THE UNIVERSITY OF NOTTINGHAM

PHD. THESIS

# RESULTS IN STOCHASTIC CONTROL: OPTIMAL PREDICTION PROBLEMS AND MARKOV DECISION PROCESSES

IKER PEREZ LOPEZ, MSC.

*Thesis submitted to the University of Nottingham for the degree of Doctor of Philosophy*

September, 2014.

*"Ez ezazue leihotik so egin, esan ziguten*
*leihoak gillotinak dira"*

## ABSTRACT

The following thesis is divided in two main topics. The first part studies variations of optimal prediction problems introduced in Shiryaev, Zhou and Xu (2008) and Du Toit and Peskir (2009) to a randomized terminal-time set up and different families of utility measures. The work presents optimal stopping rules that apply under different criteria, introduces a numerical technique to build approximations of stopping boundaries for fixed terminal time problems and suggest previously reported stopping rules extend to certain generalizations of measures.

The second part of the thesis is concerned with analysing optimal wealth allocation techniques within a defaultable financial market similar to Bielecki and Jang (2007). It studies a portfolio optimization problem combining a continuous time jump market and a defaultable security; and presents numerical solutions through the conversion into a Markov Decision Process and characterization of its value function as a unique fixed point to a contracting operator. This work analyses allocation strategies under several families of utilities functions, and highlights significant portfolio selection differences with previously reported results.

## ACKNOWLEDGEMENTS

I would first like to express my gratitude and appreciation to my thesis supervisors Huiling Le and David Hodge, for offering me the support, inspiration and constructive criticism that has made this work possible. I also wish to show my gratitude to Goran Peskir and Phil O'Neill, for acting as examiners and contributing to a really motivating and inspiring discussion. In addition, I am thankful to Theodore Kypraios, Sergey Utev, Jake Carson, Phillip Paine, Kamonrat Suphawan and fellow postgraduate students who once transmitted to me the enthusiasm for research.

I am specially grateful to the School of Mathematical Sciences and its staff at the University of Nottingham, for the financial and administrative aid offered to me during these years.

Finally, special thanks go to my mother, sister and friends for the moral support and sacrifices made on my behalf, and to Jyothika, whose everyday presence, positive attitude and encouragement I am deeply indebted to.

# CONTENTS

CHAPTER 1

# INTRODUCTION

Let us assume that you are, free of charge, offered to take part in a very simple game involving two dice. You are invited to roll these dice up to $N$ times and, at any point prior to the last roll, you are allowed to stop playing and cash-in an amount of money equivalent to the last roll number. You are therefore a *decision maker* within a game and your task is to deduce what the optimal strategy to follow is, with aims of maximising the cash reward.

Such a simple yet captivating brainteaser embodies the essence of what probabilistic problems within the scope of this thesis are trying to achieve. As a decision maker, you want to be capable of determining what the optimal decision to make is, whenever your judgement is to affect the outcome of interest. Indeed, you should certainly keep on rolling whenever the current sum of the dice is lower than the expected reward should you choose to continue.

The present thesis deals with *stochastic control* problems that aim to determine optimal strategies to follow, in situations where outcomes are partly random and partly under our control. The work is divided into two main blocks and covers control problems derived from the theory of optimal stopping and Markov decision processes, fields that have found vast applications in diverse areas such as finance, statistics, machine learning and economics. This work is motivated by two different kinds of financial problems; on the one hand, prediction problems that aim to identify the optimal time for a *stop* action to be taken, maximizing the value of a given reward process; on the other hand, wealth allocation problems that aim to maximize the expected terminal wealth of a financial portfolio, adopting optimal allocation strategies over a given set of financial products.

In what follows we separately discuss the scope and relevance of each of these fields, we review previously published research of interest, and we finally present the line of work developed in Chapters 3, 4 and 6 of this thesis.

## 1.1 Part 1: Optimal Stopping and Prediction Problems

Optimal stopping theory studies the problem of choosing the optimal time to take a particular action, with aims of maximizing or minimizing an expected reward/cost. Its use is widespread within some areas of statistics, economics, and mathematical finance. These problems may relate to either discrete or continuous time cases; in this work we focus on the latter case. In order to formally introduce the definition of a *stopping problem* we first present the notion of a *stopping time*.

**Definition 1.1.1.** Given a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, a random variable $\tau : \Omega \to [0, \infty)$ is said to be an $\mathcal{F}_t$*-stopping time* provided $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$.

Intuitively we say that $\tau$ is a stopping time if the event $\{\tau \leq t\}$ can be determined with the knowledge available up to time $t$. All decisions in stopping problems must be based on the information available prior to the present time and no anticipation is allowed.

**Definition 1.1.2.** Let $T$ be a time horizon and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space. Denote by $G = (G_t)_{t\geq 0}$ an $\mathcal{F}_t$-adapted gain process. An *optimal stopping problem* is the problem of identifying the stopping time $\tau^*$ so that

$$V = \mathbb{E}[G_{\tau^*}] = \sup_{0\leq\tau\leq T} \mathbb{E}[G_\tau] \,. \tag{1.1.1}$$

Here, $V$ denotes the value function, which models the optimal expected reward/cost. An infinite time horizon is allowed, and usually leads to a subset of stopping problems for which explicit solutions are often attainable. In this first part of this thesis we will first present a variation of a previously explored fixed horizon stopping problem, and will afterwards direct our attention to a randomized terminal time set up, in order to investigate ways of overcoming complications arising under the original set up.

Optimal stopping problems have been a major object of study for approximately 60 years now and there exists currently a wide collection of techniques for approaching them; these are determined by the nature of the underlying process. Techniques that take advantage of the unconditional finite-dimensional distribution of a gain process are categorized within the subgroup of techniques referred to as the *Martingale Approach*, with the *Snell envelope* being its most important concept (cf. [55]). Snell was the first to characterize the solution to a discrete-time stopping problem as the minimal supermartingale dominating its gain process .

On the other hand, techniques that exploit the analytical structure of conditional transition functions are referred to as the *Markovian Approach*; these attempt to study optimal stopping problems through functions of initial points in a state space. Such an approach deals with the extension of problem (1.1.1) to a state space $(E, \mathcal{B})$, where $G_t = G(X_t)$ defines the Markovian representation for a measurable function $G$, with a Markovian family of processes $((X_t)_{t\geq 0}, (\mathcal{F}_t)_{t\geq 0}, (\mathbb{P}_t)_{x\in E})$. Here, $X = (X_t)_{t\geq 0}$ is a Markov process with values in $E$. This approach leads to problems

$$V(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_x[G(X_\tau)] \ ,$$

where $\mathbb{P}_x(X_0 = x) = 1$ and $\mathcal{T} = \{\tau : 0 \leq \tau \leq T\}$. If the Markov representation of the problem is valid, analytical tools provided by the theory of Markov processes can often be utilized; it is through the infinitesimal generator of the underlying process that a close link between optimal stopping and free-boundary problems can be established. Problems addressed in next chapters admit such a representation.

This relation between optimal stopping and free-boundary problems was explored by Mikhalevich (cf. [42]) and several other authors during the 1960s. Work during this decade includes that of McKean (cf. [40]), who first transformed into a free-boundary problem the optimal stopping problem linked to pricing an *American Call* option. The success of mathematical finance would later on attract further attention to optimal stopping; see [49] for an extensive overview of the theory and applications of optimal stopping and free-boundary problems.

### Optimal Prediction Problems

*Optimal Prediction* problems can be defined as a subgroup of optimal stopping problems concerned with stopping a certain stochastic process as close as possible to its ultimate maximum, over a pre-defined period of time and with no anticipation allowed. These problems are of great theoretical interest and find applications within fields of financial engineering. Well-known discrete-time variants of these problems include the *Secretary* and *House Selling* problems (cf. [9]).

In this work we concern ourselves with continuous-time variants of these problems. A continuous time optimal prediction problem was first studied by Graversen, Peskir and Shiryaev for the case of a Brownian motion $B = (B_t)_{t \geq 0}$ (cf. [29]). The authors analysed the optimal stopping problem

$$V = \inf_{\tau \in [0,1]} \mathbb{E}[(B_\tau - \max_{0 \leq s \leq 1} B_s)^2] \, .$$

An explicit solution to the problem was obtained through the method of time change. Afterwards, Du Toit and Peskir continued in [20] their study considering an extension to the case of a drifted Brownian motion $B^\lambda = (B^\lambda_t)_{t \geq 0}$; they presented a solution to the problem

$$V = \inf_{\tau \in [0,1]} \mathbb{E}[(B^\lambda_\tau - \max_{0 \leq s \leq 1} B^\lambda_s)^2] \, .$$

In particular, the stopping rules obtained for both cases above were defined as the first entry time of an underlying stochastic process to some stopping region; the process accounted for the distance between the Brownian motion and its running maximum.

Within a financial context and considering a geometric Brownian motion $Z$, Du Toit and Peskir (cf. [21]), Shiryaev, Xu and Zhou (cf. [54]) and Zhou, Dai, Jin and Zhong (cf. [60]) derived results on the stopping problems

$$V_1 = \inf_{\tau \in [0,T]} \mathbb{E}\Big[\frac{M_T}{Z_\tau}\Big] \quad \text{and} \quad V_2 = \sup_{\tau \in [0,T]} \mathbb{E}\Big[\frac{Z_\tau}{M_T}\Big] \, , \tag{1.1.2}$$

where $M_T$ stands for the maximum of $Z$ over the entire time interval $[0, T]$. The use of probabilistic techniques in [21] and [54], and a PDE approach in [60] enabled the authors to derive the so-called *Bang-Bang* strategies, defining a goodness index through parameters describing the dynamics of $Z$ and categorizing most processes as either *good* (never to stop) or *bad* (immediate stop). Problems $V_1$ and $V_2$ surprisingly led to different optimal stopping rules for a given subset of parameters; in this case, the solution to $V_1$ was given as a time-dependent optimal stopping boundary for an underlying stochastic process to cross.

More recently, Espinosa and Touzi (cf. [26]) and Elie and Espinosa (cf. [25]) have addressed optimal stopping problems for a more general family of mean reverting diffusions with similar financial motivations. In their case the terminal time bounding the time space is random and it is given by the first hitting time of a diffusion process to 0. In [26] the solution to the optimal stopping problem $\inf_{\tau\in[0,\theta]} \mathbb{E}[U(X_\tau - \max_{0\leq s\leq\theta} X_s)]$ is defined as the first crossing time of a time dependent boundary by some underlying stochastic process, where $X$ stands for some mean reverting diffusion and $U$ is an increasing and convex loss function; $\theta$ is the first hitting time of $X$ to 0. On the other hand, [25] provides a solution to the problem

$$\inf_{\tau\in[0,\theta]} \mathbb{E}\Big[\Big(\frac{\max_{0\leq s\leq\theta} X_s - X_\tau}{\max_{0\leq s\leq\theta} X_s}\Big)^2\Big] .$$

In this case, results are consistent with those in [21], [54] and [60], and a restrictive time dependent stopping boundary is defined, so that immediate stop is close to optimal.

The first part of this thesis makes use of the extensive collection of optimal stopping techniques under a Markovian approach reviewed in [49] and explored in [17, 21, 25, 54, 60] and references therein. Chapter 2 offers an introduction to the notation and approach to optimal stopping problems for continuous time processes, and summarizes a set of results of use in order to study prediction problems to follow. Next, Chapter 3 analyses a variation of problems (1.1.2), focused on minimizing a non-linear utility function of the ratio between a geometric Brownian motion and its absolute maximum over the entire time interval. The Chapter offers optimal stopping strategies that apply under certain restrictions regarding the parameters describing the model, and characterizes the resulting value functions. The solutions show consistency with the previous research aforementioned and suggest only ever stopping *bad* processes, defined in terms of a relation between parameters in the stochastic process. In addition, the work discusses the intractability of the stopping problem via reduction to a free boundary problem, in cases when optimal stopping times are expected to respond to departures of a diffusion from the origin value 0.

Chapter 4, on the other hand, brings together the theory in [49] and randomization techniques examined in [17], [2] and [31] in order to analyse generalizations of optimal prediction problems to families of utility functions covering wider cases not presented in the literature. This is done in an extended time-randomized context, where the stopping terminal deadline is random and independent of the state of the diffusion of interest. In this work, we derive a family of stopping problems which are time-independent with the underlying diffusion being two-dimensional. We discuss the existence of optimal stopping boundaries and obtain complete solutions as the unique solution to a boundary value problem. Our results allow for us to computationally build numerical approximations of fixed terminal time set-up optimal stopping problems and suggest the possibility of extending optimal stopping rules defined in [21] to a more general family of power utility measures. The results on this work have been submitted for publication to *SIAM Journal on Control and Optimization*.

## 1.2 Part 2: Markov Decision Processes and Finance

Markov decision processes (MDPs) provide a mathematical framework for modelling decision making in situations where outcomes are partly random and partly under the control of a decision maker. They are useful for the study of diverse optimization problems generally solved via dynamic programming, and have found applications in diverse areas such as epidemic processes, queueing systems, machine learning and economics.

A Markov decision process is in essence a discrete-time stochastic control process, it allows for a generalization to a continuous time set up, this however requires a significant amount of additional theory and is out of the scope of this thesis. The most common set up of an MDP consists of a system evolving over *discrete-time* points and controlled by a set of *sequential decisions*. Transitions of the *system state* are random and *Markovian*, meaning transition to future states is independent of past history. Given a current system state, a decision maker chooses an *admissible action*, generally influencing the transition to a new state according to some stochastic law. The decision maker receives *rewards* according to his choices of controls for every system state, and aims to optimally control the system evolution process. The general optimization criterion is to maximise the expected value of the sum of random rewards. Figure 1.1 presents the schematic evolution of an MDP.

**Definition 1.2.1.** A *Markov decision process* is a sequence of random variables $X = (X_n)_{n\geq 0}$ describing the stochastic evolution of a system state. It is modelled by a 4-tuple $(E, \mathcal{A}, Q_\cdot(\cdot,\cdot), R_\cdot(\cdot,\cdot))$ where:

- $E$ denotes the *state space* that process $X$ takes values on.
- $\mathcal{A}$ denotes the *action space*. For any specific $x \in E$ at time $n$, only a subclass of actions $D_n(x) \subseteq \mathcal{A}$ may be admissible.
- $Q_n(B|x,a)$ is the *stochastic transition kernel*; it models the probability for action $a \in \mathcal{A}$ in state $x \in E$ at time $n$, to lead to some state $y \in B$ at time $n+1$, for $B \subseteq E$.
- $R_n(x,a)$ is the *one-stage reward* of the system at time $n$, if the current state is $x$ and action $a \in \mathcal{A}$ is taken.

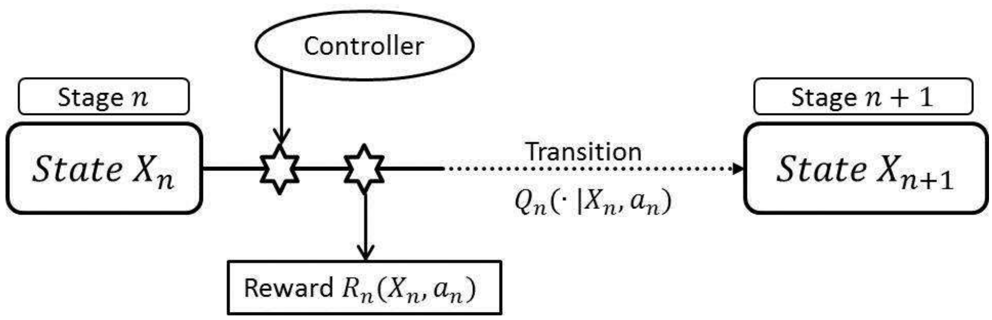


**Figure 1.1:** Schematic representation of the evolution of a Markov Decision Process

In order to introduce the concept of a *Markov decision problem* we first define the notion of a *Markov policy*. Loosely, this is a sequence of decision rules $\pi = (\pi_n(\cdot))_{n\geq 0}$

with $\pi_n(x) \in D_n(x)$; it determines the action taken by the controller for all $x \in E$ and time $n \geq 0$. If $\pi = (\pi(\cdot))_{n\geq 0}$, i.e. if $\pi_n(\cdot)$ is independent of $n$ for all $n \geq 0$, the policy is named *stationary* and is independent of time evolution; stationary policies are fundamental to the theory of infinite horizon MDPs and are essential to the work in this thesis.

**Definition 1.2.2.** A *Markov decision problem* is the problem of identifying the optimal Markovian policy $\pi$ which will maximize, over an horizon $N$, the expected sum of rewards given by

$$\mathbb{E}^{\pi}\Big[\sum_{k=0}^{N-1} R_k(X_k, \pi_k(X_k))\Big] , \tag{1.2.1}$$

where the expectation is taken over the probability distribution induced by policy $\pi$.

Equation (1.2.1) is usually referred to as the *total reward criterion* and, as mentioned, infinite time horizon $N$ is allowed. It is possible, and sometimes convenient, to extend the scope of policies $\pi$ to history-dependent (non-Markovian) policies. As well as that, we note that it is possible to include a discounting parameter in the characterization of (1.2.1). In general, there exist several different characterizations of optimality criteria and vast variations on formulations of discrete-time Markov decision processes, including problems with constraints, partial state observations, average reward criteria and so on. We will however restrict ourselves to the theory relevant to the second part of this thesis.

Markov decision processes were known at least as early as the 1950s with the work of Bellman (cf. [3]). In his work, Bellman develops functional equations for finding optimal policies through the introduction of the concepts of state, action and transition. Substantial research establishing the importance of the model resulted later from Howard's book (cf. [32]) published in 1960. The foundations on Markov decision models, and the formalization of the model in use up to these days, are due to Dubins and Savage (cf. [22]) and Blackwell (cf. [7]) respectively. Dubins and Savage analysed a gambling model whose underlying ideas are very similar to MDPs in terms of structure. On the other hand, Blackwell first established a generalized description of action sets, rewards and transition probabilities and emphasized the importance of stationary policies in his work.

Another important work of special relevance to this part of the thesis is that of Bertsekas and Shreve (cf. [4]), which provides detailed analysis on the probabilistic structure and measurability questions for the generalized Borel model for MDPs. For a detailed introduction and extensive overview of these problems and their theory, along with further references, we refer to the work of Puterman (see [51]).

**MDPs and Wealth Optimization Problems**

*Wealth optimization* or *portfolio optimization* problems are widely studied topics within the subject of financial engineering. Their concern is on choosing the *optimal* proportions of various assets to be held in a financial portfolio, according to some chosen performance criterion. This criterion usually combines considerations of the expected value of the portfolio's return, its dispersion and some measures of financial risk.

Let $T$ be a finite time horizon and denote by $X = (X_t)_{t\geq 0}$ a continuous time stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Assume that $X$ describes the evolution of a wealth process dependent on an allocation *strategy* or *policy*, taking values on a set $\Pi$. In the second part of this thesis we concern ourselves with a variation of a optimization problem of the form

$$V(t,x) = \sup_{\pi\in\Pi} \mathbb{E}[U(X_T^\pi)|X_t^\pi = x] \, , \tag{1.2.2}$$

for all $(t,x) \in [0,T] \times \mathbb{R}_+$. Here, the supremum is taken over all admissible policies in $\Pi$, and function $U$ is the *utility* determining a certain performance criterion.

Research within the field of portfolio optimization was triggered during the late 60s with the work of Merton (cf. [41]), who made use of stochastic control techniques for maximizing expected discounted utilities of consumption. Later, his work was extended to different default-free frameworks where market uncertainty was mainly modelled by continuous processes with Brownian components, such work includes that of Fleming and Pang (cf. [28]), Karatzas and Shreve (cf. [34]) and Pham (cf. [50]), among others.

In the last decade, it is the optimal investment linked to defaultable claims that has attracted attention. High yield corporate bonds offer attractive risk-return profiles and have become popular in comparison to stocks or default-free bonds; recent work in the area includes that of Bielecki and Jang (cf. [5]), Bo et al. (cf. [8]), Lakner and Liang (cf. [37]) and Capponi and Figueroa-López (cf. [16]). Authors Bielecki and Jang (cf. [5]) first considered a market including a defaultable bond, a risk-free account and a stock driven by Brownian dynamics, and analysed optimal asset allocations for a variation of problem (1.2.2) with a risk averse CRRA utility, given by

$$V(t,x,h) = \sup_{\pi\in\Pi} \mathbb{E}\left[\frac{(X_T^\pi)^\gamma}{\gamma}\Big| X_t^\pi = x, H_t = h\right], \quad \text{with } 0<\gamma<1, \tag{1.2.3}$$

for all $(t,x,h) \in [0,T] \times \mathbb{R}_+ \times \{0,1\}$; here $h$ denotes the current value of a *default process* $H = (H_t)_{t\geq 0}$ that models the state of the defaultable bond under the intensity based approach to credit risk (see [6]). For this matter, the authors assumed constant parameters governing the system and default intensity, and derived closed form solutions for the optimal allocations, pointing out that investment on defaultable securities is only justified under the presence of reasonable interest premiums. In addition, since a Brownian asset is invariant to default event risk, their results allocate it a constant fraction of wealth in a similar fashion to [41].

Bo et al. (cf. [8]) approached a perpetual allocation problem for an investor with logarithmic utility, considering a defaultable perpetual bond along with a traditional stock and a risk-free account in a similar manner to [5]. Their work modelled stochastically the intensities and premium process including a common Brownian factor, and postulated the price process of the defaultable bond based on heuristic arguments instead of arbitrage-free arguments. Their results establish, in the same fashion to Bielecki and Jang, monotonicity conditions on the optimal investment on defaultable bonds with respect to the risk premium and recovery of wealth at default.

More recently, Lakner and Liang (cf. [37]) employed duality theory to obtain similar optimal allocation strategies in a 2-way market, including a continuous time money market account and a defaultable bond whose prices can jump; and Capponi and Figueroa-López (cf. [16]) extended previous work in [5, 8, 37] to a defaultable market with different economical regimes, where a defaultable bond, a money market and a stock are all dependent on a finite state continuous time Markov process $Y = (Y_t)_{t\geq 0}$; in their work they obtained explicit solutions to the optimization problem

$$V(t,x,h;y) = \sup_{\pi\in\Pi} \mathbb{E}[U(X_T^\pi)|X_t^\pi = x, H_t = h, Y_t = y]$$

with logarithmic and risk averse CRRA utilities, for all $(t,x,h) \in [0,T] \times \mathbb{R}_+ \times \{0,1\}$ and $y \in \{y_1,...y_N\}$. A numerical economic analysis highlighted the preference of investors to buy defaultable bonds when the macroeconomic regimes yield high expected returns and the planning horizon is large.

Results in the literature do however primarily relate to markets incorporating Brownian-driven assets and are limited with regards to the choices of utility functions that they provide solutions for. The work in the second part of the thesis incorporates the presence of a defaultable bond in a finite horizon market with a bank account and a continuous-time jump asset driven by a *piecewise deterministic Markov process* (see [1]). In this circumstance, it is possible to build a bridge between a problem formulated in continuous time and the theory of discrete-time MDPs, reducing the optimization problem to a discrete-time model by considering an embedded state process. Similar financial markets, in absence of the defaultable claim, have previously been explored by Kirch and Runggaldier (cf. [35]) and Bäuerle and Rieder (cf. [11]). Authors Kirch and Runggaldier (cf. [35]) presented an algorithm for the evaluation of hedging strategies for European claims, addressing the optimization problem

$$V(t,x,s) = \min_{\pi\in\Pi} \mathbb{E}[l(F(S_T) - x - \int_t^T \pi_s \mathrm{d}S_s)|X_t^\pi = x, S_t = s] \,,$$

which aims to minimize the expected value of a convex loss function $l$ of the hedging error of a claim with payoff $F$, for all $(t,x,s) \in [0,T] \times \mathbb{R}_+^2$. Here, $S$ is an asset whose dynamics are driven by a geometric Poisson process and $X^\pi$ is the available capital under $\pi$. Strategies in $\Pi$ are given by units held in the risky asset at different times.

On the other hand, Bäuerle and Rieder (cf. [11]) considered the general portfolio utility maximization problem (1.2.2). In their case, the wealth process $X$ reflects the evolution of wealth in a portfolio mixing a bank account and a generalized family of pure jump models; in addition, utility $U$ is any increasing a concave function. The authors make use of the embedding procedure previously explored by Almudevar (cf.

[1]) in order to convert the problem into a discrete-time MDP, and offer a proof for the validity of value iteration and policy improvement algorithms to approximate optimal allocation policies.

The second part of this thesis makes use of results on credit risk presented in [6] along with the theory for MDPs reviewed in [51] and [13]. Chapter 5 offers an introduction to the notation and approach to discrete-time Markov decission processes, and summarizes a set of results of use in order to analyse an MDP derived from a portfolio optimization problem in Chapter 6. Here, the work of Bäuerle and Rieder in [11–13] is extended to the context of defaultable markets explored in [5, 8, 16, 37] and references therein. Model parameters within a pure jump asset can be determined so that a Brownian market is approximated and such an approach overcomes the need to assume any particular form for the utility function. Furthermore, it provides means of analysing portfolio strategies incorporating illiquid markets. Through the conversion of the optimization problem into a Markov decision process (MDP), its value function is characterized as the unique fixed point to a dynamic programming operator and optimal wealth allocations are numerically approximated through value iteration.

In order for such a characterization to hold, default intensities and interest rates are assumed constant in a similar manner to that in [5]. However, an extension to Markov modulated regimes similar to [16] is discussed in the closing section. Our numerical analysis explores the dependence of optimal portfolio selections on the *risk premium* and different parameters describing the system, and extends the work in [5, 8, 16, 37] to more general families of logarithmic and exponential utility functions. The results highlight the nature of the significantly different allocation procedures under an exponential family of utilities, and the existence of a dependency on optimal stock allocation to default event, in a model with short selling restrictions. The results on this work are currently being edited and will soon be submitted for publication.

Part I

# Results Associated to Optimal Prediction Problems

CHAPTER 2

# RESULTS ON OPTIMAL STOPPING AND A RANDOMIZATION TECHNIQUE

We begin the first part of this thesis offering an introduction to the notation and approach to optimal stopping problems for continuous time processes. We summarize a set of results under a Markovian approach presented in the theory in [49]; these will be of use in order to analyse optimal prediction problems presented in the next two chapters. In addition, we introduce some results on free-boundary problems and a finite horizon randomization technique playing a key role in following work.

