$

where $\quad E = e_1 + e_2 + \dots e_n, \quad E' = \epsilon_1 + \epsilon_2 + \dots + \epsilon_n.$

Hence, since E and E' are alternate numbers, any term in which they occur more than once vanishes.

Hence $\quad D = (x-a)^n + a(x-a)^{n-1}\Sigma\,\{EE'\Pi e_k\epsilon_k\}$
$\quad (k = 1, 2 \dots i-1, i+1 \dots n);$

$$
\begin{aligned}
\therefore\ D &= (x-a)^n + na(x-a)^{n-1}\\
&= \{x+(n-1)a\}(x-a)^{n-1};
\end{aligned}
$$

for $\quad Ee_1 \dots e_{i-1}e_{i+1} \dots e_n = e_ie_1 \dots e_{i-1}e_{i+1} \dots e_n$
$\quad = (-1)^{i-1}e_1e_2 \dots e_n;$

and so $\quad E'\epsilon_1 \dots \epsilon_{i-1}\epsilon_{i+1} \dots \epsilon_n = (-1)^{i-1}\epsilon_1\epsilon_2 \dots \epsilon_n.$

The last theorem of III. 25 can also be extended to determinants of higher class, for a cubic determinant we may state it as follows: If all the elements in the i^{th} stratum are equal to a_i, with the exception of that which lies in the leading diagonal, whose value is x_i, then the value of the determinant is

$$f + \Sigma a_n f'(x_r)$$

with the notation given in III. 25.

CHAPTER VIII.

APPLICATIONS TO THE THEORY OF EQUATIONS AND OF ELIMINATION.

1. IF we have n linear equations between n quantities $x_1, x_2 \dots x_n$, namely,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= u_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= u_2 \qquad (1). \\ \dots\dots\dots\dots\dots\dots\dots\dots & \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= u_n, \end{aligned}$$

the determinant $A = | a_{ik} |$ is called the determinant of the system. If A does not vanish we can at once determine the variables. For if we multiply the above equations by $A_{1k}, A_{2k} \dots A_{nk}$ respectively and add, then all the terms on the left vanish, with the exception of those multiplying x_k, which together give A (III. 11). Hence

$$Ax_k = u_1 A_{1k} + u_2 A_{2k} + \dots + u_n A_{nk} \qquad (k = 1, 2 \dots n).$$

The expression on the right is the expansion of the determinant, obtained by writing $u_1, u_2 \dots u_n$ for the elements of the k^{th} column.

2. It is interesting to compare with this the solution by alternate numbers.

Multiply the given system (1) by $e_1, e_2 \dots e_n$ and add; then if

$$e_1 a_{1k} + e_2 a_{2k} + \dots + e_n a_{nk} = A_k$$
$$e_1 u_1 + e_2 u_2 + \dots + e_n u_n = U,$$

we have

$$A_1 x_1 + A_2 x_2 + \dots + A_n x_n = U.$$

Multiply both sides of this equation by $A_1 \dots A_{k-1}\, A_{k+1} \dots A_n$, and we get

$$A_1 \dots A_{k-1} A_{k+1} \dots A_n A_k x_k = A_1 \dots A_{k-1} A_{k+1} \dots A_n U,$$

or $$A_1 \dots A_n x_k = A_1 \dots A_{k-1}\, U A_{k+1} \dots A_n,$$

and writing the products of alternate numbers as determinants we get the same solution as before.

3. If in the equations (1) the quantities u on the right vanish, we have the system of n homogeneous linear equations

$$a_{i1} x_1 + a_{i2} x_2 + \dots a_{in} x_n = u_i = 0 \qquad (i = 1, 2 \dots n).$$

We may regard these as equations to find $\frac{x_1}{x_n}, \frac{x_2}{x_n} \dots \frac{x_{n-1}}{x_n}$.

Taking any $n-1$ of the equations, by Art. 1 we can determine the ratios. These values, if the equations are consistent, must satisfy the remaining equation. This condition is

$$A = 0.$$

For if we multiply the equations by $A_{1k}, A_{2k} \dots A_{nk}$, as before and add, we get

$$x_k \mid a_{ki} \mid = 0.$$

If then the equations are to be satisfied by other than zero values of the variables we must have

$$A = 0.$$

If this be true any one of the equations is a consequence of all the rest, viz. we have

$$u_1 A_{1k} + u_2 A_{2k} + \dots + u_n A_{nk} = 0.$$

Where the u's now stand for the linear functions, that is to say, any one of the u's is expressible linearly in terms of the remaining ones, provided the quantities A_{ik} do not all vanish.

4. If the condition of the preceding paragraph holds we have

$$\frac{x_1}{A_{k1}} = \frac{x_2}{A_{k2}} = \dots = \frac{x_n}{A_{kn}}.$$

For if we substitute the values $x_i = \lambda A_{ki}$ all the equations except the k^{th} are satisfied by III. 11, and the k^{th} is also true since $A = 0$.

5. Returning again to the equations of Art. 1. Any new linear function v of the x's can be expressed linearly in terms of the u's.

For if
$$v = b_1 x_1 + b_2 x_2 + \dots + b_n x_n,$$
$$u_1 = a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$u_n = a_{n1} x_1 + a_{n2} x_2 + \dots + a_{nn} x_n,$$

we may regard these as $n+1$ equations between the $n+1$ quantities
$$-1,\ x_1,\ x_2 \dots x_n.$$

Hence, by Art. 3, we must have

$$\begin{vmatrix} v, & b_1, & b_2 & \dots & b_n \\ u_1, & a_{11}, & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ u_n, & a_{n1}, & a_{n2} & \dots & a_{nn} \end{vmatrix} = 0,$$

or
$$-Av = \begin{vmatrix} 0, & b_1, & b_2 & \dots & b_n \\ u_1, & a_{11}, & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ u_n, & a_{n1}, & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

6. If we have between n variables $x_1, x_2 \dots x_n$, the m equations
$$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = 0$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n = 0$$
where m is greater than n. Then if these equations are to be true for other than zero values of the variables, if we take any n of them their determinant must vanish by Art. 3.

This condition is represented by

$$\left\Vert \begin{matrix} a_{11}, & a_{12} & \dots & a_{1n} \\ a_{21}, & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1}, & a_{m2} & \dots & a_{mn} \end{matrix} \right\Vert = 0;$$

which means that each of the system of m_n determinants, got by selecting any n rows of elements from the array and forming a determinant with them, is to vanish. The expression on the left is frequently called a matrix.

7. The system of linear congruences

$$\left.\begin{array}{c} a_{11}x_1 + \dots + a_{1n}x_n \equiv u_1 \\ \dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + \dots + a_{nn}x_n \equiv u_n \end{array}\right\} \pmod{p},$$

first considered by Gauss, has been solved as follows by Studnička.

Let $A = | a_{ik} |$,

and let g_k be the greatest common measure of the numbers

$$A_{1k}, A_{2k} \dots A_{nk}.$$

Then, as in Art. 1, we have for all values of k from 1 to n,

$$\frac{A}{g_k} x_k \equiv \frac{1}{g_k}(u_1 A_{1k} + u_2 A_{2k} + \dots + u_n A_{nk}) \pmod{p}.$$

The advantage of the rule is that if we observe that one of the minors of a column is unity, or if two of them are prime to each other, then, for that column, $g_k = 1$.

8. The solution of the system in Art. 1 assumes different forms according to the nature of the coefficients a_{ik}. If

$$a_{ik} = -a_{ki} \text{ and } a_{ii} = 0,$$

so that the determinant of the system is skew symmetrical; first, if n is even, if we multiply the equations by

$$[2 \dots k-1,\ k+1 \dots n],\ [3 \dots k-1,\ k+1 \dots n,\ 1] \dots$$
$$[1 \dots k-1,\ k+1 \dots n-1],$$

and add, the coefficient of x_k is

$$a_{1k}[2 \dots k-1,\ k+1 \dots n] + a_{2k}[3 \dots k-1,\ k+1 \dots n,\ 1]$$
$$+ \dots + a_{nk}[1 \dots k-1,\ k+1 \dots n-1]$$
$$= -[k,\ 1 \dots k-1,\ k+1 \dots n] = (-1)^k [1,\ 2 \dots n],$$

while the coefficient of x_i is

$$-[i,\ 1 \dots k-1,\ k+1 \dots n] = 0.$$

Thus

$$(-1)^k x_k [1,\ 2 \dots n] = u_1 [2 \dots k-1,\ k+1 \dots n]$$
$$+ u_2 [3 \dots k-1,\ k+1 \dots n,\ 1] \dots + u_n [1 \dots k-1,\ k+1 \dots n-1].$$

But if n is odd, then $A = 0$ (VI. 8) and $x_1, x_2 \dots x_n$ in general are infinite, but bear fixed ratios to each other. If however

$$u_1 A_{1k} + u_2 A_{2k} + \dots + u_n A_{nk} = 0,$$

or $\quad u_1 [2 \dots n] + u_2 [3 \dots n,\ 1] + \dots + u_n [1,\ 2 \dots n-1] = 0$

(VI. 15), one equation of the system is superfluous, and the system of the remaining equations can be solved as above.

9. In Art. 3 we have the first example of the process of elimination; namely, we have found a condition, independent of the variables, which must hold if a certain given number of equations are to exist between these variables. When r homogeneous equations hold between r variable quantities, (or what is the same thing, r non-homogeneous equations between $r-1$ quantities) it is always possible to establish an equation $R=0$ between the coefficients of these equations alone. Then R is called the resultant or eliminant of the system of equations.

When the equations are two in number the most direct process is Sylvester's dialytic method. Let the two equations be

$$\left.\begin{array}{l} 0 = a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m \\ 0 = b_0 + b_1 x + b_2 x^2 + \ldots + b_n x^n \end{array}\right\} \ldots\ldots\ldots\ldots\ldots(1).$$

If we multiply the first equation by $1, x, x^2 \ldots x^{n-1}$ we get $n-1$ new equations, and from the second by multiplying by $1, x, x^2 \ldots x^{m-1}$ we get $m-1$ new equations, viz. we have now the system

$$\begin{array}{l} 0 = a_0 + a_1 x + a_2 x^2 + \ldots \\ 0 = \phantom{a_0 + {}} a_0 x + a_1 x^2 + \ldots \\ 0 = \phantom{a_0 + a_1 x + {}} a_0 x^2 + \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0 = b_0 + b_1 x + b_2 x^2 + \ldots \\ 0 = \phantom{b_0 + {}} b_0 x + b_1 x^2 + \ldots \\ 0 = \phantom{b_0 + b_1 x + {}} b_0 x^2 + \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

of $m+n$ equations satisfied by the same values of x as the given equations (1) and linear and homogeneous in the $m+n$ quantities

$$1,\ x,\ x^2 \ldots x^{m+n-1}.$$

Hence, by Art. 3, the determinant of the system must vanish, or

$$R = \left|\begin{array}{llll} a_0, & a_1, & a_2 & \ldots\ldots \\ & a_0, & a_1 & \ldots\ldots \\ & & a_0 & \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots\ldots \\ b_0, & b_1, & b_2 & \ldots\ldots \\ & b_0, & b_1 & \ldots\ldots \\ & & b_0 & \ldots\ldots \end{array}\right| = 0.$$

A determinant of order $m+n$. Since there are n rows of a's, and m of b's, the resultant is of order n in the coefficients of the first equation, and of order m in the coefficients of the second.

10. If the coefficients $a_m, a_{m-1}, a_{m-2} \dots b_n, b_{n-1}, b_{n-2} \dots$ are functions of y and z of degrees $0, 1, 2 \dots$, it can be proved that the resultant is of order mn in y and z. This will be the case if every term in R has the sum of the complements of the suffixes equal to mn.

If we change y and z into yt and zt respectively, the value of R is now

$$R' = \begin{vmatrix} a_0 t^m, & a_1 t^{m-1}, & a_2 t^{m-2} \dots \\ & a_0 t^m, & a_1 t^{m-1} \dots \\ \dots\dots & \dots\dots & \dots\dots \\ b_0 t^n, & b_1 t^{n-1}, & b_2 t^{n-2} \dots \\ & b_0 t^n, & b_1 t^{n-1} \dots \\ \dots\dots & \dots\dots & \dots\dots \end{vmatrix}.$$

Observe that the separate elements and therefore each term of R' is multiplied by a power of t equal to the complement of the suffix.

Now, multiply the first n rows by

$$t^{n-1},\ t^{n-2} \dots t,\ 1,$$

and the last m by

$$t^{m-1},\ t^{m-2}, \dots t,\ 1.$$

Then R' is multiplied by a power of t, whose exponent is

$$\frac{m(m-1)}{2} + \frac{n(n-1)}{2}.$$

But now the first column of R' divides by t^{m+n-1}, the second by t^{m+n-2}, and so on. Thus $R' \div R$ is equal to a power of t whose exponent is

$$\frac{(m+n)(m+n-1)}{2} - \frac{m(m-1)}{2} - \frac{n(n-1)}{2} = mn.$$

Thus every term in R' must divide by t^{mn}, which proves the theorem. Functions, such that the sum of the suffixes, or of their complements, of the elements in each term is constant, are sometimes called *isobaric*, and the constant sum is called the weight.

11. We may consider the question in another way.

If
$$\phi(x) = b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n$$
$$= b_n (x - \beta_1)(x - \beta_2) \dots (x - \beta_n) \dots\dots\dots\dots (1)$$

is an equation whose roots are $\beta_1, \beta_2 \dots \beta_n$, the function

$$f(x) = u = a_0 + a_1x + a_2x^2 + \dots + a_mx^m \dots\dots\dots\dots (2)$$

has n values corresponding to the different values of x given by (1). These n values are the roots of an equation of the n^{th} degree, which we now proceed to find. Multiply the equations (1) and (2) by the same powers of x as in Art. 9, and we have the $m+n$ equations

$$\begin{aligned}
0 &= a_0 - u + a_1x + a_2x^2 + \dots \\
0 &= \quad (a_0 - u)\, x + a_1x^2 + \dots \\
&\qquad\qquad (a_0 - u)\, x^2 + \dots \\
&\dots\dots\dots\dots\dots\dots \\
0 &= \quad b_0 + b_1x + b_2x^2 + \dots \\
0 &= \qquad\quad b_0x + b_1x^2 + \dots \\
0 &= \quad \dots\dots\dots\dots\dots\dots
\end{aligned}$$

Eliminating between these the quantities

$$x^{m+n-1} \dots x,\ 1,$$

we get

$$\begin{vmatrix}
a_0 - u, & a_1, & a_2 & \dots \\
 & a_0 - u, & a_1 & \dots \\
 & & a_0 - u & \dots \\
 & \dots & \dots & \dots \\
b_0, & b_1, & b_2 & \dots \\
 & b_0, & b_1 & \dots \\
 & \dots & \dots & \dots
\end{vmatrix} = 0,$$

an equation of the n^{th} degree to find u, the roots of which are

$$f(\beta_1),\ f(\beta_2) \dots f(\beta_n).$$

The product of the roots being equal to the constant term

$$(-1)^n b_n^m f(\beta_1) f(\beta_2) \dots f(\beta_n) = (-1)^n R,$$

where R has the meaning in Art. 9. Thus

$$R = b_n^m f(\beta_1) f(\beta_2) \dots f(\beta_n).$$

In the same way we may shew that

$$R = (-1)^{mn} (a_m^n)\, \phi(\alpha_1)\, \phi(\alpha_2) \dots \phi(\alpha_m)$$

if $\alpha_1 \dots \alpha_m$ are the roots of (2).

12. If the two functions ϕ and f of the preceding article are a function and its differential coefficient, then R is called the discriminant of the function, and its vanishing is the condition that the function should have equal roots. If

$$\begin{aligned} f(x) &= a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \\ &= a_n (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n) \\ f'(x) &= \quad a_1 + 2a_2 x + \ldots + na_n x^{n-1} \\ R &= a_n^{n-1} f'(\alpha_1) f'(\alpha_2) \ldots f'(\alpha_n) \\ &= \begin{vmatrix} a_1, & 2a_2, & 3a_3 & \ldots \\ & a_1, & 2a_2 & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_0, & a_1, & a_2 & \ldots \\ & a_0, & a_1 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}, \end{aligned}$$

having n rows of the first, and $n - 1$ of the second kind.

If we multiply the last row by n, and subtract it from the n^{th}, this becomes

$$0 \ldots 0, \; -na_0, \; -(n-1) a_1 \ldots \; -a_{n-1}, \; 0.$$

Thus the determinant reduces into the product of a_n by a determinant of order $2n - 2$, which we shall call Δ.

Also

$$\begin{aligned} f'(\alpha_1) &= \quad a_n \quad (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3) \ldots (\alpha_1 - \alpha_n) \\ f'(\alpha_2) &= (\alpha_2 - \alpha_1) \quad a_n \quad (\alpha_2 - \alpha_3) \ldots (\alpha_2 - \alpha_n) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f'(\alpha_n) &= (\alpha_n - \alpha_1)(\alpha_n - \alpha_2)(\alpha_n - \alpha_3) \ldots \quad a_n; \end{aligned}$$

$$\therefore f'(\alpha_1) f'(\alpha_2) \ldots f'(\alpha_n) = (-1)^{\frac{n(n-1)}{2}} a^n{}_n \, \zeta(\alpha_1, \alpha_2 \ldots \alpha_n)$$

where $\zeta(\alpha_1 \ldots \alpha_n)$ means the product of the squares of the differences of all the roots. Thus

$$\Delta = (-1)^{\frac{n(n-1)}{2}} a_n^{2n-2} \zeta(\alpha_1, \alpha_2 \ldots \alpha_n).$$

13. The artifice employed in eliminating x between two equations may sometimes be employed for the case of more equations than two, as in the following examples due to Prof. Cayley.

Let $\quad x+y+z=0, \quad x^2=a, \quad y^2=b, \quad z^2=c;$
multiply the first equation by 1, yz, zx, xy, and reduce by means of the other three, then we get

$$\begin{array}{llll} & x+y & +\ z & =0 \\ xyz & & +cy+bz & =0 \\ xyz+cx & & +az & =0 \\ xyz+bx+ay & & & =0, \end{array}$$

whence, eliminating xyz, x, y, z, we get

$$\begin{vmatrix} \cdot, & 1, & 1, & 1 \\ 1, & \cdot, & c, & b \\ 1, & c, & \cdot, & a \\ 1, & b, & a, & \cdot \end{vmatrix} = 0.$$

Or if we multiply the equation by x, y, z, xyz, and eliminate 1, yz, zx, xy, we get

$$\begin{vmatrix} \cdot, & a, & b, & c \\ a, & \cdot, & 1, & 1 \\ b, & 1, & \cdot, & 1 \\ c, & 1, & 1, & \cdot \end{vmatrix} = 0.$$

Again, if we are given the equations

$$x+y+z=0, \quad x^3=a, \quad y^3=b, \quad z^3=c,$$

if we multiply the first equation by

$$x, \ y, \ z, \ y^2z^2, \ z^2x^2, \ x^2y^2, \ x^2yz, \ y^2zx, \ z^2xy,$$

and reduce by the last three we can eliminate

$$x^2, y^2, z^2, yz, zx, xy, xy^2z^2, yz^2x^2, zx^2y^2$$

between the resulting equations, giving

$$\begin{vmatrix} 1, & \cdot, & \cdot, & \cdot, & 1, & 1, & \cdot, & \cdot, & \cdot \\ \cdot, & 1, & \cdot, & 1, & \cdot, & 1, & \cdot, & \cdot, & \cdot \\ \cdot, & \cdot, & 1, & 1, & 1, & \cdot, & \cdot, & \cdot, & \cdot \\ \cdot, & c, & b, & \cdot, & \cdot, & \cdot, & 1, & \cdot, & \cdot \\ c, & \cdot, & a, & \cdot, & \cdot, & \cdot, & \cdot, & 1, & \cdot \\ b, & a, & \cdot, & \cdot, & \cdot, & \cdot, & \cdot, & \cdot, & 1 \\ \cdot, & \cdot, & \cdot, & a, & \cdot, & \cdot, & \cdot, & 1, & 1 \\ \cdot, & \cdot, & \cdot, & \cdot, & b, & \cdot, & 1, & \cdot, & 1 \\ \cdot, & \cdot, & \cdot, & \cdot, & \cdot, & c, & 1, & 1, & \cdot \end{vmatrix} = 0.$$

Other forms of the resultant can also be obtained.

14. The resultant of the quadric

$$u = a_{11} x_1^2 + \dots + 2a_{ik} x_i x_k + \dots = 0 \quad \dots\dots\dots\dots\dots (1),$$

and of the $n-1$ linear equations

$$\begin{array}{l} v_1 = c_{11} x_1 + c_{12} x_2 + \dots + c_{1n} x_n = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (2) \\ v_{n-1} = c_{n-11} x_1 + c_{n-12} x_2 + \dots + c_{n-1n} x_n = 0 \end{array}$$

can be readily expressed by determinants.

By Euler's theorem for homogeneous functions we can write the first equation in the form

$$x_1 \frac{du}{dx_1} + x_2 \frac{du}{dx_2} + \dots + x_n \frac{du}{dx_n} = 2u = 0 \quad \dots\dots\dots (3).$$

Then if in equation (3) we do not consider the variables implicitly contained in the differential coefficients, (1) and (2) being n equations, between $x_1 \dots x_n$, (3) must be identical with

$$\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_{n-1} v_{n-1} = 0 \quad \dots\dots\dots\dots\dots\dots (4)$$

by Art. 3. Equating coefficients in (3) and (4) we must have

$$\left.\begin{array}{l} a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = \lambda_1 c_{11} + \lambda_2 c_{21} + \dots + \lambda_{n-1} c_{n-11} \\ a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n = \lambda_1 c_{12} + \lambda_2 c_{22} + \dots + \lambda_{n-1} c_{n-12} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1} x_1 + a_{n2} x_2 + \dots + a_{nn} x_n = \lambda_1 c_{1n} + \lambda_2 c_{2n} + \dots + \lambda_{n-1} c_{n-1n} \end{array}\right\} (5),$$

the equations (5) together with (2) form a system of $2n-1$ equations between $x_1, x_2 \dots x_n, \lambda_1, \lambda_2 \dots \lambda_{n-1}$, hence their determinant must vanish by Art. 3. Thus

$$\left|\begin{array}{llll} a_{11} \dots a_{1n}, & c_{11} \dots c_{n-11} \\ \dots\dots\dots\dots & \dots\dots\dots\dots \\ a_{n1} \dots a_{nn}, & c_{1n} \dots c_{n-1n} \\ c_{11} \dots c_{1n} & \\ \dots\dots\dots\dots & \\ c_{n-11} \dots c_{n-1n} & \end{array}\right| = 0,$$

the blank space being filled with zeros. This result is due to Versluijs. a_{ik} and a_{ki} mean the same thing, viz. half the coefficient of $x_i x_k$.

15. If we seek to solve the system of equations

$$x + y = a \quad x^2 + y^2 = b^2,$$

we do so by establishing the new linear equation

$$x - y = \pm\sqrt{2b^2 - a^2}.$$

Following up this idea Baur has solved the non-homogeneous system of an n-ary quadric and $n-1$ linear equations between the variables; viz. let the system be

$$a_{11}x_1^2 + \ldots + 2a_{ik}x_i x_k + \ldots = u \quad \ldots\ldots\ldots\ldots\ldots(1),$$

$$\begin{aligned} c_{11}\, x_1 + \ldots \quad & + c_{1n}\, x_n = y_1 \\ c_{21}\, x_1 + \ldots \quad & + c_{2n}\, x_n = y_2 \\ \ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2) \\ c_{n-1\,1} x_1 + \ldots \quad & + c_{n-1\,n} x_n = y_{n-1}. \end{aligned}$$

Then we wish to establish a new linear equation

$$c_{n1}x_1 + \ldots + c_{nn}x_n = y_n \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

so that if we determine the values of $x_1 \ldots x_n$ in terms of $y_1 \ldots y_n$ from (2) and (3), and substitute their values in (1), the result shall only contain y_n in the form y_n^2. We are to have then

$$u = y_n^2 + \Sigma b_{ik} y_i y_k \qquad (i, k = 1,\ 2 \ldots n-1) \ldots\ldots\ldots\ldots(4).$$

Now if $\qquad C = |\, c_{ik} \,|$

we have $\qquad Cx_i = C_{1i}y_1 + C_{2i}y_2 + \ldots + C_{ni}y_n \ldots\ldots\ldots\ldots\ldots\ldots(5).$

Hence, differentiating (4) partially with respect to y_n, we get

$$2y_n = \frac{du}{dx_1} \cdot \frac{dx_1}{dy_n} + \frac{du}{dx_2} \cdot \frac{dx_2}{dy_n} + \ldots + \frac{du}{dx_n} \cdot \frac{dx_n}{dy_n},$$

or, by aid of (5), if $\qquad u_i = \frac{1}{2}\dfrac{du}{dx_i},$

$$Cy_n = u_1 C_{n1} + u_2 C_{n2} + \ldots + u_n C_{nn}$$

$$= \begin{vmatrix} c_{11}, & c_{12} & \ldots & c_{1n} \\ c_{21}, & c_{22} & \ldots & c_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ c_{n-1\,1}, & c_{n-1\,2} & \ldots & c_{n-1\,n} \\ u_1, & u_2 & \ldots & u_n \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Substituting for the differential coefficients their values we determine the form of the equation (3). We have still to determine the value of y_n. To do this we introduce the $n(n-1)$ quantities

$$\begin{matrix} e_{11}, & e_{12} & \dots & e_{1n} \\ e_{21}, & e_{22} & \dots & e_{2n} \\ \dots & \dots & \dots & \dots \\ e_{n-1\,1}, & e_{n-1\,2} & \dots & e_{n-1\,n}, \end{matrix}$$

such that
$$e_{r1}a_{1k} + e_{r2}a_{2k} + \dots + e_{rn}a_{nk} = c_{rk};$$
and hence
$$Ae_{ri} = c_{r1}A_{i1} + c_{r2}A_{i2} + \dots + c_{rn}A_{in},$$
where
$$A = |\, a_{ik} \,|.$$

Thus

$$A \begin{vmatrix} e_{11}, & e_{12} & \dots & e_{1n} \\ \dots & \dots & \dots & \dots \\ e_{n-1\,1}, & e_{n-1\,2} & \dots & e_{n-1\,n} \\ x_1, & x_2 & \dots & x_n \end{vmatrix} = \begin{vmatrix} c_{11} & \dots & c_{1n} \\ \dots & \dots & \dots \\ c_{n-1\,1} & \dots & c_{n-1\,n} \\ u_1 & \dots & u_n \end{vmatrix} = Cy_n \quad \dots\dots(7).$$

Now from the product of (6) and (7),

$$\therefore\ C^2y_n^2 = A \begin{vmatrix} e_{11} & \dots & e_{1n} \\ \dots & \dots & \dots \\ e_{n-1\,1} & \dots & e_{n-1\,n} \\ x_1 & \dots & x_n \end{vmatrix} \cdot \begin{vmatrix} c_{11} & \dots & c_{1n} \\ \dots & \dots & \dots \\ c_{n-1\,1} & \dots & c_{n-1\,n} \\ u_1 & \dots & u_n \end{vmatrix}$$

$$= A \begin{vmatrix} B_{11}, & B_{12} & \dots & B_{1\,n-1}, & y_1 \\ \dots & \dots & \dots & \dots & \dots \\ B_{n-1\,1}, & B_{n-1\,2} & \dots & B_{n-1\,n-1}, & y_{n-1} \\ y_1, & y_2 & \dots & y_{n-1}, & u \end{vmatrix} \quad \dots\dots\dots\dots\dots\dots(8),$$

where
$$B_{rs} = c_{r1}e_{s1} + c_{r2}e_{s2} + \dots + c_{rn}e_{sn},$$
$$\begin{aligned} AB_{rs} &= c_{r1}(c_{s1}A_{11} + c_{s2}A_{12} + \dots) \\ &\quad + c_{r2}(c_{s1}A_{21} + c_{s2}A_{22} + \dots) \\ &\quad + \dots \\ &= -\begin{vmatrix} 0, & c_{r1}, & c_{r2} & \dots & c_{rn} \\ c_{s1}, & a_{11}, & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ c_{sn}, & a_{n1}, & a_{n2} & \dots & a_{nn} \end{vmatrix} = AB_{sr}. \end{aligned}$$

On the right-hand side of (8) all the quantities are known from (1) and (2). Thus Cy_n is known; substitute its value in the left of (6) and we have the required equation (3), which with the equations (2) forms a system of n linear equations sufficient to determine the quantities $x_1 \dots x_n$.

16. The equation

$$\begin{vmatrix} a_{11}-\lambda, & a_{12}, & a_{13} \dots & a_{1n} \\ a_{21}, & a_{22}-\lambda, & a_{23} \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1}, & a_{n2}, & a_{n3} \dots & a_{nn}-\lambda \end{vmatrix} = 0$$

(where $a_{ik}=a_{ki}$) formed by taking λ from each of the leading elements of a symmetrical determinant is of considerable importance in analysis. The following proof that its roots are real is due to Sylvester. If we denote the left-hand side of the equation by $\phi(\lambda)$ we have

$$\phi(-\lambda) = \begin{vmatrix} a_{11}+\lambda, & a_{12} & \dots & a_{1n} \\ a_{21}, & a_{22}+\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1}, & a_{n2} & \dots & a_{nn}+\lambda \end{vmatrix},$$

and hence

$$\phi(\lambda)\,\phi(-\lambda) = \begin{vmatrix} c_{11}-\lambda^2, & c_{12} & \dots & c_{1n} \\ c_{21}, & c_{22}-\lambda^2 & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{n1}, & c_{n2} & \dots & c_{nn}-\lambda^2 \end{vmatrix}$$

where $$c_{rs} = a_{r1}a_{s1} + a_{r2}a_{s2} + \dots + a_{rn}a_{sn}$$

the λ disappears, because $a_{rs}=a_{sr}$. Hence, expanding the right-hand side by Art. 22 of Chap. III.,

$$\phi(\lambda)\,\phi(-\lambda) = C - \lambda^2\Sigma C_1 + \lambda^4\Sigma C_2 - \dots + (-\lambda^2)^n.$$

Now, by IV. 9, C_1, C_2 ... are all sums of squares, the coefficient of each power of λ being the sum of squares is positive. Hence, if we equate the right-hand side of this last equation to zero by Des Cartes' rule it cannot have a negative root. Thus λ cannot be of the form $\beta\sqrt{-1}$. In order to shew that it cannot have the form $\alpha+\beta\sqrt{-1}$ we have only to write $a_{11}-\alpha=a_{11}'$, &c., and the case is reduced to the preceding.

17. The proof might also be conducted symbolically as follows.

Putting $$A^{(n)} = | a_{ik} | \qquad D_2^{(n)} = \lambda^n$$
in the result of III. 21,
$$\phi(\lambda) = (A - \lambda)^n$$
$$\phi(-\lambda) = (A + \lambda)^n;$$
$$\therefore \ \phi(\lambda)\,\phi(-\lambda) = (A^2 - \lambda^2)^n \ \ldots\ldots\ldots\ldots\ldots\ldots \ (1)$$
where the indices within the brackets mean actual powers.

On expanding (1) the coefficients of the powers of λ are even powers of A, or, passing from the symbolic to the real expansion, are the sums of squares of minors, and are hence positive. The remainder of the proof is as before.

18. We shall conclude this chapter by giving Fürstenau's method of approximating to the least roots of equations, following Baltzer's modification of it.

Let the equation be
$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n = 0 \ldots\ldots\ldots\ldots (1).$$
We shall suppose that all the roots are real and unequal.

The system of p equations
$$f(x) = 0, \quad xf(x) = 0 \ \ldots \ x^{p-1} f(x) = 0 \ldots\ldots\ldots\ldots\ldots (2)$$
is linear with respect to $1, x, x^2 \ldots x^{n+p-1}$, hence we can eliminate any $p-1$ successive quantities, say
$$x^{k+1}, \ x^{k+2} \ldots x^{k+p-1}.$$

For this purpose we multiply the p equations (2) by the complements of the elements in the first column of
$$R_{k,\,p} = \left| \begin{array}{llll} a_k, & a_{k+1}, & a_{k+2} & \cdots \\ a_{k-1}, & a_k, & a_{k+1} & \cdots \\ a_{k-2}, & a_{k-1}, & a_k & \cdots \\ \multicolumn{4}{l}{\cdots\cdots\cdots\cdots\cdots\cdots} \end{array} \right| .$$

The suffixes of R mean that the determinant (which is orthosymmetrical) begins with a_k, and is of order p. If r is greater than n, or negative, $a_r = 0$. Adding the equations so multiplied we get
$$0 = \phi_k(x) + b_1 x^{k+p} + b_2 x^{k+p+1} + \ldots + b_{n-k} x^{n+p-1} \ \ldots\ldots\ldots (3),$$

which is satisfied by a root of (1). Here

$$\phi_k(x) = \begin{vmatrix} a_0 + a_1 x + a_2 x^2 + \dots + a_k x^k, & a_{k+1}, & a_{k+2} \dots \\ a_0 x + a_1 x^2 + \dots + a_{k-1} x^k, & a_k, & a_{k+1} \dots \\ \dots\dots & & \end{vmatrix}$$

$$= \begin{vmatrix} a_0, & a_{k+1}, & a_{k+2} \dots \\ 0, & a_k, & a_{k+1} \dots \\ \dots\dots & & \end{vmatrix} + x \begin{vmatrix} a_1, & a_{k+1}, & a_{k+2} \dots \\ a_0, & a_k, & a_{k+1} \dots \\ \dots\dots & & \end{vmatrix}$$

$$+ \dots + x^k \begin{vmatrix} a_k, & a_{k+1} & \dots \\ a_{k-1}, & a_k & \dots \\ \dots\dots & & \end{vmatrix} \quad \dots\dots(4).$$

If now $x_{k+1}, x_{k+2} \dots x_n$ be roots of the equation (1) we have the $n-k$ identical equations

$$0 = \phi_k(x_{k+1}) + b_1 x_{k+1}^{k+p} + b_2 x_{k+1}^{k+p+1} + \dots + b_{n-k} x_{k+1}^{n+p-1}$$

$$\dots\dots\dots\dots\dots\dots$$

$$0 = \phi_k(x_n) + b_1 x_n^{k+p} + b_2 x_n^{k+p+1} + \dots + b_{n-k} x_n^{n+p-1}.$$

From these, by aid of (3), eliminating $b_1 \dots b_{n-k}$ we get

$$0 = \begin{vmatrix} \phi_k(x), & x^{k+p}, & x^{k+p+1} & \dots & x^{n+p-1} \\ \phi_k(x_{k+1}), & x_{k+1}^{k+p}, & x_{k+1}^{k+p+1} & \dots & x_{k+1}^{n+p-1} \\ \dots\dots & & & & \\ \phi_k(x_n), & x_n^{k+p}, & x_n^{k+p+1} & \dots & x_n^{n+p-1} \end{vmatrix}.$$

Or

$$0 = \begin{vmatrix} \dfrac{\phi_k(x)}{x^{k+p}}, & 1, & x & \dots & x^{n-k-1} \\ \dfrac{\phi_k(x_{k+1})}{x_{k+1}^{k+p}}, & 1, & x_{k+1} & \dots & x_{k+1}^{n-k-1} \\ \dots\dots & & & & \end{vmatrix}.$$

Expand according to the elements of the first column and then multiply up by x^{k+p}, and we get

$$0 = c_k \phi_k(x) + c_{k+1} \left(\frac{x}{x_{k+1}}\right)^{k+p} \phi_k(x_{k+1}) + \dots + c_n \left(\frac{x}{x_n}\right)^{k+p} \phi_k(x_n)$$

where $c_k \dots$ are independent of p.

This equation is satisfied by all the roots of (1), and if $x_{k+1} \dots x_n$ be the $n-k$ roots of greatest absolute magnitude, when p increases indefinitely the remaining roots of (1) are by the last equation those of

$$\phi_k(x) = 0.$$

Hence, if x_1 is the least root of (1), x_1, x_2 the two least roots, $x_1, x_2 \dots x_k$ the k least in absolute magnitude, then

$$x_1 = -a_0 \lim. \left(\frac{R_{1,\,p-1}}{R_{1,\,p}}\right)$$

$$x_1 x_2 = a_0 \lim. \left(\frac{R_{2,\,p-1}}{R_{2,\,p}}\right)$$

....

$$x_1 x_2 \dots x_k = (-1)^k a_0 \lim. \left(\frac{R_{k,\,p-1}}{R_{k,\,p}}\right).$$

To establish this rule completely as one of practical utility it would be necessary to shew, for instance, that x_1 lies between two successive convergents, obtained by taking two successive values of p, and that these convergents approached x_1, and did not recede from it. The method has been extended by Fürstenau and Nägelsbach to the case where the roots are not all unequal, and also to the case of imaginary roots, but the discussion of these points must be omitted here.

CHAPTER IX.

RATIONAL FUNCTIONAL DETERMINANTS.

1. If we have a series of n quantities $x, y, z \dots u, t$ we shall denote the product of all the $\frac{1}{2} n (n-1)$, differences obtained by subtracting from each number all that follow it, by

$$\zeta^{\frac{1}{2}} (x, y, z \dots u, t).$$

So that

$$\begin{aligned} \zeta^{\frac{1}{2}} (x, y, z \dots u, t) = (x-y)\,(x-z)\ \dots\ &(x-t) \\ (y-z) \dots\ &(y-t) \\ \dots\dots\dots\ & \\ &(u-t). \end{aligned}$$

This function $\zeta^{\frac{1}{2}} (x, y, z \dots u, t)$ is an alternating function of all the quantities $x, y, z \dots t$; viz. on interchanging any two of these it changes its sign, but not its absolute magnitude. It is thus of the nature of a square root, having two values equal in absolute magnitude, but opposite in sign. This is conveniently indicated by the index $\frac{1}{2}$. The product of the squares of the differences will be denoted by $\zeta (x, y, z \dots u, t)$, and is a symmetrical function. This notation is Sylvester's.

2. We have

$$\begin{vmatrix} x^{n-1}, & x^{n-2} & \dots & x, & 1 \\ y^{n-1}, & y^{n-2} & \dots & y, & 1 \\ \dots & \dots & \dots & \dots & \dots \\ t^{n-1}, & t^{n-2} & \dots & t, & 1 \end{vmatrix} = \zeta^{\frac{1}{2}} (x, y, z \dots t).$$

For the determinant on the left vanishes if any two of the quantities $x, y \dots t$ become equal, because then two rows become identical. Thus the determinant divides by the difference between each pair of the letters, being a rational function. Hence it contains $\zeta^{\frac{1}{2}}(x, y \dots t)$ as a factor. But the leading term in the determinant is $x^{n-1} y^{n-2} \dots u \,.\, 1$, which is also a term in $\zeta^{\frac{1}{2}}(x \dots t)$ with its proper sign. Thus the theorem follows.

3. Every alternating function of $x \dots t$ divides by $\zeta^{\frac{1}{2}}(x \dots t)$, for on interchanging two variables the function changes sign, and hence vanishes if they become equal, thus it divides by their difference, and therefore by $\zeta^{\frac{1}{2}}(x \dots t)$.

4. If $f_i(x)$ be a function of the i^{th} degree in x, the coefficient of whose highest term is unity, we have

$$\begin{vmatrix} f_{n-1}(x), & f_{n-2}(x) & \dots & f_1(x), & 1 \\ f_{n-1}(y), & f_{n-2}(y) & \dots & f_1(y), & 1 \\ \dots & \dots & \dots & \dots & \dots \\ f_{n-1}(t), & f_{n-2}(t) & \dots & f_1(t), & 1 \end{vmatrix} = \zeta^{\frac{1}{2}}(x, y \dots t).$$

For if we subtract the last column, multiplied by a proper number, from the last but one, the elements in this column become $x, y \dots t$. Now multiply the last two columns by the proper numbers, and subtract their sum from the last column but two, the elements of that column now become $x^2, y^2 \dots t^2$. Proceed in this way and we reduce the determinant on the right to that in Art. 2.

If the coefficients of the highest powers of x were not unity, the determinant is equal to $\zeta^{\frac{1}{2}}(x, y \dots t)$ multiplied by the product of the highest coefficients in the separate functions.

For example, if

$$f_i(x) = \frac{x(x-1)\dots(x-i+1)}{i!} = x_i.$$

$$\begin{vmatrix} x_{n-1}, & x_{n-2} & \dots & x_1, & 1 \\ y_{n-1}, & y_{n-2} & \dots & y_1, & 1 \\ \dots & \dots & \dots & \dots & \dots \\ t_{n-1}, & t_{n-2} & \dots & t_1, & 1 \end{vmatrix} = \frac{\zeta^{\frac{1}{2}}(x, y \dots t)}{(n-1)!\,(n-2)!\dots 2!}.$$

The denominator can also be written

$$2^{n-2} \,.\, 3^{n-3} \dots (n-2)^2 \,.\, (n-1).$$

5. If $$f_i(x) = a_{1i}x^{n-1} + a_{2i}x^{n-2} + \ldots + a_{ni},$$
we see by the theorem for multiplying two determinants (IV. 3) that

$$\begin{vmatrix} f_1(x_1), f_2(x_1) \ldots f_n(x_1) \\ f_1(x_2), f_2(x_2) \ldots f_n(x_2) \\ \ldots\ldots\ldots\ldots \\ f_1(x_n), f_2(x_n) \ldots f_n(x_n) \end{vmatrix} = \begin{vmatrix} a_{11} \ldots a_{1n} \\ \ldots\ldots \\ a_{n1} \ldots a_{nn} \end{vmatrix} \begin{vmatrix} x_1^{n-1}, x_1^{n-2} \ldots 1 \\ \ldots\ldots\ldots \\ x_n^{n-1}, x_n^{n-2} \ldots 1 \end{vmatrix}$$
$$= |a_{ik}|\, \zeta^{\frac{1}{2}}(x_1, x_2 \ldots x_n).$$

If $f_i(x_k) = (x_k - y_i)^{n-1}$

$$|a_{ik}| = \begin{vmatrix} 1, c_1(-y_1), c_2(-y_1)^2 \ldots (-y_1)^{n-1} \\ 1, c_1(-y_2), c_2(-y_2)^2 \ldots (-y_2)^{n-1} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ 1, c_1(-y_n), c_2(-y_n)^2 \ldots (-y_n)^{n-1} \end{vmatrix}$$
$$= C\zeta^{\frac{1}{2}}(y_1, y_2 \ldots y_n),$$

where C is the product of all the binomial coefficients of order $n-1$.

