Asphericity of Length Six Relative Group Presentations

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Abstract

Combinatorial group theory is a part of group theory that deals with groups given by presentations in terms of generators and defining relations. Many techniques both algebraic and geometric are used in dealing with problems in this area. In this thesis, we adopt the geometric approach. More specifically, we use so-called pictures over relative presentations to determine the asphericity of such presentations. We remark that if a relative presentation is aspherical then group theoretic information can be deduced.

In Chapter 1, the concept of relative presentations is introduced and we state the main theorems and some known results.

In Chapter 2, the concept of pictures is introduced and methods used for checking asphericity are explained.

Excluding four unresolved cases, the asphericity of the relative presentation $\mathcal{P}= \langle G, x | x^m gxh \rangle$ for $m \geq 2$ is determined in Chapter 3. If $H = \langle g, h \rangle \leq G$, then the unresolved cases occur when H is isomorphic to C_5 or C_6.

The main work is done in Chapter 4, in which we investigate the asphericity of the relative presentation $\mathcal{P}= \langle G, x | xaxbxcxdxexf \rangle$, where the coefficients $a, b, c, d, e, f \in G$ and $x \notin G$ and prove the theorems stated in Chapter 1.

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Contents

Chapter 1

Introduction

1.1 Relative group presentations

The notion of relative group presentations was introduced by Bogley and Pride in [5], who also introduced the notion of asphericity for relative group presentations.

Definition 1.1.1. *A **relative group presentation** is a presentation of the form $\mathcal{P}= \langle G, \boldsymbol{x} | \boldsymbol{r} \rangle$, where G is a group and $\boldsymbol{x}$ is a set disjoint from G. Denoting the free group on $\boldsymbol{x}$ by $\langle \boldsymbol{x} \rangle$, $\boldsymbol{r}$ is a set of cyclically reduced words in the free product $G * \langle \boldsymbol{x} \rangle$. The group defined by $\mathcal{P}$ is $\hat{G}= G * \langle \boldsymbol{x} \rangle / N$, where N is the normal closure in $G * \langle \boldsymbol{x} \rangle$ of $\boldsymbol{r}$.*

Remark 1.1.2. *In the above definition, G and $\langle \boldsymbol{x} \rangle$ are called the factors of the free product $G * \langle \boldsymbol{x} \rangle$. If $R \in \boldsymbol{r}$ then $R = \hat{g}_1 \hat{g}_2 \dots \hat{g}_n$, where no successive $\hat{g}_i$, $\hat{g}_{i+1}$ (subscripts mod n) are in the same factor. Also, $\hat{g}_1$ and $\hat{g}_n$ are in different factors.*

Example. Let G be any group. Then the following is an example of a relative presentation: $\langle G, x, y | x^{-1} g_1 x y g_2, x g_1 x g_2, x^2 g_1 y^2 g_2 y \rangle$, where $g_1, g_2 \in G$.

The presentation $\mathcal{P}= \langle G, \mathbf{x}|\mathbf{r}\rangle$ is said to be *slender* if for each $R \in \mathbf{r}$, $R^* \cap \mathbf{r} = R$, where R^* is the set of all cyclic permutations of $R \cup R^{-1}$; and $\mathcal{P}$ is *orientable* if it is slender and no element of $\mathbf{r}$ is a cyclic permutation of it's inverse. Moreover $\mathcal{P}$ is said to be *aspherical* if every spherical picture over it contains a dipole. The notions of spherical picture and dipole will be explained later on.

The group theoretic problem that is of interest in the study of relative presentations is to determine how G interacts with $\mathbf{r}$ to determine the structure of $\hat{G}$. As an example if $\mathcal{P}$ is aspherical then some group theoretic consequences about $\hat{G}$ can be deduced. More precisely the following theorem holds.

Theorem 1.1.3. *[5] Let $\hat{G}$ be the group defined by $\mathcal{P}= \langle G, \boldsymbol{x}|\boldsymbol{r}\rangle$. If $\mathcal{P}$ is an* ***orientable*** *and* ***aspherical*** *relative presentation, then the following hold:*

1. *The natural homomorphism $G \to \hat{G}$ is injective.*

2. *For $R \in \boldsymbol{r}$, write $R = (\dot{R})^n$, where $\dot{R}$ is not a proper power, and n is a positive integer. Let C_R be the subgroup of $\hat{G}$ generated by $\dot{R}N$. Then any finite subgroup of $\hat{G}$ is contained in a conjugate of G or a conjugate of one of the subgroups $C_R(R \in \boldsymbol{r})$.*

3. *The homology and cohomology [18] of $\hat{G}$ in dimensions ≥ 3 is determined by that of G and the subgroups $C_R(R \in \boldsymbol{r})$. In particular, there are isomorphisms $G_n(\hat{G}, -) \cong G_n(G, -) \oplus \big(\bigoplus_{R\in\boldsymbol{r}} G_n(C_R, -)\big)$*
   $G^n(\hat{G}, -) \cong G^n(G, -) \oplus \big(\prod_{R\in\boldsymbol{r}} G^n(C_R, -)\big)$
   for all $n \geq 3$.

There has been much interest in determining asphericity of $\mathcal{P}$ particularly when $\mathbf{x} = \{x\}$ and $\mathbf{r} = \{r\}$ both consist of a single element. Thus $r = x^{\varepsilon_1}g_1 \dots x^{\varepsilon_k}g_k$

where $g_i \in G$, $\varepsilon_i = \pm 1$ and $g_i = 1$ implies $\varepsilon_i + \varepsilon_{i+1} \neq 0$ $(1 \leq i \leq k$, subscripts mod $k)$. If $k \leq 3$ or $r \in \{\ xg_1xg_2xg_3xg_4,\ xg_1xg_2xg_3x^{-1}g_4,\ xg_1xg_2xg_3xg_4xg_5,$ $(xg_1)^{l_1}(xg_2)^{l_2}(xg_3)^{l_3}(l_i > 1, 1 \leq i \leq 3)\}$, then the asphericity of $\mathcal{P}$ has been determined (modulo some exceptional cases) in [5], [10], [4], [1], [2], [12] and [19]. This list includes $x^m gx^{-1}h$ $(g, h \in G\backslash\{1\})$ for $1 < m \leq 3$, and when $m \geq 4$ asphericity (modulo exceptional cases) has been determined in [9].

In Chapter 3 we consider $x^m gxh$ $(g, h \in G\backslash\{1\})$, where $m \geq 2$. If $m = 2$ then a complete classification of when $\mathcal{P}$ is aspherical has been obtained in [5]. Modulo some exceptions the cases $m = 3$ and $m = 4$ were determined in [4] and [12] respectively.

In Chapter 4 is the main work of the thesis where we consider $r = xg_1 \ldots xg_k$. The cases k=3, 4 and 5 were considered in [5], [4] and [12] respectively. Thus we consider k=6 and determine when $\mathcal{P}$ is aspherical with some exceptional cases. The special case when G is torsion-free was studied in [13], [14] for k=6.

1.2 Statement of results

1.2.1 $r = x^m gxh$ $(m \geq 5)$

Before we state our main result in Chapter 3, we list the following exceptional cases (observe that the exceptional cases and the results are modulo $(g, h) \leftrightarrow (h^{-1}, g^{-1})$ [see Remark 3.1.1]).

(**E1**) $g = h^2$, $|h| = 5$.
(**E2**) $g \in \{h^2, h^3, h^4\}$, $|h| = 6$.

Theorem 1.2.1. *Let $\mathcal{P}$ be the relative presentation $\mathcal{P}$ =$\langle G, x | x^m gxh \rangle$, where*

*$m \geq 5$, $x \notin G$, $g, h \in G \backslash \{1\}$. Suppose that none of the conditions in **(E1)** or **(E2)** holds. Then $\mathcal{P}$ is aspherical if and only if (modulo $(g,h) \leftrightarrow (h^{-1}, g^{-1})$) none of the following holds:*

1. *$g = h^{\pm 1}$ and g has finite order.*
2. *$g = h^2$ and $|h| \leq 4$.*
3. *$\frac{1}{|g|} + \frac{1}{|gh^{-1}|} + \frac{1}{|h|} > 1$, where $\frac{1}{\infty} := 0$.*
4. *$|g| = 2$, $|h| = 3$ and $[g, h] = 1$ ($H = gp\{g, h\} \cong C_2 \times C_3$).*

1.2.2 $r = xaxbxcxdxexf$

We consider the relative presentation $\mathcal{P}= \langle G, x | xaxbxcxdxexf \rangle$, where the coefficients $a, b, c, d, e, f \in G$ and $x \notin G$. We first consider the case $a = c = e$.

Theorem 1.2.2. *$\mathcal{P}= \langle G, x | xaxbxaxdxaxf \rangle$ is aspherical if and only if $\acute{\mathcal{P}}= \langle G, x | xba^{-1}xda^{-1}xfa^{-1} \rangle$ is aspherical.*

Since a complete classification of when $\acute{\mathcal{P}}$ is aspherical is obtained in [5], we will assume that $a = c = e$ does *not* hold in what follows.

Remarks 1.2.3.

1. *All exceptional cases and results are stated up to so-called equivalence (which will be described in full detail in Subsection 4.1.2).*
2. *The exceptional cases and results are listed in terms of which pairs of a, b, c, d, e, f are equal in G. For example **a=d, b=e only** means that no other pairs from a, b, c, d, e, f are considered. Clearly a=b=d, c=f only is the same as a=b, a=d, b=d, c=f only.*

Definition 1.2.4. *Throughout what follows we define the subgroup H of G by $H = \langle ba^{-1}, ca^{-1}, da^{-1}, ea^{-1}, fa^{-1}\rangle$.*

Definition 1.2.5. *A group T with the presentation $\langle x, y | x^n, y^m, (xy)^l\rangle$, where $\frac{1}{n} + \frac{1}{m} + \frac{1}{l} > 1$ is called a finite triangle group. If $(n, m, l) = (2, 2, k < \infty)$, (2, 3, 2), (2, 3, 3), (2, 3, 4), (2, 3, 5)(respectively), then $T \cong D_{2k}$, S_3, A_4, S_4, A_5 (respectively), where $D_{2k} = \langle x, y | x^2, y^2, (xy)^k\rangle$ $(k < \infty)$.*

Theorem 1.2.6. *If H is neither finite triangle nor finite cyclic, then $\mathcal{P}$ is aspherical unless one of the following holds:*

1. *(a=d, b=e only) and $|ca^{-1}(fa^{-1})^{-1}| < \infty$.*
2. *(a=b=d, c=f only) and $|ea^{-1}| < \infty$.*
3. *(a=b=d=e only) and $|ca^{-1}(fa^{-1})^{-1}| < \infty$.*

This theorem shows a similarity with the case $r = xaxbxcxd$ and a difference with the case $r = xaxbxcxdxe$ as can be seen from the following theorems.

Theorem 1.2.7. *[4, Theorems 2,3,4(special cases)] Consider the relative presentation $\mathcal{L} = \langle G, x | xaxbxcxd\rangle$, where the coefficients $a, b, c, d \in G$ and $x \notin G$. If $H = \langle ba^{-1}, ca^{-1}, da^{-1}\rangle$ is neither finite triangle nor finite cyclic, then $\mathcal{L}$ is aspherical unless (a=c only) and $|ba^{-1}(da^{-1})^{-1}| < \infty$.*

Theorem 1.2.8. *[12, Theorem A] Consider the relative presentation $\acute{\mathcal{L}} = \langle G, x | xaxbxcxdxe\rangle$, where the coefficients $a, b, c, d, e \in G$ and $x \notin G$. If $H = \langle ba^{-1}, ca^{-1}, da^{-1}, ea^{-1}\rangle$ is neither finite triangle nor finite cyclic, then $\acute{\mathcal{L}}$ is aspherical.*

Remark 1.2.9. *Observe that $\langle ba^{-1}, ca^{-1}, da^{-1}, ea^{-1}\rangle = \langle ab^{-1}, bc^{-1}, cd^{-1}, de^{-1}\rangle$.*

For the next theorem, we list the following exceptional cases:

(**E1**) (a=b, c=e, d=f only), $(|ca^{-1}|, |da^{-1}|) = \{(2,3)\}$ and $\frac{1}{|ca^{-1}|} + \frac{1}{|ca^{-1}da^{-1}|} + \frac{1}{|da^{-1}|} > 1$, where $\frac{1}{\infty} := 0$ ($H \cong S_3, A_4, S_4, A_5$).

(**E2**) (a=b=d, c=e only), $(ca^{-1})^2 = (fa^{-1})^2 = 1$ and $|ca^{-1}fa^{-1}| \in \{2,3\}$ ($H \cong D_4, D_6$).

Theorem 1.2.10. *Assume that none of the conditions in (**E1**) or (**E2**) holds. If H is finite triangle, then $\mathcal{P}$ is aspherical unless one of the following holds:*

1. *(a=b, c=d, e=f only) and* $\frac{1}{|ca^{-1}|} + \frac{1}{|ca^{-1}(ea^{-1})^{-1}|} + \frac{1}{|ea^{-1}|} > 1$.
2. *(a=b, c=e, d=f only),* $(ca^{-1})^2 = (da^{-1})^2$ *and* $|ca^{-1}da^{-1}| < \infty$.
3. *(a=d, b=e only) and* $|ca^{-1}(fa^{-1})^{-1}| < \infty$.
4. *(a=b=c, e=f only) and* $\frac{1}{|da^{-1}|} + \frac{1}{|da^{-1}(ea^{-1})^{-1}|} + \frac{1}{|ea^{-1}|} > 1$.
5. *(a=b=d, c=f only) and* $|ea^{-1}| < \infty$.
6. *(a=b=c=d only) and* $\frac{1}{|ea^{-1}|} + \frac{1}{|ea^{-1}(fa^{-1})^{-1}|} + \frac{1}{|fa^{-1}|} > 1$.
7. *(a=b=d=e only) and* $|ca^{-1}(fa^{-1})^{-1}| < \infty$.

For the next theorem, we list the following exceptional cases:

(**E3**) (a=b, c=d, e=f only), $ca^{-1} = (ea^{-1})^{-1}$ and $|ea^{-1}| = 3$ ($H \cong C_3$).

(**E4**) (a=b, c=e, d=f only) and one of the following holds.

(i) $da^{-1} = (ca^{-1})^2$ and $|ca^{-1}| \in \{4,6\}$ ($H \cong C_4, C_6$).

(ii) $da^{-1} = (ca^{-1})^{-1}$ and $4 \leq |ca^{-1}| < \infty$ ($H \cong C_k, 4 \leq k < \infty$).

(iii) $ca^{-1} = (da^{-1})^2$ and $|da^{-1}| \in \{4,6\}$ ($H \cong C_4, C_6$).

(iv) $(ca^{-1})^{-1} \in \{(da^{-1}(ca^{-1})^{-1})^3, (da^{-1}(ca^{-1})^{-1})^4\}$ and $|da^{-1}(ca^{-1})^{-1}| = 6$ ($H \cong C_6$).

(**E5**) (a=b=c, e=f only) and one of the following holds.

(i) $da^{-1} \in \{(ea^{-1})^{-1}, (ea^{-1})^2\}$ and $|ea^{-1}| \in \{4,6\}$ ($H \cong C_4, C_6$).

(ii) $(|da^{-1}|, |ea^{-1}|) \in \{(2,3),(3,2)\}$ and $[da^{-1}, ea^{-1}] = 1$ ($H \cong C_2 \times C_3$).

(iii) $ea^{-1} \in \{(da^{-1})^2, (da^{-1})^3, (da^{-1})^4\}$ and $|da^{-1}| = 6$ ($H \cong C_6$).

(iv) $da^{-1} \in \{(ea^{-1})^3, (ea^{-1})^4\}$ and $|ea^{-1}| = 6$ ($H \cong C_6$).

(**E6**) (a=b=d, c=e only) and one of the following holds.

(i) $ca^{-1} = (fa^{-1})^{-1}$ and $|fa^{-1}| \in \{4,5,6\}$ ($H \cong C_4, C_5, C_6$).

(ii) $ca^{-1} = (fa^{-1})^2$ and $|fa^{-1}| = 4$ or $fa^{-1} = (ca^{-1})^2$ and $|ca^{-1}| \in \{4,6\}$ ($H \cong C_4, C_6$).

(**E7**) (a=b=d, e=f only), $ca^{-1} \in \{(ea^{-1})^2, (ea^{-1})^3\}$ and $|ea^{-1}| = 4$ or $ea^{-1} = (ca^{-1})^2$ and $|ca^{-1}| = 4$ ($H \cong C_4$).

(**E8**) (a=b=c=d only) and one of the following holds.

(i) $ea^{-1} = (fa^{-1})^2$ and $|fa^{-1}| \in \{5,6\}$ ($H \cong C_5, C_6$).

(ii) $ea^{-1} \in \{(fa^{-1})^3, (fa^{-1})^4\}$ and $|fa^{-1}| = 6$ ($H \cong C_6$).

Theorem 1.2.11. *Assume that none of the conditions in (**E3**)-(**E8**) holds. If H is cyclic, then $\mathcal{P}$ is aspherical unless one of the following holds:*

1. *(a=b, c=e, d=f only), $da^{-1} = (ca^{-1})^2$ and $|ca^{-1}| = 3$.*

2. *(a=d, b=e only) and $|ca^{-1}(fa^{-1})^{-1}| < \infty$.*

3. *(a=b=c, d=f only), $ea^{-1} = (da^{-1})^{-1}$ and $|da^{-1}| < \infty$.*

4. *(a=b=c, e=f only) and the following holds.*
   $da^{-1} = (ea^{-1})^2$ and $|ea^{-1}| = 3$ or $ea^{-1} = (da^{-1})^2$ and $|da^{-1}| \in \{3,4\}$.

5. *(a=b=c, d=e=f only) and $|da^{-1}| < \infty$.*

6. *(a=b=d, c=e only) and $ca^{-1} = (fa^{-1})^{-1}$ where $|fa^{-1}| = 3$.*

7. *(a=b=d, c=f only) and $|ea^{-1}| < \infty$.*

8. *(a=b=d, e=f only) and $ca^{-1} = (ea^{-1})^{-1}$ where $|ea^{-1}| = 3$.*

9. *(a=b=d, c=e=f only) and $|ca^{-1}| < \infty$.*

10. *(a=b=c=d only) and one of the following holds.*
    (i) $ea^{-1} = (fa^{-1})^{\pm 1}$ *and* $|fa^{-1}| < \infty$.
    (ii) $ea^{-1} = (fa^{-1})^2$ *and* $|fa^{-1}| = 4$.
    (iii) $|ea^{-1}| = 2$, $|fa^{-1}| = 3$ *and* $[ea^{-1}, fa^{-1}] = 1$ *(*$H \cong C_2 \times C_3$*).*

11. *(a=b=d=e only) and* $|ca^{-1}(fa^{-1})^{-1}| < \infty$.

Corollary 1.2.12. *Assume that* H *is cyclic and* $|H| > 6$. *Also assume the following does not hold: (a=b, c=e, d=f only),* $da^{-1} = (ca^{-1})^{-1}$ *and* $6 < |ca^{-1}| < \infty$ *(*$H \cong C_k, 6 < k < \infty$*). Then* $\mathcal{P}$ *is aspherical unless one of the following holds:*

1. *(a=d, b=e only) and* $|ca^{-1}(fa^{-1})^{-1}| < \infty$.

2. *(a=b=c, d=f only),* $ea^{-1} = (da^{-1})^{-1}$ *and* $6 < |da^{-1}| < \infty$.

3. *(a=b=c, d=e=f only) and* $6 < |da^{-1}| < \infty$.

4. *(a=b=d, c=e only) and* $ca^{-1} = (fa^{-1})^{-1}$ *where* $6 < |fa^{-1}| < \infty$.

5. *(a=b=d, c=f only) and* $|ea^{-1}| < \infty$.

6. *(a=b=d, c=e=f only) and* $6 < |ca^{-1}| < \infty$.

7. *(a=b=c=d only),* $ea^{-1} = (fa^{-1})^{\pm 1}$ *and* $6 < |fa^{-1}| < \infty$.

8. *(a=b=d=e only) and* $|ca^{-1}(fa^{-1})^{-1}| < \infty$.

1.3 Some known results

Here we list some of the known results that we have used in our work.

Lemma 1.3.1. *[4, Lemma 3] The relative presentation* $\langle G, x | xg_1 \ldots xg_n \rangle$ *is aspherical if* $xg_1 \ldots xg_n$ *is not a proper power in the free product* $G * \langle x \rangle$ *and* $g_1, \ldots, g_n$ *are all contained in an infinite cyclic subgroup of* G.

Lemma 1.3.2. *[19, Lemma 1] The relative presentation* $\langle G, x | x^k (xg)^m \rangle$*, where* $k, m > 1$ *is aspherical if and only if* g *has infinite order.*

Lemma 1.3.3. *The relative presentation* $\langle G, x | x^m g \rangle$ *(*$m > 1$ *and* $g \neq 1$*) is aspherical if and only if* g *has infinite order.*

Proof. If g has infinite order then the result follows from Lemma 1.3.1, while if g has finite order then so has x. Now since $m > 1$, the result follows from Lemma 2.2.9.

Chapter 2

Method of Proof

2.1 Pictures over relative presentations

Throughout this chapter we mean by $\mathcal{P}$ the presentation $\mathcal{P}= \langle G, \mathbf{x}|\mathbf{r}\rangle$, except when otherwise stated. The definitions of this section are taken from [5]. For more details, the reader is referred to [5] and [4].

A **picture** $\mathbb{P}$ is a finite collection of pairwise disjoint discs $\{D_1, \ldots, D_m\}$ in the interior of a disc D^2, together with a finite collection of pairwise disjoint simple arcs $\{\alpha_1, \ldots, \alpha_n\}$ embedded in the closure of $D^2 - \bigcup_{i=1}^m D_i$ in such a way that each arc meets $\partial D^2 \cup \bigcup_{i=1}^m D_i$ transversely in it's end points. The *boundary* of $\mathbb{P}$ is the circle ∂D^2, denoted by $\partial\mathbb{P}$. For $1 \leq i \leq m$, the *corners* of D_i are the closures of the connected components of $\partial D_i - \bigcup_{j=1}^n \alpha_j$, where ∂D_i is the boundary of D_i. The *regions* Δ of $\mathbb{P}$ are the closures of the connected components of $D^2 - \left(\bigcup_{i=1}^m D_i \cup \bigcup_{j=1}^n \alpha_j\right)$. An *inner region* of $\mathbb{P}$ is a simply connected region of $\mathbb{P}$ that does not meet $\partial\mathbb{P}$. The picture $\mathbb{P}$ is *non-trivial* if $m \geq 1$, is *connected* if $\bigcup_{i=1}^m D_i \cup \bigcup_{j=1}^n \alpha_j$ is connected, and is *spherical* if it is non-trivial and if none of the arcs meets the boundary of D^2. The number of edges in $\partial\Delta$ is called the *degree* of the region Δ and is denoted by $d(\Delta)$. If $\mathbb{P}$ is a spherical picture, the number of different discs to which a disc D_i is connected is called the *degree* of D_i, denoted by $d(D_i)$. The discs of a spherical picture $\mathbb{P}$ are also called *vertices* of $\mathbb{P}$.

Suppose that the picture $\mathbb{P}$ is labelled in the following sense: each arc α_j is equipped with a normal orientation, indicated by a short arrow meeting the arc transversely, and labelled by an element of $\mathbf{x} \cup \mathbf{x}^{-1}$. Each corner of $\mathbb{P}$ is oriented *clockwise* (with respect to D^2) and labelled by an element of G. If κ is a corner of a disc D_i of $\mathbb{P}$, then $W(\kappa)$ will be the word obtained by reading in a clockwise order the labels on the arcs and corners meeting ∂D_i beginning with the label on the first arc we meet as we read the clockwise corner κ. If we cross an arc labelled x in the direction of it's normal orientation, we read x, else we read x^{-1}. A *picture over* $\mathcal{P}$ is a picture $\mathbb{P}$ labelled in such a way the following are satisfied:

1. For each corner κ of $\mathbb{P}$, $W(\kappa) \in \mathbf{r}^*$, the set of all cyclic permutations of elements of $\mathbf{r} \cup \mathbf{r}^{-1}$ which begin with a member of $\mathbf{x} \cup \mathbf{x}^{-1}$.

2. If $g_1, ..., g_n$ is the sequence of corner labels encountered in *anticlockwise* traversal of the boundary of an inner region Δ of $\mathbb{P}$, then the product $g_1 g_2 ... g_n$=1 in G. We say that $g_1 g_2 ... g_n$ is the label of Δ, denoted by $l(\Delta)$ $= g_1 g_2 ... g_n$.

A connected spherical picture $\mathbb{P}$ over $\mathcal{P}$ is called *strictly spherical* if the product of the corner labels of any region of $\mathbb{P}$ defines the identity in G.

Example 1. *Let* $\mathcal{P}$=$\langle C_3, x | x^3 g x^2 g x g \rangle$, *where* g *generates* C_3. *Then a strictly spherical picture over* $\mathcal{P}$ *is given by Figure 2.1, where* g^{-1} *is denoted by* $\bar{g}$.

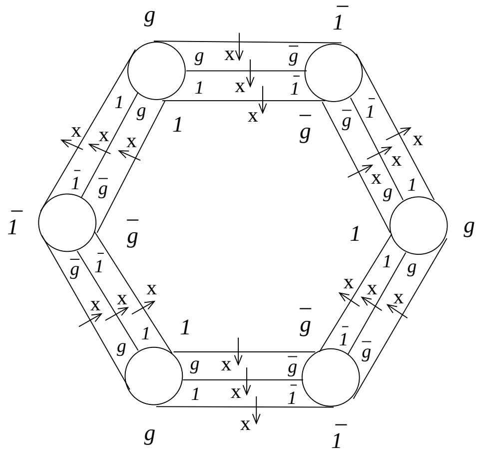


Figure 2.1: An example of a strictly spherical picture.

A *dipole* in a labelled picture $\mathbb{P}$ over $\mathcal{P}$ consists of corners κ and κ' of $\mathbb{P}$ together with an arc α joining the two corners such that κ and κ' belong to the same region and such that if $W(\kappa)= Sg$ where $g \in G$ and S begins and ends with a member of $\mathbf{x} \cup \mathbf{x}^{-1}$, then $W(\kappa')= S^{-1}g^{-1}$ (see Figure 2.2). The picture $\mathbb{P}$ is *reduced* if it does *not* contain a dipole (for more details see [7]).

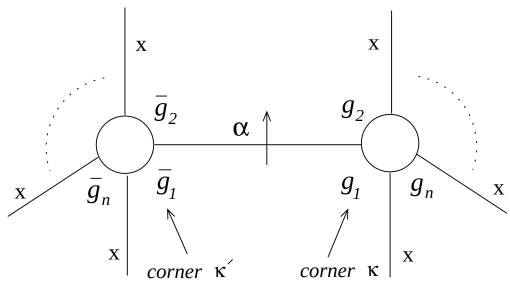


Figure 2.2: A dipole in $\mathbb{P}$.

We are now ready to state the following important definition.

Definition 2.1.1. *A relative presentation $\mathcal{P}$ is called* ***aspherical*** *if every connected spherical picture over $\mathcal{P}$ contains a dipole. If $\mathcal{P}$ is not aspherical then there is a non-trivial reduced spherical picture over $\mathcal{P}$.*

Remark 2.1.2. *In our proofs, amongst the finite set of all spherical pictures with a fixed minimal number of discs(vertices), we consider a spherical picture $\mathbb{P}$ over the given presentation.*

Definition 2.1.3. *(Bridge moves in a picture). Let Δ be a region of $\mathbb{P}$ with $l(\Delta) = uw_1vw_2$, where w_1 and w_2 are products of corner labels, such that $uw_1v = 1$. Then a bridge move can be done by a cut across e_1 and e_2 as shown in Figure 2.3 (observe that this changes $\mathbb{P}$).*

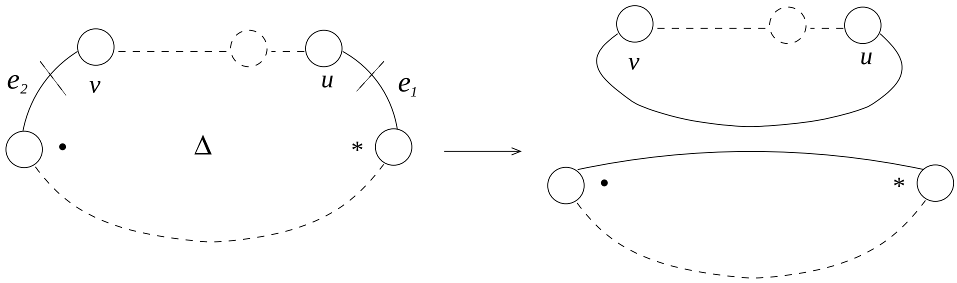


Figure 2.3: Bridge moves in pictures.

2.2 Methods used for checking asphericity

In this section $\mathcal{P}= \langle G, \mathbf{x}|\mathbf{r}\rangle$ is assumed to be an orientable relative presentation (note that both $\langle G, x|x^m gxh\rangle (m \geq 5)$ and $\langle G, x|xaxbxcxdxexf\rangle$ are orientable). Amongst the methods used for checking asphericity, in our results we mainly used weight test and curvature distribution method. To explain some of these methods the definition of star graph is needed.

Definition 2.2.1. *(star graph $\mathcal{P}^{st}$ of $\mathcal{P}$) The* ***star graph*** *$\mathcal{P}^{st}$ of a relative presentation $\mathcal{P}$ is a graph whose vertex set is $\boldsymbol{x} \cup \boldsymbol{x}^{-1}$ and edge set is $\boldsymbol{r}^*$. For $R \in \boldsymbol{r}^*$, write $R = Sg$ where $g \in G$ and S begins and ends with a member of $\boldsymbol{x} \cup \boldsymbol{x}^{-1}$. The initial and terminal functions are given as follows: $\iota(R)$ is the first symbol of S, and $\tau(R)$ is the inverse of the last symbol of S. The inverse edge*

$\bar{R}$ of R is obtained from $S^{-1}g^{-1}$ in the same way. For orientable presentation $S^{-1}g^{-1} \neq Sg$. The labelling function on the edges is defined by $\lambda(R) = g^{-1}$ and is extended to paths in the normal way, where a path in $\mathcal{P}^{st}$ is a sequence of edges, for which we write $p = e_1 \dots e_l$, $l \geq 1$, such that for $1 \leq i \leq l$, e_{i+1} begins where e_i ends. The path is closed if it begins and ends at the same vertex. A closed path p is cyclically reduced if no cyclic permutation of $e_1 \dots e_l$ contains the subword ee^{-1}. A non-empty cyclically reduced cycle (closed path) in $\mathcal{P}^{st}$ will be called ***admissible*** *if it has a trivial label in G.*

Remark 2.2.2. *The way that $\mathcal{P}^{st}$ is defined ensures that the label of each inner region of a reduced picture over $\mathcal{P}$ yields an admissible cycle in $\mathcal{P}^{st}$.*

Example 2. *Let $\mathcal{P}=\langle G, x | xaxbxcxdxexf\rangle$. Then $\mathcal{P}^{st}$ is given by Figure 2.4, where $\alpha \leftrightarrow a$, $\beta \leftrightarrow b$, $\gamma \leftrightarrow c$, $\delta \leftrightarrow d$, $\varepsilon \leftrightarrow e$ and $\zeta \leftrightarrow f$. If $ab^{-1} = dc^{-1}$ then $\alpha\beta^{-1}\gamma\delta^{-1}$ is admissible.*

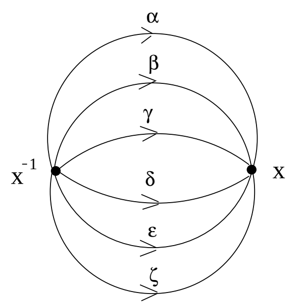


Figure 2.4: Star graph of $\langle G, x | xaxbxcxdxexf\rangle$.

2.2.1 Small cancellation hypothesis.

Let m be a positive integer. An *m-wheel* over $\mathcal{P}$ (see Figure 2.5) is a non-trivial connected picture $\mathbb{O}$ over $\mathcal{P}$ that has discs $\{\Delta_0, \Delta_1, \dots \Delta_m\}$ such that the following are satisfied:

1. Each arc of $\mathbb{O}$ meets a disc Δ_i for some $i \in \{1, \dots, m\}$.

2. Each arc of $\mathbb{O}$ either meets Δ_0 or $\partial\mathbb{O}$.

3. Each disc of $\mathbb{O}$ has a corner which lies in a region of $\mathbb{O}$ that meets $\partial\mathbb{O}$.

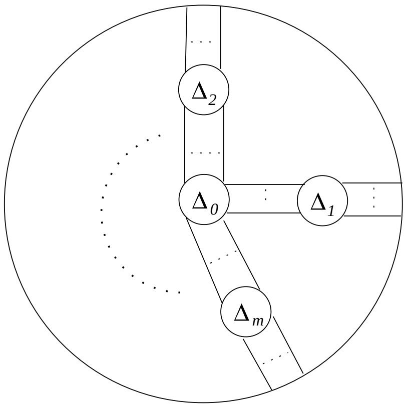


Figure 2.5: A typical m-wheel.

Definition 2.2.3. *Let p be a positive integer. Then we say that the presentation $\mathcal{P}$ satisfies $C(p)$ if there are no reduced m-wheels over $\mathcal{P}$ for $m < p$.*

Definition 2.2.4. *Let q be a positive integer. Then we say that the presentation $\mathcal{P}$ satisfies $T(q)$ if there are no admissible cycles in $\mathcal{P}^{st}$ of length k such that $3 \leq k < q$.*

Theorem 2.2.5. *(for example see Theorem 2.2 in [5]) If $\mathcal{P}$ satisfies $C(p)$ and $T(q)$ where $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, then $\mathcal{P}$ is aspherical.*

Example 3. *Let $\mathcal{P}=\langle G, x | xaxbxcxdxexf \rangle$. Then $\mathcal{P}$ satisfies $T(4)$ (admissible cycles can only have even length). If $a = b$ and $c = f$ are the only allowed equalities amongst a, b, c, d, e, f, then $\mathcal{P}$ also satisfies $C(4)$ [see Lemma 4.1.4]. Therefore in this case $\mathcal{P}$ is aspherical.*

2.2.2 Weight test

This method is due to Bogley and Pride [5]. A *weight function* θ is a real-valued function on the set of edges of $\mathcal{P}^{st}$ that satisfies $\theta(Sg) = \theta(S^{-1}g^{-1})$ where $Sg = R \in \mathbf{r}^*$. The weight of a closed cycle is the sum of the weights of the constituent edges. A weight function is *weakly aspherical* if the following conditions hold:

1. Let $R \in \mathbf{r}^*$, with $R = x_1^{\varepsilon_1} g_1 \ldots x_n^{\varepsilon_n} g_n$. Then

$$\sum_{i=1}^{n} (1 - \theta(x_i^{\varepsilon_i} g_i \ldots x_n^{\varepsilon_n} g_n x_1^{\varepsilon_1} g_1 \ldots x_{i-1}^{\varepsilon_{i-1}} g_{i-1})) \geq 2.$$

2. Each admissible cycle in $\mathcal{P}^{st}$ has weight at least 2.

Theorem 2.2.6. *[5, Lemma 1.7] If $\mathcal{P}^{st}$ admits a weakly aspherical weight function and if the natural map of G into $\hat{G}$ is an embedding, then $\mathcal{P}$ is aspherical.*

Remark 2.2.7. *In [15] Levin asserts that any relative presentation of the form $\mathcal{P} = \langle G, x | xg_1xg_2...xg_n \rangle$ is injective. Therefore, in our case if $\mathcal{P}^{st}$ admits a weakly aspherical weight function, then $\mathcal{P}$ is aspherical.*

Example 4. *Let $\mathcal{P} = \langle G, x, s | xgxs^{-1}, xsxhsh \rangle$. Figure 4.8 shows $\mathcal{P}^{st}$, where the edges α, β, γ, δ, ε, ζ and η (respectively) are labelled by 1, 1, h, 1, g, 1 and h (respectively).*

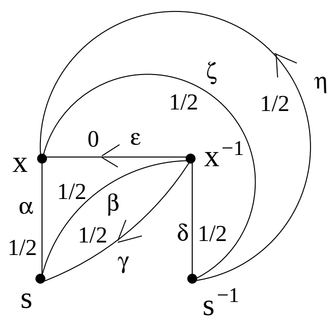


Figure 2.6: Star graph of $\langle G, x, s | xgxs^{-1}, xsxhsh \rangle$.

Assign θ to $\mathcal{P}^{st}$ with $\theta(\alpha) = \theta(\beta) = \theta(\gamma) = \theta(\delta) = \theta(\zeta) = \theta(\eta) = \frac{1}{2}$, and $\theta(\varepsilon) = 0$. Assume that g and h are non-trivial elements of G and $g \neq h$. Now let $R_1 = xgxs^{-1}$ and $R_2 = xsxhsh$. Then $1 - \theta(xgxs^{-1}) + 1 - \theta(xs^{-1}xg) + 1 - \theta(s^{-1}xgx) = 1 - \theta(\varepsilon) + 1 - \theta(\delta) + 1 - \theta(\alpha) = 2$. Also, $1 - \theta(xsxhsh) + 1 - \theta(sxhshx) + 1 - \theta(xhshxs) + 1 - \theta(shxsxh) = 1 - \theta(\beta) + 1 - \theta(\zeta) + 1 - \theta(\gamma) + 1 - \theta(\eta) = 2$. So Condition 1 of weakly aspherical function is satisfied. Now any admissible cycle of length ≥ 4 not including the edge ε has weight at least 2. However, the assumptions on g and h ensures Condition 2 for admissible cycles of length less than 4 and for the cycles of length ≥ 4 that include the edge ε. Thus θ is weakly aspherical and so $\mathcal{P}$ is aspherical.

2.2.3 Curvature test

Let $\mathbb{P}$ be any spherical picture over $\mathcal{P}$. An *angle function* α on $\mathbb{P}$ is a real-valued function on the set of corners of $\mathbb{P}$. Associated to α is a curvature function c defined on the discs(vertices) v of $\mathbb{P}$ by

$$c(v) = 2\pi - \sum_{\kappa \subseteq \partial v} \alpha(\kappa)$$

and on faces(regions) Δ of $\mathbb{P}$ by

$$c(\Delta) = 2\pi - \sum_{\kappa \subseteq \partial\Delta} (\pi - \alpha(\kappa)),$$

where κ denotes a corner in the boundary of a vertex v or a region Δ. Observe that $c(\Delta) = (2 - n)\pi + \sum_{\kappa \subseteq \partial\Delta} \alpha(\kappa)$, where n is the degree of the region Δ. Normally in our calculation of $c(\Delta)$ we use the last formula.

Lemma 2.2.8. *(Fundamental curvature formula) Let $\mathbb{P}$ be a connected, simply connected spherical picture. Then $c(\mathbb{P}) = \sum_{v} c(v) + \sum_{\Delta} c(\Delta) = 4\pi$, where the sum is taken over all the vertices and regions of $\mathbb{P}$.*

Proof. Let V, E and F (respectively) denote the number of vertices, edges and faces of $\mathbb{P}$ (respectively). Observing that $\mathbb{P}$ has twice as many corners as edges, then

$$\begin{aligned} c(\mathbb{P}) &= \sum_{v} c(v) + \sum_{\Delta} c(\Delta) \\ &= \sum_{v} \Big(2\pi - \sum_{\kappa \subseteq\, \partial v} \alpha(\kappa)\Big) + \sum_{\Delta} \Big(2\pi - \sum_{\kappa \subseteq\, \partial \Delta} \big(\pi - \alpha(\kappa)\big)\Big) \\ &= \sum_{v} 2\pi - \sum_{\kappa \subseteq\, \mathbb{P}} \alpha(\kappa) + \sum_{\Delta} 2\pi - \sum_{\kappa \subseteq\, \mathbb{P}} \pi + \sum_{\kappa \subseteq\, \mathbb{P}} \alpha(\kappa) \\ &= 2\pi V + 2\pi F - \pi 2E = 2\pi(V - E + F) = 4\pi. \end{aligned}$$

As a consequence of Lemma 2.2.8, for any angle function on any connected spherical picture, some vertex or region has positive curvature. The curvature method is used as follows. Assign to the corners of $\mathbb{P}$ an angle function in such a way that the sum of the angles of the corners of any vertex v of $\mathbb{P}$ is exactly 2π, that is $c(v) = 0$ for any vertex v of $\mathbb{P}$. In this case we say that $\mathbb{P}$ is flat at v. Thus there exists a region Δ of $\mathbb{P}$ with positive curvature. Since Δ is a region in $\mathbb{P}$ with $c(\Delta) > 0$, the possibilities of the labels of the corners of Δ can be calculated. Now since $l(\Delta) = 1$ in G, we get some restrictions on elements of G.

2.2.4 Curvature distribution method

This method is due to Edjvet [10] and it is applicable after the curvature test. Let $\mathbb{P}$ be a reduced spherical picture over $\mathcal{P}$. As above, our method of associating angles ensures that vertices have zero curvature and so $\sum c(\Delta) = 4\pi$, where the sum is taken over all the regions Δ of $\mathbb{P}$. The next step is to locate each Δ satisfying $c(\Delta) > 0$ and distribute $c(\Delta)$ to a neighbouring regions $\hat{\Delta}$ of Δ which have negative curvature. For such regions $\hat{\Delta}$ define $c^*(\hat{\Delta})$ to equal $c(\hat{\Delta})$ plus all the positive curvature $\hat{\Delta}$ receives during this process. Our strategy is to show that the positive curvature can be sufficiently compensated by the negative curvature by showing that $c^*(\hat{\Delta}) \leq 0$. Since the total curvature of $\mathbb{P}$ is at most $\sum c^*(\hat{\Delta})$, this yields a contradiction which implies that $\mathcal{P}$ is aspherical.

2.2.5 Finiteness of element x

Lemma 2.2.9. *Consider the relative presentation $\mathcal{P}$=$\langle G, x | x^{\varepsilon_1} g_1 x^{\varepsilon_2} g_2 \ldots x^{\varepsilon_n} g_n \rangle$, where the relator is not a proper power. Assuming that $\mathcal{P}$ is an orientable presentation, if $l = \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_n \neq \pm 1$ and x has finite order, then $\mathcal{P}$ is not aspherical.*

Proof. Let $\hat{G}$ be the group defined by $\mathcal{P}$. Now Theorem 1.1.3 says that if $\mathcal{P}$ is aspherical then every finite subgroup of $\hat{G}$ is contained in a $\hat{G}$-conjugate of G. Let G^c denote the normal closure of G in $\hat{G}$. Now observe that the factor group $\hat{G}/G^c$ is cyclic generated by xG^c of order l (if l=0 then it has infinite order). Also observe that since $l \neq \pm 1$, then clearly $x \notin G^c$. Therefore the $gp\{x\}$ is not contained in any $\hat{G}$-conjugate of G and so $\mathcal{P}$ is not aspherical.

Example 5. *Let $\mathcal{P}$= $\langle G, x | x^m g x g^{-1} \rangle$, where $m \geq 2$ and $g^k = 1$. Then $x^{-m} = gxg^{-1}$ implies $x = x^{(-m)^k}$ (see proof of Lemma 3.2.2). Thus x has finite order and so $\mathcal{P}$ is not aspherical.*

2.2.6 Change of presentations

As an example of this method, we consider the relative presentation $\mathcal{P}_1$=$\langle G, y | ayaycydycyfy \rangle$. Apply the following changes on $\mathcal{P}_1$ so that we get $\acute{\mathcal{P}}_1$: $\mathcal{P}_1$= $\langle G, y | ayaycydycyfy \rangle$= $\langle G, y, x | x(ay)^{-1}, x^2 ca^{-1} x da^{-1} x ca^{-1} x fa^{-1} x \rangle$= $\langle G, x | x^3 ca^{-1} x da^{-1} x ca^{-1} x fa^{-1} \rangle$= $\langle G, x | x^3 gxhxgxk \rangle$= $\langle G, x, s | xgxs^{-1},\ x^2 shsk \rangle$= $\acute{\mathcal{P}}_1$, where $g = ca^{-1}$, $h = da^{-1}$ and $k = fa^{-1}$. We aim to show that if $\acute{\mathcal{P}}_1$ is aspherical then so is $\mathcal{P}_1$.

Assume that $\mathbb{P}$ is a reduced spherical picture over $\mathcal{P}_1$. Convert $\mathbb{P}$ to $\acute{\mathbb{P}}$ by transforming each vertex of $\mathbb{P}$ as shown in Figures 2.7 and 2.8 (the first step of this transformation for negative vertices is given by Figure 2.8, where the rest of the transformation can be done similarly as positive vertices). One can observe that $\acute{\mathbb{P}}$ is a reduced spherical picture over $\acute{\mathcal{P}}_1$.

Remarks 2.2.10.

1. *Let* $l(\Delta) = w_1 w_2 \dots w_{2k}$ *be the label of a region* Δ *in* $\mathbb{P}$*, where* w_1 *is a corner of a positive disc in* $\partial\Delta$*. After applying the above process,* Δ *transforms to another region, say* $\acute{\Delta}$*. Observe that* $l(\acute{\Delta}) = w_1 a^{-1} a w_2 w_3 a^{-1} a w_4 \dots w_{2k}$ *and so* $l(\acute{\Delta}) = 1$*.*
2. *Observe that* $\mathcal{P}_1$ *is the presentation we dealt with in Case 1 (see Subsection 4.2.1). Similarly the result holds for other presentations with similar changes (such as Case 4(see Subsection 4.2.2)).*

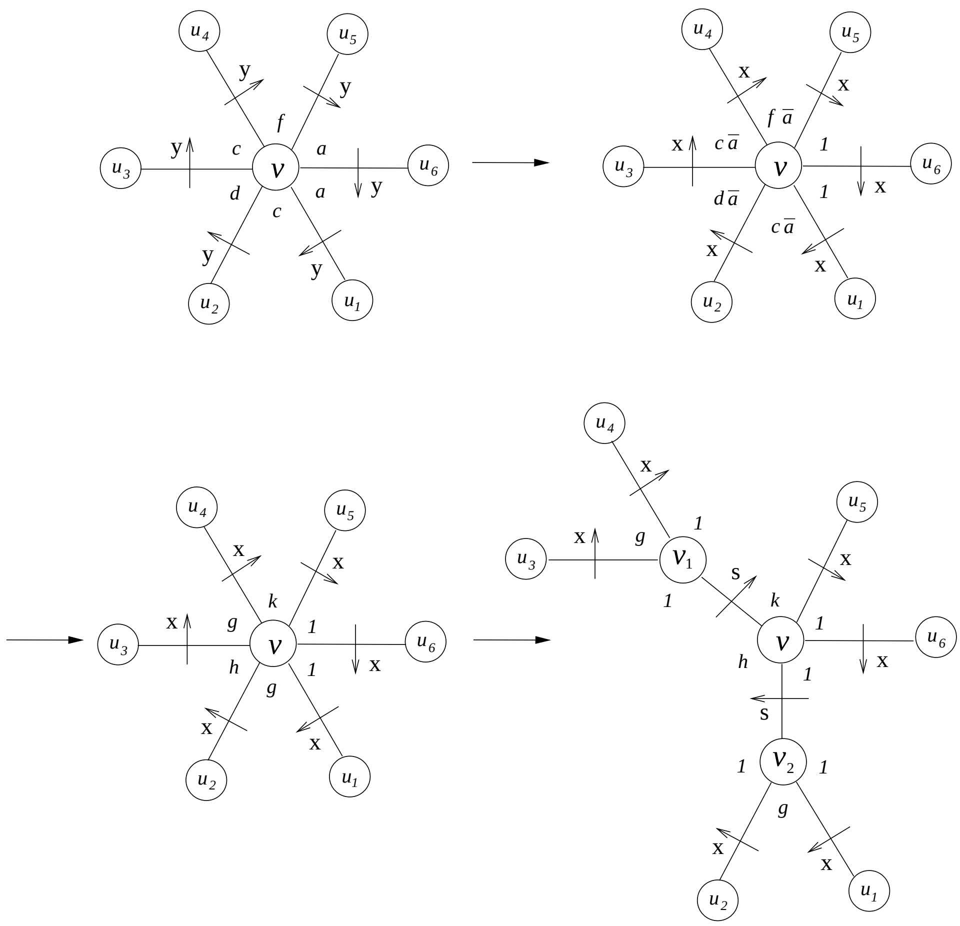


Figure 2.7: converting positive vertices of $\mathbb{P}$ to vertices of $\acute{\mathbb{P}}$.

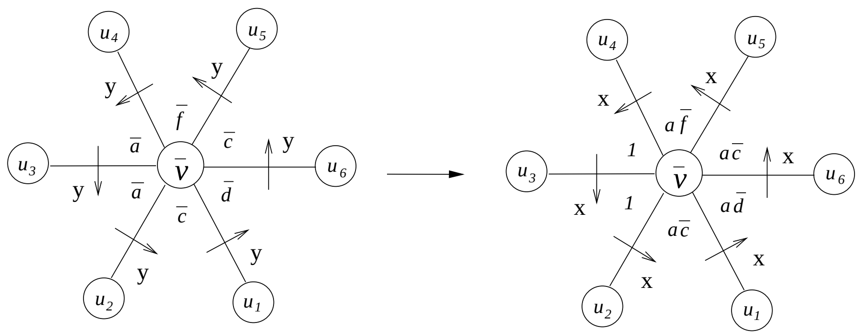


Figure 2.8: converting negative vertices of $\mathbb{P}$.

2.3 Notation

In this section we clarify some of the notation we have used throughout our work.

- If there are $m-1$ consecutive regions of degree 2, then the m arcs in the boundary of these regions constitute an *m-bond* (see Figure 2.9). We will refer to 1-bond as single bond.

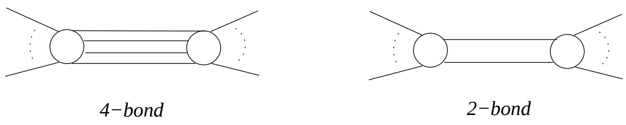


Figure 2.9: Examples of m-bond.

- kk^{-1}-bond is a bond of the form given by Figure2.9. For simplicity the labels of 11^{-1}-bond will be omitted.

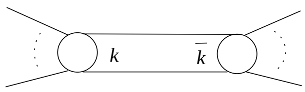


Figure 2.10: kk^{-1}-bond.

- The degree of a region Δ is the number of corners included in Δ. A region of degree n will be denoted by *n-region*.
- *w-corner* will denote a corner with label w.
- *U-region*, where U is a word in the alphabet $\{g^{\pm 1}, h^{\pm 1}, 1\}$ is a region Δ such that Δ has label U and we write $l(\Delta) = U$.
- $l(\Delta) = w\{\bar{u}, \bar{v}\}$ means $l(\Delta) = w\bar{u}$ or $w\bar{v}$.
- $c(n_1, n_2, \ldots, n_k)$ is the curvature of a region Δ of degree k such that n_i ($1 \leq i \leq k$) are the degrees (greater than 2) of the corners of Δ.

Chapter 3

The Asphericity of a Family of Relative presentations

3.1 Statement of Result

In this chapter we study the asphericity of the relative presentation $\mathcal{P}$= $\langle G, x | x^m gxh \rangle$ for $m \geq 5$, where $x \notin G$, $g, h \in G$ and $g, h \neq 1$ (For if $g = 1$ or $h = 1$, then it is done by Lemma 1.3.3). If $m = 1$, then the presentation $\langle G, x | xgxh \rangle$ =$\langle G, t | t^2 g^{-1} h \rangle$ is aspherical if and only if $|g^{-1}h| = \infty$. For $m = 2, 3$ and 4 we list the results below.

Remark 3.1.1. *Observe that $x^m gxh = 1$ if and only if $x^{-m} h^{-1} x^{-1} g^{-1} = 1$, so replacing x^{-1} by x, g^{-1} by g and h^{-1} by h we can work modulo $(g, h) \leftrightarrow (h^{-1}, g^{-1})$. For example, the result that we obtain for $\mathcal{P}$ when $g = h^2$ and $h^5 = 1$ is the same when $h = g^2$ and $g^5 = 1$.*

Theorem 3.1.2. *[5, Theorem 3.1(special case)] The relative presentation $\mathcal{P}_2$ =$\langle G, x | x^2 gxh \rangle$ is aspherical if and only if none of the following holds:*

1. *Let* $H = gp\{g, h\}$, $p > 2$ *and* $0 \leq k < p$. *Then one of the following holds:*
   (*i*) $H = gp\{g\}$, $|g| = p$ *and* $gh^{-1} = g^{-k}$;
   (*ii*) $H = gp\{gh^{-1}\}$, $|gh^{-1}| = p$ *and* $h = (gh^{-1})^k$;
   (*iii*) $H = gp\{h\}$, $|h| = p$ *and* $g^{-1} = h^k$;
   where either $k = 1$; *or* $p = k + 2$; *or* $p = 2k + 1$; *or* $p = 6$ *and* $k = 2, 3$.

2. $\frac{1}{|g|} + \frac{1}{|gh^{-1}|} + \frac{1}{|h|} > 1$, *where* $\frac{1}{\infty} := 0$.

Before stating the next result we list the following exceptional cases.

(**T1**) $g \in \{h^2, h^3, h^4\}$ and $|h| = 6$.
(**T2**) $|g| = 2$, $|h| = 4$ and $H = gp\{g, h\} \cong C_2 \times C_4$.
(**T3**) $|g| = 2$, $|h| = 5$ and $H = gp\{g, h\} \cong C_2 \times C_5 \cong C_{10}$.

Theorem 3.1.3. *[4, Theorem 4]. Let* $\mathcal{P}_3$ *be the relative presentation* $\mathcal{P}_3$ $=\langle G, x | x^3gxh\rangle$, *where* $x \notin G$, $g, h \in G$ *and* $2 \leq |g|, |h| \leq \infty$ *in* $H = gp\{g, h\}$. *Suppose that none of the exceptional conditions (**T1**)-(**T3**) holds. Then* $\mathcal{P}_3$ *is aspherical if and only if none of the following holds:*

1. $g = h^{\pm 1}$ *and* g *has finite order.*

2. $g = h^2$ *and* $|h| \in \{4, 5\}$.

3. $\frac{1}{|g|} + \frac{1}{|gh^{-1}|} + \frac{1}{|h|} > 1$, *where* $\frac{1}{\infty} := 0$.

4. $|g| = 2$, $|h| = 3$ *and* $H = gp\{g, h\} \cong C_2 \times C_3 \cong C_6$.

Remark 3.1.4. *By following the proof of Lemma 3.2.6(3), Lemma 8(3) in [4] can be amended as follows: if* $|g|$=*2*, $|gh^{-1}| \geq 4$, $|h| \geq 4$ *and* $g \neq h^2$, *then* $\mathcal{P}_3$ *is aspherical even if* $[g, h] = 1$; *and so in the exceptional cases (**T2**) and (**T3**) above* $\mathcal{P}_3$ *is also aspherical.*

Theorem 3.1.5. *[12, Theorem 2] The relative presentation $\mathcal{P}_4 = \langle G, x | x^4 gxh \rangle$ is aspherical unless one of the following holds:*

1. $g = h^{\pm 1}$ *and* $|g| < \infty$.

2. $g = h^2$ *and* $|h| \in \{4, 5\}$.

3. $\frac{1}{|g|} + \frac{1}{|gh^{-1}|} + \frac{1}{|h|} > 1$, *where* $\frac{1}{\infty} := 0$.

4. $H = gp\{g, h\}$ *is cyclic of order 6 generated by* gh^{-1}.

5. $H = gp\{g, h\}$ *is cyclic of order 6 generated by* g *or* h.

In Cases 1, 2, 3 and 4, the presentation $\mathcal{P}_4$ is not aspherical.

We now state our main theorem in this chapter. We list the following as exceptional cases.

(**E1**) $g = h^2$ and $|h| \in \{5, 6\}$.
(**E2**) $g = h^3$ and $|h| = 6$.
(**E3**) $g = h^4$ and $|h| = 6$.

Theorem 3.1.6. *(**Main Theorem**) Let $\mathcal{P}$ be the relative presentation $\mathcal{P} = \langle G, x | x^m gxh \rangle$, where $m \geq 5$, $x \notin G$, $g, h \in G$ and $2 \leq |g|, |h| \leq \infty$ in $H = gp\{g, h\}$. Suppose that none of the exceptional conditions (**E1**)-(**E3**) holds. Then $\mathcal{P}$ is aspherical if and only if (modulo $(g, h) \leftrightarrow (h^{-1}, g^{-1})$) none of the following holds:*

1. $g = h^{\pm 1}$ *and* g *has finite order.*

2. $g = h^2$ *and* $|h| = 4$.

3. $\frac{1}{|g|} + \frac{1}{|gh^{-1}|} + \frac{1}{|h|} > 1$, *where* $\frac{1}{\infty} := 0$.

4. $|g| = 2$, $|h| = 3$, $H = gp\{g, h\} \cong C_2 \times C_3$.

The proof of Theorem 3.1.6 is given in Section 3.2. For the rest of this section we give results for the relative presentation $\mathcal{Q} = \langle G, x | x^n g x^{-1} h \rangle$, where $x \notin G$, $g, h \in G$. If n=1, $\mathcal{Q}$ is not aspherical if and only if $|g| = |h| < \infty$ in $H = gp\{g, h\}$, where $\mathcal{Q}$ defines an HNN- extension. For n=2, 3, 4, respectively, the asphericity of $\mathcal{Q}$ has been determined modulo some open cases in [10], [2] and [9] respectively. We list the results below.

Theorem 3.1.7. *[10, Theorem 1.1(special case)] Consider the relative presentation* $\mathcal{Q} = \langle G, x | x^2 g x^{-1} h \rangle$, *where* $x \notin G$, $g, h \in G$, $1 < |h| \leq |g| < \infty$ *and* $(|g|, |h|) \notin \{(8, 4), (9, 3)\}$. *Then* $\mathcal{Q}$ *is aspherical if and only if none of the following holds:*

1. $\frac{1}{|g|} + \frac{1}{|gh^{-1}|} + \frac{1}{|h|} > 1$, *where* $\frac{1}{\infty} := 0$.

2. $gh = 1$.

3. $g^2 h = 1$ *or* $gh^2 = 1$.

4. $|h| = 2$ *and* $gh = hg$.

5. $|g| = 3$, $|h| = 2$ *and* $(gh)^2 (g^{-1} h)^2 = 1$.

6. $|g| = |h| = 3$ *and* $gh = hg$.

7. $|g| = 6$ *and* $h = g^2$.

8. $|g| = |h| = 7$ *and either* $h = g^2$ *or* $g = h^2$.

9. $|g| = |h| = 9$ *and either* $h = g^2$ *or* $g = h^2$.

Remark 3.1.8. *There is no loss in assuming $|h| \leq |g|$ in the above theorem; and if $|g| = \infty$ then $\mathcal{Q}$ is aspherical [5]. (Indeed, $x^n g x^{-1} h = 1$ if and only if $h^{-1} x g^{-1} x^{-n} = 1$ if and only if $x^{-n} h^{-1} x g^{-1} = 1$ so replacing x^{-1} by x it follows $x^n h x^{-1} g = 1$, where $|g| \leq |h|$).*

In the next theorem, we will refer to the following as open cases.

(**T**) $|g| = 2, |h| = 4$ and $[g, h] = 1$.

(**T1**) $g = h^2$ and $|h| = 6$.

(**T2**) $g = h^4$ and $|h| = 6$.

(**T3**) $g = h^4$ and $|h| = 8$.

Theorem 3.1.9. *[2, Theorem 1.1, Theorem 1.2(special case)] Let $\mathcal{Q}$ $=\langle G, x | x^3 g x^{-1} h\rangle$, where $x \notin G$, $g, h \in G \backslash \{1\}$. Let $H = gp\{g, h\}$. Then (modulo $(g, h) \leftrightarrow (h^{-1}, g^{-1})$) the following hold:*

1. *Assume H is non-cyclic and the open case (**T**) does not hold. Then $\mathcal{Q}$ is aspherical if and only if none of the following holds:*
   (i) $|g| = |h| = 2$ and $gh = hg$.
   (ii) $\frac{1}{|g|} + \frac{1}{|gh^{-1}|} + \frac{1}{|h|} > 1$, where $\frac{1}{\infty} := 0$.