Let $X = (X_t)_{t\geq 0}$ denote a Markovian process taking values in a measurable space $(E, \mathcal{B})$ and defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}_x)$, which satisfies the usual conditions of completeness and right-continuity. It is assumed that $E = \mathbb{R}^n$ for some $n \geq 0$ and $\mathcal{B}$ is the Borel $\sigma$-algebra on $E$. Under probability measure $\mathbb{P}_x$, process $X$ starts at $x \in E$ and is right-continuous; in addition

$$X_{\tau_n} \xrightarrow{n\to\infty} X_\tau \quad \mathbb{P}_x\text{-a.s.}\,,$$

for all sequence of stopping times such that $\tau_n \uparrow \tau$ as $n \to \infty$. It is also assumed that the mapping $x \mapsto \mathbb{P}_x(F)$ is measurable for all $F \in \mathcal{F}$.

We recall that a *Markovian method* of solution deals with optimal stopping problems of the form

$$V(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_x[G(X_\tau)]\,, \tag{2.0.1}$$

where $\mathcal{T} = \{\tau : 0 \leq \tau \leq T\}$ and the expectation is taken with respect to $\mathbb{P}_x$. Here, it is assumed that function $G : E \to \mathbb{R}$ satisfies

$$\mathbb{E}_x\Big[\sup_{0\le t\le T} |G(X_t)|\Big] < \infty$$

for all $x \in E$, and we recall that infinite time horizon is allowed. Under this setting, a decision on whether to *stop* or *continue* observing process $X$ evolve in time depends only on its present state, not on its past. Thus, it poses a stopping problem of a random path in the state space $E$.

The general theory of optimal stopping in [49] defines the notion of a *stopping set* $D$, along with a complementary *continuation set* $C$, so that stopping is optimal whenever the current state of the diffusion of interest falls within $D$. It holds that $E = D \cup C$ and $D \cap C = \emptyset$, so that

$$\tau^* = \tau_D = \inf\{t \ge 0 \;:\; X_t \in D\} \tag{2.0.2}$$

stands for the optimal stopping time offering a solution to (2.0.1), if any. In most optimal stopping problems, heuristic arguments about the shape of $D$ make it possible to guess its generalized mathematical representation; this ability is crucial in solving these problems. In these cases, $\tau^*$ will attain a supremum and the first part of the problem is reduced to identifying the shape of $D$; having to additionally compute the value function $V(x)$ as explicitly as possible.

## 2.1 General Results on Optimal Stopping

The following results refer to the theory of optimal stopping for continuous time Markovian processes and can be found in [49] (Chapter 1, subsection 2.2). We summarize this theory for future reference and all results will be stated without proof.

In what follows no different treatment of *finite horizon* and *infinite horizon* stopping problems is necessary. We note that whenever $T < \infty$, time evolution and its closeness to $T$ is a factor of importance and therefore stopping problem $V$ in (2.0.1) should be reformulated as

$$V(t,x) = \sup_{0\le\tau\le T-t} \mathbb{E}_{t,x}[G(t+\tau, X_{t+\tau})]\,, \tag{2.1.1}$$

for all $(t, x) \in [0, T] \times E$; here, the expected value is taken with respect to a measure $\mathbb{P}_{t,x}$ such that $\mathbb{P}_{t,x}(X_t = x) = 1$. The following results are obtained for problem (2.0.1) with $T = \infty$ and extend to the finite horizon case upon noting that the state space $E$ admits the representation $\mathbb{R}^+ \times E$, and process $Y_t = (t, X_t)$ is Markovian. Moreover, it holds that $V(T, x) = G(T, x)$ for all $x \in [0, T] \times E$ and therefore $\tau_D \leq T$ is finite.

From now on we assume $T = \infty$ unless otherwise specified. For problem $V$ in (2.0.1), we define the stopping set

$$D = \{x \in E \ : \ V(x) = G(x)\} \, , \tag{2.1.2}$$

and continuation set

$$C = \{x \in E \ : \ V(x) > G(x)\} \, . \tag{2.1.3}$$

We observe that if the *value function* $V$ is lower semicontinuous and the *gain function* $G$ is upper semicontinuous, then $C$ is open and it follows that $D$ is closed. In this case, $\tau_D$ in (2.0.2) is an $\mathcal{F}_t$-stopping time since both $X$ and $(\mathcal{F}_t)_{t \geq 0}$ are right continuous.

**Definition 2.1.1.** Let $F : E \to \mathbb{R}$ be a measurable function so that $F(X_\tau) \in L^1(\mathbb{P}_x)$ for all stopping times $\tau \in \mathcal{T}$. Function $F$ is said to be *superharmonic* if

$$\mathbb{E}_x F(X_\tau) \leq F(x) \, ,$$

for all $x \in E$.

The following result lists necessary conditions for the existence of an optimal stopping time and settles the optimality of $\tau_D$ in $V$ under the definition of $D$ in (2.1.2).

**Theorem 2.1.1.** *Assume there exists an optimal stopping time $\tau^*$ in problem* (2.0.1)*, so that*

$$V(x) = \mathbb{E}_x G(X_{\tau^*})$$

*for all $x \in E$. Then, value function $V$ is the smallest superharmonic function dominating the gain function $G$ on the state space $E$. In addition, if $V$ is lower semicontinuous and $G$ is upper semicontinuous, then*

- *stopping time $\tau_D$ with $D$ given by* (2.1.2) *is such that $\tau_D \leq \tau^*$ and is optimal in* (2.0.1)*;*

- *the stopped process* $(V(X_{t\wedge\tau_D}))_{t\geq 0}$ *is a right-continuous martingale under* $\mathbb{P}_x$ *for all* $x \in E$.

In addition, the following complementary result provides sufficient condition for the existence of an optimal stopping time in problem $V$.

**Theorem 2.1.2.** *Assume there exists a smallest superharmonic function* $\hat{V}$ *that dominates the gain function* $G$ *on* $E$ *in the stopping problem* (2.0.1). *Assume as well that* $\hat{V}$ *is lower semicontinuous and* $G$ *is upper semicontinuous.*

*Set* $D = \{x \in E \,:\, \hat{V}(x) = G(x)\}$ *and let* $\tau_D$ *be defined by* (2.0.2). *Then,*

- $\hat{V} = V$ *and* $\tau_D$ *is optimal in* (2.0.1) *if* $\mathbb{P}_x(\tau_D < \infty) = 1$ *for all* $x \in E$*;*
- *there is no optimal stopping time (with probability* 1*) in* (2.0.1) *if* $\mathbb{P}_x(\tau_D < \infty) < 1$ *for some* $x \in E$.

We note that condition $\mathbb{P}_x(\tau_D < \infty) = 1$ is always satisfied whenever $T < \infty$, since $(T, x) \in D$ for all $x \in E$. In this case, Theorem 2.1.2 is particularly useful since it justifies the existence of an optimal stopping time identified with $\tau_D$ in (2.0.2). These results apply whenever one can prove from the definition of $V$ that it is lower semicontinuous. The following corollary presents a way of tackling stopping problems fitting this criteria.

**Corollary 2.1.3** (Existence of a Stopping Time)**.**

**Infinite Horizon**. *Consider optimal stopping problem* (2.0.1) *and assume that* $V$ *is lower semicontinuous and* $G$ *is upper semicontinuous. If* $\mathbb{P}_x(\tau_D < \infty) = 1$ *for all* $x \in E$*, then the optimal stopping time is given by* $\tau_D$*, with* $D$ *as in* (2.1.2)*. If* $\mathbb{P}_x(\tau_D < \infty) < 1$ *for some* $x \in E$*, then there is no optimal stopping time with probability* 1.

**Finite Horizon**. *Consider optimal stopping problem* (2.1.1) *and assume that* $V$ *is lower semicontinuous and* $G$ *is upper semicontinuous. Then the optimal stopping time is given by* $\tau_D$*, with* $D$ *as in* (2.1.2).

It has therefore been shown that optimal stopping problem $V$ in (2.0.1) is equivalent to the problem of finding the smallest superharmonic function $\hat{V}$ that dominates the gain function $G$ on the state space $E$. We note that while $V$ poses a maximization problem, same arguments apply to optimal stopping problems linked to minimization problems. In these cases, the focus will be given to finding the biggest subharmonic function dominated by the gain function over the entire state space. Results on the existence of an optimal stopping time will in this case follow from the upper semicontinuity of the value function.

## 2.2 FREE-BOUNDARY PROBLEMS

A traditional way of finding superharmonic (or subharmonic) functions dominating (or dominated by) the gain function $G$ is making use of solutions to *free-boundary problems*. These are differential equations to be solved for both an unknown function and domain; the segment of the boundary of the domain is the unknown *free boundary*. Well-known free-boundary problems include the *Stefan* and *obstacle* problems (see [58] and [15]).

Consider the maximization problem (2.0.1), due to the Markovian structure of the process $X$, it is possible to set up a link between problem $V$ and a deterministic equation that governs $X$ in mean. This link takes the form of a partial differential equation when $X$ is continuous, or a partial integro-differential equation when $X$ is a jump process. The basic idea of this approach is that the smallest superharmonic function $\hat{V}$ dominating $G$ solves

$$\mathcal{A}_X \hat{V}(x) \leq 0 \quad \text{for all} \quad x \in E \quad (\hat{V} \text{ minimal}), \tag{2.2.1}$$

$$\hat{V}(x) > G(x) \quad \text{for all} \quad x \in C, \tag{2.2.2}$$

$$\hat{V}(x) = G(x) \quad \text{for all} \quad x \in D, \tag{2.2.3}$$

where $\mathcal{A}_X$ stands for the infinitesimal operator of the Markovian process $X$, given by

$$\mathcal{A}_X f(x) = \lim_{t \to 0} \frac{\mathbb{E}_x[f(X_t)] - f(x)}{t},$$

and acting on suitable functions $f : \mathbb{R}^n \to \mathbb{R}$. It is important to observe that both $\hat{V}$ and $C$ (or $D$) are unknown in the system of equations (2.2.1)-(2.2.3), and need to be determined. In addition, we note that in the problems analysed in this thesis the infinitesimal generator of $X$ has a differential form.

Assuming the gain function $G$ is smooth enough in a neighbourhood of $\partial C$, condition (2.2.1) above is in general split into two different conditions, these depend on the nature of the state process $X$ and stopping set $C$ and give rise to the concepts of *continuous fit* and *smooth fit*. Roughly speaking, if process $X$ immediately enters the interior of $D$ after starting at $\partial C$, value function $\hat{V}$ solves

$$\begin{aligned}
&\mathcal{A}_X \hat{V}(x) = 0 && \text{for all } x \in C \, , \\
&\hat{V}(x) = G(x) && \text{for all } x \in D \, , \\
&\frac{\partial \hat{V}}{\partial x}\Big|_{\partial C} = \frac{\partial G}{\partial x}\Big|_{\partial C} && \text{(smooth fit)} \, .
\end{aligned}$$

On the other hand, if process $X$ does not immediately enter the interior of $D$ after starting at $\partial C$, value function $\hat{V}$ solves

$$\begin{aligned}
&\mathcal{A}_X \hat{V}(x) = 0 && \text{for all } x \in C \, , \\
&\hat{V}(x) = G(x) && \text{for all } x \in D \, , \\
&\hat{V}\big|_{\partial C} = G\big|_{\partial C} && \text{(continuous fit)} \, .
\end{aligned}$$

We note that condition $\mathcal{A}_X \hat{V}(x) = 0$ for all $x \in C$ in the system of equations above is linked to the statement that $(V(X_{t\wedge\tau_D}))_{t\geq 0}$ is a martingale in Theorem 2.1.1. The conditions of smooth and continuous fit will be formalized and further developed in this thesis when necessary for our purposes.

## 2.3 The Supremum Functional and the Neumann Problem

Assume from now on that that the state space is given by $E = \mathbb{R}$. Let $S_t$ define the *supremum process* of $X$, given by

$$S_t = \sup_{0\leq s\leq t} X_s \, . \tag{2.3.1}$$

Free-boundary problems linked to the supremum process are of particular interest in this work; we therefore revise some results in [49] (Chapter 3) in relation to $S$, these will provide a base of analysis for Chapter 4 of this thesis. Under the settings introduced at the beginning of this chapter, we consider the optimal stopping problem

$$V = \sup_{\tau \in \mathcal{T}} \mathbb{E}[G(X_t, S_t)] \,, \tag{2.3.2}$$

where $\mathcal{T} = \{\tau : 0 \leq \tau \leq \infty\}$. We note that the process $(X_t, S_t)_{t \geq 0}$ is Markovian and therefore previous analysis within this chapter applies.

Results presented here extend to a finite horizon terminal time set-up, upon careful reformulation of the state process in (2.3.2). This usually increases the degree of complexity of problem $V$ due to the importance of the time variable (time remaining decreases as the state evolves). Such results do however fall outside the scope of this thesis.

We note that process $S$ is strictly increasing whenever $S_t = X_t$, and constant at times when the values of $S$ and $X$ differ. Its characterization allowing for arbitrary starting points $s \in E$ and $s \geq x$ is given by

$$S_t^s = s \vee S_t \,,$$

for all $t \geq 0$. The extension of (2.3.2) to an arbitrary starting point in $\{(x,s) \in \mathbb{R}^2 : s \geq x\}$ is given by

$$V(x,s) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_{x,s}[G(X_t, S_t)] \,, \tag{2.3.3}$$

where the expectation is taken with respect to a probability measure for which $\mathbb{P}(X_0 = x, S_0 = s) = 1$.

It is important to note that the dimension of the extended optimal stopping problem (2.3.3) will be that of the minimal underlying Markovian process that leads to a solution; this could be smaller than the dimension of the initial process $(X_t, S_t)_{t \geq 0}$. The dimension of a stopping problem is in general a complicated thing to determine.

**The Neumann Boundary Condition**

The Neumann boundary condition is a type of boundary condition that when imposed on an ordinary or a partial differential equation, it specifies the values that the derivative of a solution is to take on the boundary of the domain.

Assume that the gain function $G : E \to \mathbb{R}$ in (2.3.3) is continuous. Furthermore, assume the existence of an optimal stopping time $\tau^* \in \mathcal{T}$ for problem $V$ in (2.3.3). It follows from Corollary 2.1.3 that $\tau^*$ is given by the first entry time of $(X_t, S_t)_{t\geq 0}$ to the closed stopping set $D$ in (2.1.2) and is denoted by $\tau_D$. Hence, $V$ admits the representation

$$V(x,s) = \mathbb{E}_{x,s}[G(X_{\tau_D}, S_{\tau_D})] \,. \tag{2.3.4}$$

We recall that the existence of a closed stopping set $D$ implies the existence of an open continuation set $C$. We furthermore make the assumption of the boundary $\partial C$ of $C$ being *regular*, in the sense that for every starting point $(x,s) \in \partial C$ the process $(X_t^x, S_t^s)$ immediately enters the stopping set $D$.

Under these conditions, if the process $(X_t)_{t\geq 0}$ is continuous, it is shown in [49] (Chapter 3, section 7) that the extended optimal stopping problem (2.3.4) solves a boundary problem with Neumann boundary condition for all $(x,s) \in \bar{C}$, i.e.

$$\mathcal{A}_X V(x,s) = 0 \qquad \text{for } x < s \text{ with } s \text{ fixed,} \tag{2.3.5}$$

$$\frac{\partial V}{\partial s}(x,s) = 0 \qquad \text{for } x = s, \tag{2.3.6}$$

$$V\big|_{\partial C} = G. \tag{2.3.7}$$

We note that equation (2.3.6) stands for the Neumann condition on the boundary alongside the diagonal $x = s$; here, the process $(X_t, S_t)_{t\geq 0}$ can be identified with the continuous process $X$.

## 2.4 A Finite Horizon Randomization Technique

The technique of terminal time randomization, modelled as a Poisson process, was first introduced within the context of optimal stopping in order to offer approximations for American option values and their selling boundaries in [17]. In this context, randomizing the horizon $T$ was done as a first step in a more general procedure; this aimed to asymptotically reduce the variance while holding the mean in the random parameter setting.

Let $V$ denote the finite horizon optimal stopping problem given in (2.1.1); we recall that this is given by

$$V(t,x) = \sup_{0 \le \tau \le T-t} \mathbb{E}_{t,x}[G(t+\tau, X_{t+\tau})] \, ,$$

for all $(t,x) \in [0,T] \times E$. In this case the flow of time affects the value of $V$ and the dimensionality of the stopping problem gains a degree of complexity. The characterization of the boundary (or boundaries) of the continuation set $C$, offering a solution to problem $V$, usually relates to a time dependent functional defining the threshold in the state space $E$ where the process $X$ takes its values.

Let $N = (N_t)_{t \ge 0}$ denote an $\mathcal{F}_t$-adapted Poisson process with jump intensity parameter $\omega$, independent to $X$. Randomizing the time horizon in problem $V$ consists on modelling $T$ as the $n^{\text{th}}$ jump $T_n$ in process $N$, and setting the jump intensity to $\omega = n/T$, for some $n \in \mathbb{N}$. Note that the asymptotic dynamics of a counting process

$$\tilde{N}_t = \frac{T}{n} N_t$$

resemble the flow of time as $n$ tends to infinity, i.e.

$$\mathrm{d}\tilde{N}_t \approx \mathrm{d}t \quad \text{as} \quad n \to \infty \, .$$

Under such characterization of the horizon deadline, and due to the exponential distribution of jump times in $N$ and its memoryless property, the closeness to time $T$ is independent on the current time $t$ and only dependent on the current state $k$ in the jump process $N$. Under these circumstances, it is possible to set up a time-independent Markovian optimal stopping problem

$$\tilde{V}(k,x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}[\tilde{G}(N_\tau, X_\tau) | \mathcal{F}_{T_k}] \tag{2.4.1}$$

for all $(k,x) \in \{0, 1, ..., n-1\} \times E$, where $\mathcal{T}$ stands for the set of all stopping times taking values in $[T_k, T_n]$. A detailed presentation of a stopping problem of this kind will be presented in the opening section of Chapter 4 .

Note that the expected value of the randomized horizon and its variance are given by

$$\mathbb{E}[T_n] = T \quad \text{and} \quad \mathrm{Var}[T_n] = \frac{T^2}{n} \, .$$

Upon satisfaction of certain continuity and measurability conditions guaranteeing the existence of a solution in (2.4.1), it is possible to build approximations to stopping rules for the original finite horizon problem $V$ increasing the amount of steps $n$ in the random horizon setting and therefore asymptotically fixing the value of $T_n$ to $T$.

CHAPTER 3

# AN OPTIMAL PREDICTION PROBLEM

Let $T > 0$ denote a given positive terminal time and $Z = (Z_t)_{0\leq t\leq T}$ define a geometric Brownian motion with drift $\mu \in \mathbb{R}$ and volatility $\sigma > 0$, given by

$$Z_t = Z_0 \exp\{\sigma B_t + (\mu - \sigma^2/2)t\}$$

on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Here $B = (B_t)_{0\leq t\leq T}$ stands for a one-dimensional standard Brownian motion with $B_0$=0 and $\{\mathcal{F}_t\}_{t\geq 0}$ is the $\mathbb{P}$-augmentation of the filtration generated by $B$. Denote the running maximum process of $Z$ as

$$M_t = \max_{0\leq s\leq t} Z_s\,, \quad t \in [0, T]\,. \tag{3.0.1}$$

Note that $M_T$ is the *ultimate maximum* value that $Z$ will reach before time $T$; due to the stochastic nature of the process, the precise time for this to happen will only be known at time $T$. In this chapter, we set up an *optimal prediction problem* and aim to identify a *stopping time* $\tau \in [0, T]$ establishing an *optimal stopping rule* that optimizes the expected value of a weight function measuring the closeness between $M_T$ and $Z$. In view of the results in the literature presented in Chapter 1, we analyse a variation of problems (1.1.2) making use of a non-linear *utility function* $U(x) = (1 - \alpha x)^2$ of $Z_\tau/M_T$ for different values of $\alpha \in (0, 1)$. We note that $\alpha < 0$ would pose maximization problem instead. This leads to the optimal prediction problem

$$V = \inf_{\tau\in\mathcal{T}} \mathbb{E}\Big[U\Big(\frac{Z_\tau}{M_T}\Big)\Big] = \inf_{\tau\in\mathcal{T}} \mathbb{E}\Big[\Big(\frac{M_T - \alpha Z_\tau}{M_T}\Big)^2\Big]\,, \tag{3.0.2}$$

where $\mathcal{T}$ stands for the set of all $\mathcal{F}_t$-stopping times $\tau \in [0, T]$.

This chapter offers optimal stopping strategies to problem (3.0.2) under some restrictions on the parameters that define the model. The problem is first modified and adapted to the natural filtration of $B$ in Section 3.1, while Section 3.2 presents a Markov representation for its extension to all possible starting points, along with the proof of existence of optimal stopping times. Section 3.3 offers partial results defining stopping rules that show consistence with previous work; and Sections 3.4 and 3.5 focus on the application of different approaches in order to provide necessary proofs. Finally, Section 3.6 offers a discussion of used methods and their drawbacks.

## 3.1 An Alternative Expression for V

The prediction problem (3.0.2) is not adapted since $M_T$ is not $\mathcal{F}_t$-measurable. Therefore, it does not fall within the scope of standard optimal stopping problems and needs to be modified. Following work in [21], for any $\lambda \in \mathbb{R}$ we let $B^\lambda = (B_t^\lambda)_{0\le t\le T}$ denote a drifted Brownian motion given by

$$B_t^\lambda = B_t + \lambda t \, ,$$

for $t \in [0, T]$. Set $\lambda = (\mu - \sigma^2/2)/\sigma$, then $M_t$ in (3.0.1) reads

$$M_t = Z_0 \exp\{\sigma S_t^\lambda\} \, , \tag{3.1.1}$$

where $S^\lambda = (S_t^\lambda)_{0\le t\le T}$ is given by $S_t^\lambda = \max_{0\le s\le t} B_s^\lambda$. Thus, $V$ in (3.0.2) is given by

$$V = \inf_{\tau\in\mathcal{T}} \mathbb{E}\big[\big(1 - \alpha e^{-\sigma(S_T^\lambda - B_\tau^\lambda)}\big)^2\big] \, . \tag{3.1.2}$$

We note (cf. [39]) that the cumulative distribution function of $S_t^\lambda$ is given by

$$F_{S_t^\lambda}(s) = \mathbb{P}(S_t^\lambda \le s) = \Phi\Big(\frac{s - \lambda t}{\sqrt{t}}\Big) - e^{2\lambda s}\Phi\Big(\frac{-s - \lambda t}{\sqrt{t}}\Big) \, , \tag{3.1.3}$$

for all $(t, s) \in \mathbb{R}_+^2$. Here $\Phi(\cdot)$ stands for the cumulative distribution function of a standard normal variable.

**Lemma 3.1.1.** *Let function $G$ be defined as*

$$G(t, x) = (1 - \alpha e^{-\sigma x})^2 + 2\sigma\alpha \int_x^\infty (e^{-\sigma z} - \alpha e^{-2\sigma z})(1 - F_{S_{T-t}^\lambda}(z))\, \mathrm{d}z \tag{3.1.4}$$

*for all $(t, x) \in [0, T] \times \mathbb{R}_+$. Then, $V$ in* (3.1.2) *may be expressed as the $\mathcal{F}_t$-measurable optimal stopping problem*

$$V = \inf_{\tau\in\mathcal{T}} \mathbb{E}[G(\tau, X_\tau)] \, , \tag{3.1.5}$$

*with process $X = (X_t)_{0\le t\le T}$ given by $X_t = S_t^\lambda - B_t^\lambda$.*

***Proof.*** A similar approach to that in [21], using deterministic times and making use of the law of total expectation, shows that for any $t \in [0, T]$

$$\begin{aligned}\mathbb{E}\Big[\big(1-\alpha e^{-\sigma(S_T^\lambda - B_t^\lambda)}\big)^2\Big] &= \mathbb{E}\big[\mathbb{E}\big[\big(1-\alpha e^{-\sigma(S_T^\lambda - B_t^\lambda)}\big)^2|\mathcal{F}_t\big]\big] \\ &= \mathbb{E}\Big[\mathbb{E}\big[\big(1-\alpha e^{-\sigma[(S_t^\lambda - B_t^\lambda)\vee(\max_{0\le s\le T-t} B_{t+s}^\lambda - B_t^\lambda)]}\big)^2|\mathcal{F}_t\big]\Big].\end{aligned}$$

The independent and stationary increments of $B_t^\lambda$ imply that

$$\Big(\max_{0\le s\le T-t} B_{t+s}^\lambda - B_t^\lambda\Big)\Big|\mathcal{F}_t \overset{\text{Law}}{=} S_{T-t}^\lambda\,,$$

so that

$$\mathbb{E}\Big[\big(1-\alpha e^{-\sigma(S_T^\lambda - B_t^\lambda)}\big)^2\Big] = \mathbb{E}\Big[\big(1-\alpha e^{-\sigma X_t}\big)^2 F_{S_{T-t}^\lambda}(X_t) + \int_{X_t}^\infty \big(1-\alpha e^{-\sigma z}\big)^2\,\mathrm{d}F_{S_{T-t}^\lambda}(z)\Big]\,.$$

Noting that $\lim_{z\to+\infty}(1-\alpha e^{-\sigma z})^2(1-F_{S_{T-t}^\lambda}(z)) = 0$, we integrate by parts the above expression to obtain

$$\begin{aligned}\mathbb{E}\Big[\big(1-\alpha e^{-\sigma(S_T^\lambda - B_t^\lambda)}\big)^2\Big] &= \mathbb{E}\Big[(1-\alpha e^{-\sigma X_t})^2 + 2\sigma\alpha\int_{X_t}^\infty (e^{-\sigma z} - \alpha e^{-2\sigma z})(1-F_{S_{T-t}^\lambda}(z))\,\mathrm{d}z\Big] \\ &= \mathbb{E}\Big[G(t, X_t)\Big]\,.\end{aligned}$$

Arguments based on each stopping time being the limit of a decreasing sequence of discrete stopping times (cf. [21] & [24]), allow for us to extend this result to all stopping times, so that

$$V = \inf_{\tau\in\mathcal{T}} \mathbb{E}[G(\tau, X_\tau)]\,,$$

completing the proof. □

Function $G$ in (3.1.4) is referred to as the *gain function* for the stopping problem $V$.