For the elements in each column of the determinant are multiplied by that power of -1, which is introduced by moving the column from its place in $\zeta^{\frac{1}{2}}$ to the place it occupies.

Thus

$$\begin{vmatrix} (x_1-y_1)^{n-1}, (x_1-y_2)^{n-1} \ldots (x_1-y_n)^{n-1} \\ (x_2-y_1)^{n-1}, (x_2-y_2)^{n-1} \ldots (x_2-y_n)^{n-1} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ (x_n-y_1)^{n-1}, (x_n-y_2)^{n-1} \ldots (x_n-y_n)^{n-1} \end{vmatrix}$$
$$= C\zeta^{\frac{1}{2}}(x_1, x_2 \ldots x_n)\,\zeta^{\frac{1}{2}}(y_1, y_2 \ldots y_n).$$

If $x_i = y_i$ this gives us $\zeta(x_1 \ldots x_n)$ in the form of a determinant.

6. We may also give still further determinant forms to the product $$\zeta^{\frac{1}{2}}(x_1, x_2 \ldots x_n)\,\zeta^{\frac{1}{2}}(y_1, y_2 \ldots y_n).$$
Thus

$$\zeta^{\frac{1}{2}}(x_1, x_2 \ldots x_n)\,\zeta^{\frac{1}{2}}(y_1, y_2 \ldots y_n) = \begin{vmatrix} x_1^{n-1} \ldots 1 \\ \ldots\ldots \\ x_n^{n-1} \ldots 1 \end{vmatrix} \begin{vmatrix} y_1^{n-1} \ldots 1 \\ \ldots\ldots \\ y_n^{n-1} \ldots 1 \end{vmatrix}$$
$$= |c_{ik}|,$$

where if we multiply by rows

$$c_{ik} = x_i^{n-1} y_k^{n-1} + x_i^{n-2} y_k^{n-2} + \dots + x_i y_k + 1$$
$$= \frac{(x_i y_k)^n - 1}{x_i y_k - 1}.$$

Or if we multiply by columns

$$c_{ik} = x_1^{n-i} y_1^{n-k} + x_2^{n-i} y_2^{n-k} + \dots + x_n^{n-i} y_n^{n-k}.$$

If we put $x_i = y_i$ and $s_i = x_1^i + x_2^i + \dots + x_n^i$ we get

$$\zeta(x_1, x_2 \dots x_n) = \begin{vmatrix} s_{2n-2}, & s_{2n-3} & \dots & s_{n-1} \\ s_{2n-3}, & s_{2n-4} & \dots & s_{n-2} \\ \dots & \dots & \dots & \dots \\ s_{n-1}, & s_{n-2} & \dots & s_0 \end{vmatrix}$$

$$= \begin{vmatrix} s_0, & s_1 & \dots & s_{n-1} \\ s_1, & s_2 & \dots & s_n \\ \dots & \dots & \dots & \dots \\ s_{n-1}, & s_n & \dots & s_{2n-2} \end{vmatrix},$$

an orthosymmetrical determinant.

7. A more general theorem is the following. Consider the array

$$\begin{matrix} x_1^{m-1}, & x_1^{m-2} & \dots & x_1, & 1 \\ x_2^{m-1}, & x_2^{m-2} & \dots & x_2, & 1 \\ \dots & \dots & \dots & \dots & \dots \\ x_n^{m-1}, & x_n^{m-2} & \dots & x_n, & 1, \end{matrix}$$

where n is greater than m. Compound it with itself, we get a determinant of the m^{th} order which is equal to the sum of the squares of the n_m determinants, obtained by taking any m different rows in the array. The determinant has for elements

$$c_{ik} = x_1^{i-1} x_1^{k-1} + \dots + x_n^{i-1} x_n^{k-1}$$
$$= s_{i+k-2}.$$

Hence, by aid of Art. 6, we get

$$\Sigma \{\zeta(x_p, x_q \dots)\} = \begin{vmatrix} s_0, & s_1 & \dots & s_{m-1} \\ s_1, & s_2 & \dots & s_m \\ \dots & \dots & \dots & \dots \\ s_{m-1}, & s_m & \dots & s_{2m-2} \end{vmatrix},$$

where $x_p, x_q \dots$ are any m of the n quantities $x_1, x_2 \dots x_n$.

8. We have clearly by Art. 2

$$\begin{vmatrix} x^n, & x^{n-1} & \dots & x, & 1 \\ \alpha_1^n, & \alpha_1^{n-1} & \dots & \alpha_1, & 1 \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_n^n, & \alpha_n^{n-1} & \dots & \alpha_n, & 1 \end{vmatrix} = \zeta^{\frac{1}{2}}(\alpha_1, \alpha_2 \dots \alpha_n) f(x)$$

where $\quad f(x) = (x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_n)$

$$= x^n - p_1 x^{n-1} + p_2 x^{n-2} - \dots + (-1)^{n-i} p_{n-i} x^i + \dots$$

Equate coefficients of x^i on both sides and we get

$$\begin{vmatrix} \alpha_1^n & \dots & \alpha_1^{i+1}, & \alpha_1^{i-1} & \dots & 1 \\ \alpha_2^n & \dots & \alpha_2^{i+1}, & \alpha_2^{i-1} & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha_n^n & \dots & \alpha_n^{i+1}, & \alpha_n^{i-1} & \dots & 1 \end{vmatrix} = \zeta^{\frac{1}{2}}(\alpha_1 \dots \alpha_n)\, p_{n-i},$$

p_{n-i} is the sum of the products $n - i$ at a time, without repetition, of the quantities $\alpha_1 \dots \alpha_n$.

9. We may write the first equation of the preceding article in the form

$$\begin{vmatrix} \alpha_1^n, & \alpha_2^n & \dots & \alpha_n^n, & x^n, & 0 \\ \alpha_1^{n-1}, & \alpha_2^{n-1} & \dots & \alpha_n^{n-1}, & x^{n-1}, & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha_1, & \alpha_2 & \dots & \alpha_n, & x, & 0 \\ 1, & 1 & \dots & 1, & 1, & 0 \\ 0, & 0 & \dots & 0, & 0, & 1 \end{vmatrix} = (-1)^n \zeta^{\frac{1}{2}}(\alpha_1, \alpha_2 \dots \alpha_n) f(x),$$

and similarly we have

$$\begin{vmatrix} \alpha_1^n, & \alpha_2^n & \dots & \alpha_n^n, & 0, & y^n \\ \alpha_1^{n-1}, & \alpha_2^{n-1} & \dots & \alpha_n^{n-1}, & 0, & y^{n-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha_1, & \alpha_2 & \dots & \alpha_n, & 0, & y \\ 1, & 1 & \dots & 1, & 0, & 1 \\ 0, & 0 & \dots & 0, & 1, & 0 \end{vmatrix} = (-1)^{n+1} \zeta^{\frac{1}{2}}(\alpha_1 \dots \alpha_n) f(y).$$

Form the product of these two determinants by rows, and we have

$$\begin{vmatrix} s_{2n}, & s_{2n-1} & \dots & s_n, & x^n \\ s_{2n-1}, & s_{2n-2} & \dots & s_{n-1}, & x^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ s_n, & s_{n-1} & \dots & s_0, & 1 \\ y^n; & y^{n-1} & \dots & 1, & 0 \end{vmatrix} = -\zeta(\alpha_1, \alpha_2 \dots \alpha_n) f(x) \cdot f(y),$$

from which by equating coefficients of the powers of x and y we get a number of theorems. s_r is now the sum of the r^{th} powers of the roots of the equation $f(x) = 0$.

10. We may extend the theorem of Art. 8 as follows: the value of the determinant

$$\begin{vmatrix} x_1^{n+r-1}, & x_1^{n+r-2} & \dots & x_1, & 1 \\ \dots & \dots & \dots & \dots & \dots \\ x_r^{n+r-1}, & x_r^{n+r-2} & \dots & x_r, & 1 \\ \alpha_1^{n+r-1}, & \alpha_1^{n+r-2} & \dots & \alpha_1, & 1 \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_n^{n+r-1}, & \alpha_n^{n+r-2} & \dots & \alpha_n, & 1 \end{vmatrix}$$

which is of the form of that in Art. 2, consists of three parts.

First the product of all the differences of all pairs of the quantities $x_1 \dots x_r$, i.e. $\zeta^{\frac{1}{2}}(x_1 \dots x_r)$, which by Art. 2 is a determinant. Secondly, the difference of all pairs of the quantities $\alpha_1 \dots \alpha_n$, i.e. $\zeta^{\frac{1}{2}}(\alpha_1 \dots \alpha_n)$. And, lastly, the product of all such quantities as

$$\begin{aligned} f(x_i) &= (x_i - \alpha_1)(x_i - \alpha_2) \dots (x_i - \alpha_n) \\ &= x_i^n - p_1 x_i^{n-1} + \dots + (-1)^{n-k} p_{n-k} x_i^k + \dots \end{aligned}$$

Hence its value is

$$\begin{vmatrix} x_1^{r-1}, & x_1^{r-2} & \dots & x_1, & 1 \\ \dots & \dots & \dots & \dots & \dots \\ x_r^{r-1}, & x_r^{r-2} & \dots & x_r, & 1 \end{vmatrix} \zeta^{\frac{1}{2}}(\alpha_1 \dots \alpha_n) f(x_1) \dots f(x_r).$$

Multiply the i^{th} row by $f(x_i)$, and then equate coefficients of $x_1^u . x_2^v . x_3^w \dots$, and we get the theorem:

If $D_{u, v, w \dots}$ is the determinant of order n formed by suppressing the columns containing the u^{th}, v^{th}, w^{th} ... powers in the array

$$\begin{matrix} \alpha_1^{n+r-1}, & \alpha_1^{n+r-2} & \dots & \alpha_1, & 1 \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_n^{n+r-1}, & \alpha_n^{n+r-2} & \dots & \alpha_n, & 1, \end{matrix}$$

then

$$D_{u, v, w \dots} = \begin{vmatrix} p_{n-u+r-1}, & p_{n-u+r-2} & \dots & p_{n-u} \\ p_{n-v+r-1}, & p_{n-v+r-2} & \dots & p_{n-v} \\ \dots & \dots & \dots & \dots \end{vmatrix} \zeta^{\frac{1}{2}}(\alpha_1, \alpha_2 \dots \alpha_n),$$

where p_k is the sum of the products k at a time of $\alpha_1 \dots \alpha_n$.

If k is negative or greater than n, $p_k = 0$, $p_0 = 1$.

11. Let us consider the determinant

$$D = \begin{vmatrix} \dfrac{1}{x_1 - \alpha_1}, & \dfrac{1}{x_1 - \alpha_2} & \dots & \dfrac{1}{x_1 - \alpha_n} \\ \dfrac{1}{x_2 - \alpha_1}, & \dfrac{1}{x_2 - \alpha_2} & \dots & \dfrac{1}{x_2 - \alpha_n} \\ \dots\dots & \dots\dots & \dots & \dots\dots \\ \dfrac{1}{x_n - \alpha_1}, & \dfrac{1}{x_n - \alpha_2} & \dots & \dfrac{1}{x_n - \alpha_n} \end{vmatrix}.$$

Multiply the i^{th} row by

$$f(x_i) = u_i = (x_i - \alpha_1)(x_i - \alpha_2) \dots (x_i - \alpha_n),$$

we get

$$u_1 u_2 \dots u_n D = \left| \frac{u_i}{x_i - \alpha_k} \right|.$$

The determinant on the right is an integral and alternating function both of the quantities $x_1 \dots x_n$ and of $\alpha_1 \dots \alpha_n$. Hence by Art. 3 it divides by

$$\zeta^{\frac{1}{2}}(x_1, x_2 \dots x_n)\, \zeta^{\frac{1}{2}}(\alpha_1, \alpha_2 \dots \alpha_n).$$

Comparing the orders of the determinant and this product we see they are the same, hence the additional factor is numerical only. To determine it, put $x_1, x_2 \dots x_n$ equal to $\alpha_1, \alpha_2 \dots \alpha_n$ respectively, all the elements except those in the leading diagonal vanish, and

$$\frac{u_i}{x_i - \alpha_i} = (x_i - \alpha_1)(x_i - \alpha_2) \dots (x_i - \alpha_{i-1})(x_i - \alpha_{i+1}) \dots (x_i - \alpha_n)$$

$$= (-1)^{i-1} (\alpha_1 - \alpha_i) \dots (\alpha_{i-1} - \alpha_i)(\alpha_i - \alpha_{i+1}) \dots (\alpha_i - \alpha_n)$$

when $x_i = \alpha_i$,

thus the determinant reduces to

$$(-1)^{\frac{n(n-1)}{2}} \zeta(\alpha_1, \alpha_2 \dots \alpha_n),$$

which determines the factor. Hence

$$D = \frac{(-1)^{\frac{n(n-1)}{2}} \zeta^{\frac{1}{2}}(x_1, x_2 \dots x_n)\, \zeta^{\frac{1}{2}}(\alpha_1, \alpha_2 \dots \alpha_n)}{u_1 u_2 \dots u_n}.$$

12. If D_{ik} is the complement of $\dfrac{1}{x_i - \alpha_k}$ in the determinant D, then D_{ik} is equal to the determinant obtained by omitting x_i and α_k on the right, multiplied by $(-1)^{i+k}$.

$$\therefore D_{ik} = (-1)^{i+k} \frac{(-1)^{\frac{n(n-1)}{2}} \zeta^{\frac{1}{2}}(x_1 \dots x_{i-1}x_{i+1} \dots x_n)\, \zeta^{\frac{1}{2}}(\alpha_1 \dots \alpha_{i-1}\alpha_{i+1} \dots \alpha_n)}{v_1 v_2 \dots v_{n-1}},$$

where

$$v_1 v_2 \dots v_{n-1} = \frac{u_1}{x_1 - \alpha_k} \cdot \frac{u_2}{x_2 - \alpha_k} \dots \frac{u_{i-1}}{x_{i-1} - \alpha_k} \cdot \frac{u_{i+1}}{x_{i+1} - \alpha_k} \dots \frac{u_n}{x_n - \alpha_k}.$$

Now if we write

$$g(z) = (z - x_1)(z - x_2) \dots (z - x_n)$$

$$\zeta^{\frac{1}{2}}(x_1 \dots x_{i-1}x_{i+1} \dots x_n)\, \zeta^{\frac{1}{2}}(\alpha_1 \dots \alpha_{i-1}\alpha_{i+1} \dots \alpha_n)$$

$$= \frac{(-1)^{i+k} \zeta^{\frac{1}{2}}(x_1 \dots x_n)\, \zeta^{\frac{1}{2}}(\alpha_1 \dots \alpha_n)}{g'(x_i) f'(\alpha_k)}$$

$$(x_1 - \alpha_k)(x_2 - \alpha_k) \dots (x_{i-1} - \alpha_k)(x_{i+1} - \alpha_k) \dots (x_n - \alpha_k) = (-1)^n \frac{g(\alpha_k)}{x_i - \alpha_k},$$

then
$$\frac{D_{ik}}{D} = -\frac{f(x_i)\, g(\alpha_k)}{f'(\alpha_k)\, g'(x_i)} \cdot \frac{1}{x_i - \alpha_k}.$$

13. The preceding article enables us to solve the system of equations

$$\frac{y_1}{x_1 - \alpha_1} + \frac{y_2}{x_1 - \alpha_2} + \dots + \frac{y_n}{x_1 - \alpha_n} = u_1$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\frac{y_1}{x_n - \alpha_1} + \frac{y_2}{x_n - \alpha_2} + \dots + \frac{y_n}{x_n - \alpha_n} = u_n,$$

viz.
$$y_k = -\frac{g(\alpha_k)}{f'(\alpha_k)} \left\{ \frac{f(x_1)}{g'(x_1)} \frac{u_1}{x_1 - \alpha_k} + \dots + \frac{f(x_n)}{g'(x_n)} \frac{u_n}{x_n - \alpha_k} \right\}.$$

In particular if $u_1 = u_2 \dots = u_n$. Since by the rule for resolving a rational fraction into partial fractions

$$\frac{f(x)}{g(x)} = \frac{(x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_n)}{(x - x_1)(x - x_2) \dots (x - x_n)}$$

$$= 1 + \Sigma \frac{f(x_i)}{g'(x_i)} \frac{1}{x - x_i},$$

we see by putting $x = \alpha_k$ in this, that

$$\frac{f(x_1)}{g'(x_1)}\frac{1}{x_1-\alpha_k} + \dots + \frac{f(x_n)}{g'(x_n)}\frac{1}{x_n-\alpha_k} = 1.$$

Hence if $u_i = 1$, $\qquad y_k = -\dfrac{g(\alpha_k)}{f'(\alpha_k)}$.

14. If in the determinant D of Art. 11 we expand each term in a series as follows

$$\frac{1}{x_i-\alpha_k} = \frac{1}{x_i} + \frac{\alpha_k}{x_i^2} + \dots + \frac{\alpha_k^p}{x_i^{p+1}} + \dots$$

we see that the term in the expansion of the determinant which multiplies $(x_1^{p+1}.\,x_2^{q+1}\dots x_n^{s+1})^{-1}$ is

$$\begin{vmatrix} \alpha_1^p, & \alpha_2^p & \dots & \alpha_n^p \\ \alpha_1^q, & \alpha_2^q & \dots & \alpha_n^q \\ \dots & \dots & \dots & \dots \\ \alpha_1^s, & \alpha_2^s & \dots & \alpha_n^s \end{vmatrix}.$$

To expand the right-hand side we have

$$\frac{1}{u_i} = \frac{1}{(x_i-\alpha_1)(x_i-\alpha_2)\dots(x_i-\alpha_n)}$$

$$= \frac{1}{x_i^n} + \frac{H_1}{x_i^{n+1}} + \dots + \frac{H_r}{x_i^{n+r}} + \dots$$

Here H_r is the sum of all the homogeneous powers and products of order r, which can be formed from the quantities $\alpha_1, \alpha_2 \dots \alpha_n$.

Now

$$\zeta^{\frac{1}{2}}(x_1, x_2 \dots x_n) = \begin{vmatrix} x_1^{n-1}, & x_1^{n-2} & \dots & x_1, & 1 \\ x_2^{n-1}, & x_2^{n-2} & \dots & x_2, & 1 \\ \dots & \dots & \dots & \dots & \dots \\ x_n^{n-1}, & x_n^{n-2} & \dots & x_n, & 1 \end{vmatrix}.$$

Multiply the i^{th} row of this determinant by the expansion of u_i^{-1}, the coefficient of $(x_1^{p+1}.\,x_2^{q+1}\dots x_n^{s+1})^{-1}$ is

$$\begin{vmatrix} H_p, & H_{p-1} & \dots & H_{p+1-n} \\ H_q, & H_{q-1} & \dots & H_{q+1-n} \\ \dots & \dots & \dots & \dots \\ H_s, & H_{s-1} & \dots & H_{s+1-n} \end{vmatrix},$$

whence we get the final equation

$$\begin{vmatrix} \alpha_1^p, & \alpha_1^q \dots & \alpha_1^s \\ \dots & \dots & \dots \\ \alpha_n^p, & \alpha_n^q \dots & \alpha_n^s \end{vmatrix} = (-1)^{\frac{n(n-1)}{2}} \begin{vmatrix} H_p, & H_q & \dots & H_s \\ \dots & \dots & \dots & \dots \\ H_{p+1-n}, & H_{q+1-n} & \dots & H_{s+1-n} \end{vmatrix} \zeta^{\frac{1}{2}}(\alpha_1 \dots \alpha_n),$$

when r is negative, $H_r = 0$; $H_0 = 1$.

15. As an example of Art. 14,

$$\begin{vmatrix} a^4, & a, & 1 \\ b^4, & b, & 1 \\ c^4, & c, & 1 \end{vmatrix} = - \begin{vmatrix} H_4, & H_1, & H_0 \\ H_3, & H_0, & 0 \\ H_2, & 0, & 0 \end{vmatrix} \begin{vmatrix} a^2, & a, & 1 \\ b^2, & b, & 1 \\ c^2, & c, & 1 \end{vmatrix}$$

$$= -(a^2 + b^2 + c^2 + bc + ca + ab)(b-c)(c-a)(a-b).$$

We may make use either of the results of Arts. 14 or 10 to evaluate determinants whose elements are sines and cosines.

For example take

$$X = \begin{vmatrix} 1, & 1, & 1, & 1 \\ \cos A, & \cos B, & \cos C, & \cos D \\ \sin A, & \sin B, & \sin C, & \sin D \\ \sin 3A, & \sin 3B, & \sin 3C, & \sin 3D \end{vmatrix}.$$

Write for the sines and cosines their exponential values, and suppose $\epsilon^{iA} = a$, &c. Then writing only the first column of the determinant

$$X = -\frac{1}{2^3} \begin{vmatrix} 1 \\ a + a^{-1} \\ a - a^{-1} \\ a^3 - a^{-3} \end{vmatrix} = -\frac{1}{2^3 (abcd)^3} \begin{vmatrix} a^3 \\ a^4 + a^2 \\ a^4 - a^2 \\ a^6 - 1 \end{vmatrix}.$$

Add the second row to the third, divide by 2 and subtract the third row from the second, thus

$$X = -\frac{1}{4(abcd)^3} \begin{vmatrix} a^3 \\ a^2 \\ a^4 \\ a^6 - 1 \end{vmatrix}.$$

Thus

$$4\,(abcd)^3\,X = \begin{vmatrix} a^2 \\ a^3 \\ a^4 \\ a^6 \end{vmatrix} + \begin{vmatrix} 1 \\ a^2 \\ a^3 \\ a^4 \end{vmatrix},$$

the first determinant

$$= a^2b^2c^2d^2 \begin{vmatrix} 1 \\ a \\ a^2 \\ a^4 \end{vmatrix} = a^2b^2c^2d^2\,\begin{aligned}(a-b)(a-c)&(a-d)(a+b+c+d)\\ (b-c)&(b-d)\\ &(c-d)\end{aligned}$$

by Art. 8. And the second, in like manner, is equal to

$$\begin{aligned}(a-b)\,(a-c)\,&(a-d)\,(bcd+acd+abd+abc)\\ (b-c)\,&(b-d)\\ &(c-d).\end{aligned}$$

Hence

$$X = \frac{(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)}{4\,a^3b^3c^3d^3} \times$$

$$[a^2b^2c^2d^2\,(a+b+c+d) + abcd\,(a^{-1}+b^{-1}+c^{-1}+d^{-1})]$$

$$= \frac{1}{4} \cdot \frac{a-b}{\sqrt{ab}} \cdots \left[\sqrt{abcd}\,(a+b+c+d) + \frac{1}{\sqrt{abcd}}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\right].$$

Hence if $\qquad 2S = A + B + C + D$

$$X = -2^5 \,.\, \Pi \sin \tfrac{1}{2}\,(A-B)\,[\cos(S+A) + \cos(S+B)$$
$$+ \cos(S+C) + \cos(S+D)].$$

16. If we differentiate the determinant of Art. 11 with respect to x_i, the elements of the i^{th} row become

$$\frac{-1}{(x_i-\alpha_1)^2},\ \frac{-1}{(x_i-\alpha_2)^2} \cdots \frac{-1}{(x_i-\alpha_n)^2}.$$

And thus

$$(-1)^n \frac{d^n D}{dx_1 dx_2 \ldots dx_n} = \left|\frac{1}{(x_i-\alpha_k)^2}\right|$$
$$= B.$$

We shall now shew that

$$\frac{B}{D} = \left\{ \begin{matrix} \frac{1}{x_1 - \alpha_1}, & \frac{1}{x_1 - \alpha_2} & \cdots & \frac{1}{x_1 - \alpha_n} \\ \frac{1}{x_2 - \alpha_1}, & \frac{1}{x_2 - \alpha_2} & \cdots & \frac{1}{x_2 - \alpha_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{x_n - \alpha_1}, & \frac{1}{x_n - \alpha_2} & \cdots & \frac{1}{x_n - \alpha_n} \end{matrix} \right\}.$$

Where $\{\ \}$ means that the function on the right is to be formed like a determinant, only all the signs are positive instead of alternating.

Multiply the i^{th} row of B by u_i^2, then

$$(u_1 u_2 \ldots u_n)^2 B = \left| \frac{u_i^2}{(x_i - \alpha_k)^2} \right| \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The determinant on the right is an integral and alternating function, both of $x_1, x_2 \ldots x_n$ and of $\alpha_1, \alpha_2 \ldots \alpha_n$, hence it divides by

$$\zeta^{\frac{1}{2}}(x_1, x_2 \ldots x_n)\, \zeta^{\frac{1}{2}}(\alpha_1 \ldots \alpha_n).$$

If the quotient is $\phi(x_1, x_2 \ldots x_n)$, this is symmetrical with regard to each of the variables, and of order $n-1$. Thus

$$\frac{B}{D} = (-1)^{\frac{n(n-1)}{2}} \frac{\phi(x_1, x_2 \ldots x_n)}{u_1 u_2 \ldots u_n}.$$

Now, by repeated use of the rule for resolving a fraction into partial fractions

$$\frac{\phi(x_1 \ldots x_n)}{f(x_1)} = \sum_i \frac{\phi(\alpha_i \ldots x_n)}{f'(\alpha_i)(x_1 - \alpha_i)},$$

$$\frac{\phi(\alpha_i, x_2 \ldots x_n)}{f(x_2)} = \sum_k \frac{\phi(\alpha_i, \alpha_k \ldots x_n)}{f'(\alpha_k)(x_2 - \alpha_k)},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

we get finally

$$\frac{\phi(x_1, x_2 \ldots x_n)}{u_1 u_2 \ldots u_n}$$

$$= \sum \frac{\phi(\alpha_i, \alpha_k \ldots \alpha_p)}{f'(\alpha_i) f'(\alpha_k) \ldots f'(\alpha_p)(x_1 - \alpha_i)(x_2 - \alpha_k) \ldots (x_n - \alpha_p)} \ldots\ldots(2).$$

Now, in the first place, in the combination $i, k \ldots p$, no repetition can occur, for in the product

$$\frac{B(u_1 \ldots u_n)^2}{\zeta^{\frac{1}{2}}(x_1 \ldots x_n)\, \zeta^{\frac{1}{2}}(\alpha_1 \ldots \alpha_n)},$$

not only B, but also $\dfrac{\{f(x_1)\}^2}{x_2 - x_1}$ vanishes if x_1 and x_2 both coincide with α_h. Hence on the right of (2) we must write for $i, k \ldots p$ all permutations of $1, 2 \ldots n$.

Now if we write $\alpha_i, \alpha_k \ldots \alpha_p$ for $x_1, x_2 \ldots x_n$ respectively, only a single term of $(u_1 \ldots u_n)^2 B$ remains, viz.

$$\pm [f'(\alpha_i), f'(\alpha_k) \ldots f'(\alpha_p)]^2,$$

while

$$\begin{aligned} \zeta^{\frac{1}{2}}(x_1, x_2 \ldots x_n) &= \zeta^{\frac{1}{2}}(\alpha_i, \alpha_k \ldots \alpha_p) \\ &= \pm\, \zeta^{\frac{1}{2}}(\alpha_1, \alpha_2 \ldots \alpha_n), \end{aligned}$$

the ambiguous sign being the same for both. Thus

$$\begin{aligned} \phi(\alpha_i, \alpha_k \ldots \alpha_p) &= \frac{[f'(\alpha_i) f'(\alpha_k) \ldots f'(\alpha_p)]^2}{\zeta^{\frac{1}{2}}(\alpha_1, \alpha_2 \ldots \alpha_n)} \\ &= (-1)^{\frac{n(n-1)}{2}} f'(\alpha_i) f'(\alpha_k) \ldots f'(\alpha_p). \end{aligned}$$

Thus

$$\frac{B}{D} = \Sigma \frac{1}{(x_1 - \alpha_i)(x_2 - \alpha_k) \ldots (x_n - \alpha_k)}:$$

where $i, k \ldots p$ is to be a permutation of $1, 2 \ldots n$. This proves the theorem as stated at the beginning.

17. The coefficients in the expansion of the rational fraction

$$\frac{1 + b_1 x + b_2 x^2 + \ldots}{1 + a_1 x + a_2 x^2 + \ldots},$$

in ascending powers of x can be represented as determinants. Viz. if the expansion is

$$1 + P_1 x + P_2 x^2 + \ldots$$

we have

$$(1 + b_1 x + b_2 x^2 + \ldots) = (1 + P_1 x + P_2 x^2 + \ldots)(1 + a_1 x + a_2 x^2 + \ldots),$$

and hence equating coefficients

$$
\begin{aligned}
&P_1 &&= b_1 - a_1\\
&a_1P_1 + P_2 &&= b_2 - a_2\\
&a_2P_1 + a_1P_2 + P_3 &&= b_3 - a_3\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots &&\\
&a_{n-1}P_1 + a_{n-2}P_2 + \dots + P_n &&= b_n - a_n,
\end{aligned}
$$

a system of equations to find P_n. The determinant of the system is unity. Hence, if after solving by VIII. 1 we move the last column to the first place, and change the sign of this column

$$
P_n = (-1)^n \begin{vmatrix} a_1 - b_1, & 1 & & & \\ a_2 - b_2, & a_1, & 1 & & \\ a_3 - b_3, & a_2, & a_1, & 1 & \\ \dots\dots & \dots & \dots & \dots & 1 \\ a_n - b_n, & a_{n-1}, & a_{n-2} & \dots\dots & a_1 \end{vmatrix}
$$

$$
= (-1)^n \begin{vmatrix} 1, & 1, & ., & ., & . \\ b_1, & a_1, & 1, & ., & . \\ b_2, & a_2, & a_1, & 1, & . \\ b_3, & a_3, & a_2, & a_1, & 1 \\ . & . & . & . & . & . \end{vmatrix};
$$

as we see by subtracting the first column from the second in the latter determinant.

CHAPTER X.

ON JACOBIANS AND HESSIANS.

1. If $y_1, y_2 \ldots y_n$ be n functions of the n independent variables $x_1, x_2 \ldots x_n$, and if

$$a_{ik} = \frac{dy_i}{dx_k},$$

then the determinant $| a_{ik} |$ is called the Jacobian of the functions $y_1 \ldots y_n$ with respect to the variables $x_1 \ldots x_n$. The name was given by Prof. Sylvester after Jacobi, who first studied these functions.

The notations

$$\frac{d(y_1, y_2 \ldots y_n)}{d(x_1, x_2 \ldots x_n)}; \quad J(y_1, y_2 \ldots y_n)$$

have been employed for Jacobians, each of which has its advantages. The first renders evident the remarkable analogy between Jacobians and ordinary differential coefficients. The second is useful when there is no doubt as to the independent variables.

If the y's are explicit functions, the Jacobian is formed by direct differentiation.

2. If the functions $y_1 \ldots y_n$ are not independent, but are connected by an equation

$$\phi(y_1, y_2 \ldots y_n) = 0,$$

the Jacobian vanishes. For if we differentiate this equation with respect to x_k, we get

$$\frac{d\phi}{dy_1}\frac{dy_1}{dx_k} + \frac{d\phi}{dy_2}\frac{dy_2}{dx_k} + \ldots + \frac{d\phi}{dy_n}\frac{dy_n}{dx_k} = 0,$$

where $k = 1, 2 \dots n$. Eliminating

$$\frac{d\phi}{dy_1}, \frac{d\phi}{dy_2} \cdots \frac{d\phi}{dy_n},$$

from these equations we get (VIII. 3)

$$\frac{d(y_1, y_2 \cdots y_n)}{d(x_1, x_2 \dots x_n)} = 0.$$

3. If the functions y are fractions with the same denominator, so that

$$y_i = \frac{u_i}{u},$$

$$u^2 \frac{dy_i}{dx_k} = u\frac{du_i}{dx_k} - u_i \frac{du}{dx_k}.$$

Thus

$$u^{2n+1}\frac{d(y_1 \cdots y_n)}{d(x_1 \dots x_n)} = \begin{vmatrix} u, & 0, & & 0 \\ u_1, & u\frac{du_1}{dx_1} - u_1\frac{du}{dx_1} & \dots & u\frac{du_1}{dx_n} - u_1\frac{du}{dx_n} \\ \dots & \dots & \dots & \dots \\ u_n, & u\frac{du_n}{dx_1} - u_n\frac{du}{dx_1} & \dots & u\frac{du_n}{dx_n} - u_n\frac{du}{dx_n} \end{vmatrix}.$$

Add the first column multiplied by $\frac{du}{dx_i}$ to the $(i+1)^{\text{st}}$ column, and we get

$$u^{2n+1}\frac{d(y_1 \cdots y_n)}{d(x_1 \dots x_n)} = \begin{vmatrix} u, & u\frac{du}{dx_1} & \dots & u\frac{du}{dx_n} \\ u_1, & u\frac{du_1}{dx_1} & \dots & u\frac{du_1}{dx_n} \\ \dots & \dots & \dots & \dots \\ u_n, & u\frac{du_n}{dx_1} & \dots & u\frac{du_n}{dx_n} \end{vmatrix},$$

whence dividing each of the last n columns by u

$$\frac{d(y_1 \cdots y_n)}{d(x_1 \dots x_n)} = \frac{1}{u^{n+1}} \begin{vmatrix} u, & \frac{du}{dx_1} & \dots & \frac{du}{dx_n} \\ u_1, & \frac{du_1}{dx_1} & \dots & \frac{du_1}{dx_n} \\ \dots & \dots & \dots & \dots \\ u_n, & \frac{du_n}{dx_1} & \dots & \frac{du_n}{dx_n} \end{vmatrix}.$$

4. The determinant on the right has been denoted by $K(u, u_1 \ldots u_n)$. It has interesting properties of its own. For example, since the Jacobian vanishes if the quantities $y_1 \ldots y_n$ are related by an equation, it follows that

$$K(u, u_1 \ldots u_n) = 0$$

if a homogeneous relation exists between $u, u_1 \ldots u_n$.

If
$$u_i = \frac{v_i}{t},$$

it is readily shewn that

$$K(u, u_1 \ldots u_n) = \frac{1}{t^{n+1}} K(v, v_1 \ldots v_n).$$

5. If the functions $y_1 \ldots y_n$ possess a common factor, so that

$$y_i = u_i u,$$

$$\frac{d(y_1 \ldots y_n)}{d(x_1 \ldots x_n)} = \frac{1}{u} \begin{vmatrix} u, & 0 & & 0 \\ u_1, & u\dfrac{du_1}{dx_1} + u_1\dfrac{du}{dx_1} & \ldots & u\dfrac{du_1}{dx_n} + u_1\dfrac{du}{dx_n} \\ \ldots & \ldots & \ldots & \ldots \\ u_n, & u\dfrac{du_n}{dx_1} + u_n\dfrac{du}{dx_1} & \ldots & u\dfrac{du_n}{dx_n} + u_n\dfrac{du}{dx_n} \end{vmatrix}.$$

In this determinant multiply the first column by $\dfrac{du}{dx_i}$, and subtract it from the $(i+1)^{\text{st}}$ column, then

$$\frac{d(y_1 \ldots y_n)}{d(x_1 \ldots x_n)} = u^{n-1} \begin{vmatrix} u, & -\dfrac{du}{dx_1} & \ldots & -\dfrac{du}{dx_n} \\ u_1, & \dfrac{du_1}{dx_1} & \ldots & \dfrac{du_1}{dx_n} \\ \ldots & \ldots & \ldots & \ldots \\ u_n, & \dfrac{du_n}{dx_1} & \ldots & \dfrac{du_n}{dx_n} \end{vmatrix}$$

$$= 2u^n \frac{d(u_1, u_2 \ldots u_n)}{d(x_1 \ldots x_n)} - u^{n-1} K(u, u_1 \ldots u_n).$$

6. If the functions $y_1 \ldots y_n$ are given only as implicit functions of $x_1 \ldots x_n$ by means of the n equations

$$F_1(y_1 \ldots y_n, x_1 \ldots x_n) = 0 \ldots F_n(y_1 \ldots y_n, x_1 \ldots x_n) = 0,$$

then

$$\frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)} = (-1)^n \frac{d(F_1 \dots F_n)}{d(x_1 \dots x_n)} \div \frac{d(F_1 \dots F_n)}{d(y_1 \dots y_n)}.$$

For if we differentiate the i^{th} of the given equations with respect to x_k we get

$$\frac{dF_i}{dy_1}\frac{dy_1}{dx_k} + \frac{dF_i}{dy_2}\frac{dy_2}{dx_k} + \dots + \frac{dF_i}{dy_n} \cdot \frac{dy_n}{dx_k} = -\frac{dF_i}{dx_k}.$$

Thus by the rule for multiplying two determinants (IV. 3)

$$(-1)^n \left|\frac{dF_i}{dx_k}\right| = \left|\frac{dF_i}{dy_k}\right| \cdot \left|\frac{dy_i}{dx_k}\right|,$$

or $$(-1)^n \frac{d(F_1 \dots F_n)}{d(x_1 \dots x_n)} = \frac{d(F_1 \dots F_n)}{d(y_1 \dots y_n)} \cdot \frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)};$$

which proves the theorem.

(i) If F_i does not contain $x_1 \dots x_{i-1}$, then in the determinant

$$\frac{d(F_1 \dots F_n)}{d(x_1 \dots x_n)}$$

all elements below the leading diagonal vanish, and it reduces to

$$\frac{dF_1}{dx_1} \cdot \frac{dF_2}{dx_2} \cdots \frac{dF_n}{dx_n}.$$

(ii) If $$F_i = -y_i + f_i(x_1 \dots x_n),$$

then $$\frac{d(F_1 \dots F_n)}{d(y_1 \dots y_n)} = (-1)^n,$$

and $$\frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)} = \frac{d(f_1 \dots f_n)}{d(x_1 \dots x_n)}.$$

(iii) If from the given system we deduce by elimination

$$\begin{aligned} y_1 &= \phi_1(x_1, x_2 \dots x_n) \\ y_2 &= \phi_2(y_1, x_2 \dots x_n) \\ y_3 &= \phi_3(y_1, y_2, x_3 \dots x_n) \\ &\dots\dots\dots\dots\dots\dots \\ y_n &= \phi_n(y_1 \dots y_{n-1}, x_n). \end{aligned}$$

Since

$$\frac{d\phi_i}{dy_1} \cdot \frac{dy_1}{dx_k} + \dots + \frac{d\phi_i}{dy_{i-1}} \cdot \frac{dy_{i-1}}{dx_k} + \frac{d\phi_i}{dx_k} = \frac{dy_i}{dx_k},$$

we have

$$\begin{vmatrix} \frac{d\phi_1}{dx_1}, & \frac{d\phi_1}{dx_2}, & \frac{d\phi_1}{dx_3} & \cdots \\ 0, & \frac{d\phi_2}{dx_2}, & \frac{d\phi_2}{dx_3} & \cdots \\ 0, & 0, & \frac{d\phi_3}{dx_3} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = \begin{vmatrix} 1, & 0, & 0 & \cdots \\ -\frac{d\phi_2}{dy_1} & 1, & 0 & \cdots \\ -\frac{d\phi_3}{dy_1}, & -\frac{d\phi_3}{dy_2}, & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} \frac{d(y_1 \ldots y_n)}{d(x_1 \ldots x_n)}.$$

It follows then that

$$\frac{d(y_1 \ldots y_n)}{d(x_1 \ldots x_n)} = \frac{d\phi_1}{dx_1} \cdot \frac{d\phi_2}{dx_2} \cdots \frac{d\phi_n}{dx_n};$$

thus if $$\frac{d(y_1 \ldots y_n)}{d(x_1 \ldots x_n)} = 0,$$

we must have $$\frac{d\phi_1}{dx_1} \frac{d\phi_2}{dx_2} \cdots \frac{d\phi_n}{dx_n} = 0,$$

i.e. we must have $$\frac{d\phi_i}{dx_i} = 0,$$

where i is some number between 1 and n. Hence ϕ_i does not contain x_i. That is to say, we have

$$y_i = \phi_i(y_1 \ldots y_{i-1}, x_{i+1} \ldots x_n)$$

now $$y_{i+1} = \phi_{i+1}(y_1 \ldots y_i, x_{i+1} \ldots x_n).$$

Eliminate x_{i+1} between these, and we obtain

$$y_{i+1} = \psi_{i+1}(y_1 \ldots y_i, x_{i+2} \ldots x_n),$$

so that y_{i+1} does not contain x_{i+1}. Similarly we can shew that y_{i+2} does not contain x_{i+2}, and so on; finally y_n is independent of x_n or

$$y_n = \psi_n(y_1 \ldots y_{n-1}).$$

So that if the Jacobian of $y_1 \ldots y_n$ vanishes these functions are not independent. This is the converse of the theorem of Art. 2.