2. *Suppose that H is cyclic and none of the open cases (**T1**)-(**T3**) is satisfied. Then $\mathcal{Q}$ is aspherical if and only if either H is infinite or H is finite and none of the following conditions holds:*
   (i) $gh^{\pm 1} = 1$;
   (ii) $|g| = |h| = 2$;
   (iii) $|g| = 2$ and $|h| = 3$;
   (iv) $4 \leq |g| \leq 5$ and $g^2 h = 1$;
   (v) $|g| = 6$ and $g^3 h = 1$.

Before stating the next result, we list the following as exceptional cases.

(**E1**) $g = h^2$ and $3 < |h| < \infty$.

(**E2**) $g = h^{-2}$ and $3 < |h| < \infty$.

(**E3**) $g^2 = h$ and $3 < |g| = |h| < \infty$.

(**E4**) $g = h^3$ and $|h| = 9$.

Theorem 3.1.10. *[9, Theorem 1.1, Theorem 1.2] Let $\mathcal{Q} = \langle G, x | x^n g x^{-1} h\rangle$, where $n \geq 4$, $x \notin G$, $g, h \in G$. The following are satisfied:*

1. *If $g = h$, then $\mathcal{Q}$ is aspherical if and only if $|g| = \infty$ in $H = gp\{g, h\}$.*

2. *Let $g \neq h$ and $3 \leq |g| \leq |h|$. Assume that none of the exceptional cases (**E1**)-(**E4**) holds. Then $\mathcal{Q}$ is aspherical if and only if none of the following conditions holds:*
   (i) $\frac{1}{|g|} + \frac{1}{|gh^{-1}|} + \frac{1}{|h|} > 1$, where $\frac{1}{\infty} := 0$;
   (ii) $g = h^{-1}$ and $|g| < \infty$.

Remark 3.1.11. *There is no loss in assuming $|g| \leq |h|$ in the above theorem.*

3.2 Proof of Theorem 3.1.6

Assume that $m \geq 5$. We first state a series of lemmas followed by their proofs. Recall that we assume $g, h \in G \backslash \{1\}$.

Lemma 3.2.1. *If $\mathcal{P}$ is not aspherical, then at least one of the following conditions holds:*

1. *$g = h^{\pm 1}$.*

2. *$g = h^2$ or $h = g^2$.*

3. $2 \in \{|g|, |h|\}$.

4. $|gh^{-1}| = 2$ *and* $3 \in \{|g|, |h|\}$.

Lemma 3.2.2. *If $g = h^{\pm 1}$, then $\mathcal{P}$ is aspherical if and only if g has infinite order.*

Lemma 3.2.3. *Let $g = h^2$. If $|h| = 4$, then $\mathcal{P}$ is not aspherical, while if $|h| > 6$, then $\mathcal{P}$ is aspherical.*

Assume for Lemmas 3.2.4 and 3.2.5 that $g \notin \{h^{\pm 1}, h^2\}$.

Lemma 3.2.4. *If $\frac{1}{|g|} + \frac{1}{|gh^{-1}|} + \frac{1}{|h|} > 1$, then $\mathcal{P}$ is not aspherical.*

Lemma 3.2.5. *If $|gh^{-1}|$ is infinite, then $\mathcal{P}$ is aspherical.*

Lemma 3.2.6. *Suppose that $|g| = 2$.*

1. *If $|gh^{-1}| = 2$ and $|h| = \infty$, then $\mathcal{P}$ is aspherical.*

2. *If $|gh^{-1}| = 3$, $|h| \geq 6$ and $\mathcal{P}$ is not aspherical, then $g = h^3$, in particular $|h| = 6$.*

3. *If $|gh^{-1}| \geq 4$, $|h| \geq 4$ and $g \neq h^2$, then $\mathcal{P}$ is aspherical.*

4. *If $|gh^{-1}| \geq 6$ and $|h| = 3$, then $\mathcal{P}$ is not aspherical if and only if $[g, h] = 1$.*

Lemma 3.2.7. *If $|g| = 3$, $|gh^{-1}| = 2$, $|h| \geq 6$ and $\mathcal{P}$ is not aspherical, then $g = h^4$ and $|h| = 6$.*

The proofs of the above lemmas are given in the coming sections. Here we assume that they are true and prove Theorem 3.1.6.

Proof of Theorem 3.1.6.

The 'only if' direction of Theorem 3.1.6 follows from Lemmas 3.2.2, 3.2.3, 3.2.4

and 3.2.6(4). For the rest of the proof, we assume that none of the Conditions (1)-(4) of Theorem 3.1.6 is satisfied. We show that either $\mathcal{P}$ is aspherical or exceptional.

If none of the conditions of Lemma 3.2.1 holds, then $\mathcal{P}$ is aspherical. Assume that Condition 1 of Lemma 3.2.1 holds. Then $|g| = \infty$ (since Condition 1 of Theorem 3.1.6 does not hold), and so $\mathcal{P}$ is aspherical by Lemma 3.2.2. So assume from now on that $g \neq h^{\pm 1}$.

If Condition 2 of Lemma 3.2.1 holds, then it can be assumed without any loss that $g = h^2$. Then $|h| \geq 5$ (by the negation of Condition 2 of Theorem 3.1.6). If $|h| \in \{5, 6\}$, then $\mathcal{P}$ is exceptional of type (**E1**); and if $|h| \geq 7$, then $\mathcal{P}$ is aspherical by Lemma 3.2.3. So assume from now on that $g \neq h^2$.

If Condition 3 of Lemma 3.2.1 holds, then it can be assumed without any loss that $|g| = 2$. Since $g \neq h$, $|gh^{-1}| \geq 2$. If $|gh^{-1}| = 2$, then $|h| = \infty$ (Condition 3 of Theorem 3.1.6) and it follows that $\mathcal{P}$ is aspherical by Lemma 3.2.6(1). If $|gh^{-1}| = 3$, then $|h| \geq 6$ (Condition 3 of Theorem 3.1.6). By Lemma 3.2.6(2), $\mathcal{P}$ is aspherical if $g \neq h^3$, while if $g = h^3$ then $\mathcal{P}$ is exceptional of type (**E2**). If $|gh^{-1}| = 4$ or 5, then $|h| \geq 4$ (Condition 3 of Theorem 3.1.6), and so $\mathcal{P}$ is aspherical by Lemma 3.2.6(3). Now, suppose that $|gh^{-1}| \geq 6$. By Lemma 3.2.5, if $|gh^{-1}| = \infty$ then $\mathcal{P}$ is aspherical, so assume otherwise. Then $|h| \geq 3$ (Condition 3 of Theorem 3.1.6). If $|h| = 3$ then $[g, h] \neq 1$, otherwise Condition 4 of Theorem 3.1.6 holds and so $\mathcal{P}$ is aspherical by Lemma 3.2.6(4). If $|h| \geq 4$, then $\mathcal{P}$ is aspherical by Lemma 3.2.6(3).

Finally, if Condition 4 of Lemma 3.2.1 is satisfied then it can be assumed without loss that $|g| = 3$ and $|gh^{-1}| = 2$. Hence, $|h| \geq 6$ (else, Condition 3 of Theorem

3.1.6 applies). If $g = h^4$ and so $h^6 = 1$ then $\mathcal{P}$ is exceptional of type (**E3**); otherwise $\mathcal{P}$ is aspherical by Lemma 3.2.7.

3.3 Construction of pictures and defined angle functions

For this section we assume $g \neq h^{\pm 1}$. Let $\mathbb{P}$ be a reduced spherical picture over $\mathcal{P}=\langle G, x | x^m gxh \rangle$. Then each vertex(disc) in $\mathbb{P}$ has one of the forms given by Figure 3.1(i) and (ii); and the the star graph $\mathcal{P}^{st}$ of $\mathcal{P}$ is given by Figure 3.1(iii). Note that when drawing figures the edge arrows and labels shown in Figure 3.1 will often be omitted.

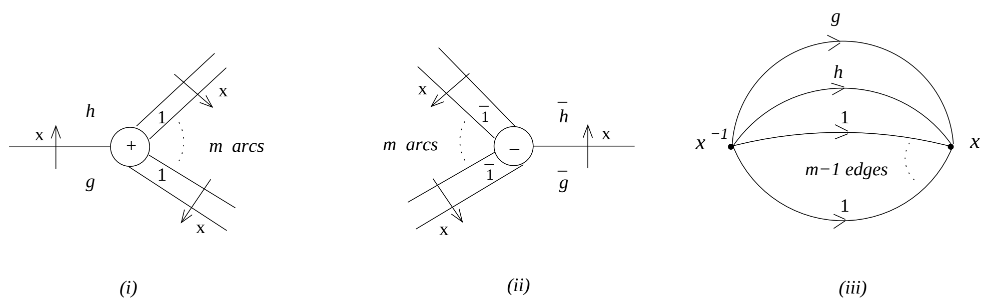


Figure 3.1: $+$ disc , - disc and $\mathcal{P}^{st}$.

Given that $g \neq h^{\pm 1}$ there are (up to inversion) only two types of $(m-1)$-bonds in a reduced picture $\mathbb{P}$ (see Figure 3.2). For simplicity, in our figures $(m-1)$-bonds will be drawn as bold 2-bonds (see Figure 3.2). Note that there are no m-bonds or $(m+1)$-bonds in $\mathbb{P}$, indeed a vertex of degree 2 can only occur in a reduced picture if $g = h$ or $g = 1$ or $h = 1$. Also, for simplicity, the vertex of degree 3 of the form shown in Figure 3.3 (i) will be drawn as shown in Figure 3.3 (ii), where $m_1 \geq 2$, $m_2 \geq 2$ and $m_1 + m_2 = m$.

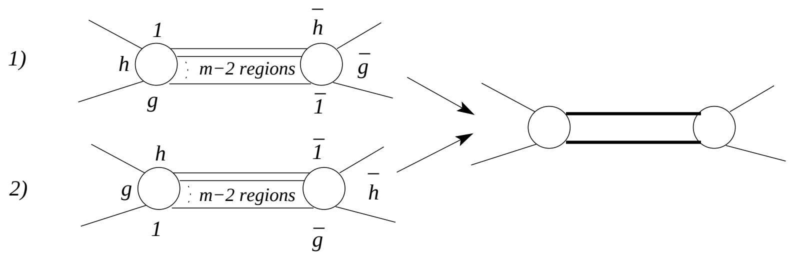


Figure 3.2: $(m-1)$-bond.

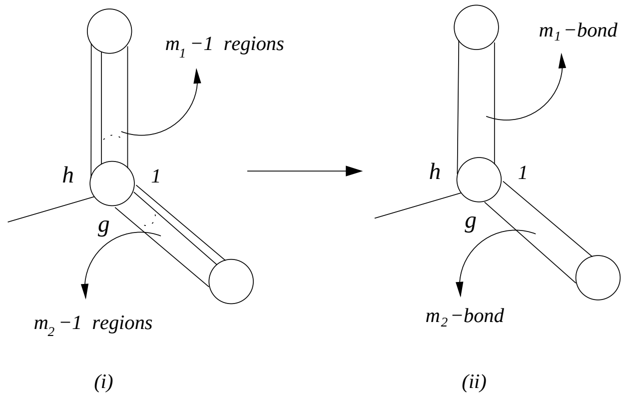


Figure 3.3: A vertex of degree 3.

Remarks 3.3.1.

1. *Each arc connects a + disc to a - disc, and so each region has even degree.*

2. *A word w obtained from reading the labels on the edges of a cyclically reduced cycle in $\mathcal{P}^{st}$ does not contain (up to cyclic permutation and inversion) gg^{-1} or hh^{-1} although it can contain 11^{-1} provided different edges in $\mathcal{P}^{st}$ are used. We will call such words w cyclically reduced.*

3. *Each region in a reduced spherical picture $\mathbb{P}$ over $\mathcal{P}$ supports a cyclically reduced word in $\{g, h, 1\}$.*

There are (up to inversion) three types of vertices of degree 3 and these are shown in Figure 3.4.

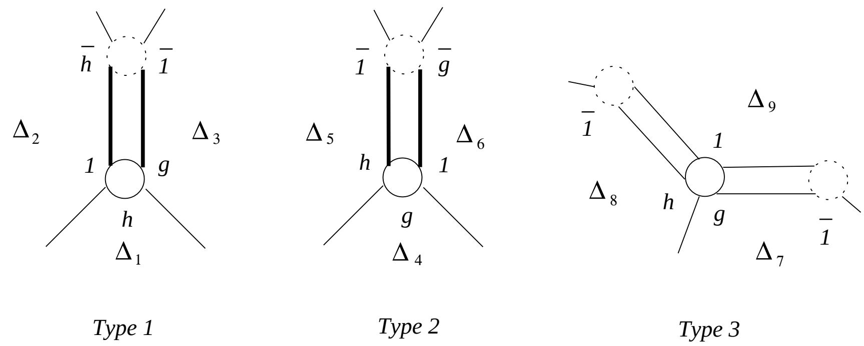


Figure 3.4: Types of vertices of degree 3.

Remark 3.3.2. *Up to inversion, each region with a vertex of degree 3 included in it's boundary could be one of Δ_i, $i \in \{1, ..., 9\}$. For example, if $\Delta = \Delta_2$, then Δ is shown in Figure 3.5.*

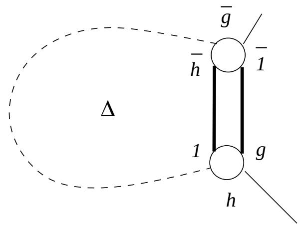


Figure 3.5: $\Delta = \Delta_2$.

For the proofs, we define the following angle functions on the vertices v of $\mathbb{P}$. The *angle function* α is defined as follows. Each corner within a 2-bond has angle zero, while each of the other corners has angle $\frac{2\pi}{d(v)}$. We will refer to α as the *standard angle function.*

The *angle function* α_1 is defined as follows. Again, corners within 2-bonds have angle zero. Each corner that is adjacent to an $(m-1)$-bond, but not within a 2-bond has angle $\frac{3\pi}{4}$. The remaining corner in a vertex with an $(m-1)$-bond has angle $\frac{\pi}{2}$ (see Figure 3.6). For vertices of degree 3 of Type 3, α_1 is given by Figure 3.6. If $d(v) \geq 4$, then each corner (not in a 2-bond) in v has angle $\frac{2\pi}{d(v)}$.

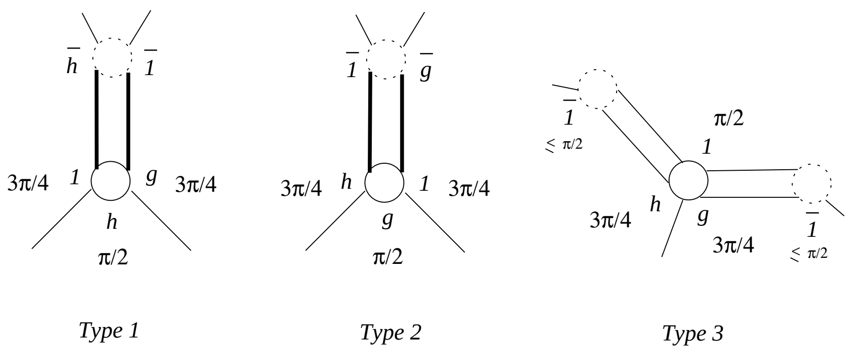


Figure 3.6: Angle function α_1 for vertices of degree 3.

Define an *angle function* α_2 on $\mathbb{P}$ as follows. Corners within 2-bonds have angle zero. In vertices of degree 3, corners labelled by $h^{\pm 1}$ have angle π, each of the other two corners has angle $\frac{\pi}{2}$ (see Figure 3.7). Corners (not within 2-bonds) in vertices of degree > 3 have angle $\frac{2\pi}{d(v)}$.

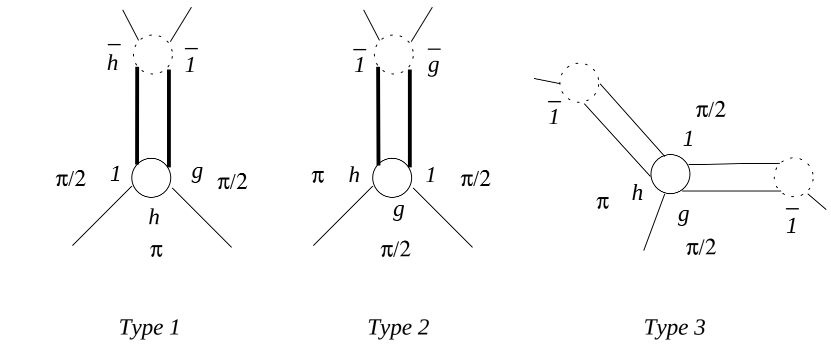


Figure 3.7: Angle function α_2 for vertices of degree 3.

Finally, the *angle function* α_3 on $\mathbb{P}$ is given as follows. Corners within 2-bonds have angle zero. For vertices of degree 3, corners labelled by $1^{\pm 1}$ have angle π, each of the other two corners has angle $\frac{\pi}{2}$ (see Figure 3.8). Corners (not in a 2-bond) in vertices of degree > 3 have angle $\frac{2\pi}{d(v)}$.

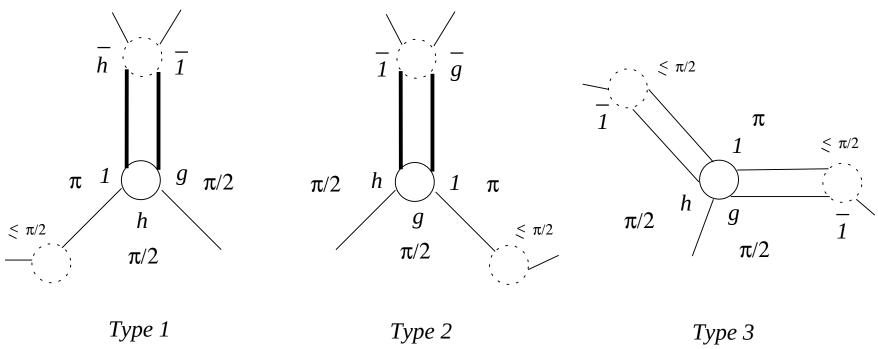


Figure 3.8: Angle function α_3 for vertices of degree 3.

Remarks 3.3.3.

1. *The corners in each 2-bond have angle 0 in each of the above angle functions. It follows that the curvature of regions of degree 2 is 0, and so we can treat each k-bond as a single bond.*

2. *By assigning the angle function* α_1 *to the corners of* $\mathbb{P}$*, the following are satisfied:*

   (i) *Since* $(2-8)\pi + 8.\frac{3\pi}{4} = 0$*, positive regions can only have degree 4 or 6.*
   (ii) *Both corners adjacent to the* $(m-1)$*-bond in a boundary of a region have angle* $\frac{3\pi}{4}$*; while the two corners adjacent to the* m_1*-bond or* m_2*-bond in a boundary of a region cannot both have angle* $\frac{3\pi}{4}$ *(see Figure 3.6).*

3. *By assigning the angle function* α_2 *to the corners of* $\mathbb{P}$*, the following are satisfied:*

   (i) *In any region* Δ *of* $\mathbb{P}$*, there are no consecutive corners with angle* π*, else* $\mathbb{P}$ *is not reduced. Hence,* $c(\Delta) \leq (2-n)\pi + \frac{n}{2}.\pi + \frac{n}{2}.\frac{\pi}{2} = \pi(\frac{8-n}{4})$*, and so positively curved regions can only be 4-regions or 6-regions.*
   (ii) *If* Δ *is a positive 4-region, then it has at least one corner labelled by* $h^{\pm 1}$ *with angle* π *(otherwise* $c(\Delta) \leq -2\pi + 4.\frac{\pi}{2} = 0$ *).*
   (iii) *If* Δ *is a positive 6-region, then it contains at least three* $h^{\pm 1}$*- corners each with angle* π *(else* $c(\Delta) \leq -4\pi + 2\pi + 4.\frac{\pi}{2} = 0$ *).*

4. *By assigning the angle function* α_3 *to the corners of* $\mathbb{P}$*, the following are satisfied:*

   (i) *There are no consecutive corners with angle* π *in the boundary of a region* Δ *of* $\mathbb{P}$ *(otherwise* $\mathbb{P}$ *is not reduced). Thus,* $c(\Delta) \leq (2-n)\pi + \frac{n}{2}.\pi + \frac{n}{2}.\frac{\pi}{2} = \pi(\frac{8-n}{4})$*, and so positive regions can only be 4-regions or 6-regions.*
   (ii) *If* Δ *is a positive 4-region then it contains at least one corner labelled by* $1^{\pm 1}$ *with angle* π *(otherwise* $c(\Delta) \leq -2\pi + 4.\frac{\pi}{2} = 0$ *).*
   (iii) *If* Δ *is a positive 6-region then it contains three occurrences of* $1^{\pm 1}$*-corners each with angle* π *(else* $c(\Delta) \leq -4\pi + 2\pi + 4.\frac{\pi}{2} = 0$ *).*

3.4 Proof of Lemma 3.2.1.

Let $\mathbb{P}$ be a reduced spherical picture over $\mathcal{P}$. It can be assumed without any loss of generality (**A**) that the number of regions of degree 4 cannot be decreased by bridge moves. Suppose that none of the Conditions 1, 2 or 3 holds. That is, $g \neq h^{\pm 1}$, $g \neq h^2$, $h \neq g^2$ and both g and h have order at least 3.

First assign the standard angle function α to the vertices of $\mathbb{P}$. By the curvature formula, there is a positively curved region Δ in $\mathbb{P}$. Also, the maximum curvature of any n-region in $\mathbb{P}$ is $\pi\left(\frac{6-n}{3}\right)$, and hence $c(\Delta) > 0$ only if $n = 4$.

A positively curved 4-region Δ has at least one vertex of degree 3. If $\Delta \in \{\Delta_i : 1 \leq i \leq 8\}$ which are shown in Figure 3.4, then at least one corner of Δ is not labelled by $1^{\pm 1}$. By considering all cyclically reduced words of length at most 4 in the alphabet $\{g^{\pm 1}, h^{\pm 1}\}$ (which are compatible with our hypotheses on g and h), we must have $l(\Delta) = (gh^{-1})^{\pm 2}$. If $\Delta = \Delta_9$ then $l(\Delta)$ gives a contradiction or Δ is the positive 4-region shown in Figure 3.9. Since $m_1 > B$ (the $*$-corner is labelled by 1^{-1} while the $\bullet$-corner is labelled by g) a sequence of bridge moves transforms Δ into a region of degree > 4 without creating a new region of degree 4. This contradicts assumption (**A**) and so by assigning α we obtain $|gh^{-1}| = 2$.

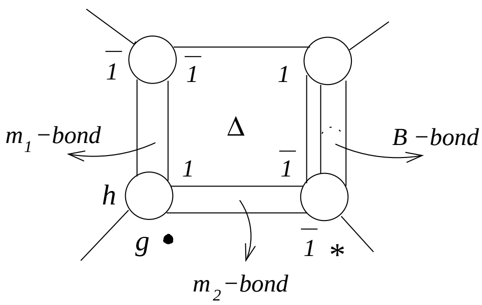


Figure 3.9: $\Delta = \Delta_9$ and $l(\Delta) = 11^{-1}11^{-1}$.

Now apply the angle function α_1. By Remark 3.3.3.(2)(i), positively curved regions can only be 4-regions or 6-regions. A positively curved 4-region Δ has at least one corner with angle $\frac{3\pi}{4}$ in it's boundary, and so $\Delta = \Delta_i$ for $i \in \{2, 3, 5, 6, 7, 8\}$. This implies that Δ has at least one corner not labelled by $1^{\pm 1}$. Also, it implies that $l(\Delta) \neq (gh^{-1})^{\pm 2}$. All other choices contradict our assumptions on g and h and so there are no positive 4-regions. It follows that Δ is a 6-region which contains at least five corners with angle $\frac{3\pi}{4}$ in it's boundary (else, $c(\Delta) \leq (2-6)\pi + 4.\frac{3\pi}{4} + 2.\frac{\pi}{2} = 0$). By Remark 3.3.3(2)($ii$) Δ contains at least two $(m-1)$-bonds in it's boundary and a third bond which is either an $(m-1)$-bond, an m_1-bond or m_2-bond. If the $(m-1)$-bonds in the boundary of Δ are inwardly oriented (that is, towards Δ), then $l(\Delta) = (g1^{-1})^{\pm 3}$, while if the $(m-1)$-bonds are oriented outward (that is, away from Δ), then $l(\Delta) = (h1^{-1})^{\pm 3}$. It follows that $|gh^{-1}| = 2$ and $3 \in \{|g|, |h|\}$ which is Condition 4, as required.

3.5 Proof of Lemma 3.2.2.

If $g = h$ then $x^m gxh = 1$ if and only if $x^{m-1}(xg)^2 = 1$. By Lemma 1.3.2, $\mathcal{P}$ is aspherical if and only if $|g| = \infty$.

If $g = h^{-1}$ and g has infinite order, then Lemma 1.3.1 applies to show that $\mathcal{P}$ is aspherical. So, we may assume that $g = h^{-1}$ and $g^k = 1$.

The relator $x^m gxh = 1$ gives that $x^m gxg^{-1} = 1 \Rightarrow x^{-m} = gxg^{-1}$. Therefore

$$x = g^k x g^{-k} = g^{k-1} gxg^{-1} g^{-(k-1)} = g^{k-1} x^{-m} g^{-(k-1)} = g^{k-2} gx^{-m} g^{-1} g^{-(k-2)} = g^{k-2} x^{(-m)^2} g^{-(k-2)} = \ldots = x^{(-m)^k},$$

and so x has finite order. Therefore, by Theorem 1.1.3, $\mathcal{P}$ is not aspherical.

3.6 Proof of Lemma 3.2.3.

Let $g = h^2$. For $|h| = 4$ there is the sphere shown in Figure 3.10. On the other hand if $|h| > 6$, then the ordinary presentation $\langle x, h | x^m h^2 x h = 1 = h^r \rangle$ is a C(4)-T(4) presentation, hence $\mathcal{P}$ is aspherical (for more details see [4]).

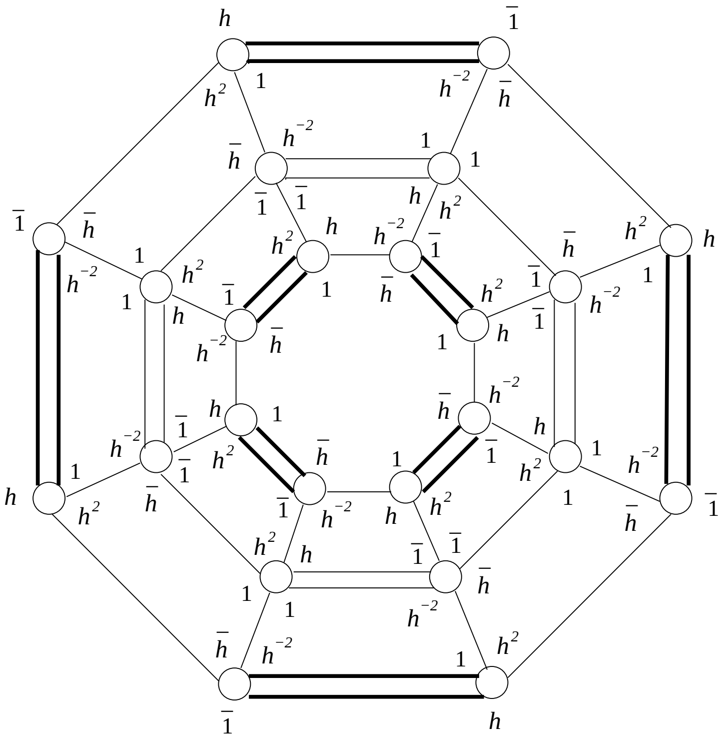


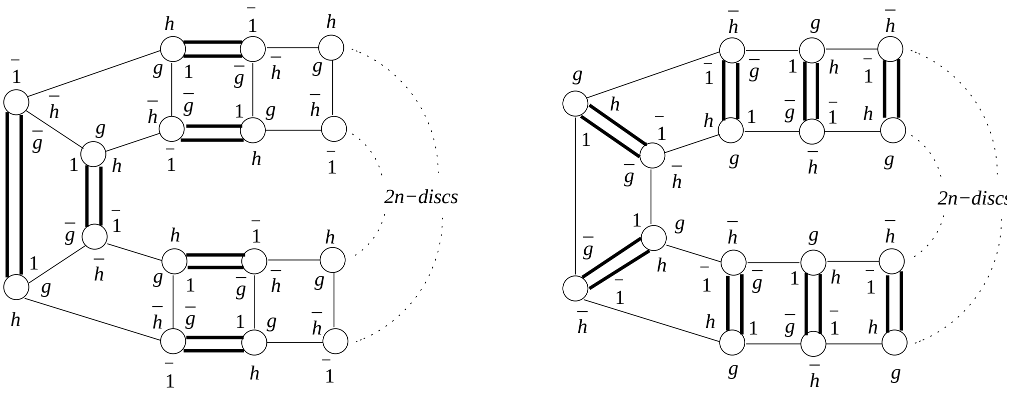
Figure 3.10: $g = h^2$ and $|h| = 4$.

3.7 Proof of Lemma 3.2.4.

We aim to construct reduced spherical pictures over $\mathcal{P}$ under the assumption $\frac{1}{|g|} + \frac{1}{|gh^{-1}|} + \frac{1}{|h|} > 1$. Without loss of generality, we may assume that $2 \leq |g| \leq |h|$ and $2 \leq |gh^{-1}|$. Thus we have the following cases, where $n \geq 2$:

$(|g|, |gh^{-1}|, |h|) \in \{$ (2, 2, n), (2, n, 2), (2, 3, 3), (2, 3, 4), (2, 3, 5), (2, 4, 3), (2, 5, 3), (3, 2, 3), (3, 2, 4), (3, 2, 5)}.

The desired spherical pictures are constructed from platonic solids (except for the cases $|g| = 2$ and $\{|gh^{-1}|, |h|\} = \{2, n\}$, where the spheres are given by Figure 3.11).



(|g|, |gh⁻¹|, |h|)= (2, 2, n) (|g|, |gh⁻¹|, |h|)= (2, n, 2)

Figure 3.11: $|g| = 2$ and $\{|gh^{-1}|, |h|\} = \{2, n\}$).

As an example, we construct the sphere illustrated in Figure 3.12 for case $(|g|, |gh^{-1}|, |h|)$ =$(2, 3, 4)$ as follows. Start with a regular tessellation T of the sphere where each vertex has degree 3 and each face has degree 4 (i.e T is a cube). Surround each vertex of T with a $(gh^{-1})^3$- region in such a way that each face of T meeting that vertex contains two discs of the surrounding $(gh^{-1})^3$-region and labelled as shown in Figure 3.13. Now join the discs v_1 and v_2 by an $(m-1)$-bond. Do the same for the discs v_3 and v_4. This creates a g^{-2}-region along the edge e. Continue in a similar way so that each edge of T supports a g^{-2}-region. Observe that this creates an h^4-region within each face of T. Similarly, we construct spheres for each of the remaining possibilities when $|g| = 2$.

If $|g| = 3$, we construct a sphere (see Figure 3.14) for the case $(|g|, |gh^{-1}|, |h|)$ $=(3, 2, 4)$. In a similar way we construct spheres for the remaining 2 cases ($|h|=$ 3 or 5). Each vertex of the tessellation has degree 3 and each face has degree 4. Surround each vertex of the tessellation with a g^{-3}-region such that the three $(m-1)$-bonds does not cross the edges of the tessellation. Within each face of the tessellation, join loose ends in such a way that each face contains an h^{-4}-region. This creates a $(gh^{-1})^{-2}$-region along each edge of the tessellation.

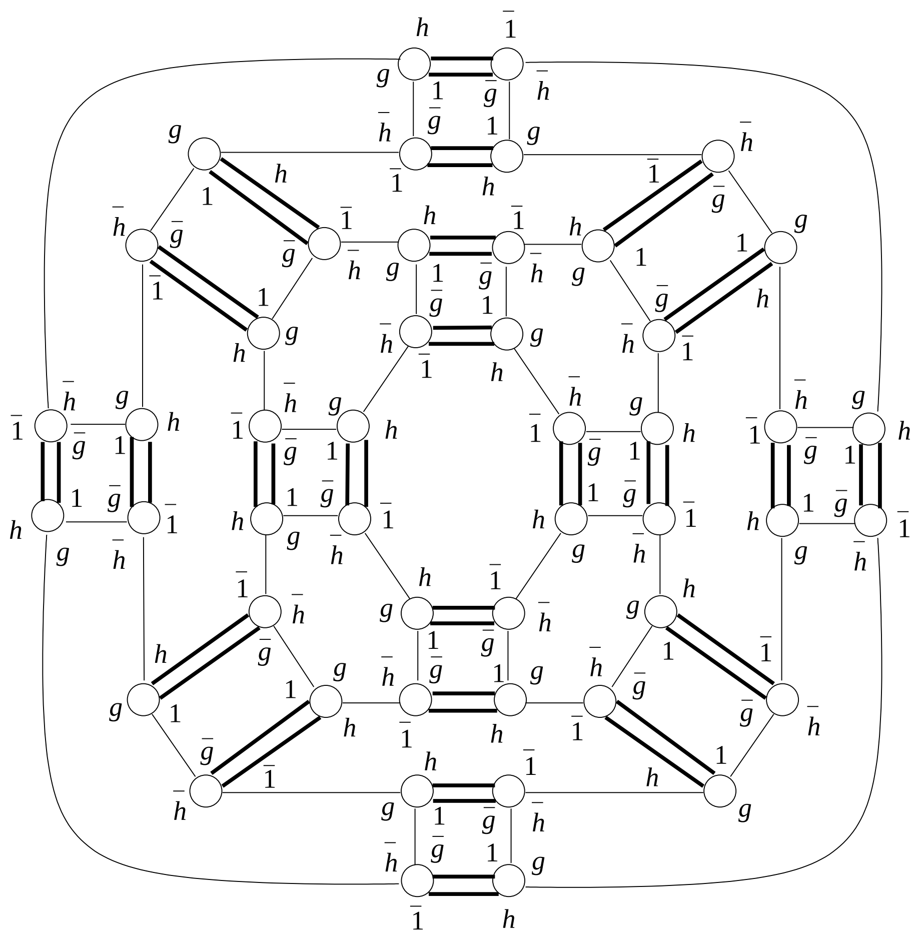


Figure 3.12: $\big(|g|, |gh^{-1}|, |h|\big) = (2, 3, 4)$.

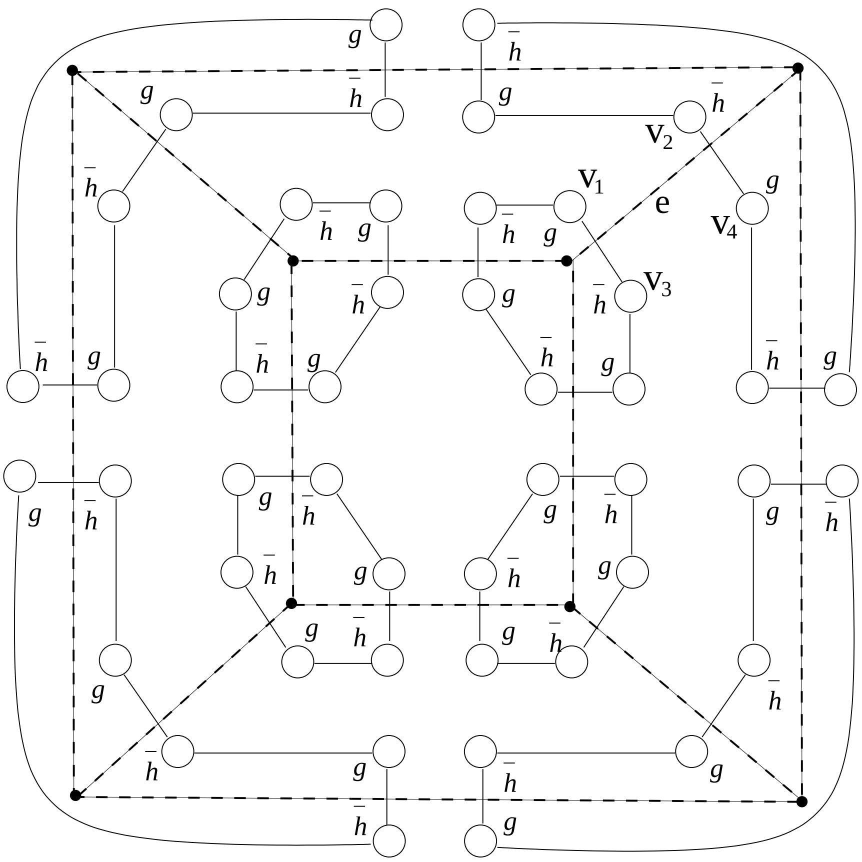


Figure 3.13: Using the tessellation of cube to construct a sphere for Case $\big(|g|, |gh^{-1}|, |h|\big) = (2, 3, 4)$.

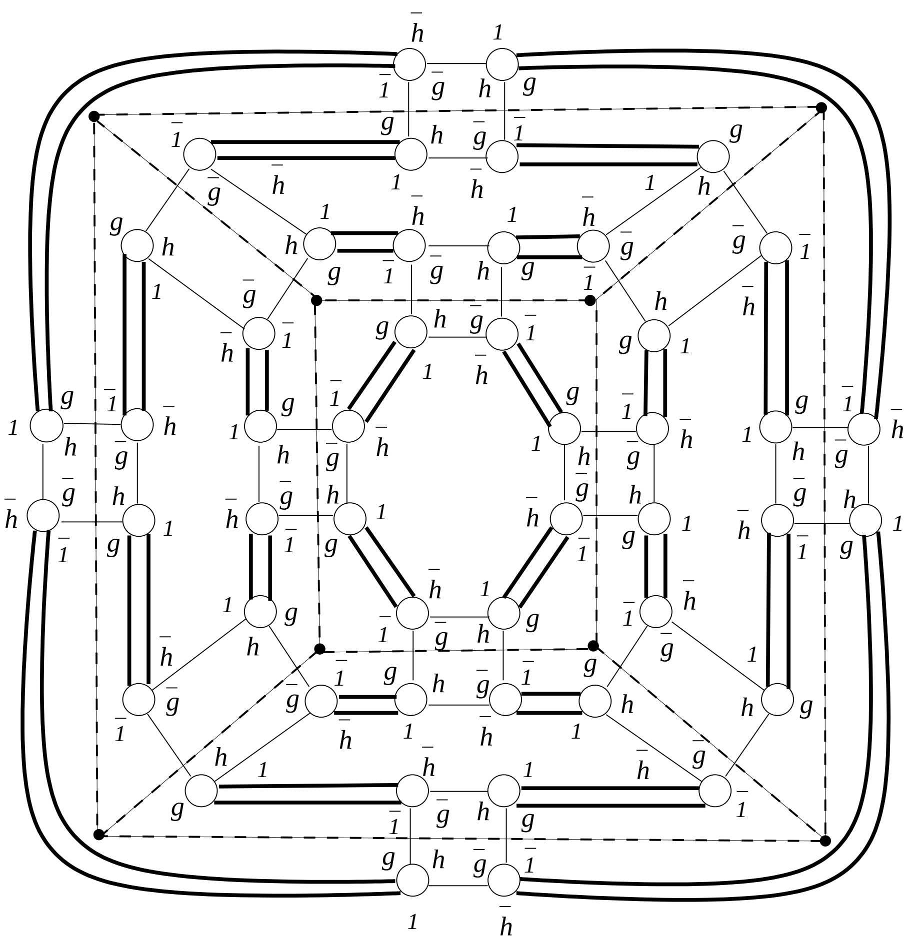


Figure 3.14: $\big(|g|, |gh^{-1}|, |h|\big) = (3, 2, 4)$.

3.8 Proof of Lemma 3.2.5.

Suppose that $|gh^{-1}|$ is infinite. If we have a relation of the form $(gh^{-1})^k g = 1$ or $h^{-1}(gh^{-1})^k = 1$ in G, then we get that $H = gp\{g, h\}$ is an infinite cyclic generated by gh^{-1}, and so $\mathcal{P}$ is aspherical by Lemma 1.3.1. Indeed, if for example $(gh^{-1})^k g = 1$, then $g = (gh^{-1})^{-k}$ and $h = (gh^{-1})^{-k-1}$, which implies that H is an infinite cyclic generated by gh^{-1}. Therefore, we may assume that there are no relations of the form $(gh^{-1})^k g = 1$ or $h^{-1}(gh^{-1})^k = 1$.

Define the following weight function θ on $\mathcal{P}^{st}$ (see Figure 3.1(iii)): $\theta(e_g)= 0= \theta(e_h)$ and $\theta(s_i) = 1$ for $(1 \leq i \leq m-1)$, where e_g, e_h, s_i $(1 \leq i \leq m-1)$ are the edges of $\mathcal{P}^{st}$ labelled g, h, 1 (respectively). Clearly Condition 1 of weakly aspherical weight function is satisfied. The assumptions on g and h imply that each admissible cycle in $\mathcal{P}^{st}$ must involve at least 2 edges labelled by the identity, and so has weight at least 2. Therefore θ is a weakly aspherical weight function which proves that $\mathcal{P}$ is aspherical (Remark 2.2.7).

3.9 Proof of Lemma 3.2.6(1): Case$(2, 2, \infty)$

In this case, $|g| = 2$, $|gh^{-1}| = 2$ and $|h| = \infty$. Suppose by way of contradiction that there is a (non-trivial reduced connected) spherical picture $\mathbb{P}$ over $\mathcal{P}$.

Assign the angle function α_2 to the corners of $\mathbb{P}$. By Remark 3.3.3 (3)(i), positive regions can only be 4-regions or 6-regions. By Remark 3.3.3 (3)(iii), positive 6-regions involve three occurrences of $h^{\pm1}$-corners and each possible label yields a contradiction. For instance, if $g = h^3$, then $h^6 = 1$, a contradiction. However, by Remark 3.3.3 (3)(ii), a positive 4-region must contain $h^{\pm1}$ forcing the label $(gh^{-1})^{\pm2}$. For instance, if $g = h^2$, then $h^4 = 1$, a contradiction. Hence, there are (up to inversion) two types of positive regions as shown in Figure 3.15, where the corners of angle π are given.

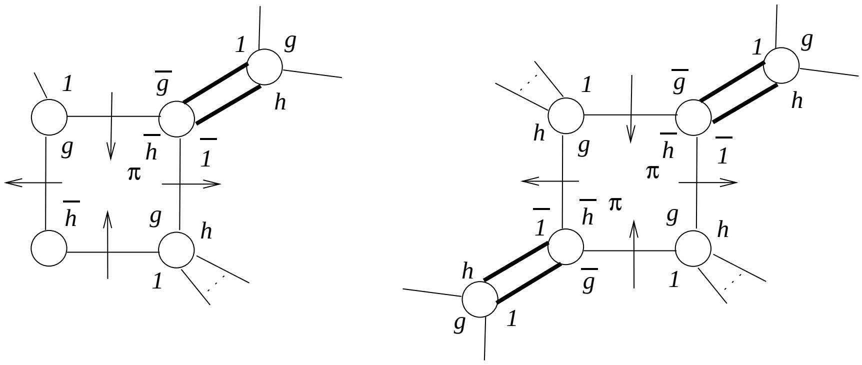


Figure 3.15: Positively curved regions in Case$(2, 2, \infty)$.

Remark 3.9.1. *Note that the maximum possible curvature is always indicated.*

We adopt the notation of [4] and define the following distribution scheme (distributing positive curvature from Δ to $\hat{\Delta}$) which is given in Figure 3.16:

$$\Gamma(\Delta, \hat{\Delta}) = \begin{cases} c(\Delta) & \text{if } 0 < c(\Delta) \leq \frac{\pi}{2} \text{ and } \Delta \text{ is separated from } \hat{\Delta} \text{ by a single} \\ & \text{bond } S \text{ that is oriented from } \Delta \text{ to } \hat{\Delta} \text{ such that } S \text{ is} \\ & \text{ adjacent to an } h^{\pm 1}\text{-corner in } \Delta \text{ with angle } \pi \\ c(\Delta)/2 & \text{if } \frac{\pi}{2} < c(\Delta) \text{ and } \Delta \text{ is separated from } \hat{\Delta} \text{ by a single} \\ & \text{bond that is oriented from } \Delta \text{ to } \hat{\Delta} \\ 0 & \text{otherwise} \end{cases}$$

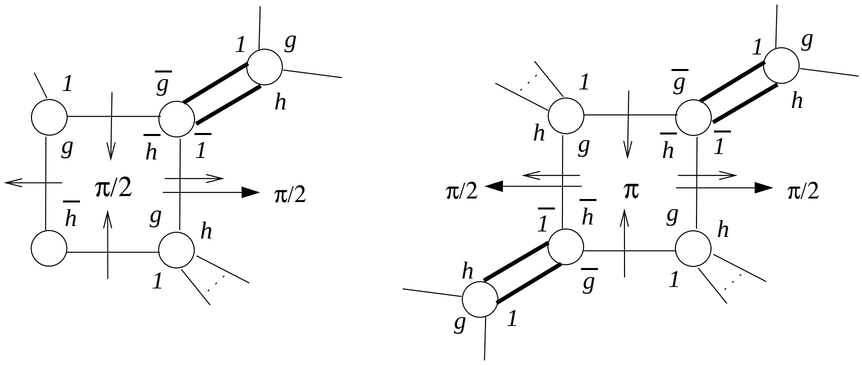


Figure 3.16: Distribution scheme in Case$(2, 2, \infty)$.

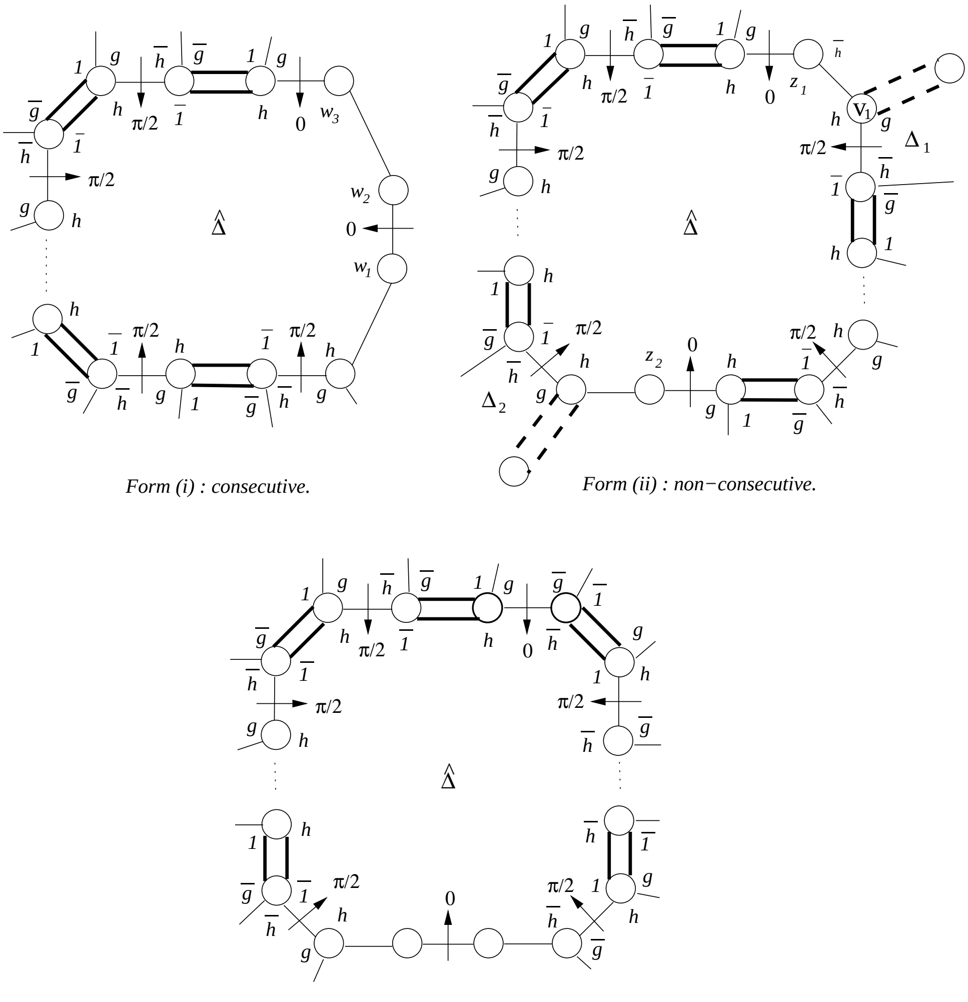
Let $\Gamma(\Delta, \hat{\Delta}) > 0$, then as shown in Figure 3.16, $(h1^{-1}h)^{\pm 1}$ is a sublabel of $\hat{\Delta}$ and so $d(\hat{\Delta}) > 4$. Let r be the number of corners of angle π in $\hat{\Delta}$. By Remark 3.3.3(3)(i), $r \leq \frac{n}{2}$, where $n = d(\hat{\Delta})$. Let c be the curvature function associated to the angle function α_2 on $\mathbb{P}$. Also let $\hat{\Delta}$ be a region such that $c^*(\hat{\Delta}) > c(\hat{\Delta})$ and $c^*(\hat{\Delta}) > 0$, where c^* is the distributed curvature function. Set $\Gamma_2 = \Gamma_2(\hat{\Delta}) = |\{\Delta : \Gamma(\Delta, \hat{\Delta}) = \frac{\pi}{2}\}| \leq \frac{n}{2}$ (since $\hat{\Delta}$ receives $\frac{\pi}{2}$ only across edges that are oriented inwards - see Figure 3.16).

Remark 3.9.2. *Assuming that both Γ_2 and r equals $\frac{n}{2}$, $c^*(\hat{\Delta}) \leq (2-n)\pi + \frac{n}{2}.\pi + \frac{n}{2}.\frac{\pi}{2} + \frac{n}{2}.\frac{\pi}{2} = 2\pi$. So, if we show that Γ_2 and r are decreased in such a way that $c^*(\hat{\Delta})$ is decreased by 2π, then $c^*(\hat{\Delta}) \leq 0$.*

If $\Gamma_2 = \frac{n}{2}$ or $\frac{n}{2} - 1$, then the labelling of $\hat{\Delta}$ implies that either $h^{\frac{n}{2}} = 1$ or $g = h^{\frac{n}{2}}$, contradicting $|h| = \infty$. Thus, we may assume that $\hat{\Delta}$ receives at most $(\frac{n}{2} - 2)\frac{\pi}{2}$ and so $c^*(\hat{\Delta}) \leq (2-n)\pi + r\pi + (n-r)\frac{\pi}{2} + (\frac{n}{2} - 2)\frac{\pi}{2} = \pi\big(2 - n + r + \frac{n}{2} - \frac{r}{2} + \frac{n}{2} - 1\big)$ $= \pi(1 - \frac{n}{4} + \frac{r}{2})$. Therefore, $c^*(\hat{\Delta}) > 0$ gives that $1 - \frac{n}{4} + \frac{r}{2} > 0$ which implies that $r > \frac{n}{2} - 2$. This means that $r = \frac{n}{2} - 1$ or $r = \frac{n}{2}$. If $r = \frac{n}{2} - 1$, then $\Gamma_2 = \frac{n}{2} - 2$ (else, $c^*(\hat{\Delta}) < 0$), while if $r = \frac{n}{2}$, then $\Gamma_2 = \frac{n}{2} - 2$ or $\Gamma_2 = \frac{n}{2} - 3$.

First assume that $r = \frac{n}{2} - 1$. The fact that $\Gamma_2 = \frac{n}{2} - 2$ means that $\hat{\Delta}$ does not receive $2.\frac{\pi}{2}$ either across consecutive or across non-consecutive inwardly oriented edges in $\partial\hat{\Delta}$ (see Figure 3.17). Form (i) in Figure 3.17 shows the first case, which yields that $l(\hat{\Delta}) = h^{\frac{n}{2}-1}w_1w_2w_3$, where $w_1, w_3 \in \{1^{-1}, g^{-1}\}$ and $w_2 \in \{1, g, h\}$. If at most one of w_1, w_2 or w_2 is g^{-1}, then $l(\hat{\Delta})$ forces $|h|$ to be finite, a contradiction. Hence, $l(\hat{\Delta})$ has the form $g^{-1}h^{\frac{n}{2}-1}g^{-1}w_2$. This implies that $h^{\frac{n}{2}-1} = 1$ or $h^{\frac{n}{2}-2} = 1$, a contradiction. In the second case, there are two possibilities for the arrangement of corner labels in $\hat{\Delta}$. One possibility is Form (ii) in Figure 3.17, the other is Form (iii) which yields that $\mathbb{P}$ is not reduced. Form (ii) gives that $l(\hat{\Delta}) = z_1h^{\alpha_1}z_2h^{\alpha_2}$, where $z_1, z_2 \in \{1^{-1}, g^{-1}\}$. If $z_1 = 1^{-1}$ or $z_2 = 1^{-1}$ then $|h| < \infty$, a contradiction, so assume otherwise. But if $z_1 = g^{-1}$ in Figure 3.17 Form (ii) then either the h-corner in the vertex v_1 has angle $\leq \frac{\pi}{2}$, or Δ_1 contains an m-bond in it's boundary and so it cannot be either of the positive regions shown in Figure 3.15 (i.e $\hat{\Delta}$ does not receive $\frac{\pi}{2}$ from Δ_1). Either way, $c^*(\hat{\Delta})$ will be decreased by $\frac{\pi}{2}$. Similarly, if $z_2 = g^{-1}$, then $c^*(\hat{\Delta})$ will be also decreased by $\frac{\pi}{2}$ and so $c^*(\hat{\Delta}) \leq 0$.

Now let $r = \frac{n}{2}$. Since $g^2{=}(gh^{-1})^2{=}1$, $hg = gh^{-1}$ and $h^{-1}g = gh$ it follows that any word in g and h can be rewritten in the form $g^{\alpha_1}h^{\alpha_2}$. If there are no occupancies of $g^{\pm 1}$ in $l(\hat{\Delta})$ then $h^{\frac{n}{2}} = 1$. If $g^{\pm 1}$ appears an odd number of times in $l(\hat{\Delta})$ then $|h| < \infty$. Also, if $g^{\pm 1}$ occurs at least four times in $l(\hat{\Delta})$, then $\Gamma_2 \leq \frac{n}{2} - 4$ and so $c^*(\hat{\Delta}) \leq 0$. Therefore $g^{\pm 1}$ appears exactly twice in $l(\hat{\Delta})$. Since $r = \frac{n}{2}$, each of these two g^{-1}-corners is adjacent to two h-corners in $\partial\hat{\Delta}$. Thus either $l(\hat{\Delta}){=}(g^{-1}h)^2(1^{-1}h)^s$ $(s \geq 1)$ which implies $|h| < \infty$, a contradiction, or arguing as in the case $z_1 = z_2 = g^{-1}$ implies that $c^*(\hat{\Delta}) \leq 0$ for this case also.



Form (i) : consecutive.

Form (ii) : non−consecutive.

Form (iii) : not reduced.

Figure 3.17: $r = \frac{n}{2} - 1$.

3.10 Proof of Lemma 3.2.6(2): Case$(2, \bar{3}, \bar{6})$

Here we assume that $|g| = 2$, $|gh^{-1}| \geq 3$ and $|h| \geq 6$. Suppose that $\mathcal{P}$ $=\langle G, x | x^m gxh \rangle$ is not aspherical. We show that $H = gp\{g, h\}$ is cyclic of order 6 generated by h and $g = h^3$. Let $\mathbb{P}$ be a reduced spherical picture over $\mathcal{P}$ to which we assign the angle function α_2. All possible labels for a positive 4-region give a contradiction since, by Remark 3.3.3 (3)(ii), each must involve $h^{\pm 1}$. For positive 6-regions, by Remark 3.3.3 (3)(iii), there are three occurrences of $h^{\pm 1}$ and the only possible labels yield $(gh^{-1})^{\pm 3} = 1$ or $g = h^3$ (and we are done). For example, if $h(g^{-1}h)^2 = 1$, then $g = (gh^{-1})^3$ and $h = (gh^{-1})^2$ implies that $h^3 = 1$, a contradiction. Therefore there is (up to inversion) only one positive region which is shown in Figure 3.18 and so $|gh^{-1}| = 3$.

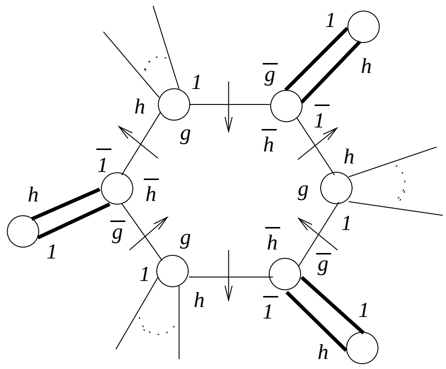


Figure 3.18: Positively curved region in Case$(2, \bar{3}, \bar{6})$.

Apply the following distribution scheme:

$$\Gamma(\Delta, \hat{\Delta}) = \begin{cases} c(\Delta)/3 & \text{if } c(\Delta) > 0 \text{ and } \Delta \text{ is separated from } \hat{\Delta} \text{ by a single} \\ & \text{bond that is oriented from } \Delta \text{ to } \hat{\Delta} \\ 0 & \text{otherwise} \end{cases}$$

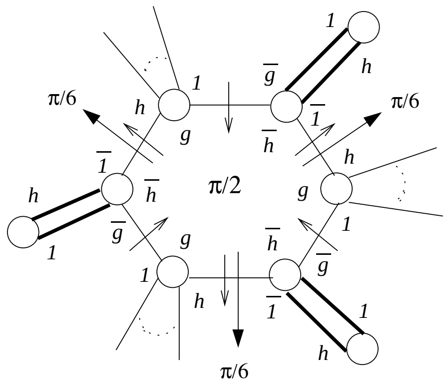


Figure 3.19: Distribution scheme in Case$(2,\bar{3},\bar{6})$.

As shown in Figure 3.19, if $\Gamma(\Delta,\hat{\Delta}) > 0$, then $(h1^{-1}h)^{\pm 1}$ is a sublabel of $\hat{\Delta}$. If the distributed curvature function is denoted by c^*, then there is a region $\hat{\Delta}$ of $\mathbb{P}$ such that $c^*(\hat{\Delta}) > c(\hat{\Delta})$ and $c^*(\hat{\Delta}) > 0$. For a fixed region $\hat{\Delta}$ set Γ_6=$\Gamma_6(\hat{\Delta}) = |\{\Delta : \Gamma(\Delta,\hat{\Delta}) = \frac{\pi}{6}\}|$.

Remarks 3.10.1.

1. *Since $\hat{\Delta}$ receives $\frac{\pi}{6}$ only across edges that are oriented towards $\hat{\Delta}$, $\Gamma_6 \leq \frac{n}{2}$.*
2. *For each $\frac{\pi}{6}$ that $\hat{\Delta}$ receives, there is an $(m-1)$-bond in the boundary of $\hat{\Delta}$ which gives $(h1^{-1})^{\pm 1}$ as a sublabel of $\hat{\Delta}$.*
3. *$l(\hat{\Delta}) = h1^{-1}hw$ and so $d(\hat{\Delta}) > 4$.*

Observe that $c^*(\hat{\Delta}) \leq (2-n)\pi + \frac{n}{2}.\pi + \frac{n}{2}.\frac{\pi}{2} + \frac{n}{2}.\frac{\pi}{6}$. The fact that $c^*(\hat{\Delta}) > 0$ gives that $\pi\big(2 - n + \frac{n}{2} + \frac{n}{4} + \frac{n}{12}\big) > 0$, which implies that $n < 12$.

Let $\hat{\Delta} = (n,r)$ denote a region of degree n with $\Gamma_6 = r$. We need to check $c^*(\hat{\Delta})$ for $\hat{\Delta}$= (n,r)= $(10,5)$, $(10,4)$, $(10,3)$, $(10,2)$, $(10,1)$, $(8,4)$, $(8,3)$, $(8,2)$, $(8,1)$,

$(6,3)$, $(6,2)$ and $(6,1)$. The region $(n,r) \neq (10,5),(8,4)$ or $(6,3)$ else it gives $h^{\pm 5}=1$, $h^{\pm 4}=1$ or $h^{\pm 3}=1$ (respectively) contradicting $|h| \geq 6$.

All possible labels for $\hat{\Delta}$=(n,r)= $(10,4),(8,3)$ or $(6,2)$ yields a contradiction or implies that $g=h^3$. For example, $\hat{\Delta}$= $(8,3)$ gives either $h^{\pm 4}=1$ or $g=h^4$: the first contradicts that $|h| \geq 6$; the second $g=h^4 \Rightarrow h^8=1$. Also, it gives that $gh^{-1}=h^3 \Rightarrow h^9=1$. But this together with $h^8=1$ implies that $h=1$, a contradiction. For $\hat{\Delta}$= $(10, r \leq 3)$, $c^*(\hat{\Delta}) \leq (2-10)\pi + 5\pi + 5.\frac{\pi}{2} + 3.\frac{\pi}{6} = 0$.