## 3.2 Extension of $V$ and Existence of an Optimal Stopping Time

We shall make use of Markovian techniques within the theory of optimal stopping presented in Chapter 2. For this, we recall from (2.1.1) that a Markovian approach to stopping problems with finite horizon deals with the extension of problem $V$ in (3.1.5) to

$$V(t,x) = \inf_{0\le\tau\le T-t} \mathbb{E}_{t,x}[G(t+\tau, X_{t+\tau})]\,, \tag{3.2.1}$$

for all $(t,x) \in [0,T] \times \mathbb{R}_+$; here, the expectation is taken with respect to a measure $\mathbb{P}_{t,x}$ such that $\mathbb{P}_{t,x}(X_t = x) = 1$. We note that the original problem (3.1.5) is obtained as the special case $V = V(0,0)$.

However, in order to redefine problem (3.2.1) in a rather tractable way, it is necessary to know how $X$ depends on its starting value $x \geq 0$. It is shown in [30] that the process $X_t = S_t^\lambda - B_t^\lambda$, with $X_0 = x \geq 0$, has the law of a Brownian motion with negative drift $-\lambda$ reflected at 0. In addition, it is known (c.f. [20]) that this shares the law of the process $X^x = (X^x)_{0 \leq t \leq T}$ defined by

$$X_t^x = x \vee S_t^\lambda - B_t^\lambda \,, \tag{3.2.2}$$

so that, for any $x \geq 0$ and $t \in [0,T]$ fixed, it holds that $X$ under $\mathbb{P}_{t,x}$ is equal in law to $X^x$ under $\mathbb{P}$.

**Lemma 3.2.1.** *The optimal stopping problem $V$ in* (3.1.5) *admits an optimal stopping time and can be extended to*

$$V(t,x) = \mathbb{E}_{t,x}[G(t+\tau_D(t,x), X_{t+\tau_D(t,x)})] = \mathbb{E}[G(t+\tau_D(t,x), X^x_{\tau_D(t,x)})] \,, \tag{3.2.3}$$

*with*

$$\tau_D(t,x) = \inf\{s \in [0,T-t] : (t+s, X_s^x) \in D\} \,, \tag{3.2.4}$$

*for all $(t,x) \in [0,T] \times \mathbb{R}_+$ and some $D \subseteq [0,T] \times \mathbb{R}_+$.*

*Note.* The subset $D \subseteq [0,T] \times \mathbb{R}_+$ will be referred to as the *stopping set*.

***Proof.*** We note from (3.1.4) that $G$ is continuous in $(t,x)$, as well as $X^x$ on $x$. In addition, we note that

$$\begin{aligned}
\mathbb{E}[G(t,X_t^x)] &= \mathbb{E}\Big[(1-\alpha e^{-\sigma X_t^x})^2 + 2\sigma\alpha \int_{X_t^x}^\infty (e^{-\sigma z} - \alpha e^{-2\sigma z})(1 - F_{S^\lambda_{T-t}}(z))\mathrm{d}z\Big] \\
&< \mathbb{E}\Big[1 + 2\sigma \int_0^\infty e^{-\sigma z}\mathrm{d}z\Big] < \infty \,,
\end{aligned}$$

for all $t \in [0,T]$. The use of the dominated convergence Theorem implies that the mapping $(t,x) \mapsto \mathbb{E}[G(t+\tau, X_\tau^x)]$ is continuous for all $\tau \in \mathcal{T}$. As a consequence, the extended $V$ given in (3.2.1) is upper semicontinuous for all $(t,x) \in [0,T] \times \mathbb{R}_+$. The interpretation of Corollary 2.1.3 for minimization stopping problems entails the existence of a stopping region $D \subseteq [0, T] \times \mathbb{R}_+$ and optimal stopping time $\tau_D$ in (3.2.4) allowing for the characterization of the extension of $V$ in (3.2.3). $\square$

Results in Section 2.1 indicate that the extended $V$ in (3.2.3) is given by the biggest subharmonic function dominated by the gain function $G$ on the entire state space. Function $G$ is a highly non-linear function at low values of $x$, as observed in Figure 3.1. Lemma A.1.1 in Appendix A shows that, for all $(t, x) \in [0, T] \times \mathbb{R}_+$, it is possible

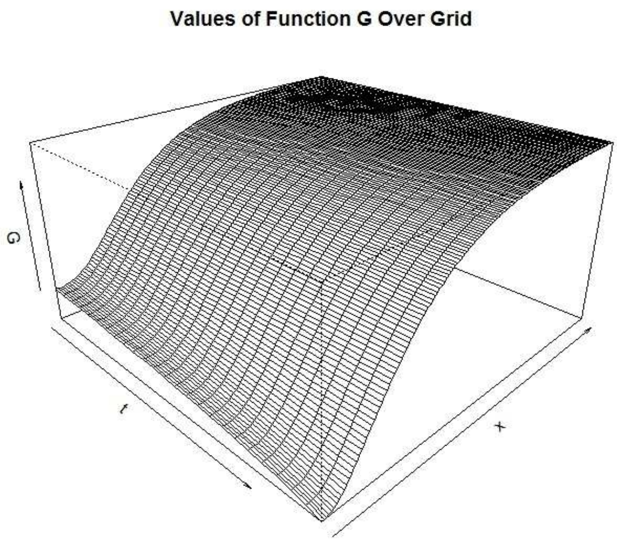


**Figure 3.1:** Value of function $G$ for $x \in [0, 4)$, with $T = 1$, $\alpha = 1$, $\lambda = -0.5$ and $\mu = 0$. A high non-linearity is observed at low values of $x$; in addition, it rapidly approaches value 1 as $x$ departs from 0, cause of the intuitive understanding of optimal stopping times coming at points of significant excursions of the process $X$ from 0.

to rewrite function $G$ as

$$\begin{aligned} G(t,x) &= 1 - \Big(2\alpha e^{-\sigma x} - \alpha^2 e^{-2\sigma x}\Big)\Phi\Big(\frac{x - \lambda(T-t)}{\sqrt{T-t}}\Big) \\ &\quad + \Big(\frac{\alpha^2\sigma}{\lambda - \sigma} e^{2(\lambda-\sigma)x} - \frac{2\alpha\sigma}{2\lambda - \sigma} e^{(2\lambda-\sigma)x}\Big)\Phi\Big(\frac{-x - \lambda(T-t)}{\sqrt{T-t}}\Big) \\ &\quad + \frac{4\alpha(\sigma - \lambda)}{2\lambda - \sigma} e^{\frac{\sigma}{2}(\sigma - 2\lambda)(T-t)}\Phi\Big(\frac{-x + (\lambda - \sigma)(T-t)}{\sqrt{T-t}}\Big) \\ &\quad + \frac{\alpha^2(\lambda - 2\sigma)}{\lambda - \sigma} e^{2\sigma(\sigma-\lambda)(T-t)}\Phi\Big(\frac{-x + (\lambda - 2\sigma)(T-t)}{\sqrt{T-t}}\Big), \end{aligned} \tag{3.2.5}$$

for all $\lambda \in \mathbb{R}/\{\frac{\sigma}{2}, \sigma\}$. If $\lambda = \frac{\sigma}{2}$ we get

$$\begin{aligned} G(t,x) &= 1 + 2\sigma\alpha\sqrt{\frac{T-t}{2\pi}}e^{-\frac{(x+\frac{\sigma}{2}(T-t))^2}{2(T-t)}} - \Big(2\alpha e^{-\sigma x} - \alpha^2 e^{-2\sigma x}\Big)\Phi\Big(\frac{x-\frac{\sigma}{2}(T-t)}{\sqrt{T-t}}\Big) \\ &\quad -\Big(2\alpha^2 e^{-\sigma x} + 2\alpha(1+\sigma x) + \sigma^2\alpha(T-t)\Big)\Phi\Big(\frac{-x-\frac{\sigma}{2}(T-t)}{\sqrt{T-t}}\Big) \\ &\quad +3\alpha^2 e^{\sigma^2(T-t)}\Phi\Big(\frac{-x-\frac{3\sigma}{2}(T-t)}{\sqrt{T-t}}\Big)\,, \end{aligned}$$

and if $\lambda = \sigma$, function $G$ is given by

$$\begin{aligned} G(t,x) &= 1 - 2\sigma\alpha^2\sqrt{\frac{T-t}{2\pi}}e^{-\frac{(x+\sigma(T-t))^2}{2(T-t)}} - \Big(2\alpha e^{-\sigma x} - \alpha^2 e^{-2\sigma x}\Big)\Phi\Big(\frac{x-\sigma(T-t)}{\sqrt{T-t}}\Big) \\ &\quad -\Big(2\alpha e^{\sigma x} - \alpha^2(1+2\sigma x) - 2\alpha^2\sigma^2(T-t)\Big)\Phi\Big(\frac{-x-\sigma(T-t)}{\sqrt{T-t}}\Big)\,. \end{aligned}$$

In addition, we recall from (2.1.2) that, for any starting $(t,x) \in [0,T] \times \mathbb{R}_+$, the optimal stopping time takes place whenever the current state of the two-dimensional Markovian process $(t+s, X_s^x)_{0\le s\le T-t}$ falls within the subset of the state space where the values of $G$ and $V$ are the same, so that the stopping set is given by

$$D = \{(t,x) \in [0,T] \times \mathbb{R}_+ : V(t,x) = G(t,x)\}\,,$$

and is a closed set. The continuation set $C$ was defined in (2.1.3) as

$$C = D^c = \{(t,x) \in [0,T] \times \mathbb{R}_+ : V(t,x) < G(t,x)\}\,.$$

We note that $D \cup C = [0,T] \times \mathbb{R}_+$. In addition, since the terminal time indicates forced stopping, so that $\{T\} \times \mathbb{R}_+$ is always a part of $D$, we have $\tau_D \le T$. Also, $V(t,x) \le G(t,x)$ for all $(t,x) \in [0,T] \times \mathbb{R}_+$, since immediate stopping is always possible in (3.2.3).

### 3.3 Characterization of the Stopping Set

The shape of the stopping set is dependent on the value of parameters $\mu$ and $\sigma$, which describe the dynamics of the process $Z$. The following results provide partial characterizations of $D$ and establish *Bang-Bang* stopping strategies for several choices of parameters, allowing for different $\alpha \in (0,1)$; we recall that Bang-Bang strategies apply whenever $[0,T) \times \mathbb{R}_+$ is fully included in either $D$ or $C$. The theorem is stated without proof and results proving its different parts are postponed to the next sections.

Techniques of use are rather direct, and a discussion on the intractability of the problem through its reduction to a free-boundary problem is included in Section 3.6.

**Theorem 3.3.1.** *The optimal stopping set $D$ for the optimal stopping problem* (3.2.3) *is partially characterized by the following expressions. If* $0 < \alpha \leq \frac{1}{2}$

$$D = \begin{cases} [0,T] \times \mathbb{R}_+ & \textit{when } \mu \leq -\frac{\sigma^2}{2}\,, \\ \{(T,x) : x \geq 0\} & \textit{when } \mu \geq \frac{\sigma^2}{2}\,. \end{cases} \tag{3.3.1}$$

*If* $\frac{1}{2} < \alpha \leq \frac{2}{3}$

$$D = \begin{cases} [0,T] \times \mathbb{R}_+ & \textit{when } \mu \leq -\frac{\sigma^2}{2}\,, \\ \{(T,x) : x \geq 0\} & \textit{when } \mu \geq \frac{3\sigma^2}{2}\,. \end{cases} \tag{3.3.2}$$

*Finally, if* $\frac{2}{3} < \alpha \leq \frac{3}{4}$

$$D = \{(T,x) : x \geq 0\} \quad \textit{when } \mu \geq \frac{3\sigma^2}{2}\,. \tag{3.3.3}$$

Note that lower values of $\alpha$ increase our ability to describe $D$. This is because the manipulation of the utility function $U$, with respect to $\alpha$, triggers several beneficial inequality properties on a differential operator closely related with the infinitesimal generator of the underlying process $X$, as we will see in Section 3.4. The following corollary follows from Theorem 3.3.1 and characterizes stopping rules for the original non-extended stopping problem $V$ in (3.1.5).

**Corollary 3.3.2.** *The optimal stopping rule for problem* (3.1.5) *is partially characterized by the following expressions. If* $0 < \alpha \leq \frac{1}{2}$

$$\tau_D(0,0) = \begin{cases} 0 & \textit{when } \mu \leq -\frac{\sigma^2}{2}\,, \\ T & \textit{when } \mu \geq \frac{\sigma^2}{2}\,. \end{cases} \tag{3.3.4}$$

*If* $\frac{1}{2} < \alpha \leq \frac{2}{3}$

$$\tau_D(0,0) = \begin{cases} 0 & \textit{when } \mu \leq -\frac{\sigma^2}{2}\,, \\ T & \textit{when } \mu \geq \frac{3\sigma^2}{2}\,. \end{cases} \tag{3.3.5}$$

*Finally, if* $\frac{2}{3} < \alpha \leq \frac{3}{4}$

$$\tau_D(0,0) = T \quad \textit{when } \mu \geq \frac{3\sigma^2}{2}\,. \tag{3.3.6}$$

In what follows, Section 3.4 introduces a direct approach to the problem based on techniques of stochastic calculus, and leads to results proving most of Theorem 3.3.1. The case $\mu \geq \frac{\sigma^2}{2}$ in equation (3.3.1) requires a different approach based on Girsanov's Theorem and techniques of change of measures; the proof of which is postponed to Section 3.5.

## 3.4 A Direct Stochastic Approach

The aim under this approach is to exploit the properties of the stochastic infinitesimal generator of the underlying process $X$, which is key in order to establish proof for most of the cases in Theorem 3.3.1. The infinitesimal generator of $X$ is known (see [21]) to act on twice differentiable functions $f$ (satisfying $f'(0) = 0$) as

$$\mathcal{A}_X f(x) = -\lambda f'(x) + \frac{1}{2} f''(x) \, . \tag{3.4.1}$$

Applying Itô formula on $G$ in (3.2.3) we get

$$\begin{aligned} V(t,x) &= G(t,x) + \mathbb{E}\Big[ \int_0^{\tau_D(t,x)} G_t(t+s, X_s^x) \mathrm{d}s + \int_0^{\tau_D(t,x)} G_x(t+s, X_s^x) \mathrm{d}X_s^x \Big] \\ &+ \mathbb{E}\Big[ \int_0^{\tau_D(t,x)} \frac{1}{2} G_{xx}(t+s, X_s^x) \mathrm{d}\langle X^x, X^x \rangle_s \Big] \, , \end{aligned} \tag{3.4.2}$$

for all $(t,x) \in [0,T] \times \mathbb{R}_+$. An application of the Itô-Tanaka formula (cf. [49], Theorem 30.9) shows that

$$\mathrm{d}X_t = -\lambda \mathrm{d}t + \mathrm{sign}(Y_t) \mathrm{d}B_t + \mathrm{d}l_t^0(Y) \, ,$$

where the process $Y$ is the unique strong solution to the stochastic differential equation

$$\mathrm{d}Y_t = -\lambda \mathrm{sign}(Y_t) \mathrm{d}t + \mathrm{d}B_t \, ,$$

with $Y_0 = 0$, and $\mathrm{d}l^0(Y)$ is the local time of $Y$ at 0 given by

$$\mathrm{d}l_t^0(Y) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t I(|Y_s| < \varepsilon) \mathrm{d}\langle Y, Y \rangle_s \, .$$

Then, we note that $\mathrm{d}\langle X^x, X^x \rangle_t = \mathrm{d}t$ and expand $\mathrm{d}X_s^x = \mathrm{d}(x \vee S_s^\lambda - B_s - \lambda s)$ in equation (3.4.2) to obtain

$$\begin{aligned} V(t,x) &= G(t,x) + \mathbb{E}\Big[ \int_0^{\tau_D(t,x)} G_t(t+s, X_s^x) + \mathcal{A}_X G(t+s, X_s^x) \mathrm{d}s \Big] \\ &+ \mathbb{E}\Big[ \int_0^{\tau_D(t,x)} G_x(t+s, X_s^x) \mathrm{d}(x \vee S_s^\lambda) - \int_0^{\tau_D(t,x)} G_x(t+s, X_s^x) \mathrm{d}B_s \Big] \, , \end{aligned}$$

for all $(t,x) \in [0,T] \times \mathbb{R}_+$. Note that $G_x(t+s, X_s^x)\mathrm{d}(x \vee S_s^\lambda)$ is always 0, since a change of value in $x \vee S_s^\lambda$ implies $X_s^x = 0$, and from $G$ in (3.1.4) we note that

$$G_x(t,x)\big|_{x=0+} = 2\alpha\sigma(e^{-\sigma x} - \alpha e^{-2\sigma x})\mathbb{P}(S_{T-t}^\lambda \le x)\big|_{x=0+} = 0\,,$$

for all $t \in [0,T]$. In addition, since $\mathbb{E}[G_x(t,x)] \le \mathbb{E}[2\sigma e^{-\sigma x}] < \infty$, the term $\int_0^r G_x(t+s, X_s^x)\mathrm{d}B_s$ is a martingale starting at 0, for all $r \ge 0$. Then, by the optional sampling theorem, the extended optimal prediction problem (3.2.3) may be expressed as

$$V(t,x) = G(t,x) + \mathbb{E}\Big[\int_0^{\tau_D(t,x)} H(t+s, X_s^x)\mathrm{d}s\Big]\,, \tag{3.4.3}$$

with

$$H(t,x) = G_t(t,x) + \mathcal{A}_X G(t,x)\,. \tag{3.4.4}$$

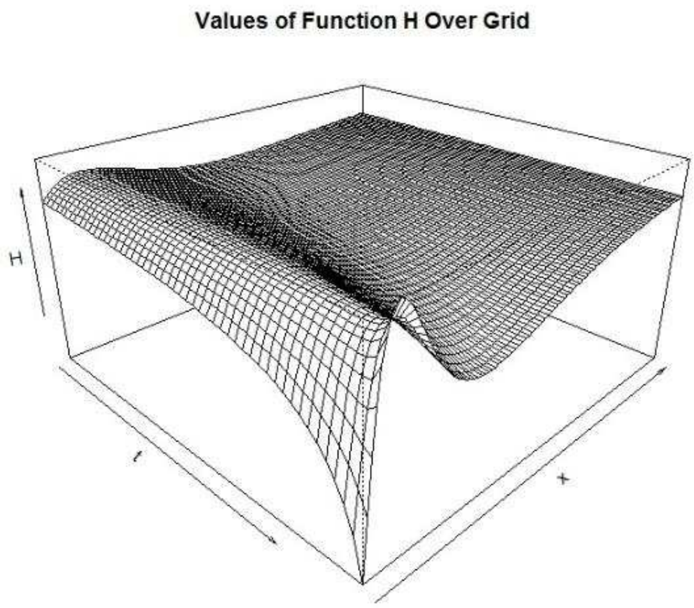


**Figure 3.2:** Value of function $H$ for $x \in [0,4)$, with $T = 5$, $\alpha = 1$, $\lambda = -0.5$ and $\mu = 0$. It is noticeable that $H$ is a non-monotone function of time and space.

Figure 3.2 presents the values of function $H$ for some choice of parameters. In addition, Lemma A.1.2 in Appendix A shows that, for all $(t,x) \in [0,T] \times \mathbb{R}_+$, $H$ is given by

$$\begin{aligned} H(t,x) &= \sigma\alpha\cdot\Bigg\{\Big[2\alpha(\sigma+\lambda)e^{-2\sigma x}-(\sigma+2\lambda)e^{-\sigma x}\Big]\Phi\Big(\frac{x-\lambda(T-t)}{\sqrt{(T-t)}}\Big) \\ &+\Big[\sigma e^{(2\lambda-\sigma)x}-2\alpha\sigma e^{2(\lambda-\sigma)x}\Big]\Phi\Big(\frac{-x-\lambda(T-t)}{\sqrt{(T-t)}}\Big) \\ &+2\alpha(\lambda-2\sigma)e^{-2\sigma(\lambda-\sigma)(T-t)}\Phi\Big(\frac{-x+(\lambda-2\sigma)(T-t)}{\sqrt{(T-t)}}\Big) \\ &-2(\lambda-\sigma)e^{-\frac{\sigma}{2}(2\lambda-\sigma)(T-t)}\Phi\Big(\frac{-x+(\lambda-\sigma)(T-t)}{\sqrt{(T-t)}}\Big)\Bigg\} \end{aligned}$$

for all $\lambda\in\mathbb{R}$.

### 3.4.1 Properties of Function H

In what follows we expose some properties of $H$ for different values of $\alpha$ that are useful in order to establish results in Theorem 3.3.1.

**Lemma 3.4.1.** *If $\alpha\le\frac{3}{4}$ and $\mu\ge\frac{3\sigma^2}{2}$, then $H$ is strictly negative for all $(t,x)\in[0,T]\times\mathbb{R}_+$.*

***Proof.*** Recall that $\lambda=(\mu-\sigma^2/2)/\sigma$, then $\mu\ge\frac{3\sigma^2}{2}$ is equivalent to $\lambda\ge\sigma$. Let function $\hat{H}$ be given by

$$\hat{H}(t,x)=\frac{1}{\sigma\alpha}\cdot H(t,x)=A(t,x)+B(t,x)+C(t,x)\,, \tag{3.4.5}$$

with

$$\begin{aligned} A(t,x) &= \Big[2\alpha(\sigma+\lambda)e^{-2\sigma x}-(\sigma+2\lambda)e^{-\sigma x}\Big]\Phi\Big(\frac{x-\lambda(T-t)}{\sqrt{(T-t)}}\Big) \\ &+ \Big[\sigma e^{(2\lambda-\sigma)x}-2\alpha\sigma e^{2(\lambda-\sigma)x}\Big]\Phi\Big(\frac{-x-\lambda(T-t)}{\sqrt{(T-t)}}\Big)\,, \\ B(t,x) &= \Big[2\alpha(\lambda-2\sigma)e^{-2\sigma(\lambda-\sigma)(T-t)}-2(\lambda-\sigma)e^{-\frac{\sigma}{2}(2\lambda-\sigma)(T-t)}\Big] \\ &\times \Phi\Big(\frac{-x+(\lambda-2\sigma)(T-t)}{\sqrt{(T-t)}}\Big)\,, \end{aligned}$$

and

$$\begin{aligned} C(t,x) &= -2(\lambda-\sigma)e^{-\frac{\sigma}{2}(2\lambda-\sigma)(T-t)} \\ &\times \Big[\Phi\Big(\frac{-x+(\lambda-\sigma)(T-t)}{\sqrt{(T-t)}}\Big)-\Phi\Big(\frac{-x+(\lambda-2\sigma)(T-t)}{\sqrt{(T-t)}}\Big)\Big]\,. \end{aligned}$$

Showing that $\hat{H}$ is strictly negative is equivalent to showing that function $H$ is strictly negative, since both $\sigma$ and $\alpha$ are strictly positive. We show that for all $(t, x) \in [0, T] \times \mathbb{R}_+$, functions $A$ and $C$ are less than or equal to 0 and function $B$ is strictly negative.

Function $C$ is obviously non-positive since $\lambda \geq \sigma$ and the term within the square brackets is positive. The inequality $A \leq 0$ is proved in different ways depending on the value of $x$. For $x \leq \frac{\log(2\alpha)}{\sigma}$ we have that

- $2\alpha(\sigma+\lambda)e^{-2\sigma x} - (\sigma+2\lambda)e^{-\sigma x} \leq 0$: Recall that $\alpha < \frac{3}{4}$ and $\sigma, x \geq 0$, then

$$\begin{aligned} 2\alpha(\sigma+\lambda)e^{-\sigma x} - (\sigma+2\lambda) \ &\leq \ \frac{3}{2}(\sigma+\lambda)e^{-\sigma x} - (\sigma+2\lambda) \leq \\ \leq \ \frac{3}{2}(\sigma+\lambda) - (\sigma+2\lambda) \ &\leq \ 0\,, \text{ since } \lambda \geq \sigma; \end{aligned}$$

- $\sigma e^{(2\lambda-\sigma)x} - 2\alpha\sigma e^{2(\lambda-\sigma)x} \leq 0$: Recall that $\sigma > 0$, then

$$1 - 2\alpha e^{-\sigma x} \ \leq 0 \ \Leftrightarrow \ x \leq \frac{\log(2\alpha)}{\sigma}\,.$$

For $x > \frac{\log(2\alpha)}{\sigma}$, a direct calculation shows that

$$\begin{aligned} A(t,x) \ &= \ \left[2\alpha(\sigma+\lambda)e^{-2\sigma x} - (\sigma+2\lambda)e^{-\sigma x}\right] \cdot \int_{-\infty}^{\frac{x-\lambda(T-t)}{\sqrt{(T-t)}}} \frac{1}{\sqrt{2\pi}} e^{\frac{-s^2}{2}} \mathrm{d}s \\ &+ \ \left[\sigma e^{(2\lambda-\sigma)x} - 2\alpha\sigma e^{2(\lambda-\sigma)x}\right] \cdot \int_{-\infty}^{\frac{-x-\lambda(T-t)}{\sqrt{(T-t)}}} \frac{1}{\sqrt{2\pi}} e^{\frac{-s^2}{2}} \mathrm{d}s\,. \end{aligned}$$

Applying the change of variable $s' = s + \frac{x}{\sqrt{T-t}}$ in the first integral and $s' = s - \frac{x}{\sqrt{T-t}}$ in the second we obtain

$$\begin{aligned} A(t,x) \ &= \ \left[2\alpha(\sigma+\lambda)e^{-2\sigma x} - (\sigma+2\lambda)e^{-\sigma x}\right] \cdot \int_{-\infty}^{-\lambda\sqrt{(T-t)}} \frac{1}{\sqrt{2\pi}} e^{\frac{-(s+\frac{x}{\sqrt{T-t}})^2}{2}} \mathrm{d}s \\ &+ \ \left[\sigma e^{(2\lambda-\sigma)x} - 2\alpha\sigma e^{2(\lambda-\sigma)x}\right] \cdot \int_{-\infty}^{-\lambda\sqrt{(T-t)}} \frac{1}{\sqrt{2\pi}} e^{\frac{-(s-\frac{x}{\sqrt{T-t}})^2}{2}} \mathrm{d}s\,. \end{aligned}$$

Hence, $A(t,x) \leq 0$ would follow from

$$2\alpha(\sigma+\lambda)e^{-\sigma x} - \sigma - 2\lambda \ \leq \ \left[2\alpha\sigma e^{-\sigma x} - \sigma\right] e^{\frac{2xs}{\sqrt{T-t}}+2\lambda x}$$

for all $s \in (-\infty, -\lambda\sqrt{T-t})$. The term within brackets on the right hand side is negative since $x > \frac{\log(2\alpha)}{\sigma}$; in addition, since $e^{\frac{2xs}{\sqrt{T-t}}+2\lambda x}$ is increasing on $s$ we set $s = -\lambda\sqrt{T-t}$, so that $A(t,x) \leq 0$ follows from

$$2\alpha(\sigma+\lambda)e^{-\sigma x} - \sigma - 2\lambda \ \leq \ 2\alpha\sigma e^{-\sigma x} - \sigma\,,$$

which holds for $\alpha \leq \frac{3}{4}$.