7. If $z_1 \ldots z_n$ are functions of $y_1 \ldots y_n$, and these again functions of $x_1 \ldots x_n$; then

$$\frac{d(z_1 \ldots z_n)}{d(x_1 \ldots x_n)} = \frac{d(z_1 \ldots z_n)}{d(y_1 \ldots y_n)} \cdot \frac{d(y_1 \ldots y_n)}{d(x_1 \ldots x_n)}.$$

For since

$$\frac{dz_i}{dx_k} = \frac{dz_i}{dy_1} \cdot \frac{dy_1}{dx_k} + \frac{dz_i}{dy_2} \cdot \frac{dy_2}{dx_k} + \ldots + \frac{dz_i}{dy_n} \cdot \frac{dy_n}{dx_k}$$

we have

$$\left| \frac{dz_i}{dx_k} \right| = \left| \frac{dz_i}{dy_k} \right| \times \left| \frac{dy_i}{dx_k} \right|,$$

which proves the theorem.

In like manner, if $z_1 \ldots z_m$ are given as functions of $y_1 \ldots y_n$, and these given as functions of $x_1 \ldots x_m$; then

$$\frac{d(z_1 \ldots z_m)}{d(x_1 \ldots x_m)} = 0, \qquad \text{if } m > n.$$

But if $m < n$

$$\frac{d(z_1 \ldots z_m)}{d(x_1 \ldots x_m)} = \Sigma \frac{d(z_1, z_2 \ldots z_m)}{d(y_t, y_u, y_v \ldots)} \cdot \frac{d(y_t, y_u, y_v \ldots)}{d(x_1, x_2 \ldots x_m)},$$

where for $t, u, v \ldots$ we take all m-ads in n (IV. 2).

8. If $f_1 \ldots f_n$ are independent functions of $x_1 \ldots x_n$, then $x_1 \ldots x_n$ are independent functions of $f_1 \ldots f_n$, and we have

$$\frac{d(f_1 \ldots f_n)}{d(x_1 \ldots x_n)} \cdot \frac{d(x_1 \ldots x_n)}{d(f_1 \ldots f_n)} = 1.$$

For differentiating f_i with respect to f_k we must consider $x_1 \ldots x_n$ to be functions of $f_1 \ldots f_n$. Thus

$$\frac{df_i}{dx_1} \cdot \frac{dx_1}{df_k} + \frac{df_i}{dx_2} \cdot \frac{dx_2}{df_k} + \ldots + \frac{df_i}{dx_n} \cdot \frac{dx_n}{df_k}$$

is equal to unity or zero, according as k is or is not equal to i. Hence

$$\left| \frac{df_i}{dx_k} \right| \cdot \left| \frac{dx_i}{df_k} \right| = 1.$$

For in the product only the elements in the leading diagonal do not vanish, and these are all equal to unity.

9. If $$A = \left| \frac{df_i}{dx_k} \right|, \qquad B = \left| \frac{dx_i}{df_k} \right|,$$

and A_{ik}, B_{ik} are the complements of $\frac{df_i}{dx_k}$ and $\frac{dx_i}{df_k}$, in these two determinants we have

$$A \frac{dx_i}{df_k} = A_{ki}, \qquad B \frac{df_i}{dx_k} = B_{ki}.$$

Also
$$A\frac{d(x_1\dots x_m)}{d(f_1\dots f_m)}=\frac{d(f_{m+1}\dots f_n)}{d(x_{m+1}\dots x_n)},$$
$$B\frac{d(f_1\dots f_m)}{d(x_1\dots x_m)}=\frac{d(x_{m+1}\dots x_n)}{d(f_{m+1}\dots f_n)}.$$

For we have just seen that

$$\frac{df_1}{dx_1}\cdot\frac{dx_1}{df_k}+\frac{df_1}{dx_2}\cdot\frac{dx_2}{df_k}+\dots+\frac{df_1}{dx_n}\cdot\frac{dx_n}{df_k}=0$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\frac{df_k}{dx_1}\cdot\frac{dx_1}{df_k}+\frac{df_k}{dx_2}\cdot\frac{dx_2}{df_k}+\dots+\frac{df_k}{dx_n}\cdot\frac{dx_n}{df_k}=1$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\frac{df_n}{dx_1}\cdot\frac{dx_1}{df_k}+\frac{df_n}{dx_2}\cdot\frac{dx_2}{df_k}+\dots+\frac{df_n}{dx_n}\cdot\frac{dx_n}{df_k}=0.$$

Multiply these equations by $A_{1i}, A_{2i}\dots A_{ni}$ respectively and add, then (III. 11).

$$A\frac{dx_i}{df_k}=A_{ki}.$$

Similarly we can shew that

$$B\frac{df_i}{dx_k}=B_{ki}.$$

Again we have (V. 6)

$$\begin{vmatrix} A_{11} & \dots & A_{1m} \\ \dots & \dots & \dots \\ A_{m1} & \dots & A_{mm}\end{vmatrix}=A^{m-1}\frac{d(f_{m+1}\dots f_n)}{d(x_{m+1}\dots x_n)}.$$

Substitute in the left for A_{ik} the value just found, and we get

$$A^m\frac{d(x_1\dots x_m)}{d(f_1\dots f_m)}=A^{m-1}\frac{d(f_{m+1}\dots f_n)}{d(x_{m+1}\dots x_n)},$$

which on dividing by A^{m-1} gives the result required.

The last equation is proved in a similar way.

10. If we suppose the functions $f_1\dots f_n$ to depend on t, we have by (III. 16)

$$\frac{dA}{dt}=\Sigma A_{ik}\frac{d^2f_i}{dt\,dx_k}\qquad(i,k=1,2\dots n),$$

and $$A_{ik} = A\,\frac{dx_k}{df_i};$$

$$\therefore\quad \frac{dA}{dt} = A\Sigma\left(\frac{d^2f_i}{dt\,dx_1}\cdot\frac{dx_1}{df_i} + \frac{d^2f_i}{dt\,dx_2}\cdot\frac{dx_2}{df_i} + \dots\right)$$
$$= A\Sigma\,\frac{d}{df_i}\left(\frac{df_i}{dt}\right),$$

or $$\frac{d}{dt}\cdot\log A = \Sigma\,\frac{d}{df_i}\left(\frac{df_i}{dt}\right).$$

A similar relation holds for B.

11. The relations between Jacobians present great resemblance to the ordinary formulæ in the differential calculus.

Thus the formulæ

$$\frac{d(z_1 \dots z_n)}{d(x_1 \dots x_n)} = \frac{d(z_1 \dots z_n)}{d(y_1 \dots y_n)}\cdot\frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)},$$
$$\frac{d(f_1 \dots f_n)}{d(x_1 \dots x_n)}\cdot\frac{d(x_1 \dots x_n)}{d(f_1 \dots f_n)} = 1,$$

are the analogues of

$$\frac{dz}{dx} = \frac{dz}{dy}\cdot\frac{dy}{dx} \text{ and } \frac{dy}{dx}\cdot\frac{dx}{dy} = 1.$$

This analogy, which was perceived by Jacobi, led Bertrand to devise a new definition of a Jacobian. Let $f_1 \dots f_n$ be n functions of the variables $x_1 \dots x_n$. Now if we give to the variables n distinct series of increments

$$\begin{matrix} d_1x_1, & d_1x_2 & \dots & d_1x_n \\ d_2x_1, & d_2x_2 & \dots & d_2x_n \\ \dots & \dots & \dots & \dots \\ d_nx_1, & d_nx_2 & \dots & d_nx_n \end{matrix} \qquad (1),$$

let the corresponding increments of the functions be

$$\begin{matrix} d_1f_1, & d_1f_2 & \dots & d_1f_n \\ d_2f_1, & d_2f_2 & \dots & d_2f_n \\ \dots & \dots & \dots & \dots \\ d_nf_1, & d_nf_2 & \dots & d_nf_n \end{matrix} \qquad (2).$$

Then just as the differential coefficient of a single function of a single variable is defined to be the limiting ratio of corresponding increments of the function and variable; the Jacobian of the functions $f_1 \dots f_n$ of the n variables $x_1 \dots x_n$ is defined to be the limiting ratio of the determinants of the systems of increments (2) and (1). That this leads to the same Jacobian as before is plain from the equation

$$d_k f_i = \frac{df_i}{dx_1} d_k x_1 + \frac{df_i}{dx_2} d_k x_2 + \dots + \frac{df_i}{dx_n} d_k x_n,$$

which gives (IV. 3)

$$| d_k f_i | = | d_k x_i | . \left| \frac{df_i}{dx_k} \right|,$$

or

$$\frac{| d_k f_i |}{| d_k x_i |} = \frac{d(f_1 \dots f_n)}{d(x_1 \dots x_n)},$$

according to our former definition.

Using this new definition we can prove all our former theorems. Let us use it to prove the first of the above equations, viz. the theorem of Art. 7. If the system of increments given to $x_1 \dots x_n$ be

$$\begin{array}{c} d_1 x_1 \ \dots \ d_1 x_n \\ \dots\dots\dots\dots\dots \\ d_n x_1 \ \dots \ d_n x_n, \end{array}$$

let the corresponding systems for $y_1 \dots y_n$ and $z_1 \dots z_n$ be

$$\begin{array}{cc} d_1 y_1 \dots d_1 y_n & d_1 z_1 \dots d_1 z_n \\ \dots\dots\dots\dots & \dots\dots\dots\dots \\ d_n y_1 \dots d_n y_n & d_n z_1 \dots d_n z_n. \end{array}$$

Then we have identically

$$\frac{| d_i z_k |}{| d_i x_k |} = \frac{| d_i z_k |}{| d_i y_k |} \cdot \frac{| d_i y_k |}{| d_i x_k |};$$

or by definition,

$$\frac{d(z_1 \dots z_n)}{d(x_1 \dots x_n)} = \frac{d(z_1 \dots z_n)}{d(y_1 \dots y_n)} \cdot \frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)}.$$

12. We can also, using alternate numbers, obtain a symbolic expression for the Jacobian, from which the ordinary results follow. Viz., $y_1 \dots y_n$, being n functions of $x_1 \dots x_n$, let

$$y = e_1 y_1 + e_2 y_2 + \dots + e_n y_n,$$
$$x = e_1 x_1 + e_2 x_2 + \dots + e_n x_n.$$

Then

$$\frac{dy}{dx_i} = e_1 \frac{dy_1}{dx_i} + e_2 \frac{dy_2}{dx_i} + \dots + e_n \frac{dy_n}{dx_i},$$

whence (I. 19)

$$\frac{dy}{dx_1} \cdot \frac{dy}{dx_2} \cdots \frac{dy}{dx_n} = \begin{vmatrix} \frac{dy_1}{dx_1} \cdots \frac{dy_1}{dx_n} \\ \cdots\cdots\cdots \\ \frac{dy_n}{dx_1} \cdots \frac{dy_n}{dx_n} \end{vmatrix}$$

$$= \frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)} \quad\dots\dots\dots\dots(1).$$

But now

$$\frac{dy}{dx_i} = \frac{dy}{dx} \cdot \frac{dx}{dx_i}$$

$$= e_i \frac{dy}{dx}.$$

Thus the above equation (1) becomes

$$\left(\frac{dy}{dx}\right)^n = \frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)}.$$

From which symbolical equation we can deduce our former theorems.

For example the equation

$$\left(\frac{dy}{dx}\right)^n \left(\frac{dx}{dy}\right)^n = 1$$

gives at once

$$\frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)} \cdot \frac{d(x_1 \dots x_n)}{d(y_1 \dots y_n)} = 1.$$

13. Jacobians occur in changing the variables in a multiple definite integral. Let us transform the integral

$$I = \iint \dots F(y_1 \dots y_n)\, dy_1 \dots dy_n$$

to an integral with respect to $x_1 \dots x_n$, the functions $y_1 \dots y_n$ being supposed given functions of $x_1 \dots x_n$.

We proceed in the manner used by Lagrange to transform a triple integral. Beginning with y_n we have to find the sum of the quantities

$$Fdy_n,$$

while $y_1, y_2 \cdots y_{n-1}$ remain constant. This gives us

$$0 = \frac{dy_1}{dx_1} dx_1 + \frac{dy_1}{dx_2} dx_2 + \ldots + \frac{dy_1}{dx_n} dx_n$$

$$0 = \frac{dy_2}{dx_1} dx_1 + \frac{dy_2}{dx_2} dx_2 + \ldots + \frac{dy_2}{dx_n} dx_n$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$dy_n = \frac{dy_n}{dx_1} dx_1 + \frac{dy_n}{dx_2} dx_2 + \ldots + \frac{dy_n}{dx_n} dx_n.$$

Solving this to find dx_n we get (VIII. 1)

$$J_{n-1}\, dy_n = J_n\, dx_n,$$

where
$$J_r = \frac{d(y_1, y_2 \cdots y_r)}{d(x_1, x_2 \ldots x_r)}.$$

Hence we must replace dy_n by $\frac{J_n}{J_{n-1}} dx_n$, and

$$I = \int \ldots F dy_1 \ldots dy_n = \int \ldots F \frac{J_n}{J_{n-1}} dy_1 \ldots dy_{n-1} dx_n,$$

the limits of x_n being determined from those of y_n.

In this integral begin by integrating with respect to y_{n-1}. We have to find the sum of the quantities $F.\frac{J_n}{J_{n-1}} dy_{n-1}$, while $y_1 \ldots y_{n-2}, x_n$ remain constant, so that

$$0 = \frac{dy_1}{dx_1} dx_1 + \ldots + \frac{dy_1}{dx_{n-1}} dx_{n-1}$$

$$0 = \frac{dy_2}{dx_1} dx_1 + \ldots + \frac{dy_2}{dx_{n-1}} dx_{n-1}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$dy_{n-1} = \frac{dy_{n-1}}{dx_1} dx_1 + \ldots + \frac{dy_{n-1}}{dx_{n-1}} dx_{n-1},$$

which gives

$$J_{n-2}\, dy_{n-1} = J_{n-1}\, dx_{n-1}.$$

Thus dy_{n-1} is to be replaced by $\frac{J_{n-1}}{J_{n-2}} dx_{n-1}$, and $F.\frac{J_n}{J_{n-1}} dy_{n-1}$

by $F.\frac{J_n}{J_{n-1}}.\frac{J_{n-1}}{J_{n-2}}dx_{n-1}$. Hence the limits being properly determined
$$I=\int\ldots F\frac{J_n}{J_{n-2}}dy_1\ldots dy_{n-2}\,dx_{n-1}\,dx_n.$$

Similarly if we began by integrating with respect to y_{n-2} we should get a system of equations which would give us

$$dy_{n-2}=\frac{J_{n-2}}{J_{n-3}}dx_{n-2},$$

and

$$I=\int\ldots F\frac{J_n}{J_{n-3}}dy_1\ldots dy_{n-3}\,dx_{n-2}\,dx_{n-1}\,dx_n.$$

Proceeding in this way we should finally obtain

$$I=\int\ldots F\frac{J_n}{J_1}dy_1\,dx_2\ldots dx_n.$$

Then we integrate with respect to y_1, subject to the equations

$$dx_2=0,\ dx_3=0\ \ldots\ dx_n=0,$$

so that we must replace dy_1 by $\frac{dy_1}{dx_1}dx_1$, i.e. J_1dx_1.

Thus
$$I=\int\ldots FJ_n\,dx_1\,dx_2\ldots dx_n$$
$$=\int\ldots F(x)\frac{d(y_1,y_2\ldots y_n)}{d(x_1,x_2\ldots x_n)}dx_1dx_2\ldots dx_n.$$

$F(x)$ being the result of substituting in F for $y_1\ldots y_n$ their values in terms of $x_1\ldots x_n$.

14. As an example let us consider the following determinant of definite integrals due to Tissot, we shall however follow Enneper's proof.

Let $a_1, a_2\ldots a_n$ be n constant quantities in ascending order of magnitude, and let

$$\phi_m(x_m)=(x_m-a_1)^{p_1}(x_m-a_2)^{p_2}\ldots(x_m-a_m)^{p_m}$$
$$(a_{m+1}-x_m)^{p_{m+1}}\ldots(a_n-x_m)^{p_n},$$

where $p_1, p_2\ldots p_n$ are either positive proper fractions or any real negative numbers. The determinant to be considered is then

$$D = \begin{vmatrix} J_{11} \dots J_{1n} \\ \dots\dots\dots \\ J_{n1} \dots J_{nn} \end{vmatrix},$$

where

$$J_{ik} = \int_{a_k}^{a_{k+1}} \frac{x_k^{i-1} \epsilon^{-x_k}}{\phi_k(x_k)} dx_k$$

$(a_{n+1} = \infty)$.

Thus

$$D = (-1)^{\frac{n(n-1)}{2}} \int_{a_1}^{a_2} dx_1 \int_{a_2}^{a_3} dx_2 \dots \int_{a_n}^{\infty} dx_n \frac{\zeta^{\frac{1}{2}}(x_1 \dots x_n) \exp.(-x_1 - \dots - x_n)}{\phi_1(x_1)\phi_2(x_2)\dots\phi_n(x_n)}$$

(exp. $u = \epsilon^u$).

Now let us introduce in place of $x_1, x_2 \dots x_n$ the n new variables $y_1 \dots y_n$, given by the equations

$$\frac{y_1}{x_1 - a_1} + \frac{y_2}{x_1 - a_2} + \dots + \frac{y_n}{x_1 - a_n} = 1,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\frac{y_1}{x_n - a_1} + \frac{y_2}{x_n - a_2} + \dots + \frac{y_n}{x_n - a_n} = 1.$$

Then by IX. 13,

$$y_k = -\frac{g(a_k)}{f'(a_k)};$$

and hence

$$\frac{dy_k}{dx_i} = \frac{y_k}{x_i - a_k}.$$

Thus by IX. 11,

$$\frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)} = y_1 \dots y_n \left| \frac{1}{x_i - a_k} \right|$$

$$= (-1)^{\frac{n(n-1)}{2}} \frac{y_1 \dots y_n \, \zeta^{\frac{1}{2}}(x_1 \dots x_n) \, \zeta^{\frac{1}{2}}(a_1 \dots a_n)}{f(x_1) f(x_2) \dots f(x_n)}.$$

Now $$y_1 \dots y_n = \frac{g(a_1) \dots g(a_n)}{\zeta(a_1 \dots a_n)} (-1)^{\frac{n(n+1)}{2}};$$

$$\therefore \qquad \frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)} = \frac{\zeta^{\frac{1}{2}}(x_1 \dots x_n)}{\zeta^{\frac{1}{2}}(a_1 \dots a_n)}.$$

Hence in the integral we replace $dx_1 \dots dx_n \zeta^{\frac{1}{2}}(x_1 \dots x_n)$ by $dy_1 \dots dy_n \zeta^{\frac{1}{2}}(a_1 \dots a_n)$.

Now if we write

$$F_m(z) = (z - a_1) \dots (z - a_m)(a_{m+1} - z) \dots (a_n - z)$$

we have

$$y_1^{p_1} y_2^{p_2} \dots y_n^{p_n} = \frac{\phi_1(x_1)\,\phi_2(x_2) \dots \phi_n(x_n)}{F_1'(a_1)^{p_1} F_2'(a_2)^{p_2} \dots F_n'(a_n)^{p_n}}.$$

Hence

$$\frac{\zeta^{\frac{1}{2}}(x_1 \dots x_n)}{\phi_1(x_1) \dots \phi_n(x_n)} dx_1 \dots dx_n$$

is replaced by

$$\frac{dy_1 \dots dy_n}{y_1^{p_1} \dots y_n^{p_n}} \frac{\zeta^{\frac{1}{2}}(a_1 \dots a_n)}{F_1'(a_1)^{p_1} \dots F_n'(a_n)^{p_n}}.$$

Again $x_1 \dots x_n$ can be regarded as the roots of the equation

$$\frac{y_1}{z - a_1} + \frac{y_2}{z - a_2} + \dots + \frac{y_n}{z - a_n} = 1,$$

the roots of which lie between a_1 and a_2; a_2 and $a_3 \dots a_n$ and ∞.

Hence $y_1 \dots y_n$ take all positive real values. Also we have

$$x_1 + x_2 + \dots + x_n = y_1 + y_2 + \dots + y_n + a_1 + \dots + a_n.$$

And our integral reduces to

$$\frac{(-1)^{\frac{n(n-1)}{2}} \zeta^{\frac{1}{2}}(a_1 \dots a_n) \exp.(-a_1 - \dots - a_n)}{F_1'(a_1)^{p_1} \dots F_n'(a_n)^{p_n}} \times$$

$$\int_0^\infty \dots \frac{\exp.(-y_1 - \dots - y_n)}{y_1^{p_1} \dots y_n^{p_n}} dy_1 \dots dy_n$$

$$= \frac{(-1)^{\frac{n(n-1)}{2}} \Gamma(1-p_1)\,\Gamma(1-p_2) \dots \Gamma(1-p_n)}{\{F_1'(a_1)^{2p_1-1} \dots F_n'(a_n)^{2p_n-1}\}^{\frac{1}{2}}} \epsilon^{-a_1-a_2-\dots-a_n}.$$

15. If u be a function of n variables $x_1, x_2 \dots x_n$ and $y_1 \dots y_n$, its differential coefficients with respect to these variables, since

$$\frac{dy_i}{dx_k} = \frac{d}{dx_k}\left(\frac{du}{dx_i}\right) = \frac{d^2u}{dx_k dx_i}$$
$$= u_{ki}.$$

The Jacobian of $y_1 \dots y_n$ is a symmetrical determinant formed from the second differential coefficients of u. This determinant is called the Hessian of u after Hesse, and is denoted by $H(u)$, so that

$$H(u) = |\, u_{ik} \,| \,.$$

The Hessian of u will vanish if the first differential coefficients of u are not independent (Art. 2).

For example, if

$$u = x_1^2 x_2^2 + x_1^2 x_3^2 + \dots + x_i^2 x_k^2 + \dots + x_{n-1}^2 x_n^2,$$

$$\frac{d^2 u}{dx_i^2} = 2\,(x_1^2 + \dots + x_{i-1}^2 + x_{i+1}^2 + \dots + x_n^2),$$

$$\frac{d^2 u}{dx_i dx_k} = 4x_i x_k;$$

$$\therefore\ H(u) = \begin{vmatrix} 2\,(x_2^2 + x_3^2 + \dots + x_n^2), & 4x_1 x_2 & \dots \\ 4x_1 x_2 & , 2\,(x_1^2 + x_3^2 + \dots + x_n^2) & \dots \\ \dots & \dots & \dots \end{vmatrix}.$$

Or dividing the i^{th} row by $2x_i$, and the k^{th} column by $2x_k$

$$H(u) = (2^n x_1 x_2 \dots x_n)^2 \begin{vmatrix} \dfrac{x_2^2 + x_3^2 \dots + x_n^2}{2x_1^2}, & 1 & \dots \\ 1 & , \dfrac{x_1^2 + x_3^2 + \dots + x_n^2}{2x_2^2} & \dots \\ \dots & \dots & \dots \end{vmatrix}.$$

This is a determinant of the form of that in III. 25. If we write

$$3\sigma = x_1^2 + x_2^2 \dots + x_n^2$$

$$v = (\sigma - x_1^2)(\sigma - x_2^2) \dots (\sigma - x_n^2)$$

$$H(u) = 6^n v \left\{1 + \tfrac{2}{3} \Sigma \frac{x_i^2}{\sigma - x_i^2}\right\}.$$

If $$u = x^2 y^2 + y^2 z^2 + z^2 x^2,$$

this gives

$$H(u) = 24\,\{9x^2 y^2 z^2 - (x^2 + y^2 + z^2)\,u\}.$$

16. Jacobians and Hessians belong to that class of functions known as covariants. That is to say, if these functions are trans-

formed by means of a linear substitution, the Jacobian of the transformed functions is equal to the Jacobian of the original function multiplied by the modulus of the substitution, and the Hessian of the transformed function equal to that of the original function multiplied by the square of the modulus.

Namely, if the variables be transformed by the substitution

$$x_i = a_{i1}\xi_1 + a_{i2}\xi_2 + \ldots + a_{in}\xi_n \ (i = 1,\ 2 \ldots n),$$

the determinant $|\, a_{ik} \,|$ is called the determinant, or modulus, of the transformation.

If the functions $y_1 \ldots y_n$ of $x_1 \ldots x_n$ when transformed by this substitution become the functions $y_1', y_2' \ldots y'_n$ of $\xi_1 \ldots \xi_n$. Since

$$\frac{dy_i'}{d\xi_k} = \frac{dy_i}{dx_1}\frac{dx_1}{d\xi_k} + \frac{dy_i}{dx_2}\frac{dx_2}{d\xi_k} + \ldots + \frac{dy_i}{dx_n}\frac{dx_n}{d\xi_k}$$
$$= \frac{dy_i}{dx_1} a_{1k} + \ldots + \frac{dy_i}{dx_n} a_{nk},$$

it follows from the multiplication theorem that

$$\frac{d(y_1' \ldots y_n')}{d(\xi_1 \ldots \xi_n)} = \frac{d(y_1 \ldots y_n)}{d(x_1 \ldots x_n)} \,|\, a_{ik} \,|,$$

which proves the theorem for Jacobians.

The theorem for Hessians follows from this, viz. if u be the original and u' the transformed function. Since the Hessian of u is the Jacobian of $\frac{du}{dx_1} \ldots \frac{du}{dx_n}$ we have

$$H(u') = \frac{d\left(\frac{du'}{d\xi_1}, \frac{du'}{d\xi_2} \ldots \frac{du'}{d\xi_n}\right)}{d(\xi_1, \xi_2 \ldots \xi_n)}$$
$$= \frac{d\left(\frac{du'}{d\xi_1} \ldots \frac{du'}{d\xi_n}\right)}{d(x_1 \ldots x_n)} \,|\, a_{ik} \,|.$$

Now
$$\frac{d^2u'}{dx_i d\xi_k} = \frac{d^2u}{d\xi_k dx_i};$$

$$\therefore\ H(u') = \frac{d\left(\frac{du}{dx_1} \ldots \frac{du}{dx_n}\right)}{d(\xi_1 \ldots \xi_n)} \,|\, a_{ik} \,|$$

$$= \frac{d\left(\frac{du}{dx_1} \dots \frac{du}{dx_n}\right)}{d(x_1 \dots x_n)} \mid a_{ik} \mid^2$$

$$= H(u) \mid a_{ik} \mid^2.$$

17. If we have n linear functions

$$y_i = b_{i1}x_1 + \dots + b_{in}x_n \; (i = 1, 2 \dots n),$$

clearly

$$\frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)} = \mid b_{ik} \mid .$$

If u is a quadric function

$$u = b_{11}x_1^2 + \dots + 2b_{ik}x_ix_k + \dots ,$$

then

$$H(u) = 2^n \mid b_{ik} \mid , (b_{ik} = b_{ki}).$$

The symmetrical determinant on the right, which is called the discriminant of the quadric, is therefore an invariant which on transformation is multiplied by the square of the modulus.

CHAPTER XI.

APPLICATIONS TO QUADRICS.

1. THE general quadric function in n variables $x_1 \dots x_n$ is denoted by

$$u = \Sigma a_{ik} x_i x_k,$$

the coefficient of x_i^2 being a_{ii}, that of $2x_i x_k$, a_{ik}, and we suppose

$$a_{ik} = a_{ki}.$$

By x. 17 the symmetrical determinant $A = | a_{ik} |$ is proportional to the Hessian of u, and is hence an invariant, it is called the discriminant. On transformation it is multiplied by the square of the modulus of transformation.

Let us write

$$u_i = \tfrac{1}{2} \frac{du}{dx_i}$$

$$= a_{i1} x_1 + a_{i2} x_2 + \dots + a_{in} x_n.$$

2. If we form a new quadric whose coefficients are the complements of a_{ik} in A, viz.

$$U = \Sigma A_{ik} y_i y_k,$$

U is called the reciprocal of the given quadric. We may also write it in the form (III. 25)

$$U = -\begin{vmatrix} 0, & y_1 & \cdots & y_n \\ y_1, & a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ y_n, & a_{n1} & \cdots & a_{nn} \end{vmatrix}.$$

Since $|A_{ik}| = A^{n-1}$, and if α_{ik} is the complement of A_{ik} in this determinant $\alpha_{ik} = a_{ik} A^{n-2}$ (v. 6), we see that we can write u in the form

$$A^{n-2}u = -\begin{vmatrix} 0, & x_1 & \dots & x_n \\ x_1, & A_{11} & \dots & A_{1n} \\ \dots & \dots & \dots & \dots \\ x_n, & A_{n1} & \dots & A_{nn} \end{vmatrix}.$$

We have also

$$Au = -\begin{vmatrix} 0, & u_1 & \dots & u_n \\ u_1, & a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ u_n, & a_{n1} & \dots & a_{nn} \end{vmatrix},$$

as we see by multiplying the last n rows by $x_1 \dots x_n$ and subtracting their sum from the first.

3. If $A = 0$, since then

$$A^2_{ik} = A_{ii} A_{kk},$$

it follows that

$$\begin{aligned} U &= \Sigma A_{ik} y_i y_k \\ &= \Sigma \sqrt{A_{ii} A_{kk}} y_i y_k \\ &= \Sigma \{\sqrt{A_{ii}} y_i\}^2 \end{aligned}$$

is a complete square, and that the lineo-linear function

$$\begin{aligned} V = -&\begin{vmatrix} 0, & x_1 & \dots & x_n \\ y_1, & A_{11} & \dots & A_{1n} \\ \dots & \dots & \dots & \dots \\ y_n, & A_{n1} & \dots & A_{nn} \end{vmatrix} \\ &= \Sigma A_{ik} y_i x_k \\ &= \Sigma \sqrt{A_{ii}} y_i \,.\, \Sigma \sqrt{A_{kk}} x_k \end{aligned}$$

is the product of two linear factors.

4. The reciprocal quadric U is the first of a series of covariant quantics. If the variables $x_1 \dots x_n$ are transformed by a linear substitution

$$x_i = c_{i1} x_1' + c_{i2} x_2' + \dots + c_{in} x_n' \dots\dots\dots\dots (1) \qquad (i = 1, 2 \dots n),$$

then the function

$$x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

becomes

$$\dots + x_i' \left(c_{1i} y_1 + c_{2i} y_2 + \dots + c_{ni} y_n\right) + \dots$$

Hence, if we have a series of quantities $y_1' \dots y_n'$ given by

$$y_i' = c_{1i} y_1 + c_{2i} y_2 + \dots + c_{ni} y_n \dots\dots\dots\dots (2) \qquad (i = 1, 2 \dots n),$$

the function $\Sigma x_i y_i$ on transformation becomes changed to $\Sigma x_i' y_i'$, and so is absolutely unchanged in form by the transformation.

Now observe that in the substitutions (1) and (2) the determinants of the transformation are identical; only the columns of the determinant of (2) coincide with the rows of the determinant of (1). Also in (1) the old variables are given in terms of the new, in (2) the new variables are given in terms of the old. The variables $x_1 \dots x_n$, $y_1 \dots y_n$ are said to be contragredient. Any function of the coefficients of u and the quantities $y_1 \dots y_n$ whose value on transformation is equal to its original value multiplied by a power of the modulus of transformation is called a contravariant.

The semi-differential coefficients $u_1 \dots u_n$ are contragredient to $x_1 \dots x_n$.

5. If the p sets of n variables

$$\begin{array}{c} y_{11} \cdots y_{n1} \\ y_{12} \cdots y_{n2} \\ \dots\dots\dots\dots \\ y_{1p} \cdots y_{np} \end{array}$$

are contragredient to the variables $x_1 \dots x_n$, then the series of determinants

$$R_p = \left| \begin{array}{l} a_{11} \dots a_{1n},\ y_{11} \cdots y_{1p} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1} \cdots a_{nn},\ y_{n1} \cdots y_{np} \\ y_{11} \cdots y_{n1} \\ \dots\dots\dots\dots \\ y_{1p} \cdots y_{np} \end{array} \right|$$

are contravariants.

For, let us consider the quadric function

$$V = \Sigma a_{ik} x_i x_k + 2t_1 (x_1 y_{11} + \dots + x_n y_{n1})$$
$$+ 2t_2 (x_1 y_{12} + \dots + x_n y_{n2}) + \dots + 2t_p (x_1 y_{1p} + \dots + x_n y_{np}),$$

where the variables $x_1 \dots x_n$, $t_1 \dots t_p$ are cogredient (i.e. transformed by the same substitution), while y_{ik} are contragredient to these.

If we regard V as a quadric in $n + p$ variables $x_1 \dots x_n$, $t_1 \dots t_p$, R_p is its discriminant. Let us transform it by means of the substitutions

$$\left.\begin{aligned} x_i &= c_{i1} x_1' + \dots + c_{in} x_n' \\ y_i' &= c_{1i} y_1 + \dots + c_{ni} y_n \end{aligned}\right\} \quad (i = 1,\ 2 \dots n)$$
$$t_k = t_k' \qquad\qquad k = 1,\ 2 \dots p.$$

Then the determinant of the transformation for x, t is

$$u = \begin{vmatrix} c_{11} \dots c_{1n}, & 0 \dots 0 \\ \dots\dots & \dots\dots \\ c_{n1} \dots c_{nn}, & 0 \dots 0 \\ 0 \dots 0, & 1 \dots 0 \\ \dots\dots & \dots\dots \\ 0 \dots 0, & 0 \dots 1 \end{vmatrix} = | c_{ik} | .$$

In the transformed function V, the terms multiplying t_i are unaltered in form. Hence, by Art. 1,

$$R_p' = \mu^2 R_p.$$

Thus R_p is a contravariant. Since on transformation it is multiplied by the square of the modulus, it does not change its sign.

6. If $p = 1$ so that we have only one system of y's, then R_1 is the reciprocal quadric. If for uniformity we denote the discriminant by R_0, we have (III. 25)

$$R_1 = - \Sigma \frac{dR_0}{da_{ik}} y_i y_k.$$

And in general we have

$$R_{p+1} = - \Sigma \frac{dR_p}{da_{ik}} y_{i,\,p+1}\, y_{k,\,p+1}.$$

Clearly

$$R_n = (-1)^n \begin{vmatrix} y_{11} & \cdots & y_{1n} \\ \cdots & \cdots & \cdots \\ y_{n1} & \cdots & y_{nn} \end{vmatrix}^2,$$

while R_p vanishes identically if p is greater than n, as we see by resolving it into the sum of products of complementary minors of order n and p. Thus we have the series of functions

$$R_0,\ R_1 \ldots R_n,$$

containing 0, 1, 2 ... n series of variables y, and of orders n, $n-1$ ... 1, 0 in the coefficients of the quadric u.

7. The determinants R_p are of great importance in the discussion of the properties of a quadric, and especially in the resolution of the quadric into the sum of squares of functions linear in the variables $x_1 \ldots x_n$.

If $u_1 \ldots u_n$ are the semi-differential coefficients of u, let us write

$$U_p = \begin{vmatrix} a_{11} \ldots a_{1n}, & y_{11} \ldots y_{1p}, & u_1 \\ \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots \\ a_{n1} \ldots a_{nn}, & y_{n1} \ldots y_{np}, & u_n \\ y_{11} \ldots y_{n1} & & \\ \cdots\cdots\cdots & & \\ y_{1p} \ldots y_{np} & & \\ u_1 \ldots u_n & & \end{vmatrix}.$$

We must remember that $U_n = 0$ identically.

Also let X_p be the determinant obtained from U_p by erasing the $(n+p+1)^{\text{st}}$ column and $(n+p)^{\text{th}}$ row; or the $(n+p+1)^{\text{st}}$ row and $(n+p)^{\text{th}}$ column.

Since in any determinant of order m, we have (v. 7)

$$D \frac{d^2D}{da_{mm}\, da_{m-1\,m-1}} = \frac{dD}{da_{m-1\,m-1}} \cdot \frac{dD}{da_{mm}} - \frac{dD}{da_{m-1\,m}} \cdot \frac{dD}{da_{m\,m-1}},$$

we get by applying this to U_p, $(m = n+p)$

$$R_p U_{p-1} - X_p^2 = R_{p-1} U_p,$$

or

$$\frac{U_{p-1}}{R_{p-1}} = \frac{X_p^2}{R_{p-1} R_p} + \frac{U_p}{R_p}.$$

In this equation write $p=n,\ n-1\ \ldots\ 1,\ 0$, and remembering that

$$U_n=0, \qquad U_0=-R_0u,$$

we get the series of equations

$$\frac{U_{n-1}}{R_{n-1}}=\frac{X_n^2}{R_{n-1}R_n}$$

$$\frac{U_{n-2}}{R_{n-2}}=\frac{X_{n-1}^2}{R_{n-2}R_{n-1}}+\frac{U_{n-1}}{R_{n-1}}$$

$$\frac{U_{n-3}}{R_{n-3}}=\frac{X_{n-2}^2}{R_{n-3}R_{n-2}}+\frac{U_{n-2}}{R_{n-2}}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{U_0}{R_0}=\frac{X_1^2}{R_0R_1}+\frac{U_1}{R_1}.$$

Thus

$$u=-\frac{X_1^2}{R_0R_1}-\frac{X_2^2}{R_1R_2}-\cdots-\frac{X_n^2}{R_{n-1}R_n}.$$

Now the quantities $X_1\ \ldots\ X_n$ are linear functions of $u_1\ \ldots\ u_n$, i.e. of $x_1\ \ldots\ x_n$; hence we have resolved the given quadric into the sum of the squares of n linear functions of the variables $x_1\ \ldots\ x_n$.

Also the number of positive squares in this sum is the number of variations in sign in the series

$$R_0,\ R_1\ \ldots\ R_n,$$

and these being unaltered in sign by a linear transformation we have the important theorem, that if a quadric be linearly transformed to the sum of n squares, the number of positive and negative squares is always the same. This theorem, due to Sylvester, has been called by him the law of inertia of quadratic forms.

8. The discussion of the preceding article, due to Darboux, requires modification in certain cases. For example, if the minors of order $p-1$ of the discriminant vanish, then all the functions $R_0\ \ldots\ R_{p-1}$ inclusive vanish. In this case Darboux has shewn that u can be resolved into the sum of $n-p$ squares, viz.

$$u=-\frac{X_{p+1}^2}{R_{p+1}R_p}-\cdots-\frac{X_{n-1}^2}{R_{n-1}R_{n-2}}-\frac{X_n^2}{R_{n-1}R_n}.$$

9. If a quadric, by means of a linear transformation, has been reduced to the sum of n squares,

$$\begin{aligned} u &= \Sigma a_{ik} x_i x_k \\ &= A_1 y_1^2 + A_2 y_2^2 + \dots + A_n y_n^2; \end{aligned}$$

the discriminant of the right-hand side being $A_1 A_2 \dots A_n$ if μ is the modulus of transformation,

$$A_1 A_2 \dots A_n = \mu^2 \mid a_{ik} \mid.$$

Two given quadrics

$$u = \Sigma a_{ik} x_i x_k, \quad v = \Sigma b_{ik} x_i x_k$$

can by a simultaneous linear transformation

$$x_i = c_{i1} y_1 + c_{i2} y_2 + \dots + c_{in} y_n \quad (i = 1, 2 \dots n)$$

be reduced, each to the sum of n squares of the same linear functions, viz.

$$\begin{aligned} u &= A_1 y_1^2 + A_2 y_2^2 + \dots + A_n y_n^2 \\ v &= s_1 A_1 y_1^2 + s_2 A_2 y_2^2 + \dots + s_n A_n y_n^2; \end{aligned}$$

for in order to determine the n^2 constants, c_{ik}, we have first $n(n-1)$ equations from the fact that the coefficients of the products $y_i y_k$ must vanish, and n additional equations from the condition that the ratio of the coefficients of y_i^2 is to be s_i, in all n^2 equations.

If we form the discriminant of $su - v$, its value for the original quadrics is

$$\mid s a_{ik} - b_{ik} \mid \qquad \text{..........(1)},$$

and for the transformed quadrics

$$A_1 \dots A_n (s - s_1)(s - s_2) \dots (s - s_n) \qquad \text{..........(2)}.$$

The ratio of the quantities (1) and (2) is μ^2; hence $s_1 \dots s_n$ are the roots of the equation

$$\Delta(s) = \mid s a_{ik} - b_{ik} \mid = 0 \qquad \text{..........(3)}.$$

10. The following resolution is due to Darboux.

If we write

$$F = su - v, \quad X_i = \tfrac{1}{2} \frac{dF}{dx_i} = s u_i - v_i \qquad \text{..........(4)},$$

we have identically by Art. 2

$$F = su - v = -\frac{1}{\Delta(s)} \begin{vmatrix} sa_{11} - b_{11} & \dots & sa_{1n} - b_{1n}, & X_1 \\ \dots & \dots & \dots & \dots \\ sa_{n1} - b_{n1} & \dots & sa_{nn} - b_{nn}, & X_n \\ X_1 & \dots & X_n & \end{vmatrix} \dots (5).$$

The determinant on the right is a function of s of order $n-1$; resolve the fraction into partial fractions, and we get

$$su - v = -\Sigma \frac{1}{\Delta'(s_i)(s - s_i)} \begin{vmatrix} s_i a_{11} - b_{11} & \dots & s_i a_{1n} - b_{1n}, & X_1 \\ \dots & \dots & \dots & \dots \\ s_i a_{n1} - b_{n1} & \dots & s_i a_{nn} - b_{nn}, & X_n \\ X_1 & \dots & X_n & \end{vmatrix} \dots (6).$$

The determinants on the right are all perfect squares by Art. 3, for they are obtained by bordering the vanishing determinant $\Delta(s_i)$. Whence

$$su - v = \Sigma \frac{U_i^2}{\Delta'(s_i)(s - s_i)},$$

where U_i is a linear function of the form

$$U_i = d_{i1} X_1 + \dots + d_{in} X_n.$$

If in the determinant (6) we replace X_i by its value from (4), and subtract from the last column the first n multiplied by $x_1 \dots x_n$, and do the same for the rows, the value of the determinant is unaltered, but X_i is replaced by $\frac{1}{2}(s - s_i)\dfrac{du}{dx_i}$.