Now since $(2-8)\pi + 3\pi + 5.\frac{\pi}{2} + 2.\frac{\pi}{6} = -\frac{\pi}{6} < 0$, $c^*(\hat{\Delta}) > 0$ for $\hat{\Delta}$= $(8, r \leq 2)$ only if it contains 4 corners with angle π (up to inversion $\hat{\Delta}$ is shown in Figure 3.20), and each possible $l(\hat{\Delta})$ yields a contradiction. For example, $h^3g^{-1}hg^{-1}=1 \Rightarrow (gh^{-1})^2h^{-2}=1 \xRightarrow{(gh^{-1})^3=1} hg^{-1}h^{-2}=1 \Rightarrow g=h^{-1}$, a contradiction.

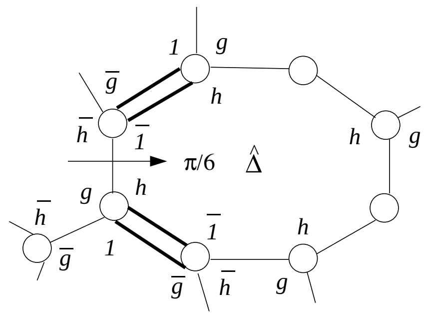


Figure 3.20: $\hat{\Delta}$= $(8, r \leq 2)$ with $c^*(\hat{\Delta}) > 0$.

This leaves $(n,r)=(6,1)$. Observe that $(2-6)\pi + \pi + 5.\frac{\pi}{2} + \frac{\pi}{6} = -\frac{\pi}{3} < 0$, and so it remains to check $c^*(\hat{\Delta})$ for $\hat{\Delta}$= $(6,1)$ with at least 2 corners with angle π. Up to inversion, $\hat{\Delta}$ has one of the forms shown in Figure 3.21 where the two corners with angle π are in bold. It follows that the label of $\hat{\Delta}$ either gives a contradiction or $g=h^3$. For example, Form (i) gives $g=h^3$, and Form (ii) gives $h^2g^{-1}hg^{-1}=1 \Rightarrow (gh^{-1})^2h^{-1}=1 \xRightarrow{(gh^{-1})^3=1} g=1$, a contradiction.

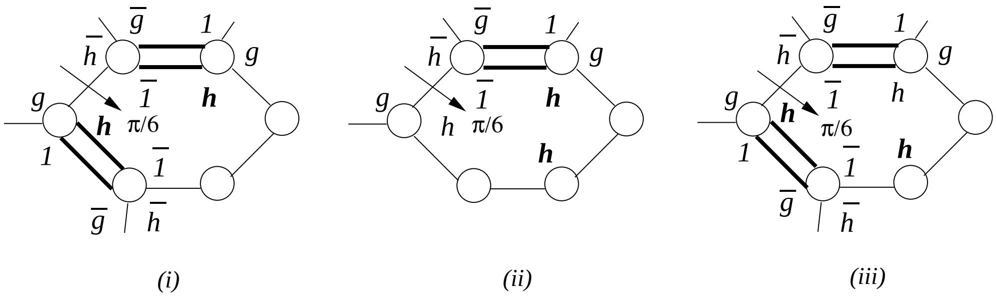


Figure 3.21: $\hat{\Delta}$= $(6, 1)$ with $c^*(\hat{\Delta}) > 0$.

3.11 Proof of Lemma 3.2.6(3): Case$(2, \bar{4}, \bar{4})$

Here we assume that $|g| = 2$, $|gh^{-1}| \geq 4$ and $|h| \geq 4$. Suppose there is a non-trivial reduced connected spherical picture $\mathbb{P}$ over $\mathcal{P}$, we get a contradiction. Assign the angle function α_2 to the corners of $\mathbb{P}$. By Remark 3.3.3(3)(i) positive region $\hat{\Delta}$ can only have degree 4 or 6. It follows from Remarks 3.3.3(3)(ii) and (iii) that $l(\hat{\Delta})$ yields a contradiction. For example, if $g = h^3$, then $(gh^{-1})^3 = 1$, which contradicts that $|gh^{-1}| \geq 4$. Therefore, in this case $\mathcal{P}$ is aspherical.

3.12 Proof of Lemma 3.2.6(4): Case$(2, \bar{6}, 3)$

In this case $|g| = 2$, $|gh^{-1}| \geq 6$ and $|h| = 3$. If $[g, h] = 1$ (which implies that $|gh^{-1}| = 6$ as will shown later), then $\mathcal{P}$ is not aspherical. This result has been obtained by Bogley and William [6] as follows: consider the group K defined by the presentation $\mathcal{K}$= $\langle b, x | x^{s-1}b^3xb^2, b^6 \rangle$ for $s \geq 2$. The following are satisfied:

1. $(i)(x^sb^2)^2 = 1$.

   (ii) $bx^sb^{-1} = x^{-s}bx^s$.

2. The presentation $\mathcal{K}$ is not aspherical.

Proof.

1. (i) $x^{s-1}b^3xb^2 = 1 \Rightarrow x^{s-1} = b^{-2}x^{-1}b^{-3} \Rightarrow x^s = b^{-2}x^{-1}b^{-3}x \Rightarrow b^2x^s = x^{-1}b^{-3}x \Rightarrow (b^2x^s)^2 = (x^{-1}b^{-3}x)^2{=}1$.

   (ii) $x^{s-1} = b^{-2}x^{-1}b^{-3} \Rightarrow x^s = xb^{-2}x^{-1}b^{-3} \Rightarrow x^sb^3 = xb^{-2}x^{-1}$. This implies $(x^sb^3)^3 = 1$.

   Now by 1)(i) $x^sb^2 = b^{-2}x^{-s}$, and so $(x^sb^3)^3 = 1 \Rightarrow x^sb^2bx^sb^2bx^sb^3 = 1$
   $\Rightarrow b^{-2}x^{-s}b^{-1}x^{-s}bx^sb^3 = 1$
   $\Rightarrow x^{-s}bx^s = (bx^{-s}b^{-1})^{-1} = bx^sb^{-1}$.

2. The abelianization $K^{ab}{=}\ \langle b, x|b^6, x^sb^{-1}\rangle{=}\ \langle x|x^{6s}\rangle$. The element b has order 6 in K^{ab}, so b has order 6 in K. By 1)(ii) $bx^sb^{-1} = x^{-s}bx^s$ and so x^s and b are conjugates in K by the element x^sb. Thus x has order $6s$. The element x can be conjugate to an element of $\langle b|b^6\rangle$ only if $s = \pm 1$. However $s \geq 2$ and so K is not aspherical by Theorem 1.1.3.

In our case, if $[g, h] = 1$, then $(gh^{-1})^6 = g^6h^{-6} = 1$. This gives that $|gh^{-1}| \leq 6$, together with the assumption $|gh^{-1}| \geq 6$ we get $|gh^{-1}| = 6$. Now $(gh^{-1})^3{=}\ g^3h^{-3} = g$. Similarly, $h = (gh^{-1})^2$. Thus, $\hat{G} = \langle G, x|x^mgxh\rangle{=}\ \langle G, x|x^m(gh^{-1})^3x(gh^{-1})^2, (gh^{-1})^6\rangle$. Thus by the above argument $\mathcal{P}\ {=}\langle G, x|x^mgxh\rangle$ is not aspherical when $[g, h] = 1$. So it can be assumed that $[g, h] \neq 1$. We prove that $\mathcal{P}\ {=}\langle G, x|x^mgxh\rangle$ is aspherical. To this end let $\mathbb{P}$ be a non-trivial reduced spherical picture over $\mathcal{P}$ with the assumption (**A**) stated in the proof of Lemma 3.2.1 and assign the angle function α_3. By Remark 3.3.3(4)(i) the degree of a positive region Δ can only be 4 or 6. If Δ is a positive 4-region with an $h^{\pm 1}$-corner, then $l(\Delta)$ yields a contradiction, so assume otherwise. If now Δ has a $g^{\pm 1}$-corner then $l(\Delta)$ yields the 4-regions shown in Figure 3.22. This leaves $l(\Delta)$=$11^{-1}11^{-1}$ which contradicts (**A**) as in the proof of Lemma 3.2.1.

If Δ is a 6-region, then either there is a contradiction or $l(\Delta) \in \{1^{-1}11^{-1}11^{-1}1,$ $1^{-1}11^{-1}g1^{-1}g,\ 1^{-1}h1^{-1}h1^{-1}h\}$. The first two cannot be positive, while the last gives the positive 6-region shown in Figure 3.22.

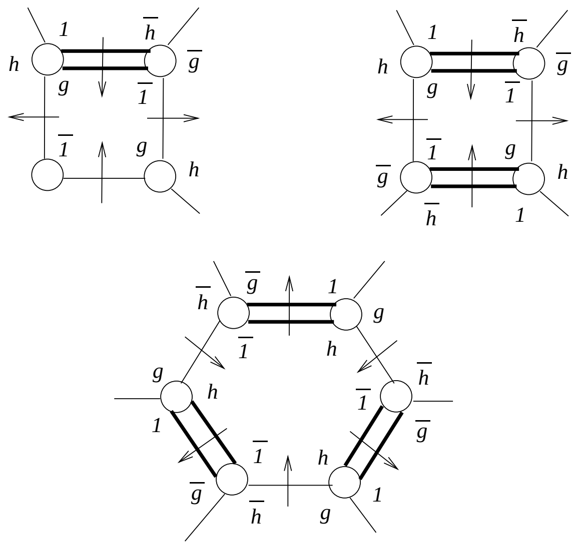


Figure 3.22: Positively curved regions in Case$(2, \bar{6}, 3)$.

Define the following distribution scheme which is given in Figure 3.23:

$$\Gamma(\Delta,\hat{\Delta}) = \begin{cases} c(\Delta)/2 & \text{if } c(\Delta)=\pi \text{ and } \Delta \text{ is separated from } \hat{\Delta} \text{ by a single} \\ & \text{bond that is oriented from } \Delta \text{ to } \hat{\Delta} \\ c(\Delta) & \text{if } 0 < c(\Delta) \leq \frac{\pi}{2},\ \Delta \text{ is separated from } \hat{\Delta} \text{ by a single} \\ & \text{bond } S \text{ that is oriented from } \Delta \text{ to } \hat{\Delta} \text{ and } S \text{ is} \\ & \text{adjacent to a 1-corner in } \Delta \text{ with angle } \pi \\ \pi/6 & \text{if } c(\Delta)=\frac{\pi}{2} \text{ and } \Delta \text{ is separated from } \hat{\Delta} \text{ by a single} \\ & \text{bond that is oriented from } \hat{\Delta} \text{ to } \Delta \\ 0 & \text{otherwise} \end{cases}$$

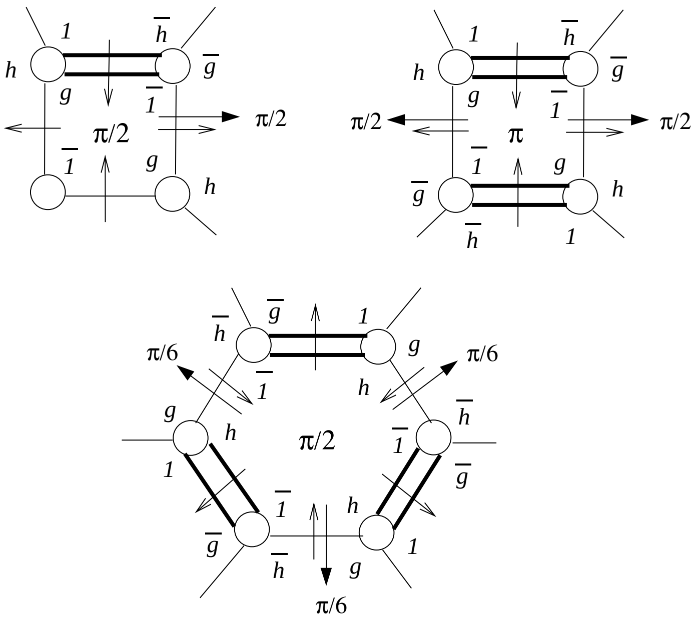


Figure 3.23: Distribution scheme in Case$(2,\bar{6},3)$.

Let r be the number of corners of angle π in Δ. Then $r \leq \frac{n}{2}$ (by Remark 3.3.3(4)(i)). Let s denote the number of pairs $(\frac{\pi}{2},\frac{\pi}{6})$ or $(\frac{\pi}{6},\frac{\pi}{2})$ such that $\hat{\Delta}$ receives $\frac{\pi}{2}$ and $\frac{\pi}{6}$ across adjacent edges in $\partial\hat{\Delta}$, with the understanding that each $\frac{\pi}{2}$ and $\frac{\pi}{6}$ that $\hat{\Delta}$ receives appears at most once in these pairs. Denote the remaining number of $\frac{\pi}{2}$ that $\hat{\Delta}$ receives by s_1. Also, let s_2 denote the remaining number of $\frac{\pi}{6}$ that $\hat{\Delta}$ receives. As an example to show how to get the values s, s_1 and s_2 see Figure 3.24.

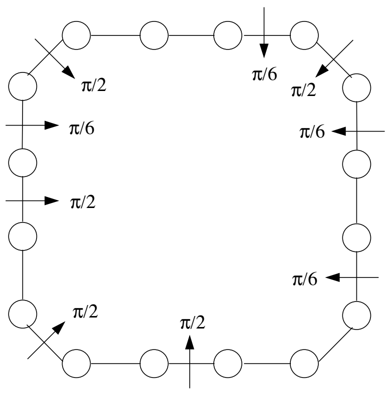


Figure 3.24: $n = 16, s = 2, s_1 = 3, s_2 = 2$.

Remarks 3.12.1.

1. *As shown in Figure 3.23,* $l(\hat{\Delta}) \in \{hg^{-1}w, h^{-1}gw\} \Rightarrow d(\hat{\Delta}) > 6$ *for otherwise* $l(\hat{\Delta})$ *yields a contradiction.*

2. *For each increase of* s, s_1 *or* s_2*, the value of* r *decreases by 1 (*$r \leq \frac{n}{2} - (s + s_1 + s_2)$*).*

3. *Note that* $s \leq \frac{n}{2}$ *and if* s_2 *increases by 1, then* s *decreases by 1 (also* $s_2 \leq \frac{n}{2}$*) and so* $s + s_2 \leq \frac{n}{2}$*.*

For a fixed region $\hat{\Delta}$ set $\Gamma_2(\hat{\Delta}) = |\{\Delta : \Gamma(\Delta, \hat{\Delta}) = \frac{\pi}{2}\}|$ and $\Gamma_6(\hat{\Delta}) = |\{\Delta : \Gamma(\Delta, \hat{\Delta}) = \frac{\pi}{6}\}|$. Let $\hat{\Delta}$ be a region such that $c^*(\hat{\Delta}) > 0$. Then $c^*(\hat{\Delta}) \leq$

$(2-n)\pi + \left[\frac{n}{2} - (s+s_1+s_2)\right]\pi + (\frac{n}{2}+s+s_1+s_2)\frac{\pi}{2} + s(\frac{1}{2}+\frac{1}{6})\pi + s_1.\frac{\pi}{2} + s_2.\frac{\pi}{6} = \frac{24-3n+2s-4s_2}{12}$, and so $c^*(\hat{\Delta}) > 0$ implies $24-3n+2s-4s_2 > 0 \Rightarrow 3n < 24-4s_2+2s \leq 24-4s_2+2(\frac{n}{2}-s_2) = 24-6s_2+n \Rightarrow 2n < 24-6s_2 \Rightarrow n < 12-3s_2 \Rightarrow n < 12$.

Let $n = 10$. Then $c^*(\hat{\Delta}) > 0 \Rightarrow 24-3(10)+2s > 4s_2 \geq 0 \Rightarrow s > 3$. If s=4 or 5, then $l(\hat{\Delta}) = (g^{-1}h)^4 g^{-1}\{1,h\}$ or $\{1,g\}h^{-1}(gh^{-1})^4$ and we get a contradiction.

This leaves $n = 8$. But checking the possible labels shows that $l(\hat{\Delta}) = hg^{-1}1g^{-1}h1^{-1}h1^{-1}$. Clearly Γ_2, $\Gamma_6 \leq 1$. Also if Γ_2=0 or Γ_6=0, then $c^*(\hat{\Delta}) \leq (2-8)\pi + 3\pi + 5.\frac{\pi}{2} + \Gamma_2.\frac{\pi}{2} + \Gamma_6.\frac{\pi}{6} \leq 0$. Thus $\Gamma_2 = \Gamma_6 = 1$ and $\hat{\Delta}$ is shown in Figure 3.25. However $r \leq 2$ and so $c^*(\hat{\Delta}) \leq -6\pi + 2\pi + 6.\frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{6} = -\frac{\pi}{3}$.

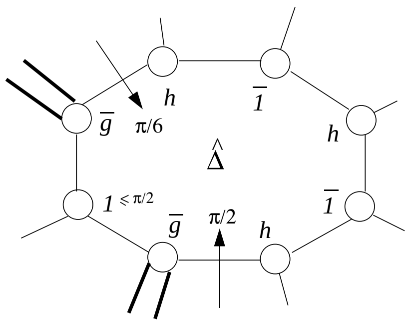


Figure 3.25: $n = 8$.

3.13 Proof of Lemma 3.2.7: Case$(3, 2, \bar{6})$

Here, we assume that $|g| = 3$, $|gh^{-1}| = 2$ and $|h| \geq 6$. Let $\mathbb{P}$ be a reduced spherical picture over $\mathcal{P}$. Let c be the curvature function associated to the angle function α_1 on $\mathbb{P}$. Observe that if $c(\hat{\Delta}) > 0$, then $l(\hat{\Delta}) \in \{1^{-1}gw, h1^{-1}w\}$ (see Figure 3.6). It follows that all positively curved regions are shown in Figure 3.26.

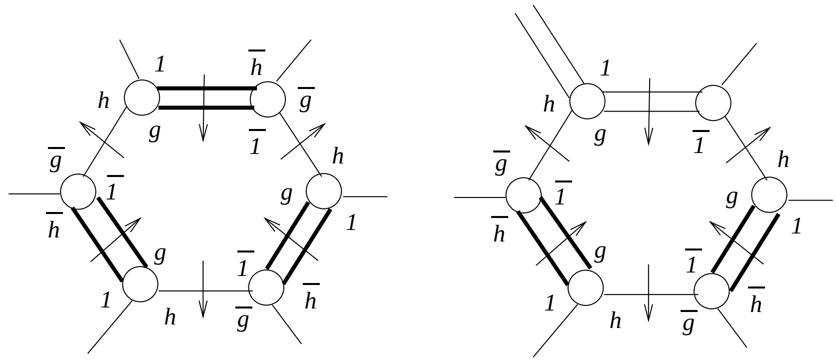


Figure 3.26: Positively curved regions in Case$(3, 2, \bar{6})$.

Define the following distribution scheme which is given in Figure 3.27:

$$\Gamma(\Delta, \hat{\Delta}) = \begin{cases} \pi/6 & \text{if } c(\Delta) = \frac{\pi}{2} \text{ and } \Delta \text{ is separated from } \hat{\Delta} \text{ by an } (m-1)\text{-bond} \\ c(\Delta)/2 & \text{if } 0 < c(\Delta) \leq \frac{\pi}{4} \text{ and } \Delta \text{ is separated from } \hat{\Delta} \text{ by an } (m-1)\text{-bond} \\ 0 & \text{otherwise} \end{cases}$$

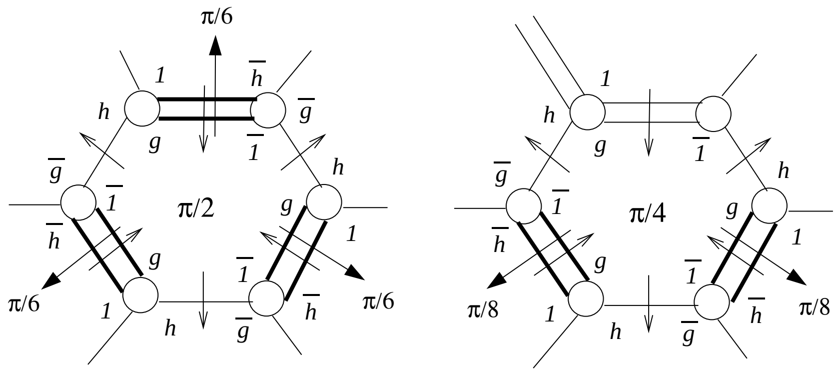


Figure 3.27: Distribution scheme in Case$(3, 2, \bar{6})$.

Let $\hat{\Delta}$ be a region such that $c^*(\hat{\Delta}) > c(\hat{\Delta})$ and $c^*(\hat{\Delta}) > 0$, where c^* is the distributed curvature function. For a fixed region $\hat{\Delta}$ set $\Gamma_6(\hat{\Delta}) = |\{\Delta : \Gamma(\Delta, \hat{\Delta}) = \frac{\pi}{6}\}|$.

Remarks 3.13.1.

1. *The region $\hat{\Delta}$ receives each $\frac{\pi}{6}$ across an $(m-1)$-bond in it's boundary which gives $(1h^{-1})^{\pm 1}$ as a sublabel of $\hat{\Delta}$.*

2. *$\hat{\Delta}$ receives $\frac{\pi}{6}$ only across edges that are oriented outwards $\hat{\Delta}$ and so $\hat{\Delta}$ does not receive $\frac{\pi}{6}$ across consecutive edges in it's boundary ($\Gamma_6 \leq \frac{n}{2}$). Also, for each $\frac{\pi}{6}$ that $\hat{\Delta}$ receives, there are two corners in $\hat{\Delta}$ with angle $\frac{3\pi}{4}$. Therefore, $\Gamma_6 \leq \frac{r}{2}$, where r is the number of corners with angle $\frac{3\pi}{4}$ in the boundary of $\hat{\Delta}$.*

3. *As shown in Figure 3.27, $l(\hat{\Delta}) = 1h^{-1}w$, which implies that $d(\hat{\Delta}) > 4$ for otherwise $l(\hat{\Delta})$ yields a contradiction.*

By using $\Gamma_6 \leq \frac{r}{2}$, $c^*(\hat{\Delta}) \leq (2-n)\pi + r.\frac{3\pi}{4} + (n-r)\frac{\pi}{2} + \frac{r}{2}.\frac{\pi}{6} = \pi\big(2-n+\frac{3r}{4}+\frac{n}{2}-\frac{r}{2}+\frac{r}{12}\big) = \pi\big(2 - \frac{n}{2} + \frac{r}{3}\big)$ and so $c^*(\hat{\Delta}) > 0$ gives that $2 - \frac{n}{2} + \frac{r}{3} > 0 \Rightarrow 12 - 3n + 2r > 0$ $\Rightarrow 2r > 3n - 12$. Since $r \leq n$, this implies that $n < 12$.

Let $\hat{\Delta} = (n, r)$ denote a region of degree n with r corners of angle $\frac{3\pi}{4}$ and assume that $c^*(\hat{\Delta}) > 0$. Since $2r > 3n - 12$ it follows that if $n = 10$ then $r = 10$; if $n = 8$ then $r = 7$ or 8; and if $n = 6$ then $r = 4, 5$ or 6.

If $(n, r) = (10, 10)$, $(8, 8)$ or $(6, 6)$ (respectively) then $l(\hat{\Delta})$ implies that $h^5 = 1$, $h^4 = 1$ or $h^3 = 1$ (respectively) contradicting $|h| \geq 6$. Now let $(n, r) = (8, 7)$. Since $c^*(\hat{\Delta}) > c(\hat{\Delta})$ it follows that $\hat{\Delta}$ contains three $(m-1)$-bonds in it's boundary and an m_1-bond (see Figure 3.28). However $l(\hat{\Delta})$ implies that $h^4 = 1$, a contradiction. In a similar way, if $(n, r) = (6, 5)$, then $h^3 = 1$, a contradiction.

This leaves $(n, r) = (6, 4)$. Up to inversion, different forms of $(n, r) = (6, 4)$ are shown in Figure 3.29 (The 4 corners with angle $\frac{3\pi}{4}$ have been highlighted). All possible labels for $\hat{\Delta}$ give either a contradiction or the exceptional case (**E3**).

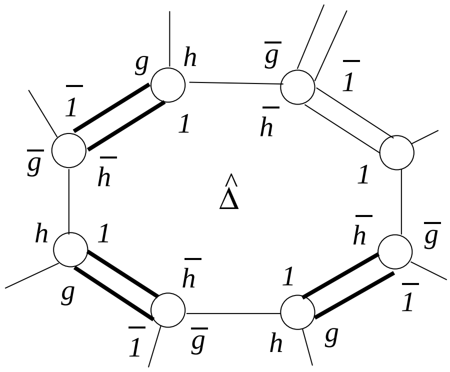


Figure 3.28: $(n, r) = (8, 7)$.

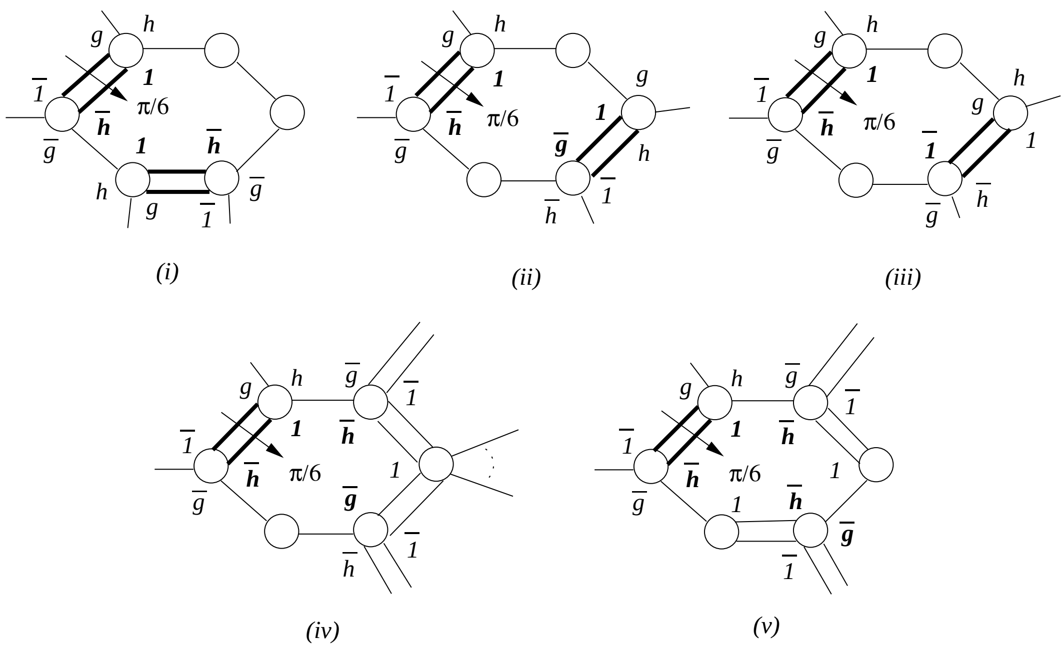


Figure 3.29: $(n, r) = (6, 4)$.

Chapter 4

Asphericity of Length Six Relative Group Presentations

4.1 Reduction to special cases

In this chapter we study the asphericity of the relative presentation $\mathcal{P}=\langle G, x | xaxbxcxdxexf\rangle$, where the coefficients $a, b, c, d, e, f \in G$ and $x \notin G$. Let $\hat{G}$ denote the group defined by $\mathcal{P}$ and let $H=\langle ba^{-1}, ca^{-1}, da^{-1}, ea^{-1}, fa^{-1}\rangle$.

4.1.1 Construction of pictures

Let $\mathbb{P}$ be a (non-trivial reduced) spherical picture over $\mathcal{P}=\langle G, x | xaxbxcxdxexf\rangle$. Then each vertex(disc) in $\mathbb{P}$ has one of the forms given by Figure 4.1(i) and (ii); and Figure 4.1(iii) shows the star graph $\mathcal{P}^{st}$ of $\mathcal{P}$ with the labels $\alpha \leftrightarrow a$, $\beta \leftrightarrow b$, $\gamma \leftrightarrow c$, $\delta \leftrightarrow d$, $\varepsilon \leftrightarrow e$ and $\zeta \leftrightarrow f$. Note that when drawing figures the edge arrows and labels shown in Figure 4.1 will often be omitted.

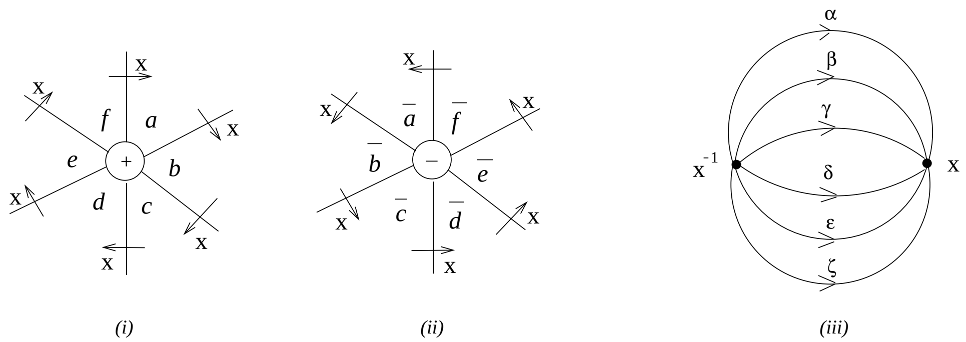


Figure 4.1: $+$ disc, - disc and $\mathcal{P}^{st}$.

Remark 4.1.1. *Each arc connects a + disc to a - disc and so each region of* $\mathbb{P}$ *has even degree (see Figure 4.1 (iii)).*

4.1.2 Reduction to special cases

Recall that $\mathcal{P}$=$\langle G, x|xaxbxcxdxexf\rangle$.

Theorem 4.1.2. *If the set* $\{a, b, c, d, e, f\}$ *contains at least five different elements, then the relative presentation* $\mathcal{P}$=$\langle G, x|xaxbxcxdxexf\rangle$ *is aspherical.*

Proof. The assumptions of Theorem 4.1.2 imply that no reduced m-wheels over $\mathcal{P}$ for $m < 4$ and so $\mathcal{P}$ satisfies $C(4)$. On the other hand, the smallest admissible cycle in $\mathcal{P}^{st}$ of length > 2 is of length 4 and so $\mathcal{P}$ satisfies $T(4)$. Thus by Theorem 2.2.5, $\mathcal{P}$ is aspherical.

Remark 4.1.3. *Observe that a relative group presentation of the form* $\langle G, x|xa_1xa_2...xa_n\rangle$ $(a_1, a_2, ...a_n \in G)$ *always satisfies* $T(4)$. *(clear from the star graph of the presentation).*

Let $r(x)$=$xaxbxcxdxexf$. By applying cyclic permutation

$$\begin{aligned} r(x) &= xaxbxcxdxexf = 1 \\ &\Leftrightarrow xfxaxbxcxdxe = 1 \\ &\Leftrightarrow xexfxaxbxcxd = 1 \\ &\Leftrightarrow xdxexfxaxbxc = 1 \\ &\Leftrightarrow xcxdxexfxaxb = 1 \\ &\Leftrightarrow xbxcxdxexfxa = 1. \end{aligned}$$

Now by applying cyclic permutation, inversion, $x \to x^{-1}$ and $g_i \to g_i^{-1}$ ($g_i \in \{a, b, c, d, e, f\}$), we get that:

$$\begin{aligned} r(x) &= xaxbxcxdxexf = 1 \\ &\to xfxexdxcxbxa = 1 \\ &\to xaxfxexdxcxb = 1 \\ &\to xbxaxfxexdxc = 1 \\ &\to xcxbxaxfxexd = 1 \\ &\to xdxcxbxaxfxe = 1. \\ &\to xexdxcxbxaxf = 1. \\ &\to xaxbxcxdxexf = 1. \end{aligned}$$

Thus, in checking the asphericity of $\mathcal{P}$, the number of cases can be reduced. For example, the case $a = b$ only is equivalent to the five cases: $b = c$ only, $c = d$ only, $d = e$ only, $e = f$ only and lastly $f = a$ only. To avoid repeating same cases and by the above equalities, we can use the table given below. For example, the case $a = b$ and $c = e$ only is equivalent to the cases: $f = a$ and $b = d$ only, $e = f$

and $a = c$ only, $d = e$ and $f = b$ only, $c = d$ and $e = a$ only, $b = c$ and $d = f$ only, $f = e$ and $d = b$ only, $a = f$ and $e = c$ only, $b = a$ and $f = d$ only, $c = b$ and $a = e$ only, $d = c$ and $b = f$ only and finally $e = d$ and $c = a$ only.

	a	b	c	d	e	f
1	f	a	b	c	d	e
2	e	f	a	b	c	d
3	d	e	f	a	b	c
4	c	d	e	f	a	b
5	b	c	d	e	f	a
6	f	e	d	c	b	a
7	a	f	e	d	c	b
8	b	a	f	e	d	c
9	c	b	a	f	e	d
10	d	c	b	a	f	e
11	e	d	c	b	a	f

It follows, in view of Theorem 4.1.2 and up to cyclic permutation, inversion, $x \to x^{-1}$ and $g_i \to g_i^{-1}$ ($g_i \in \{a, b, c, d, e, f\}$), that there are 33 cases to be considered according to the coincidences amongst a, b, c, d, e and f. These cases are listed below.

- a=b, c=d only,
- a=b, c=e only,
- a=b, c=f only,
- a=b, d=e only,

- a=b, c=d, e=f only,
- a=b, c=e, d=f only,
- a=b, c=f, d=e only,
- a=c, b=d only,
- a=c, b=e only,
- a=c, d=f only,
- a=c, b=e, d=f only,
- a=d, b=e only,
- a=d, b=e , c=f only,
- a=b=c only,
- a=b=c, d=f only,
- a=b=c, e=f only,
- a=b=c, d=e=f only,
- a=b=d only,
- a=b=d, c=e only,
- a=b=d, c=f only,
- a=b=d, e=f only,
- a=b=d, c=e=f only,
- a=c=e only,
- a=c=e, b=d only,

- a=c=e, b=d=f only,
- a=b=c=d only,
- a=b=c=d, e=f only,
- a=b=c=e only,
- a=b=c=e, d=f only,
- a=b=d=e only,
- a=b=d=e, c=f only,
- a=b=c=d=f only,
- a=b=c=d=e=f.

Lemma 4.1.4. *For six of the above cases namely $a = b, c = d$ only, $a = b, c = f$ only, $a = b, d = e$ only, $a = c, d = f$ only, $a = b = c$ only and $a = b = d$ only, $\mathcal{P}$ is aspherical.*

Proof. Remark 4.1.3 says that the presentation $\mathcal{P}$ satisfies the condition $T(4)$. In fact $\mathcal{P}$ also satisfies $C(4)$, and so by Theorem 2.2.5, $\mathcal{P}$ is aspherical. Indeed, in the case $a = b$, $c = f$ only, for example, if v is a vertex in a reduced spherical picture $\mathbb{P}$ over $\mathcal{P}$, then v is adjacent to at least four other different vertices and so $d(v) \geq 4$ (see Figure 4.2). Thus $\mathcal{P}$ satisfies $C(4)$ in this case. Similar arguments apply for the remaining five cases which completes the proof.

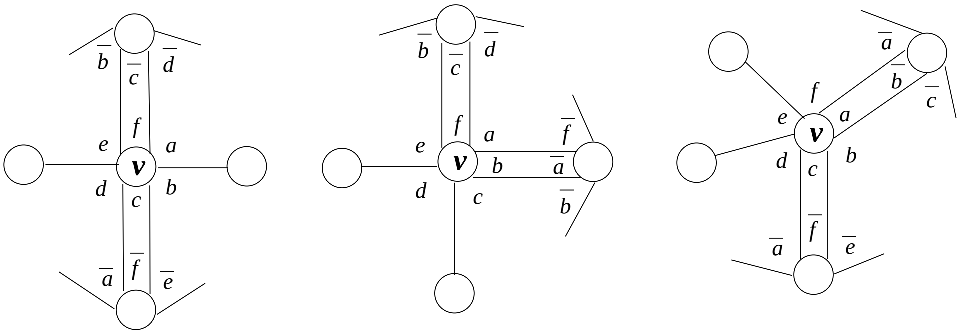


Figure 4.2: $\mathcal{P}$ satisfies $C(4)$ in the case $a = b$, $c = f$ only.

The above lemma leaves the following 27 cases to be considered:

- a=b, c=e only,
- a=b, c=d, e=f only,
- a=b, c=e, d=f only,
- a=b, c=f, d=e only,
- a=c, b=d only,
- a=c, b=e only,
- a=c, b=e, d=f only,
- a=d, b=e only,
- a=d, b=e , c=f only,
- a=b=c, d=f only,
- a=b=c, e=f only,
- a=b=c, d=e=f only,
- a=b=d, c=e only,

- a=b=d, c=f only,
- a=b=d, e=f only,
- a=b=d, c=e=f only,
- a=c=e only,
- a=c=e, b=d only,
- a=c=e, b=d=f only,
- a=b=c=d only,
- a=b=c=d, e=f only,
- a=b=c=e only,
- a=b=c=e, d=f only,
- a=b=d=e only,
- a=b=d=e, c=f only,
- a=b=c=d=f only,
- a=b=c=d=e=f.

However we collect the last 11 cases into 3 cases and so we have the the following 19 cases:

Case 1: a=b, c=e only,

Case 2: a=b, c=d, e=f only,

Case 3: a=b, c=e, d=f only,

Case 4: a=b, c=f, d=e only,

Case 5: a=c, b=d only,

Case 6: a=c, b=e only,

Case 7: a=c, b=e, d=f only,

Case 8: a=d, b=e only,

Case 9: a=d, b=e , c=f only,

Case 10: a=b=c, d=f only,

Case 11: a=b=c, e=f only,

Case 12: a=b=c, d=e=f only,

Case 13: a=b=d, c=e only,

Case 14: a=b=d, c=f only,

Case 15: a=b=d, e=f only,

Case 16: a=b=d, c=e=f only,

Case 17: a=c=e,

Case 18: a=b=c=d,

Case 19: a=b=d=e.

These 19 cases have been partitioned into Groups I, II, III and IV. Group I includes the cases in which $\mathcal{P}$ is aspherical, Group III includes the cases in which the results have been obtained by Theorem 4.4.1. The cases with some exceptional subcases are included in Group IV. Finally, the remaining cases are included in Group II. These groups are listed below.

- Group I: Cases 1, 4, 5, 6, 7 and 9.
- Group II: Cases 8, 10, 12, 14 and 16.
- Group III: Cases 17 and 19.
- Group IV: Cases 2, 3, 11, 13, 15 and 18.

In each case 1-19, $\mathcal{P}_i$ will denote the presentation $\mathcal{P}$ in which x is replaced by y.

4.2 Group I

The first group contains the cases: 1, 4, 5, 6, 7 and 9. In each of these cases $\mathcal{P}$ is aspherical.

4.2.1 Case 1: (a=b, c=e only)

$\mathcal{P}_1$= $\langle G, y | ayaycydycyfy \rangle$= $\langle G, y, x | x(ay)^{-1}, x^2ca^{-1}xda^{-1}xca^{-1}xfa^{-1}x \rangle$= $\langle G, x | x^3ca^{-1}xda^{-1}xca^{-1}xfa^{-1} \rangle$= $\langle G, x | x^3gxhxgxk \rangle$, where $g = ca^{-1}$, $h = da^{-1}$ and $k = fa^{-1}$ (and so by assumption, $g, h, k \in G \backslash \{1\}$, $g \neq h$, $h \neq k$ and $g \neq k$).

Lemma 4.2.1. *$\mathcal{P}_1$ is aspherical.*

Proof. $\mathcal{P}_1$=$\langle G, x | x^3gxhxgxk \rangle$= $\langle G, x, s | xgxs^{-1}, x^2shsk \rangle$. The star graph $\mathcal{P}_1^{st}$ of $\mathcal{P}_1$ is given by Figure 4.3, where the edges α, β, γ, δ, ε, ζ and η (respectively) are labelled by 1, 1, h, 1, g, 1 and k (respectively).

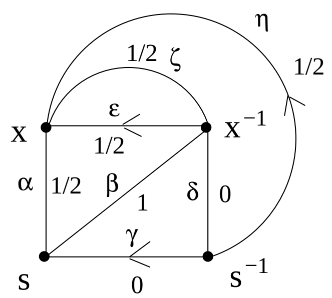


Figure 4.3: Star graph of $\langle G, x, s | xgxs^{-1}, x^2shsk \rangle$.

By assigning the following weights to the edges of $\mathcal{P}_1^{st}$, we get a weakly aspherical weight function: $\theta(\alpha) = \theta(\varepsilon) = \theta(\zeta) = \theta(\eta) = \frac{1}{2}$, $\theta(\beta) = 1$ and $\theta(\gamma) = \theta(\delta) = 0$. To see this let $R_1 = xgxs^{-1}$ and $R_2 = x^2shsk$. Then $1 - \theta(s^{-1}xgx) + 1 - \theta(xgxs^{-1})$+ $1 - \theta(xs^{-1}xg) = 1 - \theta(\alpha) + 1 - \theta(\varepsilon)$ $+1 - \theta(\delta)$= $\frac{1}{2}$+ $\frac{1}{2}$+ 1=2. Also for R_2, $1-\theta(x^2shsk)$+ $1-\theta(xshskx)$+ $1-\theta(shskx^2)$+ $1-\theta(skx^2sh)$= $1-\theta(\zeta)$+ $1-\theta(\beta)$+ $1-\theta(\gamma)$+ $1-\theta(\eta)$ = $(1-\frac{1}{2})$+ $(1-1)$ +$(1-0)$+$(1-\frac{1}{2})$=2. That is, the first condition of weakly aspherical weight function holds. Moreover, admissible cycles of weight less than 2 yield the relators g, h, k, gh^{-1}, gk^{-1} and hk^{-1}, all of which contradict the assumptions. Thus the second condition of weakly aspherical weight function is also satisfied and so $\mathcal{P}_1$ is aspherical.

4.2.2 Case 4: (a=b, d=e, c=f only)

$\mathcal{P}_4 = \langle G, y | ayaycydydycy \rangle = \langle G, y, x | x(ay)^{-1}, x^2ca^{-1}xda^{-1}xda^{-1}xca^{-1}x \rangle$
$= \langle G, x | x^3ca^{-1}xda^{-1}xda^{-1}xca^{-1} \rangle = \langle G, x | x^3gxhxhxg \rangle$, where $g = ca^{-1}$ and $h = da^{-1}$ (and so by assumption, $g, h \in G \backslash \{1\}$ and $g \neq h$).

Lemma 4.2.2. *$\mathcal{P}_4$ is aspherical.*

Proof. $\mathcal{P}_4 = \langle G, x | x^3gxhxhxg \rangle = \langle G, x, s | xgxs^{-1}, xshxhs \rangle$. Then the star complex $\mathcal{P}_4^{st}$ of $\mathcal{P}_4$ has the form shown in Figure 4.4, where the edges α, β, γ, δ, ε, ζ and η (respectively) are labelled by 1, 1, h, 1, g, h and 1 (respectively).

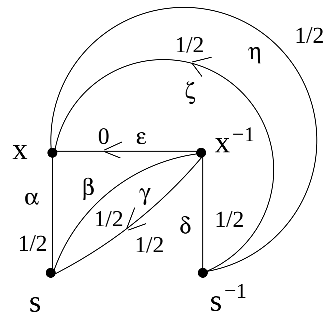


Figure 4.4: Star graph of $\langle G, x, s | xgxs^{-1}, xshxhs \rangle$.

We obtain a weakly aspherical weight function by giving the following weights to the edges of $\mathcal{P}_4^{st}$: $\theta(\alpha) = \theta(\beta) = \theta(\gamma) = \theta(\delta) = \theta(\zeta) = \theta(\eta) = \frac{1}{2}$ and $\theta(\varepsilon) = 0$. Indeed, the first condition of weight test holds (as in Case 1) and there are no possible admissible cycles of $\mathcal{P}_4^{st}$ have weight less than 2.

4.2.3 Case 5: (a=c, b=d only)

$\mathcal{P}_5= \langle G, y | aybyaybyeyfy \rangle= \langle G, y, x | x(ay)^{-1},\ x^2ba^{-1}x^2ba^{-1}xea^{-1}xfa^{-1} \rangle= \langle G, x |$ $x^2gx^2gxhxk \rangle$, where $g = ba^{-1}$, $h = ea^{-1}$ and $k = fa^{-1}$.

Lemma 4.2.3. $\mathcal{P}_5$ *is aspherical.*

Proof. Let $\mathbb{P}$ be a reduced spherical picture over $\mathcal{P}_5$. Then each vertex(disc) in $\mathbb{P}$ has one of the forms given by Figure 4.5. There are (up to inversion) two types of vertices of degree 3 in $\mathbb{P}$ and these are shown in Figure 4.6. Assign the angle function α to the corners of $\mathbb{P}$ as follows: corners within 2-bonds have angle zero. In vertices of degree 3, corners labelled by $1^{\pm 1}$ and $g^{\pm 1}$ have angle π. Each of the remaining corners has angle $\frac{\pi}{2}$ (see Figure 4.6). If $d(v) \geq 4$, then each corner in v not in a 2-bond has angle $\frac{2\pi}{d(v)}$.

Remark 4.2.4. *By assigning α to the corners of $\mathbb{P}$, the following are satisfied:*

(i) There are no consecutive corners with angle π in the boundary of a region Δ of $\mathbb{P}$ (observe that each of the vertices u and v has degree at least 4).

(ii) Since $(2-8)\pi + 4\pi + 4.\frac{\pi}{2} = 0$, positive regions can only have degree 4 or 6.

Let Δ be a positive region. Then $l(\Delta) = h^{-1}11^{-1}w_1$ or $g^{-1}gk^{-1}w_2$ and so $d(\Delta) > 4$. If $d(\Delta) = 6$ then Δ contains three corners each with angle π, for otherwise $c(\Delta) \leq -4\pi + 2\pi + 4.\frac{\pi}{2} = 0$. However each of the $*$-corner and the $\bullet$-corner in Figure 4.6 has angle $\leq \frac{\pi}{2}$. Thus there are no positive regions in Case 5 and so $\mathcal{P}_5$ is aspherical.

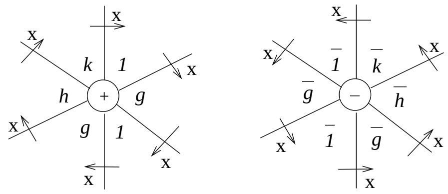


Figure 4.5: $+$ disc and - disc in Case 5.

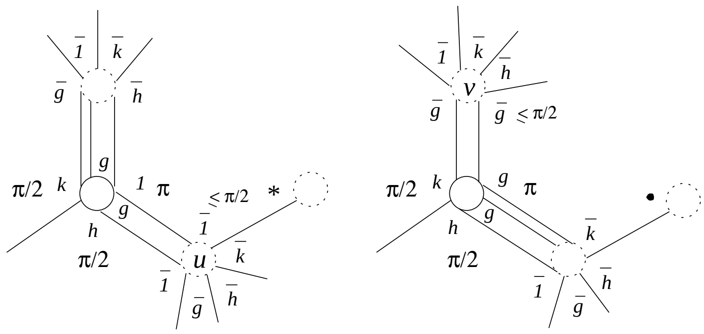


Figure 4.6: Types of vertices of degree 3 and defined angle function.

4.2.4 Case 6: (a=c, b=e only)

$\mathcal{P}_6 = \langle G, y | aybyaydybyfy \rangle = \langle G, y, x | x(ay)^{-1}, xba^{-1}x^2da^{-1}xba^{-1}xfa^{-1}x \rangle = \langle G, x | x^2ba^{-1}x^2da^{-1}xba^{-1}xfa^{-1} \rangle = \langle G, x | x^2gx^2hxgxk \rangle$, where $g = ba^{-1}$, $h = da^{-1}$ and $k = fa^{-1}$ (and so by assumption, $g, h, k \in G \backslash \{1\}$, $g \neq h$, $h \neq k$ and $g \neq k$).

Lemma 4.2.5. *$\mathcal{P}_6$ is aspherical.*

Proof. $\mathcal{P}_6 = \langle G, x | x^2gx^2hxgxk \rangle = \langle G, x, s | xgxs^{-1}, xsxhsk \rangle$. The edges of $\mathcal{P}_6^{st}$ (given by Figure 4.7) α, β, γ, δ, ε, ζ and η (respectively) have the labels 1, 1, h, 1, g, 1 and k (respectively).

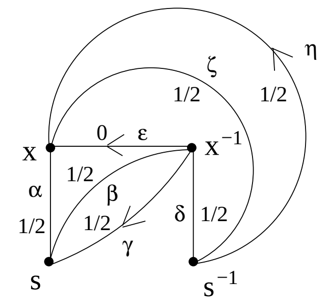


Figure 4.7: Star graph of $\langle G, x, s | xgxs^{-1}, xsxhsk \rangle$.

Assign to $\mathcal{P}_6^{st}$ the weight function θ in the following way: $\theta(\alpha) = \theta(\beta) = \theta(\gamma) = \theta(\delta) = \theta(\zeta) = \theta(\eta) = \frac{1}{2}$, and $\theta(\varepsilon) = 0$. Then θ is weakly aspherical.

4.2.5 Case 7: (a=c, b=e, d=f only)

$\mathcal{P}_7$= $\langle G, y | aybyaydybydy \rangle$= $\langle G, y, x | x(ay)^{-1}, xba^{-1}x^2da^{-1}xba^{-1}xda^{-1}x \rangle$= $\langle G, x | x^2ba^{-1}x^2da^{-1}xba^{-1}xda^{-1} \rangle$= $\langle G, x | x^2gx^2hxgxh \rangle$, where $g = ba^{-1}$ and $h = da^{-1}$ (and so by assumption, $g, h \in G \backslash \{1\}$ and $g \neq h$).

Lemma 4.2.6. $\mathcal{P}_7$ *is aspherical.*

Proof. $\mathcal{P}_7$=$\langle G, x | x^2gx^2hxgxh \rangle$= $\langle G, x, s | xgxs^{-1}, xsxhsh \rangle$. Figure 4.8 shows $\mathcal{P}_7^{st}$, where the edges α, β, γ, δ, ε, ζ and η (respectively) are labelled by 1, 1, h, 1, g, 1 and h (respectively).

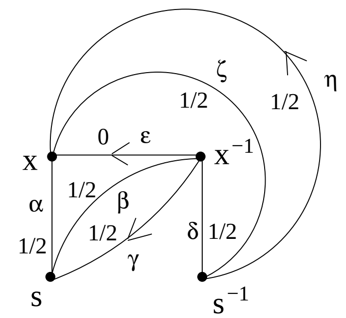


Figure 4.8: Star graph of $\langle G, x, s | xgxs^{-1}, xsxhsh \rangle$.

Assign θ to $\mathcal{P}_7^{st}$ with $\theta(\alpha) = \theta(\beta) = \theta(\gamma) = \theta(\delta) = \theta(\zeta) = \theta(\eta) = \frac{1}{2}$, and $\theta(\varepsilon) = 0$. Then θ is weakly aspherical.

4.2.6 Case 9: (a=d, b=e, c=f only)

$\mathcal{P}_9 = \langle G, y | aybycyaybycy \rangle = \langle G, y, | (aybycy)^2 \rangle$.

Lemma 4.2.7. $\mathcal{P}_9$ *is aspherical.*

Proof. Assume by way of contradiction that $\mathbb{P}$ is a non-trivial reduced spherical picture over $\mathcal{P}_9$. Thus $\mathbb{P}$ contains a region, say Δ, of positive curvature. Therefore $\partial\Delta$ involves at least one vertex v of degree 3, for otherwise $c(\Delta) = c(4, 4, 4, 4) = 0$. However, any possibility for v gives a dipole. For example if one of the a-corners of v is in a 2-bond, then the dipole is shown in Figure 4.9.

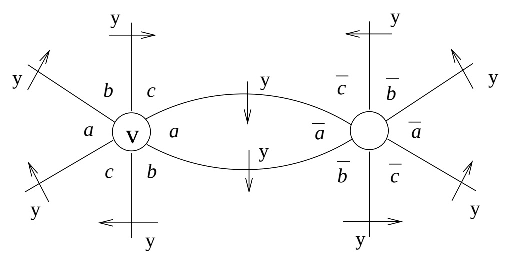


Figure 4.9: Example of a dipole arises from possible positive regions.

4.3 Group II

Here we consider the Cases: 8, 10, 12, 14 and 16.

4.3.1 Case 8: (a= d, b= e only)

$\mathcal{P}_8 = \langle G, y | aybycyaybyfy \rangle = \langle G, y, x | x(ay)^{-1}, xba^{-1}xca^{-1}x^2ba^{-1}xfa^{-1}x \rangle = \langle G, x | x^2ba^{-1}xca^{-1}x^2ba^{-1}xfa^{-1} \rangle$.

Lemma 4.3.1. $\mathcal{P}_8$ *is aspherical if and only if* $|cf^{-1}|$ *is infinite.*

Proof. $\mathcal{P}_8 = \langle G, x | x^2kxgx^2kxh \rangle$, where $k = ba^{-1}$, $g = ca^{-1}$ and $h = fa^{-1}$ (and so $gh^{-1} = cf^{-1}$). Then by assumption, $g, h, k \in G \backslash \{1\}$, $g \neq h$, $h \neq k$ and $g \neq k$. If $|gh^{-1}| < \infty$ then a non-trivial reduced spherical picture over $\mathcal{P}_8$ is given by Figure 4.10, and so assume $|gh^{-1}| = \infty$.

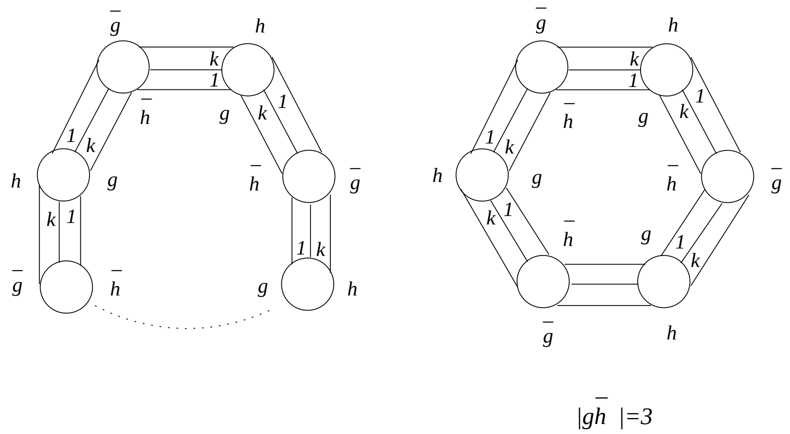


Figure 4.10: $\mathcal{P}_8$; $|gh^{-1}| < \infty$.

Moreover $\mathcal{P}_8 = \langle G, x | x^2kxgx^2kxh \rangle = \langle G, x, s | x^2kxs^{-1}, sgsh \rangle$. The star graph $\mathcal{P}_8^{st}$ of $\mathcal{P}_8$ is shown in Figure 4.11 with labels $\alpha \leftrightarrow 1$, $\beta \leftrightarrow h$, $\gamma \leftrightarrow g$, $\delta \leftrightarrow 1$, $\varepsilon \leftrightarrow k$ and $\zeta \leftrightarrow 1$.

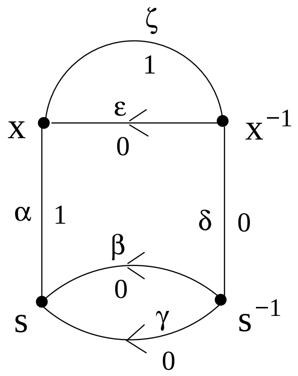


Figure 4.11: Star graph of $\langle G, x, s | x^2kxs^{-1}, sgsh \rangle$.

Define the following weight function: $\theta(\alpha) = \theta(\zeta) = 1$ and $\theta(\beta)= \theta(\gamma)= \theta(\delta)= \theta(\varepsilon) = 0$. The possible relations with weight less than 2 have the forms

$$gk^{-1}(gh^{-1})^l = 1,\ hk^{-1}(hg^{-1})^r = 1 \text{ and } (gh^{-1})^s k^{\pm 1} = 1 \text{ for some integers } l, r, s > 0.$$

The weight function θ is weakly aspherical except if at least one of the above three relations holds. By each of these relations if either g or h belongs to the group generated by gh^{-1} then $gp\{g, h, k\}= gp\{gh^{-1}\}$.

Let $\mathbb{P}$ be a non-trivial reduced spherical picture over $\mathcal{P}_8 = \langle G, x | x^2kxgx^2kxh \rangle$. Then each vertex(disc) in $\mathbb{P}$ has one of the forms given by Figure 4.12(i) and (ii).

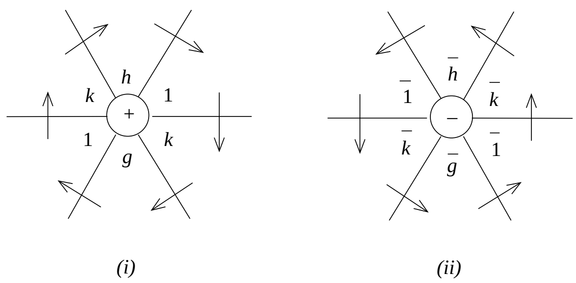


Figure 4.12: $+$ disc, - disc.

Up to inversion, the types of vertices of degree 2 and 3 in $\mathbb{P}$ are given by Figure 4.13.

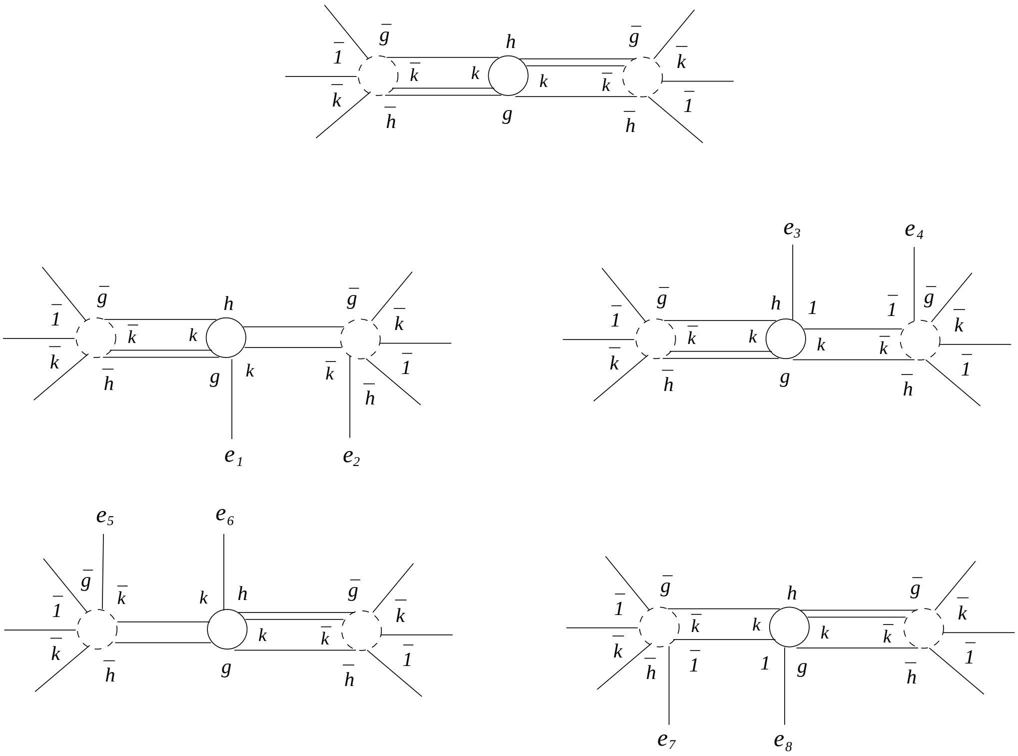


Figure 4.13: Types of vertices of degree 2 and 3.

Now assume (**A**) that the number of 3-bonds in $\mathbb{P}$ is maximal(see Remark 2.1.2).

Remarks 4.3.2.