In order to prove that $B(t,x) < 0$ we show that

$$2\alpha(\lambda - 2\sigma)e^{-2\sigma(\lambda-\sigma)(T-t)} < 2(\lambda - \sigma)e^{-\frac{\sigma}{2}(2\lambda-\sigma)(T-t)}$$

for different values of $\lambda$. Recall that $\lambda \geq \sigma$, then

- if $\lambda \leq 2\sigma$ the inequality is obvious, since the term on the left hand side is strictly negative while the term in the right and side is strictly positive;
- in case $\lambda > 2\sigma$, both expressions are positive and the result follows from

  $$2\alpha(\lambda - 2\sigma) < 2(\lambda - 2\sigma) < 2(\lambda - \sigma) \ , \quad \text{since } \alpha \leq \frac{3}{4},$$

  and

  $$-2\sigma(\lambda - \sigma)(T - t) \leq -\frac{\sigma}{2}(2\lambda - \sigma)(T - t) \ \Leftrightarrow \ 3\sigma \ \leq \ 2\lambda \ ,$$

therefore completing the proof. $\square$

**Lemma 3.4.2.** *If $\alpha \leq \frac{2}{3}$ and $\mu \leq -\frac{\sigma^2}{2}$, then $H$ is strictly positive for all $(t,x) \in [0,T] \times \mathbb{R}_+$.*

***Proof.*** Note that $\mu \leq -\frac{\sigma^2}{2}$ is equivalent to $\lambda \leq -\sigma$. We make use of function $\hat{H}$ in (3.4.5) and rewrite it as

$$\hat{H}(t,x) = A(t,x) + B(t,x) \ ,$$

with

$$\begin{aligned} A(t,x) &= I(x) \cdot \Big[\Phi\Big(\frac{x - \lambda(T-t)}{\sqrt{(T-t)}}\Big) - \Phi\Big(\frac{-x - \lambda(T-t)}{\sqrt{(T-t)}}\Big)\Big] \\ &+ \Big[I(x) + II(x)\Big] \cdot \Phi\Big(\frac{-x - \lambda(T-t)}{\sqrt{(T-t)}}\Big) \ , \end{aligned}$$

and

$$\begin{aligned} B(t,x) &= 2\alpha(\lambda - 2\sigma)e^{-2\sigma(\lambda-\sigma)(T-t)} \ \cdot \ \Phi\Big(\frac{-x + (\lambda - 2\sigma)(T-t)}{\sqrt{(T-t)}}\Big) \\ &- 2(\lambda - \sigma)e^{-\frac{\sigma}{2}(2\lambda-\sigma)(T-t)} \ \cdot \ \Phi\Big(\frac{-x + (\lambda - \sigma)(T-t)}{\sqrt{(T-t)}}\Big) \ . \end{aligned}$$

Functions $I(x)$ and $II(x)$ are given by

$$\begin{aligned} I(x) &= 2\alpha(\sigma+\lambda)e^{-2\sigma x} - (\sigma+2\lambda)e^{-\sigma x} , \\ II(x) &= \sigma e^{(2\lambda-\sigma)x} - 2\alpha\sigma e^{2(\lambda-\sigma)} . \end{aligned}$$

The strict positivity of $H$ would follow from the strict positivity of $\hat{H}$. We show that for $\lambda \leq -\sigma$ and $\alpha \leq \frac{2}{3}$, function $A$ is strictly positive while function $B$ is greater than or equal to 0 for all $(t,x) \in [0,T] \times \mathbb{R}_+$. In order to show that $A(t,x) > 0$, we separately show that both $I$ and $I + II$ are strictly positive, since the remainder terms in the equation are clearly positive.

- $I(x) > 0$ is equivalent to

  $$\alpha < \frac{e^{\sigma x}(\sigma+2\lambda)}{2(\sigma+\lambda)} ,$$

  since $\lambda \leq -\sigma$. We note that the right hand side is increasing on $\lambda$, since

  $$\frac{d}{\mathrm{d}\lambda}\frac{e^{\sigma x}(\sigma+2\lambda)}{2(\sigma+\lambda)} = \frac{\sigma e^{\sigma x}}{2(\sigma+\lambda)^2} > 0 .$$

  Therefore

  $$\frac{e^{\sigma x}(\sigma+2\lambda)}{2(\sigma+\lambda)} \geq \lim_{\lambda\to-\infty}\frac{e^{\sigma x}(\sigma+2\lambda)}{2(\sigma+\lambda)} = e^{\sigma x} \geq 1 > \alpha .$$

- $I(x) + II(x) > 0$ is, on the other hand, equivalent to

  $$\alpha < \frac{1}{2}\Big(\frac{e^{\sigma x}[(\sigma+2\lambda) - \sigma e^{2\lambda x}]}{\sigma+\lambda-\sigma e^{2\lambda x}}\Big) ,$$

  since $\lambda \leq -\sigma$. We note that the right hand side is increasing on $x$, since

  $$\begin{aligned} \frac{d}{\mathrm{d}x}\frac{1}{2}\Big(\frac{e^{\sigma x}[(\sigma+2\lambda) - \sigma e^{2\lambda x}]}{\sigma+\lambda-\sigma e^{2\lambda x}}\Big) &= \frac{[\sigma+2\lambda-\sigma e^{2\lambda x}]\cdot[\sigma+\lambda-\sigma e^{2\lambda x}]}{(\sigma+\lambda-\sigma e^{2\lambda x})^2} \\ &\quad + \frac{2\lambda^2 e^{2\lambda x}}{(\sigma+\lambda-\sigma e^{2\lambda x})^2} > 0 . \end{aligned}$$

  Therefore

  $$\frac{1}{2}\Big(\frac{e^{\sigma x}[(\sigma+2\lambda) - \sigma e^{2\lambda x}]}{\sigma+\lambda-\sigma e^{2\lambda x}}\Big) \geq \frac{1}{2}\Big(\frac{e^{\sigma x}[(\sigma+2\lambda) - \sigma e^{2\lambda x}]}{\sigma+\lambda-\sigma e^{2\lambda x}}\Big)\Big|_{x=0} = 1 > \alpha .$$

In order to show that $B(t,x) \geq 0$ we use a direct calculation. We note that the inequality would follow from

$$\begin{aligned} & 2\alpha(\lambda-2\sigma)e^{-2\sigma(\lambda-\sigma)(T-t)} \cdot \int_{-\infty}^{\frac{-x+(\lambda-2\sigma)(T-t)}{\sqrt{(T-t)}}} \frac{1}{\sqrt{2\pi}}e^{\frac{-s^2}{2}}\mathrm{d}s \\ \geq\ & 2(\lambda-\sigma)e^{-\frac{\sigma}{2}(2\lambda-\sigma)(T-t)} \cdot \int_{-\infty}^{\frac{-x+(\lambda-\sigma)(T-t)}{\sqrt{(T-t)}}} \frac{1}{\sqrt{2\pi}}e^{\frac{-s^2}{2}}\mathrm{d}s . \end{aligned}$$

The change of variable $s' = s - \sigma\sqrt{T-t}$ on the right hand side integral leads to

$$
\begin{aligned}
&2\alpha(\lambda - 2\sigma)e^{-2\sigma(\lambda-\sigma)(T-t)} \cdot \int_{-\infty}^{\frac{-x+(\lambda-2\sigma)(T-t)}{\sqrt{(T-t)}}} \frac{1}{\sqrt{2\pi}} e^{\frac{-s^2}{2}} \mathrm{d}s \\
&\geq\ 2(\lambda - \sigma)e^{-\frac{\sigma}{2}(2\lambda-\sigma)(T-t)} \cdot \int_{-\infty}^{\frac{-x+(\lambda-2\sigma)(T-t)}{\sqrt{(T-t)}}} \frac{1}{\sqrt{2\pi}} e^{\frac{-s^2}{2}} e^{-\frac{\sigma^2(T-t)}{2}} e^{-s\sigma\sqrt{T-t}} \mathrm{d}s\ .
\end{aligned}
$$

Thus, the non-negativity of $B$ would follow from

$$
\alpha(\lambda - 2\sigma)e^{-2\sigma(\lambda-\sigma)(T-t)} \geq (\lambda - \sigma)e^{-\frac{\sigma}{2}(2\lambda-\sigma)(T-t)} e^{-\frac{\sigma^2(T-t)}{2}} e^{-s\sigma\sqrt{T-t}}\ ,
$$

for all $s \in (-\infty, \frac{-x+(\lambda-2\sigma)(T-t)}{\sqrt{(T-t)}})$. Since $\lambda \leq -\sigma$, the right hand side of the above expression is always negative and increasing on $s$; substituting $s$ for $\frac{-x+(\lambda-2\sigma)(T-t)}{\sqrt{(T-t)}}$, $B(t,x) \geq 0$ would follow from

$$
\alpha(\lambda - 2\sigma) \geq (\lambda - \sigma)e^{\sigma x}\ . \tag{3.4.6}
$$

We note that the term in the right hand side is always negative and decreasing on $x \geq 0$; so that

$$
(\lambda - \sigma) \geq (\lambda - \sigma)e^{\sigma x}\ .
$$

Thus, to show that (3.4.6) holds, we require to prove that $\alpha(\lambda - 2\sigma) \geq \lambda - \sigma$, or equivalently

$$
\alpha \leq \frac{\lambda - \sigma}{\lambda - 2\sigma}\ . \tag{3.4.7}
$$

Finally, since

$$
\frac{d}{\mathrm{d}\lambda} \frac{\lambda - \sigma}{(\lambda - 2\sigma)} = -\frac{\sigma}{(\lambda - \sigma)^2} < 0\,,
$$

the right hand side of (3.4.7) is decreasing on $\lambda$ and therefore

$$
\frac{\lambda - \sigma}{\lambda - 2\sigma} \geq \frac{\lambda - \sigma}{\lambda - 2\sigma}\Big|_{\lambda=-\sigma} = \frac{2}{3}\ .
$$

The inequality $B(t,x) \geq 0$ follows from the assumption $\alpha \leq \frac{2}{3}$. □

### 3.4.2 Partial Proof of Theorem 3.3.1

We now use results in Lemmas 3.4.1 and 3.4.2 in order to establish different parts of Theorem 3.3.1.

***Proof of case*** $\mu \geq \frac{3\sigma^2}{2}$ ***in*** (3.3.2) ***and*** (3.3.3). We show that in this case the optimal stopping set $D$ is given $\{T\} \times \mathbb{R}_+$. Recalling expression for $V$ in (3.4.3) and noting the strict negativity result for $H$ in Lemma 3.4.1, we conclude that for all $(t,x) \in [0,T) \times \mathbb{R}_+$ an infimum is attained at time $\tau_D(t,x) = T - t$. Therefore, it holds $V(t,x) < G(t,x)$ for all $(t,x) \in [0,T) \times \mathbb{R}_+$ and $(t,x) \in C$. For $(t,x) \in \{T\} \times \mathbb{R}_+$, we get $V(t,x) = G(t,x)$ and $(t,x) \in D$. □

***Proof of case*** $\mu \leq -\frac{\sigma^2}{2}$ ***in*** (3.3.1) ***and*** (3.3.2). We show that in this case the optimal stopping set $D$ is given $[0,T] \times \mathbb{R}_+$. Recovering expression for $V$ in (3.4.3), and noting the strict result for $H$ in Lemma (3.4.2), we conclude that for all $(t,x) \in [0,T) \times \mathbb{R}_+$ it holds $\tau_D(t,x) = t$ and $(t,x) \in D$, since otherwise $V(t,x) > G(t,x)$, which is a contradiction. □

## 3.5 STOPPING TIMES FOR $\alpha \leq \frac{1}{2}$ AND $\mu \geq \frac{\sigma^2}{2}$

We now make use of the moment generating function of $X^x$, allowing for similar probabilistic techniques to that in [54] (Section 5) to be applied, and setting the remaining results in Theorem 3.3.1. Lemma A.2.1 in Appendix A shows that the moment-generating function of $X^x$ is given by

$$\begin{aligned} M_{X_t^x}(s) \;=\; \mathbb{E}[e^{sX_t^x}] \;&=\; e^{s(x+\sigma t(\frac{s}{2}\sigma-\lambda))} \cdot \Phi\Big(\frac{x-\sigma t(\lambda-\sigma s)}{\sigma\sqrt{t}}\Big) \\ &+\; \frac{\sigma s}{\sigma s-2\lambda} e^{(\frac{2x\lambda}{\sigma}+\frac{\sigma^2 s^2 t}{2}-(\lambda\sigma t+x)s)} \cdot \Phi\Big(-\frac{x+\sigma t(\lambda-\sigma s)}{\sigma\sqrt{t}}\Big) \\ &-\; \frac{2\lambda}{\sigma s-2\lambda} \cdot \Phi\Big(-\frac{x-\lambda\sigma t}{\sigma\sqrt{t}}\Big) \end{aligned} \tag{3.5.1}$$

for all $t \in [0,T]$ and $s \in (-\infty,0) \cup (0,\frac{2\lambda}{\sigma})$. In addition, $M_{X_t^x}(0) = 1$.

In order to prove case $\mu \geq \frac{\sigma^2}{2}$ in equation (3.3.1), without loss of generality we fix $\sigma = 1$ and show that the result holds whenever $\mu \geq \frac{1}{2}$. This is allowed by the scaling property of Brownian motion, since a time change $B_{\sigma^2 t}$ with terminal time $\frac{T}{\sigma^2}$ and $\sigma > 0$ recovers the original stopping problem. We continue introducing some preliminary results.

**Lemma 3.5.1.** *If $\mu = \frac{1}{2}$ and $\alpha \leq \frac{1}{2}$, the following inequalities hold*

$$\begin{aligned} \mathbb{E}[G(T, X_T^x)] &= G(0,x) \quad \textit{for } x = 0\,, \textit{ and} &(3.5.2)\\ \mathbb{E}[G(T, X_T^x)] &< G(0,x) \quad \textit{for } x > 0\,. &(3.5.3)\end{aligned}$$

***Proof.*** Note that $\mu = \frac{1}{2}$ is equivalent to $\lambda = 0$. From expression for $G$ in (3.1.4) we obtain

$$\mathbb{E}[G(T, X_T^x)] = \mathbb{E}[(1 - \alpha e^{-(X_T^x)})^2] = 1 - 2\alpha M_{X_T^x}(-1) + \alpha^2 M_{X_T^x}(-2)\,,$$

with $M_{X_t^x}(s)$ as in (3.5.1). Then, noting expression for $G$ in (3.2.5), for $\lambda = 0$ we get

$$\begin{aligned} G(0,x) - \mathbb{E}[G(T, X_T^x)] &= (\alpha^2 e^{-2x} - 2\alpha e^{-x})\Phi\Big(\frac{x}{\sqrt{T}}\Big) + (2\alpha e^{-x} - \alpha^2 e^{-2x})\Phi\Big(-\frac{x}{\sqrt{T}}\Big)\\ &+ 2\alpha e^{\frac{T}{2}-x}\Phi\Big(\frac{x-T}{\sqrt{T}}\Big) - \alpha^2 e^{-2(x-T)}\Phi\Big(\frac{x-2T}{\sqrt{T}}\Big)\\ &+ (2\alpha e^{x+\frac{T}{2}} - 4\alpha e^{\frac{T}{2}})\Phi\Big(-\frac{x+T}{\sqrt{T}}\Big)\\ &+ (2\alpha^2 e^{2T} - \alpha^2 e^{2x+2T})\Phi\Big(-\frac{x+2T}{\sqrt{T}}\Big)\,. \end{aligned} \tag{3.5.4}$$

Substituting $x = 0$ in (3.5.4) above cancels all terms out and yields result (3.5.2).

In order to prove the second result (3.5.3) we first define the auxiliary function

$$\rho(x) = \frac{e^{2x}}{\alpha}\big(G(0,x) - \mathbb{E}[G(T, X_T^x)]\big)\,.$$

We will show that $\rho(x) > 0$ for all $x > 0$, which is equivalent to (3.5.3). We have seen that $\rho(0) = 0$ and the proof would follow from $\frac{\mathrm{d}\rho(x)}{\mathrm{d}x} > 0$ for all $x > 0$. We note that

$$\begin{aligned} \frac{\mathrm{d}\rho(x)}{\mathrm{d}x} &= 2e^x\Big[\Phi\Big(-\frac{x}{\sqrt{T}}\Big) - \Phi\Big(\frac{x}{\sqrt{T}}\Big)\Big] + 2e^{\frac{T}{2}+x}\Phi\Big(\frac{x-T}{\sqrt{T}}\Big)\\ &+ \big[6e^{3x+\frac{T}{2}} - 8e^{2x+\frac{T}{2}}\big]\Phi\Big(-\frac{x+T}{\sqrt{T}}\Big) + 4\alpha e^{2x+2T}(1-e^{2x})\Phi\Big(-\frac{x+2T}{\sqrt{T}}\Big)\,, \end{aligned}$$

so that the result would follow from

$$\begin{aligned} 0 &< \Phi\Big(-\frac{x}{\sqrt{T}}\Big) - \Phi\Big(\frac{x}{\sqrt{T}}\Big) + e^{\frac{T}{2}}\Phi\Big(\frac{x-T}{\sqrt{T}}\Big)\\ &+ \big[3e^{2x} - 4e^x\big]e^{\frac{T}{2}}\Phi\Big(-\frac{x+T}{\sqrt{T}}\Big) + 2\alpha e^{x+2T}(1-e^{2x})\Phi\Big(-\frac{x+2T}{\sqrt{T}}\Big)\,. \end{aligned}$$

It is shown in [54] (Section 5) that

$$\Phi\Big(-\frac{x}{\sqrt{T}}\Big) - \Phi\Big(\frac{x}{\sqrt{T}}\Big) + e^{\frac{T}{2}}\Phi\Big(\frac{x-T}{\sqrt{T}}\Big) + \big[e^{2x} - 2e^x\big]e^{\frac{T}{2}}\Phi\Big(-\frac{x+T}{\sqrt{T}}\Big)$$

is strictly positive for all $x > 0$. Thus, the problem can be simplified to showing that

$$0 \leq 2\big[e^{2x} - e^{x}\big]e^{\frac{T}{2}}\Phi\Big(-\frac{x+T}{\sqrt{T}}\Big) + \ 2\alpha e^{x+2T}(1-e^{2x})\Phi\Big(-\frac{x+2T}{\sqrt{T}}\Big)$$

for all $x > 0$, i.e

$$0 \leq \int_{-\infty}^{-\frac{x+T}{\sqrt{T}}} \big[e^{x}-1\big]e^{\frac{T}{2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\mathrm{d}y + \alpha\int_{-\infty}^{-\frac{x+2T}{\sqrt{T}}} \big[1-e^{2x}\big]e^{\frac{T}{2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\mathrm{d}y \tag{3.5.5}$$

for all $x > 0$. The change of variable $y' = y - \sqrt{T}$ in the first integral implies that (3.5.5) follows from

$$0 \leq (e^{x}-1)e^{-y\sqrt{T}} + \alpha e^{2T}(1-e^{2x})$$

for all $y \in (-\infty, -\frac{x+2T}{\sqrt{T}})$. This is a decreasing function on $y$. We therefore show that

$$(e^{x}-1)e^{x+2T} + \alpha e^{2T}(1-e^{2x}) \geq 0 \Leftrightarrow \alpha \leq \frac{(e^{x}-1)e^{x}}{e^{2x}-1}\,.$$

This holds true for $\alpha \leq \frac{1}{2}$, since

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{(e^{x}-1)e^{x}}{e^{2x}-1} = \frac{e^{x}+e^{-x}-2}{(e^{x}-e^{-x})^2} \geq \frac{2(\cosh(x)-1)}{(e^{x}-e^{-x})^2} \geq 0\,,$$

and, using L'Hôpital's rule

$$\frac{(e^{x}-1)e^{x}}{e^{2x}-1} \geq \lim_{x\to 0}\frac{(e^{x}-1)e^{x}}{e^{2x}-1} = \frac{1}{2}\,.$$

□

**Lemma 3.5.2.** *If $\mu > \frac{1}{2}$ and $\alpha \leq \frac{1}{2}$, the following inequality holds*

$$\mathbb{E}[G(T, X_T^x)] \quad < \quad G(0,x) \quad \textit{for } x \geq 0\,. \tag{3.5.6}$$

***Proof.*** We note that $\mu > \frac{1}{2}$ is equivalent to $\lambda > 0$ and recall from the proof of Lemma 3.1.1 that

$$G(t,x) = \mathbb{E}[(1-\alpha e^{-(x\vee S^{\lambda}_{T-t})})^2|\mathcal{F}_t]\,,$$

for all $(t,x) \in [0,T]\times\mathbb{R}_+$. Thus, noting (3.1.4) we have

$$\begin{aligned} G(0,x) - \mathbb{E}[G(T,X_T^x)] &= \mathbb{E}\big[(1-\alpha e^{-(x\vee S_T^{\lambda})})^2 - (1-\alpha e^{-(x\vee S_T^{\lambda}-B_T^{\lambda})})^2\big] \\ &= \alpha\mathbb{E}\big[e^{-(x\vee S_T^{\lambda})}(\alpha e^{-(x\vee S_T^{\lambda})}[1-e^{2B_T^{\lambda}}] + 2[e^{B_T^{\lambda}}-1])\big]\,. \end{aligned}$$

We make use of Girsanov's theorem and define a new probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, with Radon-Nikodym derivative

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = e^{\frac{1}{2}\lambda^2 t - \lambda B_t^\lambda} \, .$$

We observe that under $\mathbb{Q}$, process $(B_t^\lambda)_{0\le t\le T}$ is a standard Brownian motion, since

$$\mathrm{d}\mathbb{Q}(B_t^\lambda \le x) = e^{\frac{1}{2}\lambda^2 t - \lambda x}\mathrm{d}\mathbb{P}(B_t^\lambda \le x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \, ,$$

for all $t \in [0, T]$.

The use of $\mathbb{Q}$ allows for a simplification of the problem with respect to $\lambda$, so that

$$G(0,x) - \mathbb{E}[G(T, X_T^x)] = \alpha \mathbb{E}_{\mathbb{Q}}\left[e^{-(x\vee S_T^\lambda)}(\alpha e^{-(x\vee S_T^\lambda)}[1 - e^{2B_T^\lambda}] + 2[e^{B_T^\lambda} - 1])e^{-\frac{1}{2}\lambda^2 T + \lambda B_T^\lambda}\right] \, .$$

Equivalently,

$$G(0,x) - \mathbb{E}[G(T, X_T^x)] = \alpha \mathbb{E}\left[e^{-(x\vee S_T)}(\alpha e^{-(x\vee S_T)}[1 - e^{2B_T}] + 2[e^{B_T} - 1])e^{-\frac{1}{2}\lambda^2 T + \lambda B_T}\right] \, ,$$

where the expectation is taken with respect to the original measure $\mathbb{P}$.

Now, we define the auxiliary function

$$\rho(\lambda) = \frac{e^{\frac{1}{2}\lambda^2 T}}{\alpha}\left(G(0,x) - \mathbb{E}[G(T, X_T^x)]\right) \, ,$$

and show that it is strictly increasing on $\lambda$, so that result (3.5.6) follows from Lemma 3.5.1. Note that

$$\frac{\mathrm{d}\rho(\lambda)}{\mathrm{d}\lambda} = \mathbb{E}\left[e^{-(x\vee S_T)}e^{\lambda B_T}B_T(\alpha e^{-(x\vee S_T)}[1 - e^{2B_T}] + 2[e^{B_T} - 1])\right] \, .$$

The equation within the expectation is always equal or greater than zero, and so its expected value is positive. This can be seen as follows.

- If $B_T = 0$, the case is trivial.
- If $B_T > 0$, we note that $x \vee S_T \ge B_T$ and prove that

  $$\alpha e^{-(x\vee S_T)}[e^{2B_T} - 1] \le \alpha e^{-B_T}[e^{2B_T} - 1] < 2[e^{B_T} - 1] \, ,$$

  or equivalently

  $$\alpha < \frac{2e^{2B_T} - 2e^{B_T}}{e^{2B_T} - 1} \, .$$

This result therefore follows from

$$\alpha < \frac{2y^2 - 2y}{y^2 - 1} \quad \text{for all} \quad y > 1 \, ,$$

which is true whenever $\alpha \leq \frac{1}{2}$, since

$$\frac{\mathrm{d}}{\mathrm{d}y} \frac{2y^2 - 2y}{y^2 - 1} = \frac{2(y-1)^2}{(y^2-1)^2} \geq 0 \, ,$$

and the function is increasing on $y$, so that and application of L'Hôpital's rule yields

$$\frac{2y^2 - 2y}{y^2 - 1} \geq \lim_{y \to 1} \frac{2y^2 - 2y}{y^2 - 1} = 1 > \alpha \, .$$

- If $B_T < 0$, we show that

$$\alpha e^{-(x \vee S_T)} [1 - e^{2B_T}] \leq \alpha [1 - e^{2B_T}] < 2[1 - e^{B_T}] \, ,$$

or equivalently

$$\alpha < \frac{2[1 - e^{B_T}]}{1 - e^{2B_T}} \, .$$

The result therefore follows from

$$\alpha < \frac{2[1-y]}{1-y^2} \quad \text{for all} \quad y \in (0,1) \, ,$$

which is true whenever $\alpha \leq \frac{1}{2}$, since

$$\frac{\mathrm{d}}{\mathrm{d}y} \frac{2[1-y]}{1-y^2} = -\frac{2(y-1)^2}{(y^2-1)^2} \leq 0 \, ,$$

and the function is decreasing on $y$. Hence, the application of L'Hôpital's rule yields

$$\frac{2[1-y]}{1-y^2} \geq \lim_{y \to 1} \frac{2[1-y]}{1-y^2} = 1 > \alpha \, .$$

□

### 3.5.1 Remainder Proof of Theorem 3.3.1

We make use of results in Lemmas 3.5.1 and 3.5.2 to establish the remaining part of Theorem 3.3.1 using a similar argument to that in [54] (Section 5).