A term is also introduced in the principal diagonal in the last place, but since its minor vanishes by (3) we may replace it by zero. Thus U_i is replaced by

$$U_i' = \tfrac{1}{2}(s - s_i)\left(d_{i1}\frac{du}{dx_1} + \dots + d_{in}\frac{du}{dx_n}\right)$$
$$= (s - s_i)\, V_i,$$

where V_i is independent of s;

$$\therefore\ su - v = \Sigma \frac{V_i^2 (s - s_i)}{\Delta'(s_i)}.$$

Equating coefficients of s we get

$$u = \Sigma \frac{V_i^2}{\Delta'(s_i)}, \quad v = \Sigma \frac{s_i V_i^2}{\Delta'(s_i)},$$

which is the required resolution.

11. An important branch of the theory of quadrics is that of their linear automorphic transformation. That is to say, as the name implies, the discussion of those linear transformations which do not alter the outward appearance of the quadric. So that if $x_1 \dots x_n$ are the original, and $y_1 \dots y_n$ the new variables,

$$\Sigma a_{ik} x_i x_k \text{ becomes } \Sigma a_{ik} y_i y_k.$$

Without entering into a discussion of the general case we shall study that particular one which gave rise to the whole theory.

In the transformation from one set of rectangular axes in space to another with the same origin, the distance of a point from the origin is the same, expressing this for the two systems

$$x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2,$$

such a transformation is linear and automorphic, and is known as an orthogonal transformation.

12. The general case of an orthogonal transformation is to determine those linear transformations which give us

$$x_1^2 + x_2^2 + \dots + x_n^2 = y_1^2 + y_2^2 + \dots + y_n^2.$$

The theory is due to Cayley, but we shall here give it as modified by Veltmann.

Let us consider the following equations

$$\left.\begin{array}{l} b_{11}x_1 + b_{12}x_2 + \dots + b_{1n}x_n = b_{11}y_1 + b_{21}y_2 + \dots + b_{n1}y_n \\ b_{21}x_1 + b_{22}x_2 + \dots + b_{2n}x_n = b_{12}y_1 + b_{22}y_2 + \dots + b_{n2}y_n \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ b_{n1}x_1 + b_{n2}x_2 + \dots + b_{nn}x_n = b_{1n}y_1 + b_{2n}y_2 + \dots + b_{nn}y_n \end{array}\right\} \dots(1),$$

where the system b_{ik} is skew, so that

$$b_{ik} = -b_{ki}, \quad b_{ii} = z \dots\dots\dots\dots\dots\dots(2).$$

The rows of coefficients on the right coincide with the columns on the left.

Let $$B = | b_{ik} | = | b_{ki} |,$$

so that B is a skew determinant, let B_{ik} be the system of first minors. Solving the system of equations (1) we get

$$y_i = c_{i1}x_1 + c_{i2}x_2 + \dots + c_{in}x_n$$
$$x_k = d_{k1}y_1 + d_{k2}y_2 + \dots + d_{kn}y_n.$$

The coefficient of x_k in y_i is given by

$$Bc_{ik} = B_{i1}b_{1k} + B_{i2}b_{2k} + \dots + B_{in}b_{nk}.$$

If

$$s = B_{i1}b_{k1} + B_{i2}b_{k2} + \dots + B_{in}b_{kn},$$

then

$$Bc_{ik} + s = 2B_{ik}b_{kk}.$$

Now $s = B$ or 0 according as i is or is not equal to k, thus

$$c_{ik} = \frac{2B_{ik}z}{B}, \quad c_{ii} = \frac{2B_{ii}z - B}{B}.$$

In the same way

$$d_{ki} = \frac{2B_{ik}z}{B}, \quad d_{ii} = \frac{2B_{ii}z - B}{B}.$$

Thus

$$c_{ik} = d_{ki},$$

and we may write

$$y_i = c_{i1}x_1 + c_{i2}x_2 + \dots + c_{in}x_n$$
$$x_i = c_{1i}y_1 + c_{2i}y_2 + \dots + c_{ni}y_n.$$

Substitute for $x_1 \dots x_n$ from the second of these systems in the first and equate coefficients of y_k and y_i on both sides, thus

$$\left.\begin{aligned} c_{i1}^2 + c_{i2}^2 + \dots + c_{in}^2 &= 1 \\ c_{i1}c_{k1} + c_{i2}c_{k2} + \dots + c_{in}c_{kn} &= 0 \end{aligned}\right\}.$$

If we substitute from the first system in the second, we get

$$\left.\begin{aligned} c_{1i}^2 + c_{2i}^2 + \dots + c_{ni}^2 &= 1 \\ c_{1i}c_{1k} + c_{2i}c_{2k} + \dots + c_{ni}c_{nk} &= 0 \end{aligned}\right\}.$$

Whence we see at once that

$$x_1^2 + x_2^2 + \dots + x_n^2 = y_1^2 + y_2^2 + \dots + y_n^2,$$

and thus the coefficients c_{ik} are those of an orthogonal substitution.

13. By the preceding article we are able to express the n^2 coefficients of an orthogonal transformation by means of the $\frac{1}{2}n(n-1)$ quantities

$$\begin{matrix} b_{12}, & b_{13} & \dots & b_{1n} \\ & b_{23} & \dots & b_{2n} \\ & \dots & \dots & \dots \\ & & & b_{n-1\,n}, \end{matrix}$$

by forming a skew determinant with these, the elements of whose leading diagonal are equal to z.

For the case $n = 2$, let

$$B = \begin{vmatrix} 1, & \lambda \\ -\lambda, & 1 \end{vmatrix} = 1 + \lambda^2;$$

the system of first minors is

$$\begin{matrix} 1, & \lambda \\ -\lambda, & 1. \end{matrix}$$

Hence the coefficients of a binary orthogonal transformation are

$$\begin{matrix} \dfrac{1-\lambda^2}{1+\lambda^2}, & \dfrac{2\lambda}{1+\lambda^2}, \\[2ex] \dfrac{-2\lambda}{1+\lambda^2}, & \dfrac{1-\lambda^2}{1+\lambda^2}. \end{matrix}$$

For a ternary orthogonal transformation

$$B = \begin{vmatrix} 1, & \nu, & -\mu \\ -\nu, & 1, & \lambda \\ \mu, & -\lambda, & 1 \end{vmatrix} = 1 + \lambda^2 + \mu^2 + \nu^2;$$

the system of first minors is

$$\begin{matrix} 1+\lambda^2, & \nu + \lambda\mu, & -\mu + \lambda\nu, \\ -\nu + \lambda\mu, & 1 + \mu^2, & \lambda + \mu\nu, \\ \mu + \lambda\nu, & -\lambda + \mu\nu, & 1 + \nu^2. \end{matrix}$$

Hence the coefficients of the ternary orthogonal transformation are

$$\begin{matrix} \dfrac{1+\lambda^2-\mu^2-\nu^2}{B}, & 2\,\dfrac{\nu+\lambda\mu}{B}, & 2\,\dfrac{-\mu+\lambda\nu}{B}, \\[2ex] 2\,\dfrac{-\nu+\lambda\mu}{B}, & \dfrac{1+\mu^2-\lambda^2-\nu^2}{B}, & 2\,\dfrac{\lambda+\mu\nu}{B}, \\[2ex] 2\,\dfrac{\mu+\lambda\nu}{B}, & 2\,\dfrac{-\lambda+\mu\nu}{B}, & \dfrac{1+\nu^2-\lambda^2-\mu^2}{B}. \end{matrix}$$

If we write

$$\lambda = \cos f \tan \tfrac{1}{2}\theta, \quad \mu = \cos g \tan \tfrac{1}{2}\theta, \quad \nu = \cos h \tan \tfrac{1}{2}\theta,$$

where $$\cos^2 f + \cos^2 g + \cos^2 h = 1,$$

and $$\therefore \quad B = \sec^2 \tfrac{1}{2}\theta,$$

we get Rodrigues' formulæ.

For the quaternary orthogonal transformation

$$B = \begin{vmatrix} 1, & a, & b, & c \\ -a, & 1, & h, & -g \\ -b, & -h, & 1 & f \\ -c, & g, & -f, & 1 \end{vmatrix}.$$

Then

$$B = 1 + a^2 + b^2 + c^2 + f^2 + g^2 + h^2 + \theta^2,$$

where $$\theta = af + bg + ch.$$

And the system of first minors is

$$\begin{aligned}
B_{11} &= 1 + f^2 + g^2 + h^2, & B_{12} &= a + f\theta - bh + cg, \\
B_{21} &= -a - f\theta + cg - bh, & B_{22} &= 1 + f^2 + b^2 + c^2, \\
B_{31} &= -b - cf - g\theta + ah, & B_{32} &= -h + fg - ab - c\theta, \\
B_{41} &= -c + bf - ag - h\theta, & B_{42} &= g + fh + b\theta - ca, \\
B_{13} &= b + g\theta - cf + ah, & B_{14} &= c + h\theta - ag + bf, \\
B_{23} &= h + fg + c\theta - ab, & B_{24} &= -g + hf - ac - b\theta, \\
B_{33} &= 1 + g^2 + c^2 + a^2, & B_{34} &= f + gh + a\theta - bc, \\
B_{43} &= -f + gh - bc - a\theta, & B_{44} &= 1 + h^2 + a^2 + b^2.
\end{aligned}$$

Thus the coefficients of the quaternary orthogonal transformation are

$$\begin{aligned}
Bc_{11} &= 1 - \theta^2 + f^2 - a^2 + g^2 - b^2 + h^2 - c^2, \\
Bc_{12} &= 2\,(a + f\theta - bh + cg), \\
Bc_{13} &= 2\,(b + g\theta - cf + ah), \\
Bc_{14} &= 2\,(c + h\theta - ag + bf), \\
&\&c.
\end{aligned}$$

14. The square of the determinant of an orthogonal substitution is unity, for

$$| c_{ik} |^2 = | d_{ik} |,$$

where $$d_{ik} = c_{1i}c_{1k} + c_{2i}c_{2k} + \ldots + c_{ni}c_{nk},$$

i.e. $$d_{ik} = 0, \quad d_{ii} = 1;$$

$$\therefore \quad | c_{ik} |^2 = 1, \text{ or } | c_{ik} | = \epsilon,$$

where ϵ means ± 1.

15. If C_{ik} is the complement of c_{ik} in C, then

$$C_{ik} = \epsilon \,.\, c_{ik}.$$

For we have the system of equations

$$\begin{aligned} c_{11}c_{1k} + \ldots + c_{n1}c_{nk} &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ c_{1k}c_{1k} + \ldots + c_{nk}c_{nk} &= 1 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ c_{1n}c_{1k} + \ldots + c_{nn}c_{nk} &= 0. \end{aligned}$$

Multiply these equations by $C_{i1}, C_{i2} \ldots C_{in}$ and add, the coefficient of c_{ik} is ϵ, the others vanish, thus

$$C_{ik} = \epsilon c_{ik}.$$

16. Any minor of the system c_{ik} is equal to its complementary minor.

For
$$\begin{vmatrix} C_{11} & \ldots & C_{1p} \\ \ldots & \ldots & \ldots \\ C_{p1} & \ldots & C_{pp} \end{vmatrix} = \epsilon^{p-1} \begin{vmatrix} c_{p+1\,p+1} & \ldots & c_{p+1\,n} \\ \ldots & \ldots & \ldots \\ c_{n\,p+1} & \ldots & c_{nn} \end{vmatrix},$$

by v. 7. But

$$\begin{vmatrix} C_{11} & \ldots & C_{1p} \\ \ldots & \ldots & \ldots \\ C_{p1} & \ldots & C_{pp} \end{vmatrix} = \epsilon^{p} \begin{vmatrix} c_{11} & \ldots & c_{1p} \\ \ldots & \ldots & \ldots \\ c_{p1} & \ldots & c_{pp} \end{vmatrix}$$

by the theorem just proved. Hence

$$\epsilon \begin{vmatrix} c_{11} & \ldots & c_{1p} \\ \ldots & \ldots & \ldots \\ c_{p1} & \ldots & c_{pp} \end{vmatrix} = \begin{vmatrix} c_{p+1\,p+1} & \ldots & c_{p+1\,n} \\ \ldots & \ldots & \ldots \\ c_{n\,p+1} & \ldots & c_{nn} \end{vmatrix}.$$

17. If $A^{(n)} = | a_{ik} |$ $B^{(n)} = | b_{ik} |$ be two determinants of orthogonal substitutions of order n, then the determinant

$$P(\lambda, \mu) = | \lambda a_{ik} + \mu b_{ik} |$$

is not altered by interchanging λ and μ.

For the symbolical expression for $P(\lambda, \mu)$ is

$$\begin{aligned} P(\lambda, \mu) &= (\lambda A + \mu B)^n \\ &= A^{(n)}B^{(n)}\left(\frac{\lambda}{B} + \frac{\mu}{A}\right)^n, \end{aligned}$$

as in v. 8. And as there proved

$$P(\lambda, \mu) = A^{(n)}B^{(n)} \left| \frac{\lambda B_{ik}}{B^{(n)}} + \frac{\mu A_{ik}}{A^{(n)}} \right|.$$

Or, if $A^{(n)} = 1 = B^{(n)}$, we have by Art. 15,

$$P(\lambda, \mu) = | \lambda b_{ik} + \mu a_{ik} |$$
$$= P(\mu, \lambda).$$

From this we see, that if from the coefficients of an orthogonal substitution of order n we subtract the corresponding coefficients of another orthogonal substitution of the same order, the determinant formed with these differences vanishes if n is odd.

18. If we take n quadrics in n variables we may conveniently represent them by the system of equations

$$u_i = \Sigma a_{ijk} x_j x_k \qquad (i, j, k = 1, 2 \ldots n).$$

With the coefficients a_{ijk} we can form a cubic determinant of order n which will be an invariant of the system of quadrics $u_1 \ldots u_n$. Zehfuss has pointed out that for three ternary quadrics this gives Aronhold's invariant, while the auxiliary expressions he gives for its calculation are the cubic minors of the second order.

For the two binary quadrics

$$ax^2 + 2bxy + cy^2,$$
$$a'x^2 + 2b'xy + c'y^2,$$

it is the harmonic invariant

$$aa' - 2bb' + cc'.$$

The general theorem is that for n, n-ary p^{tics} the determinant of class $(p+1)$, which can be formed with their coefficients, is an invariant of the system. By allowing all the quantics to become identical we get an invariant of a single quantic when it is of even order.

CHAPTER XII.

DETERMINANTS OF FUNCTIONS OF THE SAME VARIABLE.

1. If $y_1, y_2 \dots y_n$ are functions of a variable x, and if

$$y_i^{(k)} = \frac{d^k y_i}{dx^k},$$

the determinant

$$\Sigma \pm y_1 y_2^{(1)} \dots y_n^{(n-1)} = \begin{vmatrix} y_1, & y_2 & \dots & y_n \\ y_1^{(1)}, & y_2^{(1)} & \dots & y_n^{(1)} \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)}, & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$$

is called the determinant of the functions $y_1, y_2 \dots y_n$, and is denoted by $D(y_1, y_2 \dots y_n)$.

2. If y is any function of x, and we multiply the above determinant by

$$\begin{vmatrix} y, & 0, & 0 & \dots 0 \\ y^{(1)}, & y, & 0 & \dots 0 \\ y^{(2)}, & 2y^{(1)}, & y & \dots 0 \\ \dots & \dots & \dots & \dots \\ y^{(n-1)}, & (n-1)_1 y^{(n-2)}, & (n-1)_2 y^{(n-3)} & \dots y \end{vmatrix} = y^n,$$

combining the columns of D with the rows of the latter, we obtain

$$D(yy_1, yy_2 \dots yy_n) = y^n D(y_1, y_2 \dots y_n).$$

In particular if we put $yy_1 = 1$ in the determinant on the left, all the elements in the first column vanish, except the first, which is unity, and the determinant reduces to the determinant of the $n-1$ functions

$$\frac{d}{dx}\left(\frac{y_2}{y_1}\right)=\frac{D(y_1,\ y_2)}{y_1^2}\ \cdots\ \frac{d}{dx}\left(\frac{y_n}{y_1}\right)=\frac{D(y_1,\ y_n)}{y_1^2}.$$

If therefore we put

$$D(y_1,\ y_2)=y_2'\ \dots\ D(y_1,\ y_n)=y_n',$$

then $$D(y_1,\ y_2\ \cdots\ y_n)=\frac{1}{y_1^{n-2}}D(y_2',\ y_3'\ \cdots\ y_n').$$

3. If the functions $y_1 \dots y_n$ are connected by any linear relation

$$c_1y_1+c_2y_2+\dots+c_ny_n=0,$$

it is plain by differentiating this $n-1$ times, and eliminating $c_1 \dots c_n$ between the original and these $n-1$ new equations that we get:

$$D(y_1,\ y_2\ \cdots\ y_n)=0.$$

Conversely if the determinant of the functions $y_1 \dots y_n$ vanishes, then they are connected by a linear equation with constant coefficients. We shall prove this by induction; we shall assume that if the determinant of $n-1$ functions vanishes, these functions are linearly connected, and we shall shew that the same is true for n functions. If y_1 does not vanish, which would be equivalent to a linear relation among the functions, it follows from the preceding article that since

$$D(y_1,\ y_2\ \cdots\ y_n)=0,$$

we must also have

$$D(y_2',\ y_3'\ \cdots\ y_n')=0.$$

Hence by hypothesis the $n-1$ functions $y_2' \dots y_n'$ are linearly connected, i.e. we have

$$c_2y_2'+c_3y_3'+\dots+c_ny_n'=0.$$

Dividing by y_1^2 we get

$$c_2\frac{d}{dx}\left(\frac{y_2}{y_1}\right)+c_3\frac{d}{dx}\left(\frac{y_3}{y_1}\right)+\dots+c_n\frac{d}{dx}\left(\frac{y_n}{y_1}\right)=0,$$

or integrating

$$c_1y_1+c_2y_2+\dots+c_ny_n=0.$$

Thus if the theorem is true for $n-1$ functions, it is true for n, but it is clearly true for two functions, and hence generally.

4. From the formula

$$D(y_1, y_2 \dots y_n) = \frac{1}{y_1^{n-2}} D(y_2', y_3' \dots y_n'),$$

it follows that

$$D(y_1, y_2, y_3) = \frac{1}{y_1} D(y_2', y_3')$$

$$D(y_1, y_2, y_4) = \frac{1}{y_1} D(y_2', y_4')$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$D(y_1, y_2, y_n) = \frac{1}{y_1} D(y_2', y_n').$$

The same formula also gives

$$D(y_2', y_3' \dots y_n') = \frac{1}{y_2'^{n-3}} D\{D(y_2', y_3'),\ D(y_2', y_4') \dots D(y_2', y_n')\}.$$

Combining these formulæ, we obtain the equation

$$D(y_1, y_2 \dots y_n) = \frac{1}{[D(y_1, y_2)]^{n-3}} D\{D(y_1, y_2, y_3), \\ D(y_1, y_2, y_4) \dots D(y_1, y_2, y_n)\}.$$

By repeated application of this method we should obtain the theorem.

If $u_1, u_2 \dots u_m,\ v_1, v_2 \dots v_n$ be functions of x, and if

$$w_i = D(u_1, u_2 \dots u_m, v_i) \quad (i = 1, 2 \dots n),$$

then $$D(u_1, u_2 \dots u_m, v_1, v_2 \dots v_n) = \frac{D(w_1, w_2 \dots w_n)}{\{D(u_1, u_2 \dots u_m)\}^{n-1}}.$$

5. A special case of this theorem is

$$D(y_1 \dots y_{k-1}, y_{k+1} \dots y_n, y_k, y)$$
$$= \frac{D\{D(y_1 \dots y_{k-1}, y_{k+1} \dots y_n, y_k)\ D(y_1 \dots y_{k-1}, y_{k+1} \dots y_n, y)\}}{D(y_1 \dots y_{k-1}, y_{k+1} \dots y_n)},$$

which we may write in the form

$$\frac{D(y_1 \dots y_n, y)\, D(y_1 \dots y_{k-1}, y_{k+1} \dots y_n)}{D(y_1 \dots y_n)\, D(y_1 \dots y_n)} = -\frac{d}{dx} \frac{D(y, y_1 \dots y_{k-1}, y_{k+1} \dots y_n)}{D(y_1 \dots y_n)}.$$

Assuming now that the functions $y_1 \dots y_n$ are independent, let us write

$$z_k = (-1)^{n+k} \frac{D(y_1 \dots y_{k-1}, y_{k+1} \dots y_n)}{D(y_1 \dots y_n)}$$

$$P(y) = (-1)^n \frac{D(y, y_1 \dots y_n)}{D(y_1 \dots y_n)}$$

$$P(y, z_k) = (-1)^{k-1} \frac{D(y, y_1 \dots y_{k-1}, y_{k+1} \dots y_n)}{D(y_1 \dots y_n)},$$

then the above equation can be written

$$z_k P(y) = \frac{d}{dx} P(y, z_k).$$

6. The determinant

$$\begin{vmatrix} y_1, & y_2 & \dots & y_n \\ y_1^{(1)}, & y_2^{(1)} & \dots & y_n^{(1)} \\ \dots & \dots & \dots & \dots \\ y_1^{(n-2)}, & y_2^{(n-2)} & \dots & y_n^{(n-2)} \\ y_1^{(k)}, & y_2^{(k)} & \dots & y_n^{(k)} \end{vmatrix}$$

vanishes if $k < n-1$, but if $k = n-1$ its value is $D(y_1, y_2 \dots y_n)$. Expanding it according to the elements of the last row we get the system of equations

$$\left.\begin{aligned} y_1 z_1 + y_2 z_2 + \dots + y_n z_n &= 0 \\ y_1^{(1)} z_1 + y_2^{(1)} z_2 + \dots + y_n^{(1)} z_n &= 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots& \\ y_1^{(n-2)} z_1 + y_2^{(n-2)} z_2 + \dots + y_n^{(n-2)} z_n &= 0 \\ y_1^{(n-1)} z_1 + y_2^{(n-1)} z_2 + \dots + y_n^{(n-1)} z_n &= 1 \end{aligned}\right\} \dots\dots\text{(A)}.$$

If we write $s_{pq} = y_1^{(p)} z_1^{(q)} + y_2^{(p)} z_2^{(q)} + \dots + y_n^{(p)} z_n^{(q)}$, we can write these more briefly

$$s_{00} = 0, \quad s_{10} = 0 \dots s_{n-20} = 0, \quad s_{n-10} = 1.$$

Now we have

$$\frac{ds_{k-10}}{dx} = s_{k0} + s_{k-11}$$

$$\frac{d^2 s_{k-20}}{dx^2} = s_{k0} + 2s_{k-11} + s_{k-22}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\frac{d^k s_{00}}{dx^k} = s_{k0} + k s_{k-11} + k_2 s_{k-22} + \dots + s_{0k}.$$

If $k < n-1$, it follows from these equations that

$$s_{\alpha\beta} = 0, \text{ if } \alpha + \beta < n - 1.$$

If $k = n-1$, it follows since

$$(1-1)^r = 1 - r + r_2 - \dots + (-1)^r = 0$$

that s_{n-10}, s_{n-21} ... are alternately equal to $+1$ and -1.

If $k = n$, we conclude in the same way that

$$s_{n0} = -s_{n-11} = s_{n-22} = \dots = (-1)^n s_{0n}.$$

Hence we get the following theorem: The expression s_{pq} is equal to zero when $p+q<n-1$, and equal to $(-1)^q$ when $p+q=n-1$.

7. Among the relations just established we have

$$\left.\begin{array}{l} z_1 y_1 + z_2 y_2 + \dots + z_n y_n = 0 \\ z_1^{(1)} y_1 + z_2^{(1)} y_2 + \dots + z_n^{(1)} y_n = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ z_1^{(n-2)} y_1 + z_2^{(n-2)} y_2 + \dots + z_n^{(n-2)} y_n = 0 \\ z_1^{(n-1)} y_1 + z_2^{(n-1)} y_2 + \dots + z_n^{(n-1)} y_n = (-1)^{n-1} \end{array}\right\} \dots\dots(B).$$

If $D(z_1, z_2 \dots z_n) = 0$ vanished, it would follow that since

$$s_{00} = 0,\ s_{01} = 0 \dots s_{0n-2} = 0,$$

then s_{0n-1} would also vanish, while its value is $(-1)^{n-1}$. Thus the functions $z_1, z_2 \dots z_n$ are not linearly connected with each other. Comparing the systems (A) and (B) it appears that the relation between $y_1 \dots y_n$ and $z_1 \dots z_n$ is a reciprocal one, if we neglect the sign when n is even. From each relation between these systems we deduce a new one by interchanging

$$y_1, y_2 \dots y_n,\quad z_1, z_2 \dots z_n$$

with $\quad (-1)^{n-1} z_1, (-1)^{n-1} z_2 \dots (-1)^{n-1} z_n, y_1, y_2 \dots y_n$.

Thus from the equation

$$z_k = (-1)^{n+k} \frac{D(y_1 \dots y_{k-1}, y_{k+1} \dots y_n)}{D(y_1, y_2 \dots y_n)}$$

we deduce

$$y_k = (-1)^{k-1} \frac{D(z_1 \dots z_{k-1}, z_{k+1} \dots z_n)}{D(z_1, z_2 \dots z_n)}.$$

In consequence of this we shall call $z_1 \dots z_n$ the conjugates of $y_1 \dots y_n$.

8. If we form the product by rows of the two following determinants

$$\begin{vmatrix} y_1 & \cdots & y_k, & y_{k+1} & \cdots & y_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ y_1^{(k-1)} & \cdots & y_k^{(k-1)}, & y_{k+1}^{(k-1)} & \cdots & y_n^{(k-1)} \\ y_1^{(k)} & \cdots & y_k^{(k)}, & y_{k+1}^{(k)} & \cdots & y_n^{(k)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)} & \cdots & y_k^{(n-1)}, & y_{k+1}^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix},$$

$$\begin{vmatrix} 1 & \cdots & 0, & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 1, & 0 & \cdots & 0 \\ z_1 & \cdots & z_k, & z_{k+1} & \cdots & z_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ z_1^{(n-k-1)} & \cdots & z_k^{(n-k-1)}, & z_{k+1}^{(n-k-1)} & \cdots & z_n^{(n-k-1)} \end{vmatrix},$$

the first of which is $D(y_1 \dots y_n)$, the second $D(z_{k+1} \dots z_n)$ we get

$$\begin{vmatrix} y_1 & \cdots & y_k, & s_{00} & \cdots & s_{0n-k-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ y_1^{(k-1)} & \cdots & y_k^{(k-1)}, & s_{k-10} & \cdots & s_{k-1n-k-1} \\ y_1^{(k)} & \cdots & y_k^{(k)}, & s_{k0} & \cdots & s_{kn-k-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)} & \cdots & y_k^{(n-1)}, & s_{n-10} & \cdots & s_{n-1n-k-1} \end{vmatrix}.$$

In this determinant the block of elements common to the first k rows and last $n-k$ columns all vanish, whence it reduces to

$$\begin{vmatrix} y_1 & \cdots & y_k \\ \cdots & \cdots & \cdots \\ y_1^{(k-1)} & \cdots & y_k^{(k-1)} \end{vmatrix} \begin{vmatrix} s_{k0} & \cdots & s_{kn-k-1} \\ \cdots & \cdots & \cdots \\ s_{n-10} & \cdots & s_{n-1n-k-1} \end{vmatrix}.$$

The first of these $= D(y_1 \dots y_k)$, in the second all the elements to the left of the second diagonal vanish, whence its value is

$$(-1)^{\frac{(n-k)(n-k+1)}{2}} s_{n-10}\, s_{n-21} \dots s_{kn-k-1}$$
$$= 1.$$

Thus we have

$$D(y_1 \dots y_n)\, D(z_{k+1} \dots z_n) = D(y_1 \dots y_k).$$

If $k = 0$ we have

$$D(y_1 \dots y_n)\, D(z_1 \dots z_n) = 1.$$

9. From this last equation we get

$$P(y) = (-1)^n \frac{D(y, y_1 \dots y_n)}{D(y_1, y_2 \dots y_n)}$$
$$= (-1)^n D(y, y_1 \dots y_n)\, D(z_1, z_2 \dots z_n).$$

Or

$$P(y) = (-1)^n \begin{vmatrix} y, & y_1 & \dots & y_n \\ y^{(1)}, & y_1^{(1)} & \dots & y_n^{(1)} \\ \dots & \dots & \dots & \dots \\ y^{(n)}, & y_1^{(n)} & \dots & y_n^{(n)} \end{vmatrix} \begin{vmatrix} 1, & 0 & \dots & 0 \\ 0, & z_1 & \dots & z_n \\ \dots & \dots & \dots & \dots \\ 0, & z_1^{(n-1)} & \dots & z_n^{(n-1)} \end{vmatrix}$$

$$= (-1)^n \begin{vmatrix} y, & s_{00}, & s_{01} & \dots & s_{0n-1} \\ y^{(1)}, & s_{10}, & s_{11} & \dots & s_{1n-1} \\ \dots & \dots & \dots & \dots & \dots \\ y^{(n)}, & s_{n0}, & s_{n1} & \dots & s_{nn-1} \end{vmatrix}.$$

Similarly we should get

$$P(z) = (-1)^n \begin{vmatrix} z, & z^{(1)} & \dots & z^{(n)} \\ s_{00}, & s_{01} & \dots & s_{0n} \\ \dots & \dots & \dots & \dots \\ s_{n-10}, & s_{n-11} & \dots & s_{n-1n} \end{vmatrix}.$$

10. These determinants occur in the theory of linear differential equations. Thus, if we have the equation

$$a_0 y + a_1 y^{(1)} + \dots + a_n y^{(n)} = 0$$

where the quantities a_0, $a_1 \dots a_n$ do not contain y. Then if $y_1 \dots y_n$ are n particular integrals, we have the n equations

$$a_0 y_i + a_1 y_i^{(1)} + \dots + a_n y_i^{(n)} = 0 \qquad (i = 1, 2 \dots n),$$

eliminating the a's we get

$$\begin{vmatrix} y, & y^{(1)} & \dots & y^{(n)} \\ y_1, & y_1^{(1)} & \dots & y_1^{(n)} \\ \dots & \dots & \dots & \dots \\ y_n, & y_n^{(1)} & \dots & y_n^{(n)} \end{vmatrix} = 0,$$

or $$D(y, y_1 \dots y_n) = 0.$$

If we solve the equations for $\frac{a_{n-1}}{a_n}$ we get

$$\begin{vmatrix} y_1, y_1^{(1)} \cdots y_1^{(n-2)}, y_1^{(n)} \\ \cdots\cdots\cdots\cdots \\ y_n, y_n^{(1)} \cdots y_n^{(n-2)}, y_n^{(n)} \end{vmatrix} \div \begin{vmatrix} y_1, y_1^{(1)} \cdots y_1^{(n-1)} \\ \cdots\cdots\cdots\cdots \\ y_n, y_n^{(1)} \cdots y_n^{(n-1)} \end{vmatrix} = -\frac{a_{n-1}}{a_n},$$

i.e. $$\frac{d}{dx} \log D(y_1, y_2 \cdots y_n) = -\frac{a_{n-1}}{a_n},$$

$$D(y_1, y_2 \cdots y_n) = \exp. \left(-\int \frac{a_{n-1}}{a_n} dx\right).$$

11. Though not immediately connected with the subject of the present chapter we shall give Hesse's solution of Jacobi's differential equation.

This equation is

$$-A_1 d\eta + A_2 d\xi + A_3 (\xi d\eta - \eta d\xi) = 0,$$

where $$A_i = a_{i1}\xi + a_{i2}\eta + a_{i3} \qquad (i = 1, 2, 3).$$

We can write the equation in the form of the determinant

$$\begin{vmatrix} \xi, & \eta, & 1 \\ d\xi, & d\eta, & 0 \\ A_1, & A_2, & A_3 \end{vmatrix} = 0.$$

Now let $\xi = \frac{x}{z}$, $\eta = \frac{y}{z}$, and the equation becomes

$$\begin{vmatrix} x, & y, & z \\ zx' - z'x, & zy' - yz', & 0 \\ A_1, & A_2, & A_3 \end{vmatrix} = 0.$$

Multiply the first row by z' and add it to the second, this divides by z, and we get

$$\begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ A_1, & A_2, & A_3 \end{vmatrix} = 0.$$

Now let us multiply this equation by

$$\begin{vmatrix} \alpha_1, & \beta_1, & \gamma_1 \\ \alpha_2, & \beta_2, & \gamma_2 \\ \alpha_3, & \beta_3, & \gamma_3 \end{vmatrix},$$

and let $$p_i = \alpha_i x + \beta_i y + \gamma_i z.$$

Also assume that

$$\lambda_i p_i = A_1\alpha_i + A_2\beta_i + A_3\gamma_i.$$

Then

$$\begin{vmatrix} p_1, & p_2, & p_3 \\ dp_1, & dp_2, & dp_3 \\ \lambda_1 p_1, & \lambda_2 p_2, & \lambda_3 p_3 \end{vmatrix} = 0,$$

i.e.

$$\begin{vmatrix} \frac{dp_1}{p_1}, & \frac{dp_2}{p_2}, & \frac{dp_3}{p_3} \\ 1, & 1, & 1 \\ \lambda_1, & \lambda_2, & \lambda_3 \end{vmatrix} = 0, \qquad \text{or} \qquad \begin{vmatrix} \log p_1, & \log p_2, & \log p_3 \\ 1, & 1, & 1 \\ \lambda_1, & \lambda_2, & \lambda_3 \end{vmatrix} = C,$$

or, as we may write it

$$p_1^{\lambda_2-\lambda_3} \cdot p_2^{\lambda_3-\lambda_1} \cdot p_3^{\lambda_1-\lambda_2} = C.$$

Since we assumed that

$$A_1\alpha_i + A_2\beta_i + A_3\gamma_i = \lambda_i p_i.$$

Equating coefficients of x, y, z

$$\alpha_1(a_{11}-\lambda) + \beta_1 a_{12} + \gamma_1 a_{13} = 0,$$

$$\alpha_1 a_{21} + \beta_1(a_{22}-\lambda) + \gamma_1 a_{23} = 0,$$

$$\alpha_1 a_{31} + \beta_1 a_{32} + \gamma_1(a_{33}-\lambda) = 0.$$

Hence eliminating α_1, β_1, γ_1, we see that λ_1, λ_2, λ_3 are the roots of the equation

$$\begin{vmatrix} a_{11}-\lambda, & a_{12}, & a_{13} \\ a_{21}, & a_{22}-\lambda, & a_{23} \\ a_{31}, & a_{32}, & a_{33}-\lambda \end{vmatrix} = 0.$$

CHAPTER XIII.

APPLICATIONS TO THE THEORY OF CONTINUED FRACTIONS.

1. THE application of the theory of determinants to continued fractions is one of its latest developments, and gives great facility in the discussion of these functions.

As usual in English mathematical works we shall denote the continued fraction

$$\cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \dots + \cfrac{b_n}{a_n}}}}$$

by

$$\frac{b_1}{a_1 +}\,\frac{b_2}{a_2 +}\,\frac{b_3}{a_3 +} \cdots \frac{b_n}{+ a_n}.$$

Such a fraction is called a descending continued fraction.

In addition to these we shall discuss a less known form of continued fractions, which, however, is historically the older form of the two, namely, the ascending continued fraction

$$\cfrac{b_1 + \cfrac{b_2 + \cdots}{a_2}}{a_1},$$

which, in an analogous manner, will be denoted by

$$\frac{b_1 +}{a_1}\,\frac{b_2 +}{a_2} \cdots \frac{+ b_n}{a_n}.$$

Our object is to establish a determinant expression for the convergents to these two forms.

2. If we write down the system of equations

$$\begin{aligned} b_1 x &= a_1 x_1 + x_2 \\ b_2 x_1 &= a_2 x_2 + x_3 \\ b_3 x_2 &= a_3 x_3 + x_4 \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

we see that

$$\frac{x_1}{x} = \frac{b_1}{a_1 + \dfrac{x_2}{x_1}}, \quad \frac{x_2}{x_1} = \frac{b_2}{a_2 + \dfrac{x_3}{x_2}} \cdots$$

Hence $\dfrac{x_1}{x}$ is the continued fraction

$$\frac{b_1}{a_1 +} \frac{b_2}{a_2 +} \cdots$$

3. If we are to determine the n^{th} convergent, i.e. the value of the fraction when we stop at $\dfrac{b_n}{a_n}$, we must suppose that x_{n+1} and all succeeding x's vanish, whence we have the system of equations

$$\begin{aligned} b_1 x &= a_1 x_1 + x_2 \\ 0 &= -b_2 x_1 + a_2 x_2 + x_3 \\ 0 &= \qquad -b_3 x_2 + a_3 x_3 + x_4 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 &= \qquad\qquad\qquad -b_n x_{n-1} + a_n x_n . \end{aligned}$$

Solving this set of equations for x_1 we get:

$$\begin{vmatrix} a_1, & 1, & 0 & \dots & & \\ -b_2, & a_2, & 1 & \dots & & \\ 0, & -b_3, & a_3 & \dots & & \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & 0 & \dots & a_{n-1}, & 1 \\ 0, & 0, & 0 & \dots & -b_n, & a_n \end{vmatrix} x_1 = \begin{vmatrix} b_1 x, & 1, & 0 & \dots & \\ 0, & a_2, & 1 & \dots & \\ 0, & -b_3, & a_3 & \dots & \\ \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & 0 & \dots -b_n, & a_n \end{vmatrix}.$$

Thus

$$\frac{x_1}{x} = b_1 \begin{vmatrix} a_2, & 1 & \dots & 0, & 0 \\ -b_3, & a_3 & \dots & 0, & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0, & 0 & \dots & a_{n-1}, & 1 \\ 0, & 0 & \dots & -b_n, & a_n \end{vmatrix} \div \begin{vmatrix} a_1, & 1, & 0 & \dots & 0, & 0 \\ -b_2, & a_2, & 1 & \dots & 0, & 0 \\ 0, & -b_3, & a_3 & \dots & 0, & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & 0 & \dots & a_{n-1}, & 1 \\ 0, & 0, & 0 & \dots & -b_n, & a_n \end{vmatrix}$$

Or
$$\frac{x_1}{x} = \frac{p_n}{q_n} \text{ say.}$$

Where

$$p_n = b_1 \begin{vmatrix} a_2, & 1, & 0, & 0 \ldots & 0, & 0 \\ -b_3, & a_3, & 1, & 0 \ldots & 0, & 0 \\ 0, & -b_4, & a_4, & 1 \ldots & 0, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & 0, & 0 \ldots & a_{n-1}, & 1 \\ 0, & 0, & 0, & 0 \ldots & -b_n, & a_n \end{vmatrix}$$

$$= a_n p_{n-1} + b_n p_{n-2},$$

if we expand (III. 24) according to the elements of the last row and column.

Similarly

$$q_n = \begin{vmatrix} a_1, & 1, & 0, & 0 \ldots & 0, & 0 \\ -b_2, & a_2, & 1, & 0 \ldots & 0, & 0 \\ 0, & -b_3, & a_3, & 1 \ldots & 0, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & 0, & 0 \ldots & a_{n-1}, & 1 \\ 0, & 0, & 0, & 0 \ldots & -b_n, & a_n \end{vmatrix}$$

$$= a_n q_{n-1} + b_n q_{n-2}.$$

Since $p_n = b_1 \dfrac{dq_n}{da_1}$, we can write the convergent in the form

$$b_1 \frac{d}{da_1} (\log q_n).$$

4. The determinants of the form q_n have been called continuants by Mr Muir. Since

$$q_n = a_n q_{n-1} + b_n q_{n-2},$$

if u_n is the number of terms in the continuant of order n

$$u_n = u_{n-1} + u_{n-2},$$

an equation of differences which gives

$$u_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Since $u_1 = 1$, $u_2 = 2$, we have

$$u_n = \{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}\} \div 2^{n+1}\sqrt{5}.$$

It is easy to shew by the binomial theorem that this number is an integer. Prof. Sylvester obtains this number in the form of the series

$$1 + (n-1) + \frac{(n-2)(n-3)}{1 \,.\, 2} + \frac{(n-3)(n-4)(n-5)}{1 \,.\, 2 \,.\, 3} + \dots$$

5. The value of the continuant q_n is the same as that of the determinant

$$q_n' = \begin{vmatrix} a_1, & c_1, & 0 & \dots & 0 \\ d_2, & a_2, & c_2 & \dots & 0 \\ 0, & d_3, & a_3 & \dots & \\ \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & 0 & \dots & a_n \end{vmatrix},$$

provided only

$$c_r d_{r+1} = - b_{r+1} \,.\, (r = 1,\ 2 \dots n-1).$$

This is clear if we expand by III. 24, according to the elements which stand in the last row and column. For then

$$\begin{aligned} q_n' &= a_n q'_{n-1} - d_n c_{n-1} q'_{n-2} \\ &= a_n q'_{n-1} + b_n q'_{n-2}, \end{aligned}$$

while $q_1' = q_1$, $q_2' = q_2$. Hence $q_n' = q_n$, the equation of differences being linear.

Thus we can also write

$$q_n = \begin{vmatrix} a_1, & -1, & 0, & 0 & \dots \\ b_2, & a_2, & -1, & 0 & \dots \\ 0, & b_3, & a_3, & -1 & \dots \\ 0, & 0, & b_4, & a_4 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}.$$

6. The value of the continued fraction is not altered if we replace

$$b_r,\quad a_r,\quad b_{r+1}$$

by

$$kb_r,\ ka_r,\ kb_{r+1}.$$

For the quotient $\frac{p_n}{q_n}$ is unaltered if we multiply numerator and denominator by any the same number. If we multiply both by k, the row

$$\dots - b_r,\ a_r,\ 1 \dots$$

in each is replaced by

$$\ldots - kb_r,\ ka_r,\ k \ldots$$

and by Art. 5, in place of the last k, we can write unity if we replace b_{r+1} by kb_{r+1}.