1. *None of the edges $e_i, 1 \leq i \leq 8$ (see Figure 4.13) is included in a 3-bond. Thus by (**A**), all vertices of degree 3 transform to vertices of degree 2.*

2. *In this case, when counting the degree n of a region Δ we ignore the vertices of degree 2 in $\partial\Delta$. Thus we could have a region of odd degree which includes at least one vertex of degree 2 in it's boundary.*

3. *In $\mathcal{P}_8 = \langle G, x | x^2kxgx^2kxh \rangle$, by symmetry, the roles of g and h can be interchanged.*

Now since $|gh^{-1}| = \infty$, a sequence of vertices of degree 2 in boundary of a region in $\mathbb{P}$ must terminate with a vertex of degree 4 at each side. Up to inversion, all the possibilities are given by Figure 4.14, where $d(v_i) \geq 4$ $(1 \leq i \leq 4)$ and each sequence includes at least one vertex of degree 2.

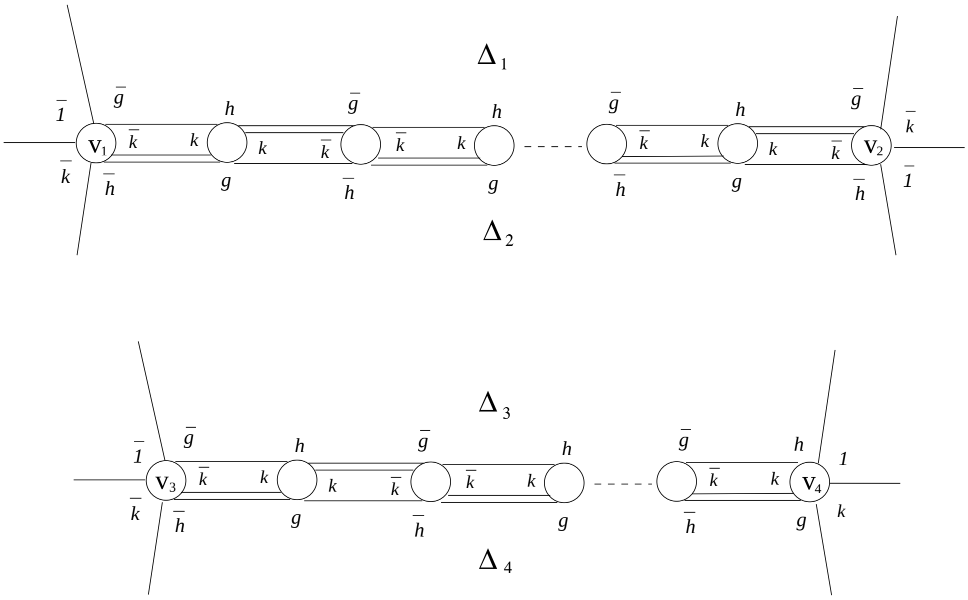


Figure 4.14: All possible sequences of vertices of degree 2 in $\partial\Delta$ and possible positive regions.

If the boundary of a region Δ contains no vertices of degree 2, then $c(\Delta) \leq (2-n)\pi + n.\frac{\pi}{2} = (2 - \frac{n}{2})\pi$. Thus $c(\Delta) > 0$ implies $n < 4$, which is not the case. Therefore a positive region is one of the Δ_i $(1 \leq i \leq 4)$ shown in Figure 4.14. Observe that each of v_i $(1 \leq i \leq 4)$ cannot be adjacent to two vertices of degree 2 in $\partial\Delta_i$ $(1 \leq i \leq 4)$. Therefore $c(\Delta_i) > 0$ implies $d(\Delta_i) = 3$ $(1 \leq i \leq 4)$. However, no two positive discs are adjacent and the same holds for negative discs. Thus $d(\Delta_i) \geq 4$ for $i = 3, 4$. Therefore there are two possibilities for regions of positive curvature given by Figure 4.15, where $d(v_i) \geq 4$ for $1 \leq i \leq 6$. The $* - corner$ is labelled by 1 or k. If $(g^{-1}h)^r g^{-1} = 1$ $(r > 0)$, then $gp\{g,h,k\} = gp\{gh^{-1}\}$. Thus by Lemma 1.3.1, $\mathcal{P}_8$ is aspherical. Hence the $* - corner$ is labelled by k.

Similarly the $\bullet - corner$ is labelled by k. Now assume $k(g^{-1}h)^r g^{-1} = 1 (r > 0)$. If $k(h^{-1}g)^s h^{-1} = 1$ or $k^{-1}(hg^{-1})^m h = 1$ $(s, m > 0)$, then $(g^{-1}h)^r g^{-1} h (g^{-1}h)^s = 1$ or $(hg^{-1})^m h (g^{-1}h)^r g^{-1} = 1$, contradicting $|gh^{-1}| = \infty$. Therefore positive regions could only be of one type. By Remark 4.3.2(3), we may assume that positive regions are of Type 1 only as shown in Figure 4.16. Distribute the curvature as shown.

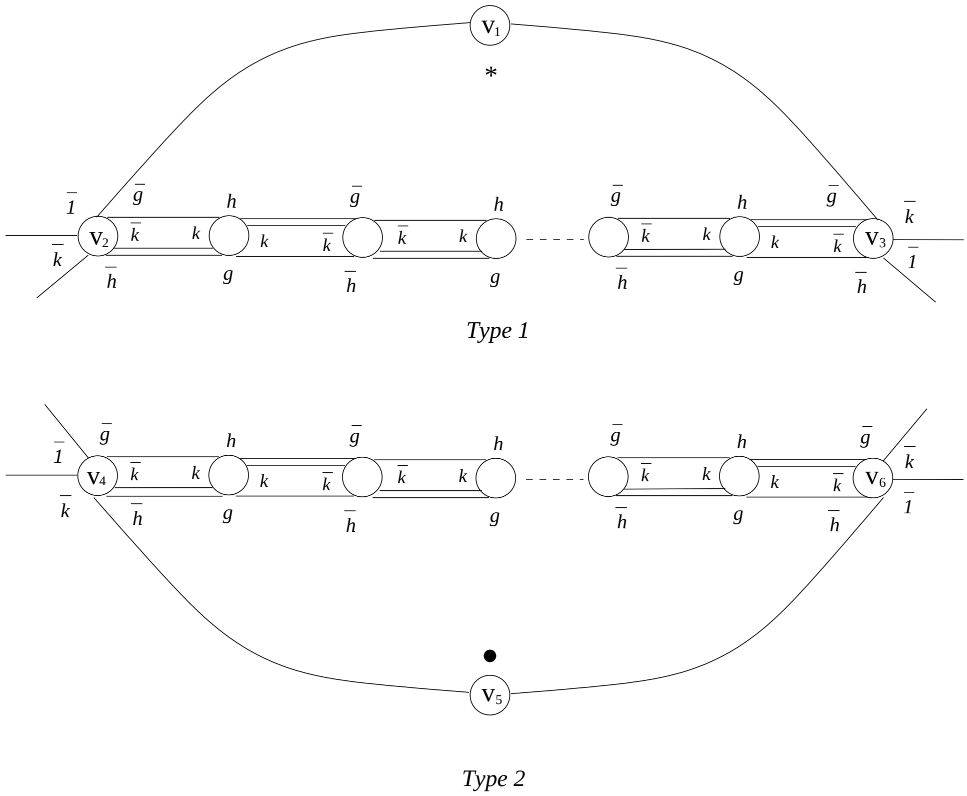


Figure 4.15: Positive regions.

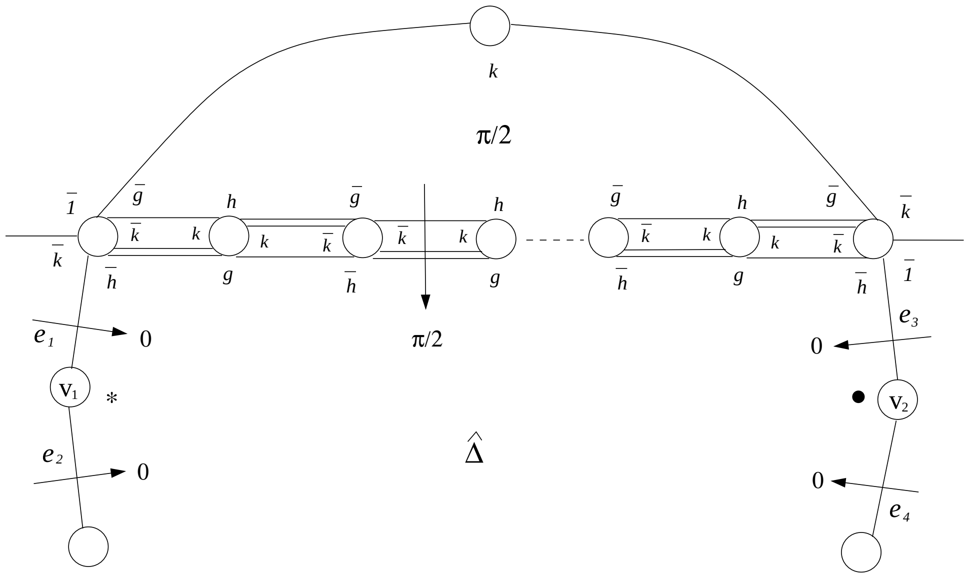


Figure 4.16: Only one type of positive region.

Remark 4.3.3. *In Figure 4.16, neither the $*-$corner nor $\bullet-$corner is labelled by h and so $\hat{\Delta}$ receives 0 across the edges e_i ($1 \leq i \leq 4$).*

Let $d(\hat{\Delta}) = n$. By inspection, in Figure 4.16, $d(v_i) \geq 4$ (i=1,2). From above $v_1 \neq v_2$ and so $n > 3$. Thus by Remark 4.3.3, $n \geq 2 + 2\Gamma_2$. Hence $c^*(\hat{\Delta}) \leq (2-n)\pi + n.\frac{\pi}{2} + \Gamma_2.\frac{\pi}{2}$ $=\pi(2 - \frac{n}{2} + \frac{\Gamma_2}{2})$. Thus $c^*(\hat{\Delta}) > 0$ implies $\Gamma_2 = 1$. Now, $c^*(\hat{\Delta}) \leq (2-n)\pi + n.\frac{\pi}{2} + [\frac{n}{3}].\frac{\pi}{2}$. Thus $c^*(\hat{\Delta}) \leq 0$ for $n \geq 5$ and so $n = 4$. The vertices v_1 and v_2 are positive discs and so cannot be adjacent. Thus at least one vertex of degree 2 lies in $\partial\hat{\Delta}$ between v_1 and v_2. By inspection, the possible labelling of $\hat{\Delta}$ implies $|gh^{-1}| < \infty$, a contradiction. Therefore if $|gh^{-1}| = \infty$ then $\mathcal{P}_8$ is aspherical. This completes the proof.

4.3.2 Case 10: (a=b=c, d=f only)

$\mathcal{P}_{10}$= $\langle G, y | ayayaydyeydy \rangle$= $\langle G, y, x | x(dy)^{-1}, ad^{-1}xad^{-1}xad^{-1}x^2ed^{-1}x^2 \rangle$= $\langle G, x | x^2ed^{-1}x^2ad^{-1}xad^{-1}xad^{-1} \rangle$.

Lemma 4.3.4. *$\mathcal{P}_{10}$ is not aspherical if and only if ed^{-1}=$(ad^{-1})^2$ and $|ad^{-1}| < \infty$.*

Proof. $\mathcal{P}_{10}= \langle G, x | x^2hx^2gxgxg\rangle = \langle G, x, s | x^2s^{-1}, shsgxgxg\rangle$, where $h = ed^{-1}$ and $g = ad^{-1}$ (and so by assumption, $g, h \in G\backslash\{1\}$ and $g \neq h$).

The star graph $\mathcal{P}_{10}^{st}$ is given by Figure 4.17, where $\alpha \leftrightarrow 1$, $\beta \leftrightarrow g$, $\gamma \leftrightarrow h$, $\delta \leftrightarrow 1$, $\varepsilon \leftrightarrow 1$, $\zeta \leftrightarrow g$ and $\eta \leftrightarrow g$.

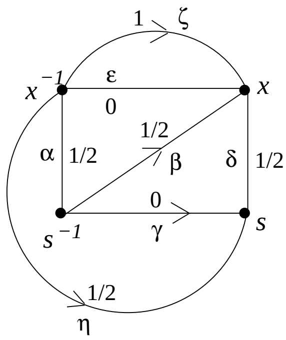


Figure 4.17: Star graph of $\langle G, x, s | x^2s^{-1}, shsgxgxg\rangle$.

Define the following weight function θ on $\mathcal{P}_{10}^{st}$: $\theta(\alpha) = \theta(\beta) = \theta(\delta) = \theta(\eta) = \frac{1}{2}$, $\theta(\gamma) = \theta(\varepsilon) = 0$ and $\theta(\zeta) = 1$. Then $\mathcal{P}_{10}$ is aspherical except if $h = g^2$ (obtained from the closed path $\varepsilon\beta^{-1}\gamma\eta^{-1}$). So assume $h = g^2$ and we may assume that $|g| < \infty$, for otherwise Lemma 1.3.1 applies and so $\mathcal{P}_{10}$ is aspherical. Figure 4.18 provides non-trivial reduced spherical pictures over $\mathcal{P}_{10}$. As an example, the case $g^5 = 1$ is done by Figure 4.19.

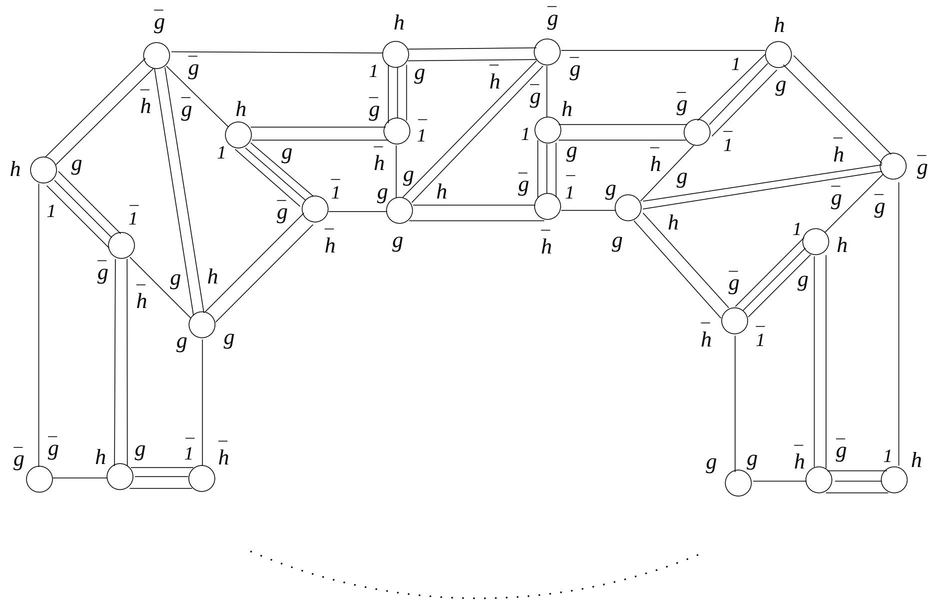


Figure 4.18: $\mathcal{P}_{10}$; $h = g^2$ and $|g| < \infty$.

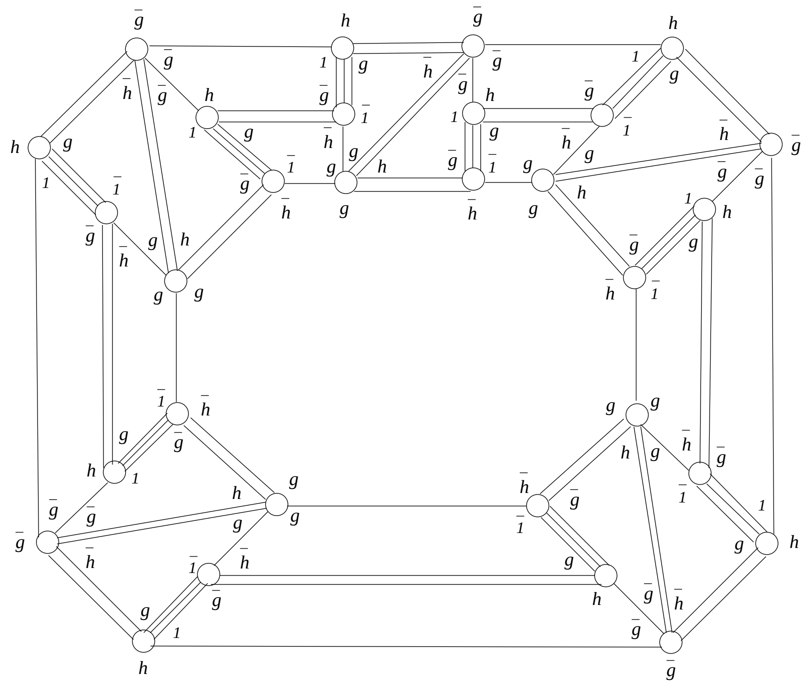


Figure 4.19: $\mathcal{P}_{10}$; $h = g^2$ and $|g| = 5$.

4.3.3 Case 12: (a=b=c, d=e=f only)

$\mathcal{P}_{12} = \langle G, y | ayayaydydydy \rangle = \langle G, y, x | x(ay)^{-1}, x^3da^{-1}xda^{-1}xda^{-1}x \rangle$
$= \langle G, x | x^4da^{-1}xda^{-1}xda^{-1} \rangle$.

Lemma 4.3.5. *$\mathcal{P}_{12}$ is aspherical if and only if $|da^{-1}| = \infty$.*

Proof. $\mathcal{P}_{12} = \langle G, x | x^4da^{-1}xda^{-1}xda^{-1} \rangle = \langle G, x | x^3(xda^{-1})^3 \rangle$. The result now follows from Lemma 1.3.2.

4.3.4 Case 14: (a= b= d, c= f only)

The relative presentation $\mathcal{P}_{14} = \langle G, y | ayaycyayeycy \rangle$
$= \langle G, y, x | x(ay)^{-1}, x^2ca^{-1}x^2ea^{-1}xca^{-1}x \rangle = \langle G, x | x^3ca^{-1}x^2ea^{-1}xca^{-1} \rangle$.

Lemma 4.3.6. *$\mathcal{P}_{14}$ is aspherical if and only if $|ea^{-1}|$ is infinite.*

Proof. $\mathcal{P}_{14} = \langle G, x | x^3gx^2hxg \rangle$, where $g = ca^{-1}$ and $h = ea^{-1}$ (and so by assumption, $g, h \in G\backslash\{1\}$ and $g \neq h$). If $|h| < \infty$ then the diagram given by Figure 4.20 shows that $\mathcal{P}_{14}$ is not aspherical.

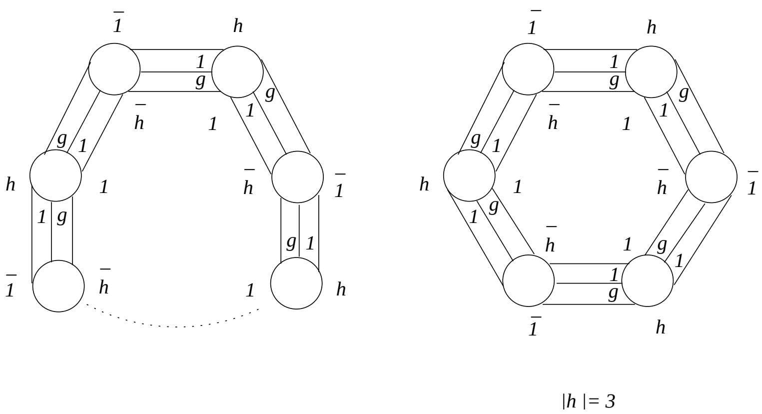


Figure 4.20: $\mathcal{P}_{14}$; $|h| < \infty$.

Now suppose that $|h|$ is infinite. The presentation $\mathcal{P}_{14} = \langle G, x | x^3gx^2hxg \rangle =$

$\langle G, x, s | x^2 s^{-1}, xsgshxg \rangle$. Then the star graph $\mathcal{P}_{14}^{st}$ is given by Figure 4.21, where $\alpha \leftrightarrow 1$, $\beta \leftrightarrow g$, $\gamma \leftrightarrow h$, $\delta \leftrightarrow 1$, $\varepsilon \leftrightarrow g$, $\zeta \leftrightarrow 1$ and $\eta \leftrightarrow 1$.

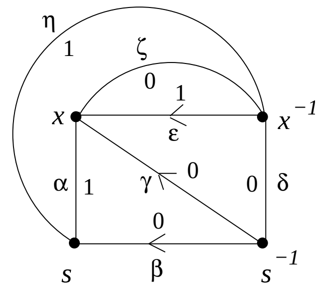


Figure 4.21: Star graph of $\langle G, x, s | x^2 s^{-1}, xsgshxg \rangle$.

We may assign the following weights to $\mathcal{P}_{14}^{st}$: $\theta(\alpha) = \theta(\varepsilon) = \theta(\eta) = 1$ and $\theta(\beta) = \theta(\gamma) = \theta(\delta) = \theta(\zeta) = 0$.

Clearly, the first condition of weight test holds. If $gh^l = 1$, for an integer $l \neq 0$, then both $g = h^{-l}$ and h are contained in an infinite cyclic group. Thus Lemma 1.3.1 applies and so $\mathcal{P}_{14}$ is aspherical. Therefore, by assigning the weight function θ, $\mathcal{P}_{14}$ is aspherical. (Note that the only possible admissible cycle of weight less than 2 gives $gh^l = 1$ and we are done).

4.3.5 Case 16: (a= b= d, c= e= f only)

The relative presentation $\mathcal{P}_{16}$= $\langle G, y | ayaycyaycycy \rangle$
$= \langle G, y, x | x(ay)^{-1}, x^2ca^{-1}x^2ca^{-1}xca^{-1}x \rangle$= $\langle G, x | x^3ca^{-1}x^2ca^{-1}xca^{-1} \rangle$.

Lemma 4.3.7. *$\mathcal{P}_{16}$ is aspherical if and only if $|ca^{-1}|$ is infinite.*

Proof. $\mathcal{P}_{16}$= $\langle G, x | x^3gx^2gxg \rangle$, where $g = ca^{-1}$. If $|g|$ is infinite then Lemma 1.3.1 applies, and so $\mathcal{P}_{16}$ is aspherical. So assume otherwise. Then the diagram given by Figure 4.22 is a reduced sphere over $\mathcal{P}_{16}$. As an example, the case $|g| = 3$ is provided.

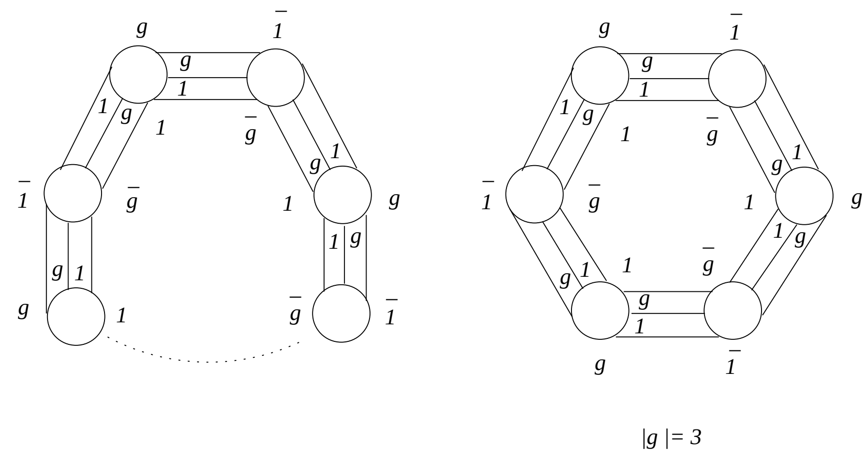


Figure 4.22: $\mathcal{P}_{16}$; $|g| < \infty$.

4.4 Group III

Here we consider the Cases: 17 and 19. Consider the group presentation $\mathcal{P}_*= \langle H, x|w(x^k, H)\rangle$, where $w(x^k, H)$ is a cyclically reduced word in x^k and the elements of the group H. Observe that $\mathcal{P}_*= \langle H, x|w(x^k, H)\rangle= \langle H, s|w(s, H)\rangle *_{s=x^k} \langle x|x^{km}\rangle$, where $m = |s|$ and $m = 0$ if and only if $|s| = \infty$ (this point of view is useful because of the following theorem)(see [17] for example for definition of free product with amalgamation).

4.4.1 Theorem 4.4.1

Theorem 4.4.1. *(i) If $\mathcal{P}_1=\langle H, s|w(s, H)\rangle$ is aspherical and $|s| = \infty$ in $\mathcal{P}_1$, then $\mathcal{P}_*$ is aspherical.*
(ii) If $\mathcal{P}_1=\langle H, s|w(s, H)\rangle$ is not aspherical, then $\mathcal{P}_*$ is not aspherical.

Proof. (i) By Tietze transformations $\mathcal{P}_*=\langle H, x, s|sx^{-k}, w(s, H)\rangle$. Let $R_1 = sx^{-k}$ and $R_2 = w(s, H)$. Assume that $\mathcal{P}_1$ is aspherical and $|s| = \infty$. Thus, any reduced sphere $\mathbb{P}$ over $\mathcal{P}_*$ should involve a disc whose label is a cyclic permutation of $R_1^{\pm 1}$.

We will refer to such discs as R_1-discs.

Observe that the following two properties are satisfied in $\mathcal{P}_*$.

Property 1: The letters $x^{\pm 1}$ appear in $R_1^{\pm 1}$ but not in $R_2^{\pm 1}$. This property implies that x-edges can only be common edges between R_1-discs.

Property 2: The letter s occurs only once in R_1 and this implies that s-edges cannot be common edges between R_1-discs, for otherwise $\mathbb{P}$ is not reduced.

Let $\mathcal{S}_{R_1}$ be the maximal set of connected (but not necessarily simply connected) subpictures $K_i, i \in \mathbb{Z}$ in $\mathbb{P}$ consists of R_1-discs and $\bar{R}_1$-discs, where $\bar{R}_1$-discs are the inverses of R_1-discs. As a result of the above properties, the boundary label of each K_i is s^{r_i} (see Figure 4.23), which implies that $|s| < \infty$, a contradiction. Hence $\mathcal{P}_*$ is aspherical.

(ii) By Tietze transformations $\mathcal{P}_*{=}\langle H, x | w(x^k, H)\rangle{=}\langle H, x, s | sx^{-k}, w(s, H)\rangle$. Let $\mathbb{P}$ be a non-trivial reduced sphere over $\mathcal{P}_1$, then $\mathbb{P}$ is also a non-trivial reduced sphere over $\mathcal{P}_*$ by using the relator $R_2 = w(s, H)$ only in $\mathcal{P}_*$.

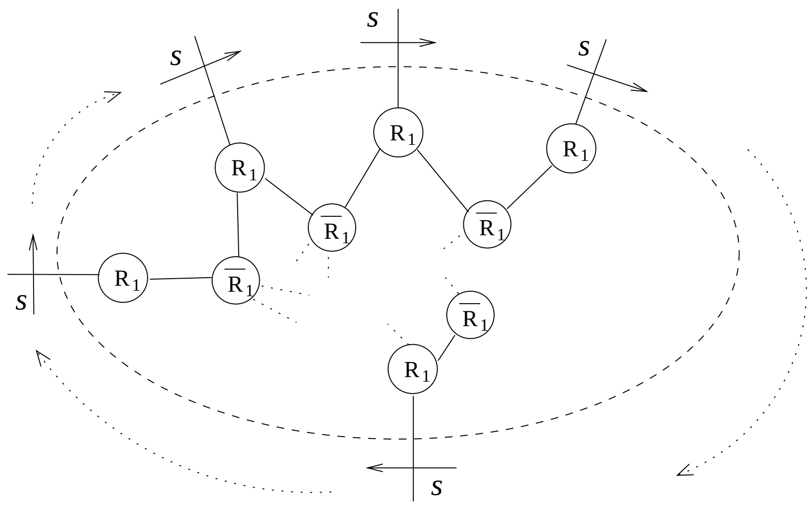


Figure 4.23: Example of K_i.

4.4.2 Case 17: (a=c=e)

$\mathcal{P}_{17}= \langle G, y|aybyaydyayfy\rangle=\langle G, y, x|x(ay)^{-1}, xba^{-1}x^2da^{-1}x^2fa^{-1}x\rangle$
$= \langle G, x|x^2gx^2hx^2k\rangle$, where $g = ba^{-1}$, $h = da^{-1}$ and $k = fa^{-1}$.

Corollary 4.4.2. *$\mathcal{P}_{17}$ is aspherical if and only if $\hat{\mathcal{P}}= \langle G, s|sgshsk\rangle$ is aspherical. (Note that a complete classification of when $\hat{\mathcal{P}}$ is aspherical is obtained in [5]).*

Proof. Observe that $\mathcal{P}_{17}= \langle G, x|x^2gx^2hx^2k\rangle= \langle G, s|sgshsk\rangle *_{s=x^2} \langle x|x^{2m}\rangle$, where $m = |s|$ and $m = 0$ if and only if $|s| = \infty$. Assume that $\hat{\mathcal{P}}$ is aspherical and so $|s| = \infty$ (see Lemma 2.2.9). Therefore the result follows from Theorem 4.4.1 (i) and (ii).

4.4.3 Case 19: (a=b=d=e)

$\mathcal{P}_{19}=\langle G, y|ayaycyayayfy\rangle= \langle G, y, x|x(ay)^{-1}, x^2ca^{-1}x^3fa^{-1}x\rangle= \langle G, x|x^3gx^3h\rangle$, where $g = ca^{-1}$ and $h = fa^{-1}$.

Corollary 4.4.3. *$\mathcal{P}_{19}$ is aspherical if and only if $|gh^{-1}|$ is infinite.*

Proof. The relative presentation $\mathcal{P}_{19}= \langle G, x|x^3gx^3h\rangle= \langle G, s|sgsh\rangle *_{s=x^3} \langle x|x^{3m}\rangle$, where $m = |s|$ and $m = 0$ if and only if $|s| = \infty$. Observe $\langle G, s|sgsh\rangle$ is aspherical if and only if $|gh^{-1}|$ is infinite. Also, note that if $\langle G, s|sgsh\rangle$ is aspherical then $|s| = \infty$ and so the result follows from Theorem 4.4.1.

4.5 Group IV

In this section we deal with the cases which includes some exceptional subcases. These cases are: 2, 3, 11, 13, 15 and 18.

4.5.1 Case 2: (a=b, c=d, e=f only)

The presentation $\mathcal{P}_2 = \langle G, y | (ay)^2(cy)^2(ey)^2 \rangle$. Let $g = ca^{-1}$ and $h = ea^{-1}$, then by assumption $g \neq h$ and $g, h \in G \backslash \{1\}$. Before stating the result, we state the following exceptional case.

(**E**) $g = h^{-1}$ and $|h| = 3$.

Proposition 4.5.1. *Suppose that (**E**) does not hold. Then $\mathcal{P}_2$ is not aspherical if and only if $\frac{1}{|a^{-1}c|} + \frac{1}{|a^{-1}e|} + \frac{1}{|c^{-1}e|} > 1$, where $\frac{1}{\infty} := 0$.*

Proof. $\mathcal{P}_2$ is a special case of the relative presentation considered in [19], where Theorem 2 says that $\mathcal{P}_2$ is aspherical except if one of the following holds.

1. $\frac{1}{|a^{-1}c|} + \frac{1}{|a^{-1}e|} + \frac{1}{|c^{-1}e|} > 1$, where $\frac{1}{\infty} := 0$.
2. $ae^{-1}ac^{-1} = 1$ and $|ac^{-1}| < \infty$ [by symmetry it is the same as either of the two cases: $ca^{-1}ce^{-1} = 1$ and $|ce^{-1}| < \infty$ or $ec^{-1}ea^{-1} = 1$ and $|ea^{-1}| < \infty$].

With the first condition it has been shown in [19] that there is a non-trivial reduced spherical picture over $\mathcal{P}_2$. Therefore we consider the second condition, that is, $\mathcal{P}_2 = \langle G, y | (ay)^2(cy)^2(ey)^2 \rangle = \langle G, y, x | x(ay)^{-1}, x^2ca^{-1}xca^{-1}xea^{-1}xea^{-1}x \rangle = \langle G, x | x^3gxgxhxh \rangle$, where $gh = 1$ and $|g| < \infty$.

Let $\mathbb{P}$ be a non-trivial reduced spherical picture over $\mathcal{P}_2 = \langle G, x | x^3gxgxhxh \rangle$. Then each vertex(disc) in $\mathbb{P}$ has one of the forms given by Figure 4.24(i) and (ii).

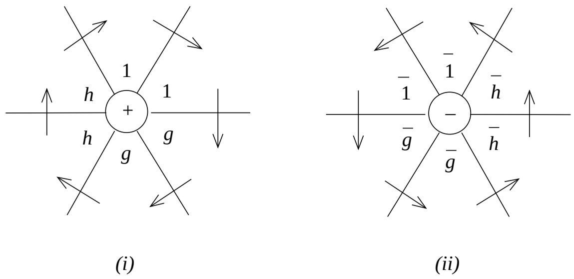


Figure 4.24: $+$ disc, - disc.

Up to inversion, the types of vertices of degree 3 in $\mathbb{P}$ are given by Figure 4.25.

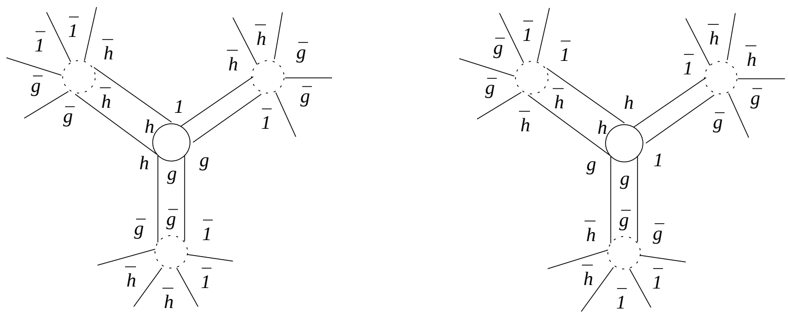


Figure 4.25: Types of vertices of degree 3.

Define an *angle function* α on $\mathbb{P}$ as follows. Corners within 2-bonds have angle zero. In vertices of degree 3, corners (not within 2-bonds) labelled by $1^{\pm 1}$ have angle π, each of the other two corners has angle $\frac{\pi}{2}$ (see Figure 4.26). If $d(v) \geq 4$, then each corner in v not in a 2-bond has angle $\frac{2\pi}{d(v)}$.

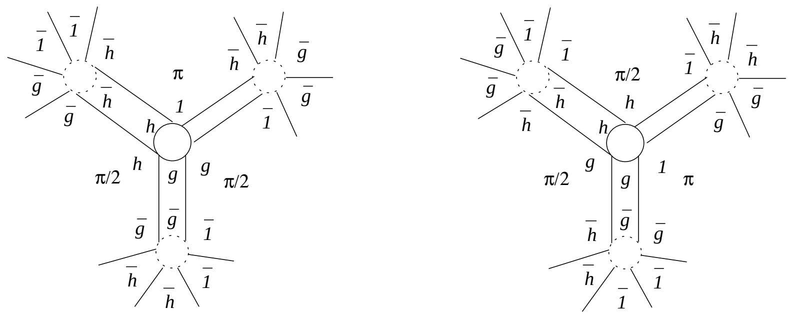


Figure 4.26: Angle function α for vertices of degree 3.

Remark 4.5.2. *By assigning the angle function α to the corners of $\mathbb{P}$, the following are satisfied:*

(i) Clearly, there are no consecutive corners with angle π in the boundary of a region Δ of $\mathbb{P}$ and so positive regions can only have degree 4 or 6.

(ii) If Δ is a positive 4-region then it contains at least one corner labelled by $1^{\pm 1}$ with angle π.

(iii) If Δ is a positive 6-region then it contains three occurrences of $1^{\pm 1}$- corners each with angle π.

By Remark 4.5.2 and Figure 4.26, if Δ is a positive region then $l(\Delta) = g^{-1}1g^{-1}w_1$ or $h^{-1}1h^{-1}w_2$ or $h^{-1}1h^{-1}1h^{-1}1$. Since $g \neq h$ the only allowed labelling of Δ implies $h^3 = 1$. Therefore if $g = h^{-1}$ and $h^3 \neq 1$ then $\mathcal{P}_2$ is aspherical.

4.5.2 Case 3: (a=b, c=e, d=f only)

$\mathcal{P}_3 = \langle G, y | ayaycydycydy \rangle = \langle G, y, x | x(cy)^{-1}, ac^{-1}xac^{-1}x^2dc^{-1}x^2dc^{-1}x \rangle =$
$\langle G, x | x^2dc^{-1}x^2dc^{-1}xac^{-1}xac^{-1} \rangle = \langle G, x | x^2gx^2gxhxh \rangle =$
$\langle G, x, s | x^2s^{-1}, sgsgxhxh \rangle$, where $g = dc^{-1}$ and $h = ac^{-1}$ (and so by assumption, $g, h \in G \backslash \{1\}$ and $g \neq h$).

Lemma 4.5.3. *Let H be the subgroup of G generated by g and h. Then $\mathcal{P}_3$ is aspherical unless one of the following holds.*

1. $g = h^{-1}$.
2. $g = h^2$ *or* $h = g^2$.
3. *H is cyclic of order 6, that is, one of the following six cases holds:*
   (i) $g = h^4$ *and* $|h| = 6$.
   (ii) $g = h^3$ *and* $|h| = 6$.
   (iii) $g = (gh^{-1})^3$, $h = (gh^{-1})^2$ *and* $|gh^{-1}| = 6$.
   (iv) $h = g^4$ *and* $|g| = 6$.
   (v) $h = g^3$ *and* $|g| = 6$.
   (vi) $h = (hg^{-1})^3$, $g = (hg^{-1})^2$ *and* $|gh^{-1}| = 6$.
4. $\frac{1}{|g|} + \frac{1}{|gh^{-1}|} + \frac{1}{|h|} > 1$.

Proof. The star graph of $\langle G, x, s | x^2 s^{-1}, sgsgxhxh\rangle$ is shown in Figure 4.27, where $\alpha \leftrightarrow 1$, $\beta \leftrightarrow g$, $\gamma \leftrightarrow g$, $\delta \leftrightarrow 1$, $\varepsilon \leftrightarrow 1$, $\zeta \leftrightarrow h$ and $\eta \leftrightarrow h$.

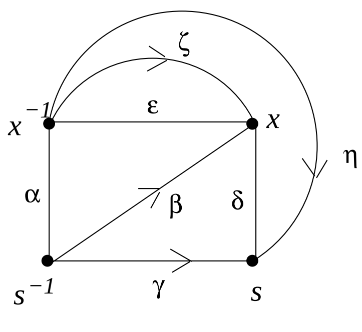


Figure 4.27: Star graph of $\langle G, x, s | x^2 s^{-1}, sgsgxhxh\rangle$.

Assign to $\mathcal{P}_3^{st}$ the following weight function θ: $\theta(\alpha) = \theta(\varepsilon) = 0$, $\theta(\beta) = \theta(\gamma) = \theta(\zeta) = \theta(\eta) = \frac{1}{2}$, and $\theta(\delta) = 1$. Then the possible relations of weight less than 2 are:

$$gh,\, h^2, h^3, g^2, g^3, hg^{\pm 2} \text{ and } gh^{\pm 2}.$$

We can assume that at least one of these relations hold in H, else θ is a weakly aspherical function.

Assume that $g \neq h^{-1}$, $g \neq h^2$ and $h \neq g^2$. We will show that either $\mathcal{P}_3$ is aspherical or one of the Conditions 3 or 4 holds. This leaves the following cases to be considered:

$$gh^2 = 1;\ g^2 = 1;\ g^3 = 1$$
$$hg^2;\ h^2 = 1;\ h^3 = 1.$$

Provided that $\theta(\alpha) = \theta(\varepsilon) = 0$ in all the weight functions used in the cases $gh^2 = 1$, $g^2 = 1$ and $g^3 = 1$ (respectively), we get similar results for the cases $hg^2 = 1$, $h^2 = 1$ and $h^3 = 1$ (respectively). This is done by exchanging the values of $\theta(\beta)$ and $\theta(\zeta)$ together with exchanging the values of $\theta(\gamma)$ and $\theta(\eta)$ in all of the defined weight functions for the cases $g^2 = 1$, $g^3 = 1$ and $gh^2 = 1$. This is because $\mathcal{P}_3^{st}$ can be viewed as shown in Figure 4.28, where both of the edges α and ε are labelled with the identity of H. Therefore the roles of g and h can be interchanged.

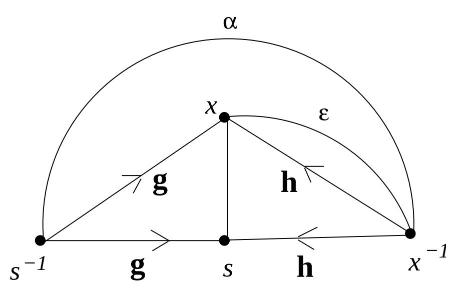


Figure 4.28: Star graph of $\langle G, x, s | x^2 s^{-1}, sgsgxhxh \rangle$.

Let $gh^2 = 1$, then define a new weight function on $\mathcal{P}_3^{st}$ with $\theta(\alpha) = \theta(\varepsilon) = 0$, $\theta(\beta) = \theta(\gamma) = \frac{1}{2}$, $\theta(\zeta) = \frac{3}{4}, \theta(\eta) = \frac{1}{4}$ and $\theta(\delta) = 1$. Then the admissible cycles of weight less than 2 give either $g^2 = 1$, $g^3 = 1$ or $(gh^{-1})^2 = 1$. In all cases H is cyclic generated by h of order $4, 6$ and 6 respectively. However, the first possibility contradicts $g \neq h^2$ and the other two possibilities are $3(i)$. Thus we may assume that $gh^2 \neq 1$ and so $hg^2 \neq 1$ by symmetry.

Now assume that $g^2 = 1$ and assign another weight function to $\mathcal{P}_3^{st}$ with $\theta(\alpha) = \theta(\gamma) = \theta(\varepsilon) = 0$, $\theta(\beta) = \theta(\delta) = 1$ and $\theta(\zeta) = \theta(\eta) = \frac{1}{2}$. Then if θ is not a weakly aspherical function one of the following holds in H:

$$h^2 = 1,\ h^3 = 1,\ (gh^{-1})^2 = 1,\ (gh^{-1})^3 = 1,\ gh^{-3} = 1 \text{ or } gh^{-1}gh^{-2} = 1.$$

If $h^2 = 1$ and $|gh^{-1}|$ is finite then $\frac{1}{|g|} + \frac{1}{|gh^{-1}|} + \frac{1}{|h|} > 1$, while if $|gh^{-1}|$ is infinite then define the following weakly aspherical weight function: $\theta(\alpha) = \theta(\gamma) = \theta(\varepsilon) = \theta(\eta) = 0$ and $\theta(\beta) = \theta(\delta) = \theta(\zeta) = 1$. Similarly, if $(gh^{-1})^2 = 1$ and $|h|$ is finite then $\frac{1}{|g|} + \frac{1}{|gh^{-1}|} + \frac{1}{|h|} > 1$, while if $|h|$ is infinite then assign to $\mathcal{P}_3^{st}$ the following weakly aspherical weight function: $\theta(\alpha) = \theta(\gamma) = \theta(\varepsilon) = \theta(\zeta) = 0$ and $\theta(\beta) = \theta(\delta) = \theta(\eta) = 1$.

If $gh^{-3} = 1$ then H is cyclic of order 6 generated by h and we get $3(ii)$. Moreover if $gh^{-1}gh^{-2} = 1$ then H is cyclic of order 6 generated by gh^{-1} and we get $3(iii)$.

If $h^3 = 1$ then assign the following weight function: $\theta(\alpha) = \theta(\gamma) = \theta(\varepsilon) = 0$, $\theta(\beta) = \theta(\delta) = 1$, $\theta(\zeta) = \frac{2}{3}$ and $\theta(\eta) = \frac{1}{3}$. Then one of the following is satisfied: $(gh^{-1})^2$, $(gh^{-1})^3$, $(gh^{-1})^4$ or $(gh^{-1})^5$. Any of these four relations imply that $\frac{1}{|g|} + \frac{1}{|gh^{-1}|} + \frac{1}{|h|} > 1$.

If $(gh^{-1})^3 = 1$ then define the following weight function with $\theta(\alpha) = \theta(\gamma) = \theta(\varepsilon) = 0$, $\theta(\beta) = \theta(\delta) = 1$, $\theta(\zeta) = \frac{1}{3}$ and $\theta(\eta) = \frac{2}{3}$. Here the possible admissible cycles of weight less than 2 are: h^2, h^3, h^4, h^5 or $g^{-1}h^3$. Any of the first four relations give that $\frac{1}{|g|} + \frac{1}{|gh^{-1}|} + \frac{1}{|h|} > 1$. The last relation imply that H is cyclic of order 6 generated by h. Thus we may assume $g^2 \neq 1$ and by symmetry $h^2 \neq 1$.

If $g^3 = 1$ then assign the following weight function such that $\theta(\alpha) = \theta(\varepsilon) = 0$, $\theta(\beta) = \theta(\zeta) = \frac{2}{3}$, $\theta(\gamma) = \theta(\eta) = \frac{1}{3}$ and $\theta(\delta) = 1$. So the only allowed relation in H is $(gh^{-1})^2$. For this case ($g^3 = 1 = (gh^{-1})^2$) define a weight function in the following way: $\theta(\alpha) = \theta(\varepsilon) = 0$, $\theta(\beta) = \theta(\eta) = \frac{2}{3}$, $\theta(\gamma) = \theta(\zeta) = \frac{1}{3}$ and $\theta(\delta) = 1$. Possible admissible cycles of weight less than 2 give the following relations: h^3, h^4, h^5. Each of which gives $\frac{1}{|g|} + \frac{1}{|gh^{-1}|} + \frac{1}{|h|} > 1$. Thus by symmetry $h^3 \neq 1$. This completes the proof.

Lemma 4.5.4. *In $\mathcal{P}_3$ the roles of h and $h^{-1}g$ can be interchanged.*

Proof. In $\mathcal{P}_3$, substitute $y = xg$. Then the relation transforms to $yg^{-1}y^2g^{-1}y^2g^{-1}hyg^{-1}h$. Using inversion, cyclic permutation and replacing y^{-1} by x we get $\acute{\mathcal{P}}_3 = \langle G, x | x^2gx^2gxh^{-1}gxh^{-1}g\rangle$. Thus all the results that apply to $\mathcal{P}_3$ also apply to $\acute{\mathcal{P}}_3$ and we can work modulo $(g, h) \leftrightarrow (g, h^{-1}g)$. This completes the proof.

Let $\mathbb{P}$ be a non-trivial reduced spherical picture over $\mathcal{P}_3 = \langle G, x | x^2gx^2gxhxh\rangle$. Then each vertex(disc) in $\mathbb{P}$ has one of the forms given by Figure 4.29(i) and (ii).

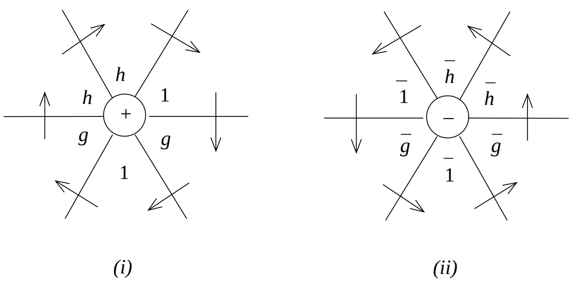


Figure 4.29: + disc and - disc.

There are (up to inversion) six types of vertices of degree 3 in $\mathbb{P}$ and these are shown in Figure 4.30.

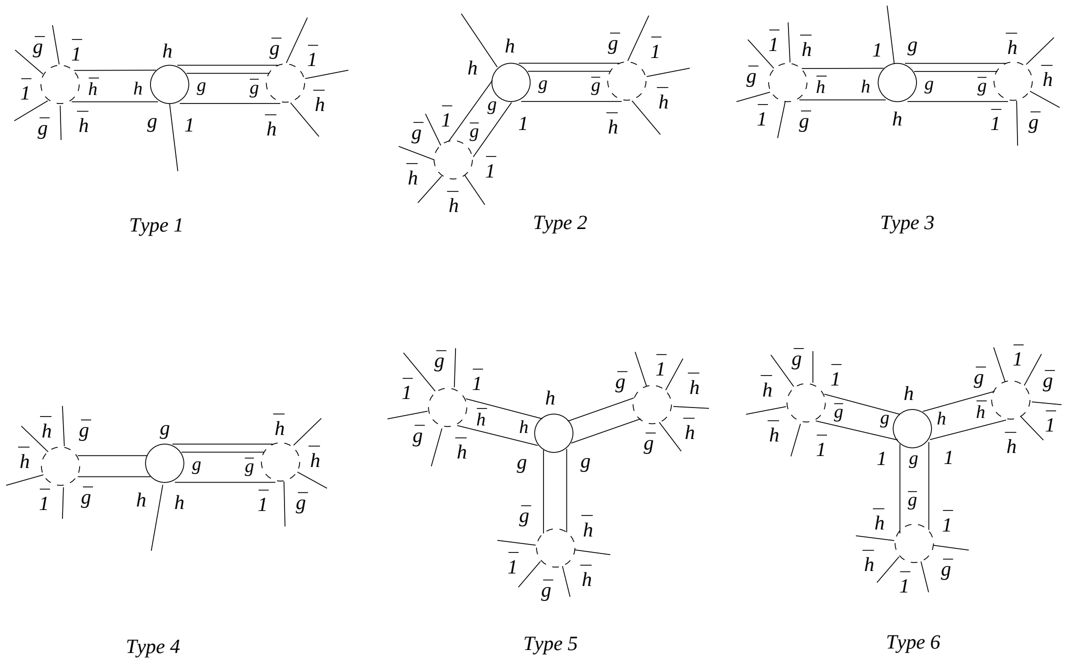


Figure 4.30: Types of vertices of degree 3.

For the proofs, define an *angle function* α on the vertices v of $\mathbb{P}$ as follows. Corners within 2-bonds have angle zero. In vertices of degree 3, corners (not within 2-bonds) labelled by $h^{\pm 1}$ have angle π except the ones in vertices of degree 3 of Types 2 and 4. In these types there are two $h^{\pm 1}$-corners not in a 2-bond. The one with angle π is the one that ensures that $(1^{-1}h)^{\pm 1}$ is a sublabel of a positive region (see Figure 4.31). Each of the other two corners in a vertex of degree 3 has angle $\frac{\pi}{2}$. If $d(v) \geq 4$, then each corner in v not in a 2-bond has angle $\frac{2\pi}{d(v)}$.

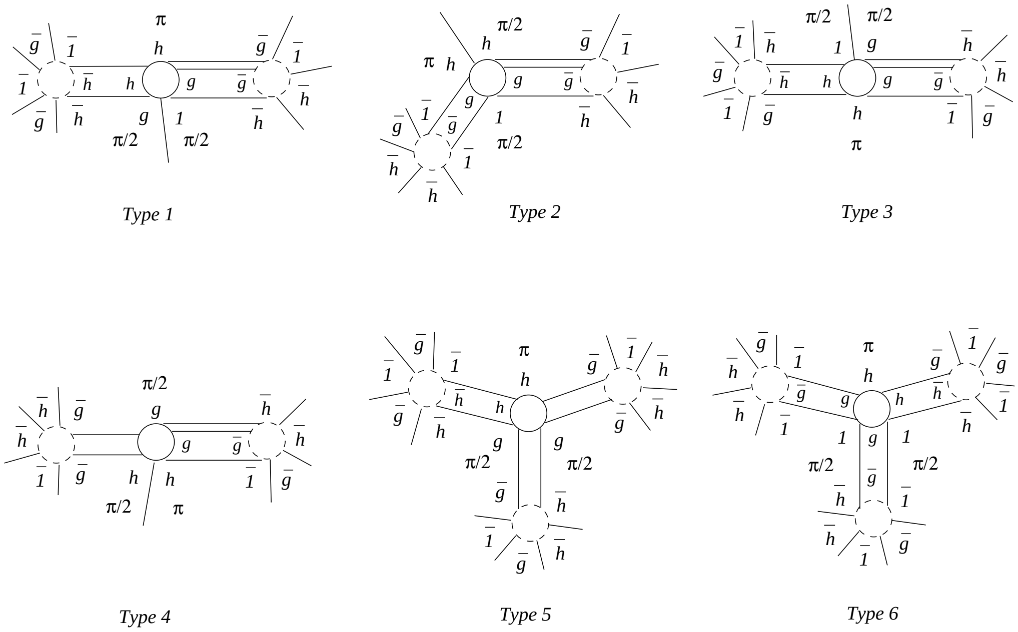


Figure 4.31: Angle function α for vertices of degree 3.

Remark 4.5.5. *By assigning the angle function α to the corners of $\mathbb{P}$, the following holds: there are no consecutive corners with angle π in the boundary of a region Δ of $\mathbb{P}$. Clearly this is true for the corners of vertices of Types 1, 3, 5 and 6. Figure 4.32 shows the result for Types 2 and 4 since consecutive corners of angle π of these two types gives that $\mathbb{P}$ is not reduced. Therefore positively curved regions can only be 4-regions or 6-regions.*

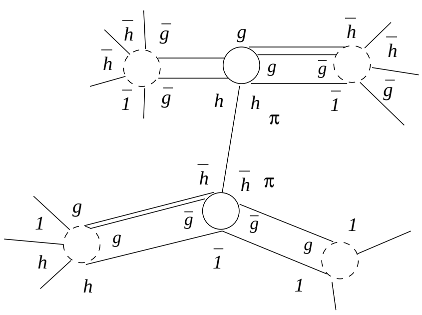


Figure 4.32: No consecutive corners with angle π.

Assume that

- (**A**) the number of 2-bonds in $\mathbb{P}$ is maximal.
- (**B**) given (**A**), the number of 4-regions with the labelling $(hg^{-1}gh^{-1})^{\pm 1}$ in $\mathbb{P}$ is maximal.

Assign α to the corners of $\mathbb{P}$. The positive 4-regions Δ imply one of the following: $g = h^{-1}$ or $h^2 = 1$ or $g = h^2$ or $l(\Delta) = 1^{-1}hh^{-1}1$. If $l(\Delta) = 1^{-1}hh^{-1}1$ then Δ could be one of the two regions given by Figure 4.33. As shown Δ_1 contradicts the assumptions. Similarly, Δ_2 also contradicts the assumptions. By Remark 4.5.5, positive 6-regions imply that $h^3 = 1$. Therefore by assigning the angle function α to the corners of $\mathbb{P}$, $\mathcal{P}_3$ is aspherical except possibly if at least one of the following is satisfied: $g = h^{-1}$ or $h^2 = 1$ or $g = h^2$ or $h^3 = 1$. Therefore, by Lemma 4.5.4, $\mathcal{P}_3$ is aspherical except possibly if at least one of the following holds: $h = g^2$ or $(g^{-1}h)^2 = 1$ or $g = h^2$ or $(g^{-1}h)^3 = 1$.

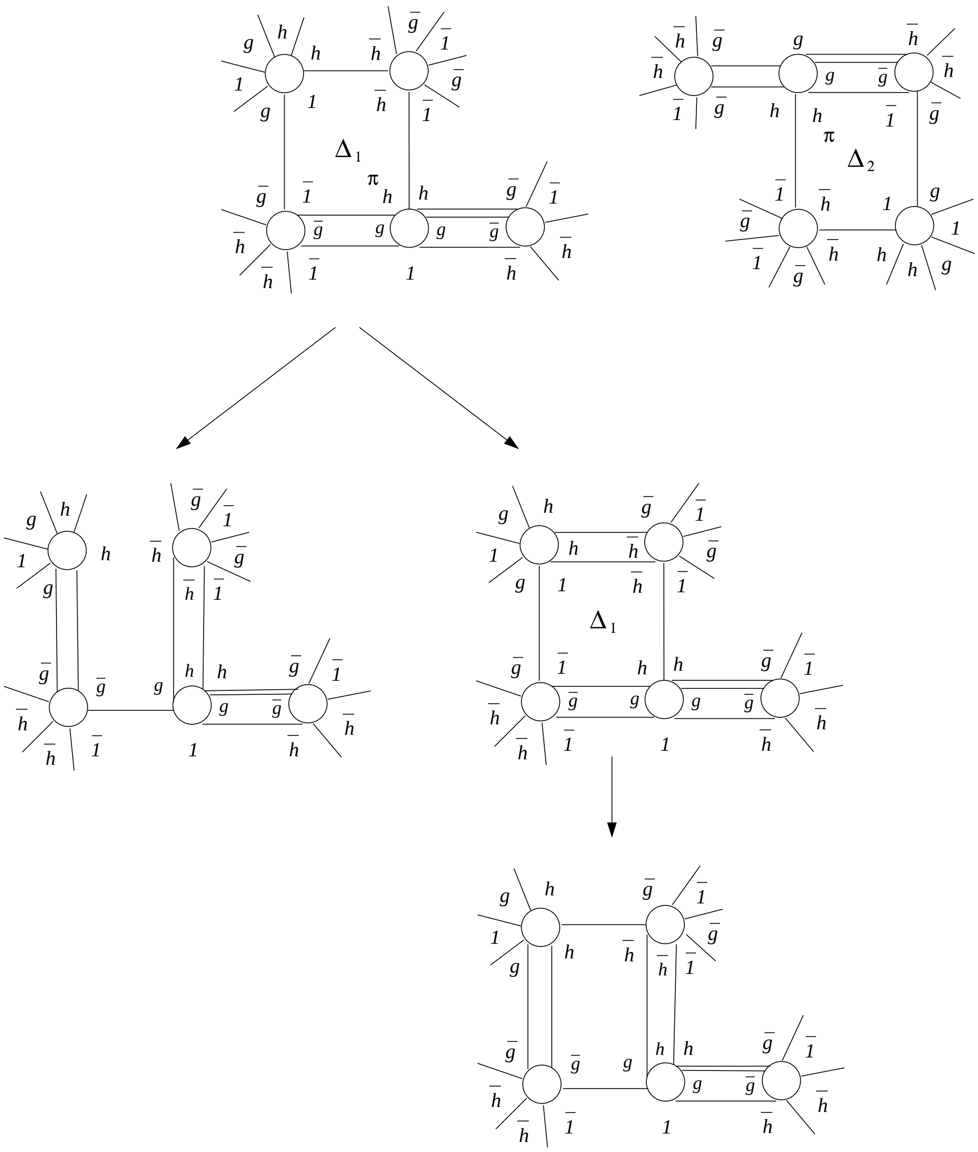


Figure 4.33: Positive regions with label $1^{-1}hh^{-1}1$ do not exist.

Lemma 4.5.6. *Consider the presentation $\mathcal{P}_3$. If $h^2 = (gh^{-1})^2 = 1$ and $|g| < \infty$, then $\mathcal{P}_3$ is not aspherical.*

Proof. Assume that $h^2 = (gh^{-1})^2 = 1$ and $|g| < \infty$. In this case non-trivial reduced spherical pictures over $\mathcal{P}_3$ are given by Figure 4.34. As an example, the case $|g| = 5$ is given by Figure 4.35.

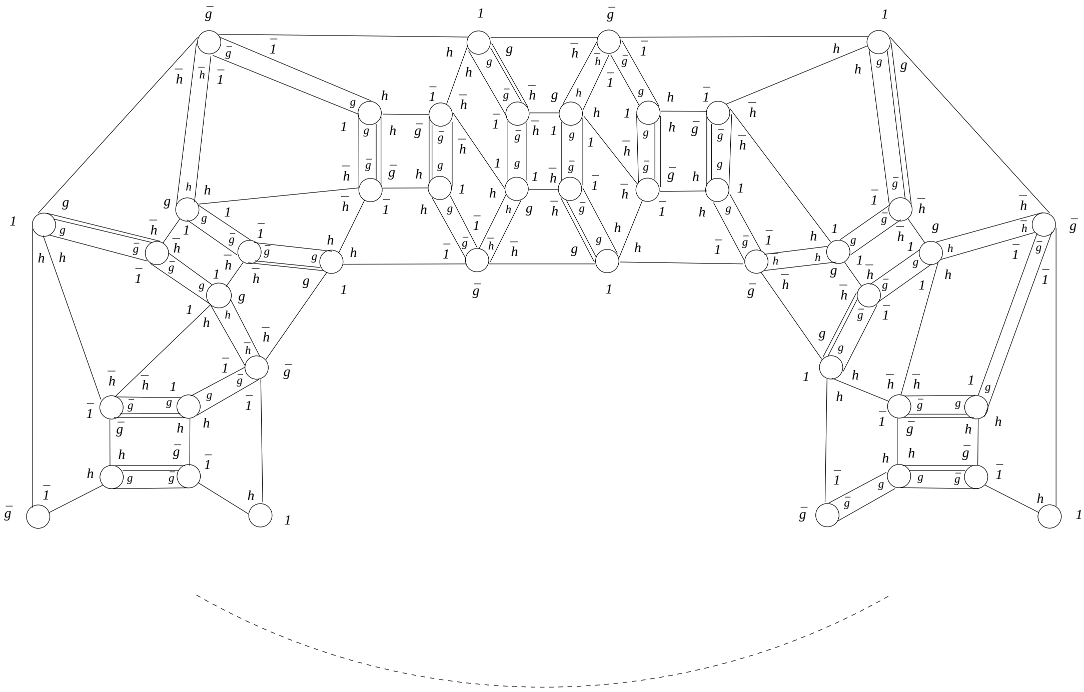


Figure 4.34: $\mathcal{P}_3$, $|h| = |gh^{-1}| = 2$ and $|g| < \infty$.