***Proof of case*** $\mu \geq \frac{\sigma^2}{2}$ ***in*** (3.3.1). We note that $T$ and $x$ are arbitrary points and recall that $\alpha \leq \frac{1}{2}$. Then, from (4.4.12) in Lemma 3.5.2 we observe that for $\mu > \frac{\sigma^2}{2}$

$$\mathbb{E}_{t,x}[G(T, X_T)] < G(t,x) \quad \text{for all } (t,x) \in [0,T) \times \mathbb{R}_+ \, ,$$

where the expectation is taken under a measure for which $\mathbb{P}(X_t = x) = 1$. It follows that for all $(t,x) \in [0,T) \times \mathbb{R}_+$

$$V(t,x) = \mathbb{E}_{t,x}[G(t + \tau_D(t,x), X_{t+\tau_D(t,x)})] \leq \mathbb{E}_{t,x}[G(T, X_T)] < G(t,x) \, ,$$

and $(t,x) \in C$ by the definition of the continuation set in (4.2.5).

If $\mu = \frac{\sigma^2}{2}$, inequality $V(t,x) < G(t,x)$ cannot be attained for $x = 0$. However, both stopping at deadline and immediate stopping are optimal, supporting result (3.3.1). To see that, note from Lemma 3.5.1 and the arbitrary nature of $T$ and $x$ that

$$\mathbb{E}_{t,x}[G(T, X_T)] \leq G(t,x) \quad \text{for all} \quad (t,x) \in [0,T) \times [0,\infty) \, .$$

Therefore, for all $x \in \mathbb{R}_+$ and stopping times $\tau \in \mathcal{T}$

$$\mathbb{E}[G(T, X_T^x) | \mathcal{F}_\tau] \leq \mathbb{E}[G(\tau, X_\tau^x)] \, ,$$

and

$$V(0,x) = \mathbb{E}[G(\tau_D(0,x), X^x_{\tau_D(0,x)})] = E[G(T, X_T^x)] \, ,$$

since $V$ is a minimization problem. This shows that both 0 and $T$ are optimal stopping strategies and settles result (3.3.1). □

## 3.6 Discussion

This chapter has analysed finite horizon stopping times that aim to minimize a choice of a non-linear utility function; we recall that this function accounts for the ratio between a running geometric Brownian motion and its absolute maximum over the total time interval. It has been shown that Bang-Bang stopping strategies are optimal under certain conditions, yielding similar results to those in [21, 25, 54, 60] and reinforcing the idea of only ever stopping *bad* processes; in our case those for which the condition $\mu \leq -\frac{\sigma^2}{2}$ is satisfied. However, complete characterizations of $D$ have not been established in all cases, a problem that responds to the complexity of function $H$ derived from the choice of non-linear utility function in $V$; in Section 3.6.2 we discuss the complications that a reduction to a free boundary problem similar to that in [21] encounters in this case.

### 3.6.1 Characterization of the Value Function

The following corollary follows from Theorem 3.3.1 and Lemma A.2.1 in Appendix A; it partially specifies the value function of the optimal stopping problem (3.2.3).

**Corollary 3.6.1.** *Value function $V$ in extended problem* (3.2.3) *is, for all $(t,x) \in [0,T] \times \mathbb{R}_+$, partially characterized by the following expressions. If* $0 < \alpha \leq \frac{1}{2}$

$$V(t,x) = \begin{cases} G(t,x) & \text{when } \mu \leq -\frac{\sigma^2}{2}\,, \\ 1 - 2\alpha M_{X^x_{T-t}}(-1) + \alpha^2 M_{X^x_{T-t}}(-2) & \text{when } \mu \geq \frac{\sigma^2}{2}\,; \end{cases}$$

*if* $\frac{1}{2} < \alpha \leq \frac{2}{3}$

$$V(t,x) = \begin{cases} G(t,x) & \text{when } \mu \leq -\frac{\sigma^2}{2}\,, \\ 1 - 2\alpha M_{X^x_{T-t}}(-1) + \alpha^2 M_{X^x_{T-t}}(-2) & \text{when } \mu \geq \frac{3\sigma^2}{2}\,; \end{cases}$$

*and if* $\frac{2}{3} < \alpha \leq \frac{3}{4}$

$$V(t,x) = 1 - 2\alpha M_{X^x_{T-t}}(-1) + \alpha^2 M_{X^x_{T-t}}(-2) \quad \text{when } \mu \geq \frac{3\sigma^2}{2}\,,$$

*where $M_{X^x_t}$ denotes the moment generating function of $X^x_t$ given in* (3.5.1).

Figure 3.3 below illustrates the gain and value functions of a stopping problem of the form (3.2.3) whose optimal strategy is to always stop at terminal time $T$. It is observed that $V < G$ for all $(t,x) \in [0,T) \times \mathbb{R}_+$ and $V = G$ for $(t,x) \in \{T\} \times \mathbb{R}_+$. We recall from the theory of optimal stopping in Chapter 2 that $V$ is the biggest subharmonic function dominated by $G$.

### 3.6.2 Reduction to a Free Boundary Problem

As explained in the introductory Chapter 2, a common technique for characterizing both the stopping set and value function is to pose a free boundary problem for $V$

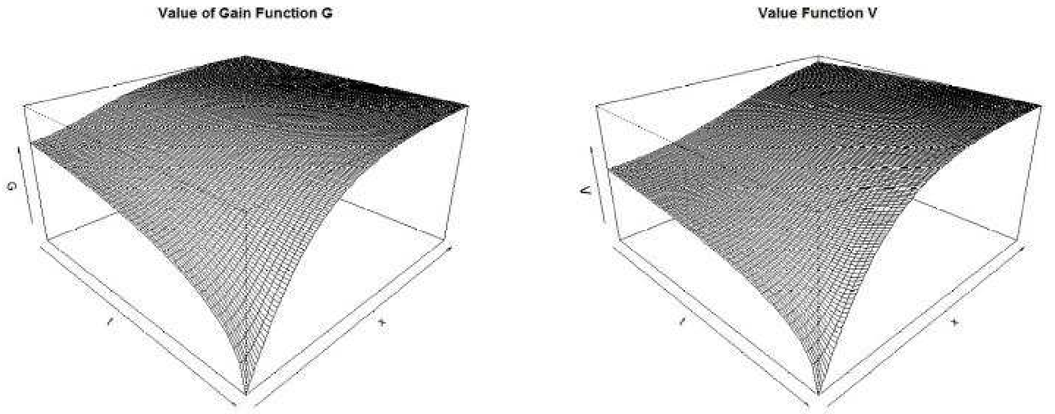


**Figure 3.3:** Gain and value functions of stopping problem (3.2.3). Here, $T = 5$, $\alpha = 0.45$, $\lambda = 0.2$ and $\sigma = 1$. It is observed that $V < G$ for all $(t, x) \in [0, T) \times \mathbb{R}_+$ and $V = G$ for $(t, x) \in \{T\} \times \mathbb{R}_+$.

to solve; this is particularly helpful when stopping times are expected to respond to departures of a diffusion of interest from a certain threshold value, and lead to the concept of *stopping boundaries*. These boundaries do by definition define the boundaries between sets $D$ and $C$. In view of (2.2.1)-(2.2.3) we note that the stopping problem must solve

$$\begin{aligned} V_t + \mathcal{A}_X V(x) &\geq 0 \quad \text{for all} \quad (t, x) \in [0, T] \times \mathbb{R}_+ \, , \\ V(x) &< G(x) \quad \text{for all} \quad (t, x) \in C \, , \\ V(x) &= G(x) \quad \text{for all} \quad (t, x) \in D \, , \end{aligned} \tag{3.6.1}$$

with $\mathcal{A}_X$ given by (3.4.1). The inclusion of the time differential of $V$ in equation (3.6.1) corresponds to the finite-horizon nature of the problem and we refer the reader to [49] for details on this matter.

Drawbacks on using such an approach in this case are revealed from a direct analysis of function $H$ in (3.4.4). For instance, when $\alpha$ is close to 1, direct examination of $H$ reveals the existence of two non-monotone functions $h_1(t)$ and $h_2(t)$ such that

$$\begin{aligned} P &= \{H > 0\} = \{(t, x) \in [0, T] \times \mathbb{R}_+ \, : \, h_2(t) < x < h_1(t)\} \, , \quad \text{and} \\ N &= \{H < 0\} = \{(t, x) \in [0, T] \times \mathbb{R}_+ \, : \, (x, t) \notin P\} \, , \end{aligned}$$

whenever $\lambda \in (-\frac{1}{2}, \lambda_*)$, as seen in the left hand side of Figure 3.4, or monotone functions $h_1(t)$, $h_2(t)$, $h_3(t)$ and $h_4(t)$, along with time points $u_1$ and $u_2$, so that

$P = P_1 \cup P_2$ and

$$P_1 = \{(t, x) \in [0, u_1] \times \mathbb{R}_+ \; : \; h_2(t) < x < h_1(t)\} \; ,$$
$$P_2 = \{(t, x) \in [u_2, T] \times \mathbb{R}_+ \; : \; h_3(t) < x < h_4(t)\} \; ,$$
$$N = \{(t, x) \in [0, T] \times \mathbb{R}_+ \; : \; (x, t) \notin P\} \; ,$$

whenever $\lambda \in (\lambda_*, \frac{1}{2})$, for some $\lambda_* \in (-\frac{1}{2}, \frac{1}{2})$, as seen in the right hand side of Figure 3.4. In this case, while arguments similar to those in Subsection 3.4.2 imply that $N \subset C$,

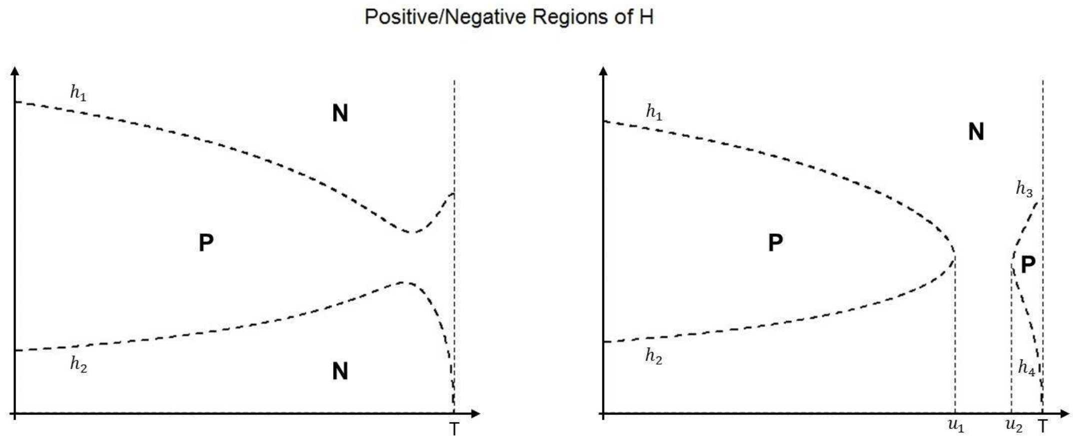


**Figure 3.4:** Positive and negative regions of function $H$, for different values of $(x, t) \in [0, T] \times \mathbb{R}_+$ and $\alpha = 1$. Left figure: $\lambda = 0.2$, $\sigma = 1$ and $T = 5$. Right figure: $\lambda = 0.22$, $\sigma = 1$, $T = 5$.

we only know that $P$ may contain a stopping set. However, the complexity of function $H$ along with the non-monotone behaviour of the mapping $t \mapsto H(t, x)$, precludes establishing the existence of such a stopping set in these cases, as well as characterising the stopping boundaries and providing results on their regularity conditions; therefore preventing the set up of a free boundary problem without resorting to purely analytic methods outside of the scope of this thesis. In addition, establishing strict inequalities for $H$ with respect to 0 results an intractable problem, so that work in this chapter yielding optimal Bang-Bang strategies is not applicable whenever $\alpha$ is close to 1.

Similarly, the analysis of the case $\mu \in (-\frac{\sigma}{2}, \frac{\sigma}{2})$ when $\alpha \leq \frac{1}{2}$ in Theorem 3.3.1, reveals that $H$ is such that there exists a continuous and decreasing function $h$, with $h(T) = 0$, so that

$$P = \{H > 0\} = \{(t, x) \in [0, T] \times \mathbb{R}_+ : x < h(t)\}, \text{ and}$$
$$N = \{H < 0\} = \{(t, x) \in [0, T] \times \mathbb{R}_+ : x > h(t)\},$$

leading to the complications discussed above and leaving the optimal strategy for this interval unsolved.

### 3.6.3 Conclusion and Future Work

Results in this section show consistency on stopping strategies to be adopted in comparison with previous work summarized in the introduction in Chapter 1, this is with independence of the choice of utility function made. However, results are unsatisfactory due to the high limitations in terms of cases covered.

Hence, tackling natural extensions of stopping problems (1.1.2), allowing for generalized families of utility functions, seems to be beyond the bounds of possibility in view of the technical complications arisen in this chapter. Thus, the randomization technique introduced in Chapter 2 that focuses on modifying the dimensionality of the stopping problem, leading to a family of simpler free boundary problems, seems a rather efficient way to obtain solutions capable of asymptotically approximating stopping rules to finite-horizon problems seemingly out of the reach of methods exposed in this chapter. This leads to the next topic of study in Chapter 4.

CHAPTER 4

# TIME-RANDOMIZED PREDICTION PROBLEMS FOR A FAMILY OF UTILITY FUNCTIONS

We continue with the notation introduced in Chapter 3 and let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space endowed with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. For fixed $n > 0$, we now denote by $T$ the waiting time to the $n^{\text{th}}$ jump $T_n$ of an $\mathcal{F}_t$-adapted Poisson process $N = (N_t)_{t\geq 0}$ with rate $\omega$, so that

$$\mathbb{P}(T \in [t, t + \mathrm{d}t)) = \frac{\omega^n t^{n-1} e^{-\omega t}}{(n-1)!} \mathrm{d}t \,.$$

We recall that $B = (B_t)_{t\geq 0}$ denotes a one-dimensional standard Brownian motion with $B_0 = 0$; we assume $B$ and $N$ to be independent. Additionally, for fixed constants $\mu$ and $\sigma$, we recall that $Z = (Z_t)_{t\geq 0}$ denotes a geometric Brownian motion given by

$$Z_t = Z_0 \exp\{\sigma B_t + (\mu - \sigma^2/2)t\} \,.$$

The running maximum processes $M = (M_t)_{t\geq 0}$ and $S = (S_t^\lambda)_{t\geq 0}$ are given by

$$M_t = \max_{0\leq s\leq t} Z_s \quad \text{and} \quad S_t^\lambda = \max_{0\leq s\leq t} B_s^\lambda \,, \quad t \geq 0 \,, \tag{4.0.1}$$

where $\lambda$ is a fixed constant and $B_t^\lambda = B_t + \lambda t$. Recall further from (3.1.3) that the distribution of $S_t^\lambda$ is given by

$$F_{S_t^\lambda}(s) = \mathbb{P}(S_t^\lambda \leq s) = \Phi\Big(\frac{s - \lambda t}{\sqrt{t}}\Big) - e^{2\lambda s}\Phi\Big(\frac{-s - \lambda t}{\sqrt{t}}\Big) \,.$$

In addition, we saw in (3.2.2) that the stochastic process $X = (X_t)_{t\geq 0}$ given by $X_t = S_t^\lambda - B_t^\lambda$ (with $X_0 = x \geq 0$), shared the stochastic law of the alternative process $X^x = (X_t^x)_{t\geq 0}$, with

$$X_t^x = x \vee S_t^\lambda - B_t^\lambda \,.$$

**Definition 4.0.1.** The family $\mathcal{U}$ consists of all $C^2$ functions $U(x)$ defined in $\mathcal{D} = [1, \infty)$ that are increasing, strictly concave or convex and satisfy

$$\mathbb{E}\big[ \sup_{0 \le t \le T_n} \big\{ U(e^{\sigma X_t^x}) + \sigma \int_0^\infty \int_{X_t^x}^\infty e^{\sigma z} U'(e^{\sigma z})(1 - F_{S_T^\lambda}(z))\, \mathrm{d}z \mathbb{P}(T_{n-N_t} \in \mathrm{d}T) \big\} \big] < \infty \,, \tag{4.0.2}$$

for all $n \ge 1$ and $x \ge 0$. In addition, they meet the following convergence and integrability criteria:

$$\lim_{x \to +\infty} e^{\alpha x} U'(e^{\alpha x}) \mathbb{P}(S_{t_0}^\lambda \ge x) = 0 \,, \tag{4.0.3}$$

$$\int_0^\infty e^{\alpha x} U'(e^{\alpha(\beta + x)}) \mathbb{P}(|B_{t_0}| \ge x) \mathrm{d}x < \infty \,, \tag{4.0.4}$$

$$\int_0^\infty e^{2\alpha x} U''(e^{\alpha(\beta + x)}) \mathbb{P}(|B_{t_0}| \ge x) \mathrm{d}x < \infty \,, \tag{4.0.5}$$

for all constants values $\alpha, \beta, t_0 \in \mathbb{R}^+$, where $U'(x)$ and $U''(x)$ are the first and second order derivatives of $U(x)$.

*Note.* Analytically testing the veracity of (4.0.2)-(4.0.5) for a given utility function can be a daunting challenge. However, Section 4.7 offers a discussion on existence and provides, by computational means, examples of utilities meeting these criteria.

For a given function $U \in \mathcal{U}$, we consider the optimal stopping problem

$$V = \inf_{\tau \in \mathcal{T}} \mathbb{E}\Big[ U\Big( \frac{M_T}{Z_\tau} \Big) \Big] \,, \tag{4.0.6}$$

where $\mathcal{T}$ stands for the set of all stopping times taking values in $[0, T]$.

This chapter will derive a family of time-independent stopping problems dependent on a 2-dimensional underlying diffusion. Complete solutions will be obtained as the unique solution to a family of free boundary problems. The detection of Bang-Bang strategies and links to work in the previous Chapter will be analysed. The final results will allow for us to computationally build numerical approximations of fixed terminal time set-up optimal stopping problems, and suggest the possibility of extending optimal stopping rules in [21] to a more general family of power utility measures.

In Section 4.1, problem (4.0.6) will be modified in order to pose a time-independent 2-dimensional optimal stopping problem fitting the general theory presented in the introduction. Section 4.2 will then provide evidence of the existence of optimal stopping times and offer a Markovian representation of the problem. The relation between the stopping problem and the stochastic infinitesimal generator of the underlying process is explored in Section 4.3, where the existence of Bang-Bang strategies and stopping boundaries is discussed and the main result in the chapter introduced. The proof of the result will be offered in Sections 4.4 and 4.5; where a family of free boundary problems is presented and its solution is derived. Finally, section 4.7 will discuss results and suggest future research directions.

## 4.1 An alternative Expression for $V$

**Lemma 4.1.1.** *For any given utility function $U \in \mathcal{U}$, let the gain function $G$ be defined as*

$$G(k,x) = \begin{cases} U(e^{\sigma x}) + \sigma \int_0^\infty \int_x^\infty e^{\sigma z} U'(e^{\sigma z})(1 - F_{S_T^\lambda}(z))\, \mathrm{d}z \mathbb{P}(T_{n-k} \in \mathrm{d}T)\,, & k < n\,, \\ \\ U(e^{\sigma x})\,, & k \geq n\,, \end{cases} \tag{4.1.1}$$

*where $T_{n-k}$ stands for the waiting time until the (n-k)$^{th}$ jump of a Poisson process with rate $\omega$, $\lambda = (\mu - \sigma^2/2)/\sigma$ and $F_{S_t^\lambda}(s)$ is as in* (3.1.3). *Then,* (4.0.6) *can be expressed as the time-independent optimal stopping problem with underlying $\mathcal{F}_t$-measurable gain function given by*

$$V = \inf_{\tau \in \mathcal{T}} \mathbb{E}[G(N_\tau, X_\tau)]]\,. \tag{4.1.2}$$

***Proof.*** The proof is similar to that in Lemma 3.1.1. We can rewrite $V$ in terms of a Brownian motion with drift $\lambda$ and its running maximum so that

$$V = \inf_{\tau \in \mathcal{T}} \mathbb{E}[U(e^{\sigma(S_T^\lambda - B_\tau^\lambda)})]\,.$$

Using deterministic times, and making use of the law of total expectation, the term involving the expected value above, restricted to the case when $\{t \leq T\}$, reads

$$\begin{aligned} \mathbb{E}[U(e^{\sigma(S_T^\lambda - B_t^\lambda)})1_{\{t \leq T\}}] &= \mathbb{E}[\mathbb{E}[U(e^{\sigma(S_T^\lambda - B_t^\lambda)})1_{\{t \leq T\}} | \mathcal{F}_t]] \\ &= \mathbb{E}[1_{\{t \leq T\}} \mathbb{E}[U(e^{\sigma((S_t^\lambda - B_t^\lambda) \vee (\max_{0 \leq s \leq T-t} B_{t+s}^\lambda - B_t^\lambda))}) | \mathcal{F}_t]]\,. \end{aligned}$$

The independent and stationary increments of $B^\lambda_t$ imply that

$$\Big(\max_{0\le s\le T-t} B^\lambda_{t+s} - B^\lambda_t\Big)\Big|\mathcal{F}_t \overset{\text{law}}{=} S^\lambda_{T-t}\,.$$

In addition, due to the memoryless property of the exponential distribution we have $T - t \overset{\text{law}}{=} T_{n-N_t}$, where $T_{n-N_t}$ stands for the waiting time until the $(n-N_t)^{\text{th}}$ jump in a Poisson process with rate $\omega$. Recalling that processes $N$ and $B$ are independent, we get

$$\begin{aligned}
\mathbb{E}[U(e^{\sigma(S^\lambda_T - B^\lambda_t)})1_{\{t\le T\}}] &= \mathbb{E}[1_{\{t=T\}}U(e^{\sigma X_t})] \\
&+ \mathbb{E}[1_{\{t<T\}}\int_0^\infty \Big\{U(e^{\sigma X_t})\mathbb{P}(S^\lambda_T \le X_t)\Big\}\mathbb{P}(T_{n-N_t}\in \mathrm{d}T)] \\
&+ \mathbb{E}[1_{\{t<T\}}\int_0^\infty \Big\{\int_{X_t}^\infty U(e^{\sigma z})f_{S^\lambda_T}(z)\mathrm{d}z\Big\}\mathbb{P}(T_{n-N_t}\in \mathrm{d}T)]\,,
\end{aligned}$$

where $f_{S^\lambda_T}(z)$ is density function of $S^\lambda_T$. Using property (4.0.3) and integrating by parts the inner integral in the last term of the right hand side we obtain

$$\begin{aligned}
\mathbb{E}[U(e^{\sigma(S^\lambda_T - B^\lambda_t)})1_{\{t\le T\}}] &= \mathbb{E}[1_{\{t\le T\}}U(e^{\sigma X_t})] \\
&+ \mathbb{E}[1_{\{t<T\}}\sigma\int_0^\infty \int_{X_t}^\infty e^{\sigma z}U'(e^{\sigma z})(1-F_{S^\lambda_T}(z))\mathrm{d}z\mathbb{P}(T_{n-N_t}\in \mathrm{d}T)] \\
&= \mathbb{E}[G(N_t, X_t)1_{\{t\le T\}}]\,.
\end{aligned}$$

As pointed out in [24] and [21], arguments based on each stopping time being the limit of a decreasing sequence of discrete stopping times, allow for us to extend this result for deterministic times to all stopping times. This implies that we may rewrite $V$ as

$$V = \inf_{\tau\in\mathcal{T}} \mathbb{E}[G(N_\tau, X_\tau)]\,,$$

completing the proof. $\square$

## 4.2 Extension of $V$ and Existence of an Optimal Stopping Time

Let us for now assume that $V$ in (4.1.2) admits an optimal stopping time, this will be shown below. Then, we let $D$ denote the stopping set of all possible states at which immediate halting results optimal in the stopping problem, so that

$$V = \mathbb{E}[G(N_{\tau_D}, X_{\tau_D})]\,, \tag{4.2.1}$$

where $\tau_D$ is defined as

$$\tau_D = \inf\{t \geq 0 \ : \ (N_t, X_t) \in D\} \ .$$

We note that the subset defined by $\{n\} \times \mathbb{R}^+$ must always be part of $D$, since the state $n$ in $N$ indicates forced stopping. This implies that $\tau_D \leq T < \infty$ almost surely.

We note that the law of $N$ started at $k$ is equal to that of $(N_t^k)_{t\geq 0}$, with $N_t^k = k + N_t$. In order to make use of optimal stopping techniques under a Markovian setting given in Chapter 2, we extend stopping problem $V$ allowing for a start at any point in the state space $(k, x) \in \{0, 1, ..., n\} \times \mathbb{R}_+$, so that

$$V(k,x) = \mathbb{E}_{k,x}[G(N_{t+\tau_D(k,x)}, X_{t+\tau_D(k,x)})|t < T] = \mathbb{E}[G(N^k_{\tau_D(k,x)}, X^x_{\tau_D(k,x)})] \ , \tag{4.2.2}$$

with

$$\tau_D(k,x) = \inf\{t \geq 0 \ : \ (N_t^k, X_t^x) \in D\} \ , \tag{4.2.3}$$

where $\mathbb{E}_{k,x}$ denotes the expectation under a Markovian probability measure for which $\mathbb{P}(N_t = k, X_t = x|t < T) = 1$. Here, $\tau_D(k,x)$ stands for the first entry time of the 2-dimensional Markovian process $Y_t^{k,x} = (N_t^k, X_t^x)$ in $D$. We note that the original problem in (4.2.1) can be retrieved as $V = V(0,0)$. Recall that the solution to our stopping problem is provided by the largest subharmonic function that is dominated by the gain function on the entire state space. The optimal stopping time is whenever the current state of the Markovian process falls within the subset of the state space where the value of the gain and dominating functions is the same. From the definition of $D$ in (2.1.2), the optimal stopping set $D$ can be defined as

$$D = \{(k,x) \in \{0, 1, ..., n\} \times \mathbb{R}^+ \ : V(k,x) = G(k,x)\} \ , \tag{4.2.4}$$

and is complemented by

$$C = D^c = \{(k,x) \in \{0, 1, ..., n\} \times \mathbb{R}^+ \ : V(k,x) < G(k,x)\} \ . \tag{4.2.5}$$

Note that if a Bang-Bang strategy were to be optimal, then $\{0, ...n-1\} \times \mathbb{R}^+$ would be fully included in either $D$ or $C$.