Since then we can write the continued fraction

$$\frac{b_1}{a_1+}\frac{b_2}{a_2+}\cdots\frac{b_n}{+a_n}$$

in the form

$$\frac{b_1\dfrac{k}{a_1}}{k+}\;\frac{b_2\dfrac{k^2}{a_1a_2}}{k+}\;\frac{b_3\dfrac{k^2}{a_2a_3}}{k+}\cdots+\frac{b_n\dfrac{k^2}{a_{n-1}a_n}}{k};$$

q_n can be written in the form of the skew determinant

$$\begin{vmatrix} k, & \alpha_1, & 0, & 0 \ldots \\ -\alpha_1, & k, & \alpha_2, & 0 \ldots \\ 0, & -\alpha_2, & k, & \alpha_3 \ldots \\ 0, & 0, & -\alpha_3, & k \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix},$$

where $$\alpha_r = \sqrt{\left(\frac{b_{r+1}k^2}{a_ra_{r+1}}\right)}.$$

Thus the convergents to a continued fraction can always be represented by the quotient of two skew determinants.

7. In any determinant D we have

$$D\frac{d^2D}{da_{11}da_{nn}} = \frac{dD}{da_{11}}\frac{dD}{da_{nn}} - \frac{dD}{da_{1n}}\frac{dD}{da_{n1}}.$$

For D take the continuant q_n (Art. 5), then

$$\frac{d^2D}{da_{11}da_{nn}} = \frac{1}{b_1}\cdot p_{n-1}, \qquad \frac{dD}{da_{nn}} = q_{n-1}, \qquad \frac{dD}{da_{11}} = \frac{1}{b_1}p_n,$$

$$\frac{dD}{da_{1n}} = b_2b_3\ldots b_n, \quad \frac{dD}{da_{n1}} = (-1)^{n-1}.$$

Thus $$q_np_{n-1} - q_{n-1}p_n = (-1)^n b_1b_2\ldots b_n.$$

8. In the case of the ascending continued fraction

$$\frac{b_1+}{a_1}\frac{b_2+}{a_2}\cdots$$

it is clear that if the n^{th} convergent be $\frac{p_n}{q_n}$, the scale of relation is

$$\frac{p_n}{q_n} = \frac{a_n p_{n-1} + b_n}{a_n q_{n-1}}.$$

Hence $$q_n = a_1 a_2 \dots a_n.$$

To determine p_n we have the system of equations:

$$\begin{array}{llllll}
p_1 & & & & & = b_1 \\
-a_2 p_1 + p_2 & & & & & = b_2 \\
& -a_3 p_2 + p_3 & & & & = b_3 \\
\multicolumn{6}{c}{\dots\dots\dots\dots\dots\dots\dots\dots} \\
& & -a_{n-1} p_{n-2} + p_{n-1} & & & = b_{n-1} \\
& & & -a_n p_{n-1} + p_n & & = b_n.
\end{array}$$

The determinant of this system is unity, all the elements to the right of the leading diagonal vanishing;

$$\therefore\ p_n = \begin{vmatrix} 1\ , & 0, & 0 \dots & 0\ , & b_1 \\ -a_2, & 1, & 0 \dots & 0\ , & b_2 \\ 0\ , & -a_3, & 1 \dots & 0\ , & b_3 \\ \dots & \dots & \dots & \dots & \dots \\ 0\ , & 0, & 0 \dots & 1\ , & b_{n-1} \\ 0\ , & 0, & 0 \dots & -a_n, & b_n \end{vmatrix}.$$

Multiply all the columns except the last by -1, and move the last column to the first place; the determinant is unchanged, thus

$$p_n = \begin{vmatrix} b_1, & -1, & 0 & \dots 0, & 0 \\ b_2, & a_2\ , & -1 & \dots 0, & 0 \\ b_3, & 0\ , & a_3 & \dots 0, & 0 \\ \dots & \dots & \dots & \dots & \dots \\ b_n, & 0\ , & 0 & \dots 0, & a_n \end{vmatrix}.$$

The n^{th} convergent to the fraction is

$$\frac{p_n}{a_1 a_2 \dots a_n}.$$

The number of terms in p_n is n.

9. By means of these determinant expressions for the convergents we can transform an ascending continued fraction into a descending continued fraction.

In the determinant p_n of the preceding article multiply the r^{th} row, beginning with the last, by b_{r-1}, and subtract from it the $(r-1)^{\text{st}}$ row multiplied by b_r, and do this for all the rows. The determinant is altered by the factor

$$k = (b_1 b_2 \dots b_{n-1})^{-1},$$

and

$$p_n = k \begin{vmatrix} b_1, & -1\ , & 0 & \dots & & & \\ 0, & a_2b_1+b_2, & -b_1 & \dots & & & \\ 0, & -a_2b_3\ , & a_3b_2+b_3 & \dots & & & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0, & 0\ , & 0 & \dots & a_{n-2}b_{n-3}+b_{n-2}, & -b_{n-3}\ , & 0 \\ 0, & 0\ , & 0 & \dots & -a_{n-2}b_{n-1}\ , & a_{n-1}b_{n-2}+b_{n-1}, & -b_{n-2} \\ 0, & 0\ , & 0 & \dots & 0\ , & -a_{n-1}b_n\ , & a_nb_{n-1}+b_n \end{vmatrix}.$$

Similarly, since

$$q_n = a_1 a_2 \dots a_n$$

$$= \begin{vmatrix} a_1, & -1, & 0 & \dots & 0\ , & 0 \\ 0, & a_2\ , & -1 & \dots & 0\ , & 0 \\ 0, & 0\ , & a_3 & \dots & 0\ , & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0, & 0\ , & 0 & \dots & a_{n-1}, & -1 \\ 0, & 0\ , & 0 & \dots & 0\ , & a_n \end{vmatrix},$$

$$q_n = k \begin{vmatrix} a_1\ , & -1\ , & 0 & \dots & & \\ -a_1b_2, & a_2b_1+b_2, & -b_1 & \dots & & \\ 0\ , & -a_2b_3\ , & a_3b_2+b_3 & \dots & & \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0\ , & 0\ , & 0 & \dots & -a_{n-1}b_n, & a_nb_{n-1}+b_n \end{vmatrix}.$$

Now on inspection it is clear that these determinants p_n and

q_n are continuants as defined in Art. 3, whose $2^{nd}, 3^{rd} \ldots (n-1)^s$ rows have been multiplied by $b_1, b_2 \ldots b_{n-2}$ respectively, also

$$p_n = b_1 \frac{dq_n}{da_1}.$$

Whence by Arts. 3 and 6

$$\frac{p_n}{q_n} = \frac{b_1}{a_1 -} \; \frac{a_1 b_2}{a_2 b_1 + b_2 -} \; \frac{a_2 b_1 b_3}{a_3 b_2 + b_3} \cdots \frac{a_{n-2} b_{n-3} b_{n-1}}{a_{n-1} b_{n-2} + b_{n-1} -} \; \frac{a_{n-1} b_{n-2} b_n}{a_n b_{n-1} + b_n},$$

which gives us a rule for transforming an ascending continued fraction into a descending continued fraction, the number of quotients in each being the same.

10. We can make immediate use of this theorem to deduce a formula of Euler's, by means of which a series can be converted into a continued fraction.

Take the series

$$S = A_1 - A_2 + A_3 - A_4 + \ldots + (-1)^{n-1} A_n$$

$$= \begin{vmatrix} A_1, & 1, & 0, & 0 & \ldots & 0 \\ A_2, & 1, & 1, & 0 & \ldots & 0 \\ A_3, & 0, & 1, & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ A_n, & 0, & 0, & 0 & \ldots & 1 \end{vmatrix},$$

as we see by subtracting from each row the one below it, beginning with the last, when the determinant reduces to its principal term. Multiplying each column after the first by -1, we reduce the determinant to the continuant for an ascending continued fraction. Thus the above series is equal to:

$$(-1)^{n-1} \frac{A_1 +}{1} \; \frac{A_2 +}{-1} \cdots \frac{A_{n-1} +}{-1} \; \frac{A_n}{-1},$$

and transforming this by the rule just obtained to a descending continued fraction

$$S = (-1)^{n-1} \frac{A_1}{1 -} \; \frac{A_2}{A_2 - A_1 +} \; \frac{A_1 A_3}{A_3 - A_2 +} \cdots \frac{A_{n-2} A_n}{A_n - A_{n-1}}$$

$$= \frac{A_1}{1 +} \; \frac{A_2}{A_1 - A_2 +} \; \frac{A_1 A_3}{A_2 - A_3 +} \cdots \frac{A_{n-2} A_n}{A_{n-1} - A_n}.$$

If the original series is

$$s = \frac{1}{A_1} - \frac{1}{A_2} + \frac{1}{A_3} \cdots$$

we can obtain its form as a continued fraction by altering the continuant to S in accordance with Art. 6, when we get

$$s = \frac{1}{A_1 +}\;\frac{A_1^2}{A_2 - A_1 +}\;\frac{A_2^2}{A_3 - A_2 +} \cdots$$

11. Various generalisations of continued fractions have been devised by Jacobi and others. The following generalisation, due to Fürstenau, is taken from a review of his memoir by Günther.

If x and y are any two real numbers, and we write

$$y = a_0 + \frac{x_1}{y_1},\quad y_1 = a_1 + \frac{x_2}{y_2},\quad y_2 = a_2 + \frac{x_3}{y_3} \cdots$$

$$x = b_0 + \frac{1}{y_1},\quad x_1 = b_1 + \frac{1}{y_2},\quad x_2 = b_2 + \frac{1}{y_3} \cdots$$

where a and b are the greatest integers contained in x and y, then on substituting we have:

$$y = a_0 + \cfrac{b_1 + \cfrac{1}{a_2 + \cfrac{b_3 + \cfrac{1}{a_4}}{a_3 + \cfrac{b_4}{a_4}}}}{a_1 + \cfrac{b_2 + \cfrac{1}{a_3 + \cfrac{b_4}{a_4}}}{a_2 + \cfrac{b_3 + \cfrac{1}{a_4}}{a_3 + \cfrac{b_4}{a_4}}}}$$

and

$$x = b_0 + \cfrac{1}{a_1 + \cfrac{b_2 + \cfrac{1}{a_3 + \cfrac{b_4 +}{a_4 +}}}{a_2 + \cfrac{b_3 + \cfrac{1}{a_4 +}}{a_3 + \cfrac{b_4}{a_4}}}}.$$

If now all that stands to the left of one of the vertical lines be called a first, second ... convergent, and if we denote the numerators of x and y by X_p, Y_p, while the denominator, which is clearly the same for both, is called N_p, we shall have

$$(Y, X, N)_{p+1} = a_{p+1}(Y, X, N)_p + b_{p+1}(Y, X, N)_{p-1} + (Y, X, N)_{p-2}.$$

Thus the equations have four instead of three terms, and we get

$$Y_p = \begin{vmatrix} a_0, & b_1, & 1, & 0 \dots & 0 \\ -1, & a_1, & b_2, & 1 \dots & 0 \\ 0, & -1, & a_2, & b_3 \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & 0, & 0 \dots & a_p \end{vmatrix}$$

$$X_p = \begin{vmatrix} b_0, & 1, & 0, & 0 \dots & 0 \\ -1, & a_1, & b_2, & 1 \dots & 0 \\ 0, & -1, & a_2, & b_3 \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & 0, & 0 \dots & a_p \end{vmatrix}$$

$$N_p = \begin{vmatrix} a_1, & b_2, & 1, & 0 \dots & 0 \\ -1, & a_2, & b_3, & 1 \dots & 0 \\ 0, & -1, & a_3, & b_4 \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & 0, & 0 \dots & a_p \end{vmatrix}$$

Corresponding to the theorem of Art. 7 we have now

$$\begin{vmatrix} Y_{p+1}, & Y_p, & Y_{p-1} \\ X_{p+1} & X_p, & X_{p-1} \\ N_{p+1}, & N_p, & N_{p-1} \end{vmatrix} = 1.$$

12. If ordinary continued fractions be called fractions of the first class, those in Art. 11 may be called fractions of the second class.

Fürstenau extends the idea still further, and summing up his results we may state them as follows: If we seek to determine n quantities x_1, $x_2 \dots x_n$ as fractions of the form

$$x_1 = \frac{X_1}{N}, \quad x_2 = \frac{X_2}{N} \dots x_n = \frac{X_n}{N}$$

each such fraction can be written as a continued fraction of the $(n-1)^{\text{th}}$ class. The p^{th} convergents to these continued fractions take the form

$$\frac{X_{p1}}{N_p},\ \frac{X_{p2}}{N_p} \dots \frac{X_{pn}}{N_p},$$

and if

$$\begin{array}{lll} a_{11} & \dots & a_{1n+1} \\ a_{21} & \dots & a_{2n+1} \\ \dots & \dots & \dots \\ a_{n+11} & \dots & a_{n+1n+1} \end{array}$$

are the quotients entering into the continued fractions, then

$$X_{pq} = a_{1p}X_{p-1q} + a_{2p}X_{p-2q} + \dots + a_{n+1p}X_{p-n-1q},$$
$$N_p = a_{1p}N_{p-1} + a_{2p}N_{p-2} + \dots + a_{n+1p}N_{p-n-1}.$$

The quotients X and N are always connected by the equation

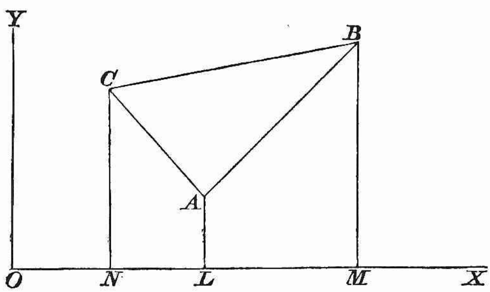
$$\begin{vmatrix} X_{p1}, & X_{p-11}, & X_{p-21} & \dots & X_{p-n1} \\ X_{p2}, & X_{p-12}, & X_{p-22} & \dots & X_{p-n2} \\ \dots & \dots & \dots & \dots & \dots \\ X_{pn}, & X_{p-1n}, & X_{p-2n} & \dots & X_{p-nn} \\ N_p, & N_{p-1}, & N_{p-2} & \dots & N_{p-n} \end{vmatrix} = (-1)^{np}.$$

The author also shews that the real roots of an equation of the n^{th} order can be represented as periodic continued fractions of the $(n-1)^{\text{th}}$ class.

CHAPTER XIV.

APPLICATIONS TO GEOMETRY.

1. THE axes being rectangular let the co-ordinates of the angular points of a triangle ABC be (x_1, y_1) (x_2, y_2) (x_3, y_3). Then if Δ is the area of the triangle it is plain from the figure that



$$\Delta = \text{trap. } BN - \text{trap. } BL - \text{trap. } CL$$

$$= \tfrac{1}{2}(y_2 + y_3)(x_2 - x_3) - \tfrac{1}{2}(y_2 + y_1)(x_2 - x_1) - \tfrac{1}{2}(y_3 + y_1)(x_1 - x_3),$$

or
$$2\Delta = y_3 x_2 - y_2 x_3 + x_3 y_1 - x_1 y_3 + x_1 y_2 - x_2 y_1$$

$$= \begin{vmatrix} 1, & 1, & 1 \\ x_1, & x_2, & x_3 \\ y_1, & y_2, & y_3 \end{vmatrix} = \begin{vmatrix} 1, & x_1, & y_1 \\ 1, & x_2, & y_2 \\ 1, & x_3, & y_3 \end{vmatrix}.$$

If the axes were oblique this would have to be multiplied by the sine of the angle between the axes. Thus

$$2\Delta = \sin(XY) \begin{vmatrix} 1, & 1, & 1 \\ x_1, & x_2, & x_3 \\ y_1, & y_2, & y_3 \end{vmatrix},$$

where (XY) is the angle between the axes. This form is however not often used, and unless the fact is specially mentioned the axes are supposed to be rectangular.

If we multiply the first row by x_1 and subtract it from the second, then the first row by y_1 and subtract it from the third, we get

$$2\Delta = \begin{vmatrix} x_2 - x_1, & x_3 - x_1 \\ y_2 - y_1, & y_3 - y_1 \end{vmatrix}.$$

It must be noticed that the area of a triangle changes sign if we alter the cyclical order of the letters. Thus ABC and ACB are equal triangles, whose areas are opposite in sign; ABC and BCA are equal in magnitude and agree in sign.

2. Let the co-ordinates of the angular points of a tetrahedron $ABCD$ be $(x_1, y_1, z_1) \dots (x_4, y_4, z_4)$. Let V be its volume.

Let Δ be the area of the triangle BCD, and let the equation of its plane be

$$(x - x_2) \cos \alpha + (y - y_2) \cos \beta + (z - z_2) \cos \gamma = 0.$$

The projection of the triangle BCD on the plane of xy is $\Delta \cos \gamma$, and the co-ordinates of its angular points are

$$(x_2, y_2)\ (x_3, y_3)\ (x_4, y_4);$$

thus, by Art. 1,

$$2\Delta \cos \gamma = \begin{vmatrix} x_3 - x_2, & x_4 - x_2 \\ y_3 - y_2, & y_4 - y_2 \end{vmatrix}.$$

Similarly we get

$$2\Delta \cos \beta = \begin{vmatrix} z_3 - z_2, & z_4 - z_2 \\ x_3 - x_2, & x_4 - x_2 \end{vmatrix}, \qquad 2\Delta \cos \alpha = \begin{vmatrix} y_3 - y_2, & y_4 - y_2 \\ z_3 - z_2, & z_4 - z_2 \end{vmatrix}.$$

If p is the perpendicular from A on the plane BCD,

$$-p = (x_1 - x_2) \cos \alpha + (y_1 - y_2) \cos \beta + (z_1 - z_2) \cos \gamma.$$

Hence

$$\begin{aligned} -6V &= -2\Delta p \\ &= 2\Delta \cos \alpha\, (x_1 - x_2) + 2\Delta \cos \beta\, (y_1 - y_2) + 2\Delta \cos \gamma\, (z_1 - z_2) \end{aligned}$$

$$= (x_1 - x_2) \begin{vmatrix} y_3 - y_2, & y_4 - y_2 \\ z_3 - z_2, & z_4 - z_2 \end{vmatrix} + (y_1 - y_2) \begin{vmatrix} z_3 - z_2, & z_4 - z_2 \\ x_3 - x_2, & x_4 - x_2 \end{vmatrix}$$

$$+ (z_1 - z_2) \begin{vmatrix} x_3 - x_2, & x_4 - x_2 \\ y_3 - y_2, & y_4 - y_2 \end{vmatrix}$$

$$= \begin{vmatrix} x_1 - x_2, & x_3 - x_2, & x_4 - x_2 \\ y_1 - y_2, & y_3 - y_2, & y_4 - y_2 \\ z_1 - z_2, & z_3 - z_2, & z_4 - z_2 \end{vmatrix} = - \begin{vmatrix} 1, & 1, & 1, & 1 \\ x_1 - x_2, & 0, & x_3 - x_2, & x_4 - x_2 \\ y_1 - y_2, & 0, & y_3 - y_2, & y_4 - y_2 \\ z_1 - z_2, & 0, & z_3 - z_2, & z_4 - z_2 \end{vmatrix}.$$

Or if in this last determinant we multiply the first row by x_2, y_2, z_2 and add it to the second, third and fourth rows respectively,

$$6V = \begin{vmatrix} 1, & 1, & 1, & 1 \\ x_1, & x_2, & x_3, & x_4 \\ y_1, & y_2, & y_3, & y_4 \\ z_1, & z_2, & z_3, & z_4 \end{vmatrix}.$$

3. If the tetrahedron be referred to oblique axes through the same origin, and if the cosines of the angles these make with the rectangular axes be given by the scheme

	X	Y	Z
x	l_1	l_2	l_3
y	m_1	m_2	m_3
z	n_1	n_2	n_3

$$x = Xl_1 + Yl_2 + Zl_3, \text{ \&c.}$$

Whence

$$\begin{vmatrix} 1, & 1, & 1, & 1 \\ x_1, & x_2, & x_3, & x_4 \\ y_1, & y_2, & y_3, & y_4 \\ z_1, & z_2, & z_3, & z_4 \end{vmatrix} = \begin{vmatrix} 1, & 1, & 1, & 1 \\ X_1, & X_2, & X_3, & X_4 \\ Y_1, & Y_2, & Y_3, & Y_4 \\ Z_1, & Z_2, & Z_3, & Z_4 \end{vmatrix} \begin{vmatrix} 1, & 0, & 0, & 0 \\ 0, & l_1, & m_1, & n_1 \\ 0, & l_2, & m_2, & n_2 \\ 0, & l_3, & m_3, & n_3 \end{vmatrix}.$$

Now let

$$D = \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}.$$

Then remembering that

$$l_1^2 + m_1^2 + n_1^2 = 1,$$

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = \cos XY, \text{ \&c.,}$$

we have

$$D^2 = \begin{vmatrix} 1, & \cos XY, & \cos XZ \\ \cos YX, & 1, & \cos YZ \\ \cos ZX, & \cos ZY, & 1 \end{vmatrix}.$$

This determinant is usually called the square of the sine of the solid angle, contained by the oblique axes in analogy with the determinant

$$\sin^2 XY = \begin{vmatrix} 1, & \cos XY \\ \cos YX, & 1 \end{vmatrix}$$

in a plane. Thus

$$D^2 = \sin^2 (XYZ).$$

And in oblique co-ordinates

$$6V = \begin{vmatrix} 1, & 1, & 1, & 1 \\ X_1, & X_2, & X_3, & X_4 \\ Y_1, & Y_2, & Y_3, & Y_4 \\ Z_1, & Z_2, & Z_3, & Z_4 \end{vmatrix} \sin (XYZ).$$

4. From the determinant expressions in Arts. 1 and 2 we can at once write down a number of geometrical relations.

If the distances x be measured along a straight line from a fixed point, we see that

$$\begin{vmatrix} 1, & x_i \\ 1, & x_k \end{vmatrix} = (x_k - x_i) = (ki)$$

is the distance between the two points marked k and i. The determinant

$$\begin{vmatrix} 1, & x_1, & 1, & x_1 \\ 1, & x_2, & 1, & x_2 \\ 1, & x_3, & 1, & x_3 \\ 1, & x_4, & 1, & x_4 \end{vmatrix}$$

vanishes identically, because it has several columns alike. Expanding it by III. 6 according to products of minors from the first two and last two columns, we get

$$(12)(34) + (13)(42) + (14)(23) = 0.$$

Or, if we call the points A, B, C, D, this is the well-known relation between the segments formed by four collinear points

$$AB \, . \, CD + AC \, . \, DB + AD \, . \, BC = 0.$$

If we expand the vanishing determinant

$$| \; 1, \; x_i, \; y_i, \; 1, \; x_i, \; y_i \; | \qquad (i = 1, 2 \ldots 6)$$

according to minors from the first three and last three columns, we get no geometrical relation, the terms cancelling each other in pairs.

But if we expand the determinant

$$|1, x_i, y_i, z_i, 1, x_i, y_i, z_i| = 0 \qquad (i = 1, 2 \ldots 8)$$

according to the products of minors from the first and last four columns we get an identical relation of thirty-five terms between the volumes of the tetrahedra, formed by eight points.

5. Again, for five points

$$\begin{vmatrix} 1, & 1, & 1, & 1, & 1 \\ 1, & 1, & 1, & 1, & 1 \\ x_1, & x_2, & x_3, & x_4, & x_5 \\ y_1, & y_2, & y_3, & y_4, & y_5 \\ z_1, & z_2, & z_3, & z_4, & z_5 \end{vmatrix} = 0.$$

If v_1 = volume of tetrahedron (2345) and we expand the determinant according to the elements of the first row, by III. 10, we get

$$v_1 + v_2 + v_3 + v_4 + v_5 = 0.$$

6. By the theorem v. 4,

$$\begin{vmatrix} 1, & 1, & 1 \\ x_1, & x_2, & y_3 \\ y_1, & y_2, & y_3 \end{vmatrix} \begin{vmatrix} 1, & 1, & 1 \\ \xi_1, & \xi_2, & \xi_3 \\ \eta_1, & \eta_2, & \eta_3 \end{vmatrix} = \begin{vmatrix} 1, & 1, & 1 \\ x_1, & \xi_1, & \xi_2 \\ y_1, & \eta_1, & \eta_2 \end{vmatrix} \begin{vmatrix} 1, & 1, & 1 \\ \xi_3, & x_2, & x_3 \\ \eta_3, & y_2, & y_3 \end{vmatrix}$$

$$+ \begin{vmatrix} 1, & 1, & 1 \\ x_1, & \xi_2, & \xi_3 \\ y_1, & \eta_2, & \eta_3 \end{vmatrix} \begin{vmatrix} 1, & 1, & 1 \\ \xi_1, & x_2, & x_3 \\ \eta_1, & y_2, & y_3 \end{vmatrix} + \begin{vmatrix} 1, & 1, & 1 \\ x_1, & \xi_3, & \xi_1 \\ y_1, & \eta_3, & \eta_1 \end{vmatrix} \begin{vmatrix} 1, & 1, & 1 \\ \xi_2, & x_2, & x_3 \\ \eta_2, & y_2, & y_3 \end{vmatrix}.$$

Or if the two sets of three points be called ABC, DEF,

$$ABC \times DEF = ADE \times FBC + AEF \times DBC + AFD \times BCE$$

is a relation between triangles.

The product of the two determinants

$$\begin{vmatrix} 1, & 1, & 1, & 1 \\ x_1, & x_2, & x_3, & x_4 \\ y_1, & y_2, & y_3, & y_4 \\ z_1, & z_2, & z_3, & z_4 \end{vmatrix} \qquad \begin{vmatrix} 1, & 1, & 1, & 1 \\ \xi_1, & \xi_2, & \xi_3, & \xi_4 \\ \eta_1, & \eta_2, & \eta_3, & \eta_4 \\ \zeta_1, & \zeta_2, & \zeta_3, & \zeta_4 \end{vmatrix}$$

can be represented either as a sum of four terms

$$\begin{vmatrix} 1, & 1, & 1, & 1 \\ x_1, & x_2, & x_3, & \xi_1 \\ y_1, & y_2, & y_3, & \eta_1 \\ z_1, & z_2, & z_3, & \zeta_1 \end{vmatrix} \begin{vmatrix} 1, & 1, & 1, & 1 \\ \xi_2, & \xi_3, & \xi_4, & x_4 \\ \eta_2, & \eta_3, & \eta_4, & y_4 \\ \zeta_2, & \zeta_3, & \zeta_4, & z_4 \end{vmatrix} + \ldots,$$

or as the sum of six terms

$$\begin{vmatrix} 1, & 1, & 1, & 1 \\ x_1, & x_2, & \xi_1, & \xi_2 \\ y_1, & y_2, & \eta_1, & \eta_2 \\ z_1, & z_2, & \zeta_1, & \zeta_2 \end{vmatrix} \begin{vmatrix} 1, & 1, & 1, & 1 \\ \xi_3, & \xi_4, & x_3, & x_4 \\ \eta_3, & \eta_4, & y_3, & y_4 \\ \zeta_3, & \zeta_4, & z_3, & z_4 \end{vmatrix} + \ldots.$$

Or calling the two sets of points $ABCD$, $EFGH$, we have the identical relations between the volumes of tetrahedra:

$$\begin{aligned} ABCD \times EFGH = {} & ABCE \times FGHD - ABCF \times GHED \\ & + ABCG \times HEFD - ABCH \times FGED \\ ABCD \times EFGH = {} & ABEF \times GHCD + ABGH \times EFCD \\ & + ABEG \times HFCD + ABHF \times EGCD \\ & + ABEH \times FGCD + ABFG \times EHCD. \end{aligned}$$

Application of Alternate Numbers in Geometry.

7. In applying alternate numbers to geometry, a number stands for a point in a flat space whose dimensions are one less than the number of units.

To begin with a plane, the units e_1, e_2, e_3 stand for the vertices of a fundamental triangle ABC. Any other number

$$P = xe_1 + ye_2 + ze_3$$

stands for some point in the plane of the triangle. It is generally convenient to assume that

$$x + y + z = 1,$$

so that x, y, z may be taken to mean the ratios of the triangles PBC, PCA, PAB to the triangle ABC, though this is not necessary.

If P and Q are two points, then

$$\frac{mP + nQ}{m + n}$$

is a point in the line PQ, dividing PQ in the ratio $m : n$. Thus $\frac{1}{2}(P + Q)$ is the middle point, and $P - Q$ the point at infinity of PQ.

Similar definitions hold for a space of three dimensions. Four points $ABCD$ being taken and represented by the units e_1, e_2, e_3, e_4 any other point in the space is represented by

$$P = xe_1 + ye_2 + ze_3 + we_4,$$

where if we choose we may write

$$x + y + z + w = 1,$$

x being the ratio of the tetrahedron $PBCD$ to $ABCD$.

And so on for a space of any number of dimensions.

Then a binary product e_re_s is a unit length measured on the line joining the points e_r, e_s or the distance between the points e_r, e_s.

A ternary product $e_re_se_t$ is a unit area measured on the plane of the points e_r, e_s, e_t, or the area of the triangle formed by the points e_r, e_s, e_t. And so on.

In a space of two dimensions the product of three points is the area of the triangle they form referred to the fundamental triangle.

Now if

$$P = x_1e_1 + y_1e_2 + z_1e_3,$$
$$Q = x_2e_1 + \dots$$
$$R = x_3e_1 + \dots$$
$$PQR = \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} e_1e_2e_3.$$

And $e_1e_2e_3 = ABC = \Delta$, the area of the fundamental triangle, so that in areal co-ordinates

$$PQR = \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} \Delta.$$

Similarly in a flat space of three dimensions if

$$e_1e_2e_3e_4 = V$$

is the volume of the fundamental tetrahedron, the volume of the tetrahedron formed by four points is

$$PQRS = \begin{vmatrix} x_1, & y_1, & z_1, & w_1 \\ x_2, & y_2, & z_2, & w_2 \\ x_3, & y_3, & z_3, & w_3 \\ x_4, & y_4, & z_4, & w_4 \end{vmatrix} V.$$

Similar definitions may be stated with reference to flat spaces of more than three dimensions.

The assumption which has been made throughout the present work, that the product of all the units of a system is unity, receives here its justification and explanation. For, geometrically speaking, the product of the units is the measure of the fundamental figure of the space considered, which is our unit of measure. In a plane, for example, it is the area of the triangle of reference, in ordinary space of three dimensions the volume of the tetrahedron of reference. It is no part of the plan of the present treatise to develop the geometrical applications of alternate numbers; for these we must refer to the memoirs and works of Grassmann and Schlegel.

Angles between straight lines. Solid angles. Spherical figures.

8. With rectangular axes let

$$\begin{array}{ll} l_1,\ m_1,\ n_1 & \lambda_1,\ \mu_1,\ \nu_1 \\ l_2,\ m_2,\ n_2 & \lambda_2,\ \mu_2,\ \nu_2 \\ \cdots\cdots\cdots & \cdots\cdots\cdots \end{array}$$

be the direction cosines of two sets of straight lines, then if

$$\cos(ik) = l_i\lambda_k + m_i\mu_k + n_i\nu_k$$

is the cosine of the angle between the i^{th} line of the first and k^{th} of the second system; if we compound the two arrays, we get the determinant

$$|\cos(ik)|.$$

Hence by IV. 2, if there are two sets of four straight lines we get

$$\begin{vmatrix} \cos(11) & \dots & \cos(14) \\ \dots & \dots & \dots \\ \cos(41) & \dots & \cos(44) \end{vmatrix} = 0 \quad\dots\dots\dots\dots(\text{i}).$$

If there are two sets of three straight lines a, b, c; f, g, h,

$$\begin{vmatrix} \cos af, & \cos ag, & \cos ah \\ \cos bf, & \cos bg, & \cos bh \\ \cos cf, & \cos cg, & \cos ch \end{vmatrix} = \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} \begin{vmatrix} \lambda_1, & \mu_1, & \nu_1 \\ \lambda_2, & \mu_2, & \nu_2 \\ \lambda_3, & \mu_3, & \nu_3 \end{vmatrix}$$

$$= \sin(abc)\sin(fgh) \quad\dots\dots\dots\dots(\text{ii}).$$

If there are only two straight lines in each set

$$\begin{vmatrix} \cos(11), & \cos(12) \\ \cos(21), & \cos(22) \end{vmatrix} = \begin{vmatrix} l_1, & m_1 \\ l_2, & m_2 \end{vmatrix} \begin{vmatrix} \lambda_1, & \mu_1 \\ \lambda_2, & \mu_2 \end{vmatrix} + \dots.$$

Now if n, ν be the directions of the shortest distances between the lines of each pair, θ, ϕ, the angles between the pairs

$$\begin{vmatrix} l_1, & m_1 \\ l_2, & m_2 \end{vmatrix} = \sin\theta\cos(nz), \text{ \&c.}$$

$$\therefore \begin{vmatrix} \cos(11), & \cos(12) \\ \cos(21), & \cos(22) \end{vmatrix} = \sin\theta\sin\phi\cos(n\nu) \quad\dots\dots(\text{iii}).$$

9. If in the relation (i) of Art. 8 the two sets of straight lines coincide with one set of straight lines a, b, c, d, we have

$$\begin{vmatrix} 1, & \cos(ab), & \cos(ac), & \cos(ad) \\ \cos(ba), & 1, & \cos(bc), & \cos(bd) \\ \cos(ca), & \cos(cb), & 1, & \cos(cd) \\ \cos(da), & \cos(db), & \cos(dc), & 1 \end{vmatrix} = 0.$$

This is the identical relation between the mutual inclination of four straight lines in space, or also the relation between the sides and diagonals of a spherical quadrilateral.

If we write $-\cos(AB)$ for $\cos(ab)$, or what comes to the same thing change the signs of the elements in the leading diagonal, it becomes the identical relation between the cosines of the dihedral angles of a tetrahedron formed by four planes A, B, C, D perpendicular to the lines a, b, c, d.

10. If the two straight lines marked 1 coincide with two straight lines u, v; while those marked 2, 3, 4 coincide with a set of oblique axes x, y, z,

$$\begin{vmatrix} \cos uv, & \cos ux, & \cos uy, & \cos uz \\ \cos xv, & 1, & \cos xy, & \cos xz \\ \cos yv, & \cos yx, & 1, & \cos yz \\ \cos zv, & \cos zx, & \cos zy, & 1 \end{vmatrix} = 0,$$

which gives the cosine of the angle between two straight lines u, v, referred to a set of oblique axes x, y, z in terms of their direction cosines.

11. As another example of the use of the same formula, let ABC, $A'B'C'$ be two spherical triangles, O, O' the centres of the small circles circumscribing them. For our two sets of straight lines take the lines joining the centre to $O'ABC$, $OA'B'C'$. Then if $OO' = \phi$, and R, R' are the radii of the circumscribing circles, we get

$$\begin{vmatrix} \cos\phi, & \cos R', & \cos R', & \cos R' \\ \cos R, & \cos(AA'), & \cos(AB'), & \cos(AC') \\ \cos R, & \cos(BA'), & \cos(BB'), & \cos(BC') \\ \cos R, & \cos(CA'), & \cos(CB'), & \cos(CC') \end{vmatrix} = 0.$$

We can write this

$$\cos\phi \sin(ABC)\sin(A'B'C') = -\cos R\cos R' \begin{vmatrix} 0, & 1 & \dots & 1 \\ 1, & \cos AA' & \dots & \cos(AC') \\ \dots & \dots & \dots & \dots \\ 1, & \cos(CA') & \dots & \cos(CC') \end{vmatrix}$$

If the angle at which the small circles cut is ψ

$$\cos\phi = \cos R \cos R' - \sin R \sin R' \cos\psi;$$

and the above formula can be written

$$(1 - \tan R \tan R' \cos\psi) \sin(ABC) \sin(A'B'C') = -\begin{vmatrix} 0, & 1 & \dots & 1 \\ 1, & \cos(AA') & \dots & \cos(AC') \\ \dots & \dots & \dots & \dots \\ 1, & \cos(CA') & \dots & \cos(CC') \end{vmatrix}.$$

If the two systems coincide $\psi = \pi$, and we get

$$\begin{vmatrix} \sec^2 R, & 1, & 1, & 1 \\ 1, & 1, & \cos c, & \cos b \\ 1, & \cos c, & 1, & \cos a \\ 1, & \cos b, & \cos a, & 1 \end{vmatrix} = 0,$$

a, b, c being the sides of the spherical triangle.

12. Similar relations can be developed in the same way for a plane.

In a plane we can shew that for two sets of three straight lines

$$\begin{vmatrix} \cos(11), & \cos(12), & \cos(13) \\ \cos(21), & \cos(22), & \cos(23) \\ \cos(31), & \cos(32), & \cos(33) \end{vmatrix} = 0,$$

and then deduce

$$\begin{vmatrix} 1, & \cos C, & \cos B \\ \cos C, & 1, & \cos A \\ \cos B, & \cos A, & 1 \end{vmatrix} = 0, \qquad \begin{vmatrix} \cos(xy), & \cos(xa), & \cos(xb) \\ \cos(ay), & 1, & \cos(ab) \\ \cos(by), & \cos(ba), & 1 \end{vmatrix} = 0,$$

similar to the equations of 9 and 10.

13. Next, let us compound two arrays

$$\begin{matrix} 1, \; l_1, \; m_1, \; n_1 \\ \dots\dots\dots \\ 1, \; l_p, \; m_p, \; n_p \end{matrix} \qquad \begin{matrix} 1, \; -\lambda_1, \; -\mu_1, \; -\nu_1 \\ \dots\dots\dots \\ 1, \; -\lambda_p, \; -\mu_p, \; -\nu_p. \end{matrix}$$

We get the determinant

$$|\, 1 - \cos(ik) \,| = |\, 2\sin^2 \tfrac{1}{2}(ik) \,|.$$

Hence, by IV. 2, for two sets of five straight lines

$$\begin{vmatrix} \sin^2 \tfrac{1}{2}(11) \ldots \sin^2 \tfrac{1}{2}(15) \\ \ldots\ldots\ldots\ldots \\ \sin^2 \tfrac{1}{2}(51) \ldots \sin^2 \tfrac{1}{2}(55) \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(\text{i}).$$

For two sets of four straight lines a, b, c, d; a', b', c', d',

$$16 \begin{vmatrix} \sin^2 \tfrac{1}{2}(aa') \ldots \sin^2 \tfrac{1}{2}(ad') \\ \ldots\ldots\ldots\ldots \\ \sin^2 \tfrac{1}{2}(da') \ldots \sin^2 \tfrac{1}{2}(dd') \end{vmatrix} = - |\, 1,\ l_i,\ m_i,\ n_i \,| \times |\, 1,\ \lambda_i,\ \mu_i,\ \nu_i \,| \quad (i = 1, 2, 3, 4) \ \ldots\ldots\ldots(\text{ii}).$$

Expanding the determinants on the right according to the elements of their first column, our determinant

$$= \{\sin (bcd) + \sin (cad) + \sin (abd) - \sin (abc)\}$$
$$\times \{\sin (b'c'd') + \sin (c'a'd') + \sin (a'b'd') - \sin (a'b'c')\}.$$

For two sets of three straight lines, our determinant is

$$\begin{vmatrix} 1 - \cos(11) \ \ldots \ 1 - \cos(13) \\ \ldots\ldots\ldots\ldots \\ 1 - \cos(31) \ \ldots \ 1 - \cos(33) \end{vmatrix},$$

or

$$\begin{vmatrix} 1, & 0 & \ldots & 0 \\ 1, & 1 - \cos(11) & \ldots & 1 - \cos(13) \\ 1, & \ldots & \ldots & \ldots \\ 1, & 1 - \cos(31) & \ldots & 1 - \cos(33) \end{vmatrix} = \begin{vmatrix} 1, & -1, & \ldots & -1 \\ 1, & -\cos(11) & \ldots & -\cos(13) \\ 1, & \ldots & \ldots & \ldots \\ 1, & -\cos(31) & \ldots & -\cos(33) \end{vmatrix}.$$

This is equal to the sum of the products of determinants of the third order taken from the two arrays. Omitting the term

$$\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} \begin{vmatrix} -\lambda_1, & -\mu_1, & -\nu_1 \\ -\lambda_2, & -\mu_2, & -\nu_2 \\ -\lambda_3, & -\mu_3, & -\nu_3 \end{vmatrix} = \begin{vmatrix} -\cos(11) \ \ldots \ -\cos(13) \\ \ldots\ldots\ldots\ldots \\ -\cos(31) \ \ldots \ -\cos(33) \end{vmatrix},$$

we get

$$\begin{vmatrix} 0, & 1 & \ldots & 1 \\ 1, & \cos(11) & \ldots & \cos(13) \\ \ldots & \ldots & \ldots & \ldots \\ 1, & \cos(31) & \ldots & \cos(33) \end{vmatrix} = |\, 1, l, m \,|\, |\, 1, \lambda, \mu \,| + |\, 1, l, n \,|\, |\, 1, \lambda, \nu \,| + |\, 1, m, n \,|\, |\, 1, \mu, \nu \,|.$$

If the straight lines be called a, b, c; a', b', c', and N_1, N_2, N_3

are the directions of the shortest distances between bc, ca, ab, we have

$$|1, l, m| = \sin(bc)\cos(N_1 z) + \sin(ca)\cos(N_2 z) + \sin(ab)\cos(N_3 z),$$
$$|1, \lambda, \mu| = \sin(b'c')\cos(N'_1 z) + \sin(c'a')\cos(N'_2 z) + \sin(a'b')\cos(N'_3 z),$$

and similarly for the other determinants. In particular, if abc lie in one plane, and $a'b'c'$ in another, the normals to the two planes being N, N', the value of the determinant is

$$\{\sin(bc)+\sin(ca)+\sin(ab)\}\{\sin(b'c')+\sin(c'a')+\sin(a'b')\}\cos(NN'),$$

viz. this

$$= -\begin{vmatrix} 0, & 1 & \dots & 1 \\ 1, & \cos(aa') & \dots & \cos(ac') \\ \dots & \dots & \dots & \dots \\ 1, & \cos(ca') & \dots & \cos(cc') \end{vmatrix} \quad \dots\dots\dots\dots (\text{iii}).$$

For two sets of two straight lines we deduce in the same way, if R, r are the directions of the external bisectors between them,

$$\begin{vmatrix} 0, & 1, & 1 \\ 1, & \cos(11), & \cos(12) \\ 1, & \cos(21), & \cos(22) \end{vmatrix} = -4\sin\frac{ab}{2}\sin\frac{a'b'}{2}.\cos(Rr).$$

14. If we compound the arrays

$$\begin{matrix} l_1, & m_1, & n_1, & 1, & 0 \\ \dots & \dots & \dots & \dots & \dots \\ l_i, & m_i, & n_i, & 1, & 0 \\ 0, & 0, & 0, & 0, & 1 \end{matrix} \qquad \begin{matrix} \lambda_1, & \mu_1, & \nu_1, & 0, & 1 \\ \dots & \dots & \dots & \dots & \dots \\ \lambda_i, & \mu_i, & \nu_i, & 0, & 1 \\ 0, & 0, & 0, & 1, & 0, \end{matrix}$$

we get the determinant

$$\begin{vmatrix} \cos(11) & \dots & \cos(1i), & 1 \\ \dots & \dots & \dots & \dots \\ \cos(i1) & \dots & \cos(ii), & 1 \\ 1 & \dots & 1, & 0 \end{vmatrix}.$$

Hence for two sets of five straight lines

$$\begin{vmatrix} \cos(11) & \dots & \cos(15), & 1 \\ \dots & \dots & \dots & \dots \\ \cos(51) & \dots & \cos(55), & 1 \\ 1 & \dots & 1 & \end{vmatrix} = 0.$$

For two sets of four lines

$$\begin{vmatrix} \cos(11) \ldots \cos(14), 1 \\ \dots\dots\dots\dots\dots\dots \\ \cos(41) \ldots \cos(44), 1 \\ 1 \quad \ldots \quad 1, \quad 0 \end{vmatrix} = - |1, l, m, n| \, |1, \mu, \lambda, \nu|,$$

and so on.