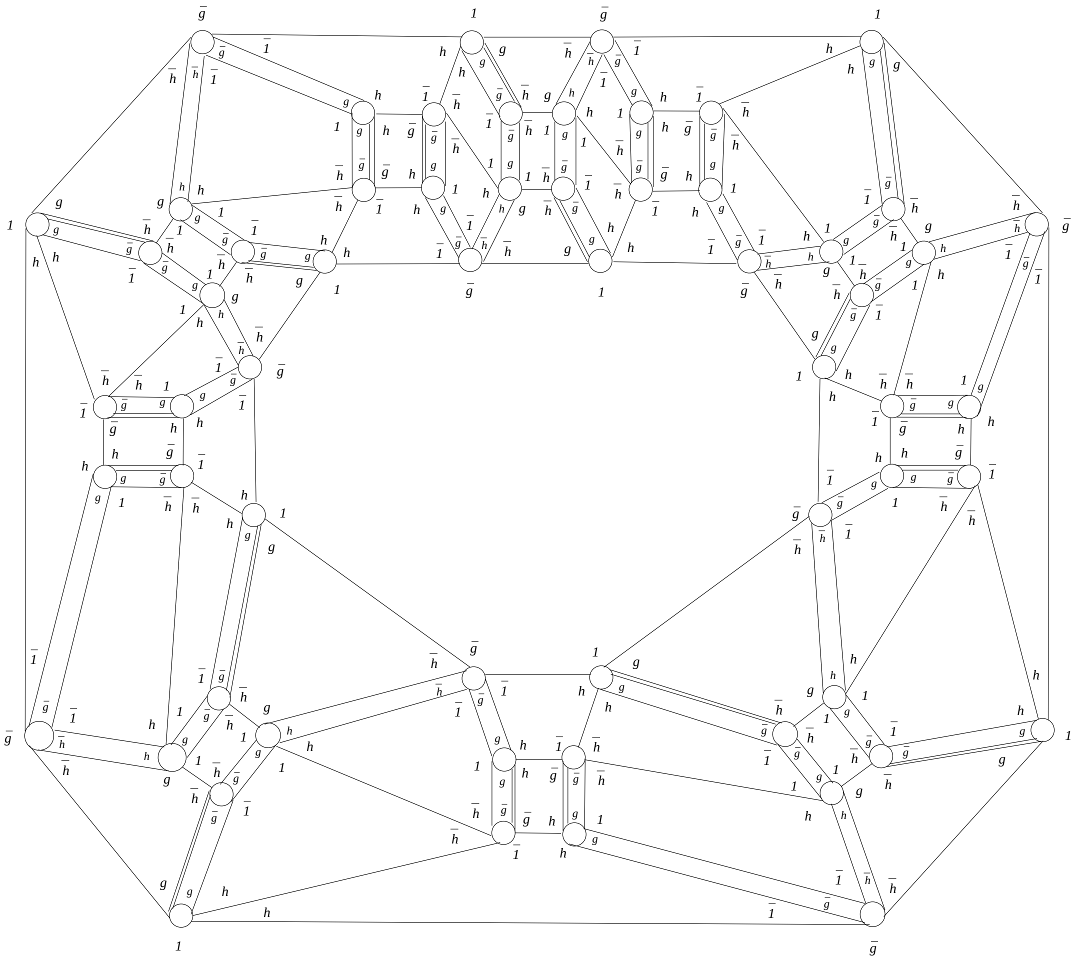


Figure 4.35: $\mathcal{P}_3$, $|h| = |gh^{-1}| = 2$ and $|g| = 5$.

Lemma 4.5.7. *Consider the presentation $\mathcal{P}_3$. Assume that $h \neq g^2$ and $g \neq h^2$. If $h^3 = (gh^{-1})^3 = 1$, then $\mathcal{P}_3$ is aspherical.*

Proof. Assign α to the corners of $\mathbb{P}$. Assume that $h \neq g^2$, $g \neq h^2$ and $h^3 = (gh^{-1})^3 = 1$. Then a positive region Δ has label $1^{-1}h1^{-1}h1^{-1}h$, where each of the h-corners has angle π and has one of the forms shown in Figure 4.36. Observe that $c(\Delta) \leq \frac{\pi}{2}$ and distribute the curvature as shown in Figure 4.37. Let $\hat{\Delta}$ be the region such that $c^*(\hat{\Delta}) > c(\hat{\Delta})$.

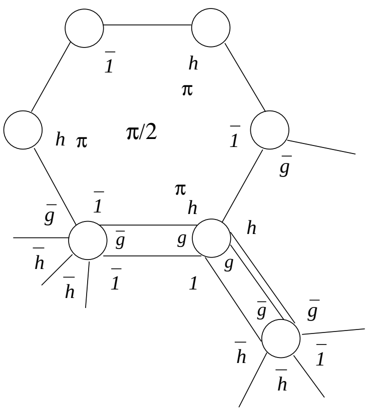


Figure 4.36: Positive regions with label $1^{-1}h1^{-1}h1^{-1}h$.

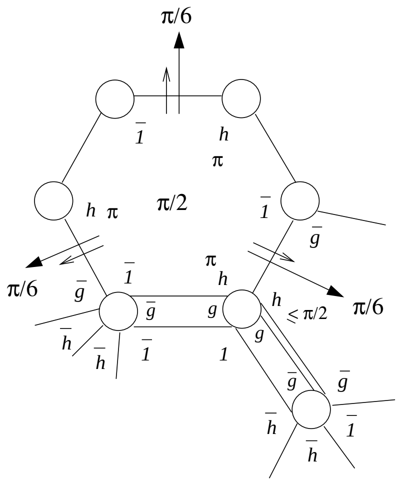


Figure 4.37: Distribution of curvature for regions with label $1^{-1}h1^{-1}h1^{-1}h$.

Remarks 4.5.8.

1. *$l(\hat{\Delta}) = g^{-1}hg^{-1}w$ and so $d(\hat{\Delta}) > 4$.*

2. *Since $\hat{\Delta}$ receives $\frac{\pi}{6}$ only across edges that are oriented towards $\hat{\Delta}$, $\Gamma_6 \leq \frac{n}{2}$.*

3. *For each $\frac{\pi}{6}$ that $\hat{\Delta}$ receives, $\Phi(\hat{\Delta})$ will be decreased by 1 ($\Phi(\hat{\Delta})$ is the number of corners of $\hat{\Delta}$ of angle π).*

Now $c^*(\hat{\Delta}) \leq (2-n)\pi + (\frac{n}{2} - \Gamma_6)\pi + (\frac{n}{2} + \Gamma_6)\frac{\pi}{2} + \Gamma_6.\frac{\pi}{6} < \pi(2 - \frac{n}{4} - \Gamma_6 + \frac{\Gamma_6}{2} + \frac{\Gamma_6}{2})$ $= \pi(2 - \frac{n}{4})$. Therefore $c^*(\hat{\Delta}) > 0$ implies $n = 6$ (Remark 4.5.8(1)). Observe that $\Phi(\hat{\Delta}) + \Gamma_6 \leq 3$ and if $\Phi(\hat{\Delta}) + \Gamma_6 \leq 2$ then $c^*(\hat{\Delta}) \leq 0$. Now if $\Gamma_6 = 3$, then $c^*(\hat{\Delta})$ $\leq -4\pi + 6.\frac{\pi}{2} + 3.\frac{\pi}{6} < 0$. Also if $\Gamma_6 = 2$, then $c^*(\hat{\Delta}) \leq -4\pi + \pi + 5.\frac{\pi}{2} + 2.\frac{\pi}{6} < 0$. Thus $\Gamma_6 = 1$ and $\Phi = 2$ (see Figure 4.38). However the $*$-corner has angle $\leq \frac{\pi}{2}$ since the $\bullet$-corner is not labelled by 1^{-1}. Thus $c^*(\hat{\Delta}) \leq 0$. This completes the proof.

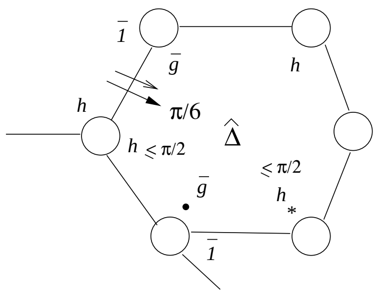


Figure 4.38: $c^*(\hat{\Delta}) \leq 0$.

Before stating the main result for Case 3 consider the following exceptional cases.

(**E1**) $g = h^{-1}$ and $|h| \in \{4, 6\}$.

(**E2**) $g = h^2$ and $4 \leq |h| < \infty$.

(**E3**) $h = g^2$ and $|g| \in \{4, 6\}$.

(**E4**) $h \in \{g^3, g^4\}$ and $|g| = 6$.

(**E5**) $(|h|, |gh^{-1}|) \in \{(2,3), (3,2)\}$ and $\frac{1}{|g|} + \frac{1}{|gh^{-1}|} + \frac{1}{|h|} > 1$.

Remark 4.5.9. *Observe that modulo $(g, h) \leftrightarrow (g, h^{-1}g)$ (see Lemma 4.5.4) the exceptional case **(E1)** is the same as **(E3)**. Also in **(E4)** it is enough to consider $h = g^3$ and in **(E5)** it is enough to consider $|h| = 2$, $|gh^{-1}| = 3$ and $\frac{1}{|g|} + \frac{1}{|gh^{-1}|} + \frac{1}{|h|} > 1$.*

Proposition 4.5.10. *Suppose that none of the exceptional cases **(E1)**- **(E5)** holds. Then $\mathcal{P}_3$ is not aspherical if and only if one of the following holds:*

1. $h^2 = (gh^{-1})^2 = 1$ *and* $|g| < \infty$.

2. $g = h^{-1}$ *and* $|h| = 3$.

Proof. If $h^2 = (gh^{-1})^2 = 1$ and $|g| < \infty$, then $\mathcal{P}_3$ is not aspherical by Lemma 4.5.6. If $g = h^{-1}$ and $|h| = 3$, then $\hat{G} = G *_{G_\circ} K$, where $G_\circ$ is the group with the presentation $\langle h | h^3 \rangle$ and K has the presentation $\langle h, x | x^2 h^{-1} x^2 h^{-1} x h x h, h^3 \rangle$. Coset enumeration shows that the order of the group K equals 55296. Also K properly contains $G_\circ$, and so $\hat{G}$ contains a finite subgroup that is not conjugate to a subgroup of G. Theorem 1.1.3 therefore implies that $\mathcal{P}_3$ is not aspherical. Assume now that neither of the conditions holds. Then it follows from the above that $\mathcal{P}_3$ is aspherical except possibly if at least one of $g = h^{-1}$, $g = h^2$, $h^2 = 1$, $h^3 = 1$ holds and at least one of $h = g^2$, $g = h^2$, $(g^{-1}h)^2 = 1$, $(g^{-1}h)^3 = 1$ holds.

Let $g = h^{-1}$. This forces $|h| = 3$, a contradiction, or $|h| \in \{4, 6\}$ which is (**E1**). So assume $g \neq h^{-1}$. Let $g = h^2$ then $|h| < 4$ yields a contradiction and $4 \leq |h| < \infty$ is (**E2**). Also if $|h| = \infty$ then $\mathcal{P}_3$ is aspherical by Lemma 1.3.1, so assume $g \neq h^2$. Let $h^2 = 1$ then either $h = g^2$ which is (**E3**) or $|gh^{-1}| \in \{2, 3\}$. If $|gh^{-1}| = 2$ then Lemma 4.5.3 implies $\frac{1}{|g|} + \frac{1}{|gh^{-1}|} + \frac{1}{|h|} > 1$ and so $|g| < \infty$, a contradiction. Now if $|gh^{-1}| = 3$ then Lemma 4.5.3 implies either $h = g^3$ which is (**E4**) or $\frac{1}{|g|} + \frac{1}{|gh^{-1}|} + \frac{1}{|h|} > 1$ which is (**E5**). So assume otherwise. Finally let $h^3 = 1$ then either $h = g^2$ which is (**E3**) or $|gh^{-1}| \in \{2, 3\}$. If $|gh^{-1}| = 2$ then Lemma 4.5.3 implies either $h = g^4$ which is (**E4**), or $\frac{1}{|g|} + \frac{1}{|gh^{-1}|} + \frac{1}{|h|} > 1$ which is (**E5**). If $|gh^{-1}| = 3$ then Lemma 4.5.7 implies $\mathcal{P}_3$ is aspherical. This completes the proof.

4.5.3 Case 11: (a=b=c, e=f only)

The relative presentation $\mathcal{P}_{11} = \langle G, y | ayayaydyeyey \rangle =$ $\langle G, y, x | x(ay)^{-1}, x^3 da^{-1} xea^{-1} xea^{-1} x \rangle = \langle G, x | x^4 gxhxh \rangle$, where $g = da^{-1}$ and $h = ea^{-1}$ (and so by assumption, $g, h \in G \backslash \{1\}$ and $g \neq h$).

Before stating the main result for Case 11, we list the following exceptional cases.

(**E1**) $g \in \{h^{-1}, h^2\}$ and $|h| \in \{4, 6\}$.

(**E2**) $H \cong C_2 \times C_3$ and one of the following holds: $|g| = 2$ and $|h| = 3$ or $|g| = 3$ and $|h| = 2$.

(**E3**) $h \in \{g^2, g^3, g^4\}$ and $|g| = 6$.

(**E4**) $g \in \{h^3, h^4\}$ and $|h| = 6$.

Proposition 4.5.11. *Let $\mathcal{P}_{11}$ be the relative presentation $\mathcal{P}_{11}=\langle G, x | x^4gxhxh\rangle$, where $g, h \in G\backslash\{1\}$. Suppose that none of the conditions in **(E1)**-**(E4)** holds. Then $\mathcal{P}_{11}$ is aspherical if and only if none of the following holds:*

1. *$g = h^2$ and $|h| = 3$ or $h = g^2$ and $|g| \in \{3, 4\}$.*
2. *$\frac{1}{|g|} + \frac{1}{|gh^{-1}|} + \frac{1}{|h|} > 1$, where $\frac{1}{\infty} := 0$.*

To prove Proposition 4.5.11, we first provide a series of lemmas.

Lemma 4.5.12. *If $\mathcal{P}_{11}$ is not aspherical, then at least one of the following conditions holds:*

1. *$g = h^{-1}$.*
2. *$g = h^2$ or $h = g^2$.*
3. *$2 \in \{|g|, |h|\}$.*
4. *$|gh^{-1}| = 2$ and $3 \in \{|g|, |h|\}$.*

Lemma 4.5.13. *Let $g = h^2$ and $|h| = 3$ or $h = g^2$ and $|g| \in \{3, 4\}$, then $\mathcal{P}_{11}$ is not aspherical.*

Lemma 4.5.14. *If $g = h^{-1}$ and $|h| \notin \{3, 4, 6\}$, then $\mathcal{P}_{11}$ is aspherical.*

Lemma 4.5.15. *If $g = h^2$ and $|h| \notin \{3, 4, 6\}$, then $\mathcal{P}_{11}$ is aspherical.*

Lemma 4.5.16. *If $h = g^2$ and $|g| \notin \{3, 4, 6\}$, then $\mathcal{P}_{11}$ is aspherical.*

Lemma 4.5.17. *If $\frac{1}{|g|} + \frac{1}{|gh^{-1}|} + \frac{1}{|h|} > 1$, then $\mathcal{P}_{11}$ is not aspherical.*

Lemma 4.5.18. *The following hold:*

1. *If $|g| = 2$, $|gh^{-1}| = 2$ and $|h| = \infty$, then $\mathcal{P}_{11}$ is aspherical.*
2. *If $|g| = 2$, $|gh^{-1}| = 3$, $|h| \geq 6$ and $g \neq h^3$ then $\mathcal{P}_{11}$ is aspherical.*
3. *If $|g| = 2$, $|gh^{-1}| \geq 4$ and $|h| \geq 4$, then $\mathcal{P}_{11}$ is aspherical.*
4. *If $|g| = 2$, $|gh^{-1}| = \infty$ and $|h| = 2$, then $\mathcal{P}_{11}$ is aspherical.*
5. *If $|g| = 2$, $|gh^{-1}| \geq 6$, $|h| = 3$ and $\mathcal{P}_{11}$ is not aspherical then $[g, h] = 1$.*
6. *If $|g| = \infty$, $|gh^{-1}| = 2$ and $|h| = 2$, then $\mathcal{P}_{11}$ is aspherical.*
7. *If $|g| \geq 6$, $|gh^{-1}| = 3$, $|h| = 2$ and $h \neq g^3$ then $\mathcal{P}_{11}$ is aspherical.*
8. *If $|g| \geq 4$, $|gh^{-1}| \geq 4$ and $|h| = 2$, then $\mathcal{P}_{11}$ is aspherical.*
9. *If $|g| = 3$, $|gh^{-1}| \geq 6$, $|h| = 2$ and $\mathcal{P}_{11}$ is not aspherical then $[g, h] = 1$.*
10. *If $|g| = 3$, $|gh^{-1}| = 2$, $|h| \geq 6$ and $\mathcal{P}_{11}$ is not aspherical, then $g = h^4$ and $|h| = 6$.*
11. *If $|g| \geq 6$, $|gh^{-1}| = 2$, $|h| = 3$ and $\mathcal{P}_{11}$ is not aspherical, then $h = g^4$ and $|g| = 6$.*

The proofs of the above lemmas are given later on. Here we assume that they are true and prove Proposition 4.5.11.

Proof of Proposition 4.5.11.

The 'only if' direction of Proposition 4.5.11 follows from Lemmas 4.5.13 and 4.5.17. For the rest of the proof, we assume that none of the conditions of Proposition 4.5.11 holds. We show that either $\mathcal{P}_{11}$ is aspherical or exceptional.

If none of the conditions of Lemma 4.5.12 holds, then $\mathcal{P}_{11}$ is aspherical. Suppose that Condition 1 of Lemma 4.5.12 holds. Then either $\mathcal{P}_{11}$ is aspherical by Lemma 4.5.14 or exceptional of type (**E1**, $g = h^{-1}$)(since Condition 1 of Proposition 4.5.11 does not hold). So assume from now on that $g \neq h^{-1}$.

If Condition 2 of Lemma 4.5.12 holds and $g = h^2$. Then $|h| \geq 4$ (by the negation of Condition 1 of Proposition 4.5.11) and so either $\mathcal{P}_{11}$ is aspherical by Lemma 4.5.15 or exceptional of type (**E1**, $g = h^2$). Moreover if $h = g^2$ then $|g| \geq 5$ (Condition 1 of Proposition 4.5.11) and so either $\mathcal{P}_{11}$ is aspherical by Lemma 4.5.16 or exceptional of type (**E3**, $h = g^2$). So assume from now on that $g \neq h^2$ and $h \neq g^2$.

Assume that Condition 3 of Lemma 4.5.12 holds and $|g| = 2$. Since $g \neq h$, $|gh^{-1}| \geq 2$. If $|gh^{-1}| = 2$ then $|h| = \infty$ (Condition 2 of Proposition 4.5.11) and it follows that $\mathcal{P}_{11}$ is aspherical by Lemma 4.5.18(1). If $|gh^{-1}| = 3$, then $|h| \geq 6$ (Condition 2 of Proposition 4.5.11). By Lemma 4.5.18(2), $\mathcal{P}_{11}$ is aspherical if $g \neq h^3$, while if $g = h^3$ then $\mathcal{P}_{11}$ is exceptional of type (**E4**, $g = h^3$). If $|gh^{-1}| = 4$ or 5 then $|h| \geq 4$ (Condition 2 of Proposition 4.5.11), and so $\mathcal{P}_{11}$ is aspherical by Lemma 4.5.18(3). Now suppose that $|gh^{-1}| \geq 6$. By Lemmas 4.5.18(4), 4.5.18(5) and 4.5.18(3), if $|gh^{-1}| = \infty$ then $\mathcal{P}_{11}$ is aspherical, so assume otherwise. Then $|h| \geq 3$ (Condition 2 of Proposition 4.5.11). If $|h| = 3$ then $[g, h] \neq 1$, otherwise $\mathcal{P}_{11}$ is exceptional of type **E2**, and so $\mathcal{P}_{11}$ is aspherical by Lemma 4.5.18(5). If $|h| \geq 4$, then $\mathcal{P}_{11}$ is aspherical by Lemma 4.5.18(3).

Assume that Condition 3 of Lemma 4.5.12 holds and $|h| = 2$. Since $g \neq h$, $|gh^{-1}| \geq 2$. If $|gh^{-1}| = 2$ then $|g| = \infty$ (Condition 2 of Proposition 4.5.11) and it follows that $\mathcal{P}_{11}$ is aspherical by Lemma 4.5.18(6). If $|gh^{-1}| = 3$, then $|g| \geq 6$ (Condition 2 of Proposition 4.5.11). By Lemma 4.5.18(7), $\mathcal{P}_{11}$ is aspherical if $h \neq g^3$, while if $h = g^3$ then $\mathcal{P}_{11}$ is exceptional of type (**E3**, $h = g^3$). If $|gh^{-1}| = 4$ or 5 then $|g| \geq 4$ (Condition 2 of Proposition 4.5.11), and so $\mathcal{P}_{11}$ is aspherical by Lemma 4.5.18(8). Now suppose that $|gh^{-1}| \geq 6$. By Lemmas 4.5.18(4), 4.5.18(9) and 4.5.18(8) , if $|gh^{-1}| = \infty$ then $\mathcal{P}_{11}$ is aspherical, so assume otherwise. Then $|g| \geq 3$ (Condition 2 of Proposition 4.5.11). If $|g| = 3$ then $[g, h] \neq 1$, otherwise $\mathcal{P}_{11}$ is exceptional of type **E2**, and so $\mathcal{P}_{11}$ is aspherical by Lemma 4.5.18(9). If $|g| \geq 4$, then $\mathcal{P}_{11}$ is aspherical by Lemma 4.5.18(8).

Finally, if Condition 4 of Lemma 4.5.12 is satisfied and $|g| = 3$ ($|gh^{-1}| = 2$). Then $|h| \geq 6$ (else, Condition 2 of Proposition 4.5.11 applies). If $g = h^4$ then $\mathcal{P}$ is exceptional of type (**E4**, $g = h^4$); otherwise $\mathcal{P}_{11}$ is aspherical by Lemma 4.5.18(10). Now if Condition 4 of Lemma 4.5.12 is satisfied and $|h| = 3$ ($|gh^{-1}| = 2$). Then $|g| \geq 6$ (else, Condition 2 of Proposition 4.5.11 applies). If $h = g^4$ then $\mathcal{P}_{11}$ is exceptional of type (**E3**, $h = g^4$); otherwise $\mathcal{P}_{11}$ is aspherical by Lemma 4.5.18(11). This completes the proof.

Let $\mathbb{P}$ be a non-trivial reduced spherical picture over $\mathcal{P}_{11} = \langle G, x | x^4 gxhxh \rangle$. Then each vertex(disc) in $\mathbb{P}$ has one of the forms given by Figure 4.39(i) and (ii); and the the star graph $\mathcal{P}_{11}^{st}$ of $\mathcal{P}_{11}$ is given by Figure 4.39(iii).

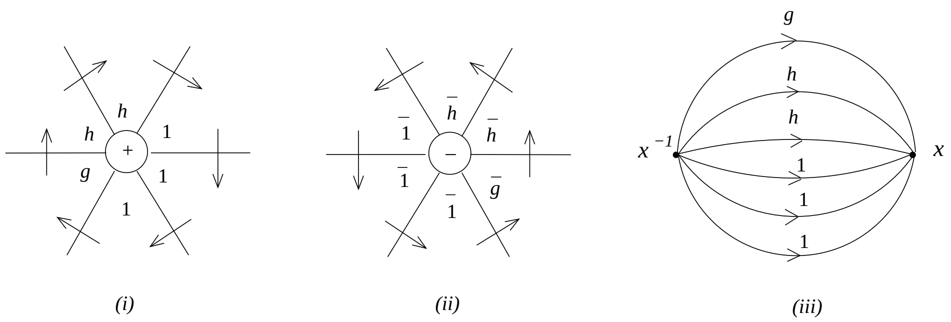


Figure 4.39: $+$ disc, - disc and $\mathcal{P}_{11}^{st}$.

There are (up to inversion) four types of vertices of degree 3 in $\mathbb{P}$ and these are shown in Figure 4.40.

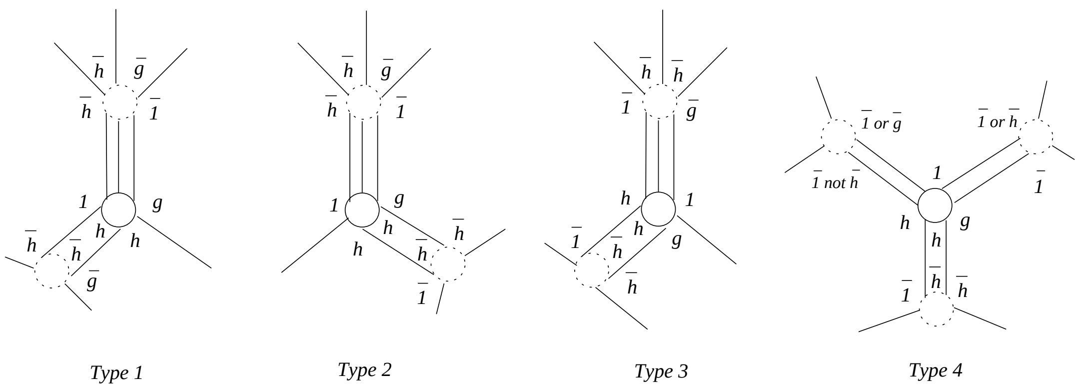


Figure 4.40: Types of vertices of degree 3.

For the proofs, we define the following angle functions on the vertices v of $\mathbb{P}$. Recall that the standard angle function α is defined as follows. Each corner within a 2-bond has angle zero, while each of the other corners has angle $\frac{2\pi}{d(v)}$.

Define an *angle function* α_1 on $\mathbb{P}$ as follows. Corners within 2-bonds have angle zero. In vertices of degree 3, corners (not within 2-bonds) labelled by $1^{\pm 1}$ have angle π, each of the other two corners has angle $\frac{\pi}{2}$ (see Figure 4.41). If $d(v) \geq 4$, then each corner in v not in a 2-bond has angle $\frac{2\pi}{d(v)}$.

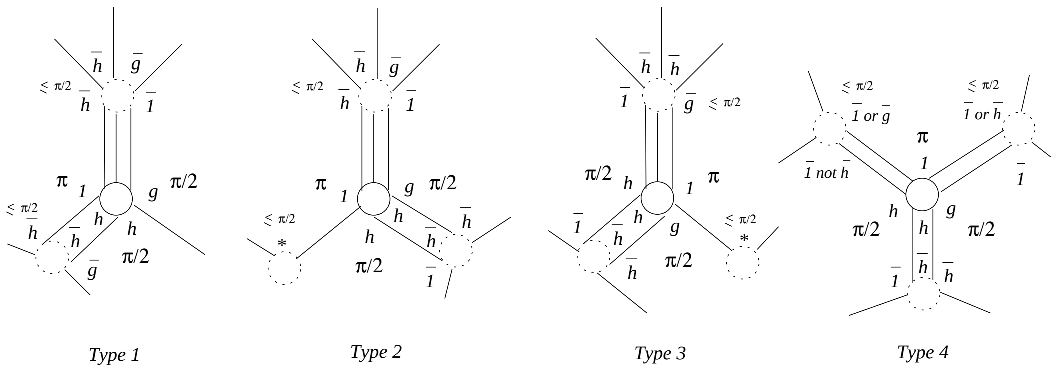


Figure 4.41: Angle function α_1 for vertices of degree 3.

The *angle function* α_2 is defined as follows. Again, corners within 2-bonds have angle zero. For vertices of degree 3, corners labelled by $g^{\pm 1}$ have angle π, each of the other two corners has angle $\frac{\pi}{2}$ (see Figure 4.42). If $d(v) \geq 4$, then each corner in v not in a 2-bond has angle $\frac{2\pi}{d(v)}$.

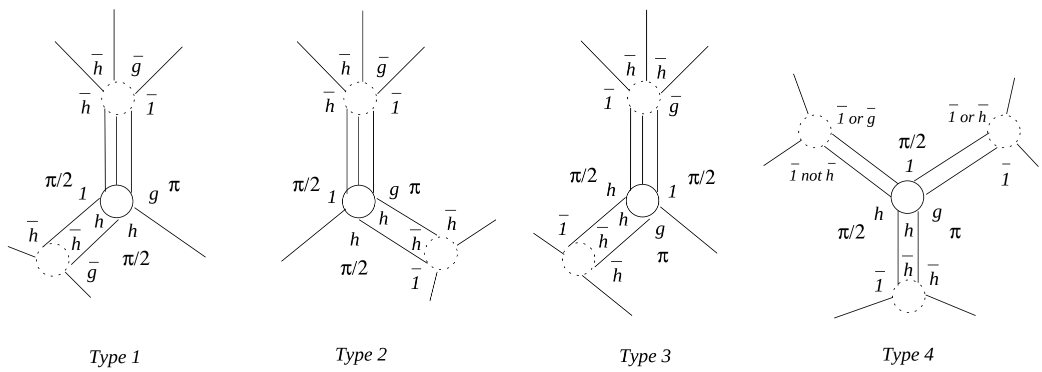


Figure 4.42: Angle function α_2 for vertices of degree 3.

Finally define the *angle function* α_3 as follows. Again, corners within 2-bonds have angle zero. In vertices of degree 3, corners labelled by $h^{\pm 1}$ have angle π, each of the other two corners has angle $\frac{\pi}{2}$ (see Figure 4.43). If $d(v) \geq 4$, then each corner in v not in a 2-bond has angle $\frac{2\pi}{d(v)}$.

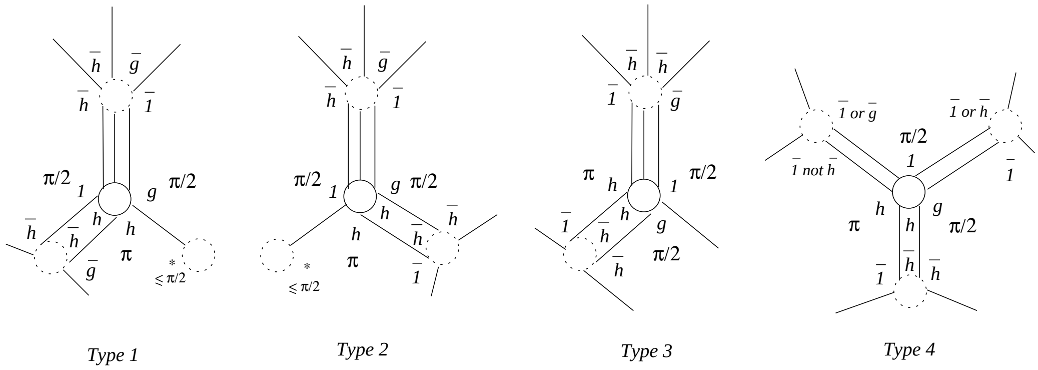


Figure 4.43: Angle function α_3 for vertices of degree 3.

Remarks 4.5.19.

1. *By assigning the angle function α_1 to the corners of $\mathbb{P}$, the following are satisfied:*

   (i) There are no consecutive corners with angle π in the boundary of a region Δ of $\mathbb{P}$. As shown in Figure 4.41, this is clear for Type 1 vertex. Now if the $*$-corner in Type 2 vertex is labelled by 1^{-1} with angle π then $\mathbb{P}$ is not reduced. The same can be deduced for Type 3 and Type 4 vertices.

   (ii) Since $(2-8)\pi + 4\pi + 4.\frac{\pi}{2} = 0$, positive regions can only have degree 4 or 6.

   (iii) If Δ is a positive 4-region then it contains at least one corner labelled by $1^{\pm 1}$ with angle π (otherwise $c(\Delta) \leq -2\pi + 4.\frac{\pi}{2} = 0$).

   (iv) If Δ is a positive 6-region then it contains three occurrences of $1^{\pm 1}$-corners each with angle π (else $c(\Delta) \leq -4\pi + 2\pi + 4.\frac{\pi}{2} = 0$).

2. *By assigning the angle function α_2 to the corners of $\mathbb{P}$, the following are satisfied:*

   (i) In any region Δ of $\mathbb{P}$, there are no consecutive corners with angle π, else $\mathbb{P}$ is not reduced (if two adjacent corners in the boundary of a region are

labelled by g and g^{-1} or vice versa then it is clear from $\mathcal{P}^{st}_{11}$ that $\mathbb{P}$ is not reduced). Hence positively curved regions can only be 4-regions or 6-regions.
(ii) If Δ is a positive 4-region, then it has at least one corner labelled by $g^{\pm 1}$ with angle π.
(iii) If Δ is a positive 6-region, then it contains at least three $g^{\pm 1}$- corners each with angle π.

3. *By assigning the angle function α_3 to the corners of $\mathbb{P}$, the following are satisfied:*

   (i) There are no consecutive corners with angle π in the boundary of a region Δ of $\mathbb{P}$ (otherwise $\mathbb{P}$ is not reduced). Thus positive regions can only be 4-regions or 6-regions.
   (ii) If Δ is a positive 4-region then it contains at least one corner labelled by $h^{\pm 1}$ with angle π.
   (iii) If Δ is a positive 6-region then it contains three occurrences of $h^{\pm 1}$-corners each with angle π.
   (iv) Note that an $h^{\pm 1}$-corner in a sublabel $(g^{-1}h1^{-1})^{\epsilon}$, $\epsilon = \pm 1$ of a label of a region in $\mathbb{P}$ cannot be of angle π. This is because no h-corner in the vertex v shown in Figure 4.44 can be in a 2-bond and so $d(v) \geq 4$. The case $\epsilon = 1$ is done in Figure 4.44 and the case $\epsilon = -1$ can be done similarly. The same applies for an $h^{\pm 1}$-corner in sublabels $(h^{-1}hh^{-1})^{\pm 1}$ and $(g^{-1}hh^{-1})^{\pm 1}$.

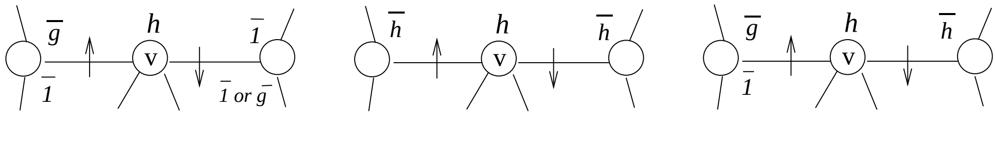


Figure 4.44: $h^{\pm 1}$-corner in $(g^{-1}h1^{-1})^{\pm 1}$, $(h^{-1}hh^{-1})^{\pm 1}$ and $(g^{-1}hh^{-1})^{\pm 1}$ has angle $\leq \frac{\pi}{2}$.

Throughout the proofs of the coming lemmas in Case 11, we assume that $\mathbb{P}$ is a non-trivial reduced spherical picture over $\mathcal{P}_{11}$. Consider the following assumption on $\mathbb{P}$:

- (**A**) the number of 4-regions of $\mathbb{P}$ is minimal.

We will adopt this assumption in some subcases and it will be clearly stated when but in general we will not insist on it. Let c be the curvature function associated to the given angle function on the vertices of $\mathbb{P}$. Let Δ denote a positive region in each subcase and let $\Gamma(\Delta, \hat{\Delta}) > 0$, where Γ is the distribution function. If the distributed curvature function is denoted by c^*, then $c^*(\hat{\Delta}) > c(\hat{\Delta})$ and $c^*(\hat{\Delta}) > 0$. Moreover, set $\Gamma_k(\hat{\Delta}) = |\{\Delta : \Gamma(\Delta, \hat{\Delta}) = \frac{\pi}{k}\}|$.

Proof of Lemma 4.5.12. Assume that $\mathcal{P}_{11}$ is not aspherical and assume that (**A**) holds. To prove Lemma 4.5.12, suppose that none of the Conditions 1, 2 or 3 holds. That is, $g \neq h^{-1}$, $g \neq h^2$, $h \neq g^2$ and both g and h have order at least 3. We must show that Condition 4 holds.

First assign the standard angle function α to the vertices of $\mathbb{P}$. By the curvature formula, there is a positively curved region Δ in $\mathbb{P}$. Also, the maximum curvature of any n-region in $\mathbb{P}$ is $\pi\left(\frac{6-n}{3}\right)$, and hence $c(\Delta) > 0$ only if $n = 4$.

A positively curved 4-region Δ has at least one vertex of degree 3. The possible labellings for a positive 4-regions are: $11^{-1}11^{-1}$, $11^{-1}hh^{-1}$, $1^{-1}1h^{-1}h$ and $gh^{-1}gh^{-1}$. Each of the first three labellings gives 4-regions that contradicts assumption (**A**) as shown in Figure 4.45 and so by assigning α we obtain $|gh^{-1}| = 2$.

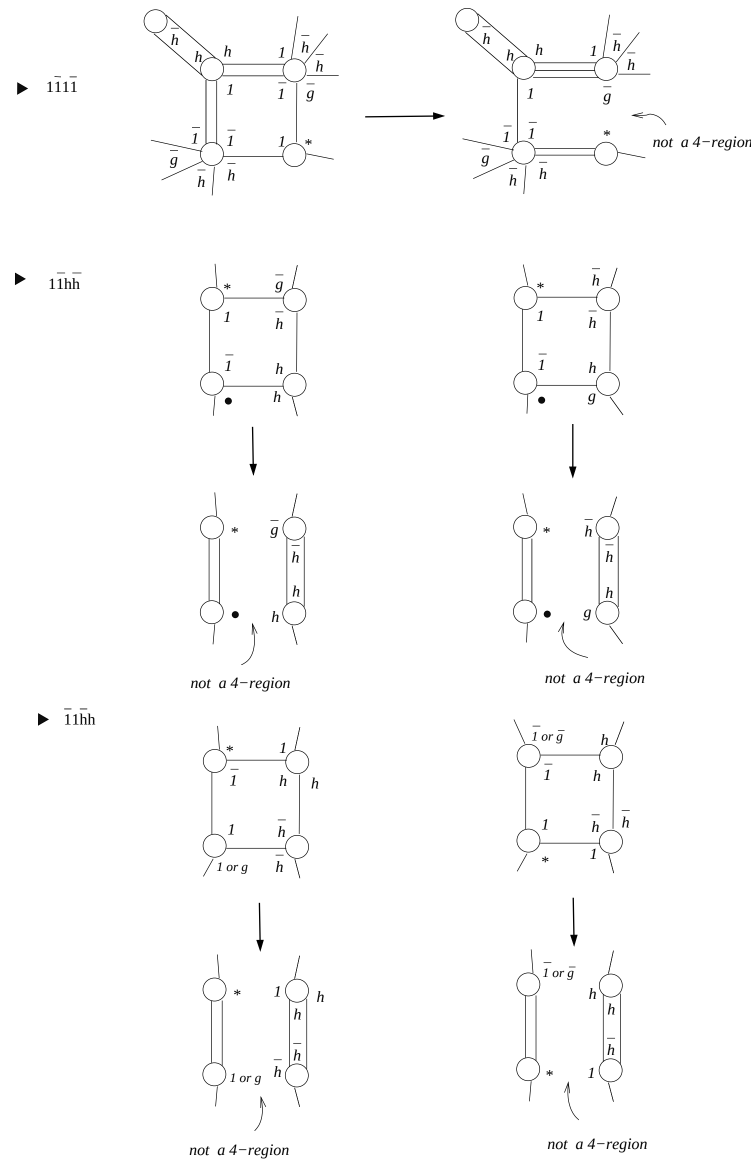


Figure 4.45: The regions $11^{-1}11^{-1}$, $11^{-1}hh^{-1}$ and $1^{-1}1h^{-1}h$.

Now use the angle function α_1 for the vertices of $\mathbb{P}$. By Remark 4.5.19.(1)(ii), positively curved regions can only be 4-regions or 6-regions. By Remark 4.5.19.(1)(iii) positively curved 4-regions do not exist. By Remark 4.5.19.(1)(iv) the possible labellings for positive 6-regions give either $g^3 = 1$ or $h^3 = 1$ (observe that the 1-corner in the sublabel $h^{-1}1g^{-1}$ cannot have angle π and so positive regions do not imply $g = h^{-2}$ or $h = g^{-2}$).

By using the standard angle function and the angle function α_1, if none of the Conditions 1, 2 or 3 holds, then $(gh^{-1})^2=1$ and one of the following is satisfied: $g^3 = 1$ or $h^3 = 1$. That is $|gh^{-1}| = 2$ and $3 \in \{|g|, |h|\}$ which is Condition 4, as required.

Proof of Lemma 4.5.13. Coset enumeration shows that the order of the group $\langle h, x | x^4h^2xhxh, h^3\rangle$ equals 342. Also the order of $\langle g, x | x^4gxg^2xg^2, g^k\rangle$ equals 342, 1275000 (respectively) for $k = 3$ and 4 (respectively).

Proof of Lemma 4.5.14 Let $g = h^{-1}$ and $|h| \notin \{3, 4, 6\}$. Let $\mathbb{P}$ be a non-trivial reduced spherical picture over $\mathcal{P}_{11}$. By assigning α_2 to the corners of $\mathbb{P}$, the only positive region is given by Figure 4.46. Distribute the curvature as shown.

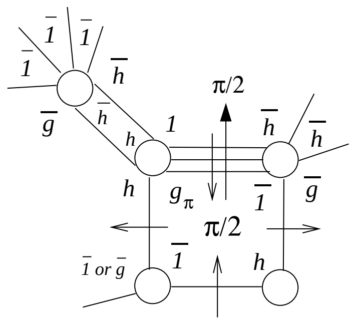


Figure 4.46: Positively curved region in Case $g = h^{-1}$ and distribution scheme.

Remarks 4.5.20.

1. *$l(\hat{\Delta}) = h^{-1}1h^{-1}w$ and so $d(\hat{\Delta}) \geq 6$.*

2. *Since $\hat{\Delta}$ receives $\frac{\pi}{2}$ only across edges that are oriented outwards $\hat{\Delta}$, $\Gamma_2 \leq \frac{n}{2}$.*

3. *For each $\frac{\pi}{2}$ that $\hat{\Delta}$ receives, $\Phi(\hat{\Delta})$ will be decreased by 1.*

Now $c^*(\hat{\Delta}) \leq (2-n)\pi + (\frac{n}{2} - \Gamma_2)\pi + (\frac{n}{2} + \Gamma_2)\frac{\pi}{2} + \Gamma_2.\frac{\pi}{2} = (2 - \frac{n}{4})\pi$. Therefore $c^*(\hat{\Delta}) > 0$ implies $n = 6$. If $\Gamma_2 = 3$ then $h^3 = 1$, a contradiction. Thus $\Gamma_2 \leq 2$ and so $\Phi(\hat{\Delta}) \geq 1$. In fact $\Gamma_2 + \Phi(\hat{\Delta}) = 3$, for otherwise $c^*(\hat{\Delta}) \leq -4\pi + 6.\frac{\pi}{2} + 2.\frac{\pi}{2} = 0$ or $c^*(\hat{\Delta}) \leq -4\pi + 5.\frac{\pi}{2} + \pi + \frac{\pi}{2} = 0$. Now if $\Gamma_2 = 2$ and $\Phi(\hat{\Delta}) = 1$ then $l(\hat{\Delta}) = h^{-1}1h^{-1}1h^{-1}g$ which implies $h^4 = 1$, a contradiction. This leaves $\Gamma_2 = 1$ and $\Phi(\hat{\Delta}) = 2$ and so the situation shown in Figure 4.47 is forced. However since e is not an hh^{-1}-bond, $\Phi(\hat{\Delta}) = 2$ is not possible. Thus $c^*(\hat{\Delta}) \leq 0$ and so $\mathcal{P}_{11}$ is aspherical.

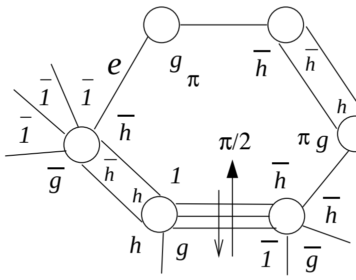


Figure 4.47: $\Gamma_2 = 1$ and $\Phi(\hat{\Delta}) = 2$ is not possible.

Proof of Lemma 4.5.15 Let $g = h^2$ and $|h| \notin \{3, 4, 6\}$. Let $\mathbb{P}$ be a non-trivial reduced spherical picture over $\mathcal{P}_{11}$. By assigning α_2 to the corners of $\mathbb{P}$, the only positive region is given by Figure 4.48. Distribute the curvature as shown.

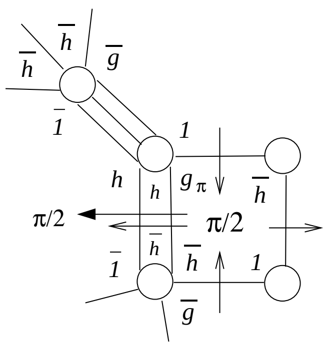


Figure 4.48: Positively curved region in Case $g = h^2$ and distribution scheme.

Remarks 4.5.21.

1. *$l(\hat{\Delta}) = 1^{-1}h1^{-1}w$ and so $d(\hat{\Delta}) \geq 6$.*

2. *Since $\hat{\Delta}$ receives $\frac{\pi}{2}$ only across edges that are oriented towards $\hat{\Delta}$, $\Gamma_2 \leq \frac{n}{2}$.*

3. *For each $\frac{\pi}{2}$ that $\hat{\Delta}$ receives, $\Phi(\hat{\Delta})$ will be decreased by 1.*

Now $c^*(\hat{\Delta}) \leq (2-n)\pi + (\frac{n}{2} - \Gamma_2)\pi + (\frac{n}{2} + \Gamma_2)\frac{\pi}{2} + \Gamma_2.\frac{\pi}{2} = (2 - \frac{n}{4})\pi$. Therefore $c^*(\hat{\Delta}) > 0$ implies $n = 6$. If $\Gamma_2 = 3$ then $h^3 = 1$, a contradiction. Thus $\Gamma_2 \leq 2$ and so $\Phi(\hat{\Delta}) \geq 1$. In fact $\Gamma_2 + \Phi(\hat{\Delta}) = 3$, for otherwise $c^*(\hat{\Delta}) \leq -4\pi + 6.\frac{\pi}{2} + 2.\frac{\pi}{2} = 0$ or $c^*(\hat{\Delta}) \leq -4\pi + 5.\frac{\pi}{2} + \pi + \frac{\pi}{2} = 0$. Now if $\Gamma_2 = 2$ and $\Phi(\hat{\Delta}) = 1$ then $l(\hat{\Delta}) = 1^{-1}h1^{-1}h1^{-1}g$ which implies $h^4 = 1$, a contradiction. This leaves $\Gamma_2 = 1$ and $\Phi(\hat{\Delta}) = 2$ and so the situation shown in Figure 4.49 is forced. However since e is not a 3-bond, $\Phi(\hat{\Delta}) = 2$ is not possible. Thus $c^*(\hat{\Delta}) \leq 0$ and so $\mathcal{P}_{11}$ is aspherical.

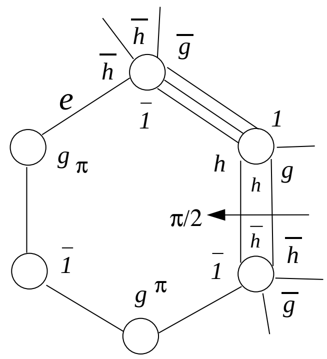


Figure 4.49: $\Gamma_2 = 1$ and $\Phi(\hat{\Delta}) = 2$ is not possible.

Proof of Lemma 4.5.16 Let $h = g^2$ and $|g| \notin \{3, 4, 6\}$. Let $\mathbb{P}$ be a non-trivial reduced spherical picture over $\mathcal{P}_{11}$. Assign α_1 to the corners of $\mathbb{P}$. By assumption (**A**), the only positive region is given by Figure 4.50. Distribute the curvature as shown.

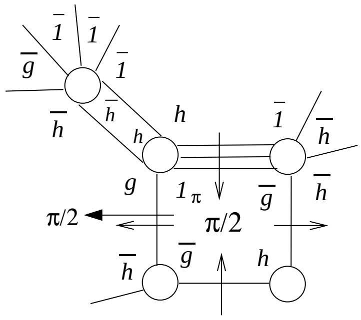


Figure 4.50: Positively curved region in Case $h = g^2$ and distribution scheme.

Remarks 4.5.22.

1. *$l(\hat{\Delta}) = h^{-1}gh^{-1}w$ and so $d(\hat{\Delta}) \geq 6$.*

2. *Since $\hat{\Delta}$ receives $\frac{\pi}{2}$ only across edges that are oriented towards $\hat{\Delta}$, $\Gamma_2 \leq \frac{n}{2}$.*

3. *For each $\frac{\pi}{2}$ that $\hat{\Delta}$ receives, $\Phi(\hat{\Delta})$ will be decreased by 1.*

Now $c^*(\hat{\Delta}) \leq (2-n)\pi + (\frac{n}{2} - \Gamma_2)\pi + (\frac{n}{2} + \Gamma_2)\frac{\pi}{2} + \Gamma_2.\frac{\pi}{2} = (2 - \frac{n}{4})\pi$. Therefore $c^*(\hat{\Delta}) > 0$ implies $n = 6$. If $\Gamma_2 = 3$ then $(gh^{-1})^3 = 1$ which implies $g^3 = 1$, a contradiction. Thus $\Gamma_2 \leq 2$ and so $\Phi(\hat{\Delta}) \geq 1$. In fact $\Gamma_2 + \Phi(\hat{\Delta}) = 3$, for otherwise $c^*(\hat{\Delta}) \leq -4\pi + 6.\frac{\pi}{2} + 2.\frac{\pi}{2} = 0$ or $c^*(\hat{\Delta}) \leq -4\pi + 5.\frac{\pi}{2} + \pi + \frac{\pi}{2} = 0$. Now if $\Gamma_2 = 2$ and $\Phi(\hat{\Delta}) = 1$ then $l(\hat{\Delta}) = h^{-1}gh^{-1}gh^{-1}1$ which implies $g^4 = 1$, a contradiction. This leaves $\Gamma_2 = 1$ and $\Phi(\hat{\Delta}) = 2$ and so the situation shown in Figure 4.51 is forced. However since e is not a 3-bond, $\Phi(\hat{\Delta}) = 2$ is not possible. Thus $c^*(\hat{\Delta}) \leq 0$ and so $\mathcal{P}_{11}$ is aspherical.

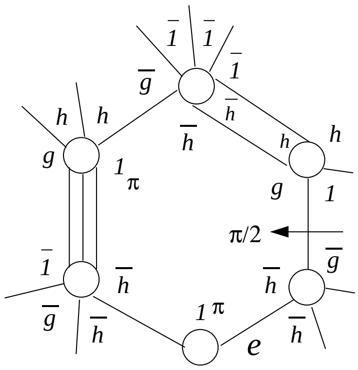


Figure 4.51: $\Gamma_2 = 1$ and $\Phi(\hat{\Delta}) = 2$ is not possible.

Proof of Lemma 4.5.17. If $\frac{1}{|g|} + \frac{1}{|gh^{-1}|} + \frac{1}{|h|} > 1$ then there are non-trivial reduced spherical pictures $\mathbb{P}$ over $\mathcal{P}_{11}$. For example if $(|g|, |gh^{-1}|, |h|) = (3, 2, 3)$ then $\mathbb{P}$ is given by Figure 4.52 (we omit the labellings of hh^{-1}-bonds). The other spheres are constructed in a similar way, we omit the details.

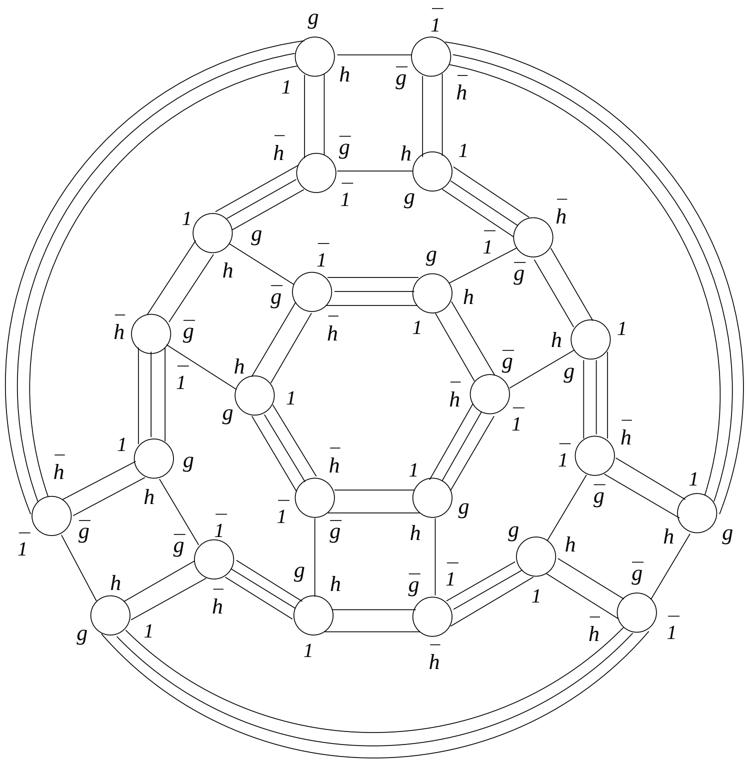


Figure 4.52: $(|g|, |gh^{-1}|, |h|) = (3, 2, 3)$.

Remark 4.5.23. *Let Δ be a region of degree n. Also let $\Phi{=}\Phi(\Delta)$ denote the number of corners of angle π in Δ. Then by assigning the angle functions α_1, α_2 and α_3 to the vertices of $\mathbb{P}$, $\Phi(\Delta) \leq \frac{n}{2}$ (Remarks 4.5.19(1)(i), (2)(i) and (3)(i)).*

Proof of Lemma 4.5.18(1): Case$(2, 2, \infty)$

Here we assume that $|g| = |gh^{-1}| = 2$ and $|h| = \infty$. We show $\mathcal{P}_{11}$ is aspherical. Assume that (**A**) holds and assign the angle function α_3 to the vertices of $\mathbb{P}$. By Remark 4.5.19(3)(i) the degree of a positive region Δ can only be 4 or 6. For positive 4-regions the allowed labelling is $g^{-1}hg^{-1}h$ which gives the positive regions shown in Figure 4.53. However the labelling for positive 6-regions contradict the assumptions. Distribute the curvature as shown.

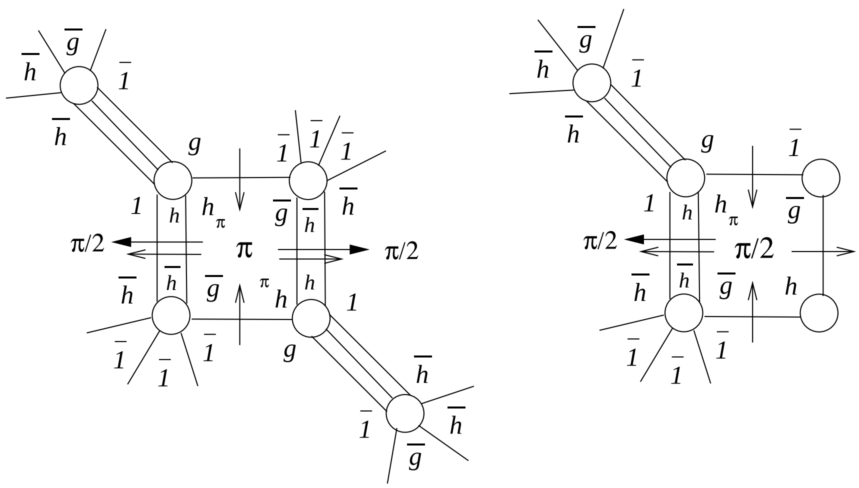


Figure 4.53: Positively curved regions in Case$(2, 2, \infty)$ and distribution scheme.

Remarks 4.5.24.

1. *Observe that for each $g^{\pm 1}$-corner in $\hat{\Delta}$, Γ_2 is decreased by 1.*

2. *Let k denote the number of $g^{\pm 1}$-corners in $\hat{\Delta}$. Since $g^2 = (gh^{-1})^2 = 1$ in $gp\{g, h\}$, k is even.*

3. Since $\hat{\Delta}$ receives $\pi/2$ only across edges that are oriented towards $\hat{\Delta}$, $\Gamma_2 \leq \frac{n}{2}$.

Now $l(\hat{\Delta}) = h^{-1}1h^{-1}w$ and so $d(\hat{\Delta}) > 4$. If $d(\hat{\Delta}) = 6$ then the only possible label is $h^{-1}1h^{-1}h1^{-1}1^{-1}h$. However this implies that $\Phi(\hat{\Delta}) = 0$ and $\Gamma_2 \leq 2$ and so $c^*(\hat{\Delta}) \leq 0$. Thus assume $d(\hat{\Delta}) \geq 8$. If $k \geq 4$ then $\Gamma_2 \leq \frac{n}{2} - 4$ and so $c^*(\hat{\Delta}) \leq (2-n)\pi + \frac{n}{2}.\pi + \frac{n}{2}.\frac{\pi}{2} + (\frac{n}{2} - 4)\frac{\pi}{2}$ =0. Thus k=0 or 2. If k=0 then either $|h| < \infty$, a contradiction or $l(\hat{\Delta})$ involves two of the sublabels $h^{-1}(11^{-1})^k h$ ($k \geq 0$) which forces $c^*(\hat{\Delta}) \leq 0$. Hence k=2. If $\Phi(\hat{\Delta}) \leq \frac{n}{2} - 2$, then $c^*(\hat{\Delta}) \leq (2-n)\pi + (\frac{n}{2}-2)\pi + (\frac{n}{2}+2)\frac{\pi}{2} + (\frac{n}{2}-2)\frac{\pi}{2} = 0$. Thus up to inversion, $l(\hat{\Delta}) = gw_1g^{\varepsilon}w_2$ where $\varepsilon = \pm 1$. Clearly then the maximum possible number of $h^{\pm 1}$-corners in either w_1 or w_2 must have angle π, say w_2. This forces $w_2 = h^{-1}$, $\varepsilon = 1$ and the situation is shown in Figure 4.54.

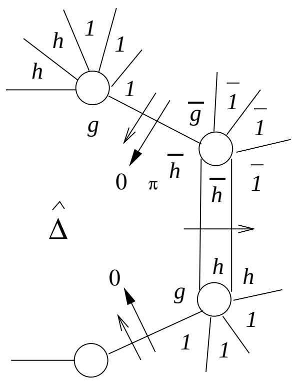


Figure 4.54: $k = 2$ and $\Phi(\hat{\Delta}) > \frac{n}{2} - 2$.

Proof of Lemma 4.5.18(2): Case$(2, 3, \bar{6})$

Here we assume that $|g| = 2$, $|gh^{-1}| = 3$, $|h| \geq 6$ and $g \neq h^3$. We show that $\mathcal{P}_{11}$ is aspherical. Assign the angle function α_3 on $\mathbb{P}$.

By Remark 4.5.19(3)(ii), the possible labels for a positive 4-region must involve $h^{\pm 1}$ and each yields a contradiction. By Remark 4.5.19(3)(iii), possible positive 6-regions give the labelling $(hg^{-1})^3 = 1$. Therefore there is only one positive region which is given by Figure 4.55. Distribute the curvature as shown.