**Lemma 4.2.1.** *The extended optimal stopping problem $V$ in* (4.2.2) *admits an optimal stopping time given by $\tau_D$ in* (4.2.3)*, for any $(k,x) \in \{0, 1, ..., n\} \times \mathbb{R}_+$.*

***Proof.*** We recall that any utility function $U \in \mathcal{U}$ is continuous. Moreover, any function $f : \mathbb{N}_0 \mapsto \mathbb{R}$, where $\mathbb{N}_0$ stands for the set of natural numbers including 0, is continuous since all sets in $\mathbb{N}_0$ are open. Therefore, we note from the expression for $G$ in (4.1.1) that the mapping $(k, x) \mapsto G(k, x)$ is continuous on $\{0, 1, ..., n\} \times \mathbb{R}^+$. Then, by the dominated convergence theorem and assumption (4.0.2), it follows that mappings of the form $(k, x) \mapsto \mathbb{E}[G(N^k_\tau, X^x_\tau)]$ are continuous and therefore upper semicontinuous over stopping times taking values in $[0, T_{n-k}]$, where $T_{n-k}$ stands for the time of the $(n-k)^{\text{th}}$ jump in a Poisson process with rate $\omega$. Moreover, the value function $V(k, x)$ is the infimum of the mapping over such stopping times and is therefore upper semicontinuous itself. The existence of an optimal stopping time and its characterization in terms of the stopping set $D$ follows from these facts, along with Corollary 2.1.3 in the introduction section. $\square$

## 4.3 Infinitesimal Generator and Solution to the Problem

The stochastic infinitesimal generator of the process $X$ was introduced in (3.4.1). We recall it acts on twice differentiable functions $f$ (satisfying $f'(0) = 0$) as

$$\mathcal{A}_X f(x) = -\lambda f'(x) + \frac{1}{2} f''(x) .$$

On the other hand, the generator of a Poisson counting process is given by

$$\begin{aligned} \mathcal{A}_N f(k) &= \lim_{t \to 0} \frac{\mathbb{E}[f(k + N_t)] - f(k)}{t} \\ &= \lim_{t \to 0} \left\{ (e^{-\omega t} - 1) \frac{f(k)}{t} + e^{-\omega t} [f(k+1)\omega + \frac{f(k+2)\omega^2 t}{2} + ...] \right\} \\ &= \omega [f(k+1) - f(k)] . \end{aligned} \tag{4.3.1}$$

Therefore, the infinitesimal generator of the 2-dimensional Markovian process $Y_t = (N_t, X_t)$ acts on suitable functions $f : \{0, ..., n-1\} \times \mathbb{R} \to \mathbb{R}$ as

$$\mathcal{A}_Y f(k, x) = \omega [f(k+1, x) - f(k, x)] - \lambda \frac{\mathrm{d} f(k, x)}{\mathrm{d}x} + \frac{1}{2} \frac{\mathrm{d}^2 f(k, x)}{\mathrm{d}x^2} . \tag{4.3.2}$$

For all $(k, x) \in \{0, ..., n-1\} \times \mathbb{R}$, an application of Itô formula on function $G$ in (4.2.2) yields

$$
\begin{aligned}
V(k,x) &= G(k,x) + \mathbb{E}\Big[\int_0^{\tau_D(k,x)} G_x(N^k_{s-}, X^x_s)\mathrm{d}(x \vee S^\lambda_s - B^\lambda_s)\Big] \\
&+ \mathbb{E}\Big[\int_0^{\tau_D(k,x)} \frac{1}{2} G_{xx}(N^k_{s-}, X^x_s)\mathrm{d}\langle X^x, X^x\rangle_s + \int_0^{\tau_D(k,x)} \Delta G(N^k_s, X^x_s)\mathrm{d}N^k_s\Big] ,
\end{aligned}
$$

with $\Delta G(N^k_s, X^x_s) = G(N^k_s, X^x_s) - G(N^k_{s-}, X^x_s)$. We recall that $\mathrm{d}\langle X^x, X^x\rangle_t = \mathrm{d}t$; then, adding and subtracting a compensator to the last integral term in the above equation, we obtain

$$
\begin{aligned}
V(k,x) &= G(k,x) + \mathbb{E}\Big[\int_0^{\tau_D(k,x)} \mathcal{A}_X G(N^k_{s-}, X^x_s)\mathrm{d}s + \int_0^{\tau_D(k,x)} G_x(N^k_{s-}, X^x_s)\mathrm{d}(x \vee S^\lambda_s)\Big] \\
&+ \mathbb{E}\Big[-\int_0^{\tau_D(k,x)} G_x(N^k_{s-}, X^x_s)\mathrm{d}B_s + \int_0^{\tau_D(k,x)} \Delta G(N^k_s, X^x_s)\mathrm{d}(N^k_s - \omega s)\Big] \\
&+ \mathbb{E}\Big[\int_0^{\tau_D(k,x)} \omega \Delta G(N^k_s, X^x_s)\mathrm{d}s\Big] ,
\end{aligned} \tag{4.3.3}
$$

Note that the processes $(N^k_s - \omega s)_{s\geq 0}$ and $(B_s)_{s\geq 0}$ are martingales; in addition, a change in value in $x \vee S^\lambda_s$ implies $X^x_s = 0$, and $G_x(k, 0) = 0$ for all $k \in \{0, ..., n-1\}$, since

$$
\begin{aligned}
G_x(k,x) &= \int_0^\infty \Big\{\frac{\mathrm{d}U(e^{\sigma x})}{\mathrm{d}x} + \sigma \frac{\mathrm{d}}{\mathrm{d}x}\int_x^\infty e^{\sigma z} U'(e^{\sigma z})(1 - F_{S^\lambda_T}(z))\,\mathrm{d}z\Big\}\mathbb{P}(T_{n-k} \in \mathrm{d}T) \\
&= \int_0^\infty \Big\{\sigma e^{\sigma x} U'(e^{\sigma x}) - \sigma e^{\sigma x} U'(e^{\sigma x})(1 - F_{S^\lambda_T}(x))\Big\}\mathbb{P}(T_{n-k} \in \mathrm{d}T) \\
&= \int_0^\infty \sigma e^{\sigma x} U'(e^{\sigma x}) F_{S^\lambda_T}(x)\mathbb{P}(T_{n-k} \in \mathrm{d}T) .
\end{aligned} \tag{4.3.4}
$$

Therefore, in a similar manner to (3.4.3), equation (4.3.3) becomes

$$
V(k,x) = G(k,x) + \mathbb{E}\Big[\int_0^{\tau_D(k,x)} \mathcal{A}_Y G(N^k_s, X^x_s)\mathrm{d}s\Big] , \tag{4.3.5}
$$

for all $(k,x) \in \{0, ..., n-1\} \times \mathbb{R}$. Moreover, differentiation of equation (4.3.4) with respect to $x$ shows that $\mathcal{A}_Y G(k,x)$ in (4.3.2) is given by

$$
\begin{aligned}
\mathcal{A}_Y G(k,x) &= \omega[G(k+1,x) - G(k,x)] - (\lambda - \frac{\sigma}{2})G_x(k,x) \\
&+ \frac{\sigma}{2} e^{\sigma x} \frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty U'(e^{\sigma x}) F_{S^\lambda_T}(x)\mathrm{d}\mathbb{P}(T_{n-k} \in \mathrm{d}T) .
\end{aligned}
$$

### 4.3.1 Bang-Bang Stopping Rules and Characterization of $V$

Noting expression (4.3.5), the following two sets play a fundamental role in the description of $C$ and $D$,

$$
\Theta = \{(k,x) \in \{0,1,...,n-1\} \times \mathbb{R}^+ : \mathcal{A}_Y G(k,x) \geq 0\} , \tag{4.3.6}
$$

$$
\Upsilon = \{(k,x) \in \{0,1,...,n-1\} \times \mathbb{R}^+ : \mathcal{A}_Y G(k,x) < 0\} . \tag{4.3.7}
$$

Let $\Delta = \sqrt{\lambda^2 + 2\omega}$ and define functions $R_1$ and $R_2$ as

$$R_1(k,x) = \int_0^x V(k+1,r)e^{-(\lambda-\Delta)r}\mathrm{d}r\ ;\ R_2(k,x) = \int_0^x V(k+1,r)e^{-(\lambda+\Delta)r}\mathrm{d}r\ . \tag{4.3.8}$$

**Lemma 4.3.1.** *Let $U \in \mathcal{U}$. If $\Upsilon = \{0,1,...,n-1\} \times \mathbb{R}^+$ or $\Theta = \{0,1,...,n-1\} \times \mathbb{R}^+$, then a Bang-Bang stopping strategy is optimal. Moreover, if $\Upsilon = \{0,1,...,n-1\} \times \mathbb{R}^+$,*

$$V(k,x) = U(e^{\sigma x})\Big(1 + \frac{\lambda+\Delta}{\Delta-\lambda}e^{-2\Delta x}\Big) + \frac{\omega}{\Delta}e^{(\lambda-\Delta)x}\Big(\frac{\lambda+\Delta}{\Delta-\lambda}R_2(k,x) + R_1(k,x)\Big)\ ,$$

*for all $(k,x) \in \{0,1,...,n-1\} \times \mathbb{R}^+$; if $\Theta = \{0,1,...,n-1\} \times \mathbb{R}^+$,*

$$V(k,x) = G(k,x)\ ,$$

*for all $(k,x) \in \{0,1,...,n-1\} \times \mathbb{R}^+$.*

***Proof.*** Note that, if $\Upsilon = \{0,1,...,n-1\} \times \mathbb{R}^+$, it follows from expression (4.3.5) that for all $(k,x) \in \{0,1,...,n-1\} \times \mathbb{R}^+$ an infimum is attained at deadline $T$, so that $V(k,x) < G(k,x)$ and according to (4.2.5) we have $(k,x) \in C$. Thus, the entire state space with the exception of $\{n\} \times \mathbb{R}^+$ is contained in the continuation set, and the explicit expression for $V$ is given as the unique solution to the set of equations

$$\mathcal{A}_Y V(k,x) = 0 \qquad \text{for all} \qquad (k,x) \in \{0,1,...,n-1\} \times \mathbb{R}^+\ , \tag{4.3.9}$$

$$\lim_{x\to 0} V_x(k,x) = 0 \qquad \text{for all} \qquad k \in \{0,1,...,n-1\}\ , \tag{4.3.10}$$

$$\lim_{x\to\infty} V(k,x) = U(e^{\sigma x}) \qquad \text{for all} \qquad k \in \{0,1,...,n-1\}\ . \tag{4.3.11}$$

Here, the differential equation (4.3.9) follows from the results on free boundary problems presented in the introductory Chapter 2. Also, equation (4.3.11) is rather obvious in view of the definition of the gain function in (4.1.1). The derivation of (4.3.10) is however necessary and provided in Lemma 4.4.4 later in this Chapter. Hence, making use of (4.3.10) and (4.3.11) as boundary conditions, the explicit solution for $V$ can be obtained solving the ordinary differential equation (4.3.9). We omit the details on this procedure here, since this approach will be exposed later with means of providing proof to the main result in the Chapter.

Finally, if $\Theta = \{0,1,...,n-1\} \times \mathbb{R}^+$, we note from the equation for $V$ in (4.3.5) that for all $(k,x) \in \{0,1,...,n-1\} \times \mathbb{R}^+$ it holds $\tau_D(k,x) = 0$, since otherwise $V(k,x) > G(k,x)$, which is contradictory. This implies that $V(k,x) = G(k,x)$ and instantaneous stopping is optimal. $\square$

It follows from Lemma 4.3.1 that for a given $U \in \mathcal{U}$

$$\begin{aligned} D &= \{n\} \times \mathbb{R}^+ && \text{if} \quad \Theta = \{0,1,...,n-1\} \times \mathbb{R}^+, \text{ and} \\ D &= \{1,2,...,n\} \times \mathbb{R}^+ && \text{if} \quad \Upsilon = \{0,1,...,n-1\} \times \mathbb{R}^+. \end{aligned}$$

### 4.3.2 Stopping Boundaries and Solution to the Problem

Should conditions of Lemma 4.3.1 not be met, we note from expression (4.3.5) that $\Upsilon \subseteq C$ always. This is observed noting that for all $(k,x) \in \Upsilon$ we have $\mathcal{A}_Y G(k,x) < 0$ and thus $\tau_D(k,x) \geq 0$. It follows that $V(k,x) < G(k,x)$ and $(k,x) \in C$. On the other hand, it is not necessarily true that $\Theta \subseteq D$.

In this case, the memoryless property of the exponential distribution poses an independent optimal stopping problem for each subsequent step in $N$ (cf. [2, 17, 31]), and may give rise to the existence of arrays of critical points in $\mathbb{R}^+$ dividing the state space $\{0,1,...,n\} \times \mathbb{R}^+$ into sets $D$ and $C$. These are referred to as *optimal stopping boundaries*, and the optimal stopping rule for a problem $V$ started at an arbitrary $(k,x) \in C$ is given by the first crossing time for process $X$ to a boundary. Formally defined as time functions (constant over time within jumps in $N$), stopping boundaries are linked to the amount of steps left to deadline in $N$ at any given point of time, and we denote them

$$\zeta_t^* = \zeta^*(n - N_t^k) \,, \tag{4.3.12}$$

for $t \geq 0$, (see example in Figure 4.1).

If set $\Theta$ in (4.3.6) is non-empty, there exist *bounding* functions $b_i : \{0,1,...,n-1\} \to \mathbb{R}_+$ that define its frontier(s) with set $\Upsilon$. In this case, a direct analysis of functions $b_i$ along with properties of operator $\mathcal{A}_Y$ in (4.3.2) can usually determine the shape of the stopping set $D$ in terms of stopping boundaries $\zeta^*$. This is the basis for the conversion of the stopping problem to an equivalent free boundary problem (for some examples see [49], Sections 7 & 8). In what follows we make the following assumption and study

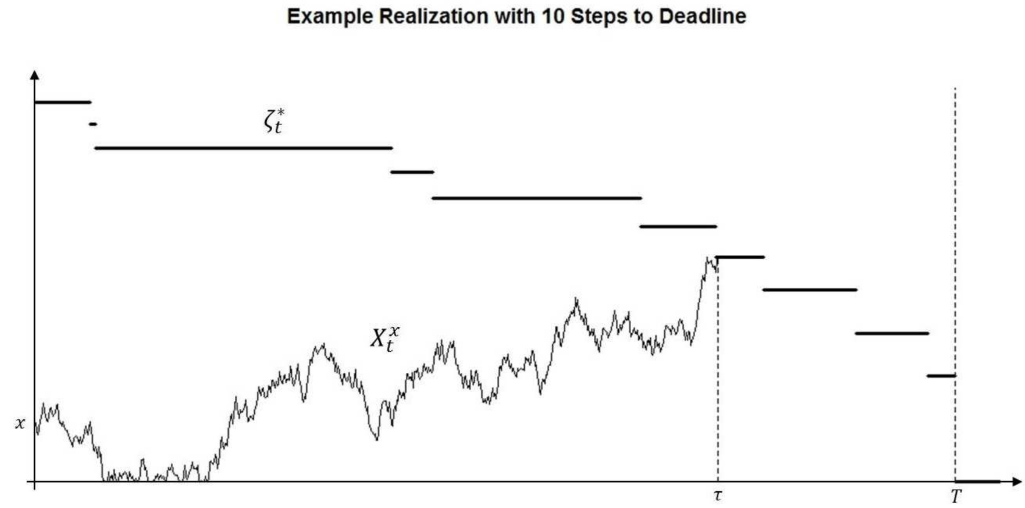


**Figure 4.1:** Example realization with n=10 and $U(x) = x$. The values of the optimal stopping boundary $\zeta^*$ are observed along with the dynamics of a process $X^x$; $\tau$ is the optimal stopping time.

the reduction of problem $V$ in (4.2.2) to a free boundary problem.

**Assumption 4.3.2.** *Sets* $\Theta$ *and* $\Upsilon$ *in* (4.3.6)-(4.3.7) *are non-empty and there exists a unique n-dimensional array b such that*

$$\begin{aligned} \Theta &= \{(k,x) \in \{0,1,...,n-1\} \times \mathbb{R}^+ : x \geq b(k)\}, \textit{ and} \\ \Upsilon &= \{(k,x) \in \{0,1,...,n-1\} \times \mathbb{R}^+ : x < b(k)\}. \end{aligned}$$

Figure 4.2 below offers examples of choices of function $U$ meeting this criteria; a short discussion on the challenges of facing more than a single bounding function is included in the discussion Section 4.7.

Under Assumption 4.3.2 and recalling that $\Upsilon \subseteq C$, it follows that $D$ includes all points $(k,x) \in \{0,1,...,n-1\} \times \mathbb{R}_+$ where $x$ lies above a boundary in (4.3.12) such that $\zeta^*(n-k) \geq b(k)$ for all $k \in \{0,1,...,n-1\}$, if any. Thus, the continuation set $C$ is defined by

$$C = \{(k,x) \in \{0,1,...,n\} \times \mathbb{R}^+ \ : \ x < \zeta^*(n-k)\};$$

equivalently

$$D = \{(k,x) \in \{0,1,...,n\} \times \mathbb{R}^+ \ : \ x \geq \zeta^*(n-k)\}.$$

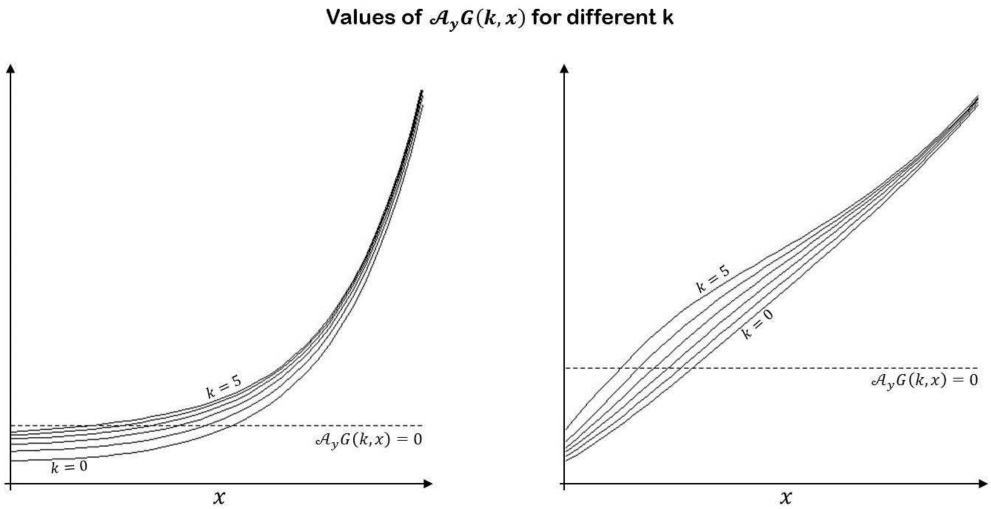


**Figure 4.2:** Numerical examples of the value of functions $\mathcal{A}_Y G(k,x)$ with respect to $x$, for different fixed values of $k$. Here, $\mu = 0.5$, $\sigma = 1$, $\omega = 4$ and $n = 5$; the left hand side plot corresponds to $U(x) = \frac{1}{2}(x^{3/2} + x^{4/3})$, on the right hand side we have $U(x) = \frac{1}{2}(x^{1/2} + x^{1/4})$.

Note that $\zeta^*_T = \zeta(n-n) = \zeta^*(0) = 0$. This definition of sets $C$ and $D$ stands in accordance with the intuitive argument suggesting that stopping times are linked to big departures of process $Z$ from its running maximum. In addition, $\zeta^*$ may take infinite values, so that if $\zeta^*(n-k) = \infty$ for all $k \in \{0,1,...,n-1\}$, it follows $\Theta \subset C$, never stopping results optimal and the corresponding value function $V$ is given by Lemma 4.3.1. Finally, for any starting point $(k,x) \in \{0,...,n\} \times \mathbb{R}_+$, the optimal stopping rule $\tau_D$ linked to $\zeta^*$ takes the form

$$\tau_D(k,x) = \inf\{t \geq 0 \; : \; X^x_t \geq \zeta^*(n - N^k_t)\} \, . \tag{4.3.13}$$

Therefore, the solution to $V$ will follow from the correct detection of the values that $\zeta^*$ takes at each step. To give the main result in this Chapter, we first define the following functions. Let $C_1$ and $C_2$, in terms of the corresponding optimal stopping boundary $\zeta^*$ and the set of parameters $(\lambda, \sigma, \omega)$, be given by

$$\begin{aligned} C_1(k) &= (G_x(k,\zeta^*(n-k)) - (\lambda-\Delta)G(k,\zeta^*(n-k))) \cdot \frac{e^{-(\lambda+\Delta)\zeta^*(n-k)}}{2\Delta} \\ &+ \frac{\omega}{\Delta} R_2(k,\zeta^*(n-k)) \, , \end{aligned} \tag{4.3.14}$$

and

$$\begin{aligned} C_2(k) &= -(G_x(k,\zeta^*(n-k)) - (\lambda+\Delta)G(k,\zeta^*(n-k))) \cdot \frac{e^{-(\lambda-\Delta)\zeta^*(n-k)}}{2\Delta} \\ &- \frac{\omega}{\Delta} R_1(k,\zeta^*(n-k)) \ . \end{aligned} \tag{4.3.15}$$

with functions $R_1$ and $R_2$ given by (4.3.8).

**Theorem 4.3.3.** *For a given $U \in \mathcal{U}$ such that Assumption 4.3.2 holds, the extended optimal stopping problem $V(k,x)$ in* (4.2.2) *can be recursively decomposed as follows,*

$$\begin{aligned} V(k,x) &= C_1(k)e^{(\lambda+\Delta)x} + C_2(k)e^{(\lambda-\Delta)x} \\ &+ \frac{\omega}{\Delta} \cdot \left\{ e^{(\lambda-\Delta)x} R_1(k,x) - e^{(\lambda+\Delta)x} R_2(k,x) \right\} , \end{aligned} \tag{4.3.16}$$

*if $x < \zeta^*(n-k)$ and $k < n$, and*

$$V(k,x) = G(k,x) \ , \tag{4.3.17}$$

*if $x \geq \zeta^*(n-k)$ or $k = n$. Function $G(t,x)$ is as described in* (4.1.1) *and functions $R_1, R_2, C_1$ and $C_2$ are given by* (4.3.8), (4.3.14) *and* (4.3.15), *respectively.*

*The value of the optimal stopping boundary $\zeta^*$, at 'n-k' steps left to deadline, can be identified as the unique positive solution to the integral equation*

$$(\lambda+\Delta)C_1(k) + (\lambda-\Delta)C_2(k) = 0 \ . \tag{4.3.18}$$

For all $x \in \mathbb{R}^+$, $V$ is known at deadline and given by $V(n,x) = G(n,x) = U(e^{\sigma x})$; thus, equation (4.3.16) provides an iterative method for finding the numerical value of $V$ at any point in the state space. Next, we provide the proof for Theorem 4.3.3 in the following two sections. As mentioned, we aim to pose a family of free boundary problems so that $V$ stands as its unique solution; this is done in Section 4.4. Results establishing (4.3.16) and (4.3.18) follow then from the application of ordinary techniques for solving linear second order differential equations with constant coefficients (see for example [57]); this part of the proof is postponed to Section 4.5. Additionally, Section 4.6 presents and discusses a direct simplification of the above result to a case with a single jump in $N$.

## 4.4 A FREE BOUNDARY PROBLEM

We recall from Chapter 2 (Section 2.2) that the optimal stopping problem $V(k,x)$ in (4.2.2) satisfies

$$\mathcal{A}_Y V(k,x) = 0 \qquad \text{for } (k,x) \in C\,, \tag{4.4.1}$$

$$V(k,x) = G(k,x), \qquad \text{for } (k,x) \in D\,, \tag{4.4.2}$$

where $\mathcal{A}_Y$ stands for the infinitesimal generator of the process $Y$ given in (4.3.2). In terms of the optimal stopping boundary $\zeta^*$, this is equivalent to

$$\omega[V(k+1,x) - V(k,x)] - \lambda V_x(k,x) + \tfrac{1}{2} V_{xx}(k,x) = 0 \quad \text{for } x < \zeta^*(n-k)\,, \tag{4.4.3}$$

$$V(k,x) = G(k,x) \qquad \text{for } x \geq \zeta^*(n-k)\,. \tag{4.4.4}$$

In the following, we show that the mapping $x \mapsto V(k,x)$ is continuous for any fixed value of $k$ in $N$. Note that twice differentiability of this mapping when restricted in $C$ follows from the general theory of Markov processes in [49] (Chapter 3, Section 7). Moreover, in view of the introductory Section 2.3 in Chapter 2 on the Neumann free boundary problem, we show that, for any $k < n$, the system of equations (4.4.3)-(4.4.4) is complemented by the following boundary conditions

$$\lim_{x \to \zeta^*(n-k)} V(k,x) = G(k, \zeta^*(n-k))\,, \tag{4.4.5}$$

$$\lim_{x \to \zeta^*(n-k)} V_x(k,x) = G_x(k, \zeta^*(n-k))\,, \qquad \textit{(smooth fit)}\,, \tag{4.4.6}$$

$$\lim_{x \to 0} V_x(k,x) = 0\,, \qquad \textit{(normal reflection)}\,. \tag{4.4.7}$$

In order to show the validity of (4.4.5)-(4.4.7), we make use of variations of the methods of solution presented in [49] (Chapter 4) and applied in [18, 20, 21, 29] among others.

### 4.4.1 Monotonicity and Continuity of V

We recall that $V(k,x) < G(k,x)$ for any $x < \zeta^*(n-k)$. Then, condition (4.4.5) will follow from continuity of the mapping $x \mapsto V(k,x)$. We start by introducing the following lemma for later use.