But these are not new theorems. In the first for example, if we expand by III. 24, according to products of elements in the last row and column, each term vanishes by Art. 8.

On Systems of Straight lines.

15. If
$$\frac{x-p}{\cos\alpha} = \frac{y-q}{\cos\beta} = \frac{z-r}{\cos\gamma}$$
be the equations of a straight line, then

$$a = \cos\alpha, \qquad b = \cos\beta, \qquad c = \cos\gamma,$$

$$f = \begin{vmatrix} q & r \\ \cos\beta, & \cos\gamma \end{vmatrix}, \quad g = \begin{vmatrix} r & p \\ \cos\gamma, & \cos\alpha \end{vmatrix}, \quad h = \begin{vmatrix} p & q \\ \cos\alpha, & \cos\beta \end{vmatrix}$$

are called the co-ordinates of the line. It is plain that

$$af + bg + ch = 0.$$

16. If the constants belonging to two straight lines be denoted by the suffixes 1 and 2, the equation of a plane through the second line, parallel to the first, is

$$\begin{vmatrix} x-p_2, & y-q_2, & z-r_2 \\ \cos\alpha_1, & \cos\beta_1, & \cos\gamma_1 \\ \cos\alpha_2, & \cos\beta_2, & \cos\gamma_2 \end{vmatrix} = 0.$$

If d be the shortest distance between the two straight lines, and θ the angle between them, it follows that

$$d\sin\theta = \begin{vmatrix} p_1-p_2, & q_1-q_2, & r_1-r_2 \\ \cos\alpha_1, & \cos\beta_1, & \cos\gamma_1 \\ \cos\alpha_2, & \cos\beta_2, & \cos\gamma_2 \end{vmatrix}$$

$$= \begin{vmatrix} p_1, & q_1, & r_1 \\ \cos\alpha_1, & \cos\beta_1, & \cos\gamma_1 \\ \cos\alpha_2, & \cos\beta_2, & \cos\gamma_2 \end{vmatrix} + \begin{vmatrix} p_2, & q_2, & r_2 \\ \cos\alpha_2, & \cos\beta_2, & \cos\gamma_2 \\ \cos\alpha_1, & \cos\beta_1, & \cos\gamma_1 \end{vmatrix}$$

$$= a_2 f_1 + b_2 g_1 + c_2 h_1 + a_1 f_2 + b_1 g_2 + c_1 h_2.$$

If the expression on the right vanishes, then either $d=0$, i.e. the two straight lines intersect, or $\sin\theta=0$ when they are parallel, and hence also meet. It is convenient to have a name for the expression on the right. If a unit force acted in one of the lines its moment about the other would be $d\sin\theta$, i.e. in terms of the co-ordinates of the lines

$$a_1f_2+b_1g_2+c_1h_2+a_2f_1+b_2g_1+c_2h_1.$$

Hence we shall call this the moment of the two straight lines. If two straight lines meet their moment vanishes.

17. Let us take two systems of straight lines whose co-ordinates are

$$\begin{array}{ll} a_1,\, b_1,\, c_1,\, f_1,\, g_1,\, h_1 & f_1',\, g_1',\, h_1',\, a_1',\, b_1',\, c_1' \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ a_i,\, b_i,\, c_i,\, f_i,\, g_i,\, h_i & f_i',\, g_i',\, h_i',\, a_i',\, b_i',\, c_i'. \end{array}$$

Then if m_{rs} denotes the moment of the line r of the first and s of the second system, by compounding the two arrays we get the determinant

$$| m_{ik} | .$$

Hence for two sets of seven straight lines

$$\begin{vmatrix} m_{11} & \dots & m_{17} \\ \cdots & \cdots & \cdots \\ m_{71} & \dots & m_{77} \end{vmatrix} = 0,$$

an identical relation between the mutual moments of two sets of seven straight lines. If the two systems coincide

$$\begin{vmatrix} 0, & m_{12} & \dots & m_{17} \\ m_{21}, & 0 & \dots & m_{27} \\ \cdots & \cdots & \cdots & \cdots \\ m_{71}, & m_{72} & \dots & 0 \end{vmatrix} = 0.$$

For two sets of six straight lines

$$\begin{vmatrix} m_{11} & \dots & m_{16} \\ \cdots & \cdots & \cdots \\ m_{61} & \dots & m_{66} \end{vmatrix} = |a_i,\, b_i,\, c_i,\, f_i,\, g_i,\, h_i| \times |f_i',\, g_i',\, h_i',\, a_i',\, b_i',\, c_i'| \qquad (i=1,\, 2 \dots 6).$$

If one of the sets of six straight lines—say the first—is met by a common transversal whose co-ordinates are a, b, c, f, g, h, we have for each of the straight lines of that system

$$af_i+bg_i+ch_i+fa_i+gb_i+hc_i=0.$$

Thus the first of the determinants on the right vanishes, and

$$\begin{vmatrix} m_{11} & \dots & m_{16} \\ \dots & \dots & \dots \\ m_{61} & \dots & m_{66} \end{vmatrix} = 0$$

is the relation between the mutual moments of the two sets of six straight lines, one set of which is met by a common transversal.

If the two sets coincide we get the identity for a system of six lines met by a common transversal.

18. If the moments of a system of forces about one set of seven lines be $m_1, m_2 \dots m_7$, and about a second set $n_1, n_2 \dots n_7$, we can establish an identity among the moments involved.

For if any force P of the system act in a line whose co-ordinates are a, b, c, f, g, h, we have

$$\begin{aligned} m_1 &= \Sigma P\, \{af_1 + bg_1 + ch_1 + fa_1 + gb_1 + hc_1\} \\ &= f_1 \Sigma Pa + g_1 \Sigma Pb + h_1 \Sigma Pc + a_1 \Sigma Pf + b_1 \Sigma Pg + c_1 \Sigma Ph, \end{aligned}$$

and six other equations for $m_2 \dots m_7$. Hence eliminating

$$\Sigma Pa,\ \Sigma Pb \dots \Sigma Ph,$$

we get

$$\begin{vmatrix} m_1, & a_1, & b_1, & c_1, & f_1, & g_1, & h_1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ m_7, & a_7, & b_7, & c_7, & f_7, & g_7, & h_7 \end{vmatrix} = 0,$$

and a similar equation for the other system. Hence each of the determinants

$$\begin{vmatrix} 0, & m_1, & a_1, & b_1, & c_1, & f_1, & g_1, & h_1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0, & m_7, & a_7, & b_7, & c_7, & f_7, & g_7, & h_7 \\ 1, & 0, & 0, & 0, & 0, & 0, & 0, & 0 \end{vmatrix} \quad \begin{vmatrix} n_1, & 0, & f_1', & g_1', & h_1', & a_1', & b_1', & c_1' \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ n_7, & 0, & f_7', & g_7', & h_7', & a_7', & b_7' & c_7' \\ 0, & 1, & 0, & 0, & 0, & 0, & 0, & 0 \end{vmatrix}$$

vanishes. Forming their product we get

$$\begin{vmatrix} m_{11} & \dots & m_{17}, & n_1 \\ \dots & \dots & \dots & \dots \\ m_{71} & \dots & m_{77}, & n_7 \\ m_1 & \dots & m_7, & 0 \end{vmatrix} = 0.$$

Tetrahedra and Triangles.

19. Let there be two systems of points in space whose co-ordinates referred to rectangular axes are (x_i, y_i, z_i), (ξ_i, η_i, ζ_i). Let us compound the two arrays

$$\begin{array}{ll} x_1, y_1, z_1, 1, 0 & -2\xi_1, -2\eta_1, -2\zeta_1, 0, 1 \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ x_i, y_i, z_i, 1, 0 & -2\xi_i, -2\eta_i, -2\zeta_i, 0, 1 \\ 0, 0, 0, 0, 1 & 0, \quad 0, \quad 0, \quad 1, 0, \end{array}$$

we obtain the determinant

$$\begin{vmatrix} c_{11} \dots c_{1i}, & 1 \\ \cdots\cdots & \\ c_{i1} \dots c_{ii}, & 1 \\ 1 \dots 1 & \end{vmatrix},$$

where $$c_{rs} = -2x_r\xi_s - 2y_r\eta_s - 2z_r\zeta_s.$$

To the r^{th} row add the last multiplied by $x_r^2 + y_r^2 + z_r^2$, and to the s^{th} column add the last multiplied by $\xi_s^2 + \eta_s^2 + \zeta_s^2$, the determinant is unaltered and its elements are now

$$\begin{aligned} d_{rs} &= x_r^2 + y_r^2 + z_r^2 - 2x_r\xi_s - 2y_r\eta_s - 2z_r\zeta_s + \xi_s^2 + \eta_s^2 + \zeta_s^2 \\ &= (x_r - \xi_s)^2 + (y_r - \eta_s)^2 + (z_r - \zeta_s)^2, \end{aligned}$$

i.e. d_{rs} is the square of the distance between the r^{th} point of the first and s^{th} point of the second system. We have then the determinant

$$\begin{vmatrix} d_{11} \dots d_{1i}, & 1 \\ \cdots\cdots & \\ d_{i1} \dots d_{ii}, & 1 \\ 1 \dots 1 & \end{vmatrix}.$$

If $i = 5$ the determinant vanishes, hence

$$\begin{vmatrix} d_{11} \dots d_{15}, & 1 \\ \cdots\cdots & \\ d_{51} \dots d_{55}, & 1 \\ 1 \dots 1 & \end{vmatrix} = 0 \quad\cdots\cdots\cdots\cdots \text{(i)}$$

is the identical relation which subsists between the lines joining two sets of five points in space. If the two systems coincide $d_{ii} = 0$, and the determinant, which is then symmetrical, gives the relation between the lines joining five points in space. The relation in this form is due to Cayley.

If $i=4$,

$$\begin{vmatrix} d_{11} \dots d_{14}, & 1 \\ \dots\dots & \\ d_{41} \dots d_{44}, & 1 \\ 1 \dots 1, & 0 \end{vmatrix} = \begin{vmatrix} x_1, & y_1, & z_1, & 1, & 0 \\ \dots & & & & \\ x_4, & y_4, & z_4, & 1, & 0 \\ 0, & 0, & 0, & 0, & 1 \end{vmatrix} \begin{vmatrix} -2\xi_1, & -2\eta_1, & -2\zeta_1, & 0, & 1 \\ \dots & & & & \\ -2\xi_4, & -2\eta_4, & -2\zeta_4, & 0, & 1 \\ 0, & 0, & 0, & 1, & 0 \end{vmatrix}$$

$$= 288VV' \dots\dots\dots\dots \text{(ii)},$$

where V, V' are the volumes of the tetrahedra formed by the two sets of four points.

If the two sets coincide in a single tetrahedron, for which a, a'; b, b'; c, c' are pairs of opposite edges,

$$288V^2 = \begin{vmatrix} 0, & c'^2, & b'^2, & a'^2, & 1 \\ c'^2, & 0, & c^2, & b^2, & 1 \\ b'^2, & c^2, & 0, & a^2, & 1 \\ a'^2, & b^2, & a^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix}.$$

If $i=3$, we have

$$\begin{vmatrix} d_{11} \dots d_{13}, & 1 \\ \dots\dots & \\ d_{31} \dots d_{33}, & 1 \\ 1 \dots 1, & 0 \end{vmatrix} = -4|x,y,1|\;|\xi,\eta,1| - 4|x,z,1|\;|\xi,\zeta,1| - 4|y,z,1|\;|\eta,\zeta,1|,$$

all the other determinants on the right vanish identically.

Now if Δ, Δ' be the areas of the triangles formed by the two sets of three points, l, m, n; λ, μ, ν the direction cosines of the normals to their planes

$$|x, y, 1| = 2 \text{ projection of } \Delta \text{ on plane } xy = 2\Delta n,$$

and similarly for the others; hence if ϕ is the angle between the planes of the triangles

$$\begin{vmatrix} d_{11} \dots d_{13}, & 1 \\ \dots\dots & \\ d_{31} \dots d_{33}, & 1 \\ 1 \dots 1, & 0 \end{vmatrix} = -16\Delta\Delta' \cos\phi \dots\dots\dots \text{(iii)}.$$

Lastly, if $i=2$,

$$\begin{vmatrix} d_{11}, & d_{12}, & 1 \\ d_{21}, & d_{22}, & 1 \\ 1, & 1, & 0 \end{vmatrix} = \begin{vmatrix} x_1, & 1, & 0 \\ x_2, & 1, & 0 \\ 0, & 0, & 1 \end{vmatrix} \begin{vmatrix} -2\xi_1, & 0, & 1 \\ -2\xi_2, & 0, & 1 \\ 0, & 1, & 0 \end{vmatrix} + \dots$$

$$= 2(x_1 - x_2)(\xi_1 - \xi_2) + 2(y_1 - y_2)(\eta_1 - \eta_2) + 2(z_1 - z_2)(\zeta_1 - \zeta_2),$$

the other terms vanish. Now if a, b be the lengths of the lines joining the points of the first and second systems and θ the angle between them,

$$\frac{x_1 - x_2}{a} \cdot \frac{\xi_1 - \xi_2}{b} + . + . = \cos\theta.$$

Hence

$$\begin{vmatrix} d_{11}, & d_{12}, & 1 \\ d_{21}, & d_{22}, & 1 \\ 1, & 1, & 0 \end{vmatrix} = 2ab\cos\theta \quad\quad\quad\quad\text{(iv)}.$$

20. If in case (iii) of Art. 19 we allow the two sets of three points to coincide with the vertices of a single triangle whose sides are a, b, c,

$$-16\Delta^2 = \begin{vmatrix} 0, & c^2, & b^2, & 1 \\ c^2, & 0, & a^2, & 1 \\ b^2, & a^2, & 0, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix}.$$

Multiply each column by abc, then

$$-16\Delta^2 a^4b^4c^4 = \begin{vmatrix} 0, & abc^3, & ab^3c, & abc \\ abc^3, & 0, & a^3bc, & abc \\ ab^3c, & a^3bc, & 0, & abc \\ abc, & abc, & abc, & 0 \end{vmatrix}.$$

Divide the first, second, and third rows and columns by bc, ca, ab respectively, then

$$-16\Delta^2 = \begin{vmatrix} 0, & c, & b, & a \\ c, & 0, & a, & b \\ b, & a, & 0, & c \\ a, & b, & c, & 0 \end{vmatrix}$$

$$= \begin{vmatrix} a, & b, & c, & 0 \\ b, & a, & 0, & c \\ c, & 0, & a, & b \\ 0, & c, & b, & a \end{vmatrix}$$

by an interchange of columns.

If in the first expression for $-16\Delta^2$ we divide the second and third columns by a^2, and then multiply the first and last rows by a^2, we get:

$$-16\Delta^2 = \begin{vmatrix} 0, & c^2, & b^2, & a^2 \\ c^2, & 0, & 1, & 1 \\ b^2, & 1, & 0, & 1 \\ a^2, & 1, & 1, & 0 \end{vmatrix}.$$

21. If in case (ii) of Art. 19 one of the sets of four points—say the first—lies in a plane, $V = 0$, and

$$\begin{vmatrix} d_{11} \dots d_{14}, & 1 \\ \dots\dots\dots\dots & \\ d_{41} \dots d_{44}, & 1 \\ 1 \dots 1 & \end{vmatrix} = 0.$$

If one of the sets in case (iii) lies in a straight line the corresponding triangle vanishes; hence

$$\begin{vmatrix} d_{11} \dots d_{13}, & 1 \\ \dots\dots\dots\dots & \\ d_{31} \dots d_{33}, & 1 \\ 1 \dots 1 & \end{vmatrix} = 0.$$

By allowing the second system to coincide with the first we get the identical relations between the lines joining four coplanar and three collinear points.

22. In the identical relation

$$\begin{vmatrix} d_{11} \dots d_{15}, & 1 \\ \dots\dots\dots\dots & \\ d_{51} \dots d_{55}, & 1 \\ 1 \dots 1 & \end{vmatrix} = 0$$

between the squares of the lines joining two sets of five points, let the fifth point of the first system be the centre of the sphere circumscribing the tetrahedron formed by the first four points of the second system, and the point 5 of the second system the centre of the sphere circumscribing the first four points of the first system. Then

$$d_{15} = d_{25} = d_{35} = d_{45} = R^2$$
$$d_{51} = d_{52} = d_{53} = d_{54} = R'^2.$$

Also, if ϕ be the angle at which the two circumscribing spheres intersect, $d_{55} = R^2 + R'^2 + 2RR' \cos\phi$.

Hence with an interchange of rows and columns

$$\begin{vmatrix} d_{11} & \dots & d_{14}, & 1, & R^2 \\ \dots & \dots & \dots & \dots & \dots \\ d_{41} & \dots & d_{44}, & 1, & R^2 \\ 1 & \dots & 1, & 0, & 1 \\ R'^2 & \dots & R'^2, & 1, & d_{55} \end{vmatrix} = 0.$$

Multiply the fifth column by R^2 and subtract it from the last, and the fifth row by R'^2 and subtract it from the last, then

$$\begin{vmatrix} d_{11} & \dots & d_{14}, & 1, & 0 \\ \dots & \dots & \dots & \dots & \dots \\ d_{41} & \dots & d_{44}, & 1, & 0 \\ 1 & \dots & 1, & 0, & 1 \\ 0 & \dots & 0, & 1, & 2RR'\cos\phi \end{vmatrix} = 0.$$

Or, resolving according to the elements of the last row and column, we have by Art. 19 (ii)

$$576\,VR\,V'R' \cos\phi = \begin{vmatrix} d_{11} & \dots & d_{14} \\ \dots & \dots & \dots \\ d_{41} & \dots & d_{44} \end{vmatrix}.$$

We see from this that so long as the circumscribing spheres remain fixed the tetrahedra can turn about in them without altering the value of the determinant on the right. The determinant vanishes if the circumscribing spheres of the two systems cut orthogonally. This relation is due to Siebeck.

23. If in Art. 22 we allow the two tetrahedra to coincide we get, since $\phi = \pi$,

$$16\,(6VR)^2 = -\begin{vmatrix} 0, & a'^2, & b'^2, & c'^2 \\ a'^2, & 0, & c^2, & b^2 \\ b'^2, & c^2, & 0, & a^2 \\ c'^2, & b^2, & a^2, & 0 \end{vmatrix}.$$

Multiply the second, third and fourth rows and columns by a^2, b^2, c^2 respectively, then

$$16\,(6VR)^2 a^4b^4c^4 = -\begin{vmatrix} 0, & (aa')^2, & (bb')^2, & (cc')^2 \\ (aa')^2, & 0, & a^2b^2c^2, & a^2b^2c^2 \\ (bb')^2, & a^2b^2c^2, & 0, & a^2b^2c^2 \\ (cc')^2, & a^2b^2c^2, & a^2b^2c^2, & 0 \end{vmatrix}.$$

Divide the second, third and fourth rows by $(abc)^2$, then multiply the first column by the same quantity,

$$16\,(6VR)^2 = -\begin{vmatrix} 0, & (aa')^2, & (bb')^2, & (cc')^2 \\ (aa')^2, & 0, & 1, & 1 \\ (bb')^2, & 1, & 0, & 1 \\ (cc')^2, & 1, & 1, & 0 \end{vmatrix}.$$

Now if we write

$$aa' = kx, \quad bb' = ky, \quad cc' = kz,$$

then if Δ is the area of the triangle, whose sides are x, y, z, we have by Art. 20,

$$(6VR)^2 = k^4\Delta^2,$$

$$6VR = k^2\Delta.$$

This triangle, whose sides are proportional to the square roots of the products of pairs of opposite sides of the tetrahedron, has many interesting relations to the tetrahedron. It is sometimes called the conjugate triangle.

Formulæ relating to the Ellipsoid.

24. If (x_i, y_i, z_i) and (ξ_i, η_i, ζ_i) be two sets of points on the ellipsoid,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Then, if d_{rs} denote the square of the distance between the r^{th} and s^{th} points of the two systems and D_{rs} the square of the parallel semidiameter, we have

$$a_{rs} = \frac{d_{rs}}{D_{rs}} = 2\left(1 - \frac{x_r\xi_s}{a^2} - \frac{y_r\eta_s}{b^2} - \frac{z_r\zeta_s}{c^2}\right).$$

Hence, if we compound the two arrays,

$$\begin{array}{cccc} \frac{x_1}{a}, & \frac{y_1}{b}, & \frac{z_1}{c}, & 1 \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots} \\ \frac{x_i}{a}, & \frac{y_i}{b}, & \frac{z_i}{c}, & 1 \end{array} \qquad \begin{array}{cccc} -\frac{2\xi_1}{a}, & -\frac{2\eta_1}{b}, & -\frac{2\zeta_1}{c}, & 2 \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots} \\ -\frac{2\xi_i}{a}, & -\frac{2\eta_i}{b}, & -\frac{2\zeta_i}{c}, & 2, \end{array}$$

we get as in the preceding articles.

For two sets of five points situated on the ellipsoid,

$$\begin{vmatrix} a_{11} \cdots a_{15} \\ \cdots\cdots\cdots \\ a_{51} \cdots a_{55} \end{vmatrix} = 0.$$

For two sets of four points forming two tetrahedra of volumes V, V',

$$\begin{vmatrix} a_{11} \dots a_{14} \\ \dots\dots\dots\dots \\ a_{41} \dots a_{44} \end{vmatrix} = -\frac{576\,VV'}{a^2b^2c^2}.$$

Similar formulæ can be established for an ellipse in a plane.

If the ellipsoid become a sphere $a = b = c = R$, and since all diameters are equal, we can replace a_{rs} by d_{rs}. Thus

$$\begin{vmatrix} d_{11} \dots d_{15} \\ \dots\dots\dots\dots \\ d_{51} \dots d_{55} \end{vmatrix} = 0$$

is an identical relation between two sets of five points on a sphere. This relation is due to Cayley.

The second relation in this case reduces to the result of Art. 22, when the two tetrahedra have the same circumscribing sphere.

25. If the points (x_i, y_i, z_i) (ξ_i, η_i, ζ_i) are not situated on the ellipsoid, then since

$$a_{rs} = \frac{d_{rs}}{D_{rs}} = \frac{(x_r - \xi_s)^2}{a^2} + \frac{(y_r - \eta_s)^2}{b^2} + \frac{(z_r - \zeta_s)^2}{c^2}$$
$$= \frac{x_r^2}{a^2} + \frac{y_r^2}{b^2} + \frac{z_r^2}{c^2} - \frac{2x_r\xi_s}{a^2} - \frac{2y_r\eta_s}{b^2} - \frac{2z_r\zeta_s}{c^2} + \frac{\xi_s^2}{a^2} + \frac{\eta_s^2}{b^2} + \frac{\zeta_s^2}{c^2};$$

if we compound the two arrays whose i^{th} rows are

$$\frac{x_i^2}{a^2} + \frac{y_i^2}{b^2} + \frac{z_i^2}{c^2},\ \frac{x_i}{a},\ \frac{y_i}{b},\ \frac{z_i}{c},\ 1,$$

$$1,\ \frac{-2\xi_i}{a},\ \frac{-2\eta_i}{b},\ \frac{-2\zeta_i}{c},\ \frac{\xi_i^2}{a^2} + \frac{\eta_i^2}{b^2} + \frac{\zeta_i^2}{c^2},$$

we get the identical relations (IV. 2)

$$\begin{vmatrix} a_{11} \dots a_{16} \\ \dots\dots\dots\dots \\ a_{61} \dots a_{66} \end{vmatrix} = 0$$

$$\begin{vmatrix} a_{11} \dots a_{15} \\ \dots\dots\dots\dots \\ a_{51} \dots a_{55} \end{vmatrix}$$

$$= \left| \frac{x_i^2}{a^2} + \frac{y_i^2}{b^2} + \frac{z_i^2}{c^2}, \frac{x_i}{a}, \frac{y_i}{b}, \frac{z_i}{c}, 1 \right|$$

$$\times \left| 1, \frac{-2\xi_i}{a}, \frac{-2\eta_i}{b}, \frac{-2\zeta_i}{c}, \frac{\xi_i^2}{a^2} + \frac{\eta_i^2}{b^2} + \frac{\zeta_i^2}{c^2} \right|$$

$$(i = 1, 2 \ldots 5),$$

for two systems of six points and five points respectively.

If in the latter equation all the points of the first system lie on the ellipsoid,

$$\left(\frac{x-p}{a}\right)^2 + \left(\frac{y-q}{b}\right)^2 + \left(\frac{z-r}{c}\right)^2 = m^2,$$

we should have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - \frac{2px}{a^2} - \frac{2qy}{b^2} - \frac{2rz}{c^2} + \frac{p^2}{a^2} + \frac{q^2}{b^2} + \frac{r^2}{c^2} = m^2$$

satisfied for each point of the system. Hence we see by eliminating

$$\frac{-2p}{a}, \frac{-2q}{a}, \frac{-2r}{a}, \frac{p^2}{a^2} + \frac{q^2}{b^2} + \frac{r^2}{c^2} - m^2$$

between these five equations, that the first determinant on the right vanishes. Hence

$$\begin{vmatrix} a_{11} \cdots a_{15} \\ \cdots\cdots\cdots \\ a_{51} \cdots a_{55} \end{vmatrix} = 0,$$

if the five points of one of the systems lie on an ellipsoid similar and similarly situated to the given one. If the ellipsoid reduce to a sphere, we get

$$\begin{vmatrix} d_{11} \cdots d_{16} \\ \cdots\cdots\cdots \\ d_{61} \cdots d_{66} \end{vmatrix} = 0,$$

an identical and homogeneous relation between the lines joining two sets of six points.

And
$$\begin{vmatrix} d_{11} \cdots d_{15} \\ \cdots\cdots\cdots \\ d_{51} \cdots d_{55} \end{vmatrix} = 0$$

for five points situated on a sphere.

26. In like manner, if for the same systems of points as in the last article we compound the arrays

$$\begin{array}{lcl} \frac{x_1}{a},\ \frac{y_1}{b},\ \frac{z_1}{c},\ 1,\ 0 & \qquad & -\frac{2\xi_1}{a},\ -\frac{2\eta_1}{b},\ -\frac{2\zeta_1}{c},\ 0,\ 1 \\ \cdots\cdots\cdots & & \cdots\cdots\cdots\cdots \\ \frac{x_i}{a},\ \frac{y_i}{b},\ \frac{z_i}{c},\ 1,\ 0 & & -\frac{2\xi_i}{a},\ -\frac{2\eta_i}{b},\ -\frac{2\zeta_i}{c},\ 0,\ 1 \\ 0,\ 0,\ 0,\ 0,\ 1, & & 0,\ 0,\ 0,\ 1,\ 0 \end{array}$$

we get the determinant

$$\begin{vmatrix} c_{11} \ldots c_{1i}, & 1 \\ \cdots\cdots\cdots & \\ c_{i1} \ldots c_{ii}, & 1 \\ 1 \ \ldots \ 1 & \end{vmatrix}$$

where $$c_{rs} = -\frac{2x_r\xi_s}{a^2} - \frac{2y_r\eta_s}{b^2} - \frac{2z_r\zeta_s}{c^2}.$$

Multiply the last column by

$$\frac{\xi_s^{\,2}}{a^2} + \frac{\eta_s^{\,2}}{b^2} + \frac{\zeta_s^{\,2}}{c^2}$$

and add it to the s^{th} column, and the last row by

$$\frac{x_r^{\,2}}{a^2} + \frac{y_r^{\,2}}{b^2} + \frac{z_r^{\,2}}{c^2}$$

and add it to the r^{th} row, then the element at the intersection of the r^{th} row and s^{th} column is

$$\left(\frac{x_r - \xi_s}{a}\right)^2 + \left(\frac{y_r - \eta_s}{b}\right)^2 + \left(\frac{z_r - \zeta_s}{c}\right)^2 = a_{rs}.$$

And hence, (IV. 2),

$$\begin{vmatrix} a_{11} \ldots a_{15}, & 1 \\ \cdots\cdots\cdots & \\ a_{51} \ldots a_{55}, & 1 \\ 1 \ \ldots \ 1 & \end{vmatrix} = 0$$

is an identical relation between any two sets of five points in space. If the ellipsoid becomes a sphere we regain Cayley's relation (Art. 19, i).

For $i = 4$, we have

$$\begin{vmatrix} a_{11} \ldots a_{14}, & 1 \\ \cdots\cdots\cdots & \\ a_{41} \ldots a_{44}, & 1 \\ 1 \ \ldots \ 1 & \end{vmatrix} = \frac{288\,VV'}{a^2b^2c^2},$$

V, V' being the volumes of the tetrahedra formed by each set of four points.

27. The polar plane of a point $P(x_r, y_r, z_r)$ with respect to the ellipsoid, is

$$\frac{xx_r}{a^2}+\frac{yy_r}{b^2}+\frac{zz_r}{c^2}=1.$$

The distance of a point $Q(\xi_s, \eta_s, \zeta_s)$ from this plane is

$$(P, Q)=-p\left(\frac{x_r\xi_s}{a^2}+\frac{y_r\eta_s}{b^2}+\frac{z_r\zeta_s}{c^2}-1\right)$$

$$\frac{1}{p^2}=\frac{x_r^2}{a^4}+\frac{y_r^2}{b^4}+\frac{z_r^2}{c^4}.$$

If (Q, P) and q denote like quantities for the point Q,

$$\frac{(P, Q)}{p}=\frac{(Q, P)}{q}=1-\frac{x_r\xi_s}{a^2}-\frac{y_r\eta_s}{b^2}-\frac{z_r\zeta_s}{c^2}.$$

This function has been called by Faure the index of the two points P and Q, denote it by I_{rs}. Then, by compounding the arrays whose i^{th} rows are

$$\frac{x_i}{a},\ \frac{y_i}{b},\ \frac{z_i}{c},\ 1; \qquad \frac{-\xi_i}{a},\ \frac{-\eta_i}{b},\ \frac{-\zeta_i}{c},\ 1,$$

we obtain

$$\begin{vmatrix} I_{11} \dots I_{15} \\ \dots\dots\dots \\ I_{51} \dots I_{55} \end{vmatrix}=0$$

$$\begin{vmatrix} I_{11} \dots I_{14} \\ \dots\dots\dots \\ I_{41} \dots I_{44} \end{vmatrix}=-\frac{36VV'}{a^2b^2c^2}.$$

28. It may be remarked that these space relations connected with an ellipsoid are not really more general than those connected with a sphere. For they are what the relations in an ordinary space become when the sphere

$$x^2+y^2+z^2=R^2$$

becomes changed by a homogeneous pure strain to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Formulæ relating to Systems of Spheres.

29. If r, s be the radii of two spheres, ϕ the angle at which they intersect, and d the distance between their centres, then

$$d^2 = r^2 + s^2 + 2rs\cos\phi.$$

The function

$$2rs\cos\phi = d^2 - r^2 - s^2$$

is of importance in the study of the mutual relations of spheres; it is called the power of the two spheres. We shall denote it by p_{rs}.

If one of the spheres, say s, becomes a point, the limit of $2rs\cos\phi$ is $d^2 \sim r^2$, i.e. the square of the tangent from the point to the sphere, or what is known as the power of the sphere at the point, or the power of the point with respect to the sphere.

If both spheres reduce to points the limit of $2rs\cos\phi$ is d^2, the square of the distance between the points.

If one of the spheres becomes a plane, and p is its distance from the centre of the other,

$$\cos\phi = \frac{p}{r}.$$

If the second sphere become a point, and p is its distance from the plane, the limit of $r\cos\phi$ is p.

30. Let (x_i, y_i, z_i) and (ξ_k, η_k, ζ_k) be the co-ordinates of the centres of two spheres of radii r_i and ρ_k, then if p_{ik} is their mutual power

$$\begin{aligned} p_{ik} &= d^2 - r_i^2 - \rho_k^2 \\ &= x_i^2 + y_i^2 + z_i^2 - r_i^2 - 2x_i\xi_k - 2y_i\eta_k - 2z_i\zeta_k + \xi_k^2 + \eta_k^2 + \zeta_k^2 - \rho_k^2. \end{aligned}$$

Hence, compounding the two arrays

$$\begin{array}{l} x_1,\ y_1,\ z_1,\ 1,\ x_1^2 + y_1^2 + z_1^2 - r_1^2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ x_i,\ y_i,\ z_i,\ 1,\ x_i^2 + y_i^2 + z_i^2 - r_i^2, \end{array}$$

and

$$\begin{array}{l} -2\xi_1, -2\eta_1, -2\zeta_1,\ \xi_1^2 + \eta_1^2 + \zeta_1^2 - \rho_1^2,\ 1 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ -2\xi_i, -2\eta_i, -2\zeta_i,\ \xi_i^2 + \eta_i^2 + \zeta_i^2 - \rho_i^2,\ 1, \end{array}$$

we see by IV. 2 that for two systems of six spheres

$$\begin{vmatrix} p_{11} \cdots p_{16} \\ \cdots\cdots \\ p_{61} \cdots p_{66} \end{vmatrix} = 0 \qquad \text{.........................(i).}$$

If $\cos\phi_{ik}$ is the cosine of the angle at which two spheres cut, we can also write this

$$|\cos\phi_{ik}| = 0 \qquad (i, k = 1, 2 \dots 6).$$

For two systems, each of five spheres,

$$\begin{vmatrix} p_{11} \cdots p_{15} \\ \cdots\cdots\cdots \\ p_{51} \cdots p_{55} \end{vmatrix} \dots\dots\dots\dots\dots\dots\dots\dots\dots\text{(ii)}$$

$$= |\, x, y, z, 1, x^2+y^2+z^2-r^2 \,| \times |\, -2\xi, -2\eta, -2\zeta, \xi^2+\eta^2+\zeta^2-\rho^2, 1 \,|.$$

If the five spheres of one of the systems—say the first—have a common radical centre, taking this for origin we should have

$$x^2+y^2+z^2-r^2 = c^2,$$

where c is the same for all the five spheres. Hence, in the first determinant on the right of (ii), the fourth and fifth columns are proportionals and the determinant vanishes.

Thus

$$\begin{vmatrix} p_{11} \cdots p_{15} \\ \cdots\cdots\cdots \\ p_{51} \cdots p_{55} \end{vmatrix} = 0 \qquad \dots\dots\dots\dots\dots\dots\dots\text{(iii)}$$

when the five spheres of one system have a common radical centre.

If the five spheres of the first system reduce to points (iii) is the condition that they should lie on a sphere.

If both systems reduce to points we regain Cayley's condition, that the five points of one system should lie on the same sphere.

31. But if neither of the determinants on the right of (ii) vanish, expand the first determinant with regard to the elements of the last column.

Then $\qquad p_i = x_i^2 + y_i^2 + z_i^2 - r_i^2$

is the power of the origin (i.e. any point) with regard to the i^{th} sphere of the first system. Then if we write 1, 2, 3, 4, 5 for the centres of the five spheres, and denote by

$$v_1 = (2345),\ v_2 = (3451),\ \&\text{c.},$$

the volumes of the tetrahedra formed by the points in brackets, and if accents denote similar quantities for the second determinant, we have in place of (ii)

$$|\, p_{ik} \,| = 288\,(v_1p_1 + v_2p_2 + \dots + v_5p_5)\,(v_1'p_1' + \dots + v_5'p_5')$$
$$(i, k = 1, 2 \dots 5).$$

Now describe about the origin a sphere of radius r, cutting the spheres $r_1 \dots r_5$ at angles $\phi_1 \dots \phi_5$.

We have, since (Art. 5)

$$v_1 + v_2 + \dots + v_5 = 0 \text{ identically},$$
$$v_1 p_1 + \dots + v_5 p_5 = v_1 (p_1 - r^2) + \dots + v_5 (p_5 - r^2)$$
$$= 2r (v_1 r_1 \cos \phi_1 + \dots + v_5 r_5 \cos \phi_5),$$

and ρ being a similar sphere for the second system,

$$| p_{ik} | = 288 \rho r \Sigma 2 v_i r_i \cos \phi_i \Sigma 2 v_i' \rho_i \cos \phi_i' \qquad (i, k = 1 \dots 5).$$

Thus $r \Sigma 2 v_i r_i \cos \phi_i$ is independent of the particular sphere r, let this be the orthotomic sphere of the first four, then this sum reduces to

$$2 v_5 r_5 R \cos (r_5 R),$$

and the second factor, in like manner, becomes

$$2 v_5' \rho_5 R' \cos (\rho_5 R').$$

Hence

$$\begin{vmatrix} p_{11} \cdots p_{15} \\ \cdots\cdots\cdots \\ p_{51} \cdots p_{55} \end{vmatrix} = 1152 v_5 v_5' r_5 \rho_5 R R' \cos (r_5 R) \cos (\rho_5 R').$$

32. For the fifth sphere of each system in this last equation take the orthotomic sphere of the first four spheres in the other system. Then in the determinant on the left all the elements in the last row and column vanish except p_{55}, and

$$p_{55} = 2 R R' \cos (R R').$$

Hence we obtain

$$\begin{vmatrix} p_{11} \cdots p_{14} \\ \cdots\cdots\cdots \\ p_{41} \cdots p_{44} \end{vmatrix} 2 R R' \cos (R R') = 1152 v_5 v_5' R^2 R'^2 \cos^2 (R R'),$$

or dividing out the common factors and writing V, V' for v_5, v_5', we get for two sets of four spheres

$$\begin{vmatrix} p_{11} \cdots p_{14} \\ \cdots\cdots\cdots \\ p_{41} \cdots p_{44} \end{vmatrix} = 576 V V' R R' \cos (R R').$$

If the spheres reduce to points we regain Siebeck's formula (Art. 22).

The determinant on the left vanishes if the orthotomic spheres of the two systems of spheres cut orthogonally.

33. To determine the meaning of the determinant

$$|p_{ik}| \quad (i, k = 1, 2, 3).$$

In the determinant of Art. 32, let the fourth sphere of each system be the plane determined by the centres of the first three spheres of the other system, then if Δ, Δ' be the areas of the triangles formed by the centres, ϕ the angle between their planes,

$$\lim. \frac{V'}{r_4'} = 3\Delta \cos\phi, \quad \lim. \frac{V}{r_4} = 3\Delta' \cos\phi.$$

Also if the radical axis of the spheres of the first system meet the plane of centres of the second system in P, whose power with reference to the spheres is p, and P', p' denote like quantities for the other system,

$$2RR' \cos(RR') = PP'^2 - p - p'.$$

Hence

$$\begin{vmatrix} p_{11} \cdots p_{13} \\ \cdots\cdots\cdots \\ p_{31} \cdots p_{33} \end{vmatrix} = 16\Delta\Delta' \cos\phi\,(PP'^2 - p - p').$$

34. If in the relations

$$\begin{vmatrix} d_{11} \ldots d_{15}, & 1 \\ \cdots\cdots\cdots & \\ d_{51} \ldots d_{55}, & 1 \\ 1 \ldots 1 & \end{vmatrix} = 0,$$

$$\begin{vmatrix} d_{11} \ldots d_{14}, & 1 \\ \cdots\cdots\cdots & \\ d_{41} \ldots d_{44}, & 1 \\ 1 \ldots 1 & \end{vmatrix} = -288VV',$$

of Art. 19, we suppose the sets of points to be the centres of our spheres.