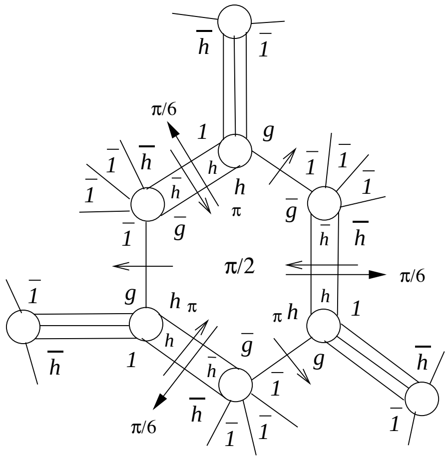


Figure 4.55: Positively curved region in Case$(2, 3, \bar{6})$ and distribution scheme.

Remarks 4.5.25.

1. *$l(\hat{\Delta}) = h^{-1}1h^{-1}w$ and so $d(\hat{\Delta}) > 4$.*

2. *Since $\hat{\Delta}$ receives $\pi/6$ only across edges that are oriented towards $\hat{\Delta}$, $\Gamma_6 \leq \frac{n}{2}$.*

Observe that $c^*(\hat{\Delta}) \leq (2-n)\pi + \frac{n}{2}.\pi + \frac{n}{2}.\frac{\pi}{2} + \frac{n}{2}.\frac{\pi}{6} = \pi\big(2 - n + \frac{n}{2} + \frac{n}{4} + \frac{n}{12}\big)$. The fact that $c^*(\hat{\Delta}) > 0$ implies that $n < 12$.

Let $\hat{\Delta}$ be a 6-region. Then the possible label is $h^{-1}1h^{-1}h1^{-1}h$. However this implies $\Phi(\hat{\Delta}) \leq 1$ and so $c^*(\hat{\Delta}) \leq 0$. Now let $\hat{\Delta}$ be an 8-region. If $\Gamma_6 = 4$ then $h^4 = 1$, a contradiction. Now $\hat{\Delta}$ involves 4 corners with angle π. For otherwise, $c^*(\hat{\Delta}) \leq -6\pi + 3\pi + 5.\frac{\pi}{2} + 3.\frac{\pi}{6} = 0$. Hence $l(\hat{\Delta}) = h^{-1}1h^{-1}w_1h^{-1}w_2h^{-1}w_3$, where $w_1, w_2, w_3 \in \{1, g, h\}$. By inspection $c^*(\hat{\Delta}) > 0$ forces $w_i = 1$ $(1 \leq i \leq 3)$ which implies $h^4 = 1$, a contradiction. Finally let $\hat{\Delta}$ be a 10-region. Assume that $\hat{\Delta}$ contains 5 corners with angle π, or else $c^*(\hat{\Delta}) \leq -8\pi + 4\pi + 6.\frac{\pi}{2} + 5.\frac{\pi}{6} < 0$. Thus as before $l(\hat{\Delta})$ implies $h^5 = 1$, a contradiction.

Proof of Lemma 4.5.18(3): Case$(2, \bar{4}, \bar{4})$

Here $|g| = 2$, $|gh^{-1}| \geq 4$ and $|h| \geq 4$. Let $\mathbb{P}$ be a reduced spherical picture over $\mathcal{P}_{11}$ and assign the angle function α_3 to $\mathbb{P}$. Then $\mathcal{P}_{11}$ is aspherical unless one of the following conditions holds: $h^2 = 1$ or $h^3 = 1$ or$(gh^{-1})^2 = 1$ or $(gh^{-1})^3 = 1$. However each of these conditions contradicts the assumptions and so $\mathcal{P}_{11}$ is aspherical.

Proof of Lemma 4.5.18(4): Case$(2, \infty, 2)$

Here $|g| = |h| = 2$ and $|gh^{-1}| = \infty$. We show $\mathcal{P}_{11}$ is aspherical. Assign the angle function α_2 to the vertices of $\mathbb{P}$. By Remark 4.5.19(2)(i) the degree of a positive region Δ can only be 4 or 6. For positive 4-regions the only allowed labelling is $g1^{-1}g1^{-1}$ which gives the two positive regions shown in Figure 4.56. However, by Remark 4.5.19(2)(iii) there are three occurrences of $g^{\pm 1}$ in the labellings of positive 6-regions, and each yields a contradiction.

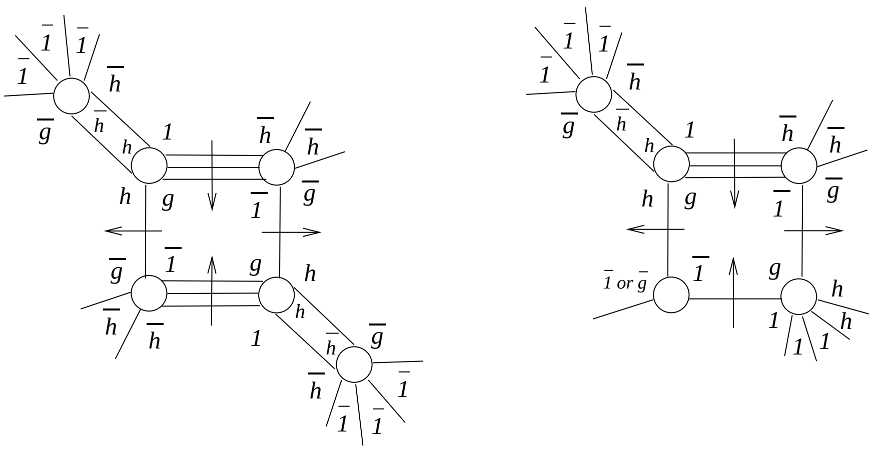


Figure 4.56: Positively curved region in Case$(2, \infty, 2)$.

Apply the following distribution scheme which is shown in Figure 4.57:

$$\Gamma(\Delta,\hat{\Delta}) = \begin{cases} c(\Delta)/2 & \text{if } \frac{\pi}{2} < c(\Delta) \leq \pi \text{ and } \Delta \text{ is separated from } \hat{\Delta} \text{ by a single} \\ & \text{bond that is oriented from } \Delta \text{ to } \hat{\Delta} \\ c(\Delta) & \text{if } 0 < c(\Delta) \leq \frac{\pi}{2} \text{ and } \Delta \text{ is separated from } \hat{\Delta} \text{ by a single} \\ & \text{ bond } S \text{ that is oriented from } \Delta \text{ to } \hat{\Delta} \text{ and } S \text{ is} \\ & \text{ adjacent to a } g^{\pm 1}\text{-corner in } \Delta \text{ with angle } \pi \\ 0 & \text{otherwise} \end{cases}$$

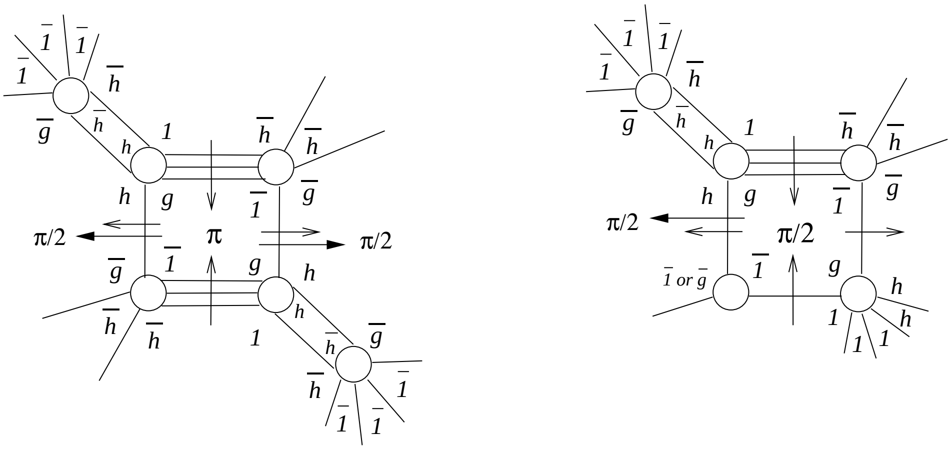


Figure 4.57: Distribution scheme in Case$(2,\infty,2)$.

Remarks 4.5.26.

1. *Since $\hat{\Delta}$ receives $\frac{\pi}{2}$ only across edges that are oriented towards $\hat{\Delta}$, $\Gamma_2 \leq \frac{n}{2}$.*

2. *Let e denote the edge across which $\hat{\Delta}$ receives curvature. One of the corners of $\hat{\Delta}$ that is adjacent to e is labelled by $h^{\pm 1}$, the other is labelled by $1^{\pm 1}$ or $g^{\pm 1}$.*

3. *Observe that $c^*(\hat{\Delta}) \leq (2-n)\pi + \frac{n}{2}.\pi + \frac{n}{2}.\frac{\pi}{2} + \frac{n}{2}.\frac{\pi}{2} = 2\pi$. This upper bound for $c^*(\hat{\Delta})$ holds by assuming that $\hat{\Delta}$ involves $\frac{n}{2}$ corners of angle $\frac{\pi}{2}$, $\frac{n}{2}$ corners of angle π and $\Gamma_2 = \frac{n}{2}$.*

4. *Since $g^2 = h^2 = 1$ and $|gh^{-1}| = \infty$, if w=1 in $gp\{g,h\}$ then a cyclic permutation of w involves either the sublabel gg twice or the sublabel hh twice or the sublabels gg and hh (for example, the relator $hghgghgh$ involves gg and hh). The possible sublabels of $l(\hat{\Delta})$ that give gg and hh (respectively) are: $g(1^{-1}1)^k1^{-1}g (k \geq 0)$ and $\{h(1^{-1}1)^r1^{-1}h, h(1^{-1}1)^rh^{-1} (r \geq 0)\}$ (respectively). Note that $g(1^{-1}1)^kg^{-1} (k \geq 0)$ cannot occur as a sublabel of a region in $\mathbb{P}$ for (up to bridge moves) $\mathbb{P}$ is not reduced.*

Claim: If $l(\hat{\Delta})$ includes one of the sublabels that give gg or hh, then $c^*(\hat{\Delta})$ is decreased at least by π.

Proof of Claim. It is enough to prove the result for the sublabels $g1^{-1}g$, $h1^{-1}h$, $h1^{-1}1h^{-1}$ and hh^{-1}.

- $g1^{-1}g$: by Remark 4.5.26(2), $\hat{\Delta}$ receives 0 across the shown edge in Figure 4.58(i). If the g-corner of vertex v has angle $\leq \frac{\pi}{2}$ then we are done. Otherwise v is given by Figure 4.58(ii). In this case $\hat{\Delta}$ receives 0 also across e and so either each of $\Phi(\hat{\Delta})$ and Γ_2 is decreased by 1 or Γ_2 is decreased by 2.

- hh^{-1}: then $\Phi(\hat{\Delta})$ is decreased by 1. By inspection (Figure 4.57) $\hat{\Delta}$ receives 0 across each of the edges e_1 and e_2 as shown in Figure 4.58(iii). Thus Γ_2 is decreased by 2.

- $h1^{-1}1h^{-1}$: then $\Phi(\hat{\Delta}) \leq \frac{n}{2} - 2$.

- $h1^{-1}h$: then $\Phi(\hat{\Delta})$ is decreased by 1. Also either Γ_2 is decreased by 1 and we are done, or the situation shown in Figure 4.58(iii) is forced. However, in the latter case Γ_2 is also decreased by 1. This is because the corner indicated by $*$ is labelled by h or g. Thus either Δ_1 is not the positive region $(g1^{-1})^2$ shown in Figure 4.56 or $d(v) \geq 4$ and so $\Gamma(\Delta_1, \hat{\Delta}) = 0$. This completes the proof of the claim.

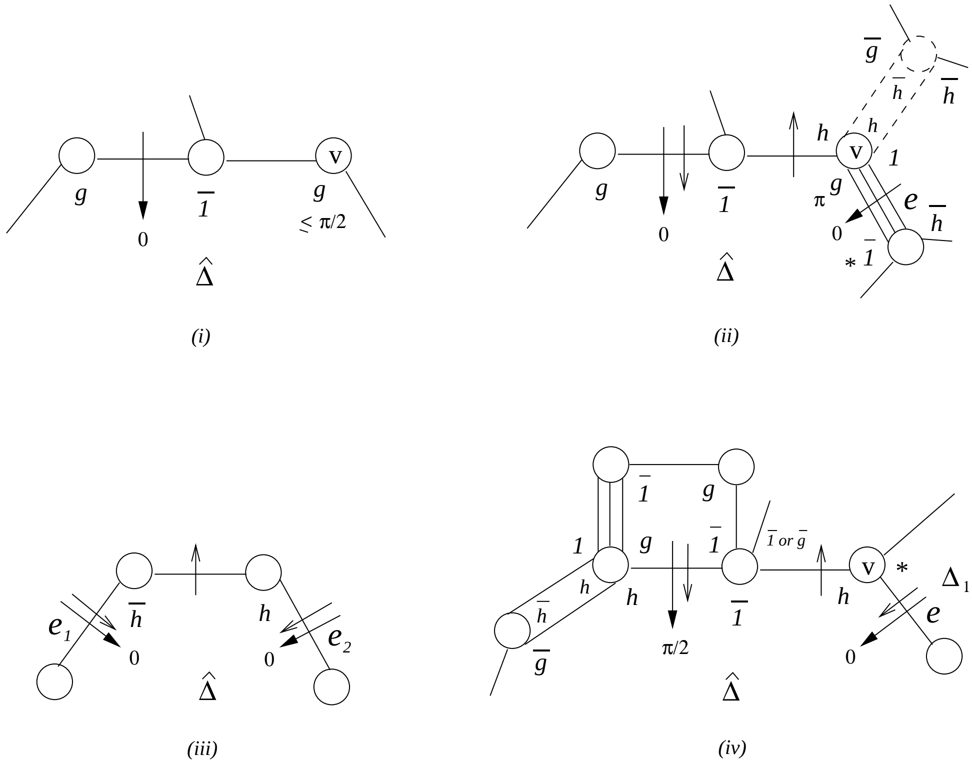


Figure 4.58: The sublabels $g1^{-1}g$, hh^{-1} and $h1^{-1}h$.

Observe that the $*$-corner in Figure 4.58(ii) is labelled by 1^{-1} and not g^{-1}. Thus if $l(\hat{\Delta})$ involves two of the sublabels $(g1^{-1}g)^{\pm 1}$ then $c^*(\hat{\Delta})$ is decreased at least by 2π. Now if hh^{-1} occurs in $l(\hat{\Delta})$ with any of the sublabels $g1^{-1}g$, hh^{-1}, $h1^{-1}1h^{-1}$ and $h1^{-1}h$ then $\Phi + \Gamma_2 \leq \frac{n}{2} - 4$ and so $c^*(\hat{\Delta}) \leq 0$. It now follows from Remarks 4.5.26(3), (4), the above claim and Figure 4.58 that $c^*(\hat{\Delta}) \leq 0$ except possibly when $l(\hat{\Delta})$ involves two of the sublabels $(h1^{-1}h)^{\pm 1}$. In this case $\hat{\Delta}$ is given by Figure 4.58(iv) so that $\hat{\Delta}$ receives 0 across the edge e as shown. Now if $\hat{\Delta}$ involves another sublabel of the form $(h1^{-1}h)^{\pm 1}$, then by the same argument $\hat{\Delta}$ receives 0 across an edge, say e_1. The concern is that e can coincide with e_1 and so $c^*(\hat{\Delta})$ is decreased by $\frac{3\pi}{2}$ not 2π. This can only occur in the case shown in Figure 4.59. However, in this case $\Phi(\hat{\Delta})$ is decreased by 3 and Γ_2 is decreased by 1. Thus $c^*(\hat{\Delta}) \leq 0$.

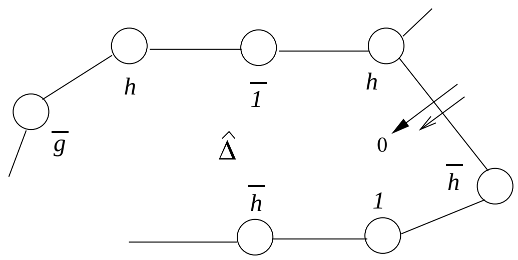


Figure 4.59: $\hat{\Delta}$ involves the sublabels $(h1^{-1}h)^{\pm 1}$ twice where the edges e and e_1 coincide.

Proof of Lemma 4.5.18(5): Case$(2, \bar{6}, 3)$

Here $|g| = 2$, $|gh^{-1}| \geq 6$ and $|h| = 3$. Assume that $[g, h] \neq 1$. Then we show $\mathcal{P}_{11}$ is aspherical. Assign the angle function α_2 on $\mathbb{P}$. By Remark 4.5.19(2)(i) the degree of a positive region Δ can only be 4 or 6. For positive 4-regions the only allowed labelling is $g1^{-1}g1^{-1}$ which gives the two positive regions shown in Figure 4.60. However, by Remark 4.5.19(2)(iii) there are three occurrences of $g^{\pm 1}$ in the labellings of positive 6-regions, and each yields a contradiction. For example, if $(gh^{-1})^2g = 1$ then $(gh^{-1})^3 = 1$, contradicting $|gh^{-1}| \geq 6$.

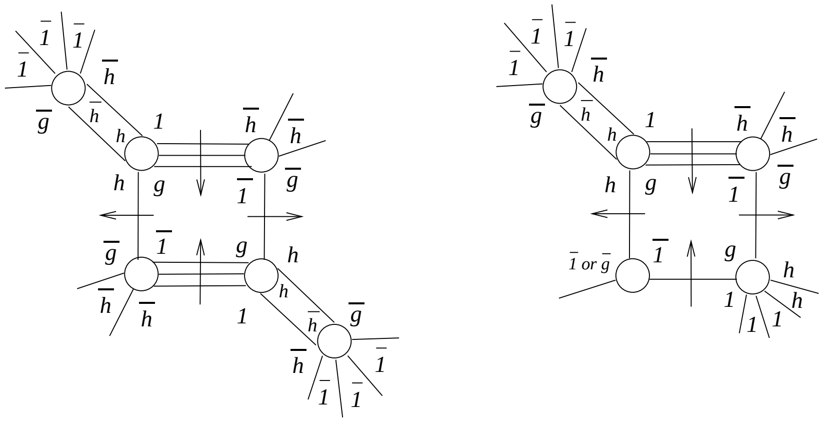


Figure 4.60: Positively curved regions in Case$(2, \bar{6}, 3)$.

Apply the following distribution scheme which is shown in Figure 4.61:

$$\Gamma(\Delta,\hat{\Delta}) = \begin{cases} c(\Delta)/3 & \text{if } c(\Delta) > 0 \text{ and } \Delta \text{ is separated from } \hat{\Delta} \text{ by a 3-bond} \\ & \text{that is oriented from } \hat{\Delta} \text{ to } \Delta \\ c(\Delta)/6 & \text{if } \frac{\pi}{2} < c(\Delta) \leq \pi \text{ and } \Delta \text{ is separated from } \hat{\Delta} \text{ by} \\ & \text{a single bond that is oriented from } \Delta \text{ to } \hat{\Delta} \\ c(\Delta)/3 & \text{if } 0 < c(\Delta) \leq \frac{\pi}{2} \text{ and } \Delta \text{ is separated from } \hat{\Delta} \text{ by} \\ & \text{ a single bond } S \text{ that is oriented from } \Delta \text{ to } \hat{\Delta} \text{ and } S \\ & \text{is adjacent to a } g^{\pm 1}\text{-corner in } \Delta \text{ with angle } \pi \\ 0 & \text{otherwise} \end{cases}$$

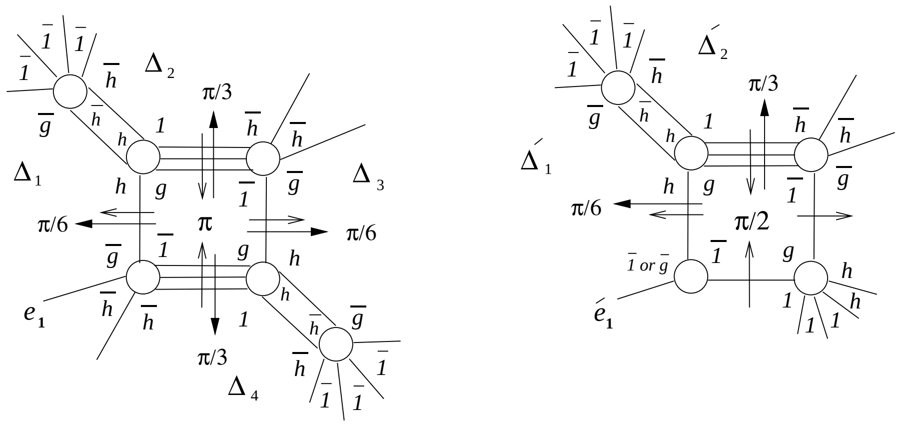


Figure 4.61: Distribution scheme in Case$(2, \bar{6}, 3)$.

Claim: Curvature cannot be distributed into $\hat{\Delta}$ across adjacent arcs in the boundary of $\hat{\Delta}$. (In particular, $\Gamma_3 + \Gamma_6 \leq \frac{n}{2}$).

Proof of Claim. Up to inversion, $\hat{\Delta}$ could be only one of the regions Δ_1, Δ_2, Δ_3, Δ_4, Δ_1' or Δ_2' that are shown in Figure 4.61. If $\hat{\Delta}= \Delta_1$ then $\Gamma(\Delta_2, \hat{\Delta}) = 0$ since Δ_2 has corner with label h^{-1}. Also, $\hat{\Delta} = \Delta_1$ does not receive curvature across e_1 for the same reason. We follow the same argument if $\hat{\Delta} = \Delta_2$, Δ_3, Δ_4

or Δ_2'. Now let $\hat{\Delta} = \Delta_1'$. Then as above $\Gamma(\Delta_2', \hat{\Delta}) = 0$. Note that the edge e_1' cannot be a 3-bond and so $\hat{\Delta} = \Delta_1'$ does not receive any curvature across e_1'. This completes the proof of the claim.

Remarks 4.5.27.

1. *If $\hat{\Delta}$ receives $\frac{\pi}{6}$ then $l(\hat{\Delta}) = \{1^{-1}, g^{-1}\}hg^{-1}w_1$, while if it receives $\frac{\pi}{3}$ then $l(\hat{\Delta}) = h^{-1}1h^{-1}w_2$. Therefore $d(\hat{\Delta}) > 4$.*
2. *By Remark 4.5.23, $\Phi(\hat{\Delta}) \leq \frac{n}{2}$. For each $\frac{\pi}{3}$ that $\hat{\Delta}$ receives, $\Phi(\hat{\Delta})$ will be decreased by 1 and so $c(\hat{\Delta})$ will be decreased by $\pi/2$.*
3. *For each $\frac{\pi}{6}$ that $\hat{\Delta}$ receives, $\Phi(\hat{\Delta})$ will be decreased by 1 or the situation shown in Figure 4.62 holds (since $d(v) = 3$ forces this situation).*

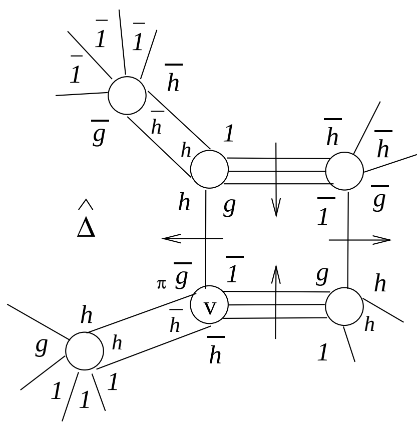


Figure 4.62: $l(\hat{\Delta}) = hg^{-1}hg^{-1}w$.

It follows from the claim and Remark 4.5.27(2), that if $d(\hat{\Delta}) = n$ then $c^*(\hat{\Delta}) \leq (2-n)\pi + \frac{n}{2}.\pi + \frac{n}{2}.\frac{\pi}{2} + \frac{n}{2}.\frac{\pi}{6} \leq 0 \Leftrightarrow 2 - \frac{n}{4} + \frac{n}{12} \leq 0 \Leftrightarrow n \geq 12$ and so it remains to check degrees 6, 8 and 10 only.

Firstly, let $\hat{\Delta}$ be a 6-region. Then the possible labellings are: $h1^{-1}h1^{-1}h1^{-1}$, $h^{-1}1h^{-1}h1^{-1}h$ and $g^{-1}hg^{-1}1h^{-1}1$. For the region labelled by $h1^{-1}h1^{-1}h1^{-1}$ note that $\Gamma_6 = 0$ by Remark 4.5.27(1). Thus $c^*(\hat{\Delta}) \leq -4\pi + 6.\frac{\pi}{2} + 3.\frac{\pi}{3} = 0$. The same applies to the region labelled by $h^{-1}1h^{-1}h1^{-1}h$. It remains to check the region $\hat{\Delta}$ shown in Figure 4.63. By Remark 4.5.27(1), $\Gamma_3 = 0$ and so $\Gamma_6 = 1$. Note that $d(v) \geq 4$ and so the g^{-1}-corner of v has angle $\leq \frac{\pi}{2}$. Thus $c^*(\hat{\Delta}) \leq -4\pi + \pi + 5.\frac{\pi}{2} + \frac{\pi}{6} = -\frac{\pi}{2} + \frac{\pi}{6} < 0$.

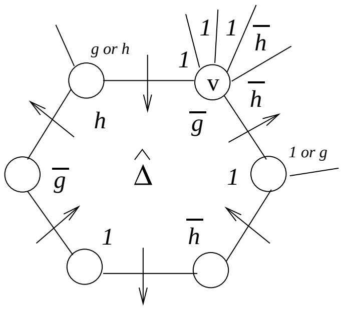


Figure 4.63: $l(\hat{\Delta}) = g^{-1}hg^{-1}1h^{-1}1$.

Now let $\hat{\Delta}$ be an 8-region. If $\Gamma_6 = 0$ then by Remark 4.5.27(2), $c^*(\hat{\Delta}) < 0$. Moreover by Remarks 4.5.27(2), (3) and the above claim, $l(\hat{\Delta}) = hg^{-1}hg^{-1}w$ or else $c^*(\hat{\Delta}) \leq 0$. However, $l(\hat{\Delta}) = hg^{-1}hg^{-1}w$ yields a contradiction. For example, if $(g^{-1}h)^3h = 1$ then $h = (g^{-1}h)^{-3}$ and $g = (g^{-1}h)^{-4}$. Since $g^2 = h^3 = 1$ this gives that $g^{-1}h = 1$, a contradiction.

Finally, let $\hat{\Delta}$ be a 10-region. Since $(2-10)\pi + 5.\pi + 5.\frac{\pi}{2} = -\frac{\pi}{2}$, it remains to check the cases $\Gamma_6 = 5$ and $\Gamma_6 = 4$. The first implies $(gh^{-1})^5 = 1$, a contradiction. By Remarks 4.5.27(2), (3) and the above claim, the second implies $(gh^{-1})^5 = 1$, a contradiction.

Proof of Lemma 4.5.18(6): Case$(\infty, 2, 2)$

Here we assume that $|h| = |gh^{-1}| = 2$ and $|g| = \infty$. We show $\mathcal{P}_{11}$ is aspherical. Assume that (**A**) holds and assign the angle function α_1 to the vertices of $\mathbb{P}$. By Remark 4.5.19(1)(ii) the degree of a positive region Δ can only be 4 or 6. For positive 4-regions the allowed labelling is $1h^{-1}1h^{-1}$. This labelling gives the positive regions shown in Figure 4.64. By Remark 4.5.19(1)(iv) the possible labellings for positive 6-regions are $11^{-1}11^{-1}11^{-1}$ and $11^{-1}1h^{-1}1h^{-1}$. However these labellings imply that $\mathbb{P}$ is not reduced.

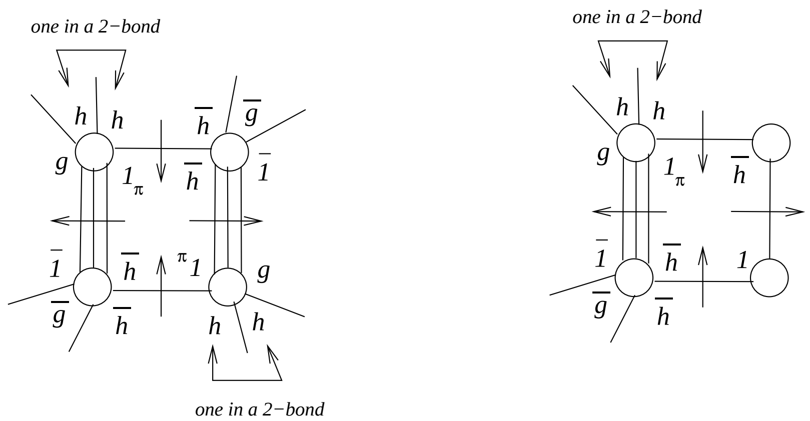


Figure 4.64: Positively curved regions in Case$(\infty, 2, 2)$.

Apply the following distribution scheme which is shown in Figure 4.65:

$$\Gamma(\Delta,\hat{\Delta}) = \begin{cases} c(\Delta)/2 & \text{if } \frac{\pi}{2} < c(\Delta) \leq \pi \text{ and } \Delta \text{ is separated from } \hat{\Delta} \text{ by} \\ & \text{ a 3-bond that is oriented from } \Delta \text{ to } \hat{\Delta} \\ c(\Delta) & \text{if } 0 < c(\Delta) \leq \frac{\pi}{2} \text{ and } \Delta \text{ is separated from } \hat{\Delta} \text{ by} \\ & \text{ a 3-bond } S \text{ that is oriented from } \Delta \text{ to } \hat{\Delta} \text{ and} \\ & \text{ } S \text{ is adjacent to an } h^{\pm 1}\text{-corner in } \Delta \text{ with angle } \pi \\ 0 & \text{otherwise} \end{cases}$$

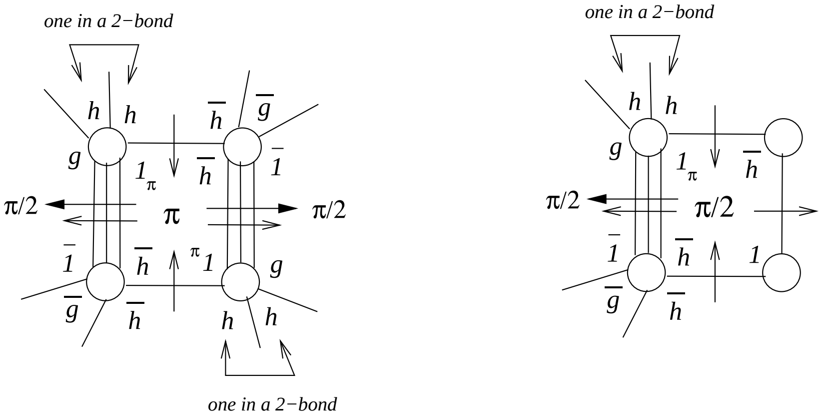


Figure 4.65: Distribution scheme in Case$(\infty, 2, 2)$.

Remarks 4.5.28.

1. *Observe that for each $h^{\pm 1}$-corner in $\hat{\Delta}$, Γ_2 is decreased by 1.*

2. *Let k denote the number of $h^{\pm 1}$-corners in $\hat{\Delta}$. Since $h^2 = (gh^{-1})^2 = 1$ in $gp\{g, h\}$, k is even.*

3. *Since $\hat{\Delta}$ receives $\pi/2$ only across edges that are oriented outwards $\hat{\Delta}$, $\Gamma_2 \leq \frac{n}{2}$.*

If $k \geq 4$ then $\Gamma_2 \leq \frac{n}{2} - 4$ and so $c^*(\hat{\Delta}) \leq (2-n)\pi + \frac{n}{2}.\pi + \frac{n}{2}.\frac{\pi}{2} + (\frac{n}{2} - 4)\frac{\pi}{2}$ $=0$. Thus k=0 or 2. However, $l(\hat{\Delta})$ should involve $h^{\pm 1}$-corners, for otherwise $\mathbb{P}$ is not reduced or $|g| < \infty$. Hence k=2. If $\Phi(\hat{\Delta}) \leq \frac{n}{2} - 2$, then $c^*(\hat{\Delta}) \leq (2-n)\pi + (\frac{n}{2} - 2)\pi + (\frac{n}{2} + 2)\frac{\pi}{2} + (\frac{n}{2} - 2)\frac{\pi}{2} = 0$. Thus up to inversion, $l(\hat{\Delta}) = hw_1h^{\varepsilon}w_2$ where $\varepsilon = \pm 1$. Clearly then the maximum possible number of $1^{\pm 1}$-corners in either w_1 or w_2 must have angle π, say w_2. This forces $w_2 = 1^{-1}$, $\varepsilon = 1$ and the situation is shown in Figure 4.66. Since $c^*(\hat{\Delta}) > c(\hat{\Delta})$, w_1 must involve at least one $g^{\pm 1}$-corner. But h^2=1 and $|g| = \infty$ imply that w_1 involves both $g-$ and $g^{-1}-$ corners and so $\mathbb{P}$ is not reduced. It follows that $c^*(\hat{\Delta}) \leq 0$.

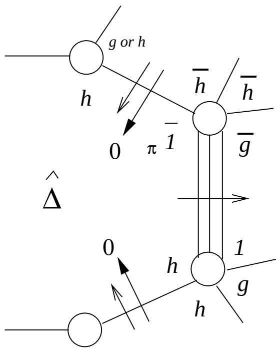


Figure 4.66: $k = 2$ and $\Phi(\hat{\Delta}) > \frac{n}{2} - 2$.

Proof of Lemma 4.5.18(7): Case$(\bar{6}, 3, 2)$

Here we assume that $|g| \geq 6$, $|gh^{-1}| = 3$, $|h| = 2$ and $h \neq g^3$. We show that $\mathcal{P}_{11}$ is aspherical. Assign the angle function α_2 on $\mathbb{P}$.

By Remark 4.5.19(2)(ii), the possible labels for a positive 4-region must involve $g^{\pm 1}$ and each yields a contradiction. By Remark 4.5.19(2)(iii), possible positive 6-regions give the labelling $(gh^{-1})^3 = 1$. Therefore there is only one positive region which is given by Figure 4.67.

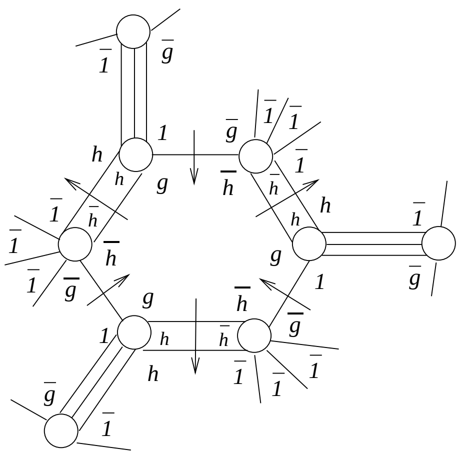


Figure 4.67: Positively curved region in Case$(\bar{6}, 3, 2)$.

Apply the following distribution scheme:

$$\Gamma(\Delta,\hat{\Delta}) = \begin{cases} c(\Delta)/3 & \text{if } c(\Delta) > 0 \text{ and } \Delta \text{ is separated from } \hat{\Delta} \text{ by a single} \\ & \text{bond that is oriented from } \hat{\Delta} \text{ to } \Delta \\ 0 & \text{otherwise} \end{cases}$$

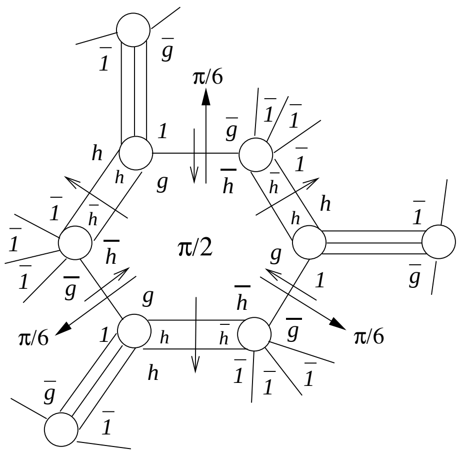


Figure 4.68: Distribution scheme in Case$(\bar{6},3,2)$.

Remarks 4.5.29.

1. *Since $\hat{\Delta}$ receives $\pi/6$ only across edges that are oriented outwards $\hat{\Delta}$, $\Gamma_6 \leq \frac{n}{2}$.*

2. *$l(\hat{\Delta}) = g^{-1}1g^{-1}w$ and so $d(\hat{\Delta}) > 4$.*

Observe that $c^*(\hat{\Delta}) \leq (2-n)\pi + \frac{n}{2}.\pi + \frac{n}{2}.\frac{\pi}{2} + \frac{n}{2}.\frac{\pi}{6} = \pi\big(2-n+\frac{n}{2}+\frac{n}{4}+\frac{n}{12}\big)$. The fact that $c^*(\hat{\Delta}) > 0$ implies that $n < 12$.

Let $\hat{\Delta}$ be a 6-region. Then each possible label yields a contradiction. So let $\hat{\Delta}$ be an 8-region. If $\Gamma_6 = 4$ then $g^4 = 1$, a contradiction. Now $\hat{\Delta}$ involves

4 corners with angle π. For otherwise, $c^*(\hat{\Delta}) \leq -6\pi + 3\pi + 5.\frac{\pi}{2} + 3.\frac{\pi}{6} = 0$. Hence $l(\hat{\Delta}) = g^{-1}1g^{-1}w_1g^{-1}w_2g^{-1}w_3$, where $w_1, w_2, w_3 \in \{1, h\}$. By inspection $c^*(\hat{\Delta}) > 0$ forces $w_i = 1$ $(1 \leq i \leq 3)$ which implies $g^4 = 1$, a contradiction.

Finally let $\hat{\Delta}$ be a 10-region. Assume that $\hat{\Delta}$ contains 5 corners with angle π, or else $c^*(\hat{\Delta}) \leq -8\pi+4\pi+6.\frac{\pi}{2}+5.\frac{\pi}{6} < 0$. Hence $l(\hat{\Delta}) = g^{-1}1g^{-1}w_1g^{-1}w_2g^{-1}w_3g^{-1}w_4$, where $w_i \in \{1, h\}$ $(1 \leq i \leq 4)$. By inspection $c^*(\hat{\Delta}) > 0$ forces $w_i = 1$ $(1 \leq i \leq 4)$ which implies $g^5 = 1$, a contradiction.

Proof of Lemma 4.5.18(8): Case$(\bar{4}, \bar{4}, 2)$

Here $|g| \geq 4$, $|gh^{-1}| \geq 4$ and $|h| = 2$. Let $\mathbb{P}$ be a reduced spherical picture over $\mathcal{P}_{11}$ and assign the angle function α_2 to $\mathbb{P}$. Then $\mathcal{P}_{11}$ is aspherical unless one of the following conditions holds: $g^2 = 1$ or $g^3 = 1$ or$(gh^{-1})^2 = 1$ or $(gh^{-1})^3 = 1$. Therefore by the assumptions $\mathcal{P}_{11}$ is aspherical.

Proof of Lemma 4.5.18(9): Case$(3, \bar{6}, 2)$

Here we assume that $|g| = 3$, $|gh^{-1}| \geq 6$ and $|h| = 2$. Assume that $[g, h] \neq 1$. Then we show $\mathcal{P}_{11}$ is aspherical. Assign the angle function α_2 on $\mathbb{P}$. Positive regions can be of degree 4 or 6 only (Remark 4.5.19(2)(i)). However, all possible labellings for positive 4-regions are not allowed, while positive 6-regions give the labelling $(g1^{-1})^3$. Thus there is only one positive region which is shown in Figure 4.69.

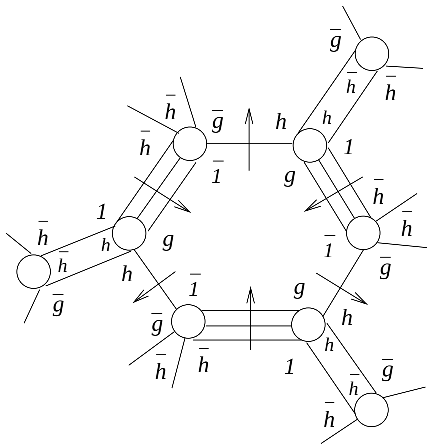


Figure 4.69: Positively curved region in Case$(3, \bar{6}, 2)$.

Apply the following distribution scheme which is shown in Figure 4.70:

$$\Gamma(\Delta, \hat{\Delta}) = \begin{cases} \frac{\pi}{6} & \text{if } c(\Delta) = \frac{\pi}{2} \text{ and } \Delta \text{ is separated from } \hat{\Delta} \text{ by a single bond} \\ & \text{that is oriented from } \Delta \text{ to } \hat{\Delta} \\ 0 & \text{otherwise} \end{cases}$$

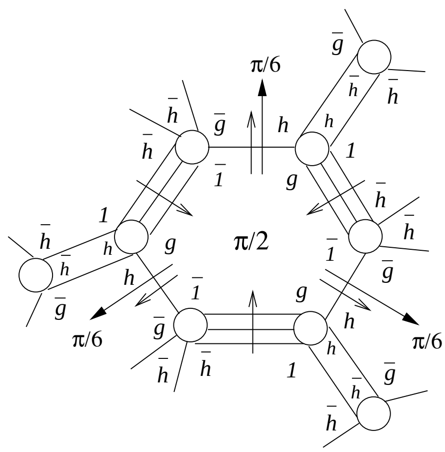


Figure 4.70: Distribution scheme in Case$(3, \bar{6}, 2)$.

Remarks 4.5.30.

1. *Since $\hat{\Delta}$ receives $\frac{\pi}{6}$ only across edges that are oriented towards $\hat{\Delta}$,* $\Gamma_6 \leq \frac{n}{2}$.

2. $l(\hat{\Delta}) = g^{-1}hg^{-1}w$ *and so* $d(\hat{\Delta}) > 4$.

Observe that $c^*(\hat{\Delta}) \leq (2-n)\pi + \frac{n}{2}.\pi + \frac{n}{2}.\frac{\pi}{2} + \frac{n}{2}.\frac{\pi}{6} = \pi\big(2 - n + \frac{n}{2} + \frac{n}{4} + \frac{n}{12}\big)$. The fact that $c^*(\hat{\Delta}) > 0$ implies that $n < 12$.

Let $\hat{\Delta}$ be a 6-region. Then all the possible labellings yield a contradiction. Now let $\hat{\Delta}$ be an 8-region. If $\Gamma_6 = 4$ then $(g^{-1}h)^4 = 1$, a contradiction. Since $(2-8)\pi + 3\pi + 5.\frac{\pi}{2} + 3.\frac{\pi}{6} = 0$, it remains to check the case in which $\hat{\Delta}$ involves 4 corners with angle π. Thus $l(\hat{\Delta})= g^{-1}hg^{-1}w_1g^{-1}w_2g^{-1}w_3$, where $w_i \in \{1, h\}$ $(1 \leq i \leq 3)$. By inspection $w_3 = h$ and since $\Phi = 4$, $l(\hat{\Delta})$ implies $(gh^{-1})^4$=1, a contradiction. It remains to consider the case when $\hat{\Delta}$ is a 10-region. If $\Gamma_6 = 5$ then $(g^{-1}h)^5 = 1$ contradicting $|gh^{-1}| \geq 6$. Since $(2-10)\pi + 4\pi + 6.\frac{\pi}{2} + 4.\frac{\pi}{6} < 0$ and $(2-10)\pi + 5\pi + 5.\frac{\pi}{2} + 3.\frac{\pi}{6} = 0$, it remains to check when $\hat{\Delta}$ includes 5 corners with angle. Thus $l(\hat{\Delta})= g^{-1}hg^{-1}w_1g^{-1}w_2g^{-1}w_3g^{-1}w_4$, where $w_i \in \{1, h\}$ $(1 \leq i \leq 4)$. As before $w_4 = h$ and $l(\hat{\Delta})$ implies $(gh^{-1})^5$=1, a contradiction.

Proof of Lemma 4.5.18(10): Case$(3, 2, \bar{6})$

Here $|g| = 3$, $|gh^{-1}| = 2$ and $|h| \geq 6$. Assume that $g \neq h^4$. Then we show $\mathcal{P}_{11}$ is aspherical. Assign the angle function α_3 on $\mathbb{P}$. Positive regions can be of degree 4 or 6 only (Remark 4.5.19(3)(i)). For positive 4-regions the possible labels give the regions shown in Figure 4.71. However, by Remark 4.5.19(3)(iii) there are three occurrences of $h^{\pm 1}$ in the labellings of positive 6-regions, and each yields a contradiction.

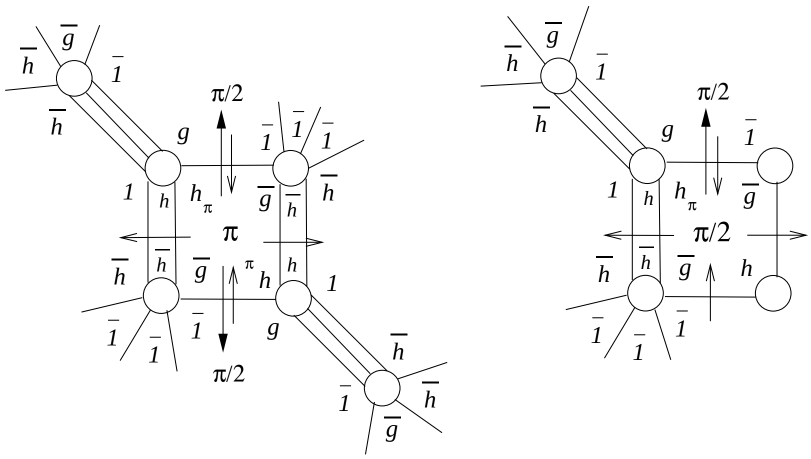


Figure 4.71: Positively curved regions in Case$(3, 2, \bar{6})$ and distribution scheme.

Remarks 4.5.31.

1. *$l(\hat{\Delta}) = 1^{-1}g1^{-1}w$ and so $d(\hat{\Delta}) > 4$.*

2. *For each $\frac{\pi}{2}$ that $\hat{\Delta}$ receives, $\Phi(\hat{\Delta})$ will be decreased by 1 and so $c(\hat{\Delta})$ will be decreased by $\frac{\pi}{2}$.*

Let $\hat{\Delta}$ be a 6-region. Then $l(\hat{\Delta}) = 1^{-1}g1^{-1}g1^{-1}g$. Observe that if $l(\hat{\Delta})$ implies $ghg^{-1}h = 1$, then $g^{-1}(g^{-1}h)^2 = 1$ which implies $g = 1$, a contradiction. Thus $l(\hat{\Delta}) = 1^{-1}g1^{-1}g1^{-1}g$ and $\Gamma_2 = 3$ for otherwise $c^*(\hat{\Delta}) \leq -4\pi + 6.\frac{\pi}{2} + 2.\frac{\pi}{2} = 0$. So the situation in Figure 4.72 is forced and $c^*(\hat{\Delta}) \leq -4\pi + 6.\frac{\pi}{2} + 3.\frac{\pi}{2} = \frac{\pi}{2}$ is distributed as shown. Note that $l(\hat{\Delta}_1) = h^{-1}1h^{-1}w$ and so $d(\hat{\Delta}_1) \geq 6$. If $d(\hat{\Delta}_1) = 6$ then the possible label is $h^{-1}1h^{-1}h1^{-1}h$. However in this case $\Gamma_2 = 0$ and $\Phi(\hat{\Delta}) \leq 1$ and so $c(\hat{\Delta}) \leq 0$.

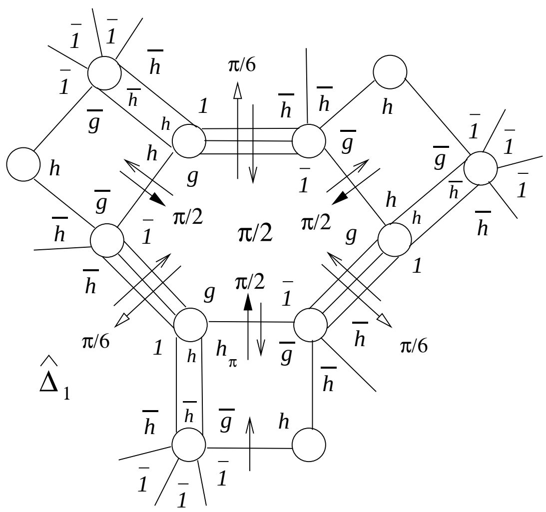


Figure 4.72: $l(\hat{\Delta}) = 1^{-1}g1^{-1}g1^{-1}g$ and $\Gamma_2 = 3$.

Now let $\hat{\Delta}$ be an 8-region. Then $c(\hat{\Delta}) \leq -4\pi + 4\pi + 4.\frac{\pi}{2} = 0$. Since $\hat{\Delta}$ does not receive curvature across adjacent edges (because distributing is always opposite the orientation of the labelling) in $\partial\hat{\Delta}$ and by Remark 4.5.31(2) it can be assumed that $\Gamma_2 = 0$. If $\Gamma_6 = 4$ then $h^4 = 1$ which contradicts $|h| \geq 6$, while if $\Gamma_6 \leq 3$ then $\Phi(\hat{\Delta}) = 4$ or else $c^*(\hat{\Delta}) \leq -6\pi + 3\pi + 5.\frac{\pi}{2} + 3.\frac{\pi}{6} = 0$. Thus as before $h^4 = 1$ is forced, a contradiction.

Let $\hat{\Delta}$ be a 10-region. As before it can be assumed that $\Gamma_2 = 0$. Assume that $\Phi(\hat{\Delta}) = 5$ since otherwise $c^*(\hat{\Delta}) \leq -8\pi + 4\pi + 6.\frac{\pi}{2} + 5.\frac{\pi}{6} < 0$. Thus $l(\hat{\Delta})$ forces $h^5 = 1$, a contradiction.

Finally assume that $d(\hat{\Delta}) = n \geq 12$. Again it can be assumed that $\Gamma_2 = 0$. Thus $c^*(\hat{\Delta}) \leq (2-n)\pi + \frac{n}{2}.\pi + \frac{n}{2}.\frac{\pi}{2} + \frac{n}{2}.\frac{\pi}{6} = \pi(2 - \frac{n}{6}) \leq 0$. Therefore $\mathcal{P}_{11}$ is aspherical except possibly if $g = h^4$.

Proof of Lemma 4.5.18(11): Case$(\bar{6}, 2, 3)$

Here $|g| \geq 6$, $|gh^{-1}| = 2$ and $|h| = 3$. Assume that $h \neq g^4$. Then we show $\mathcal{P}_{11}$ is aspherical. Assign the angle function α_2 on $\mathbb{P}$. Positive regions can be of degree 4 or 6 only (Remark 4.5.19(2)(i)). For positive 4-regions the possible labels give the regions shown in Figure 4.73. However, by Remark 4.5.19(2)(ii) there are three occurrences of $g^{\pm 1}$ in the labellings of positive 6-regions, and each yields a contradiction.

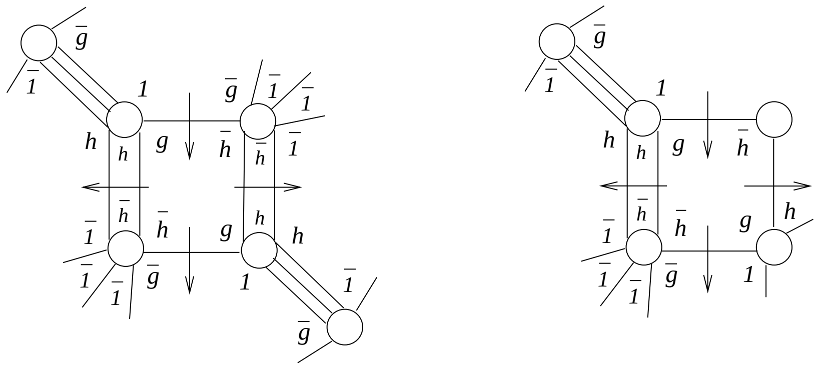


Figure 4.73: Positively curved regions in Case($\bar{6}, 2, 3$).

Apply the following distribution scheme which is shown in Figure 4.74:

$$\Gamma(\Delta, \hat{\Delta}) = \begin{cases} c(\Delta)/2 & \text{if } \frac{\pi}{2} < c(\Delta) \leq \pi \text{ and } \Delta \text{ is separated from } \hat{\Delta} \text{ by} \\ & \text{a 2-bond that is oriented from } \Delta \text{ to } \hat{\Delta} \\ c(\Delta) & \text{if } 0 < c(\Delta) \leq \frac{\pi}{2} \text{ and } \Delta \text{ is separated from } \hat{\Delta} \text{ by} \\ & \text{a 2-bond } S \text{ that is oriented from } \Delta \text{ to } \hat{\Delta} \text{ and } S \\ & \text{is adjacent to a } g^{\pm 1}\text{-corner in } \Delta \text{ with angle } \pi \\ 0 & \text{otherwise} \end{cases}$$

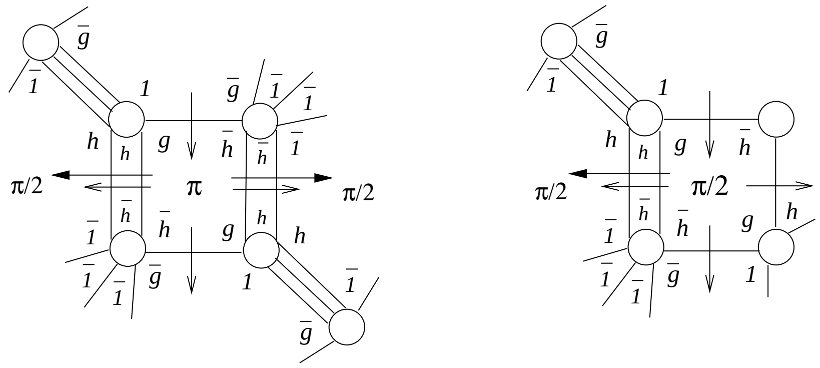


Figure 4.74: Distribution scheme in Case$(\bar{6}, 2, 3)$.

Remarks 4.5.32.

1. *$l(\hat{\Delta}) = 1^{-1}h1^{-1}w$ and so $d(\hat{\Delta}) > 4$.*

2. *For each $\frac{\pi}{2}$ that $\hat{\Delta}$ receives, $\Phi(\hat{\Delta})$ will be decreased by 1 and so $c(\hat{\Delta})$ will be decreased by $\frac{\pi}{2}$.*

Let $\hat{\Delta}$ be a 6-region. Then $l(\hat{\Delta}) = 1^{-1}h1^{-1}h1^{-1}h$ or $1^{-1}h1^{-1}1h^{-1}1$. For the first possibility, $c(\hat{\Delta}) \leq -4\pi + 6.\frac{\pi}{2} = -\pi$ and so if $\Gamma_2 = 2$ then $c^*(\hat{\Delta}) \leq 0$. Thus $\hat{\Delta}$ should have the form shown in Figure 4.75 and $c^*(\hat{\Delta}) \leq -\pi + 3.\frac{\pi}{2} = \frac{\pi}{2}$ is distributed as shown. Note that $l(\hat{\Delta}_1) = 1g^{-1}1\{g^{-1}, h^{-1}\}w$ and so $d(\hat{\Delta}_1) > 6$. If $l(\hat{\Delta}) = 1^{-1}h1^{-1}1h^{-1}1$ then $\Gamma_6 = 0$ and $\Gamma_2 \leq 2$ and so $c^*(\hat{\Delta}) \leq -4\pi + 6.\frac{\pi}{2} + 2.\frac{\pi}{2} = 0$.

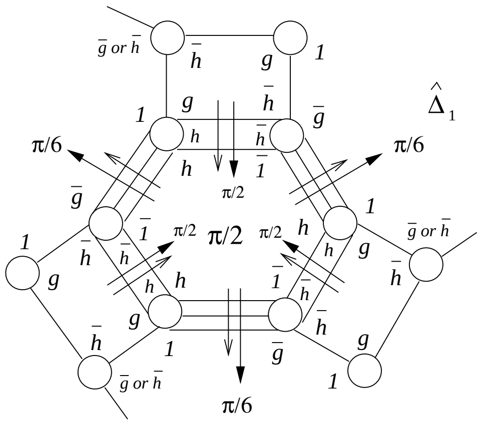


Figure 4.75: $l(\hat{\Delta}) = 1^{-1}h1^{-1}h1^{-1}h$ and $\Gamma_2 = 3$.

Now let $\hat{\Delta}$ be an 8-region. Then $c(\hat{\Delta}) \leq -4\pi + 4\pi + 4.\frac{\pi}{2} = 0$. Since $\hat{\Delta}$ does not receive curvature across adjacent edges (because distributing is always in the same orientation of the labelling) in $\partial\hat{\Delta}$ and by Remark 4.5.32(2) it can be assumed that $\Gamma_2 = 0$. If $\Gamma_6 = 4$ then $g^4 = 1$ which contradicts $|g| \geq 6$, while if $\Gamma_6 \leq 3$ then $\hat{\Delta}$ contains 4 corners with angle π or else $c^*(\hat{\Delta}) \leq -6\pi + 3\pi + 5.\frac{\pi}{2} + 3.\frac{\pi}{6} = 0$. Thus $l(\hat{\Delta}) = 1g^{-1}1g^{-1}w_1g^{-1}w_2g^{-1}$, where $w_1, w_2 \in \{1, h\}$. However, each of these possibilities contradicts the assumptions.

Let $\hat{\Delta}$ be a 10-region. As above it can be assumed that $\Gamma_2 = 0$. Assume that $\hat{\Delta}$ involves 5 corners with angle π since otherwise $c^*(\hat{\Delta}) \leq -8\pi + 4\pi + 6.\frac{\pi}{2} + 5.\frac{\pi}{6} < 0$. Thus $l(\hat{\Delta}) = 1g^{-1}1g^{-1}w_1g^{-1}w_2g^{-1}w_3g^{-1}$, where $w_1, w_2, w_3 \in \{1, h\}$. Each of these possibilities yields a contradiction.

Finally assume that $d(\hat{\Delta}) = n \geq 12$. Again it can be assumed that $\Gamma_2 = 0$. Thus $c^*(\hat{\Delta}) \leq (2-n)\pi + \frac{n}{2}.\pi + \frac{n}{2}.\frac{\pi}{2} + \frac{n}{2}.\frac{\pi}{6} = \pi(2 - \frac{n}{6}) \leq 0$. Therefore $\mathcal{P}_{11}$ is aspherical except possibly if $h = g^4$.

4.5.4 Case 13: (a=b=d, c=e only)

The relative presentation $\mathcal{P}_{13}= \langle G, y|ayaycyaycyfy\rangle=$ $\langle G, y, x|x(ay)^{-1}, x^2ca^{-1}x^2ca^{-1}xfa^{-1}x\rangle= \langle G, x|x^3gx^2gxh\rangle$, where $g = ca^{-1}$ and $h = fa^{-1}$ (and so by assumption, $g, h \in G\backslash\{1\}$ and $g \neq h$).

Let $\mathbb{P}$ be a non-trivial reduced spherical picture over $\mathcal{P}_{13} =\langle G, x|x^3gx^2gxh\rangle$. Then each vertex(disc) in $\mathbb{P}$ has one of the forms given by Figure 4.76(i) and (ii); and the star graph $\mathcal{P}_{13}^{st}$ of $\mathcal{P}_{13}$ is given by Figure 4.76(iii).

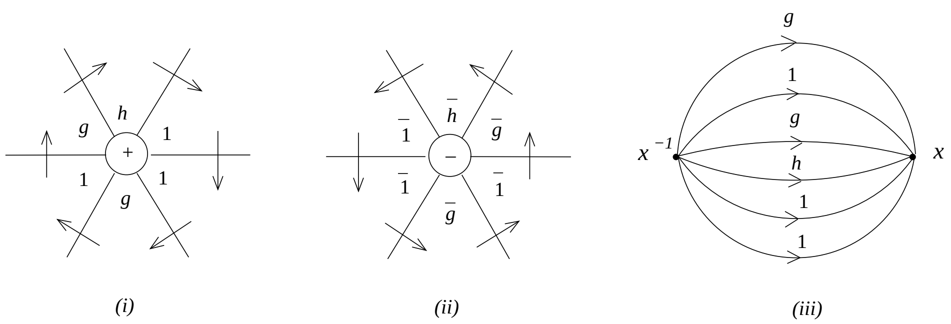


Figure 4.76: + disc, - disc and $\mathcal{P}_{13}^{st}$.

There are (up to inversion) four types of vertices of degree 3 in $\mathbb{P}$ which are shown in Figure 4.77.