**Lemma 4.4.1.** *Let $U \in \mathcal{U}$ be an strictly-convex function and fix $t \geq 0$ and $x \in \mathbb{R}^+$. Then, the random variable $e^{\sigma X_t^x} U'(e^{\sigma X_t^x})$ has finite expectation.*

***Proof.*** Note first that

$$\begin{aligned} X_t^x = x \vee S_t^\lambda - B_t^\lambda &\leq \max(x + |\lambda| t + |B_t|, \max_{0 \leq s \leq t} \{\lambda s + B_s\} - \lambda t + |B_t|) \\ &\leq \max(x + |\lambda| t + |B_t|, \max_{0 \leq s \leq t} |B_s| + |\lambda| t + |B_t|) \\ &\leq x + 2 \max_{0 \leq s \leq t} |B_s| + |\lambda| t \,. \end{aligned}$$

Since $U$ is a non-decreasing and convex function,

$$\begin{aligned} 0 \leq \mathbb{E}[e^{\sigma X_t^x} U'(e^{\sigma X_t^x})] &\leq \mathbb{E}[e^{\sigma(x + 2\max_{0 \leq s \leq t} |B_s| + |\lambda| t)} U'(e^{\sigma(x + 2\max_{0 \leq s \leq t} |B_s| + |\lambda| t)})] \\ &= -\int_0^\infty e^{\sigma w(z)} U'(e^{\sigma w(z)}) \mathrm{d}\mathbb{P}(\max_{0 \leq s \leq t} |B_s| \geq z) \,, \end{aligned}$$

where $w(z) = x + |\lambda| t + 2z$. Integrating by parts the above yields

$$\begin{aligned} \mathbb{E}[e^{\sigma X_t^x} U'(e^{\sigma X_t^x})] \leq & -[e^{\sigma w(z)} U'(e^{\sigma w(z)}) \mathbb{P}(\max_{0 \leq s \leq t} |B_s| \geq z)]\Big|_0^\infty \\ &+ \int_0^\infty [e^{\sigma w(z)} U'(e^{\sigma w(z)}) + \sigma e^{2\sigma w(z)} U''(e^{\sigma w(z)})] \mathbb{P}(\max_{0 \leq s \leq t} |B_s| \geq z) \mathrm{d}z \,. \end{aligned}$$

Recall (cf. [21]) that $\mathbb{P}(\max_{0 \leq s \leq t} |B_s| \geq z) \leq 2\mathbb{P}(|B_t| \geq z)$. Noting conditions (4.0.4) and (4.0.5), it follows that

$$\mathbb{E}[e^{\sigma X_t^x} U'(e^{\sigma X_t^x})] < \infty.$$

□

**Lemma 4.4.2.** *Fix $U \in \mathcal{U}$ and $k \leq n$, the mapping $x \mapsto V(k, x)$ is non-decreasing and continuous in $\mathbb{R}^+$.*

***Proof.*** The proof is split up in two parts. To start with, we show that function $G$ defined by (4.1.1), to which $V$ relates, is non-decreasing in $x$. We note that if $k = n$, the monotonicity of $G$ follows from $G_x(k, x) = \sigma e^{\sigma x} U'(e^{\sigma x}) \geq 0$. If $k < n$, we recall from (4.3.4) that

$$G_x(k, x) = \int_0^\infty \sigma e^{\sigma x} U'(e^{\sigma x}) F_{S_T^\lambda}(x) \mathbb{P}(T_{n-k} \in \mathrm{d}T) \geq 0 \,, \tag{4.4.8}$$

for all $x \in \mathbb{R}_+$.

We now show that $V$ is non-decreasing in $x$. If $k = n$, then $(k, x) \in D$ and so $V = G$. Therefore $V(k, x) = U(e^{\sigma x})$, which is a non-decreasing function of $x$. If $k < n$, take values $x, y \in \mathbb{R}^+$ with $x \leq y$ and set $\tau_x = \tau_D(k, x)$ and $\tau_y = \tau_D(k, y)$, where $\tau_D(k, \cdot)$ is given by (4.3.13). Since the subset $\{n\} \times \mathbb{R}^+$ is included in $D$, we have $\tau_x, \tau_y \leq T$ almost surely. According to the definition of $V(k, x)$ in (4.2.2), the infimum is attained at time $\tau_x$ and we have

$$V(k, x) = \mathbb{E}[G(N^k_{\tau_x}, X^x_{\tau_x})] \leq \mathbb{E}[G(N^k_{\tau_y}, X^x_{\tau_y})] ,$$

implying that

$$V(k, y) - V(k, x) = \mathbb{E}[G(N^k_{\tau_y}, X^y_{\tau_y}) - G(N^k_{\tau_x}, X^x_{\tau_x})] \geq \mathbb{E}[G(N^k_{\tau_y}, X^y_{\tau_y}) - G(N^k_{\tau_y}, X^x_{\tau_y})] .$$

Recalling that $G(k, x)$ is non-decreasing on $x$, and noting that $X^y_{\tau_y} \geq X^x_{\tau_y}$, we get

$$V(k, y) \geq V(k, x) ,$$

settling the result on monotonicity for $V$.

We continue showing that the mapping $x \mapsto V(k, x)$ is continuous on $x$ for any fixed $k \leq n$. If $k = n$, the value function is reduced to $U(e^{\sigma x})$, which is continuous in $x$ by assumption. If $k < n$, following the previous arguments, we note that for $x \leq y$

$$0 \leq V(k, y) - V(k, x) \leq \mathbb{E}[G(N^k_{\tau_x}, X^y_{\tau_x}) - G(N^k_{\tau_x}, X^x_{\tau_x})] .$$

Since $G(k, x)$ is continuous in $x$, for any fixed value of $k$, the mean value theorem gives

$$0 \leq V(k, y) - V(k, x) \leq \mathbb{E}[(X^y_{\tau_x} - X^x_{\tau_x})G_x(N^k_{\tau_x}, \nu)] .$$

where $X^x_{\tau_x} \leq \nu \leq X^y_{\tau_x}$. Moreover, noting that $X^y_{\tau_x} - X^x_{\tau_x} = y \vee S^\lambda_{\tau_x} - B^\lambda_{\tau_x} - x \vee S^\lambda_{\tau_x} + B^\lambda_{\tau_x} \leq y - x$, we have

$$0 \leq V(k, y) - V(k, x) \leq (y - x)\mathbb{E}[G_x(N^k_{\tau_x}, \nu)] .$$

In order to further simplify the above, we recall result (4.3.4) and note that

$$\begin{aligned} G_x(N^k_{\tau_x}, \nu) &\leq \sigma e^{\sigma\nu} U'(e^{\sigma\nu}) \int_0^\infty F_{S^\lambda_T}(\nu) \mathbb{P}(T_{n-N^k_{\tau_x}} \in \mathrm{d}T) \\ &\leq \sigma e^{\sigma\nu} U'(e^{\sigma\nu}) \int_0^\infty \mathbb{P}(T_{n-N^k_{\tau_x}} \in \mathrm{d}T) \;=\; \sigma e^{\sigma\nu} U'(e^{\sigma\nu}) , \end{aligned}$$

if $N^k_{\tau_x} < n$. Also, it is trivial that $G_x(N^k_{\tau_x}, \nu) \leq \sigma e^{\sigma\nu} U'(e^{\sigma\nu})$ if $N^k_{\tau_x} = n$.

If $U$ is concave, then $U'$ is a non-increasing function. Noting that $\nu \leq X^y_{\tau_x}$, we obtain

$$0 \leq V(k,y) - V(k,x) \leq \sigma c(y-x)\mathbb{E}[e^{\sigma X^y_{\tau_x}}] \,. \tag{4.4.9}$$

for some constant value $c > 0$. If $U$ is convex, then $U'$ is a non-decreasing function and so

$$0 \leq V(k,y) - V(k,x) \leq \sigma(y-x)\mathbb{E}[e^{\sigma X^y_{\tau_x}} U'(e^{\sigma X^y_{\tau_x}})] \,. \tag{4.4.10}$$

Note that the integrability of $e^{\sigma X^y_{\tau_x}} U'(e^{\sigma X^y_{\tau_x}})$ for convex functions $U$ follows from Lemma 4.4.1. We refer to [21] for a probabilistic proof on the integrability of the term $e^{\sigma X^y_{\tau_x}}$. Finally, take the limit as $|y-x| \to 0$ in (4.4.9) and (4.4.10) above to conclude that $x \mapsto V(k,x)$ for $k \in \{0,...,n\}$ are continuous mappings in $\mathbb{R}^+$, therefore concluding the proof. □

### 4.4.2 The Condition of Smooth Fit

**Lemma 4.4.3** (Principle of Smooth Fit)**.** *For any fixed $k < n$, $V_x(k,x)$ exists and is continuous at $\zeta^*(n-k)$. In addition, it holds $V_x(k,\zeta^*(n-k)) = G_x(k,\zeta^*(n-k))$.*

*Note.* We can observe an example of the smooth pasting of the value function $V$ to the gain function $G$ in Figure 4.3.

***Proof.*** Let $\varepsilon > 0$ and $\tau_\varepsilon = \tau_D(k,\zeta^*(n-k)-\varepsilon)$. From the definition of $C$ in (4.2.5) we note that $V(k,\zeta^*(n-k)-\varepsilon) < G(k,\zeta^*(n-k)-\varepsilon)$. Also, it is always optimal to halt while in $D$, so that $V(k,\zeta^*(n-k)) = G(k,\zeta^*(n-k)) < \mathbb{E}[G(N^k_{\tau_\varepsilon}, X^{\zeta^*(n-k)}_{\tau_\varepsilon})]$. This implies

$$G(k,\zeta^*(n-k)) - G(k,\zeta^*(n-k)-\varepsilon) \leq V(k,\zeta^*(n-k)) - V(k,\zeta^*(n-k)-\varepsilon) \,, \tag{4.4.11}$$

and

$$V(k,\zeta^*(n-k)) - V(k,\zeta^*(n-k)-\varepsilon) \leq \mathbb{E}[G(N^k_{\tau_\varepsilon}, X^{\zeta^*(n-k)}_{\tau_\varepsilon}) - G(N^k_{\tau_\varepsilon}, X^{\zeta^*(n-k)-\varepsilon}_{\tau_\varepsilon})] \,. \tag{4.4.12}$$

From (4.4.12), we derive making use of the mean value theorem, that for all fixed $k$

$$V(k,\zeta^*(n-k)) - V(k,\zeta^*(n-k)-\varepsilon) \quad \leq \quad \mathbb{E}[(X^{\zeta^*(n-k)}_{\tau_\varepsilon} - X^{\zeta^*(n-k)-\varepsilon}_{\tau_\varepsilon})G_x(N^k_{\tau_\varepsilon},\nu)] \,,$$

where $X_{\tau_\varepsilon}^{\zeta^*(n-k)-\varepsilon} \le \nu \le X_{\tau_\varepsilon}^{\zeta^*(n-k)}$. Recall that for any $U$ in $\mathcal{U}$ the term $G_x(k,x)$ is positive for all $(k,x) \in \{0,1,...,n\} \times \mathbb{R}_+$. Note also that

$$X_{\tau_\varepsilon}^{\zeta^*(n-k)} - X_{\tau_\varepsilon}^{\zeta^*(n-k)-\varepsilon} = \zeta^*(n-k) \vee S_{\tau_\varepsilon}^{\lambda} - (\zeta^*(n-k)-\varepsilon) \vee S_{\tau_\varepsilon}^{\lambda} \le \varepsilon \, .$$

Thus,

$$V(k,\zeta^*(n-k)) - V(k,\zeta^*(n-k)-\varepsilon) \quad \le \quad \varepsilon \cdot \mathbb{E}[G_x(N_{\tau_\varepsilon}^k, \nu)] \, . \tag{4.4.13}$$

Since $V$ is twice differentiable in $C$, dividing the terms in equations (4.4.11) and (4.4.13) by $\varepsilon$ and taking the limit as $\varepsilon \to 0$ leads to

$$G_x(k,\zeta^*(n-k)) \le V_x(k,\zeta^*(n-k)) \le \lim_{\varepsilon\to 0} \mathbb{E}[G_x(N_{\tau_\varepsilon}^k, \nu)] \, . \tag{4.4.14}$$

Moreover

$$\begin{aligned}
\tau_\varepsilon &= \tau_D(k,\zeta^*(n-k)-\varepsilon) = \inf\{s \ge 0 \ : \ X_s^{\zeta^*(n-k)-\varepsilon} \ge \zeta^*(n-N_s^k)\} \\
&= \inf\{s \ge 0 \ : \ (\zeta^*(n-k)-\varepsilon) \vee S_s^{\lambda} - B_s - \lambda s \ge \zeta^*(n-N_s^k)\} \\
&\le \inf\{s \ge 0 \ : \ -B_s \ge \varepsilon + \lambda s + \zeta^*(n-N_s^k) - \zeta^*(n-k)\} \xrightarrow{\varepsilon\to 0} 0 \, ,
\end{aligned}$$

since a Poisson process is right-continuous. This implies that $\nu \xrightarrow{\varepsilon\to 0} \zeta^*(n-k)$, since $X_{\tau_\varepsilon}^{\zeta^*(n-k)-\varepsilon} \le \nu \le X_{\tau_\varepsilon}^{\zeta^*(n-k)}$. Therefore, by (4.4.14), the fact that $V_x(k,\zeta^*(n-k)) = G_x(k,\zeta^*(n-k))$ follows from the right-continuity of Poisson processes.

Next, we show that for any fixed $k < n$, $V_x(k,x)$ is continuous at $\zeta^*(n-k)$. For this, we take $\delta > 0$, and in a similar fashion as before, for any $\varepsilon \in (0,\delta)$ we have

$$\begin{gathered}
V(k,\zeta^*(n-k)-\delta+\varepsilon) - V(k,\zeta^*(n-k)-\delta) \le \\
\mathbb{E}[G(N_{\tau_\delta}^k, X_{\tau_\delta}^{\zeta^*(n-k)-\delta+\varepsilon}) - G(N_{\tau_\delta}^k, X_{\tau_\delta}^{\zeta^*(n-k)-\delta})] \, ,
\end{gathered}$$

so that

$$V(k,\zeta^*(n-k)-\delta+\varepsilon) - V(k,\zeta^*(n-k)-\delta) \le \varepsilon \mathbb{E}[G_x(N_{\tau_\delta}^k, \nu)] \, ,$$

where $X_{\tau_\delta}^{\zeta^*(n-k)-\delta} \le \nu_2 \le X_{\tau_\delta}^{\zeta^*(n-k)-\delta+\varepsilon}$. Since $\tau_\delta \xrightarrow{\delta\to 0} 0$ we have $\nu_2 \xrightarrow{\varepsilon\to 0} X_{\tau_\delta}^{\zeta^*(n-k)-\delta}$. Now, dividing the above by $\varepsilon$ and taking the limit as $\varepsilon \to 0$ we obtain

$$V_x(k,\zeta^*(n-k)-\delta) \le \mathbb{E}[G_x(N_{\tau_\delta}^k, X_{\tau_\delta}^{\zeta^*(n-k)-\delta})] \, ,$$

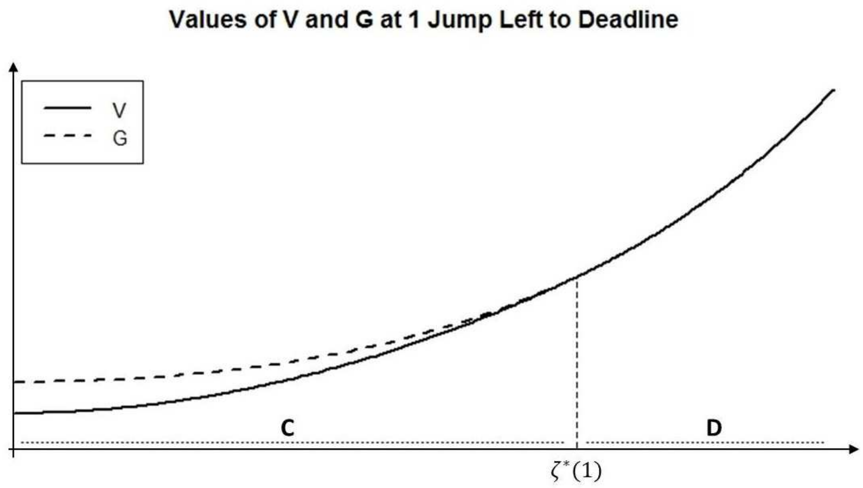


**Figure 4.3:** Values of $V$ and $G$ for different $x$ and fixed single jump to deadline; here, $U(x) = x$, $\lambda = 0$ and $\sigma = \omega = 1$. The smooth pasting of $V$ to $G$ is observed. The stopping set $D$ is the area for which $V(n-1,x) = G(n-1,x)$.

so that

$$\limsup_{\delta\to 0} V_x(k,\zeta^*(n-k)-\delta) \quad \le \quad G_x(k,\zeta^*(n-k))\,.$$

To show that the reverse inequality holds, taking $\varepsilon > 0$, arguments seen before imply that

$$\begin{aligned}&V(k,\zeta^*(n-k)-\delta) - V(k,\zeta^*(n-k)-\delta-\varepsilon) \ge \\ &\mathbb{E}[G_x(N^k_{\tau_\delta},\nu_3)(X^{\zeta^*(n-k)-\delta}_{\tau_\delta} - X^{\zeta^*(n-k)-\delta-\varepsilon}_{\tau_\delta})]\,,\end{aligned}$$

for some $\nu_3 \in [X^{\zeta^*(n-k)-\delta}_{\tau_\delta}, X^{\zeta^*(n-k)-\delta-\varepsilon}_{\tau_\delta}]$. If we divide the above by $\varepsilon$ and take the limit as $\varepsilon \to 0$, the left hand side tends to $V_x(k,\zeta^*(n-k)-\delta)$ and the right hand side does so to

$$\begin{aligned}&\frac{1}{\varepsilon}\mathbb{E}[G_x(N^k_{\tau_\delta},\nu_3)(X^{\zeta^*(n-k)-\delta}_{\tau_\delta} - X^{\zeta^*(n-k)-\delta-\varepsilon}_{\tau_\delta})] \\ = \quad &\mathbb{E}[G_x(N^k_{\tau_\delta},\nu_3)\frac{(\zeta^*(n-k)-\delta)\vee S^\lambda_{\tau_\delta} - (\zeta^*(n-k)-\delta-\varepsilon)\vee S^\lambda_{\tau_\delta}}{\varepsilon}] \\ \xrightarrow{\varepsilon\to 0} \quad &\mathbb{E}[G_x(N^k_{\tau_\delta},X^{\zeta^*(n-k)-\delta}_{\tau_\delta})I_{\{S^\lambda_{\tau_\delta}<\zeta^*(n-k)-\delta\}}]\,,\end{aligned}$$

implying that

$$V_x(k,\zeta^*(n-k)-\delta) \ge \mathbb{E}[G_x(N^k_{\tau_\delta},X^{\zeta^*(n-k)-\delta}_{\tau_\delta})I_{\{S^\lambda_{\tau_\delta}<\zeta^*(n-k)-\delta\}}]\,.$$

Hence, due the right continuity of Poisson processes we get

$$\liminf_{\delta \to 0} V_x(k, \zeta^*(n-k) - \delta) \geq G_x(k, \zeta^*(n-k)) ,$$

concluding the proof. $\square$

### 4.4.3 The Condition of Normal Reflection

**Lemma 4.4.4** (Normal Reflection)**.** *For any fixed $k < n$, $\lim_{x \to 0} V_x(k, x) = 0$.*

***Proof.*** Fix $k < n$. If $\zeta^*(n-k) = 0$, then $V_x(k, x) = G_x(k, x)$ and from equation (4.3.4) we observe that $\lim_{x \to 0} V_x(k, x) = 0$. If $\zeta^*(n-k) > 0$, we apply Itô's formula for non-continuous semimartingales to $V(N_t^k, X_t^0)$, while $(N_t^k, X_t^0)$ is in the continuation set $C$, so that

$$\begin{aligned} V(N_t^k, X_t^0) &= V(k, 0) + \int_0^t V_x(N_s^k, X_s^0) \mathrm{d}X_s^0 + \frac{1}{2} \int_0^t V_{xx}(N_s^k, X_s^0) \mathrm{d}[X^0]_s \\ &+ \sum_{s \leq t} \Delta[V(N_s^k, X_s^0)] , \end{aligned}$$

where, for any multi dimensional Markovian process $(Y_t)_{t \geq 0}$, $\Delta[V(Y_t)] = V(Y_t) - V(Y_{t^-})$, and $Y_{t^-}$ stands for the left-continuous left-limit process of $Y$ in $t$. We note that function $V$ is twice differentiable in $C$ and the limit $\lim_{x \to 0} V_x(k, x)$ exists.

We recall from Section 3.4 that $dX_s^0 = dS_s^\lambda - \lambda ds - dB_s$ is a generalized Itô process so that $[X^0]_s = (\int_0^s \mathrm{d}B_r)^2 = \int_0^s \mathrm{d}r = s$. We plug these expressions appropriately in the previous equation to obtain

$$\begin{aligned} V(N_t^k, X_t^0) - V(k, 0) &= \int_0^t \mathcal{A}_X V(N_s^k, X_s^0) \mathrm{d}s + \int_0^t V_x(N_s^k, X_s^0) \mathrm{d}S_s^\lambda \\ &- \int_0^t V_x(N_s^k, X_s^0) \mathrm{d}B_s + \sum_{s \leq t} \Delta[V(N_s^k, X_s^0)] , \end{aligned} \tag{4.4.15}$$

where the operator $\mathcal{A}_X$ is the infinitesimal generator of the process $X$ given by (3.4.1). Now, we note that jumps in process $N_t^k$ are of size 1. Therefore, the last term in the right hand side of equation (4.4.15) can be modified as follows

$$\sum_{s \leq t} \Delta[V(N_s^k, X_s^0)] = \int_0^t [V(N_{s^-}^k + 1, X_s^0) - V(N_{s^-}^k, X_s^0)] \mathrm{d}N_s^k ,$$

implying that

$$
\begin{aligned}
V(N_t^k, X_t^0) - V(k,0) &= \int_0^t \mathcal{A}_X V(N_s^k, X_s^0) \mathrm{d}s + \int_0^t V_x(N_s^k, X_s^0) \mathrm{d}S_s^\lambda \\
&- \int_0^t V_x(N_s^k, X_s^0) \mathrm{d}B_s \\
&+ \int_0^t [V(N_{s-}^k + 1, X_s^0) - V(N_{s-}^k, X_s^0)] \mathrm{d}N_s^k ,
\end{aligned}
$$

and thus

$$
\begin{aligned}
\frac{\mathbb{E}[V(N_t^k, X_t^0)] - V(k,0)}{t} &= \frac{\mathbb{E}[\int_0^t \mathcal{A}_X V(N_s^k, X_s^0) \mathrm{d}s]}{t} + \frac{\mathbb{E}[\int_0^t V_x(N_s^k, X_s^0) \mathrm{d}S_s^\lambda]}{t} \\
&+ \frac{\mathbb{E}[\int_0^t [V(N_{s-}^k + 1, X_s^0) - V(N_{s-}^k, X_s^0)] \mathrm{d}N_s^k]}{t} ,
\end{aligned} \tag{4.4.16}
$$

for all $t \geq 0$.

Hence, we take on both sides of (4.4.16) the limit as $t \to 0$ to obtain

$$
\begin{aligned}
&\mathcal{A}_Y V(k,0) = \\
&\quad \mathcal{A}_X V(k,0) + V_x(k,0^+) \cdot \lim_{t \to 0} \frac{\mathbb{E}[S_t^\lambda]}{t} + [V(k+1,0) - V(k,0)] \cdot \lim_{t \to 0} \frac{\mathbb{E}[N_t^k - k]}{t} .
\end{aligned} \tag{4.4.17}
$$

Here, for fixed $t > 0$, the random variable $N_t^k - k$ follows a Poisson distribution with rate $\omega t$. Thus, we have $\frac{\mathbb{E}[N_t^k - k]}{t} \xrightarrow{t \to 0} \omega$. Therefore, equation (4.4.17) becomes

$$
\mathcal{A}_Y V(k,0) = \mathcal{A}_X V(k,0) + V_x(k,0^+) \cdot \lim_{t \to 0} \frac{\mathbb{E}[S_t^\lambda]}{t} + \mathcal{A}_N V(k,0) , \tag{4.4.18}
$$

where $\mathcal{A}_N$ stands for the infinitesimal generator of the process $N_t$ described in (4.3.1). Recalling that $\mathcal{A}_Y = \mathcal{A}_X + \mathcal{A}_N$, equation (4.4.18) reduces to

$$
V_x(k,0^+) \cdot \lim_{t \to 0} \frac{\mathbb{E}[S_t^\lambda]}{t} = 0 .
$$

Note that

$$
\mathbb{E}[S_t^\lambda] \geq \mathbb{E}[S_t] - |\lambda| t = \sqrt{t} \mathbb{E}[|B_t|] - |\lambda| t
$$

for all $t \geq 0$, due to $S_t \overset{\text{law}}{=} |B_t| \overset{\text{law}}{=} \sqrt{t} |B_1|$. Thus, dividing the above by $t$ and letting $t \to 0$ yields

$$
\lim_{t \to 0} \frac{\mathbb{E}[S_t^\lambda]}{t} > \lim_{t \to 0} \frac{\mathbb{E}[|B_t|]}{\sqrt{t}} - |\lambda| = \infty,
$$

so that it holds $V_x(k,0^+) = 0$ and the proof is completed. □

## 4.5 A Unique Solution to the Boundary Problem

For all $k \in \{0, ..., n-1\}$, the free boundary problem linked to $V(k,x)$ in (4.2.2) comprises equations (4.4.3) and (4.4.4), along with boundary conditions (4.4.5)-(4.4.7). In this section, we use techniques for solving linear second order differential equations and prove the validity and uniqueness of expressions (4.3.16)-(4.3.18), therefore setting the proof of Theorem 4.3.3.