Then if we multiply the last column by ρ_i^2 and subtract it from the i^{th} column, and the last row by r_k^2 and subtract it from the k^{th} row, we get the relations

$$\begin{vmatrix} p_{11} \cdots p_{15}, & 1 \\ \cdots\cdots\cdots & \\ p_{51} \cdots p_{55}, & 1 \\ 1 \ldots 1 & \end{vmatrix} = 0,$$

$$\begin{vmatrix} p_{11} & \cdots & p_{14}, & 1 \\ \cdots & \cdots & \cdots & \cdots \\ p_{41} & \cdots & p_{44}, & 1 \\ 1 & \cdots & 1 & \end{vmatrix} = -288VV',$$

which give relations between the mutual powers of two sets of five and four spheres.

35. Another element connected with two spheres is the length of their common tangent. For two spheres of radii r, s the distance between whose centres is d and which cut at an angle ϕ, the square of the length of the common tangent is given by

$$\begin{aligned} t &= d^2 - (r-s)^2 \\ &= 2rs\cos^2 \tfrac{1}{2}\phi. \end{aligned}$$

If one sphere reduce to a point, t is the power of that point with respect to the other sphere. If both spheres reduce to points, t is the square of the distance between them.

36. Using the same notation as in Art. 30, if t_{ik} is the square of the tangent common to the two spheres

$$\begin{aligned} t_{ik} &= (x_i - \xi_k)^2 + (y_i - \eta_k)^2 + (z_i - \zeta_k)^2 - (r_i - \rho_k)^2 \\ &= x_i^2 + y_i^2 + z_i^2 - r_i^2 - 2x_i\xi_k - 2y_i\eta_k - 2z_i\zeta_k + 2r_i\rho_k + \xi_k^2 + \eta_k^2 + \zeta_k^2 - \rho_k^2. \end{aligned}$$

Hence, compounding the two arrays

$$\begin{array}{c} x_1,\ y_1,\ z_1,\ r_1,\ 1,\ x_1^2 + y_1^2 + z_1^2 - r_1^2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ x_i,\ y_i,\ z_i,\ r_i,\ 1,\ x_i^2 + y_i^2 + z_i^2 - r_i^2 \\ 0,\ 0,\ 0,\ 0,\ 0,\qquad 1 \end{array}$$

$$\begin{array}{c} -2\xi_1,\ -2\eta_1,\ -2\zeta_1,\ 2\rho_1,\ \xi_1^2 + \eta_1^2 + \zeta_1^2 - \rho_1^2,\ 1 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ -2\xi_i,\ -2\eta_i,\ -2\zeta_i,\ 2\rho_i,\ \xi_i^2 + \eta_i^2 + \zeta_i^2 - \rho_i^2,\ 1 \\ 0,\quad 0,\quad 0,\quad 0,\qquad 1,\qquad 0, \end{array}$$

we get for two systems of six spheres the identity

$$\begin{vmatrix} t_{11} & \cdots & t_{16}, & 1 \\ \cdots & \cdots & \cdots & \cdots \\ t_{61} & \cdots & t_{66}, & 1 \\ 1 & \cdots & 1 & \end{vmatrix} = 0.$$

For two systems of five spheres we should get

$$\begin{vmatrix} t_{11} \dots t_{15}, 1 \\ \dots\dots\dots\dots \\ t_{51} \dots t_{55}, 1 \\ 1 \dots 1 \end{vmatrix} = 576\,(v_1 r_1 + \dots + v_5 r_5)\,(v_1'\rho_1 + \dots + v_5'\rho_5),$$

using the notation of Art. 31.

If t_5 is the angle at which the plane of similitude of the first four spheres of the first system cuts each of these spheres, and $(r_5 t_5)$ the angle at which it cuts the fifth sphere, and similarly for the second system, we can reduce this to the form

$$\begin{vmatrix} t_{11} \dots t_{15}, 1 \\ \dots\dots\dots\dots \\ t_{51} \dots t_{55}, 1 \\ 1 \dots 1 \end{vmatrix} = 576 v_5 r_5 v_5' \rho_5 \left(1 - \frac{\cos(r_5 t_5)}{\cos t_5}\right)\left(1 - \frac{\cos(\rho_5 \tau_5)}{\cos \tau_5}\right).$$

Hence the determinant vanishes if one of the systems of five spheres has a common plane of similitude.

For two sets of four spheres, after some reduction we can prove that

$$\begin{vmatrix} t_{11} \dots t_{14}, 1 \\ \dots\dots\dots\dots \\ t_{41} \dots t_{44}, 1 \\ 1 \dots 1 \end{vmatrix} = 288 vv' \left(1 - \frac{\cos\phi}{\cos t \cos\tau}\right),$$

where ϕ is the angle between the planes of similitude of the two systems, and t, τ the angles at which they cut their sets of spheres.

37. By compounding the arrays whose i^{th} rows are

$$x_i,\ y_i,\ z_i,\ r_i,\ 1,\ x_i^2 + y_i^2 + z_i^2 - r_i^2$$

and $$-2\xi_i,\ -2\eta_i,\ -2\zeta_i,\ 2\rho_i,\ \xi_i^2 + \eta_i^2 + \zeta_i^2 - \rho_i^2,\ 1,$$

we get the homogeneous relation between the sets of tangents common to two sets of seven spheres

$$\begin{vmatrix} t_{11} \dots t_{17} \\ \dots\dots\dots \\ t_{71} \dots t_{77} \end{vmatrix} = 0.$$

38. We may make use of this last relation to solve the problem: Determine the equation of the sphere having with five given spheres tangents of the same length.

Let the equations of the five given spheres be

$$S_1 = 0 \ldots\ldots S_5 = 0.$$

Take these for the first five of each set of spheres in Art. 38, let the sixth sphere be the one required, and the seventh a point on the sixth.

Then we shall have

$$t_{67} = 0, \quad t_{7i} = S_i, \quad t_{6i} = k,$$

and the equation is

$$\begin{vmatrix} 0, & t_{12}, & t_{13}, & t_{14}, & t_{15}, & 1, & S_1 \\ t_{21}, & 0, & t_{23}, & t_{24}, & t_{25}, & 1, & S_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ t_{51}, & t_{52}, & t_{53}, & t_{54}, & 0, & 1, & S_5 \\ 1, & 1, & 1, & 1, & 1, & 0, & 0 \\ S_1, & S_2, & S_3, & S_4, & S_5, & 0, & 0 \end{vmatrix} = 0.$$

This is apparently of the fourth order, but by means of the sixth rows and columns we can get rid of the terms of the second degree in the seventh row and column.

39. All the equations of this section relating to spheres are capable of numerous and varied applications, some of these will be found in the examples, and others in the memoirs of Bauer, Darboux and Frobenius.

EXAMPLES.

PROVE the following relations : 1—5.

1.
$$\begin{vmatrix} (b+c)^2, & ab\ , & ac \\ ab\ , & (c+a)^2, & bc \\ ac\ , & bc\ , & (a+b)^2 \end{vmatrix} = 2abc\,(a+b+c)^3,$$

$$\begin{vmatrix} (b+c)^2, & c^2\ , & b^2 \\ c^2\ , & (c+a)^2, & a^2 \\ b^2\ , & a^2\ , & (a+b)^2 \end{vmatrix} = 2\,(bc+ca+ab)^3.$$

2.
$$\begin{vmatrix} 1\ , & 1\ , & 1 \\ \tan A\ , & \tan B, & \tan C \\ \sin 2A, & \sin 2B, & \sin 2C \end{vmatrix} = 0,$$

if A, B, C are the angles of a triangle.

3.
$$\begin{vmatrix} 1, & x, & (a+x)\sqrt{(c+x)} \\ 1, & y, & (a+y)\sqrt{(c+y)} \\ 1, & z, & (a+z)\sqrt{(c+z)} \end{vmatrix} = 0,$$

if $$\tan^{-1}\sqrt{\left(\frac{a-c}{c+x}\right)} + \tan^{-1}\sqrt{\left(\frac{a-c}{c+y}\right)} + \tan^{-1}\sqrt{\left(\frac{a-c}{c+z}\right)} = 0.$$

4.
$$\begin{vmatrix} 1\ , & \cos\alpha\ , & \cos(\alpha+\beta), & \cos(\alpha+\beta+\gamma), & \cos(\alpha+\beta+\gamma+\delta) \\ \cos\alpha\ , & 1\ , & \cos\beta\ , & \cos(\beta+\gamma)\ , & \cos(\beta+\gamma+\delta) \\ \cos(\alpha+\beta)\ , & \cos\beta\ , & 1\ , & \cos\gamma\ , & \cos(\gamma+\delta) \\ \cos(\alpha+\beta+\gamma)\ , & \cos(\beta+\gamma)\ , & \cos\gamma\ , & 1\ , & \cos\delta \\ \cos(\alpha+\beta+\gamma+\delta), & \cos(\beta+\gamma+\delta), & \cos(\gamma+\delta), & \cos\delta\ , & 1 \end{vmatrix} = 0.$$

5.
$$\begin{vmatrix} a+b+c+d, & a-b-c+d, & a-b+c-d \\ a-b-c+d, & a+b+c+d, & a+b-c-d \\ a-b+c-d, & a+b-c-d, & a+b+c+d \end{vmatrix}$$
$$= 16\,(bcd+acd+abd+abc).$$

6. If a, b, c are the sides of a triangle of area Δ, $2s = a + b + c$, then

$$\begin{vmatrix} (b+c)^2, & ab\ , & ac\ , & a \\ ab\ , & (c+a)^2, & bc\ , & b \\ ac\ , & bc\ , & (a+b)^2, & c \\ a\ , & b\ , & c & \end{vmatrix} = -16s\Delta(a^2r_1 + b^2r_2 + c^2r_3),$$

r_1, r_2, r_3 being the radii of the escribed circles.

If the elements in the principal diagonal are $(b-c)^2$, &c., the other elements being as before, the value of the determinant is

$$-16\frac{\Delta^3}{s}\left(\frac{a^2}{r_1} + \frac{b^2}{r_2} + \frac{c^2}{r_3}\right),$$

$$\begin{vmatrix} (b+c)^2, & ab\ , & ac\ , & a \\ ab\ , & (c+a)^2, & bc\ , & b \\ ac\ , & bc\ , & (a+b)^2, & c \\ 1\ , & 1\ , & 1 & \end{vmatrix} = -16s\Delta(ar_1 + br_2 + cr_3),$$

$$\begin{vmatrix} (b+c)^2, & ab\ , & ac\ , & 1 \\ ab\ , & (c+a)^2, & bc\ , & 1 \\ ac\ , & bc\ , & (a+b)^2, & 1 \\ 1\ , & 1\ , & 1 & \end{vmatrix} = 16\Delta^2 - 20abcs.$$

7. If $S = a_1 + a_2 + \ldots + a_n$, $A_i = S - a_i$, prove the following theorems:

$$\begin{vmatrix} x - A_1, & a_2 & \ldots & a_n \\ a_1\ , & x - A_2 & \ldots & a_n \\ \cdots & \cdots & \cdots & \cdots \\ a_1\ , & a_2 & \ldots & x - A_n \end{vmatrix} = x(x-S)^{n-1},$$

$$\begin{vmatrix} x - a_1, & A_2 & \ldots & A_n \\ A_1\ , & x - a_2 & \ldots & A_n \\ \cdots & \cdots & \cdots & \cdots \\ A_1\ , & A_2 & \ldots & x - a_n \end{vmatrix} = \{x + (n-2)S\}(x-S)^{n-1}.$$

8. The determinant

$$\begin{vmatrix} a, & b, & b, & b \ldots\ldots \\ a, & b, & a, & a \ldots\ldots \\ b, & b, & a, & b \ldots\ldots \\ a, & a, & a, & b \ldots\ldots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

(the diagonal consisting of a and b alternately and each row being filled up with the other letter) is equal to

$$(-1)^{n-1}(n-1)(a-b)^{2n}.$$

The determinant is supposed to have $2n$ rows.

9. If in a determinant all the minors of the second order are divisible by the same quantity p, then the minors of the m^{th} order are divisible by p^{m-1}.

p^m

10. If in a determinant of the n^{th} order there be a block of p by q elements all of which are divisible by a, the determinant is divisible by a^{p+q-n}.

11. Prove the theorems :

$$\begin{vmatrix} a, & b, & c, & d, & \dots \\ a, & a+b, & a+b+c, & a+b+c+d, & \dots \\ a, & 2a+b, & 3a+2b+c, & 4a+3b+2c+d, & \dots \\ a, & 3a+b, & 6a+3b+c, & 10a+6b+3c+d, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} = a^n,$$

$$\begin{vmatrix} a, & b, & c, & d \dots \\ a, & a+b, & a+2b+c, & a+3b+3c+d \dots \\ a, & 2a+b, & 4a+4b+c, & 8a+12b+6c+d \dots \\ a, & 3a+b, & 9a+6b+c, & 27a+27b+9c+d \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = a^n\, 1^{n-1}.\, 2^{n-2}.\, 3^{n-3} \dots (n-1),$$

where $a, b, c, d \dots$ are any quantities whatever, and n is the order of the determinant. In the first determinant each row after the first is obtained from the preceding by the rule that the r^{th} element of any row is the sum of the first r elements of the preceding row. In the second determinant the r^{th} element of any row is the sum of the first r elements of the preceding row multiplied respectively by the coefficients in the expansion of $(1+x)^{r-1}$.

12. If $D = \begin{vmatrix} a, & b, & c, & d \dots \\ -a, & b, & p, & q \dots \\ -a, & -b, & c, & r \dots \\ -a, & -b, & -c, & d \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$ (n rows),

then $$D = 2^{n-1}\, abcd \dots$$

The elements of the first row and leading diagonal are $a, b, c, d \dots$; in each column the elements below the leading diagonal are equal to the element in the first row but of opposite sign, the others are any whatever.

13. If $D = \begin{vmatrix} \cos n\alpha_0, & \cos(n-1)\alpha_0 \dots \cos \alpha_0, & 1 \\ \cos n\alpha_1, & \cos(n-1)\alpha_1 \dots \cos \alpha_1, & 1 \\ \dots & \dots & \dots \\ \cos n\alpha_n, & \cos(n-1)\alpha_n \dots \cos \alpha_n, & 1 \end{vmatrix}$,

$$D_1 = \begin{vmatrix} \cos^n\alpha_0, & \cos^{n-1}\alpha_0 & \dots & \cos\alpha_0, & 1 \\ \cos^n\alpha_1, & \cos^{n-1}\alpha_1 & \dots & \cos\alpha_1, & 1 \\ \dots & \dots & \dots & \dots & \dots \\ \cos^n\alpha_n, & \cos^{n-1}\alpha_n & \dots & \cos\alpha_n, & 1 \end{vmatrix},$$

$$D_2 = \begin{vmatrix} \sin(n+1)\alpha_0, & \sin n\alpha_0 & \dots & \sin\alpha_0 \\ \sin(n+1)\alpha_1, & \sin n\alpha_1 & \dots & \sin\alpha_1 \\ \dots & \dots & \dots & \dots \\ \sin(n+1)\alpha_n, & \sin n\alpha_n & \dots & \sin\alpha_n \end{vmatrix},$$

then

$$\frac{D}{D_1} = 2^{\frac{n(n-1)}{2}}, \qquad \frac{D_2}{D_1} = 2^{\frac{n(n+1)}{2}} \sin\alpha_0 \sin\alpha_1 \dots \sin\alpha_n.$$

14. If $b_{ik} = (a_{i1} + a_{i2} + \dots + a_{in}) - a_{ik}$, then

$$\begin{vmatrix} b_{11} & \dots & b_{1n} \\ \dots & \dots & \dots \\ b_{n1} & \dots & b_{nn} \end{vmatrix} = (-1)^{n-1}(n-1) \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}.$$

But if $\qquad b_{ik} = (a_{i1} + a_{i2} + \dots + a_{in}) - 2a_{ik}$,

$$\begin{vmatrix} b_{11} & \dots & b_{1n} \\ \dots & \dots & \dots \\ b_{1n} & \dots & b_{nn} \end{vmatrix} = n(-2)^{n-1} \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}.$$

15. Prove that every power of a symmetrical determinant is again a symmetrical determinant.

16. If for each element a_{ik} of a determinant A we write in turn $a_{ik} + c$, we get n^2 new determinants. If these be taken as the elements of another determinant its value will be

$$(Ac)^{n-1}(c+S),$$

where S is the sum of all the elements of A.

17. If $\qquad u = (X_1 - a_1b_1)(X_2 - a_2b_2) \dots (X_n - a_nb_n)$,

prove that the value of the determinant

$$\begin{vmatrix} X_1, & a_2b_1, & a_3b_1 & \dots & a_nb_1 \\ a_1b_2, & X_2, & a_3b_2 & \dots & a_nb_2 \\ a_1b_3, & a_2b_3, & X_3 & \dots & a_nb_3 \\ \dots & \dots & \dots & \dots & \dots \\ a_1b_n, & a_2b_n, & a_3b_n & \dots & X_n \end{vmatrix}$$

is $$u\left\{1 + \frac{a_1b_1}{X_1 - a_1b_1} + \dots + \frac{a_nb_n}{X_n - a_nb_n}\right\},$$

and the value of

$$\begin{vmatrix} 0, & a_1, & a_2 & \dots & a_n \\ b_1, & X_1, & a_2b_1 & \dots & a_nb_1 \\ b_2, & a_1b_2, & X_2 & \dots & a_nb_2 \\ \dots & \dots & \dots & \dots & \dots \\ b_n, & a_1b_n, & a_2b_n & \dots & X_n \end{vmatrix}$$

is

$$-u\left\{\frac{a_1b_1}{X_1-a_1b_1}+\dots+\frac{a_nb_n}{X_n-a_nb_n}\right\}.$$

18. If $u=(x-2a_1)(x-2a_2)\dots(x-2a_n)$, prove the following theorems:

$$\begin{vmatrix} (x-a_1)^2, & a_2^2, & a_3^2 & \dots \\ a_1^2, & (x-a_2)^2, & a_3^2 & \dots \\ a_1^2, & a_2^2, & (x-a_3)^2 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = x^{n-1}u\left\{x+\Sigma\frac{a_i^2}{x-2a_i}\right\}$$

$$\begin{vmatrix} 0, & 1, & 1, & 1 & \dots \\ 1, & (x-a_1)^2, & a_2^2, & a_3^2 & \dots \\ 1, & a_1^2, & (x-a_2)^2, & a_3^2 & \dots \\ 1, & a_1^2, & a_2^2, & (x-a_3)^2 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} = -x^{n-1}\frac{du}{dx},$$

$$\begin{vmatrix} (x-a_1)^2, & a_1a_2, & a_1a_3 & \dots \\ a_1a_2, & (x-a_2)^2, & a_2a_3 & \dots \\ a_1a_3, & a_2a_3, & (x-a_3)^2 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = x^{n-1}u\left\{x+\Sigma\frac{a_i^2}{x-2a_i}\right\}$$

$$\begin{vmatrix} 0, & a_1, & a_2, & a_3 & \dots \\ a_1, & (x-a_1)^2, & a_1a_2, & a_1a_3 & \dots \\ a_2, & a_1a_2, & (x-a_2)^2, & a_2a_3 & \dots \\ a_3, & a_1a_3, & a_2a_3, & (x-a_3)^2 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} = -x^{n-1}u\Sigma\frac{a_i^2}{x-2a_i}.$$

And if

$$D=\begin{vmatrix} (x-a_1)^2, & a_2^2 & \dots & a_n^2, & b_1, & 1 \\ a_1^2, & (x-a_2)^2 & \dots & a_n^2, & b_2, & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_1^2, & a_2^2 & \dots & (x-a_n)^2, & b_n, & 1 \\ b_1, & b_2 & \dots & b_n, & & \\ 1, & 1 & \dots & 1 & & \end{vmatrix},$$

then

$$\frac{D}{x^{n-2}u} = \left\{\frac{1}{x-2a_1} + \dots + \frac{1}{x-2a_n}\right\}\left\{\frac{b_1^{\ 2}}{x-2a_1} + \dots + \frac{b_n^{\ 2}}{x-2a_n}\right\}$$
$$- \left\{\frac{b_1}{x-2a_1} + \dots + \frac{b_n}{x-2a_n}\right\}^2.$$

19. Prove that, if $S = x + y + z + u$,

$$\begin{vmatrix} (S-u)^2, & x^2, & y^2, & z^2 \\ u^2, & (S-x)^2, & y^2, & z^2 \\ u^2, & x^2, & (S-y)^2, & z^2 \\ u^2, & x^2, & y^2, & (S-z)^2 \end{vmatrix} = 2S^5 xyzu \left\{\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{u} - \frac{4}{S}\right\}$$

$$\begin{vmatrix} 0, & 1, & 1, & 1, & 1 \\ 1, & (S-u)^2, & x^2, & y^2, & z^2 \\ 1, & u^2, & (S-x)^2, & y^2, & z^2 \\ 1, & u^2, & x^2, & (S-y)^2, & z^2 \\ 1, & u^2, & x^2, & y^2, & (S-z)^2 \end{vmatrix} = S^3 \{x^2(y+z+u) + y^2(x+z+u) + z^2(x+y+u) + u^2(x+z+y) + 2xyz + 2xzu + 2yzu + 2xyu - x^3 - y^3 - z^3 - u^3\}.$$

20. If $X = \operatorname{cn} x \operatorname{dn} x$, &c. prove that

$$\begin{vmatrix} \operatorname{sn} x, & \operatorname{sn}^3 x, & X \\ \operatorname{sn} y, & \operatorname{sn}^3 y, & Y \\ \operatorname{sn} z, & \operatorname{sn}^3 z, & Z \end{vmatrix} = \operatorname{sn}(y-z)\operatorname{sn}(z-x)\operatorname{sn}(x-y)\operatorname{sn}(x+y+z)M,$$

where

$$M = 1 - k^2\{\operatorname{sn}^2 y \operatorname{sn}^2 z + \operatorname{sn}^2 z \operatorname{sn}^2 x + \operatorname{sn}^2 x \operatorname{sn}^2 y\}$$
$$+ k^2(1+k^2)\operatorname{sn}^2 x \operatorname{sn}^2 y \operatorname{sn}^2 z - k^2 \operatorname{sn} x \operatorname{sn} y \operatorname{sn} z\,(YZ \operatorname{sn} x + ZX \operatorname{sn} y + XY \operatorname{sn} z).$$

21. If $\operatorname{sn} x \operatorname{cn} x \operatorname{dn} x = X$, &c. prove that

$$\begin{vmatrix} 1, & \operatorname{sn}^2 x, & \operatorname{sn}^4 x, & X \\ 1, & \operatorname{sn}^2 y, & \operatorname{sn}^4 y, & Y \\ 1, & \operatorname{sn}^2 z, & \operatorname{sn}^4 z, & Z \\ 1, & \operatorname{sn}^2 u, & \operatorname{sn}^4 u, & U \end{vmatrix} = 0,$$

provided

$$x + y + z + u = 2pK + 2qiK',$$

p, q being integers.

22. If $$S_{ij} = a_{ij} + a_{i+hj+h} - a_{i+hj} - a_{ij+},$$

then

$$\begin{vmatrix} S_{11}, & S_{12} & \dots & S_{1k-h} \\ S_{21}, & S_{22} & \dots & S_{2k-h} \\ \dots & \dots & \dots & \dots \\ S_{k-h1}, & S_{k-h2} & \dots & S_{k-hk-h} \end{vmatrix}$$

is the sum of all the minors of order $k-h$ of the determinant $A=|a_{ik}|$; excepting always in such sum those determinants and their complements of order h which in their formation have two row or column suffixes congruent with regard to the modulus h.

23. If

$$D_n = \begin{vmatrix} 0, & 1, & 1, & 1, & 1 & \dots \\ 1, & 0, & x, & 0, & 0 & \dots \\ 1, & y, & 0, & x, & 0 & \dots \\ 1, & 0, & y, & 0, & x & \dots \\ 1, & 0, & 0, & y, & 0 & \dots \\ \dots & & & & & \end{vmatrix} \quad (n \text{ rows}),$$

where all elements are zeros, with the exception of the border, and two lines of elements one on each side of the principal diagonal, prove that

$$D_{2n} = -xy\,D_{2n-2} - \frac{x^{2n-1}+y^{2n-1}}{x+y},$$

$$D_{2n+1} = -xy\,D_{2n-1} + \frac{x^{2n}+y^{2n}}{x+y} - \frac{2\,(-xy)^n}{x+y},$$

and hence that

$$D_{2n} = -\left\{\frac{x^n-(-y)^n}{x+y}\right\}^2,$$

$$D_{2n+1} = \frac{x^{2n+1}+y^{2n+1}-(2n+1)\,(x+y)\,(-xy)^n}{(x+y)^2}.$$

24. If

$$D_n = \begin{vmatrix} c, & a, & c, & c, & c & \dots \\ b, & c, & a, & c, & c & \dots \\ c, & b, & c, & a, & c & \dots \\ c, & c, & b, & c, & a & \dots \\ c, & c, & c, & b, & c & \dots \\ \dots & & & & & \end{vmatrix} \quad (n \text{ rows}),$$

where all the elements are c with the exception of two lines, one on either side of the principal diagonal, prove that

$$D_{2n-1} = c\left\{\frac{(a-c)^n-(c-b)^n}{a+b-2c}\right\}^2.$$

Find also the value of D_{2n}.

25. If

$$D_n = \begin{vmatrix} 0, & 1, & 1, & 1, & 1 & \dots \\ 1, & c, & a, & 0, & 0 & \dots \\ 1, & b, & c, & a, & 0 & \dots \\ 1, & 0, & b, & c, & a & \dots \\ 1, & 0, & 0, & b, & c & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} \quad (n \text{ rows}),$$

(where, with the exception of the border, the elements in the leading diagonal are c, in the lines on either side of it a and b, the rest are zero), then

$$D_n - cD_{n-1} + abD_{n-2} = \frac{(-a)^{n-1} + (-b)^{n-1}}{a+b+c}$$

$$- \frac{c}{a+b+c} \cdot \frac{u^{n-1} - v^{n-1}}{u-v}$$

$$+ \frac{2ab}{a+b+c} \cdot \frac{u^{n-2} - v^{n-2}}{u-v},$$

where u and v are the roots of the equation

$$z^2 - cz + ab = 0.$$

Hence shew that

$$D_n = \frac{u^n + v^n}{(a+b+c)^2} - \frac{nc(u^n + v^n)}{(a+b+c)(u-v)^2}$$

$$+ \frac{2abn}{a+b+c} \cdot \frac{u^{n-1} + v^{n-1}}{(u-v)^2} - \frac{(-a)^n + (-b)^n}{(a+b+c)^2}.$$

26. The value of the determinant

$$\begin{vmatrix} u_1, & u_2 & \dots & u_n \\ u_n, & u_1 & \dots & u_{n-1} \\ u_{n-1}, & u_n & \dots & u_{n-2} \\ \dots & \dots & \dots & \dots \\ u_2, & u_3 & \dots & u_1 \end{vmatrix}.$$

(i) If $u_r = a + (r-1)b$ is

$$\frac{2a + (n-1)b}{2}(-nb)^{n-1}.$$

(ii) If $u_r = x^{r-1}$ is $(1-x^n)^{n-1}$.

(iii) If $v_r = r^2$ is

$$(-1)^{n-1}\frac{(n+1)(2n+1)n^{n-2}}{12}\{(n+2)^n - n^n\}.$$

(iv) If $u_r = \cos\{a+(r-1)b\}$ is

$$\frac{[\cos a - \cos(a+nb)]^n - [\cos(a-b) - \cos\{a+(n-1)b\}]^n}{2(1-\cos nb)}.$$

(v) If $u_r = \sin\{a+(r-1)b\}$ we must change the cosines in the numerator of (iv) into sines.

(vi) If $u_r = x^{r-1} + x^{r+n-1} + x^{r+2n-1} + \dots$ ad inf., is

$$(1-x^n)^{-1}.$$

27. The solution of the partial differential equation

$$\begin{vmatrix} D_1, & D_2 & \dots & D_n \\ D_n, & D_1 & \dots & D_{n-1} \\ \dots & \dots & \dots & \dots \\ D_2, & D_3 & \dots & D_1 \end{vmatrix} u = 0,$$

where $$D_r = \frac{d}{dx_r}$$

is $$u = \Sigma F(x_2 - \omega x_1,\quad x_3 - \omega^2 x_1 \dots x_n - \omega^{n-1} x_1),$$

the functions being arbitrary and the summation extending to all values of ω being roots of the equation $x^n - 1 = 0$.

28. If in an orthosymmetrical determinant of order n (VI. 20),

$$a_k = \frac{(1-q^\alpha)(1-q^{\alpha+1})\dots(1-q^{\alpha+k-2})}{(1-q^\gamma)(1-q^{\gamma+1})\dots(1-q^{\gamma+k-2})},$$

the value of the determinant is equal to

$$\left(\frac{1-q^\alpha}{1-q^\gamma}\right)^{n-1}\left(\frac{1-q^{\alpha+1}}{1-q^{\gamma+1}}\right)^{n-2}\dots\left(\frac{1-q^{\alpha+n-2}}{1-q^{\gamma+n-2}}\right)$$

multiplied by a fraction whose numerator is

$$(-1)^{\frac{n(n-1)}{2}} q^{\frac{n(n-1)(n-2)}{3}} (1-q)^{n-1}(1-q^2)^{n-2}\dots(1-q^{n-1})$$
$$\times (q^\gamma - q^\alpha)^{n-1}(q^{\gamma+1}-q^\alpha)^{n-2}\dots(q^{\gamma+n-2}-q^\alpha),$$

and denominator

$$(1-q^\gamma)(1-q^{\gamma+1})^2\dots(1-q^{\gamma+n-2})^{n-1}$$
$$\times(1-q^{\gamma+n-1})^{n-1}(1-q^{\gamma+n})^{n-2}\dots(1-q^{\gamma+2n-3}).$$

29. The value of the determinant

$$D = \begin{vmatrix} 0\quad, & a_1+a_2, & a_1+a_3 & \dots \\ a_2+a_1, & 0\quad, & a_2+a_3 & \dots \\ a_3+a_1, & a_3+a_2, & 0 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} \ (n \text{ rows}),$$

the elements in the leading diagonal being zero, that in the i^{th} row and j^{th} column $a_i + a_j$, is given by

$$(-1)^n D = 2^n a_1 a_2 \ldots a_n \left(1 - n - \tfrac{1}{4}\Sigma \frac{(a_i - a_k)^2}{a_i a_k}\right),$$

where i, k are all duads from 1, 2 ... n.

30. The value of the cubic determinant of order n, such that

$$a_{ijk} = a_i + a_j + a_k, \qquad a_{iii} = 0,$$

is given by

$$\frac{(-1)^n D}{3^n a_1 a_2 \ldots a_n} = 1 - n - \tfrac{2}{9}\Sigma \frac{(a_i - a_k)^2}{a_i a_k}.$$

And if

$$a_{ijk} = \cos(a_i + a_j + a_k), \quad a_{iii} = 0,$$

$$\frac{(-1)^{n-1} D}{\cos 3a_1 \cos 3a_2 \ldots \cos 3a_n} = n - 1 + 2\Sigma \frac{\cos(a_i + a_k)\sin^2(a_i - a_k)}{\cos 3a_i \cos 3a_k},$$

where i, k are all duads from 1, 2 ... n.

31. If $A = |a_{ik}|$, $B = |b_{ik}|$ are two determinants of orders n and m respectively, we can form a new square array of $(nm)^2$ elements as follows. Repeat the array b_{ik}, n times in a row, and take n such rows, so that B is repeated like the squares on a chess-board. Then multiply each of the elements of that block which stands in the i^{th} row and k^{th} column by a_{ik}. The determinant of the resulting array is equal to $A^m B^n$.

Example:

$$A = \begin{vmatrix} a, & b \\ c, & d \end{vmatrix}; \quad B = \begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix},$$

$$\begin{vmatrix} a\alpha, & a\beta, & b\alpha, & b\beta \\ a\gamma, & a\delta, & b\gamma, & b\delta \\ c\alpha, & c\beta, & d\alpha, & d\beta \\ c\gamma, & c\delta, & d\gamma, & d\delta \end{vmatrix} = A^2 B^2.$$

32. If $a, b \ldots l$; $\alpha, \beta \ldots \lambda$ are any two sets of n quantities, and

$$d_{ik} = (a_i - \alpha_k)^r + (b_i - \beta_k)^r + \ldots + (l_i - \lambda_k)^r,$$

prove that

$$\begin{vmatrix} d_{11} \ldots d_{1s} \\ \ldots\ldots\ldots\ldots \\ d_{s1} \ldots d_{ss} \end{vmatrix} = 0, \text{ if } s = n(r-1) + 3,$$

$$\begin{vmatrix} d_{11} & \dots & d_{1s}, & 1 \\ \multicolumn{4}{c}{\dots\dots\dots\dots} \\ d_{s1} & \dots & d_{ss}, & 1 \\ 1 & \dots & 1 & \end{vmatrix} = 0, \text{ if } s = n(r-1)+2.$$

33. $\left\{ \text{In this and the next five questions} \right.$

$$\left. m_k = \frac{m(m-1)(m-2)\dots(m-k+1)}{1.2.3\dots k} \right\}.$$

The determinant

$$\begin{vmatrix} m_p, & m_{p+1} & \dots & m_u \\ (m+1)_p, & (m+1)_{p+1} & \dots & (m+1)_u \\ \dots & \dots & \dots & \dots \\ (m+r-1)_p, & (m+r-1)_{p+1} & \dots & (m+r-1)_u \\ (m+r)_p, & (m+r)_{p+1} & \dots & (m+r)_u \\ (m+r+s)_p, & (m+r+s)_{p+1} & \dots & (m+r+s)_u \\ (m+r+s+1)_p, & (m+r+s+1)_{p+1}, & \dots & (m+r+s+1)_u \\ \dots & \dots & \dots & \dots \\ (m+r+s+t)_p, & (m+r+s+t)_{p+1} & \dots & (m+r+s+t)_u \end{vmatrix},$$

where $u = p + r + t + 1$ (the suffixes p, $p+1 \dots u$ of the rows are consecutive, but m, $m+1 \dots m+r$, $m+r+s \dots m+r+s+t$ form two groups of consecutive numbers), is equal to the product of the two fractions

$$\frac{m_p(m+1)_p \dots (m+r)_p (m+r+s)_p \dots (m+r+s+t)_p}{p_p(p+1)_p \dots u_p}$$

$$\frac{(r+s)_{r+1}(r+s+1)_{r+1} \dots (r+s+t)_{r+1}}{(r+1)_{r+1}(r+2)_{r+1} \dots (r+t+1)_{r+1}}.$$

34. The determinant

$$\begin{vmatrix} m_p, & m_{p+1} & \dots & m_{p+s}, & m_{p+s+v} & \dots & m_{p+s+v+u} \\ (m+1)_p, & (m+1)_{p+1} & \dots & (m+1)_{p+s}, & (m+1)_{p+s+v} & \dots & (m+1)_{p+s+v+u} \\ (m+2)_p, & (m+2)_{p+1} & \dots & (m+2)_{p+s}, & (m+2)_{p+s+v} & \dots & (m+2)_{p+s+v+u} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ (m+r)_p, & (m+r)_{p+1} & \dots & (m+r)_{p+s}, & (m+r)_{p+s+v} & \dots & (m+r)_{p+s+v+u} \end{vmatrix},$$

where $r = s + u + 1$ (the suffixes p, $p+1 \dots p+s$, $p+s+v \dots p+s+v+u$ form two groups of consecutive numbers, while m, $m+1 \dots m+r$ are consecutive), is equal to the product of the two fractions

$$\frac{m_p(m+1)_p \dots (m+r)_p}{p_p(p+1)_p \dots (p+s)_p(p+s+v)_p \dots (p+s+v+u)_p}$$

$$\frac{(m-p)_{v-1}(m-p+1)_{v-1} \dots (m-p+u)_{v-1}}{(v-1)_{v-1}v_{v-1}(v+1)_{v-1} \dots (v+u-1)_{v-1}}.$$

35. Prove that

$$\begin{vmatrix} x^n, & p_0, & p_1 & \dots & p_{r-1} \\ (x+1)^n, & (p+1)_0, & (p+1)_1 & \dots & (p+1)_{r-1} \\ (x+2)^n, & (p+2)_0, & (p+2)_1 & \dots & (p+2)_{r-1} \\ \dots & \dots & \dots & \dots & \dots \\ (x+r)^n, & (p+r)_0, & (p+r)_1 & \dots & (p+r)_{r-1} \end{vmatrix}$$

vanishes if $n < r$, but is equal to $(-1)^n n!$ if $n = r$. If $n > r$ the determinant reduces to a function of x of order $n - r$.

36. Prove that

$$\begin{vmatrix} x^n, & p_1 & \dots & p_r \\ (x+1)^n, & (p+1)_1 & \dots & (p+1)_r \\ (x+2)^n, & (p+2)_1 & \dots & (p+2)_r \\ \dots & \dots & \dots & \dots \\ (x+r)^n, & (p+r)_1 & \dots & (p+r)_r \end{vmatrix} = (x-p)^n$$

for all positive values of n less than r.

37. Prove that

$$\begin{vmatrix} p_0, & p_1 & \dots & p_{r-1} & n^m \\ (p+1)_0, & (p+1)_1 & \dots & (p+1)_{r-1}, & (n+1)^m \\ (p+2)_0, & (p+2)_1 & \dots & (p+2)_{r-1}, & (n+2)^m \\ \dots & \dots & \dots & \dots & \dots \\ (p+r)_0, & (p+r)_1 & \dots & (p+r)_{r-1}, & (n+r)^m \end{vmatrix} = \Delta^r n^m.$$

38. Prove that the value of the determinant

$$\begin{vmatrix} (m-p)\,m_p, & n\,m_{p+1}, & q\,m_{p+2}, & t\,m_{p+3} \\ (m-p+1)(m+1)_p, & (n+1)(m+1)_{p+1}, & (q+1)(m+1)_{p+2}, & (t+1)(m+1)_{p+3} \dots \\ (m-p+2)(m+2)_p, & (n+2)(m+2)_{p+1}, & (q+2)(m+2)_{p+2}, & (t+2)(m+2)_{p+3} \dots \\ \dots & \dots & \dots & \dots \\ (m-p+r)(m+r)_p, & (n+r)(m+r)_{p+1}, & (q+r)(m+r)_{p+2}, & (t+r)(m+r)_{p+3} \dots \end{vmatrix}$$

is $\dfrac{m_p\,(m+1)_p \dots (m+r)_p}{p_p\,(p+1)_p \dots (p+r)_p}(m-p)(m-p+1)(m-p+2)\dots(m-p+r)$,

and so is independent of the quantities n, q, t...

39. If $A = |\, a_{ik} \,|$; $B = |\, b_{ik} \,|$ are two determinants of order n, and

$$f(x) = |\, a_{ik} + x\,b_{ik} \,|,$$

prove that

$$f(x)f(-x) = AB\,|\, H_{ik} - K_{ik}x^2 \,|,$$

where the quantities H_{ik}, K_{ik} satisfy the equations

$$H_{r1}K_{1r} + H_{r2}K_{2r} + \dots + H_{rn}K_{nr} = 1,$$
$$H_{r1}K_{1s} + H_{r2}K_{2s} + \dots + H_{rn}K_{ns} = 0.$$

40. With the same notation as in the preceding question, prove that if

$$P(\lambda, \mu) = | \lambda a_{ik} + \mu b_{ik} |,$$

then

$$P(\lambda,\mu) = A \begin{vmatrix} \mu H_{11} + \lambda, & \mu H_{12} & \dots & \mu H_{1n} \\ \mu H_{21}, & \mu H_{22} + \lambda & \dots & \mu H_{2n} \\ \dots & \dots & \dots & \dots \\ \mu H_{n1}, & \mu H_{n2} & \dots & \mu H_{nn} + \lambda \end{vmatrix}$$
$$= B \begin{vmatrix} \lambda K_{11} + \mu, & \lambda K_{12} & \dots & \lambda K_{1n} \\ \lambda K_{21}, & \lambda K_{22} + \mu & \dots & \lambda K_{2n} \\ \dots & \dots & \dots & \dots \\ \lambda K_{n1}, & \lambda K_{n2} & \dots & \lambda K_{n} + \mu \end{vmatrix}.$$

41. If $F(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n$,

prove that

$$P = \begin{vmatrix} x, & 0, & 0 & \dots & 0, & \frac{a_n}{a_0} \\ -1, & x, & 0 & \dots & 0, & \frac{a_{n-1}}{a_0} \\ 0, & -1, & x & \dots & 0, & \frac{a_{n-2}}{a_0} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & 0 & \dots & x, & \frac{a_2}{a_0} \\ 0, & 0, & 0 & \dots & -1, & \frac{a_1}{a_0} + x \end{vmatrix} = \frac{F(x)}{a_0},$$

$$Q = \begin{vmatrix} 1 + \frac{a_{n-1}}{a_n} x, & -x, & 0 & \dots & 0, & 0 \\ \frac{a_{n-2}}{a_n} x, & 1, & -x & \dots & 0, & 0 \\ \frac{a_{n-3}}{a_n} x, & 0, & 1 & \dots & 0, & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{a_1}{a_n} x, & 0, & 0 & \dots & 1, & -x \\ \frac{a_0}{a_n} x, & 0, & 0 & \dots & 0, & 1 \end{vmatrix} = \frac{F(x)}{a_n}.$$

If P_{rs}, Q_{rs} be the coefficients of homologous elements in P and Q,

$$a_0 P_{rr} x + a_n Q_{rr} = F(x)$$
$$a_0 P_{rs} x + a_n Q_{rs} = 0.$$

Also, if to the elements of P we add the homologous elements of Q multiplied by y, the resulting determinant is equal to

$$\frac{F(x)\,F(y)}{a_0 a_n}.$$

42. Prove the formula for the change of the independent variable in the determinant of n functions

$$\Sigma \pm y_1 \frac{dy_2}{dx}\frac{d^2y_3}{dx^2}\cdots\frac{d^{n-1}y_n}{dx^{n-1}}$$

$$=\left(\frac{dt}{dx}\right)^{\frac{n(n+1)}{2}}\Sigma \pm y_1 \frac{dy_2}{dt}\frac{d^2y_3}{dt^2}\cdots\frac{d^{n-1}y_n}{dt^{n-1}}.$$

43. Let a_1, a_2, a_3 ... be a series of n positive numbers, and let s_r be the sum of the divisors of r selected from the terms of this series, this sum being supposed to vanish for all values of r which have no divisors in the above series. Then if

$$D_n=\begin{vmatrix} s_1s_{n-1}+s_n, & -s_1, & -s_2, & -s_3 & \cdots & -s_{n-2} \\ s_1s_{n-2}+s_{n-1}, & n-1, & -s_1, & -s_2 & \cdots & -s_{n-3} \\ s_1s_{n-3}+s_{n-2}, & 0, & n-2, & -s_1 & \cdots & -s_{n-4} \\ s_1s_{n-4}+s_{n-3}, & 0, & 0, & n-3 & \cdots & -s_{n-5} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ s_1s_1+s_2, & 0, & 0, & 0 & \cdots & 2 \end{vmatrix},$$

the number of positive integral solutions of the equation

$$a_1x_1+a_2x_2+a_3x_3+\ldots=n$$

is

$$\frac{D_n}{n!}.$$

44. If s_r is the sum of all the divisors of r, then the determinant

$$\begin{vmatrix} s_{n-1}-s_n, & s_1, & s_2, & s_3 & \cdots & s_{n-3}, & s_{n-2} \\ s_{n-2}-s_{n-1}, & n-1, & s_1, & s_2 & \cdots & s_{n-4}, & s_{n-3} \\ s_{n-3}-s_{n-2}, & 0, & n-2, & s_1 & \cdots & s_{n-5}, & s_{n-4} \\ s_{n-4}-s_{n-3}, & 0, & 0, & n-3 & \cdots & s_{n-6}, & s_{n-5} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ s_2-s_3, & 0, & 0, & 0 & \cdots & 3, & s_1 \\ s_1-s_2, & 0, & 0, & 0 & \cdots & 0, & 2 \end{vmatrix}$$

is equal to $(-1)^k n!$ when n is of the form $\frac{1}{2}(3k^2\pm k)$, but vanishes for other values of n.