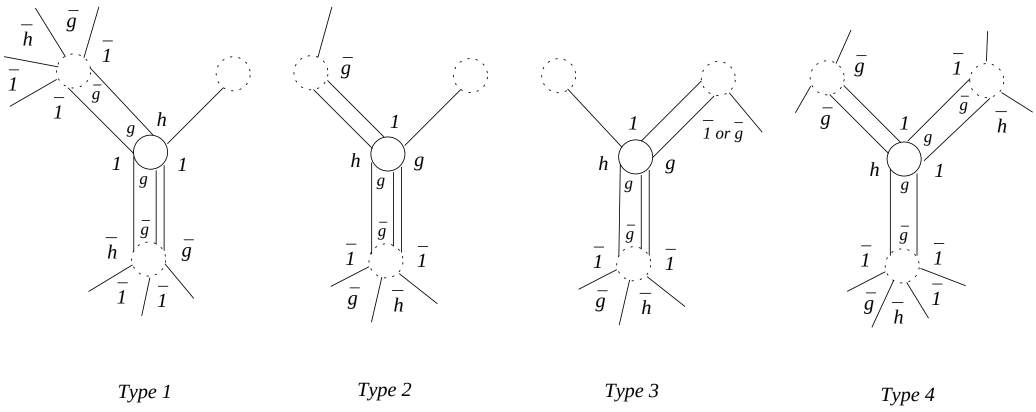


Figure 4.77: Types of vertices of degree 3.

Define an *angle function* $\tilde{\alpha}_1$ as follows. Corners within 2-bonds have angle zero. In vertices of degree 3, corners labelled by $h^{\pm 1}$ have angle π, each of the other two corners has angle $\frac{\pi}{2}$ (see Figure 4.78). If $d(v) \geq 4$, then each corner in v not in a 2-bond has angle $\frac{2\pi}{d(v)}$.

The *angle function* $\tilde{\alpha}_2$ is defined as follows. Again, corners within 2-bonds have angle zero. In each of vertices of degree 3 of Types 1 and 4, the 1-corner in a sub-label $h^{-1}11^{-1}$ has angle π, each of the other two corners has angle $\frac{\pi}{2}$ (see Figure 4.79). However in each of vertices of degree 3 of Types 2 and 3, the g-corner has angle π, each of the other two corners has angle $\frac{\pi}{2}$. If $d(v) \geq 4$, then each corner in v not in a 2-bond has angle $\frac{2\pi}{d(v)}$.

The *angle function* $\tilde{\alpha}_3$ is defined as follows. Again, corners within 2-bonds have angle zero. In vertices of degree 3, corners labelled by $h^{\pm 1}$ have angle π except for Type 3 vertices, where $g^{\pm 1}$-corner has angle π. Each of the two remaining corners has angle $\frac{\pi}{2}$ (see Figure 4.80). If $d(v) \geq 4$, then each corner in v not in a 2-bond has angle $\frac{2\pi}{d(v)}$.

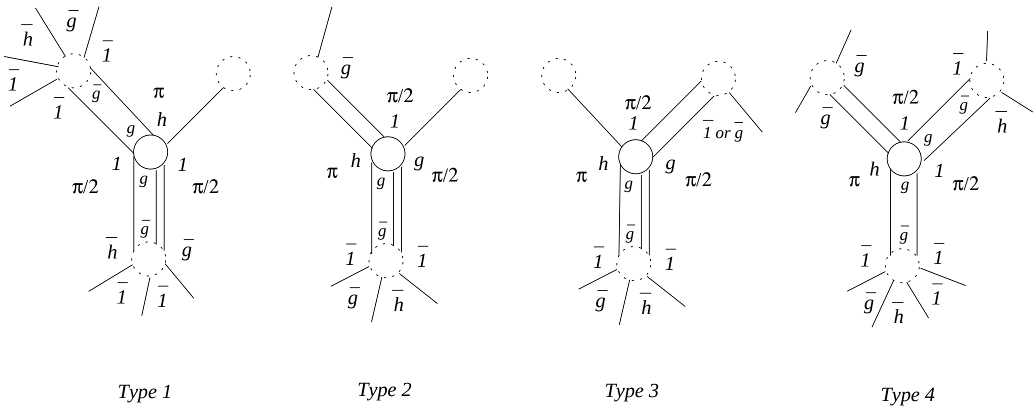


Figure 4.78: Angle function $\tilde{\alpha}_1$ of vertices of degree 3.

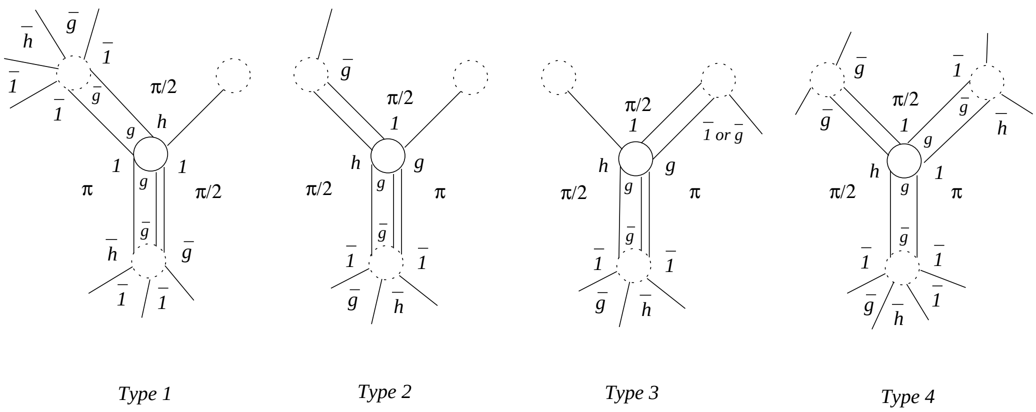


Figure 4.79: Angle function $\tilde{\alpha}_2$ of vertices of degree 3.

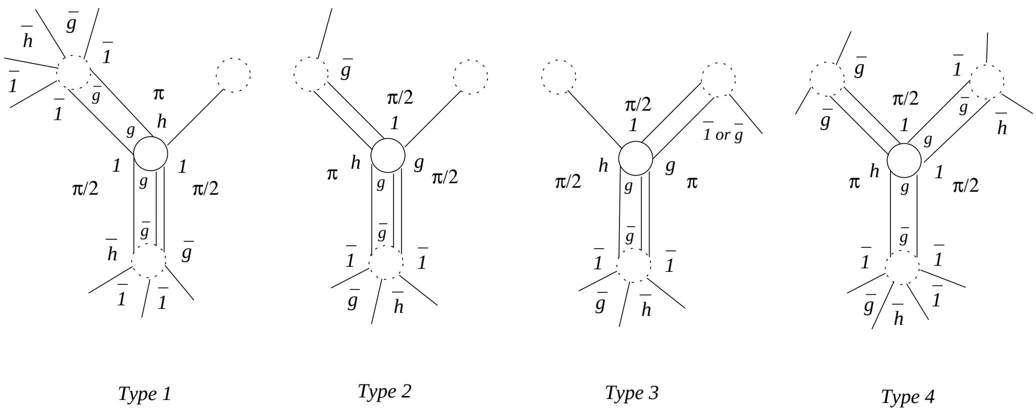


Figure 4.80: Angle function $\tilde{\alpha}_3$ of vertices of degree 3.

Remarks 4.5.33.

1. *By inspection, any region* Δ *with vertex of degree 3 in* $\partial\Delta$ *has at least one* $1^{\pm 1}$*- corner.*

2. *By assigning the angle function* $\tilde{\alpha}_1$ *to the corners of* $\mathbb{P}$*, the following are satisfied:*

   (i) *There are no consecutive corners with angle* π *in the boundary of a region* Δ *of* $\mathbb{P}$ *(otherwise* $\mathbb{P}$ *is not reduced). Thus positive regions can only be 4-regions or 6-regions.*

   (ii) *If* Δ *is a positive 4-region then it contains at least one corner labelled by* $h^{\pm 1}$ *with angle* π*.*

   (iii) *If* Δ *is a positive 6-region then it contains three occurrences of* $h^{\pm 1}$*-corners each with angle* π*.*

   (iv) *As shown in Figure 4.78, for each h-corner with angle* π *in* Δ*, the previous corner is labelled by* 1^{-1} *(anticlockwise direction).*

3. *By assigning the angle function* $\tilde{\alpha}_2$ *to the corners of* $\mathbb{P}$*, the following are satisfied:*

   (i) *By inspection, a* $1^{\pm 1}$*-corner with angle* π *cannot be adjacent to a g-corner with angle* π *in* $\partial\Delta$*. Also since* $d(v) \geq 4$ *in Figure 4.81, there are no consecutive corners with angle* π *in* $\partial\Delta$*. Thus positive regions can only be 4-regions or 6-regions.*

   (ii) *If* Δ *is a positive 4-region, then it contains at least one corner labelled by* $1^{\pm 1}$ *or* $g^{\pm 1}$ *with angle* π*.*

   (iii) *If* Δ *is a positive 6-region, then it contains three corners where each has label* $1^{\pm 1}$ *or* $g^{\pm 1}$ *with angle* π*.*

4. *By assigning the angle function* $\tilde{\alpha}_3$ *to the corners of* $\mathbb{P}$*, the following are satisfied:*

   (i) *As shown in Figure 4.80, the g-corner with angle* π *cannot be adjacent* *to an h^{-1}-corner with angle π. Thus there are no consecutive corners with angle π in the boundary of a region Δ of $\mathbb{P}$ (otherwise $\mathbb{P}$ is not reduced) and so positive regions can only be 4-regions or 6-regions.*

(ii) Observe that the g-corner with angle π lies in between a 2-bond and a 3-bond.

(iii) As shown in Figure 4.82, if Δ involves a g-corner with angle π, then the $*$-corner has angle $\leq \frac{\pi}{2}$. This is because the edge e is a single bond.

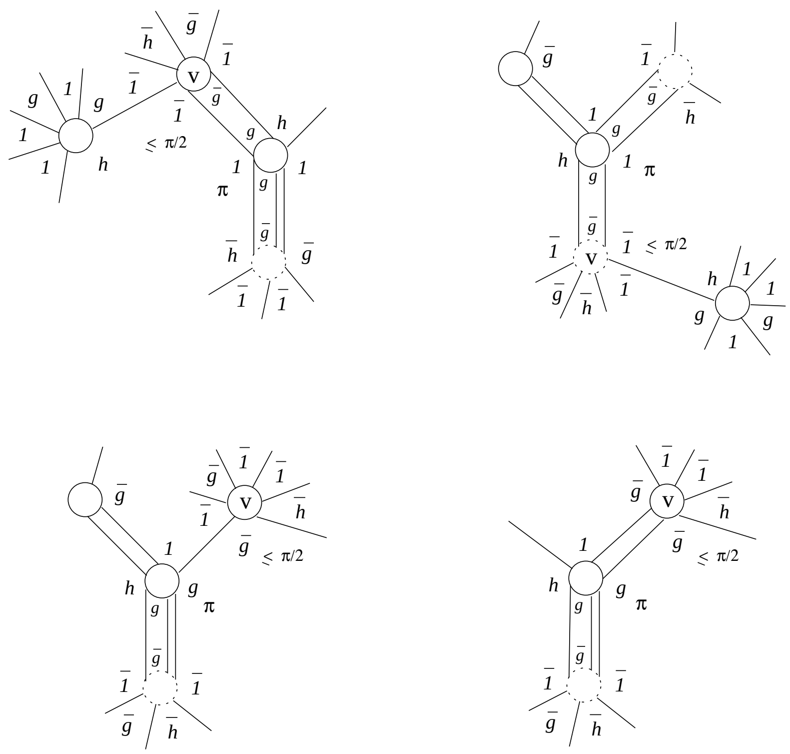


Figure 4.81: $\tilde{\alpha}_2$; no consecutive corners with angle π in $\partial\Delta$.

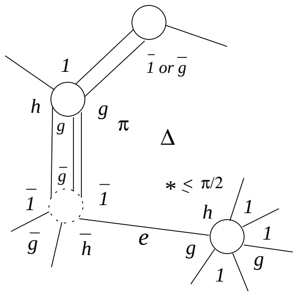


Figure 4.82: $\tilde{\alpha}_3$; g-corner with angle π.

Lemma 4.5.34. *Let $g = h^{-1}$. If $|h| = 3$, then $\mathcal{P}_{13}$ is not aspherical.*

Proof. Let $|h| = 3$. Coset enumeration shows that the order of the group $\langle h, x | x^3h^{-1}x^2h^{-1}xh, h^3\rangle$ equals 1026 and so $\mathcal{P}_{13}$ is not aspherical.

Remark 4.5.35. *From now on we may assume that if $g = h^{-1}$ then $|h| \neq 3$ unless otherwise is stated.*

Lemma 4.5.36. *If $\mathcal{P}_{13}$ is not aspherical, then at least one of the following conditions holds:*

1. $g = h^{-1}$.
2. $g = h^2$.
3. $h^2 = 1$.
4. $h^3 = 1$ *and* $g^2 = 1$.
5. $h^3 = 1$ *and* $h = g^2$.

Proof of Lemma 4.5.36. Let $\mathbb{P}$ be a non-trivial reduced spherical picture over $\mathcal{P}_{13}$. Assume that $g \neq h^{-1}$, $g \neq h^2$ and $h^2 \neq 1$. Also assume (**A**) that the number of 2-bonds of $\mathbb{P}$ is maximal. First assign the standard angle function to the corners of $\mathbb{P}$. By Remark 4.5.33(1), the possible labellings of a positive region Δ imply that $g^2 = 1$ or $h = g^2$ or $l(\Delta) \in \{1^{-1}11^{-1}1, 1^{-1}gg^{-1}1, 1^{-1}1g^{-1}g\}$. However, by bridge moves (see Figure 4.83) and assumption (**A**), we can exclude the last three labels.

Now assign the angle function $\tilde{\alpha}_1$ to the corners of $\mathbb{P}$. By Remark 4.5.33 (2)(i), we check only for regions of degree 4 and 6 only. By the assumptions above there are no positive regions of degree 4. Moreover by Remarks 4.5.33(1) and (2)(iii), the possible labellings for 6-regions imply $h^3 = 1$ and the result follows.

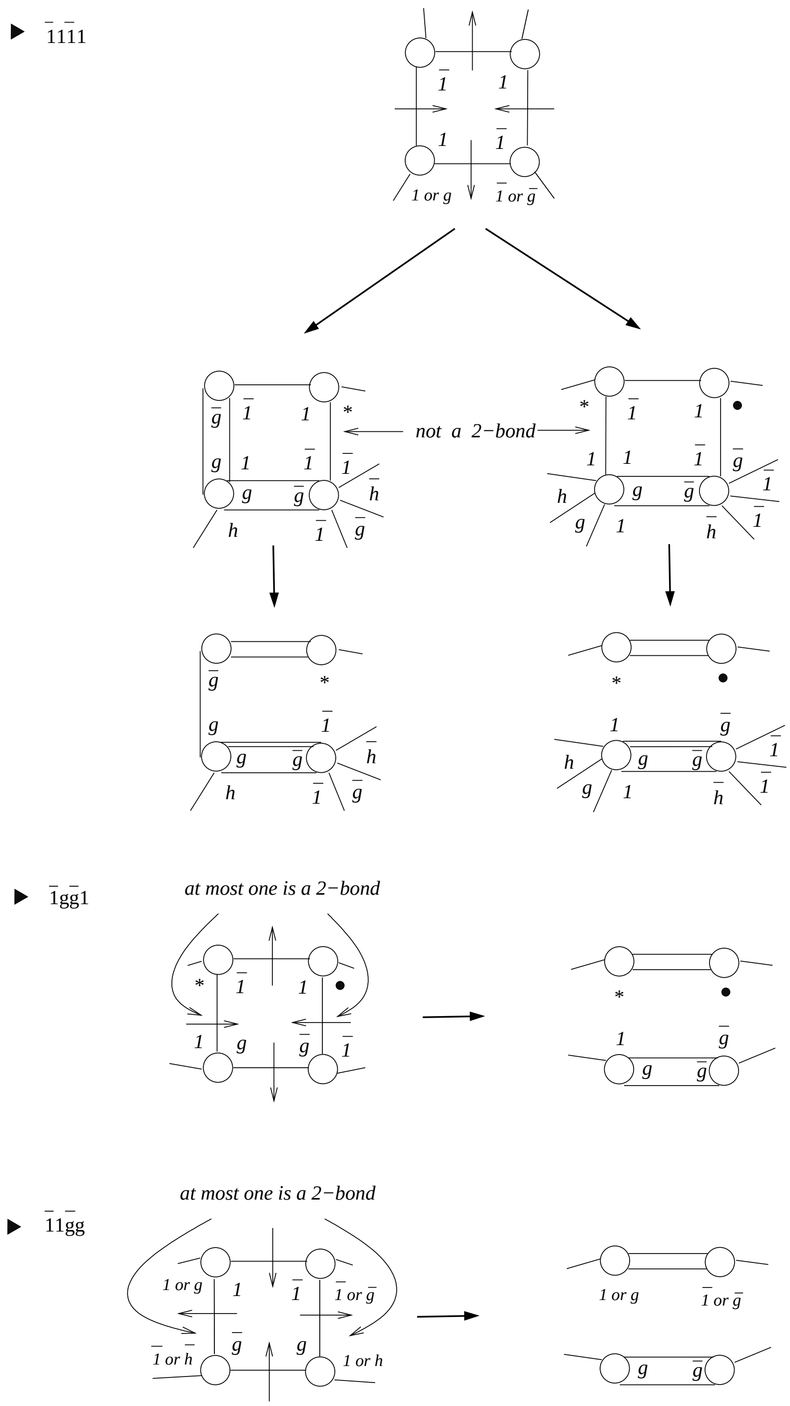


Figure 4.83: Regions with labels $1^{-1}11^{-1}1, 1^{-1}gg^{-1}1$ and $1^{-1}1g^{-1}g$.

Lemma 4.5.37. *If $\mathcal{P}_{13}$ is not aspherical, then at least one of the following conditions holds:*

1. $g = h^{-1}$ *and* $|h| \in \{4, 5, 6\}$.

2. $h = g^2$.

3. $g^2 = 1$ *and* $g = h^2$.

4. $g^2 = 1$ *and* $h^2 = 1$.

Proof of Lemma 4.5.37. Let $\mathbb{P}$ be a non-trivial reduced spherical picture over $\mathcal{P}_{13}$. Assign the angle function $\tilde{\alpha}_2$ to the corners of $\mathbb{P}$ and assume (**A**) holds. Let Δ be a positive region. Then $l(\Delta) = g1^{-1}w_1$ or $h^{-1}11^{-1}w_2$. By Remark 4.5.33(3)(i), n=4 or 6. If $n = 4$ then $l(\Delta) = g1^{-1}1g^{-1}$ or $l(\Delta)$ implies $g = h^{-1}$ or $h = g^2$ or $g^2 = 1$. Now if n=6 then the possible labelling implies $g^3 = 1$. However by (**A**) and bridge moves the region labelled $g1^{-1}1g^{-1}$ does not exist (Figure 4.84). Therefore $\mathcal{P}_{13}$ is aspherical except if $g = h^{-1}$ or $h = g^2$ or $g^2 = 1$ or $g^3 = 1$. Observe that these conditions cannot occur together. For example if $h = g^2$ and $g^3 = 1$, then $h = g^{-1}$ and $|h| = 3$, a contradiction.

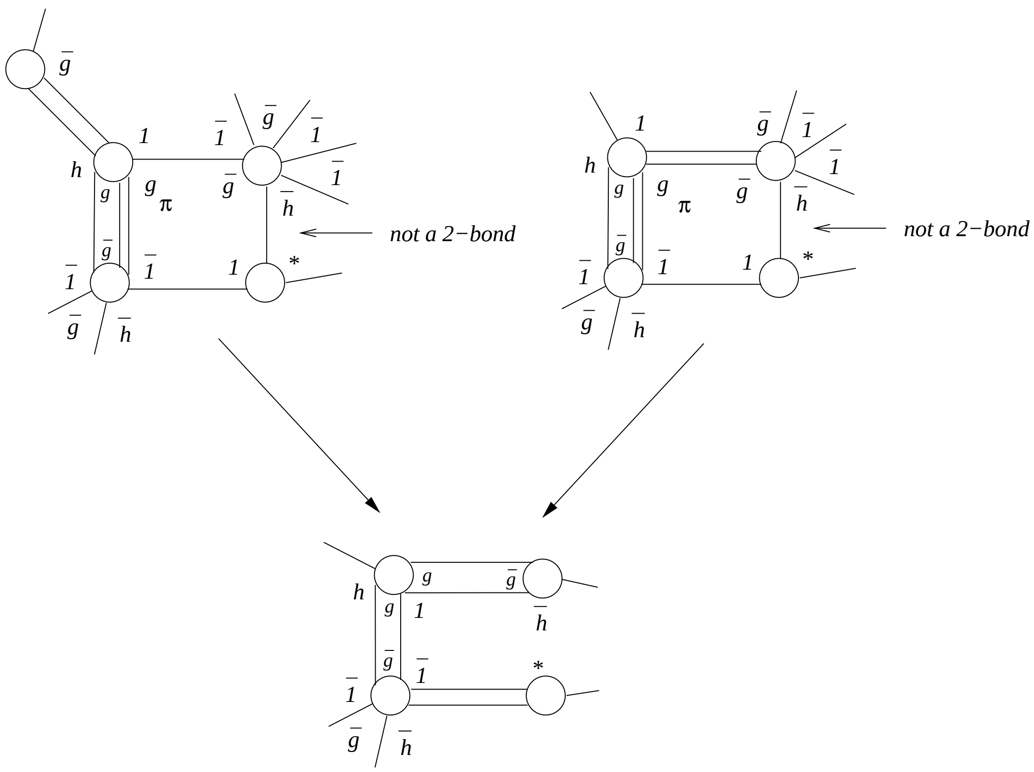


Figure 4.84: The region labelled $g1^{-1}1g^{-1}$ does not exist.

Assume that $g^3 = 1$. Then there is only one positive region shown in Figure 4.85. Distribute the curvature as shown.

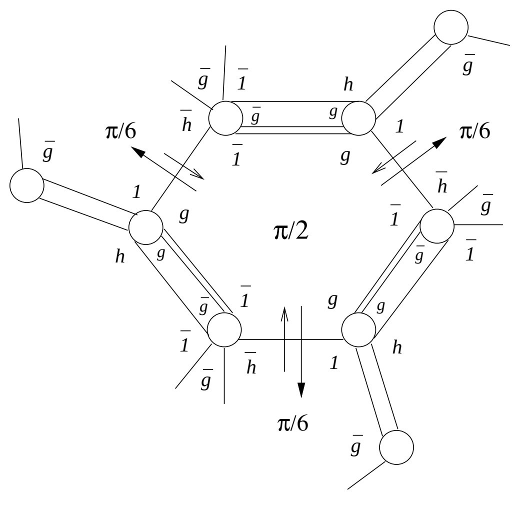


Figure 4.85: Positive region in case $g^3 = 1$.

Remarks 4.5.38.

1. *$l(\hat{\Delta}) = g^{-1}1h^{-1}w$ and so $n = d(\hat{\Delta}) > 4$.*

2. *Since $\hat{\Delta}$ receives $\frac{\pi}{6}$ only across edges that are oriented outwards $\hat{\Delta}$, $\Gamma_6 \leq \frac{n}{2}$.*

3. *For each $\frac{\pi}{6}$ that $\hat{\Delta}$ receives, $\Phi(\hat{\Delta})$ will be decreased by 1.*

Now $c^*(\hat{\Delta}) \leq (2-n)\pi + (\frac{n}{2} - \Gamma_6)\pi + (\frac{n}{2} + \Gamma_6)\frac{\pi}{2} + \Gamma_6.\frac{\pi}{6} < \pi(2 - \frac{n}{4} - \Gamma_6 + \frac{\Gamma_6}{2} + \frac{\Gamma_6}{2})$ $= \pi(2 - \frac{n}{4})$. Therefore $c^*(\hat{\Delta}) > 0$ implies $n = 6$ (Remark 4.5.38(1)). By inspection $\Gamma_6 < 3$. Now since $(2-6)\pi + \pi + 5.\frac{\pi}{2} + 2.\frac{\pi}{6} < 0$, $\hat{\Delta}$ contains at least two corners with angle π. This implies $l(\hat{\Delta}) = g^{-1}1h^{-1}1g^{-1}1$ or $g^{-1}1h^{-1}11^{-1}1$ or $g^{-1}1h^{-1}g1^{-1}1$, contradiction. Thus if $g^3 = 1$ then $\mathcal{P}_{13}$ is aspherical.

Now assume that $g = h^{-1}$. Then the positive regions are shown in Figure 4.86. Distribute the curvature as shown.

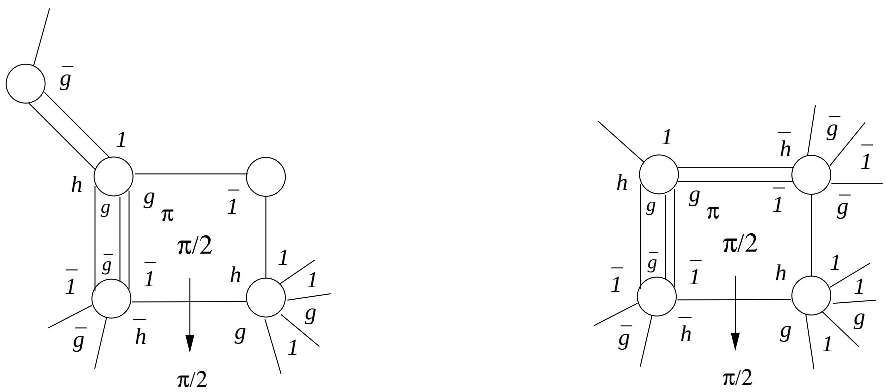


Figure 4.86: Positive regions in case $g = h^{-1}$.

Remarks 4.5.39.

1. *$l(\hat{\Delta}) = gh^{-1}w$.*

2. *Since $\hat{\Delta}$ receives $\frac{\pi}{2}$ only across edges that are oriented outwards $\hat{\Delta}$, $\Gamma_2 \leq \frac{n}{2}$.*

3. *For each $\frac{\pi}{2}$ that $\hat{\Delta}$ receives, $\Phi(\hat{\Delta})$ will be decreased by 1.*

As above n=4 or 6. If n=4 then $|h| = 4$. Let n=6. If $\Gamma_2 = 3$ then $|h| = 6$. If $\Gamma_2 = 2$ then $|h| \in \{4, 5, 6\}$. Thus $\Gamma_2 = 1$ and $\hat{\Delta}$ contains at least two corners with angle π, for otherwise $c^*(\hat{\Delta}) \leq 0$. Assume that (**B**) subject to (**A**), the number of gg^{-1}-bonds is maximal. Thus the $*$-corner in Figure 4.87 cannot have angle π, for otherwise a cut across e_1 (single bond) and e_2 increases the number of gg^{-1}-bonds without decreasing the number of 2-bonds, a contradiction. Therefore $l(\hat{\Delta}) = gh^{-1}11^{-1}g1^{-1}$ is forced. Thus $|h| = 3$, a contradiction and the result follows.

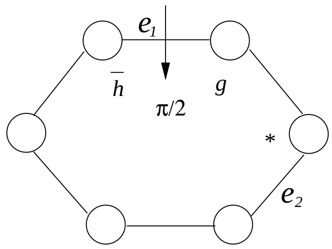


Figure 4.87: $*$-corner cannot have angle π.

Assume that $g^2 = 1$. Then the positive regions are shown in Figure 4.88. Distribute the curvature as shown.

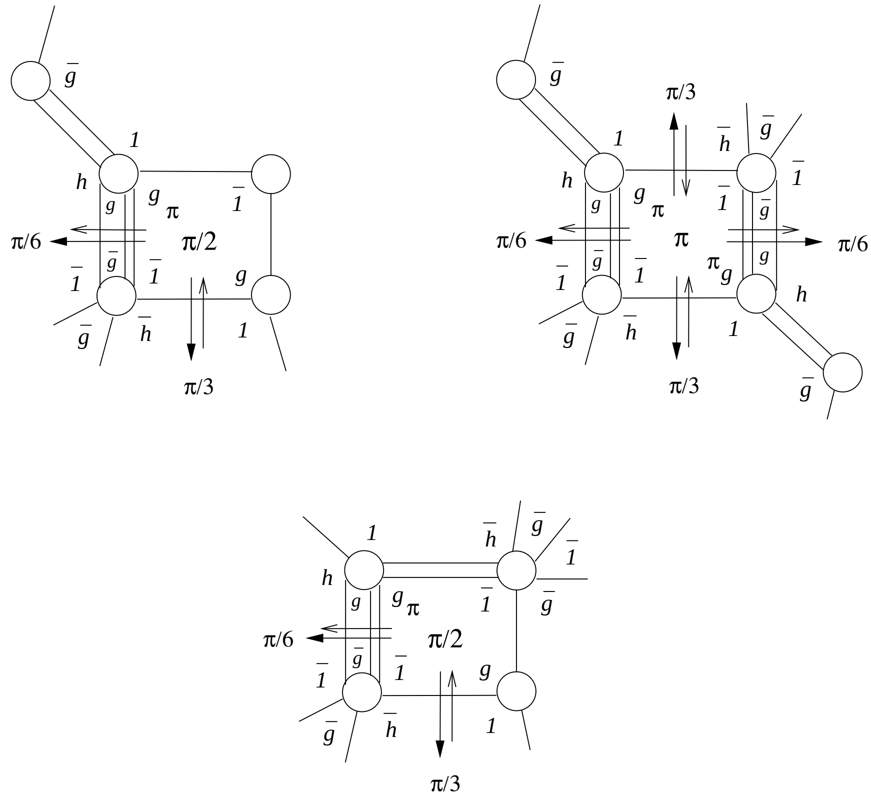


Figure 4.88: Positive regions in case $g^2 = 1$.

Remarks 4.5.40.

1. *Since* $\hat{\Delta}$ *receives* $\frac{\pi}{6}$ *only across edges that are oriented towards* $\hat{\Delta}$, $\Gamma_6 \leq \frac{n}{2}$.

2. *Since* $\hat{\Delta}$ *receives* $\frac{\pi}{3}$ *only across edges that are oriented outwards* $\hat{\Delta}$, $\Gamma_3 \leq \frac{n}{2}$.

3. *By inspection, for each* $\frac{\pi}{3}$ *that* $\hat{\Delta}$ *receives,* $\Phi(\hat{\Delta})$ *will be decreased by 1.*

4. *Observe that* $\hat{\Delta}$ *receives* $\frac{\pi}{6}$ *only across* 3*-bonds, while it receives* $\frac{\pi}{3}$ *only across* 1*-bonds.*

5. *As shown in Figure 4.89, the maximum curvature that* $\hat{\Delta}$ *receives across adjacent edges is* $\frac{\pi}{6} + \frac{\pi}{3}$*. This is because* $\hat{\Delta}$ *receives 0 across e, since e̋ is not a* 3*-bond (see Figure 4.88). Also,* $\hat{\Delta}$ *receives 0 across é, since é is not a* 3*-bond.*

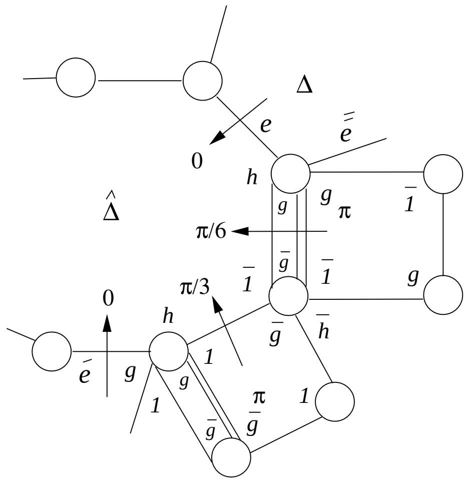


Figure 4.89: Maximum curvature that $\hat{\Delta}$ receives in one row.

Observe that $(2-n)\pi + \frac{n}{2}.\pi + \frac{n}{2}.\frac{\pi}{2} > (2-n)\pi + (\frac{n}{2} - \Gamma_3)\pi + (\frac{n}{2} + \Gamma_3)\frac{\pi}{2} + \Gamma_3.\frac{\pi}{3}$ $= (2-n)\pi + \frac{n}{2}.\pi + \frac{n}{2}.\frac{\pi}{2} - \Gamma_3.\frac{\pi}{2} + \Gamma_3.\frac{\pi}{3}$. By this observation and by Remark 4.5.40(3), we may assume that $\Gamma_3 = 0$ to get an upper bound for $n = d(\hat{\Delta})$ such that $c^*(\hat{\Delta}) > 0$. Now either for each $\frac{\pi}{6}$ that $\hat{\Delta}$ receives, $\Phi(\hat{\Delta})$ will be decreased by 1, or the situation shown in Figure 4.90 is forced. Also, observe that $(2-n)\pi + \frac{n}{2}.\pi + \frac{n}{2}.\frac{\pi}{2} + [\frac{\Gamma_6}{2}].\frac{\pi}{6} > (2-n)\pi + (\frac{n}{2} - \Gamma_6)\pi + (\frac{n}{2} + \Gamma_6)\frac{\pi}{2} + \Gamma_6.\frac{\pi}{6}$. By this observation, we may assume that for each $\frac{\pi}{6}$ that $\hat{\Delta}$ receives, the situation shown in Figure 4.90 holds. Therefore, $c^*(\hat{\Delta}) \leq (2-n)\pi + \frac{n}{2}.\pi + \frac{n}{2}.\frac{\pi}{2} + \frac{n}{4}.\frac{\pi}{6}$. Thus $c^*(\hat{\Delta}) > 0$ implies $n \leq 8$ and $l(\hat{\Delta}) = 1h^{-1}w_1$ or $1^{-1}hw_2$.

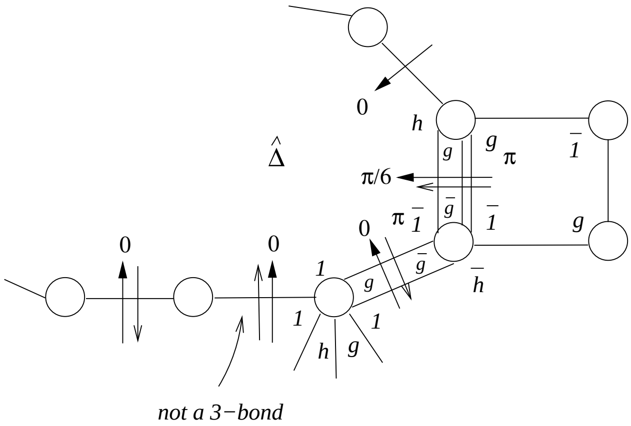


Figure 4.90: $\hat{\Delta}$ receives $\frac{\pi}{6}$ without decreasing $\Phi(\hat{\Delta})$.

If $n = 4$ then $h^2 = 1$ or $g = h^2$. Now let $n = 6$. Assume that $\Gamma_6 = 0$ and so $\Gamma_3 > 1$. If $\Gamma_3 = 3$ then by Remark 4.5.40(3), $c^*(\hat{\Delta}) \leq -4\pi + 6.\frac{\pi}{2} + 3.\frac{\pi}{3} = 0$. Now if $\Gamma_3 = 2$ then $l(\hat{\Delta}) = 1h^{-1}1h^{-1}w$. Also $\hat{\Delta}$ contains one corner with angle π, for otherwise $c^*(\hat{\Delta}) \leq -4\pi + 6.\frac{\pi}{2} + 2.\frac{\pi}{3} < 0$. Thus $l(\hat{\Delta}) \in \{1h^{-1}1h^{-1}11^{-1}, 1h^{-1}1h^{-1}g1^{-1}, 1h^{-1}1h^{-1}1g^{-1}\}$. In each case $l(\hat{\Delta})$ implies either $h^2 = 1$ or $g = h^2$. Now let $\Gamma_3 = 1$. Then $l(\hat{\Delta}) = 1h^{-1}w$. Moreover $\hat{\Delta}$ contains two corners with angle π, for otherwise $c^*(\hat{\Delta}) \leq -4\pi + \pi + 5.\frac{\pi}{2} + \frac{\pi}{3} < 0$. Thus $l(\hat{\Delta}) \in \{1h^{-1}11^{-1}1g^{-1}, 1h^{-1}1g^{-1}1g^{-1}, 1h^{-1}g1^{-1}1g^{-1}, 1h^{-1}g1^{-1}g1^{-1}, 1h^{-1}11^{-1}g1^{-1}\}$. However, each possibility yields a contradiction. Therefore $\Gamma_6 \geq 1$ and so $l(\hat{\Delta}) = 1^{-1}hw$. Then $l(\hat{\Delta})$ implies $h^2 = 1$ or $g = h^2$ or $l(\hat{\Delta}) \in \{1^{-1}h1^{-1}gh^{-1}g$, $1^{-1}h1^{-1}h1^{-1}h$, $1^{-1}h1^{-1}hg^{-1}h$, $1^{-1}hg^{-1}1h^{-1}g$, $1^{-1}hg^{-1}h1^{-1}g$, $1^{-1}hg^{-1}h1^{-1}h$, $1^{-1}hg^{-1}hg^{-1}1$, $1^{-1}hg^{-1}hg^{-1}h\}$. Note that the maximum curvature that $\hat{\Delta}$ could receive is $2.\frac{\pi}{3} + 2.\frac{\pi}{6}$. Thus if $\Phi(\hat{\Delta}) = 0$ then $c^*(\hat{\Delta}) \leq -4\pi + 6.\frac{\pi}{2} + 2.\frac{\pi}{3} + 2.\frac{\pi}{6} = 0$. If $l(\hat{\Delta}) \in \{1^{-1}h1^{-1}h1^{-1}h$, $1^{-1}h1^{-1}hg^{-1}h$, $1^{-1}hg^{-1}h1^{-1}h$, $1^{-1}hg^{-1}hg^{-1}h\}$, then $\Phi(\hat{\Delta}) = 0$ and we are done. If $l(\hat{\Delta}) \in \{1^{-1}h1^{-1}gh^{-1}g$, $1^{-1}hg^{-1}1h^{-1}g$, $1^{-1}hg^{-1}h1^{-1}g$, $1^{-1}hg^{-1}hg^{-1}1\}$, then $\Gamma_3 \leq 1$, $\Gamma_6 = 1$ and $\Phi(\hat{\Delta}) \leq 1$. Thus $c^*(\hat{\Delta}) \leq -4\pi + 5.\frac{\pi}{2} + \pi + \frac{\pi}{3} + \frac{\pi}{6} = 0$. This completes the case $n = 6$.

Finally let $n = 8$. If $\Gamma_6 = 4$ then by Remark 4.5.40(5), $\Gamma_3 = 0$. Also

$l(\hat{\Delta}) = 1^{-1}h1^{-1}h1^{-1}h1^{-1}h$ implies $\Phi(\hat{\Delta}) = 0$. Thus $c^*(\hat{\Delta}) \leq -6\pi + 8.\frac{\pi}{2} + 4.\frac{\pi}{6} < 0$. If $\Gamma_6 = 3$ then $l(\hat{\Delta}) = 1^{-1}h1^{-1}h1^{-1}hw$ and so $\Phi(\hat{\Delta}) \leq 2$. Also by Remark 4.5.40(5), $\Gamma_3 \leq 1$. Thus $c^*(\hat{\Delta}) \leq -6\pi + 6.\frac{\pi}{2} + 2\pi + 3.\frac{\pi}{6} + \frac{\pi}{3} < 0$. So assume $\Gamma_6 \leq 2$. If $\Gamma_6 = 0$ then by Remark 4.5.40(3) we are done. Moreover, if $\Gamma_3 = 4$ then as before $c^*(\hat{\Delta}) \leq -6\pi + 8.\frac{\pi}{2} + 4.\frac{\pi}{3} < 0$. Also if $\Gamma_3 = 3$ then $\Gamma_6 \leq 1$ and $\Phi(\hat{\Delta}) \leq 1$ and so $c^*(\hat{\Delta}) \leq 0$. Moreover, if $\Gamma_3 = 2$ then $c^*(\hat{\Delta}) \leq -6\pi + 6.\frac{\pi}{2} + 2\pi + 2.\frac{\pi}{3} + 2.\frac{\pi}{6} = 0$. So assume $\Gamma_3 \leq 1$. If $\Gamma_3 = 1$ then $\hat{\Delta}$ contains three corners with angle π, for otherwise $c^*(\hat{\Delta}) \leq -6\pi + 4.\frac{\pi}{2} + 2\pi + \frac{\pi}{3} + 2.\frac{\pi}{6} < 0$. Also since $-6\pi + 5.\frac{\pi}{2} + 3\pi + \frac{\pi}{3} + \frac{\pi}{6} = 0$, then $\Gamma_6 = 2$. By inspection the $*$-corner in Figure 4.91 (i) cannot be a 1-corner with angle π. So assume it is a g-corner with angle π. The situation in Figure 4.91(ii) is forced. However $\Phi(\hat{\Delta}) \neq 3$. Thus each of the $\bullet$ -corners in Figure 4.91 (i) must have angle π. Here if the $*$-corner has label h, then $\Phi(\hat{\Delta}) < 3$, once more a contradiction. Thus $\hat{\Delta}$ receives 0 across the edge e and $2.\frac{\pi}{6}$ as shown in Figure 4.91 (iii). However $\Phi(\hat{\Delta}) < 3$, a contradiction. This leaves $\Gamma_3 = 0$ and $\Gamma_6 = 1$ or 2. Then $\hat{\Delta}$ contains four corners with angle π, for otherwise $c^*(\hat{\Delta}) \leq -6\pi + 5.\frac{\pi}{2} + 3\pi + 2.\frac{\pi}{6} < 0$. However, this is impossible since the $*$-corner in Figure 4.91 (iv) cannot be with angle π. Therefore if $g^2 = 1$ and $\mathcal{P}_{13}$ is not aspherical then either $h^2 = 1$ or $g = h^2$. This completes the case $g^2 = 1$ and the proof of Lemma 4.5.37.

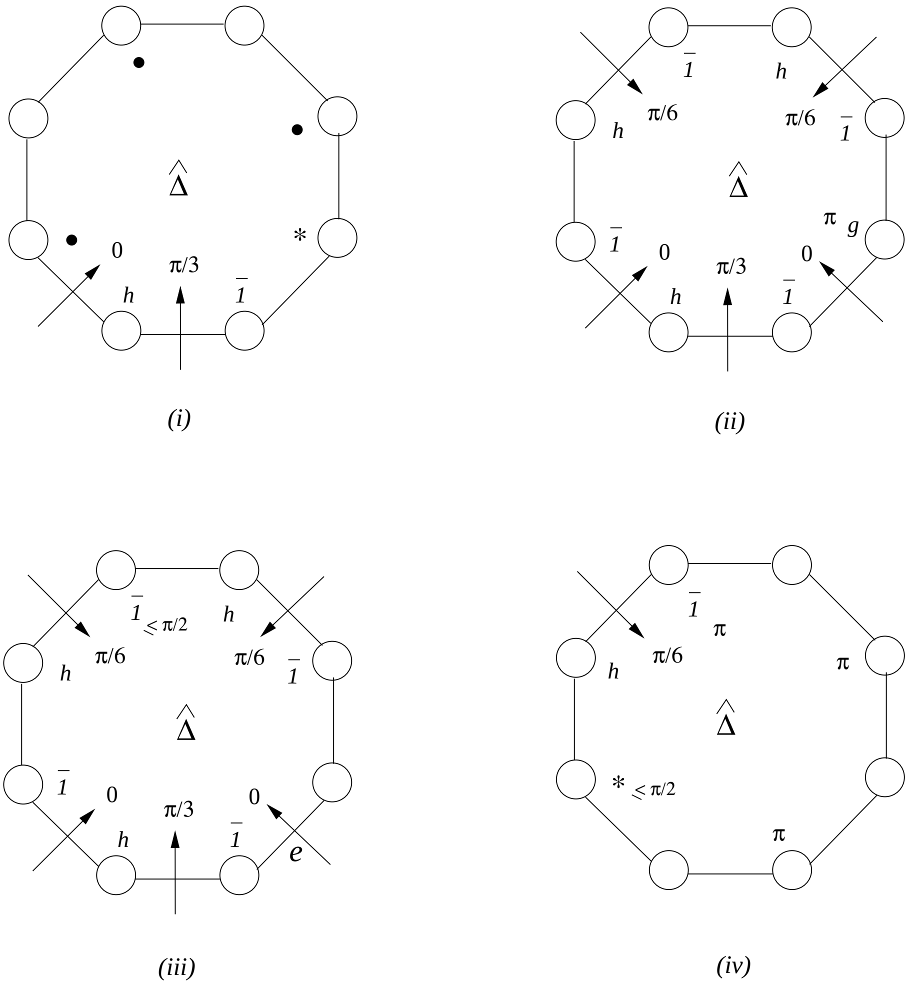


Figure 4.91: $g^2 = 1$; $\Gamma_3 \leq 1$ and $\Gamma_6 = 1$ or 2.

Remark 4.5.41. *Now $g = h^{-1}$ and $|h| = 3$ is allowed.*

Lemma 4.5.42. *If $\mathcal{P}_{13}$ is not aspherical, then at least one of the following conditions holds:*

1. $g = h^{-1}$ *and* $|h| \in \{3, 4, 5, 6\}$.
2. $g = h^2$ *and* $|h| = 4$ *or* $h = g^2$ *and* $|g| \in \{4, 6\}$.
3. $g^2 = 1$ *and* $h^2 = 1$.

Proof of Lemma 4.5.42. By Lemma 4.5.34, if $g = h^{-1}$ and $|h| = 3$ then $\mathcal{P}_{13}$ is not aspherical. Now by comparing the conditions in Lemma 4.5.36 and Lemma 4.5.37 we get the result.

Lemma 4.5.43. *Assume that $g^2 = h^2 = 1$, then $\mathcal{P}_{13}$ is aspherical except possibly when $(gh^{-1})^2 = 1$ or $(gh^{-1})^3 = 1$.*

Proof of Lemma 4.5.43. Let $\mathbb{P}$ be a non-trivial reduced spherical picture over $\mathcal{P}_{13}$. Assign the angle function $\tilde{\alpha}_3$ to the corners of $\mathbb{P}$. Assume (**A**) that the number of 4-regions of $\mathbb{P}$ is maximal, (**B**) subject to (**A**), the number of 6-regions of $\mathbb{P}$ is maximal and (**C**) subject to (**B**), the number of 8-regions of $\mathbb{P}$ is maximal. If Δ is a positive region then $l(\Delta) = 1^{-1}hw_1$ or $\{g^{-1}, 1^{-1}\}g1^{-1}w_2$. By Remark 4.5.33 (4)(*iii*), if Δ is a 6-region then $h^3 = 1$, a contradiction. It follows that Δ is given by Figure 4.92. Distribute the curvature as shown in Figure 4.93, with the understanding that $l(\acute{\Delta}) \neq 1h^{-1}1h^{-1}$. It is enough to consider the Regions (*i*), (*ii*) and (*iii*) in Figure 4.92, since Region (*iv*) is treated by symmetry as Region (*i*).

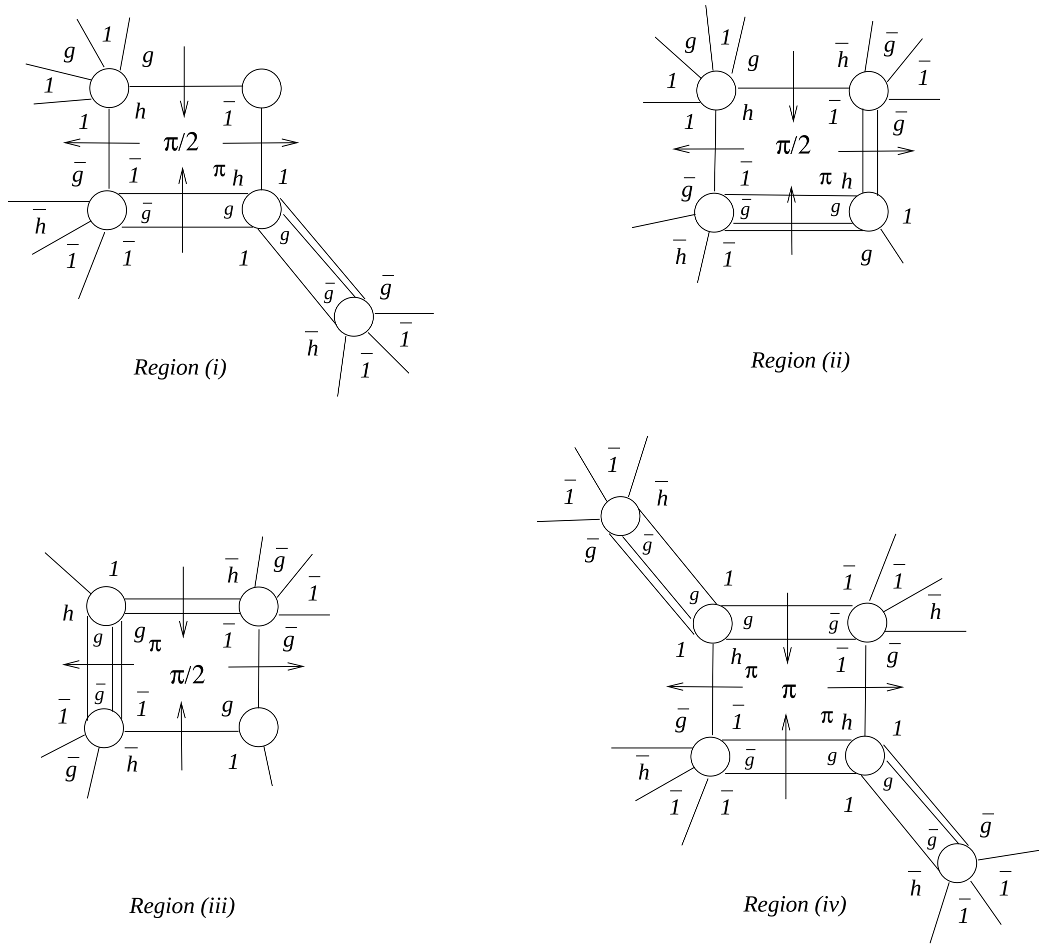


Figure 4.92: Positive regions in case $g^2 = h^2 = 1$.

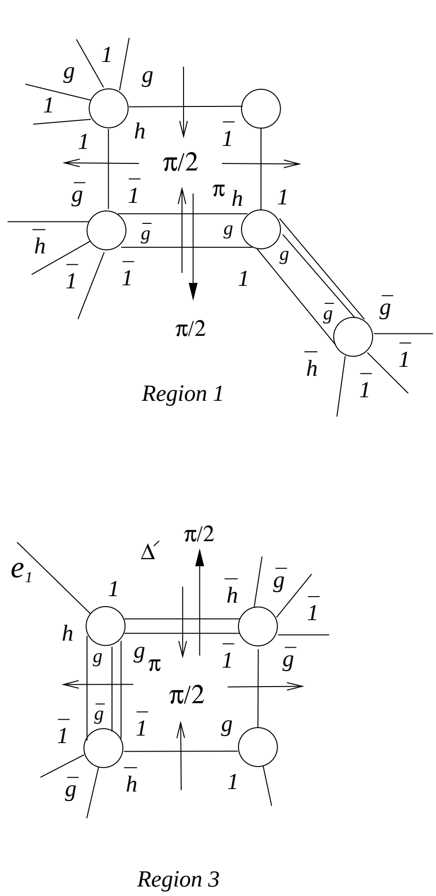


Figure 4.93: Distribution scheme in case $g^2 = h^2 = 1$.

Also consider the positive regions shown in Figure 4.94, where in Region 5 $l(\Delta) = h^{-1}1h^{-1}11^{-1}1$ or $h^{-1}1h^{-1}1g^{-1}g$. Distribute the curvature as shown (these regions appear when we dealt with the case $n = 6$). Observe that the curvature of Region 5 is distributed in a same way as Region 4. Also, up to inversion, curvature of Region 6 is distributed in a same way as Region 1.

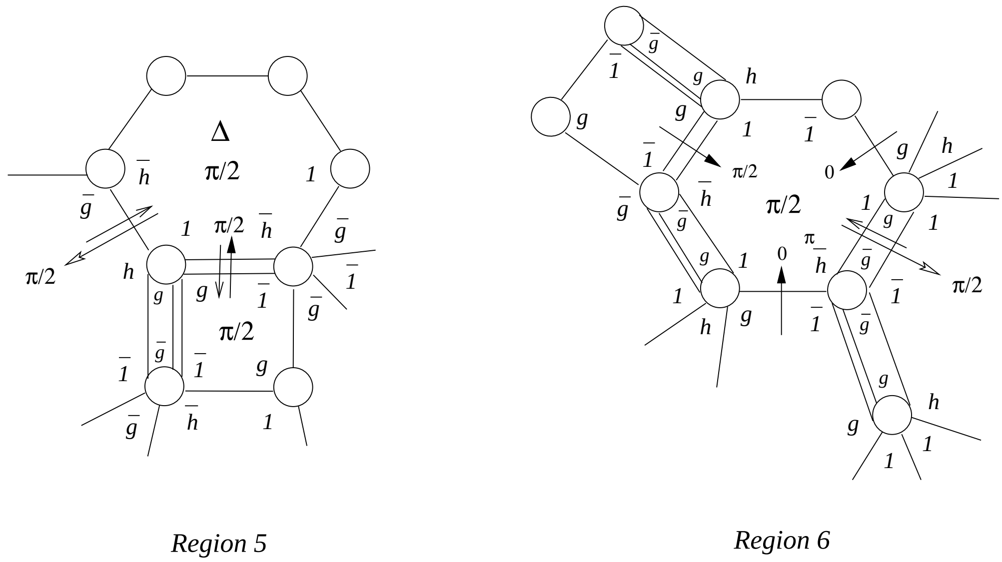


Figure 4.94: Additional positive regions.

Remarks 4.5.44.

1. *Note that $\hat{\Delta}$ receives curvature only across edges that are oriented outwards $\hat{\Delta}$, and so $\Gamma_2 \leq \frac{n}{2}$.*

2. *Observe that for each $\frac{\pi}{2}$ that $\hat{\Delta}$ receives, $\Phi(\hat{\Delta})$ is decreased by 1, or the situation shown in Figure 4.95 is forced (in this case $\hat{\Delta}$ receives curvature from Region 3).*

3. *Observe that in Figure 4.95, e_2 is not a 3-bond. Also e_3 is not a 11^{-1}-bond. Thus $\hat{\Delta}$ does not receive $\frac{\pi}{2}$ across either e_1 or e_3 without $\Phi(\hat{\Delta})$ being decreased by 1 in each case. Therefore there are no consecutive outward edges in $\partial\hat{\Delta}$ such that $\hat{\Delta}$ receives $\frac{\pi}{2}$ across each of them and $\Phi(\hat{\Delta})$ does not decrease by 1.*

4. *If $\hat{\Delta}$ receives curvature across e_3 (Figure 4.95), then it must be from a Region 1 or 6 and this is shown in Figure 4.96.*

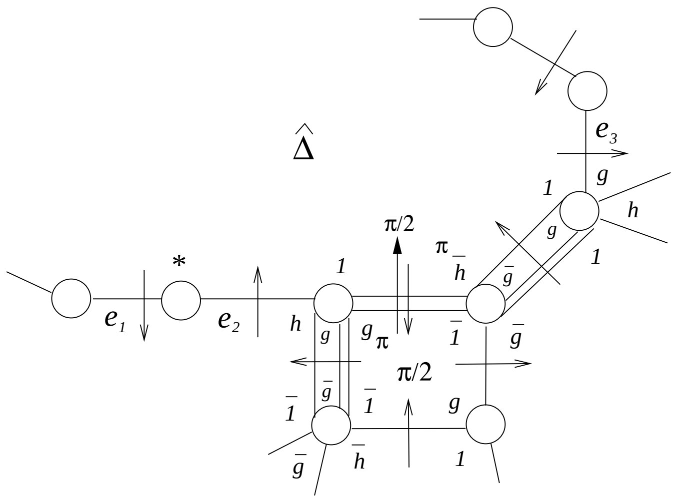


Figure 4.95: The case when $\hat{\Delta}$ receives $\frac{\pi}{2}$ without decreasing $\Phi(\hat{\Delta})$.

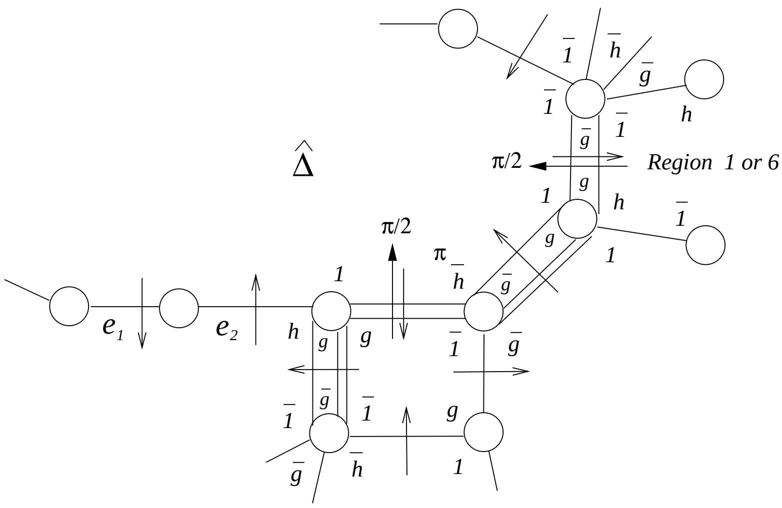


Figure 4.96: $\hat{\Delta}$ receives curvature across e_3 in Figure 4.95.

By Remarks 4.5.44, to find an upper bound for n, we may assume that $\hat{\Delta}$ receives curvature from Region 3 only without decreasing $\Phi(\hat{\Delta})$. Therefore $c^*(\hat{\Delta}) \leq (2-n)\pi + \frac{n}{2}.\pi + \frac{n}{2}.\frac{\pi}{2} + \frac{n}{4}.\frac{\pi}{2}$. So $c^*(\hat{\Delta}) > 0$ implies $2 - \frac{n}{8} > 0$ and so $n \leq 14$. However if $n = 14$ then $c^*(\hat{\Delta}) \leq -12\pi + 7\pi + 7.\frac{\pi}{2} + 3.\frac{\pi}{2} = 0$. Thus $n \leq 12$.

Let $n = 12$. Observe that $2\pi - \frac{n}{8}.\pi - \frac{\pi}{2} > 0$ implies that $n < 12$. Thus by Remarks 4.5.44, $l(\hat{\Delta}) \in \{1h^{-1}11^{-1}1h^{-1}11^{-1}1h^{-1}11^{-1}, 1h^{-1}1w_1 1h^{-1}1w_2 1h^{-1}1w_3, 1h^{-1}1w_4 1h^{-1}1w_5 1h^{-1}11^{-1}, \ 1h^{-1}1w_6 1h^{-1}11^{-1}1h^{-1}11^{-1}\}$, where each of the corners labelled $w_i (1 \leq i \leq 6)$ must have angle π. However the region $1h^{-1}11^{-1}1h^{-1}11^{-1}1h^{-1}11^{-1}$ implies $h^3 = 1$, a contradiction. Consider $\hat{\Delta}$ of Figure 4.97 and assume that the $*$-corner is labelled by w_i. If $w_i = g^{-1}$, then the edge e is a single bond. Thus by Remark 4.5.33(4)(ii) , the $*$-corner has angle $\leq \frac{\pi}{2}$. So assume $w_i = h^{-1}$ as shown. Then $d(v) \geq 4$ and once more the $*$-corner has angle $\leq \frac{\pi}{2}$. This contradiction completes the case.

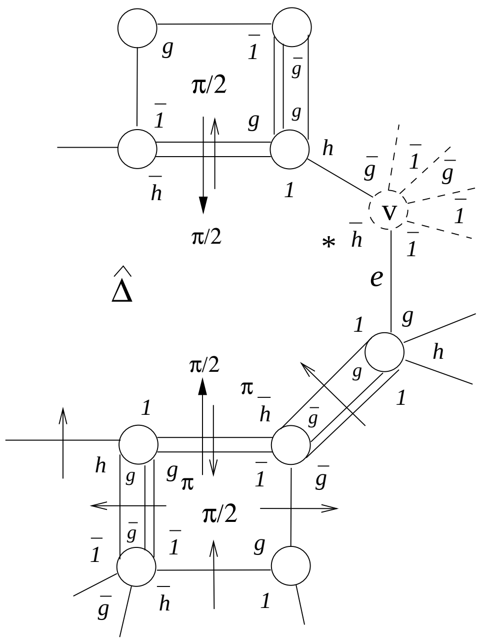


Figure 4.97: $n = 12$.

Let $n = 4$. Then $l(\hat{\Delta}) \in \{h^{-1}11^{-1}w, \, gh^{-1}w, \, 1h^{-1}w, \, 1^{-1}hg^{-1}w\}$ and this gives a contradiction (since it is assumed that $l(\acute{\Delta}) \neq 1h^{-1}1h^{-1}$ for Region 3).

Now let $n = 6$. If $\hat{\Delta}$ receives curvature from Region 1 or 6, then $l(\hat{\Delta}) = h^{-1}11^{-1}w$. The possible labelling is $h^{-1}11^{-1}1h^{-1}1$ and $\hat{\Delta}$ is shown in Figure 4.98(i). If $\hat{\Delta}$

receives curvature across e_1, then e_1 is not a single bond. Thus a cut across e_1 and e_3 would increase the number of 4-regions, a contradiction, so assume $\hat{\Delta}$ receives 0 across e_1. Now if the $*$-corner has angle π, then $l(v)$ must be as shown in Figure 4.98(ii) and so the $\bullet$-corner must have angle $\leq \frac{\pi}{2}$ (e_2 is a single bond). Hence $\Phi(\hat{\Delta}) = 1$ and so if $\Gamma_2 = 1$ then $c^*(\hat{\Delta}) \leq 0$. Thus $\hat{\Delta}$ is shown in Figure 4.99, where $c^*(\hat{\Delta}) \leq \frac{\pi}{2}$. Distribute the curvature as shown (note that $\hat{\Delta}$ is the same as Region 6).

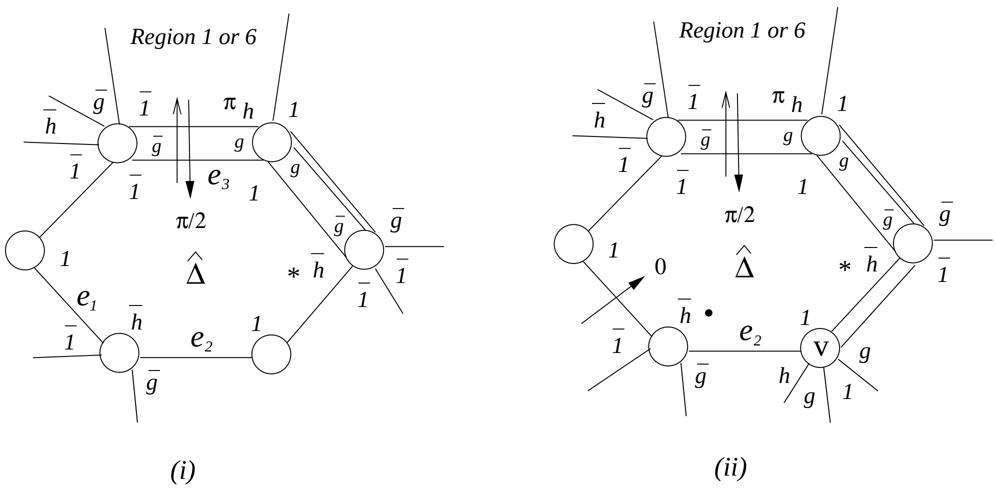


Figure 4.98: $\hat{\Delta}$ receives curvature from Region 1 or Region 6.

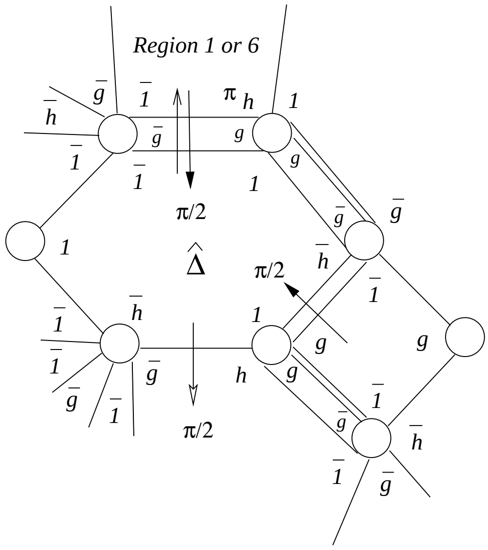


Figure 4.99: Distribution if $0 < c^*(\hat{\Delta}) \leq \frac{\pi}{2}$.

Now if $\hat{\Delta}$ receives curvature from Region 2, then $l(\hat{\Delta}) = gh^{-1}w$. Thus $l(\hat{\Delta}) \in \{gh^{-1}1g^{-1}gg^{-1},\ gh^{-1}1h^{-1}g1^{-1},\ gh^{-1}gg^{-1}hg^{-1},\ gh^{-1}1h^{-1}1g^{-1}\}$. For the possibility $gh^{-1}1g^{-1}gg^{-1}$, $\Gamma_2 = 1$ and $\Phi(\hat{\Delta}) \leq 1$. Also if $l(\hat{\Delta}) = gh^{-1}1h^{-1}g1^{-1}$, then $\Gamma_2 \leq 2$ and $\Phi(\hat{\Delta}) = 0$. Moreover, if $l(\hat{\Delta}) = gh^{-1}gg^{-1}hg^{-1}$, then $\Gamma_2 \leq 2$ and $\Phi(\hat{\Delta}) = 0$. Therefore in all these cases $c^*(\hat{\Delta}) \leq 0$. This leaves $l(\hat{\Delta}) = gh^{-1}1h^{-1}1g^{-1}$. Then $\Gamma_2 \leq 2$ and $\Phi(\hat{\Delta}) \leq 2$. However if the $*$-corner in Figure 4.100 has angle π, then the $\bullet$-corner must have angle $\leq \frac{\pi}{2}$ (e_1 is forced to be a single bond as shown) and so $\Phi(\hat{\Delta}) \leq 1$. Thus $\hat{\Delta}$ must receive curvature across e_2 and $c^*(\hat{\Delta}) \leq \frac{\pi}{2}$ (for otherwise $\Gamma_2 = 1$ and $\Phi(\hat{\Delta}) \leq 1$). Distribute the curvature as shown (note that $\hat{\Delta}$ is the same as Region 5).

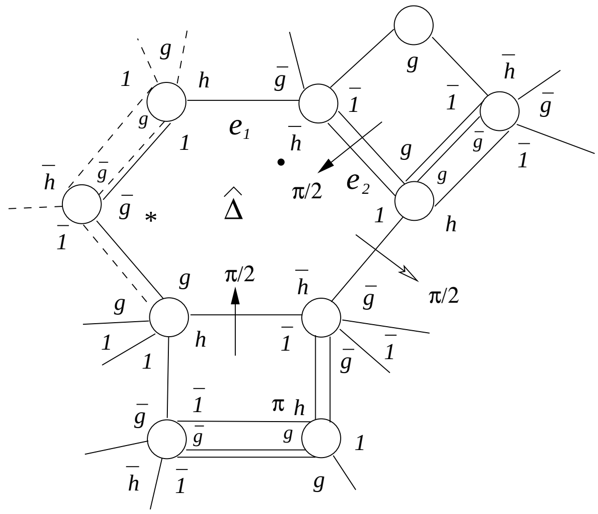


Figure 4.100: $\hat{\Delta}$ receives curvature from Region 2.

Now if $\hat{\Delta}$ receives curvature from Region 3, then $l(\hat{\Delta}) = 1h^{-1}w$. Hence $l(\hat{\Delta}) \in \{1h^{-1}1h^{-1}gg^{-1}, 1h^{-1}g1^{-1}gh^{-1}, 1h^{-1}gg^{-1}gg^{-1}, 1h^{-1}gg^{-1}1h^{-1}, 1h^{-1}11^{-1}1h^{-1}, 1h^{-1}1h^{-1}11^{-1}, 1h^{-1}1g^{-1}gh^{-1}\}$. If $l(\hat{\Delta}) = 1h^{-1}1h^{-1}gg^{-1}$ or $1h^{-1}g1^{-1}gh^{-1}$, then either $\Gamma_2 = 2$ and $\Phi(\hat{\Delta}) = 0$ or $\Gamma_2 = 1$ and $\Phi(\hat{\Delta}) \leq 1$. Moreover if $l(\hat{\Delta}) = 1h^{-1}gg^{-1}gg^{-1}$, then $\Gamma_2 = 1$ and $\Phi(\hat{\Delta}) = 0$, while if $l(\hat{\Delta}) = 1h^{-1}gg^{-1}1h^{-1}$, then $\Gamma_2 \leq 2$ and $\Phi(\hat{\Delta}) = 0$ (observe that the edge e_1 in Figure 4.93 is a single bond). Therefore in these cases $c^*(\hat{\Delta}) \leq 0$. Now let $l(\hat{\Delta}) = 1h^{-1}11^{-1}1h^{-1}$. Then $\Gamma_2 \leq 3$ and $\Phi(\hat{\Delta}) \leq 1$ (observe that the $*$-corner in Figure 4.101 has angle $\leq \frac{\pi}{2}$). However if $c^*(\hat{\Delta}) \leq \frac{\pi}{2}$, then distribute the curvature as shown in Figure 4.101 (note that $\hat{\Delta}$ is the same as Region 5). So assume that $\Gamma_2 = 3$ and so $\hat{\Delta}$ is given by Figure 4.102. But now a cut across e_1 and e_2 increases the number of 4-regions, a contradiction.

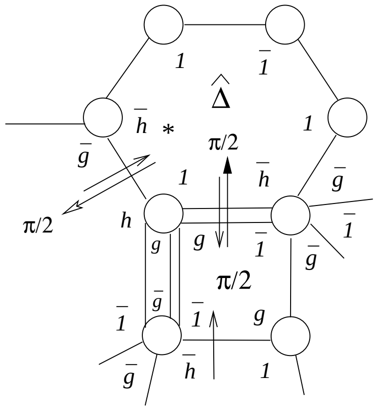


Figure 4.101: $\hat{\Delta}$ receives curvature from Region 3.

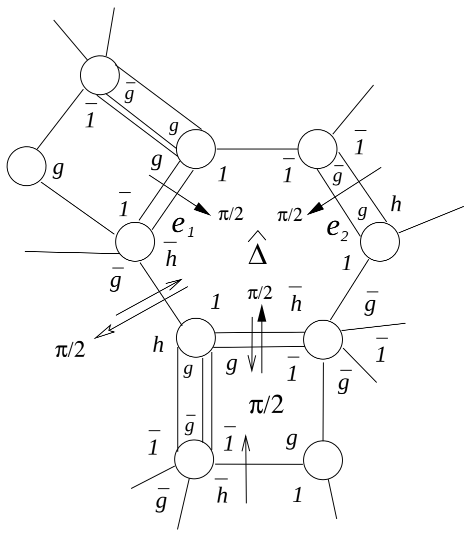


Figure 4.102: $l(\hat{\Delta}) = 1h^{-1}11^{-1}1h^{-1}$ and $\Gamma_2 = 3$.

Now if $l(\hat{\Delta}) = 1h^{-1}1h^{-1}11^{-1}$, then $\Gamma_2 \leq 3$ and $\Phi(\hat{\Delta}) \leq 2$. If $\hat{\Delta}$ receives curvature across e_1 in Figure 4.103 then we are are back to the previous case, so assume otherwise. If $\Gamma_2 = 2$ then $\hat{\Delta}$ is given by Figure 4.103, and so a cut across e_2 and e_3 contradicts (**A**). Thus $\Gamma_2 = 1$ and $\Phi(\hat{\Delta}) = 2$ and so $\hat{\Delta}$ is given by Figure 4.104. Distribute the curvature as shown (note that $\hat{\Delta}$ is the same as Region 6).

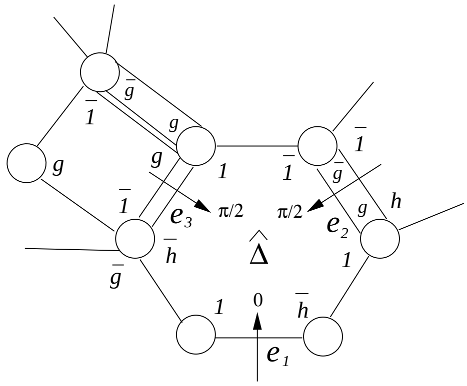


Figure 4.103: $\hat{\Delta}$ receives curvature from Region 3.

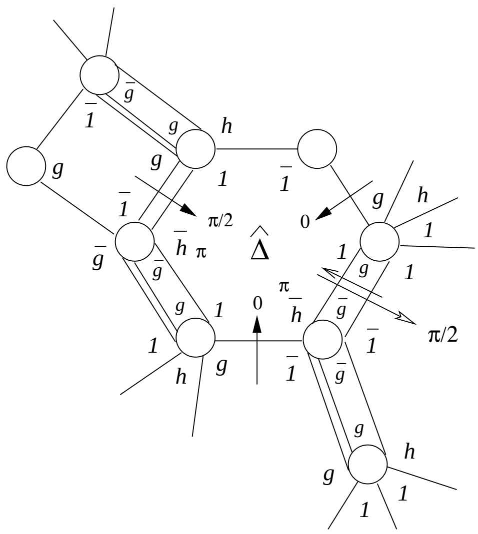


Figure 4.104: $l(\hat{\Delta}) = 1h^{-1}1h^{-1}11^{-1}$.

Finally if $l(\hat{\Delta}) = 1h^{-1}1g^{-1}gh^{-1}$, then $\Gamma_2 \leq 2$ and $\Phi(\hat{\Delta}) \leq 2$ (observe that the $*$-corner in Figure 4.105 has angle $\leq \frac{\pi}{2}$). However if $c^*(\hat{\Delta}) \leq \frac{\pi}{2}$, then distribute the curvature as shown in Figure 4.105 (note that $\hat{\Delta}$ is the same as Region 5). So assume $\Gamma_2 = 2$ and $\Phi(\hat{\Delta}) = 2$. However, since e in Figure 4.106 is a single bond, $\Phi(\hat{\Delta}) \leq 1$, a contradiction.

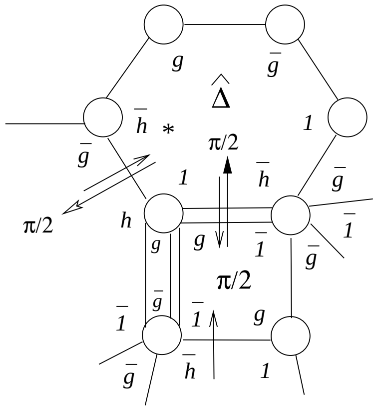


Figure 4.105: $\hat{\Delta}$ receives curvature from Region 3.

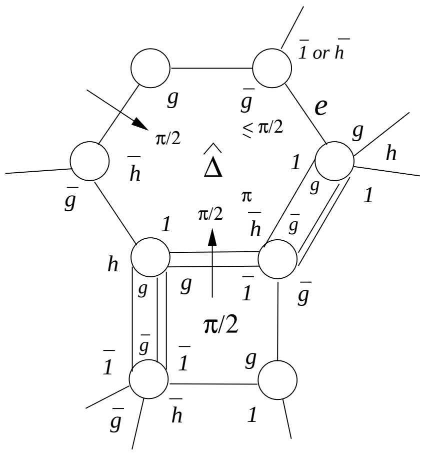


Figure 4.106: $l(\hat{\Delta}) = 1h^{-1}1g^{-1}gh^{-1}$ and $\Phi(\hat{\Delta}) \leq 1$.

Now if $\hat{\Delta}$ receives curvature from Region 4 or 5, then $l(\hat{\Delta}) = 1^{-1}hg^{-1}w$. Thus $l(\hat{\Delta}) = 1^{-1}hg^{-1}1g^{-1}h$ or $1^{-1}hg^{-1}g1^{-1}h$. If $l(\hat{\Delta}) = 1^{-1}hg^{-1}1g^{-1}h$, then $\Gamma_2 \leq 2$ and $\Phi(\hat{\Delta}) = 0$ and so $c^*(\hat{\Delta}) \leq 0$. Also if $l(\hat{\Delta}) = 1^{-1}hg^{-1}g1^{-1}h$, then $\Gamma_2 = 1$ (since e_2 in Figure 4.93 is not a 11^{-1}-bond) and $\Phi(\hat{\Delta}) \leq 1$. Again this implies $c^*(\hat{\Delta}) \leq 0$. This completes the case $n = 6$.

If for each $\frac{\pi}{2}$ that $\hat{\Delta}$ receives, $\Phi(\hat{\Delta})$ is decreased by 1, then $c^*(\hat{\Delta}) \leq (2-n)\pi + (\frac{n}{2} - \Gamma_2)\pi + (\frac{n}{2} + \Gamma_2)\frac{\pi}{2} + \Gamma_2.\frac{\pi}{2} = (2 - \frac{n}{4})\pi$. Thus $c^*(\hat{\Delta}) > 0$ implies $n < 8$. Thus for n=8 or 10, $\hat{\Delta}$ is shown in Figure 4.95 and so $l(\hat{\Delta}) = 1h^{-1}1w$.

Now let $n = 8$. If $\hat{\Delta}$ receives $2.\frac{\pi}{2}$ without decreasing $\Phi(\hat{\Delta})$ by 1 for each $\frac{\pi}{2}$, then either $l(\hat{\Delta}) = 1h^{-1}11^{-1}h1^{-1}w$ and so $\mathbb{P}$ is not reduced, or $l(\hat{\Delta}) = 1h^{-1}1w_1 1h^{-1}1w_1$ (for otherwise $l(\hat{\Delta}) \neq 1$). Since $(gh^{-1})^2 \neq 1$, either $w_1 = h^{-1}$ or 1^{-1}. However if $w_1 = h^{-1}$ then a cut across e_1 and e_3 contradicts (**A**) (see Figure 4.107). Moreover if $w_1 = 1^{-1}$ then $\hat{\Delta}$ receives curvature across at least one of e_2 or e_4, for otherwise $c^*(\hat{\Delta}) \leq -6\pi + 2\pi + 6.\frac{\pi}{2} + 2.\frac{\pi}{2} = 0$. By symmetry say across e_2 and so e_2 is not a single bond. Thus a cut across e_2 and e_3 contradicts (**B**). Therefore assume that $\hat{\Delta}$ receives at most $\frac{\pi}{2}$ without decreasing $\Phi(\hat{\Delta})$ by 1.

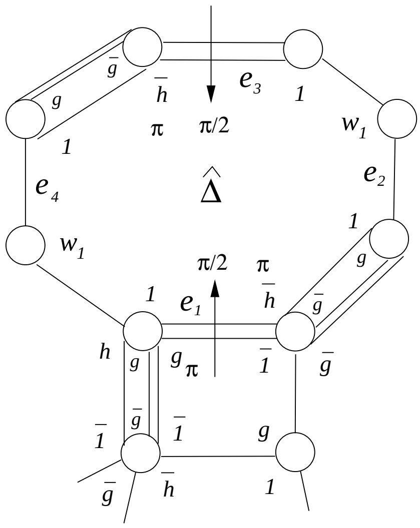


Figure 4.107: $\hat{\Delta}$ receives $2.\frac{\pi}{2}$ without decreasing $\Phi(\hat{\Delta})$ by 1 for each $\frac{\pi}{2}$.

Now note that the edge e_2 in Figure 4.95 is a single bond. Then by inspection the $*$-corner has angle $\leq \frac{\pi}{2}$. Thus $\hat{\Delta}$ must receive curvature across e_1, for otherwise $c^*(\hat{\Delta}) \leq -6\pi + 3\pi + 5.\frac{\pi}{2} + \frac{\pi}{2} = 0$ (once again Remark 4.5.44(2) ensures that $c^*(\hat{\Delta}) \leq 0$). Therefore $l(\hat{\Delta}) \in \{gh^{-1}1h^{-1}1w_1,\ hg^{-1}1h^{-1}1w_2,\ 1h^{-1}1h^{-1}1w_3,\ 1^{-1}hg^{-1}1h^{-1}1w_4\}$. Hence $l(\hat{\Delta}) \in \{gh^{-1}1h^{-1}1g^{-1}gg^{-1},\ gh^{-1}1h^{-1}1g^{-1}11^{-1},\ gh^{-1}1h^{-1}11^{-1}g1^{-1},\ gh^{-1}1h^{-1}11^{-1}1g^{-1},$

$1h^{-1}1h^{-1}1h^{-1}1h^{-1}$, $1h^{-1}1h^{-1}11^{-1}11^{-1}$, $1h^{-1}1h^{-1}1g^{-1}g1^{-1}$, $1h^{-1}1h^{-1}1g^{-1}1g^{-1}$, $1h^{-1}1h^{-1}11^{-1}gg^{-1}\}$. If $l(\hat{\Delta}) \in \{gh^{-1}1h^{-1}1g^{-1}gg^{-1}, gh^{-1}1h^{-1}1g^{-1}11^{-1}, 1h^{-1}1h^{-1}1g^{-1}g1^{-1}\}$, then $\Gamma_2 = 2$ and $\Phi(\hat{\Delta}) \leq 2$. Moreover if $l(\hat{\Delta}) = 1h^{-1}1h^{-1}11^{-1}11^{-1}$ or $1h^{-1}1h^{-1}11^{-1}gg^{-1}$, then $\Gamma_2 \leq 3$ and $\Phi(\hat{\Delta}) = 1$. In all these cases $c^*(\hat{\Delta}) \leq 0$. Now if $l(\hat{\Delta}) = gh^{-1}1h^{-1}11^{-1}g1^{-1}$ or $gh^{-1}1h^{-1}11^{-1}1g^{-1}$, $\Gamma_2 \leq 3$ and $\Phi(\hat{\Delta}) \leq 2$. If $\Phi(\hat{\Delta}) = 2$ and $l(\hat{\Delta}) = gh^{-1}1h^{-1}11^{-1}g1^{-1}$, then $\hat{\Delta}$ is shown in Figure 4.108, where the edge e_1 is not a single bond (Remark 4.5.33 (4)(ii)). Thus a cut across e_1 and e_2 increases the number of 6-regions (without affecting the number of 4-regions). Also, if $\Gamma_2 = 3$, $\Phi(\hat{\Delta}) = 2$ and $l(\hat{\Delta}) = gh^{-1}1h^{-1}11^{-1}1g^{-1}$, then $\hat{\Delta}$ is shown in Figure 4.109. As before a cut across e_3 and e_4 in Figure 4.109 yields a contradiction so $\Gamma_2 + \Phi(\hat{\Delta}) \leq 4$ and $c^*(\hat{\Delta}) < 0$.

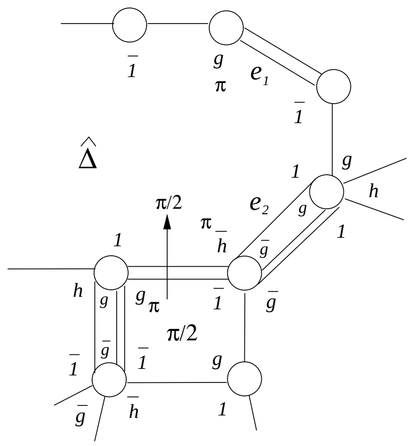


Figure 4.108: $l(\hat{\Delta}) = gh^{-1}1h^{-1}11^{-1}g1^{-1}$.

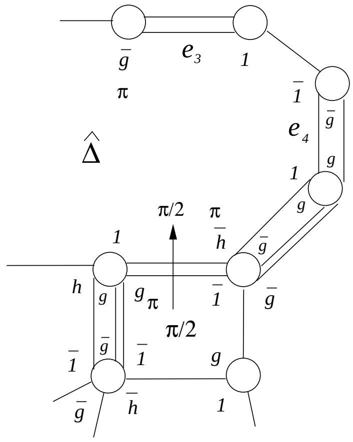


Figure 4.109: $l(\hat{\Delta}) = gh^{-1}1h^{-1}11^{-1}1g^{-1}$.

Finally, if $l(\hat{\Delta}) = 1h^{-1}1h^{-1}1h^{-1}1h^{-1}$ or $1h^{-1}1h^{-1}1g^{-1}1g^{-1}$, then a cut across e_1 and e_2 in Figure 4.110 contradicts (**A**). This completes the case $n = 8$.

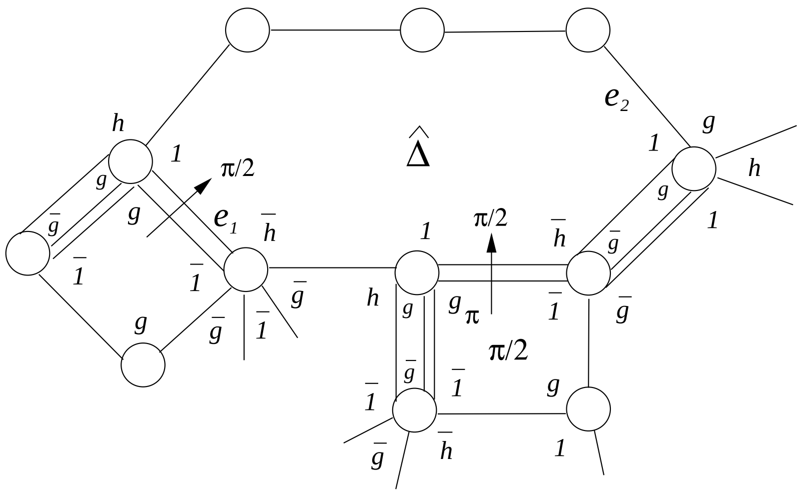
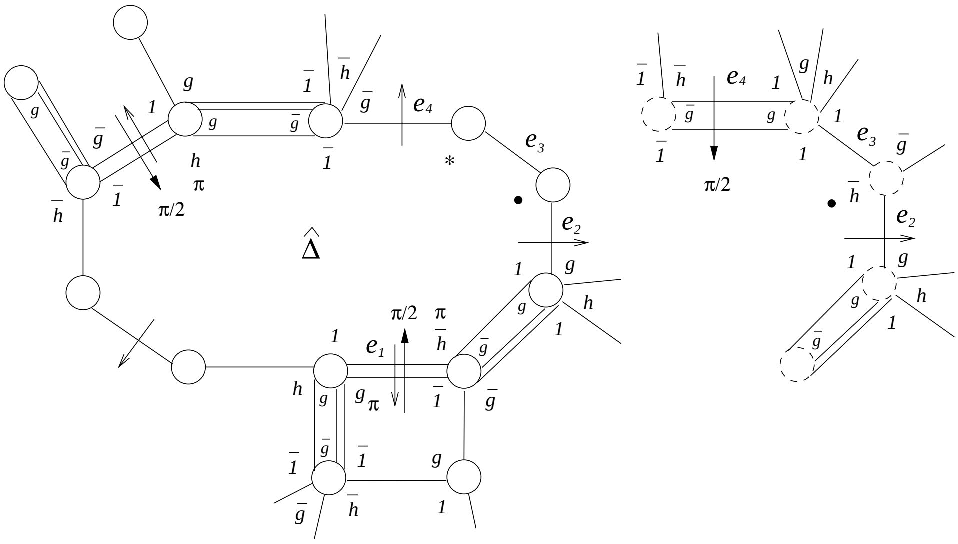


Figure 4.110: $l(\hat{\Delta}) = 1h^{-1}1h^{-1}1h^{-1}1h^{-1}$ or $1h^{-1}1h^{-1}1g^{-1}1g^{-1}$.

This leaves $n = 10$. Observe that $\hat{\Delta}$ receives $2.\frac{\pi}{2}$ without decreasing $\Phi(\hat{\Delta})$, for otherwise $-8\pi + 5\pi + 5.\frac{\pi}{2} + \frac{\pi}{2} = 0$ and Remark 4.5.44(2) ensures that $c^*(\hat{\Delta}) \leq 0$. Now by Remark 4.5.44(3), $\hat{\Delta}$ receives these $2.\frac{\pi}{2}$ across non-consecutive outward oriented edges in $\partial(\hat{\Delta})$. Then $\hat{\Delta}$ is one of the regions shown in Figures 4.111 and

4.112. Consider the first case (Figure 4.111). In this case Remark 4.5.33 (4)(i) implies that $\Phi(\hat{\Delta}) \leq 4$. Observe that $\Phi(\hat{\Delta}) + \Gamma_2 \leq 7$ and if $\Phi(\hat{\Delta}) + \Gamma_2 \leq 6$, then $c^*(\hat{\Delta}) \leq 0$. Now if $\hat{\Delta}$ receives curvature across e_2, then the situation is shown in Figure 4.96. Therefore the $\bullet$-corner has label 1^{-1} and so the $*$-corner cannot have label h which implies $\Phi(\hat{\Delta}) + \Gamma_2 \leq 6$. Therefore assume that the $\bullet$-corner has angle π and $\hat{\Delta}$ receives curvature across e_4. So the $\bullet$-corner has label h^{-1}(note that if it has label g^{-1} then e_2 is a single bond) and $*$-corner has label 1 (see Figure 4.111 (ii)). However the $\bullet$-corner has angle $\leq \frac{\pi}{2}$ since e_3 is a single bond, a contradiction.



(i) (ii)

Figure 4.111: $\hat{\Delta}$ for $n = 10$.

Now let $\hat{\Delta}$ be as shown in Figure 4.112. Observe that the $\bullet$-corner has angle $\leq \frac{\pi}{2}$. Indeed, if it has label g^{-1} then e_1 is a single bond. Also if it is an h^{-1}-corner then e_2 is a single bond. Therefore $\hat{\Delta}$ must receive curvature across e_1 (see Figure 4.96), for otherwise $\Phi(\hat{\Delta}) + \Gamma_2 \leq 6$ and so $c^*(\hat{\Delta}) \leq 0$. In this case $l(\hat{\Delta}) = 1h^{-1}11^{-1}1h^{-1}1w$ and so a cut across e_1 and e_3 contradicts (**C**). This completes the case $n = 10$.

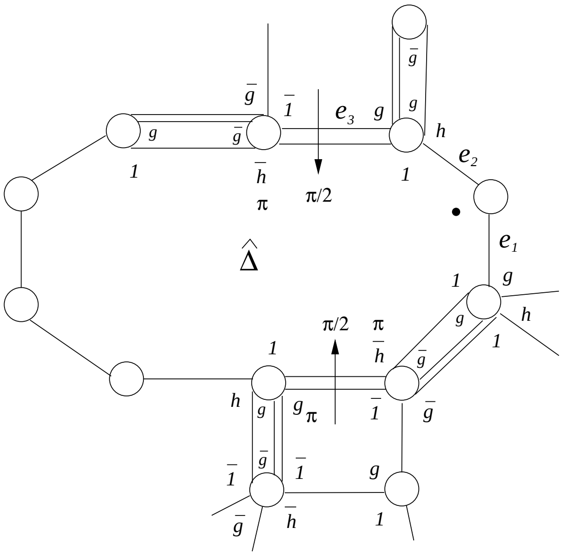


Figure 4.112: $\hat{\Delta}$ for $n = 10$.

Before stating the main result for case 13, we list the following exceptional cases.

(**E1**) $g = h^{-1}$ and $|h| \in \{4, 5, 6\}$.

(**E2**) $g = h^2$ and $|h| = 4$ or $h = g^2$ and $|g| \in \{4, 6\}$.

(**E3**)$g^2 = h^2 = 1$ and $|gh^{-1}| \in \{2, 3\}$.

Proposition 4.5.45. *Consider the presentation $\mathcal{P}_{13}$. Suppose that none of the conditions in (**E1**)-(**E3**) holds. Then $\mathcal{P}_{13}$ is not aspherical if and only if $g = h^{-1}$ and $|h| = 3$.*

Proof of Proposition 4.5.45. The ‘ if’ direction of Proposition 4.5.45 follows from Lemma 4.5.34. Now by comparing the conditions in Lemma 4.5.42 and Lemma 4.5.43 we get the result.

4.5.5 Case 15: (a=b=d, e=f only)

The relative presentation $\mathcal{P}_{15}= \langle G, y|ayaycyayeyey\rangle=$ $\langle G, y, x|x(ay)^{-1}, x^2ca^{-1}x^2ea^{-1}xea^{-1}x\rangle= \langle G, x|x^3gx^2hxh\rangle$, where $g = ca^{-1}$ and $h = ea^{-1}$ (and so by assumption, $g, h \in G\backslash\{1\}$ and $g \neq h$).

Let $\mathbb{P}$ be a non-trivial reduced spherical picture over $\mathcal{P}_{15}$ =$\langle G, x|x^3gx^2hxh\rangle$. Then each vertex(disc) in $\mathbb{P}$ has one of the forms given by Figure 4.113(i) and (ii); and the the star graph $\mathcal{P}^{st}_{15}$ of $\mathcal{P}$ is given by Figure 4.113(iii).

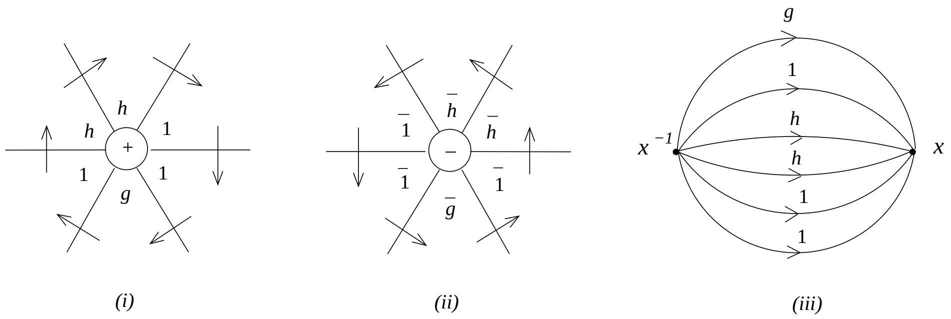


Figure 4.113: + disc, - disc and $\mathcal{P}^{st}_{15}$.

There is (up to inversion) only one type of vertex of degree 3 in $\mathbb{P}$ which is shown in Figure 4.114.

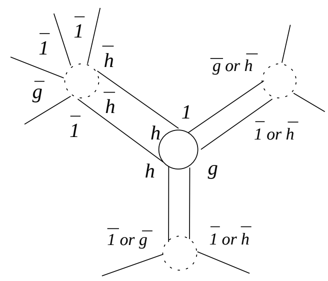


Figure 4.114: Type of vertices of degree 3.

Define an *angle function* $\acute{\alpha}_1$ on $\mathbb{P}$ as follows. Corners within 2-bonds have angle zero. In vertices of degree 3, corners (not within 2-bonds) labelled by $1^{\pm 1}$ or $h^{\pm 1}$ have angle $\frac{3\pi}{4}$, the remaining corner (labelled by $g^{\pm 1}$) has angle $\frac{\pi}{2}$ (see Figure 4.115). If $d(v) \geq 4$, then each corner in v not in a 2-bond has angle $\frac{2\pi}{d(v)}$.

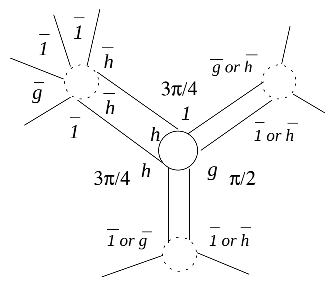


Figure 4.115: Angle function $\acute{\alpha}_1$ for vertex of degree 3.

Define another *angle function* $\acute{\alpha}_2$ on $\mathbb{P}$ as follows. Corners within 2-bonds have angle zero. In vertices of degree 3, corners (not within 2-bonds) labelled by $g^{\pm 1}$ have angle π, each of the other two corners (labelled by $1^{\pm 1}$ and $h^{\pm 1}$) has angle $\frac{\pi}{2}$ (see Figure 4.116). If $d(v) \geq 4$, then each corner in v not in a 2-bond has angle $\frac{2\pi}{d(v)}$.

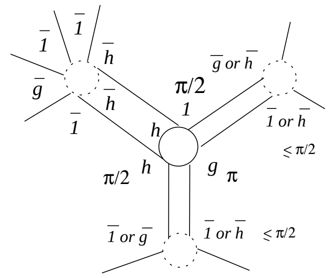


Figure 4.116: Angle function $\acute{\alpha}_2$ for vertex of degree 3.

Remarks 4.5.46.

1. *By assigning the angle function $\acute{\alpha}_1$ to the corners of $\mathbb{P}$, the following are satisfied:*

(i) There are no consecutive corners with angle $\frac{3\pi}{4}$ *in the boundary of a region* Δ *of* $\mathbb{P}$. *This is because* $d(v_1) \geq 4$ *(see Figure 4.117). Also, the* $*$*-corner in vertex* v_i *is labelled by* $g^{\pm 1}$ *or* $d(v_i) \geq 4$ *(i=2,3). Either way implies that the* $*$*-corner has angle* $\leq \frac{\pi}{2}$.

(ii) Note that $(2-n)\pi + \frac{n}{2}.\frac{3\pi}{4} + \frac{n}{2}.\frac{\pi}{2} > 0$ *implies that* $n \leq 4$ *and so positive regions can only have degree 4.*

2. *By assigning the angle function* $\acute{\alpha}_2$ *to the corners of* $\mathbb{P}$, *the following are satisfied:*

   (i) Let Δ *be a region of degree* n. *Also let* $\Phi(\Delta)$ *denote the number of corners of angle* π *in* Δ. *There are no consecutive corners with angle* π *in the boundary of a region* Δ *of* $\mathbb{P}$ *(see Figure 4.116). Thus* $\Phi(\Delta) \leq \frac{n}{2}$.

   (ii) If Δ *is a positive 4-region, then it has at least one corner labelled by* $g^{\pm 1}$ *with angle* π.

   (iii) If Δ *is a positive 6-region, then it contains at least three* $g^{\pm 1}$*- corners each with angle* π.

   (iv) As shown in Figure 4.118, in the sublabel $(g1^{-1}g)^{\pm 1}$, *at least one of the* $g^{\pm 1}$*-corners has angle* $\leq \frac{\pi}{2}$. *Same holds for the* $g^{\pm 1}$*-corners in the sublabel* $(gh^{-1}g)^{\pm 1}$.

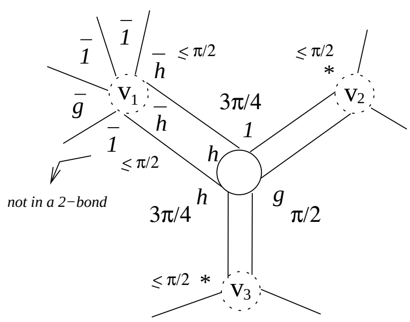


Figure 4.117: Angle function $\acute{\alpha}_1$; no consecutive corners with angle $\frac{3\pi}{4}$ in $\partial\Delta$.

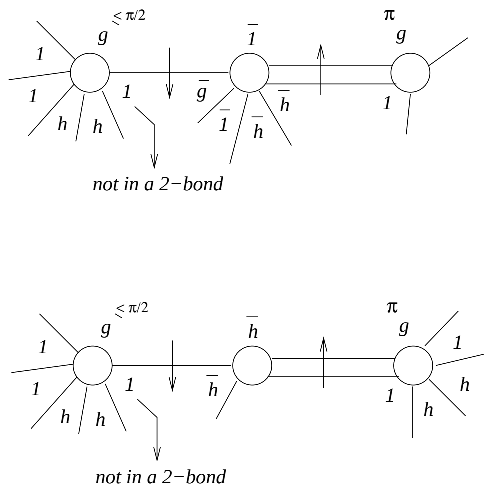


Figure 4.118: Angle function $\acute{\alpha}_2$; sublabels $(g1^{-1}g)^{\pm 1}$ and $(gh^{-1}g)^{\pm 1}$.

Before stating the result for Case 15 consider the following exceptional cases.

(**E**) $g \in \{h^{-1}, h^2\}$ and $|h| = 4$ or $h = g^2$ and $|g| = 4$.

Proposition 4.5.47. *Let $\mathcal{P}_{15}$ be the relative presentation $\mathcal{P}_{15}$=$\langle G, x | x^3 g x^2 h x h\rangle$, where $x \notin G$, $g, h \in G\backslash\{1\}$. Suppose that (**E**) does not hold. Then $\mathcal{P}_{15}$ is not aspherical if and only if $g = h^{-1}$ and $|h| = 3$.*

Lemma 4.5.48. *Let $g = h^{-1}$. If $|h| = 3$, then $\mathcal{P}_{15}$ is not aspherical. Otherwise $\mathcal{P}_{15}$ is aspherical or $\mathcal{P}_{15}$ is exceptional of type (**E**).*

Proof. Let $|h| = 3$. Coset enumeration shows that the order of the group $\langle h, x | x^3 h^{-1} x^2 h x h, h^3\rangle$ equals 666 and so $\mathcal{P}_{15}$ is not aspherical. Now let $|h| \neq 3$. Assume by way of contradiction that $\mathbb{P}$ is a non-trivial reduced spherical picture over $\mathcal{P}_{15}$. Assign the angle function $\acute{\alpha}_2$ to the corners of $\mathbb{P}$. Positive 4-regions imply that $g = h^{-1}$ or $g = h^2$ or $h = g^2$ or $g^2 = 1$ or $(gh^{-1})^2 = 1$. By Remarks 4.5.46 (2)(*iii*) and (*iv*), no positive 6-regions exist.

Now if $g = h^{-1}$ and $g = h^2$ or $h = g^2$ or $g^2 = 1$, then we get a contradiction. Also $g = h^{-1}$ and $(gh^{-1})^2 = 1$ imply (**E**). Therefore the only possible positive region is given by Figure 4.119. Distribute the curvature as shown in Figure 4.120.

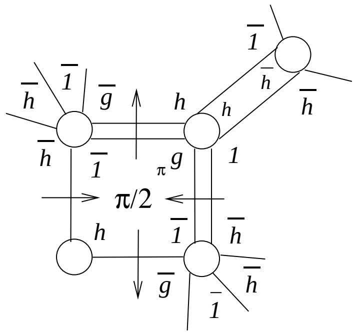


Figure 4.119: $g = h^{-1}$; the only possible positive region.

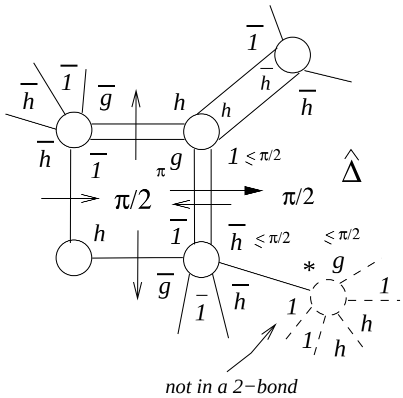


Figure 4.120: $g = h^{-1}$; distribution of curvature.

Remarks 4.5.49.

1. *As shown in Figure 4.120, if the* $*$*-corner is labelled by* g*, then this corner has angle* $\leq \frac{\pi}{2}$*.*

2. $l(\hat{\Delta}) = h^{-1}1h^{-1}w$ *and so* $d(\hat{\Delta}) > 4$*.*

3. *For each* Γ_2*,* $\Phi(\Delta)$ *is decreased by 1.*

Now $c^*(\hat{\Delta}) \leq (2-n)\pi+(\frac{n}{2}-\Gamma_2)\pi+(\frac{n}{2}+\Gamma_2)\frac{\pi}{2}+\Gamma_2.\frac{\pi}{2}= \pi(2-\frac{n}{4})$. Therefore $c^*(\hat{\Delta}) > 0$ implies $n < 8$ and so $n = 6$. Now if $\Gamma_2 = 3$ then $h^3 = 1$ and we are done. If $\Gamma_2 = 2$ then $l(\hat{\Delta}) = (h^{-1}1)^2h^{-1}\{1, g, h\}$. This implies one of the following: $h^2 = 1$, a contradiction, or $h^3 = 1$ or $l(\hat{\Delta}) = (h^{-1}1)^2h^{-1}g$. However if $l(\hat{\Delta}) = (h^{-1}1)^2h^{-1}g$ then by Remark 4.5.49(1), $\Phi(\hat{\Delta}) = 0$. Thus $c^*(\hat{\Delta}) \leq -4\pi + 6.\frac{\pi}{2} + 2.\frac{\pi}{2} = 0$. Therefore assume that $\Gamma_2 = 1$ and $\Phi(\hat{\Delta}) = 2$, for otherwise $c^*(\hat{\Delta}) \leq -4\pi + \pi + 5.\frac{\pi}{2} + \frac{\pi}{2} = 0$. However by Remark 4.5.49(1), $\Phi(\hat{\Delta}) = 2$ is impossible. Therefore $c^*(\hat{\Delta}) \leq 0$ and so $\mathcal{P}_{15}$ is aspherical.

Remark 4.5.50. *In view of Lemma 4.5.48, we may assume that $g \neq h^{-1}$ until otherwise stated.*

Lemma 4.5.51. *If $g = h^2$ then either $\mathcal{P}_{15}$ is exceptional of type (**E**) or $\mathcal{P}_{15}$ is aspherical.*

Proof. Assign the angle function $\acute{\alpha}_2$ to the corners of $\mathbb{P}$. Positive 4-regions imply that $g = h^2$ or $g = h^{-1}$ or $h = g^2$ or $g^2 = 1$ or $(gh^{-1})^2 = 1$. By Remarks 4.5.46 (2)(iii) and (iv), no positive 6-regions exist. Now assume that $g = h^2$. If $h = g^2$ then $g = h^{-1}$. Moreover if $g^2 = 1$ then $h^4 = 1$ which gives (**E**). Also if $(gh^{-1})^2 = 1$ then $g = 1$, a contradiction. Therefore there is (up to inversion) only one type of positive region which is shown in Figure 4.121. Distribute the curvature as shown.

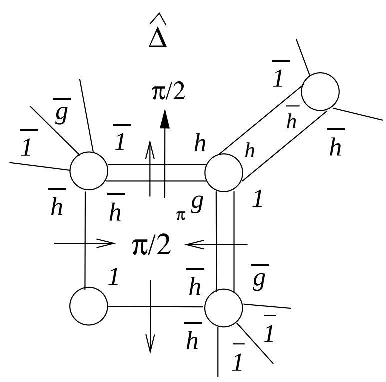


Figure 4.121: $g = h^2$.

Remarks 4.5.52.

1. *$l(\hat{\Delta}) = 1^{-1}h1^{-1}w$ and so $d(\hat{\Delta}) > 4$.*

2. *For each Γ_2, $\Phi(\hat{\Delta})$ is decreased by 1.*

As above, $c^*(\hat{\Delta}) > 0$ implies $n = 6$. Now if $\Gamma_2 = 3$ then $h^3 = 1$, contradicting $g \neq h^{-1}$. Moreover if $\Gamma_2 = 2$ then $l(\hat{\Delta}) = (1^{-1}h)^2 1^{-1}\{1, g, h\}$. This implies $h^2 = 1$ or (**E**) or $h^3 = 1$. Therefore $\Gamma_2 = 1$ and $\Phi(\hat{\Delta}) = 2$, for otherwise $c^*(\hat{\Delta}) \leq -4\pi + \pi + 5.\frac{\pi}{2} + \frac{\pi}{2} = 0$. Hence $l(\hat{\Delta}) = 1^{-1}h1^{-1}g\{1^{-1}, h^{-1}\}g$. However, by Remark 4.5.46 (2)(iv), $\Phi(\hat{\Delta}) \leq 1$. Therefore $\mathcal{P}_{15}$ is aspherical.

Remark 4.5.53. *In view of Lemma 4.5.51, we may assume that $g \neq h^2$ until otherwise stated.*

Lemma 4.5.54. *If $\mathcal{P}_{15}$ is not aspherical then $h^2 = 1$ together with one of the following holds: $h = g^2$ or $g^2 = 1$ or $(gh^{-1})^2 = 1$.*

Proof. Assume that $\mathbb{P}$ is a non-trivial reduced spherical picture over $\mathcal{P}_{15}$. Assign $\acute{\alpha}_1$ to the corners of $\mathbb{P}$. By Remark 4.5.46 (1)(ii), positive regions have degree 4. Moreover a positively curved 4-region Δ has at least one corner with angle $\frac{3\pi}{4}$ and so $l(\Delta) = h^{-1}1\{g^{-1}, h^{-1}\}w_1$ or $\{1^{-1}, g^{-1}\}h1^{-1}w_2$. Thus $l(\Delta)$ implies $h^2 = 1$. Now assign the angle function $\acute{\alpha}_2$ to the corners of $\mathbb{P}$. A positive 4-region implies that $h = g^2$ or $g^2 = 1$ or $(gh^{-1})^2 = 1$. By Remarks 4.5.46 (2)(iii) and (iv), no positive 6-region exists and the result follows.

Lemma 4.5.55. *If $\mathcal{P}_{15}$ is not aspherical then $h^2 = 1$ and $h = g^2$.*

Proof. Assign $\acute{\alpha}_2$ to the corners of $\mathbb{P}$. Positive regions imply that $h = g^2$ or $g^2 = 1$ or $(gh^{-1})^2 = 1$ and these regions are shown in Figure 4.122. If $h = g^2$

and $g^2 = 1$ or $h = g^2$ and $(gh^{-1})^2 = 1$, then $h = 1$, a contradiction. Suppose that at least one of the following holds: $g^2 = 1$ or $(gh^{-1})^2 = 1$. Then the possible positive regions are of Types 1 and 2 only. Distribute the curvature as shown in Figure 4.123. Thus $l(\hat{\Delta}) = g^{-1}h1^{-1}w_1$ or $h^{-1}1g^{-1}w_2$ and so $d(\hat{\Delta}) > 4$.

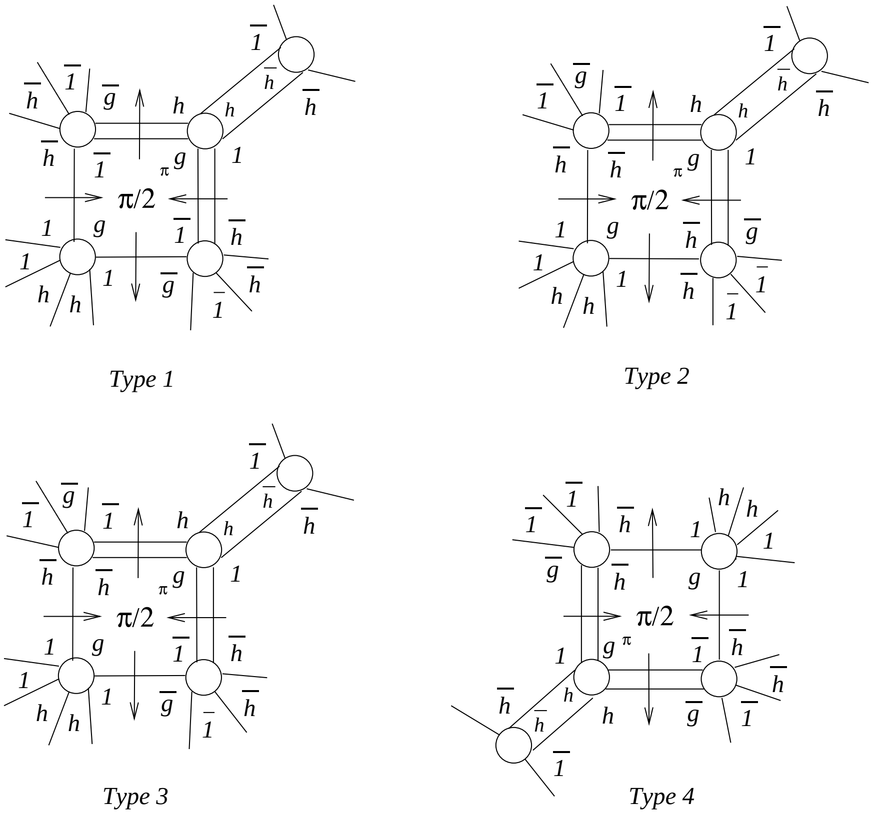
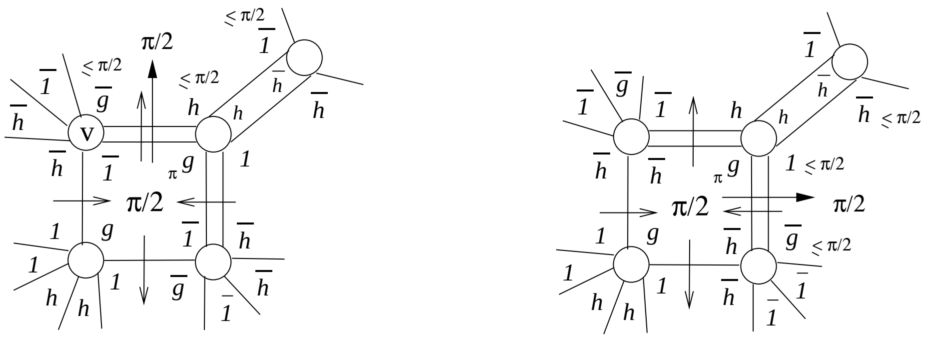


Figure 4.122: $\acute{\alpha}_2$; possible positively curved regions.



Type 1 Type 2

Figure 4.123: Distribution scheme.

Remark 4.5.56. *If $\hat{\Delta}$ receives $\frac{\pi}{2}$ across consecutive edges in $\partial\hat{\Delta}$, then the situation shown in Figure 4.124 is forced. By inspection, $\hat{\Delta}$ receives 0 across the four edges shown in Figure 4.124. Also observe that $\Phi(\hat{\Delta})$ is decreased by 2.*

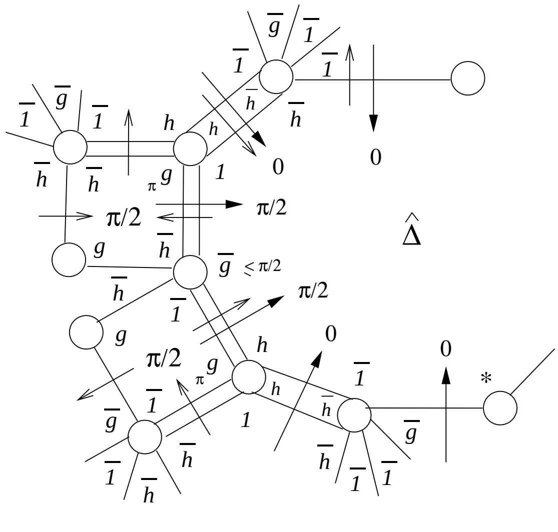


Figure 4.124: $\hat{\Delta}$ receives $\frac{\pi}{2}$ across consecutive edges in $\partial\hat{\Delta}$.

Observe that in Figure 4.123, $d(v) \geq 4$ and so the g^{-1}-corner of v has angle $\leq \frac{\pi}{2}$. By Remark 4.5.56 and Figure 4.123, $\Phi(\hat{\Delta})$ is decreased by 1 for each Γ_2. Thus as before $c^*(\hat{\Delta}) > 0$ implies $d(\hat{\Delta}) = 6$. Assume that $\hat{\Delta}$ receives $\frac{\pi}{2}$ across consecutive edges in $\partial\hat{\Delta}$. Then the degree of the vertex with $*$-corner in Figure 4.124 is at least four. So this $*$-corner has angle $\leq \frac{\pi}{2}$. Thus $c^*(\hat{\Delta}) \leq -4\pi + 6.\frac{\pi}{2} + 2.\frac{\pi}{2} = 0$. Now assume that $\hat{\Delta}$ receives $\frac{\pi}{2}$ from positive regions of Types 1 and 2 across non-consecutive edges in $\partial\hat{\Delta}$. Then the only possibility for $\hat{\Delta}$ is shown in Figure 4.125. Thus $c^*(\hat{\Delta}) \leq -4\pi + 6.\frac{\pi}{2} + 2.\frac{\pi}{2} = 0$. It remains to check for the possibility when $\hat{\Delta}$ receives $\frac{\pi}{2}$ from only one of the Types 1 or 2 of positive regions. In both cases, by inspection, $\Gamma_2 = 1$ and the number of corners of angle π is at most 1. Hence $c^*(\hat{\Delta}) \leq -4\pi + \pi + 5.\frac{\pi}{2} + \frac{\pi}{2} = 0$. Therefore $\mathcal{P}_{15}$ is aspherical. By Lemma 4.5.54 we get the result.

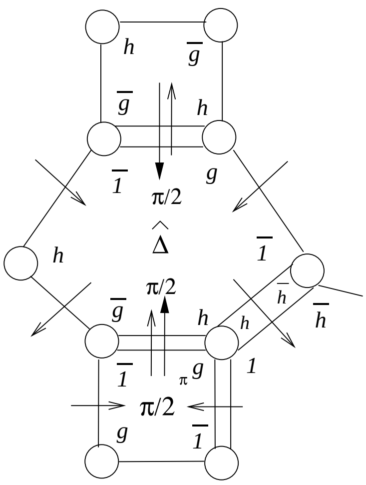


Figure 4.125: $\hat{\Delta}$ receives $\frac{\pi}{2}$ across non-consecutive edges in $\partial\hat{\Delta}$.

Remark 4.5.57. *$g = h^{-1}$ or $g = h^2$ is now allowed.*

Proof of Proposition 4.5.47. The 'if' direction follows from Lemma 4.5.48. Assume $\mathcal{P}_{15}$ is not aspherical. If $g = h^{-1}$ then we are done by Lemma 4.5.48. So assume from now on that $g \neq h^{-1}$. Now assume that $g = h^2$. Then by Lemma 4.5.51, $\mathcal{P}_{15}$ is exceptional of type (**E**). So assume from now on that $g \neq h^2$. By

Lemma 4.5.55, $\mathcal{P}_{15}$ is exceptional of type (**E**).

4.5.6 Case 18: (a=b= c=d)

The relative presentation $\mathcal{P}_{18}= \langle G, y|ayayayayeyfy\rangle=$
$\langle G, y, x|x(ay)^{-1}, x^4ea^{-1}xfa^{-1}x\rangle= \langle G, x|x^5gxh\rangle$, where $g = ea^{-1}$ and $h = fa^{-1}$ (with the assumptions $g, h \in G\backslash\{1\}$). Thus this case is done by Theorem 3.1.6 (see [3]).

Bibliography

[1] A. Ahmad, " The application of pictures to decision problems and relative presentations", Ph.D.thesis (University of Glasgow 1996).

[2] A. Ahmad, M. Al-Mulla, and M. Edjvet, "Asphericity of length four relative group presentations", preprint (2013).

[3] S. M. Aldwaik, and M. Edjvet, "On the asphericity of a family of positive relative group presentations", preprint (2015).

[4] Y. G. Baik, W. A. Bogley, and S. J. Pride, "On the asphericity of length four relative group presentations", *Internat. J. Algebra Comput.* **7** (1997), 227–312.

[5] W. A. Bogley, and S. J. Pride, "Aspherical relative presentations", *Proc. Edinburgh Math. Soc.* **35** (1992), 1–39.

[6] W. A. Bogley, and G.Williams, private communication.

[7] N. Brady, T. Riley, and H. Short, *The geometry of the word problem for finitely generated groups*, (Birkhäuser-Verlag, 2007).

[8] D. J. Collins, and J. Huebschmann, "Spherical diagrams and identities among relations", *Math. Ann.* **261** (1982), 155–183.

[9] P. J. Davidson, "On the asphericity of a family of relative group presentations", *Internat. J. Algebra Comput.* **19** (2009), 159–189.

[10] M. Edjvet, "On the asphericity of one-relator relative presentations", *Proc. Roy. Soc. Edinburgh* **124A** (1994), 713–728.

[11] GAP Group 2014. GAP-Groups, Algorithms and Programming. Version 4.7.4. http://www.gap-system.org.

[12] J. Howie, and V. Metaftsis, "On the asphericity of length five relative group presentations", *Proc. London Math. Soc.* **82** (2001), 173–194.

[13] S. K. Kim, "Asphericity of length 6 equations over torsion-free groups", Ph.D.thesis (Oregon State University 2003).

[14] S. K. Kim, "On the asphericity of length-6 relative group presentations with torsion-free coefficients", *Proc. Edinb. Math. Soc.* **51** (2008), 201–214.

[15] F. Levin, "Solutions of equations over groups", *Bull. Amer. Math. Soc.* **68** (1962), 603–604.

[16] R. C. Lyndon and P. E. Schupp, *Combinatorial group theory*, Ergebnisse der Mathmatik und ihrer Grenzgebiete, (Berlin Springer Verlag, 1977).

[17] W. Magnus, A. Karrass, and D. Solitar, *Combinatorial group theory: presentations of groups in terms of generators and relations*, (Interscience Publishers, 1966).

[18] W. S. Massey, *Algebraic topology: an introduction*, (Springer New York, 1977).

[19] V. Metaftsis, "On the asphericity of relative group presentations of arbitrary length", *Internat. J. Algebra Comput.* **13** (2003), 323–339.