45. Let (m, n) denote the greatest common divisor of the integral numbers m and n; and let $\psi(m)$ be the number of numbers not surpassing m and prime to m; the symmetrical determinant

$$D_m = \Sigma \pm (1, 1)(2, 2) \dots (m, m)$$

is equal to

$$\psi(1)\,\psi(2)\,\psi(3) \dots \psi(m).$$

46. If A is a skew determinant of order n in which the principal diagonal elements are equal to z, and A_{ik} its system of first minors, prove that

$$A_{r1}A_{s1} + A_{r2}A_{s2} + \dots + A_{rn}A_{sn}$$

is equal to Aw_{rs} if n is even, and to $\frac{A}{z}w_{rs}$ if n is odd.

47. If $f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n = 0$ has for its roots $b_1, b_2 \dots b_n$, prove that

$$f(x) = \begin{vmatrix} x, & b_1, & b_1 & \dots & b_1, & b_1 \\ b_1, & x, & b_2 & \dots & b_2, & b_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b_{n-1}, & b_{n-1}, & b_{n-1} & \dots & x, & b_n \\ 1, & 1, & 1 & \dots & 1, & 1 \end{vmatrix}.$$

And if s_r is the sum of the r^{th} powers of the roots

$$\begin{vmatrix} x^n, & x^{n-1} & \dots & x, & 1 \\ s_n, & s_{n-1} & \dots & s_1, & s_0 \\ s_{n+1}, & s_n & \dots & s_2, & s_1 \\ \dots & \dots & \dots & \dots & \dots \\ s_{2n-1}, & s_{2n-2} & \dots & s_n, & s_{n-1} \end{vmatrix} = (-1)^{\frac{n(n-1)}{2}}\zeta(b_1, b_2 \dots b_n)f(x).$$

48. Prove that

$$\begin{vmatrix} a_1^r, & a_1^{n-2} & \dots & a_1, & 1 \\ a_2^r, & a_2^{n-2} & \dots & a_2, & 1 \\ \dots & \dots & \dots & \dots & \dots \\ a_n^r, & a_n^{n-2} & \dots & a_n, & 1 \end{vmatrix} = \zeta^{\frac{1}{2}}(a_1, a_2 \dots a_n)\,H_{r-n+1},$$

H_p being the sum of the homogeneous powers and products of order p of

$$a_1, a_2 \dots a_n.$$

49. If $$a_{rs} = \frac{1}{x_s - a_r}, \qquad \alpha_{rs} = \frac{1}{(x_s - a_r)^2},$$

prove that the value of the determinant of order $2n$

$$\begin{vmatrix} a_{11}, & \alpha_{11} & \dots & a_{n1}, & \alpha_{n1} \\ a_{12}, & \alpha_{12} & \dots & a_{n2}, & \alpha_{n2} \\ \dots & \dots & \dots & \dots & \dots \\ a_{1,2n}, & \alpha_{1,2n} & \dots & a_{n,2n}, & \alpha_{n,2n} \end{vmatrix}$$

is
$$(-1)^n \frac{\zeta^2(a_1, a_2 \ldots a_n)\, \zeta^{\frac{1}{2}}(x_1, x_2 \ldots x_{2n})}{[\phi(a_1)\,\phi(a_2)\ldots\phi(a_n)]^2},$$
where
$$\phi(x) = (x - x_1)(x - x_2)\ldots(x - x_n).$$

50. Prove that the value of the determinant of order $2n+1$ whose i^{th} row is
$$1,\ \sin a_i,\ \cos a_i,\ \sin 2a_i,\ \cos 2a_i \ldots \sin na_i,\ \cos na_i,$$
is
$$2^{2n^2}\, \Pi \sin \tfrac{1}{2}(a_i - a_k),$$
where i, k are all duads from $1, 2 \ldots n\,(i > k)$.

Also that the value of the determinant of order $2n$ whose i^{th} row is
$$\sin a_i,\ \cos a_i,\ \sin 2a_i,\ \cos 2a_i \ldots \sin na_i,\ \cos na_i,$$
is
$$2^{2n^2-2n+1}\, \Pi \sin \tfrac{1}{2}(a_i - a_k)\, S,$$
where
$$S = \Sigma \cos \tfrac{1}{2}(a_1 + a_2 + \ldots + a_n - a_{n+1} \ldots - a_{2n})$$
is formed by dividing the $2n$ angles into two sets of n in all possible ways and taking the cosine of half the difference of the sums of these sets.

51. If
$$A = \begin{vmatrix} \dfrac{1}{a_1 - x_1}, & \dfrac{1}{a_2 - x_1} & \cdots & \dfrac{1}{a_n - x_1}, & 1 \\ \dfrac{1}{a_1 - x_2}, & \dfrac{1}{a_2 - x_2} & \cdots & \dfrac{1}{a_n - x_2}, & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{1}{a_1 - x_{n+1}}, & \dfrac{1}{a_2 - x_{n+1}} & \cdots & \dfrac{1}{a_n - x_{n+1}}, & 1 \end{vmatrix},$$
prove that
$$A = (-1)^n \frac{\zeta^{\frac{1}{2}}(a_1, a_2 \ldots a_n)\, \zeta^{\frac{1}{2}}(x_1, x_2 \ldots x_{n+1})}{\phi(a_1)\,\phi(a_2)\ldots\phi(a_n)},$$
where
$$\phi(x) = (x - x_1)(x - x_2)\ldots(x - x_{n+1}).$$

If B is the determinant obtained from A by writing $(a_r - x_s)^2$ in place of $(a_r - x_s)$, prove that
$$\frac{A}{B} = \left\{ \begin{matrix} \dfrac{1}{a_1 - x_1}, & \dfrac{1}{a_2 - x_1} & \cdots & \dfrac{1}{a_n - x_1}, & 1 \\ \dfrac{1}{a_1 - x_2}, & \dfrac{1}{a_2 - x_2} & \cdots & \dfrac{1}{a_n - x_2}, & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{1}{a_1 - x_{n+1}}, & \dfrac{1}{a_2 - x_{n+1}} & \cdots & \dfrac{1}{a_n - x_{n+1}}, & 1 \end{matrix} \right\},$$

the function on the right being formed like a determinant, with all the signs positive instead of alternating.

52. If $\alpha, \beta \ldots \lambda$; $\alpha', \beta' \ldots \lambda'$ are two sets each of n quantities, and C_r is the product of all the binomial coefficients in the expansion of $(1+x)^r$, prove the following equalities:

$$\begin{vmatrix} (\alpha-\alpha')^n, & (\alpha-\beta')^n \ldots (\alpha-\lambda')^n \\ (\beta-\alpha')^n, & (\beta-\beta')^n \ldots (\beta-\lambda')^n \\ \cdots\cdots & \cdots\cdots \\ (\lambda-\alpha')^n, & (\lambda-\beta')^n \ldots (\lambda-\lambda')^n \end{vmatrix} = \frac{C_n}{n!} \zeta^{\frac{1}{2}}(\alpha, \beta \ldots \lambda)\, \zeta^{\frac{1}{2}}(\alpha', \beta' \ldots \lambda')\, I,$$

where

$$I = \begin{Bmatrix} \alpha-\alpha', & \alpha-\beta' \ldots \alpha-\lambda' \\ \beta-\alpha', & \beta-\beta' \ldots \beta-\lambda' \\ \cdots\cdots & \cdots\cdots \\ \lambda-\alpha', & \lambda-\beta' \ldots \lambda-\lambda' \end{Bmatrix}.$$

If
$$u = (x-\alpha y)(x-\beta y) \ldots (x-\lambda y),$$
$$v = (x-\alpha' y)(x-\beta' y) \ldots (x-\lambda' y),$$
$$I = (12)^n uv,$$

using the notation of invariants,

$$\begin{vmatrix} (\alpha-\alpha')^n \ldots (\alpha-\lambda')^n, & (\alpha-x)^n \\ \cdots\cdots & \cdots \\ (\lambda-\alpha')^n \ldots (\lambda-\lambda')^n, & (\lambda-x)^n \\ (x-\alpha')^n \ldots (x-\lambda')^n, & \end{vmatrix} = (-1)^n C_n \zeta^{\frac{1}{2}}(\alpha, \beta \ldots \lambda)\, \zeta^{\frac{1}{2}}(\alpha', \beta' \ldots \lambda')\, uv,$$

$$\begin{vmatrix} (\alpha-\alpha')^{n+1} \ldots (\alpha-\lambda')^{n+1}, & (\alpha-x)^{n+1} \\ \cdots\cdots & \cdots \\ (\lambda-\alpha')^{n+1} \ldots (\lambda-\lambda')^{n+1}, & (\lambda-x)^{n+1} \\ (x-\alpha')^{n+1} \ldots (x-\lambda')^{n+1} & \end{vmatrix} = (-1)^n \frac{C_{n+1}}{(n+1)!} \zeta^{\frac{1}{2}}(\alpha, \beta \ldots \lambda) \times \zeta^{\frac{1}{2}}(\alpha', \beta' \ldots \lambda')\, I \,.\, uv,$$

where

$$I = \begin{Bmatrix} \alpha-\alpha' \ldots \alpha-\lambda', & \alpha-x \\ \beta-\alpha' \ldots \beta-\lambda', & \beta-x \\ \cdots\cdots & \cdots \\ \lambda-\alpha' \ldots \lambda-\lambda', & \lambda-x \\ x-\alpha' \ldots x-\lambda' & \end{Bmatrix} = -(12)^{n-1} uv.$$

Again,

$$\begin{vmatrix} (\alpha-\alpha')^n \ldots (\alpha-\lambda')^n, & 1 \\ \cdots\cdots & \cdots \\ (\lambda-\alpha')^n \ldots (\lambda-\lambda')^n, & 1 \\ 1 \quad \ldots \quad 1 & \end{vmatrix} = (-1)^{n+1} C_n \zeta^{\frac{1}{2}}(\alpha \ldots \lambda)\, \zeta^{\frac{1}{2}}(\alpha' \ldots \lambda'),$$

$$\begin{vmatrix} (a-a')^{n+1} \ldots (a-\lambda')^{n+1}, & 1 \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ (\lambda-a')^{n+1} \ldots (\lambda-\lambda')^{n+1}, & 1 \\ 1 \quad \ldots \quad 1 & \end{vmatrix} = (-1)^n \frac{C_{n+1}}{(n+1)!} \zeta^{\frac{1}{2}}(a \ldots \lambda)\, \zeta^{\frac{1}{2}}(a' \ldots \lambda') I',$$

$$I' = \left\{\begin{matrix} a-a' \ldots a-\lambda', & 1 \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ \lambda-a' \ldots \lambda-\lambda', & 1 \\ 1 \qquad 1 & \end{matrix}\right\}$$

$$= (12)^{n-1} \frac{du}{dx} \cdot \frac{dv}{dx}.$$

53. Let there be two systems of binary n-tics $u_1 \ldots u_n$; $v_1 \ldots v_n$ where

$$u_i = a_{0i} x^n + n_1 a_{1i} x^{n-1} y + n_2 a_{2i} x^{n-2} y^2 + \ldots + a_{ni} y^n,$$

$$v_i = b_{0i} x^n + n_1 b_{1i} x^{n-1} y + n_2 b_{2i} x^{n-2} y^2 + \ldots + b_{ni} y^n.$$

And let (i, k) be the lineo-linear invariant of u_i and v_k, so that

$$(i, k) = a_{0i} b_{nk} - n_1 a_{1i} b_{n-1k} + n_2 a_{2i} b_{n-2k} - \ldots \pm a_{ni} b_{0k}.$$

Prove that

$$\begin{vmatrix} (1, 1) & \ldots & (1, n+2) \\ \ldots & \ldots & \ldots \\ (n+2, 1) & \ldots & (n+2, n+2) \end{vmatrix} = 0,$$

$$\begin{vmatrix} (1, 1) & \ldots & (1, n+1) \\ \ldots & \ldots & \ldots \\ (n+1, 1) & \ldots & (n+1, n+1) \end{vmatrix} = C_n \begin{vmatrix} a_{01}, & a_{11} \ldots a_{n1} \\ \ldots & \ldots \\ a_{0n+1}, & a_{1n+1} \ldots a_{nn+1} \end{vmatrix} \begin{vmatrix} b_{01}, & b_{11} \ldots b_{n1} \\ \ldots & \ldots \\ b_{0n+1}, & b_{1n+1} \ldots b_{nn+1} \end{vmatrix}.$$

54. If $a_1, a_2 \ldots a_n$ are the roots of the equation

$$x^n + p_1 x^{n-1} + \ldots + p_n = 0,$$

prove that

$$\frac{d(p_1, p_2 \ldots p_n)}{d(a_1, a_2 \ldots a_n)} = (-1)^{\frac{1}{2}n(n-1)} \zeta^{\frac{1}{2}}(a_1, a_2 \ldots a_n).$$

55. If $\quad u_1 = \dfrac{x_1}{x_n}, \quad u_2 = \dfrac{x_2}{x_n} \ldots u_{n-1} = \dfrac{x_{n-1}}{x_n},$

x_n being a function of $x_1, x_2 \ldots x_{n-1}$ given by

$$x_1^2 + x_2^2 + \ldots + x_{n-1}^2 + x_n^2 = 1,$$

prove that

$$\frac{d(u_1, u_2 \ldots u_{n-1})}{d(x_1, x_2 \ldots x_{n-1})} = \frac{1}{x_n^{n+1}}.$$

56. If $u_n = (x+y+z)^n + (x-y-z)^n + (-x+y-z)^n + (-x-y+z)^n$, prove that the Hessian of u_n is

$$u_{2-n}(x^4+y^4+z^4-2x^2y^2-2y^2z^2-2z^2x^2)^{n-2}$$

multiplied by a numerical factor.

57. If

$$F = u_1 u_2 \dots u_n,$$

where $u_1, u_2 \dots u_n$ are linear functions of the n variables $x_1, x_2 \dots x_n$, prove that

$$F^2 H(\log F) = (-1)^n \left[\frac{d(u_1, u_2 \dots u_n)}{d(x_1, x_2 \dots x_n)}\right]^2.$$

Also that

$$\begin{vmatrix} F, & \dfrac{dF}{dx_1} & \dots & \dfrac{dF}{dx_n} \\ \dfrac{dF}{dx_1}, & \dfrac{d^2F}{dx_1^2} & \dots & \dfrac{d^2F}{dx_1 dx_n} \\ \dots & \dots & \dots & \dots \\ \dfrac{dF}{dx_n}, & \dfrac{d^2F}{dx_n dx_1} & \dots & \dfrac{d^2F}{dx_n^2} \end{vmatrix} = (-1)^n F^{n-1} \left[\frac{d(u_1 \dots u_n)}{d(x_1 \dots x_n)}\right]^2.$$

58. If u_1, u_2, u_3 be three functions of x, y, and if

$$v_1 = \frac{d(u_2, u_3)}{d(x, y)}, \quad v = \frac{d(u_3, u_1)}{d(x, y)}, \quad v_3 = \frac{d(u_1, u_2)}{d(x, y)},$$

$$w_1 = \frac{d(v_2, v_3)}{d(x, y)}, \text{ \&c.},$$

prove that

$$\frac{w_1}{u_1} = \frac{w_2}{u_2} = \frac{w_3}{u_3}.$$

59. If u_1, u_2, u_3, u_4 are four functions of x, y, and if

$$v_1 = \begin{vmatrix} \dfrac{d^2u_2}{dx^2}, & \dfrac{d^2u_3}{dx^2}, & \dfrac{d^2u_4}{dx^2} \\ \dfrac{d^2u_2}{dxdy}, & \dfrac{d^2u_3}{dxdy}, & \dfrac{d^2u_4}{dxdy} \\ \dfrac{d^2u_2}{dy^2}, & \dfrac{d^2u_3}{dy^2}, & \dfrac{d^2u_4}{dy^2} \end{vmatrix},$$

and v_2, v_3, v_4 similar determinants formed from u_3, u_4, u_1, &c., then from v_1, v_2, v_3, v_4 we can form four new functions w_1, w_2, w_3, w_4 in the same way as we obtained $v_1 \dots v_4$ from $u_1 \dots u_4$. Prove that

$$\frac{w_i}{u_i} = \mu \begin{vmatrix} \frac{d^3u_1}{dx^3}, & \frac{d^3u_2}{dx^3}, & \frac{d^3u_3}{dx^3}, & \frac{d^3u_4}{dx^3} \\ \frac{d^3u_1}{dx^2\,dy}, & \frac{d^3u_2}{dx^2\,dy}, & \frac{d^3u_3}{dx^2\,dy}, & \frac{d^3u_4}{dx^2\,dy} \\ \frac{d^3u_1}{dx\,dy^2}, & \frac{d^3u_2}{dx\,dy^2}, & \frac{d^3u_3}{dx\,dy^2}, & \frac{d^3u_4}{dx\,dy^2} \\ \frac{d^3u_1}{dy^3}, & \frac{d^3u_2}{dy^3}, & \frac{d^3u_3}{dy^3}, & \frac{d^3u_4}{dy^3} \end{vmatrix}$$

where μ is a numerical factor.

60. For the n^2 functions u_{ik} $(i, k = 1, 2 \dots n)$ of the variables $x_1, x_2 \dots x_n$, prove that the cubic determinant whose elements are

$$\frac{du_{ik}}{dx_j} \quad (i, j, k = 1, 2 \dots n)$$

is a covariant.

61. For the n functions $u_1 \dots u_n$ of the variables $x_1 \dots x_n$, prove that the cubic determinant whose elements are

$$\frac{d^2u_i}{dx_j\,dx_k} \quad (i, j, k = 1, 2 \dots n)$$

is a covariant.

62. If the function u of the variables $x_1 \dots x_n$ be transformed by the linear substitution

$$x_i = b_{i1}y_1 + b_{i2}y_2 + \dots + b_{i\,n-1}y_{n-1}$$

to a function v of $n-1$ variables, prove that

$$H(v) = -\begin{vmatrix} 0, & B_1 & \dots & B_n \\ B_1, & u_{11} & \dots & u_{1n} \\ \dots & \dots & \dots & \dots \\ B_n, & u_{n1} & \dots & u_{nn} \end{vmatrix},$$

where $u_{ik} = \dfrac{d^2u}{dx_i\,dx_k}$, and $(-1)^i B_i$ is the determinant obtained by suppressing the i^{th} row in the array formed by the quantities b_i.

63. If $$u = \Sigma a_{ik}x_ix_k \quad (i, k = 1, 2 \dots n),$$
and

$$D_r = \begin{vmatrix} a_{11} & \dots & a_{1r} \\ \dots & \dots & \dots \\ a_{r1} & \dots & a_{rr} \end{vmatrix},$$

prove that the substitution

$$x_r = y_r + \frac{1}{D_r}\frac{dD_{r+1}}{da_{rr+1}}y_{r+1} + \ldots + \frac{1}{D_{n-1}}\frac{dD_n}{da_{rn}}y_n$$

reduces the given quadric to the sum of the n squares

$$u = \Sigma \frac{D_r}{D_{r-1}} y_r^2 \qquad (r = 1, 2 \ldots n).$$

64. If u and v are two n-ary quadrics and U, V their reciprocals, prove that we can by the same linear substitution change u into AV and v into BU; A and B are the discriminants of u and v. The determinant C of the substitution is the geometric mean between the discriminants of U and V. If C be regarded as the discriminant of a quadric W, we can by the same linear substitution reduce the three quadrics U, V, W to the sum of squares. The coefficient of any term in W so transformed is the geometric mean between the homologous coefficients in U and V.

65. If to the leading elements of the determinant of an orthogonal substitution of order n we add the quantities $a_1, a_2 \ldots a_n$, or the quantities $\frac{1}{a_1}, \frac{1}{a_2} \ldots \frac{1}{a_n}$, the resulting determinants are equal if

$$a_1 a_2 \ldots a_n = 1.$$

66. If c_{ik} are the coefficients of an orthogonal substitution (modulus unity) of order n, prove that

$$D = \begin{vmatrix} c_{11} - 1, & c_{12} & \ldots & c_{1n} \\ c_{21}, & c_{22} - 1 & \ldots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n1}, & c_{n2} & \ldots & c_{nn} - 1 \end{vmatrix}$$

is equal to zero if n is odd; but if n is even its value is

$$2^n \frac{[A]}{A},$$

where A is the skew determinant from which the orthogonal substitution is derived, and $[A]$ the same determinant with the elements in the leading diagonal zero.

If D_{ii} is the coefficient of one of the leading terms in D, prove that when n is even

$$2D_{ii} = -D.$$

67. If
$$| c_{ik} | = \epsilon$$
is the determinant of an orthogonal substitution, the equation
$$\begin{vmatrix} c_{11} + x, & c_{12} & \dots & c_{1n} \\ c_{21}, & c_{22} + x & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{n1}, & c_{n2} & \dots & c_{nn} + x \end{vmatrix} = 0$$
is a reciprocal one. If n is odd it has one real root $-\epsilon$; if n is even and $\epsilon = -1$ it has the two real roots ± 1. The rest are all imaginary.

68. The maxima and minima values of
$$u = \Sigma a_{ik} x_i x_k,$$
subject to the conditions
$$v = \Sigma b_{ik} x_i x_k$$
$$c_{11} x_1 + c_{12} x_2 + \dots + c_{1n} x_n = 0$$
$$\dots\dots\dots\dots\dots\dots$$
$$c_{n-21} x_1 + c_{n-22} x_2 + \dots + c_{n-2n} x_n = 0$$
are given by the equation
$$\begin{vmatrix} b_{11} u - a_{11} v & \dots & b_{1n} u - a_{1n} v, & c_{11}, & c_{21} & \dots & c_{n-21} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_{n1} u - a_{n1} v & \dots & b_{nn} u - a_{nn} v, & c_{1n}, & c_{2n} & \dots & c_{n-2n} \\ c_{11} & \dots & c_{1n} & & & & \\ \dots & \dots & \dots & & & & \\ c_{n-21} & \dots & c_{n-2n} & & & & \end{vmatrix} = 0.$$

69. The values of $x_1, x_2 \dots x_m$ which satisfy the equations
$$a_{11} x_1 + a_{21} x_2 + \dots + a_{m1} x_m = 0$$
$$\dots\dots\dots\dots\dots\dots$$
$$a_{1r-1} x_1 + a_{2r-1} x_2 + \dots + a_{mr-1} x_m = 0$$
$$a_{1r} x_1 + a_{2r} x_2 + \dots + a_{mr} x_m = 1$$
$$a_{1r+1} x_1 + a_{2r+1} x_2 + \dots + a_{mr+1} x_m = 0$$
$$\dots\dots\dots\dots\dots\dots$$
$$a_{1n} x_1 + a_{2n} x_2 + \dots + a_{mn} x_m = 0$$
and make $x_1^2 + x_2^2 + \dots + x_m^2$ a minimum are
$$\frac{1}{2C} \frac{dC}{da_{1r}}, \quad \frac{1}{2C} \frac{dC}{da_{2r}} \dots \frac{1}{2C} \frac{dC}{da_{mr}},$$
where C is the determinant whose elements are given by
$$c_{ik} = a_{1i} a_{1k} + a_{2i} a_{2k} + \dots + a_{mi} a_{mk}.$$

70. The value of the integral

$$\iint \dots x_i x_j \, dx_1 \, dx_2 \dots dx_n,$$

taken for all values of the variables such that

$$\Sigma a_{ij} x_i x_j = < 1,$$

the quadric being a definite positive form (i.e. incapable of becoming negative), is

$$\frac{(\Gamma \tfrac{1}{2})^n}{\Gamma(\tfrac{1}{2}n + 2)} \frac{A_{ij}}{2A^{\frac{3}{2}}},$$

where $A = | a_{ik} |$ is the discriminant of the quadric.

71. The value of the integral

$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \epsilon^{-u} \cos(b_1 x_1 + b_2 x_2 + \dots + b_n x_n) \, dx_1 \, dx_2 \dots dx_n,$$

where

$$u = \Sigma a_{ik} x_i x_k,$$

is

$$\sqrt{\left(\frac{\pi^n}{A}\right)} \epsilon^{-\frac{v}{4}},$$

where

$$v = -\frac{1}{A} \begin{vmatrix} 0, & b_1, & b_2 & \dots & b_n \\ b_1, & a_{11}, & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ b_n, & a_{n1}, & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

In this question and the next u is supposed to be incapable of becoming negative.

72. The value of the integral

$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} v\epsilon^{-u} \, dx_1 \, dx_2 \dots dx_n,$$

where

$$v = \Sigma b_{ik} x_i x_k, \qquad u = \Sigma a_{ik} x_i x_k,$$

is

$$\sqrt{\left(\frac{\pi^n}{A^3}\right)} \frac{S}{2},$$

where S is the sum of the n determinants obtained by substituting for each column of A in succession the corresponding column of the discriminant of v.

73. Let $a_1, a_2 \dots a_{2n+1}$ be $2n+1$ real and different numbers in ascending order of magnitude, and let

$$P(x) = (x - a_1)(x - a_3) \dots (x - a_{2n+1})$$
$$Q(x) = (x - a_2)(x - a_4) \dots (x - a_{2n}) A$$
$$R(x) = P(x) Q(x),$$

A being a positive number. Then if

$$K_{rs} = \tfrac{1}{2}\int_{a_{2s-1}}^{a_{2s}} \frac{P(x)\,dx}{(x - a_{2r-1})\sqrt{R(x)}}, \quad L_{rs} = \tfrac{1}{2}\frac{Q(a_{2r-1})}{P'(a_{2r-1})}\int_{a_{2s-1}}^{a_{2s}} \frac{P(x)\,dx}{(x - a_{2r-1})^2\sqrt{R(x)}}$$

(these are the complete Abelian integrals of the first and second species), and if also

$$k_{rs} = \frac{\sqrt{-1}}{2}\int_{a_{2s}}^{a_{2s+1}} \frac{P(x)\,dx}{(x - a_{2r-1})\sqrt{R(x)}}, \quad l_{rs} = \frac{\sqrt{-1}}{2}\frac{Q(a_{2r-1})}{P'(a_{2r-1})}\int_{a_{2s}}^{a_{2s+1}} \frac{P(x)\,dx}{(x - a_{2r-1})^2\sqrt{R(x)}}$$

then

$$D = \begin{vmatrix} K_{11}, & L_{11} & \dots & K_{n1}, & L_{n1} \\ k_{11}, & l_{11} & \dots & k_{n1}, & l_{n1} \\ \dots & \dots & \dots & \dots & \dots \\ K_{1n}, & L_{1n} & \dots & K_{nn}, & L_{nn} \\ k_{1n}, & l_{1n} & \dots & k_{nn}, & l_{nn} \end{vmatrix} = \left(\frac{\pi}{2}\right)^n.$$

Prove also that

$$\frac{dD}{dK_{rs}} - \frac{dD}{dK_{rs-1}} = -\left(\frac{\pi}{2}\right)^{n-1} l_{rs-1}, \quad \frac{dD}{dL_{rs}} - \frac{dD}{dL_{rs-1}} = \left(\frac{\pi}{2}\right)^{n-1} k_{rs-1}$$

$$\frac{dD}{dk_{rs}} - \frac{dD}{dk_{rs-1}} = -\left(\frac{\pi}{2}\right)^{n-1} L_{rs-1}, \quad \frac{dD}{dl_{rs}} - \frac{dD}{dl_{rs-1}} = \left(\frac{\pi}{2}\right)^{n-1} K_{rs-1}.$$

74. Prove that the value of the continued fraction

$$\frac{a}{a+1-}\;\frac{b}{b+1-}\;\frac{c}{c+1-}\text{ ad. inf.}$$

is unity.

75. Prove that the product of the two continued fractions

$$a - 1 + \frac{1^2}{2(a-1)^2+}\;\frac{3^2}{2(a-1)^2+}\dots$$

$$a + 1 + \frac{1^2}{2(a+1)^2+}\;\frac{3^2}{2(a+1)^2+}\dots$$

is a^2.

76. If u_n is the number of terms in a determinant of order n which do not contain any element from the principal diagonal, prove that

$$u_n = nu_{n-1} + (-1)^n,$$

and hence that $\frac{u_n}{n!}$ is the coefficient of x^n in the expansion of $\frac{\epsilon^{-x}}{1-x}$.

77. If u_n is the number of terms in a symmetrical determinant of order n, prove that

$$u_n - nu_{n-1} + \frac{(n-1)(n-2)}{2} u_{n-3} = 0.$$

Also that $\frac{u_n}{n!}$ is the coefficient of x^n in the expansion of

$$\frac{\epsilon^{\frac{1}{2}x+\frac{1}{4}x^2}}{\sqrt{(1-x)}}.$$

78. If $[1 . 3 . 5 \ldots (2n-1)]\, u_n$ is the number of terms in a skew determinant of order $2n$, prove that

$$u_n = (2n-1)\, u_{n-1} - (n-1)\, u_{n-2}.$$

Shew also that $\frac{u_n}{2^n n!}$ is the coefficient of x^n in the expansion of

$$\sqrt[4]{\left\{\frac{\epsilon^x}{1-x}\right\}}.$$

79. If A is the area of a quadrilateral, the co-ordinates of whose angular points are $(x_1, y_1) \ldots (x_4, y_4)$, then

$$2A = \begin{vmatrix} 1, & 0, & x_1, & y_1 \\ 0, & 1, & x_2, & y_2 \\ 1, & 0, & x_3, & y_3 \\ 0, & 1, & x_4, & y_4 \end{vmatrix} = \begin{vmatrix} x_3 - x_1, & y_3 - y_1 \\ x_4 - x_2, & y_4 - y_2 \end{vmatrix}.$$

The area of a quadrilateral inscribed in a circle in terms of its sides is given by

$$16A = -\begin{vmatrix} -a, & b\,, & c\,, & d \\ b\,, & -a, & d\,, & c \\ c\,, & d\,, & -a, & b \\ d\,, & c\,, & b\,, & -a \end{vmatrix}.$$

80. If the planes

$$a_i x + b_i y + c_i z + d_i = 0 \qquad (i = 1, 2, 3, 4, 5)$$

touch the same sphere, then

$$|\, a_i, b_i, c_i, d_i, u_i \,| = 0 \qquad (i = 1, 2 \ldots 5),$$

where

$$u_i^2 = a_i^2 + b_i^2 + c_i^2.$$

81. A quadric of revolution passes through five points $P_1 \dots P_5$. The distances of these points from a focus being $r_1 \dots r_5$.

If V_1 = volume of tetrahedron $P_2\, P_3\, P_4\, P_5$, &c., prove that

$$V_1 r_1 + V_2 r_2 + \dots + V_5 r_5 = 0.$$

82. Let V, V' be the volumes, A, B, C, D; a, b, c, d the areas of the faces of two tetrahedra whose angular points are numbered 1, 2, 3, 4. Also let P_{ik} be the perpendicular from the point i of the first tetrahedron on the face opposite the point k of the second, and p_{ik} a like quantity for the other tetrahedron. Prove that

$$| P_{ik} | \times | p_{ik} | = \frac{(VV')^4}{ABCD\,abcd} \quad (i,\ k = 1,\ 2,\ 3,\ 4).$$

83. If A, B, C, D are the directions of four forces in equilibrium, and if AB is the moment of the lines A and B, &c., prove that

$$\begin{vmatrix} 0\ , & BA, & CA, & DA \\ AB, & 0\ , & CB, & DB \\ AC, & BC, & 0\ , & DC \\ AD, & BD, & CD, & 0 \end{vmatrix} = 0.$$

If a, b, c, d are the magnitudes of the forces

$$a = \sqrt{(BC\,.\,CD\,.\,DB)}, \text{ \&c.}$$

84. In Siebeck's determinant, XIV. 22, prove that

$$\frac{dD}{dd_{ik}} = 288vv',$$

where v is the volume of the tetrahedron formed by the face opposite the point i of the first tetrahedron and the centre of the sphere circumscribing the second tetrahedron, and similarly for v'.

85. If in a system of five points d_{ik} is the square of the line joining the i^{th} and k^{th} points, and r is a sixth point of the system, prove that

$$\begin{vmatrix} d_{r1}^{\ 2}\ , & d_{r1}d_{r2} + d_{12} \dots & d_{r1}d_{r5} + d_{15}, & d_{r1} + 1 \\ d_{r1}d_{r2} + d_{12}, & d_{r2}^{\ 2} \quad \dots & d_{r2}d_{r5} + d_{25}, & d_{r2} + 1 \\ \dots\dots & \dots\dots & \dots\dots & \dots \\ d_{r1}d_{r5} + d_{15}, & d_{r2}d_{r5} + d_{25} \dots & d_{r5}^{\ 2}\ , & d_{r5} + 1 \\ d_{r1} + 1\ , & d_{r2} + 1 \quad \dots & d_{r5} + 1 & 1 \end{vmatrix} = 0.$$

86. If in a system of seven straight lines, m_{ik} is the moment of the i^{th} and k^{th} lines, and r is an eighth line, prove that

$$\begin{vmatrix} m_{r1}^2, & m_{r1}m_{r2}+m_{12} & \cdots & m_{r1}m_{r7}+m_{17} \\ m_{r1}m_{r2}+m_{12}, & m_{r2}^2 & \cdots & m_{r2}m_{r7}+m_{27} \\ \cdots & \cdots & \cdots & \cdots \\ m_{r1}m_{r7}+m_{17}, & m_{r2}m_{r7}+m_{27} & \cdots & m_{r7}^2 \end{vmatrix} = 0.$$

87. Having given two tetrahedra whose angular points are marked 1, 2, 3, 4, let d_{ik} denote the square of the distance between the i^{th} point of the first and k^{th} point of the second tetrahedron. Prove the following relations:

(i) For two points P, Q the distances of P from the angular points of the first tetrahedron being a_i, of Q from those of the second b_i, and $d = PQ^2$,

$$\begin{vmatrix} d, & 1, & b_1 & \cdots & b_4 \\ 1, & 0, & 1 & \cdots & 1 \\ a_1, & 1, & d_{11} & \cdots & d_{14} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_4, & 1, & d_{41} & \cdots & d_{44} \end{vmatrix} = 0.$$

(ii) For the point P and a plane, q_i being the distances of the vertices of the second tetrahedron from the plane, p the distance of P from the plane,

$$\begin{vmatrix} p, & 0, & q_1 & \cdots & q_4 \\ 1, & 0, & 1 & \cdots & 1 \\ a_1, & 1, & d_{11} & \cdots & d_{14} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_4, & 1, & d_{41} & \cdots & d_{44} \end{vmatrix} = 0.$$

(iii) For two planes, p_i, q_i being the perpendiculars from the angular points of the tetrahedra on them, ϕ the angle between the planes,

$$\begin{vmatrix} -\frac{1}{2}\cos\phi, & 0, & q_1 & \cdots & q_4 \\ 0, & 0, & 1 & \cdots & 1 \\ p_1, & 1, & d_{11} & \cdots & d_{14} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_4, & 1, & d_{41} & \cdots & d_{44} \end{vmatrix} = 0.$$

88. For a system of six and a second system of five spheres, if p_{ik} is the power of the i^{th} and k^{th} spheres,

$$\begin{vmatrix} 1, & p_{11} & \cdots & p_{15} \\ \cdots & \cdots & \cdots & \cdots \\ 1, & p_{61} & \cdots & p_{65} \end{vmatrix} = 0.$$

89. The equation

$$\begin{vmatrix} 0\,, & S_1, & S_2, & S_3, & S_4 \\ S_1, & 0, & t_{12}, & t_{13}, & t_{14} \\ \dots & \dots & \dots & \dots & \dots \\ S_4, & t_{41}, & t_{42}, & t_{43}, & 0 \end{vmatrix} = 0$$

represents two spheres touching the given spheres $S_1 = 0 \ldots S_4 = 0$; t_{ik} is the square of the common tangent to the i^{th} and k^{th} spheres.

90. Prove that for any five spheres $S_1 = 0 \ldots S_5 = 0$,

$$\begin{vmatrix} 0, & 1\,, & 1 & \ldots 1\,, & 1 \\ 1, & 0\,, & t_{12} & \ldots t_{15}\,, & S_1 \\ 1, & t_{21}, & 0 & \ldots t_{25}, & S_2 \\ \dots & \dots & \dots & \dots & \dots \\ 1, & t_{51}, & t_{52} & \ldots 0\,, & S_5 \\ 1, & S_1, & S_2 & \ldots S_5, & 0 \end{vmatrix} = 0.$$

91. The index of two points being defined as in XIV. 27, the index of two planes D, D' is obtained by taking in the planes the points abc, $a'b'c'$ and forming the determinant

$$I_{DD'} = \frac{1}{4abc\,.\,a'b'c'} \begin{vmatrix} I_{aa'}, & I_{ab'}, & I_{ac'} \\ I_{ba'}, & I_{bb'}, & I_{bc'} \\ I_{ca'}, & I_{cb'}, & I_{cc'} \end{vmatrix};$$

and the index of two lines γ, γ' by taking in the lines two points ab, $a'b'$ and forming the determinant

$$I_{\gamma\gamma'} = \frac{1}{ab\,.\,a'b'} \begin{vmatrix} I_{aa'}, & I_{ab'} \\ I_{ba'}, & I_{bb'} \end{vmatrix}.$$

Prove that for two groups of planes numbered 1 ... 5

$$\begin{vmatrix} I_{11} & \ldots & I_{15} \\ \dots & \dots & \dots \\ I_{51} & \ldots & I_{55} \end{vmatrix} = 0,$$

$$\begin{vmatrix} I_{11} & \ldots & I_{14} \\ \dots & \dots & \dots \\ I_{41} & \ldots & I_{44} \end{vmatrix} = -\frac{1}{(abc)^6} \frac{(3V)^3}{2ABCD} \cdot \frac{(3V')^3}{2A'B'C'D'},$$

where a, b, c are now the semiaxes of the ellipsoid, V, V' the volumes, and $A \ldots A' \ldots$ the faces of the tetrahedra formed by the planes.

Prove also that for two groups of lines passing through the points P, P'

$$\begin{vmatrix} I_{11} & \ldots & I_{14} \\ \ldots & \ldots & \ldots \\ I_{41} & \ldots & I_{44} \end{vmatrix} = 0,$$

$$\begin{vmatrix} I_{11} & \ldots & I_{13} \\ \ldots & \ldots & \ldots \\ I_{31} & \ldots & I_{33} \end{vmatrix} = -\frac{\sin(123)\sin(1'2'3')}{(abc)^2} I_{PP'}^{\,2}.$$

92. If between the points of two surfaces we establish the correspondence

$$\xi = \phi(x, y, z), \quad \eta = \psi(x, y, z), \quad \zeta = \chi(x, y, z),$$

prove that the ratio of corresponding elements of the surfaces is given by

$$\frac{d\sigma}{ds} = -\begin{vmatrix} \frac{d\xi}{dx}, & \frac{d\xi}{dy}, & \frac{d\xi}{dz}, & \alpha \\ \frac{d\eta}{dx}, & \frac{d\eta}{dy}, & \frac{d\eta}{dz}, & \beta \\ \frac{d\zeta}{dx}, & \frac{d\zeta}{dy}, & \frac{d\zeta}{dz}, & \gamma \\ a, & b, & c & \end{vmatrix},$$

where (a, b, c), (α, β, γ) are the direction cosines of the normal to ds and $d\sigma$.

LIST OF MEMOIRS AND WORKS RELATING TO DETERMINANTS.

Albeggiani, M. Sviluppo di un determinante ad elementi binomii. *Giornale*, x. 279. (1872.)

——— Sviluppo di un determinante ad elementi polinomi. *Ibid.* XIII. 1. (1875.)

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CAMBRIDGE: PRINTED BY C. J. CLAY, M.A. AT THE UNIVERSITY PRESS.