First, we recall that $\zeta^*$ takes value 0 for $k = n$, so that instantaneous stopping results optimal. This, along with condition (4.4.4), establishes (4.3.17) in the decomposition of $V(k,x)$ whenever $x \geq \zeta^*(n-k)$ or $k = n$. In order to prove the complimentary case when $x < \zeta^*(n-k)$ and $k < n$, we provide a standard solution for $V(k,x)$ in the ordinary differential equation (4.4.3). This is a linear second order differential equation with constant coefficients, we rewrite it as

$$\frac{1}{2}V_{xx}(k,x) - \lambda V_x(k,x) - \omega V(k,x) = -\omega V(k+1,x) .$$

General theory of differential equations in [57] suggests the solution to this equation is of the form $V(k,x) = V_H(k,x) + V_P(k,x)$. Here, $V_H$ is the solution to the associated homogeneous equation and $V_P$ is the non-homogeneous particular solution. Thus, we use the method of variation of parameters on the system of equations

$$\begin{aligned} V_{xx}(k,x) - 2\lambda V_x(k,x) - 2\omega V(k,x) &= 0 , && (4.5.1) \\ V_{xx}(k,x) - 2\lambda V_x(k,x) - 2\omega V(k,x) &= -2\omega V(k+1,x) . && (4.5.2) \end{aligned}$$

### 4.5.1 Solution to the Homogeneous Equation

The solution to $V_H$ is of the form $V_H(k,x) = C_1(k)V_1(k,x) + C_2(k)V_2(k,x)$, with both $V_1(k,x)$ and $V_2(k,x)$ being exponentials of the form $e^{rx}$, plugging these into equation (4.5.1) leads to the characteristic equation

$$r^2 e^{rx} - 2\lambda r e^{rx} - 2\omega e^{rx} = 0 \quad \Rightarrow \quad r^2 - 2\lambda r - 2\omega = 0 ,$$

solved by

$$r = \lambda + \sqrt{\lambda^2 + 2\omega} \quad \text{and} \quad r = \lambda - \sqrt{\lambda^2 + 2\omega} ,$$

so that

$$V_H(k,x) = C_1(k)e^{(\lambda+\sqrt{\lambda^2+2\omega})x} + C_2(k)e^{(\lambda-\sqrt{\lambda^2+2\omega})x}\,, \tag{4.5.3}$$

where $C_1(k)$ and $C_2(k)$ are both functions of $k$.

### 4.5.2 Particular Solution

The method of variation of parameters tells us that we ought to look for a solution such that $V_P(k,x) = u_1(k,x)e^{(\lambda+\sqrt{\lambda^2+2\omega})x} + u_2(k,x)e^{(\lambda-\sqrt{\lambda^2+2\omega})x}$, where $u_1$ and $u_2$ are the solutions to

$$\begin{cases} u_1'(k,x)e^{(\lambda+\sqrt{\lambda^2+2\omega})x} + u_2'(k,x)e^{(\lambda-\sqrt{\lambda^2+2\omega})x} = 0\,, \\ \\ u_1'(k,x)\big(e^{(\lambda+\sqrt{\lambda^2+2\omega})x}\big)' + u_2'(k,x)\big(e^{(\lambda-\sqrt{\lambda^2+2\omega})x}\big)' = -2\omega V(k+1,x)\,. \end{cases}$$

Equivalently,

$$\begin{cases} u_1(k,x) = \displaystyle\int \frac{2\omega V(k+1,x)e^{(\lambda-\sqrt{\lambda^2+2\omega})x}}{\mathcal{W}(e^{(\lambda+\sqrt{\lambda^2+2\omega})x}, e^{(\lambda-\sqrt{\lambda^2+2\omega})x})}\mathrm{d}x\,, \\ \\ u_2(k,x) = -\displaystyle\int \frac{2\omega V(k+1,x)e^{(\lambda+\sqrt{\lambda^2+2\omega})x}}{\mathcal{W}(e^{(\lambda+\sqrt{\lambda^2+2\omega})x}, e^{(\lambda-\sqrt{\lambda^2+2\omega})x})}\mathrm{d}x\,, \end{cases}$$

where $\mathcal{W}(f,g)$ stands for the Wronskian determinant of functions $f$ and $g$, given by

$$\mathcal{W}(f,g)(x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = (fg' - f'g)(x)\,,$$

so that

$$\mathcal{W}(e^{(\lambda+\sqrt{\lambda^2+2\omega})x}, e^{(\lambda-\sqrt{\lambda^2+2\omega})x}) = -2e^{2\lambda x}\sqrt{\lambda^2+2\omega}\,.$$

Hence, we have

$$\begin{aligned} u_1(k,x) &= -\int \frac{\omega\tilde{V}(k+1,x)e^{(\lambda-\sqrt{\lambda^2+2\omega})x}}{e^{2\lambda x}\sqrt{\lambda^2+2\omega}}\mathrm{d}x \\ &= -\frac{\omega}{\sqrt{\lambda^2+2\omega}}\int V(k+1,x)e^{-(\lambda+\sqrt{\lambda^2+2\omega})x}\mathrm{d}x\,, \end{aligned}$$

and

$$\begin{aligned} u_2(k,x) &= \int \frac{\omega\tilde{V}(k+1,x)e^{(\lambda+\sqrt{\lambda^2+2\omega})x}}{e^{2\lambda x}\sqrt{\lambda^2+2\omega}}\mathrm{d}x \\ &= \frac{\omega}{\sqrt{\lambda^2+2\omega}}\int V(k+1,x)e^{-(\lambda-\sqrt{\lambda^2+2\omega})x}\mathrm{d}x\,, \end{aligned}$$

so that $V_P$ is given by

$$\begin{aligned} V_P(k,x) &= u_1(k,x)e^{(\lambda+\sqrt{\lambda^2+2\omega})x} + u_2(k,x)e^{(\lambda-\sqrt{\lambda^2+2\omega})x} \\ &= \frac{\omega}{\Delta}\Big(e^{(\lambda-\Delta)x}R_1(k,x) - e^{(\lambda+\Delta)x}R_2(k,x)\Big) \,, \end{aligned}$$

with $\Delta = \sqrt{\lambda^2+2\omega}$ and functions $R_1$ and $R_2$ given in (4.3.8). This, along with (4.5.3) shows

$$V(k,x) = C_1(k)e^{(\lambda+\Delta)x} + C_2(k)e^{(\lambda-\Delta)x} + \frac{\omega}{\Delta}\Big(e^{(\lambda-\Delta)x}R_1(k,x) - e^{(\lambda+\Delta)x}R_2(k,x)\Big) \,, \tag{4.5.4}$$

the complete general solution for $V(k,x)$ in (4.4.3).

### 4.5.3 Determining a Unique Solution

We use boundary conditions (4.4.5)-(4.4.6) to derive functions $C_1$ and $C_2$ in terms of the optimal stopping boundary $\zeta^*$. We recall that functions $R_1$ and $R_2$ in (4.3.8) are such that

$$\frac{\mathrm{d}R_1(k,x)}{\mathrm{d}x} = V(k+1,x)e^{-(\lambda-\Delta)x} \quad \text{and} \quad \frac{\mathrm{d}R_2(k,x)}{\mathrm{d}x} = V(k+1,x)e^{-(\lambda+\Delta)x} \,.$$

Then, from (4.5.4) above we have

$$\begin{aligned} V_x(k,x) &= (\lambda+\Delta)C_1(k)e^{(\lambda+\Delta)x} + (\lambda-\Delta)C_2(k)e^{(\lambda-\Delta)x} \\ &+ \frac{\omega}{\Delta}\Big((\lambda-\Delta)e^{(\lambda-\Delta)x}R_1(k,x) - (\lambda+\Delta)e^{(\lambda+\Delta)x}R_2(k,x)\Big) \,. \end{aligned} \tag{4.5.5}$$

Note that $x \mapsto V(k,x)$ is a continuous mapping; then, the general form solutions for $V$ and $V_x$ in (4.5.4) and (4.5.5), along with (4.4.5)-(4.4.6), gives

$$\begin{aligned} G(k,\zeta^*(n-k)) &= C_1(k)e^{(\lambda+\Delta)\zeta^*(n-k)} + C_2(k)e^{(\lambda-\Delta)\zeta^*(n-k)} \\ &+ \frac{\omega}{\Delta}e^{(\lambda-\Delta)\zeta^*(n-k)}R_1(k,\zeta^*(n-k)) \\ &- \frac{\omega}{\Delta}e^{(\lambda+\Delta)\zeta^*(n-k)}R_2(k,\zeta^*(n-k)) \,, \end{aligned} \tag{Eq. A}$$

and

$$\begin{aligned} G_x(k,\zeta^*(n-k)) &= (\lambda+\Delta)C_1(k)e^{(\lambda+\Delta)\zeta^*(n-k)} + (\lambda-\Delta)C_2(k)e^{(\lambda-\Delta)\zeta^*(n-k)} \\ &+ \frac{\omega}{\Delta}(\lambda-\Delta)e^{(\lambda-\Delta)\zeta^*(n-k)}R_1(k,\zeta^*(n-k)) \\ &- \frac{\omega}{\Delta}(\lambda+\Delta)e^{(\lambda+\Delta)\zeta^*(n-k)}R_2(k,\zeta^*(n-k)) \,. \end{aligned} \tag{Eq. B}$$

Here, taking $A \cdot (\lambda - \Delta) - B$ and $A \cdot (\lambda + \Delta) - B$ yields

$$\begin{aligned}
C_1(k) &= \frac{\omega}{\Delta} R_2(k, \zeta^*(n-k)) \\
&+ \left(\tilde{G}_x(k, \zeta^*(n-k)) - (\lambda - \Delta)\tilde{G}(k, \zeta^*(n-k))\right) \cdot \frac{e^{-(\lambda+\Delta)\zeta^*(n-k)}}{2\Delta} ,
\end{aligned}$$

and

$$\begin{aligned}
C_2(k) &= -\frac{\omega}{\Delta} R_1(k, \zeta^*(n-k)) \\
&- \left(\tilde{G}_x(k, \zeta^*(n-k)) - (\lambda + \Delta)\tilde{G}(k, \zeta^*(n-k))\right) \cdot \frac{e^{-(\lambda-\Delta)\zeta^*(n-k)}}{2\Delta} ,
\end{aligned}$$

settling (4.3.14) and (4.3.15).

Finally, note that $\lim_{x \to 0} R_i(k, x) = 0$ for $i \in \{1, 2\}$. Then, normal reflection condition in (4.4.7) yields integral equation in (4.3.18) for optimal boundary $\zeta^*$ to solve, i.e.

$$V_x(k, x) \xrightarrow{x \to 0} (\lambda + \Delta) C_1(k) + (\lambda - \Delta) C_2(k) = 0,$$

which settles the last result in Theorem 4.3.3.

## 4.6 Exponential Terminal Time

In the special case when $n = 1$, so that $T$ is an exponentially distributed random variable, the stopping boundary will take a constant value. This is a common set up within stopping problems with financial applications and is known as *canadization* of the terminal time (see for example [17, 36]). Its advantage lies in that the dimensionality of the problem is reduced to 1, so that the existence of a stopping set is easy to justify. In this case, a simplified result derived from Theorem 4.3.3 above can be offered.

Let the gain function $G$ be given by

$$G(x) = U(e^{\sigma x}) + \sigma\omega \int_0^\infty \int_x^\infty e^{\sigma z} U'(e^{\sigma z})(1 - F_{S_T^\lambda}(z))\, \mathrm{d}z e^{-\omega T} \mathrm{d}T \,. \tag{4.6.1}$$

In addition, define functions $R_1$ and $R_2$ as

$$R_1(x) = \int_0^x U(e^{\sigma r}) e^{-(\lambda-\Delta) r} \mathrm{d}r \quad \text{and} \quad R_2(x) = \int_0^x U(e^{\sigma r}) e^{-(\lambda+\Delta) r} \mathrm{d}r \,, \tag{4.6.2}$$

and let functions $C_1$ and $C_2$, in terms of the corresponding constant stopping boundary $\zeta^*$ and parameters $(\lambda, \sigma, \omega)$, be given by

$$C_1 = \frac{\omega}{\Delta} R_2(\zeta^*) + (G_x(\zeta^*) - (\lambda - \Delta) G(\zeta^*)) \cdot \frac{e^{-(\lambda+\Delta)\zeta^*}}{2\Delta} , \tag{4.6.3}$$

and

$$C_2 = -\frac{\omega}{\Delta}R_1(\zeta^*) - (G_x(\zeta^*) - (\lambda+\Delta)G(\zeta^*)) \cdot \frac{e^{-(\lambda-\Delta)\zeta^*}}{2\Delta}\,, \tag{4.6.4}$$

with $\Delta = \sqrt{\lambda^2 + 2\omega}$.

**Corollary 4.6.1.** *Let $U \in \mathcal{U}$ be such that Assumption 4.3.2 is satisfied, then the underlying extended optimal stopping problem $V(x)$, derived from setting $k = 0$ and $n = 1$ in* (4.2.2), *is such that*

$$V(x) = C_1 e^{(\lambda+\Delta)x} + C_2 e^{(\lambda-\Delta)x} + \frac{\omega}{\Delta} \cdot \left\{ e^{(\lambda-\Delta)x} R_1(x) - e^{(\lambda+\Delta)x} R_2(x) \right\}, \tag{4.6.5}$$

*if $x < \zeta^*$, and*

$$V(x) = G(x)\,, \tag{4.6.6}$$

*if $x \geq \zeta^*$, where $\zeta^*$ stands for the optimal stopping boundary. Function $G(x)$ is given by* (4.6.1), *and functionals $R_1, R_2, C_1$ and $C_2$ are given by* (4.6.2), (4.6.3) *and* (4.6.4) *respectively. Finally, $\zeta^*$ can be identified as the only positive solution to the integral equation*

$$(\lambda+\Delta)C_1 + (\lambda-\Delta)C_2 = 0\,. \tag{4.6.7}$$

## 4.7 Discussion

This Chapter has analysed, in a time-randomized context, generalizations of optimal prediction problems to a family of utility functions meeting certain conditions. A time-independent family of stopping problems has been derived and the existence of optimal stopping boundaries discussed, characterizing boundaries and optimal value functions for problems meeting certain criteria relating the operator $\mathcal{A}_Y$ in (4.3.2) and the gain function $G$. Results in Theorem 4.3.3 allow for the iterative computation of a stopping boundary $\zeta^*$ associated to the problem of optimally halting a stochastic process $Z$ driven by a geometric Brownian motion with drift $\mu$ and variance $\sigma$.

In this case, we recall that the optimal stopping time comes at the first hitting time of the diffusion $X^0$ in (3.2.2) to the boundary $\zeta^*$. We note that different sets of parameters $(\mu, \sigma)$ defining $Z$ must be linked to different boundaries; these can be either more permissive, allowing for a broader range of values of $X_t$ not to fall in the stopping set $D$, or more restrictive, reducing the values of $\zeta^*$ and therefore forcing the halt at a slight take off of $X$ from 0.

### 4.7.1 Existence

For a choice of utility function $U \in \mathcal{U}$, analytically tackling the veracity of conditions in Definition 4.0.1 and Assumption 4.3.2 can be a daunting challenge, especially due to the complexity of the integral term in (4.0.2) and equation $\mathcal{A}_Y$ in (4.3.2). A computational approach to these conditions does on the other hand show that several families of functions meet these requirements. An example is given by the family of combined power utility functions, i.e. functions $U(x)$ of the form

$$U(x) = \sum_{i=1}^{m} \alpha_i x^{\delta_i} \ ,$$

with $m \geq 0$, $0 \leq \delta_i < 1$ for all $i \in \{1, 2, ..., m\}$ (strictly concave) or $1 < \delta_i$ for all $i \in \{1, 2, ..., m\}$ (strictly convex); $0 \leq \alpha_i \leq 1$ and $\sum_{i=1}^{m} \alpha_i = 1$. Direct numerical analysis of functions $\mathcal{A}_Y G(k, x)$ in (4.3.5) reveals the existence of common properties for measures of this kind. For all $(k, x) \in \{1, 2, ...n\} \times \mathbb{R}^+$ there exist $u_1, u_2 \in \mathbb{R}_+$ with $u_1 < u_2$ so that

$$\begin{aligned} \mathcal{A}_Y G(k, x) &< 0 \qquad \text{if } \mu \geq u_2 \ , \text{ and} \\ \mathcal{A}_Y G(k, x) &> 0 \qquad \text{if } \mu \leq u_1 \ . \end{aligned}$$

Moreover, for any $\mu \in (u_1, u_2)$ conditions in Assumption (4.3.2) are met. It follows that for this family of functions the optimal stopping set $D$ can partially be defined as

$$D = \{n\} \times \mathbb{R}^+ \qquad \text{if } \mu \geq u_2 \ , \tag{4.7.1}$$

$$D = \{0, 1, ..., n\} \times \mathbb{R}^+ \qquad \text{if } \mu \leq u_1 \ , \text{ and} \tag{4.7.2}$$

$$D = \{(k, x) \in \{0, 1, ..., n\} \times \mathbb{R}^2 \ : \ x > \zeta^*(n-k)\} \qquad \text{if } \mu \in (u_1, u_2) \ . \tag{4.7.3}$$

We recall from the introduction to this thesis that it was introduced in [54] and later on extended in [21], that under a fixed terminal time set-up, the problem of optimizing the ratio within a geometric Brownian motion and its ultimate maximum until deadline (or its inverse), led to the categorization of processes into first 2 and later on 3 different groups. The solution to our randomized problem for the family of combined power utility functions suggests that we can make use of a virtually similar categorization to that in [21] and [54], as exposed in the set of equations (4.7.1)-(4.7.3).

### 4.7.2 Multiple Bounding Functions

It is possible to extend the scope of this chapter to functions $U$ in $\mathcal{U}$ not meeting conditions listed in Assumption 4.3.2. The shape of sets $\Theta$ and $\Upsilon$ in (4.3.6) and (4.3.7) is dependent on the choice of $U$, and this alters the construction of the stopping set $D$. For instance, the choice of squared logarithmic utility function $U(x) = \log^2(x)$, leading to the randomized terminal time optimal stopping problem

$$\inf_{\tau\in[0,T]} \mathbb{E}[(B^\lambda_\tau - \max_{0\le s\le T} B^\lambda_s)^2] \ , \tag{4.7.4}$$

shows the existence of two bounding points for the set $\Theta$ at any fixed value of $k$, when $\lambda > 0$ and is close to 0 (see Figure 4.4). This is consistent with results in

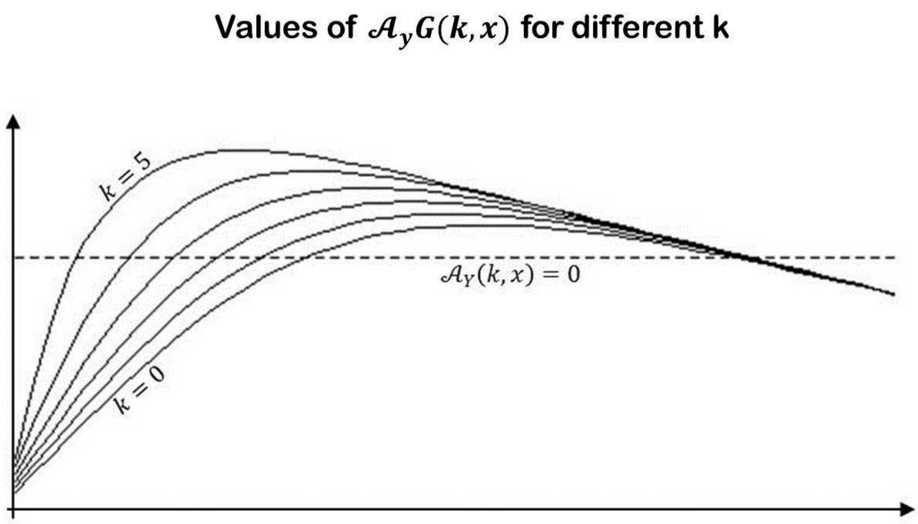


**Figure 4.4:** Value of function $\mathcal{A}_Y G(k,x)$ for varying $x$ and different fixed values of $k$. Here, $\lambda > 0$ and close to 0; in addition, $\sigma = 1$, $\omega = 3$, $n = 6$ and $U(x) = \log^2(x)$. We notice the existence of two bounding functions for sets $\Theta$ and $\Upsilon$.

[18] that analyse the stopping problem (4.7.4) under a fixed terminal time $T$. Such observation suggests that there may exist two stopping boundaries, which in turn shifts

the reduction procedure to an alternative free boundary problem to that in (4.4.3) and (4.4.4).

### 4.7.3 Approximating a Fixed Terminal Time

Finally, results in this chapter also make it possible to build numerical approximations of some fixed terminal time set-up optimal prediction problems. We recall from Section 2.4 that if a randomized $T$ is modelled by the $n^{\text{th}}$ jump in a Poisson process with rate $\omega = n/T_*$, for fixed $T_*$, then it holds

$$\mathbb{E}[T] = T_* \quad \text{and} \quad V[T] = \frac{T_*}{n} ,$$

so that $T \to T_*$ as $n \to \infty$. Thus, it is possible to make use of low-variance Gamma distributed estimates to the true deadlines. Figure 4.5 presents an approximation of the stopping boundary for the fixed terminal time problem $V$ with $U(x) = x$ as first analysed in [21], when $\lambda = -0.25$.


**Figure 4.5:** Estimate of continuous optimal stopping boundary for fixed terminal time $T_* = 10$, with $\lambda = -0.25$ and $\sigma = 1$. The amount of breaks used to build this estimate is 60, so that $\omega = 6$. Time $\tau$ stands for the optimal stopping time to this process $X^x$.

PART II

# APPLICATION OF MARKOV DECISION PROCESSES TO FINANCE

CHAPTER 5

# RESULTS ON MARKOV DECISION PROCESSES

We begin the second part of this thesis offering an introduction to the notation and approach to discrete-time Markov decision processes (MDPs). We will also summarize a set of results presented in the theory in [51] and [13], which will be of use in analysing an MDP derived from a portfolio optimization problem in Chapter 6. In addition, we will introduce and describe computational procedures that aim to approximate optimal solutions within the context of these problems.

In what follows, we will be restricting ourselves to the theory strictly relevant to this thesis, and we therefore present results that apply to infinite horizon models under non-discounted expected total reward criterion. Additionally, we focus on *Markovian decision rules* that comply with the Markovian set up of our problem, and we ignore the existence of *history-dependent decision rules* that allow for a formulation in greater generality.

## 5.1 THE BOREL MODEL

Let $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$ denote a probability space where:

- The sample space $\Omega$ is given by $\{E \times \mathcal{A}\}^\infty$, with $E$ and $\mathcal{A}$ to be defined below.
- The $\sigma$-algebra $\mathcal{B}(\Omega)$ is given by the Borel measurable subsets of $\{E \times \mathcal{A}\}^\infty$.
- Measure $\mathbb{P}$ is a probability measure on $\mathcal{B}(\Omega)$.

In addition, let $X = (X_n)_{n\geq 0}$ and $Y = (Y_n)_{n\geq 0}$ be sequences of random variables taking values in $E$ and $\mathcal{A}$ respectively, so that

$$X_n(\omega) = x_n \quad \text{and} \quad Y_n(\omega) = \alpha_n \ ,$$

for all $n \geq 0$. Here, an element $\omega \in \Omega$ is a sequence of observations

$$\omega = (x_0, \alpha_0, x_1, \alpha_1, ...) \ ,$$

taking values in $\{E \times \mathcal{A}\}^\infty$.

A technical formulation of an MDP in this thesis, extending the presentation in the introduction section, consists of a *history process* $Z = (Z_n)_{n\geq 0}$, describing the joint stochastic evolution of *states* $X_n \in E$ and *actions* $Y_n \in \mathcal{A}$ in a *system*, defined by

$$Z_0(\omega) = x_0 \quad \text{and} \quad Z_n(\omega) = (x_0, \alpha_0, x_1, \alpha_1, ..., x_n) \ ,$$

for $n \geq 1$. It incorporates the following components:

- A Borel space $(E, \mathcal{B}(E))$ of the *state space* $E$ and its Borel subsets $\mathcal{B}(E)$. We assume $E$ to be an unbounded subset of the Euclidean space.
- A Borel space $(\mathcal{A}, \mathcal{B}(\mathcal{A}))$ of the *action space* $\mathcal{A}$ and its Borel subsets $\mathcal{B}(\mathcal{A})$, along with a collection of *admissible action* sets $D(x) \in \mathcal{B}(\mathcal{A})$ such that there exists a measurable function $f : E \to \mathcal{A}$, with $f(x) \in D(x)$ for all $x \in E$.
- A family of measures $\mathbb{P}(D(x))$ on the Borel subsets $\mathcal{B}(D(x))$, for all $x \in E$.
- Conditional *transition probability functions* $Q_n(\cdot|x, \alpha)$, which for all $n \geq 0$ satisfy
  - i) $Q_n(B|\cdot, \cdot)$ is measurable with respect to $\mathcal{B}(E \times D(x))$, for $B \in \mathcal{B}(E)$.
  - i) $Q_n(B|\cdot, \cdot)$ is integrable with respect to each $\mathcal{P} \in \mathbb{P}(D(x))$, for all $x \in E$ and $B \in \mathcal{B}(E)$.
- Real-valued *reward functions* $R_n(x, \alpha)$, which for all $n \geq 0$ satisfy
  - i) $R_n(\cdot, \cdot)$ is measurable with respect to $\mathcal{B}(E \times D(x))$.
  - i) $R_n(\cdot, \cdot)$ is integrable with respect to each $\mathcal{P} \in \mathbb{P}(D(x))$, for all $x \in E$.

Finally, it includes a set of Markovian *one-step deterministic policies* or *decision rules* $F$ given by

$$F_D = \{f : E \to \mathcal{A} \mid f \text{ measurable and } f(x) \in D(x) \text{ for all } x \in E\} ,$$

and a set of Markovian *one-step randomized policies* or *decision rules* given by

$$F_R = \{f : E \to \mathbb{P}(\mathcal{A}) \text{ with } f(x) = \mathcal{P}_x \in \mathbb{P}(\mathcal{A}) \mid \mathcal{P}_x(D(x)) = 1 \text{ for all } x \in E\} ,$$

so that for all $f \in F_R$

$$R_n(x_n, f(x_n)) = \int_{D(x)} R_n(x_n, \alpha)\mathcal{P}_{x_n}(\mathrm{d}\alpha) ,$$

and

$$Q_n(B|x_n, f(x_n)) = \int_{D(x)} Q_n(B|x_n, \alpha)\mathcal{P}_{x_n}(\mathrm{d}\alpha) ,$$

for $B \in \mathcal{B}(E)$. We note that $F_D \subseteq F_R$ always, since deterministic policies can be attained as randomized policies with density atoms.

A set of deterministic *Markovian policies* $\Pi_D$ is defined by the Cartesian product of the corresponding decision set $F_D$; analogously, randomized Markovian policies $\Pi_R$ are defined by the Cartesian product of $F_R$, so that $\Pi_D \subseteq \Pi_R$. Within the scope of this thesis, a *Markov decision problem* deals with the problem of identifying the optimal deterministic Markovian policy $\pi = (f_0, f_1, ...) \in \Pi_D$, if any, that maximizes the expected sum of rewards, which in view of (1.2.1) is given by

$$\mathbb{E}_x^{\pi}\Big[\sum_{k=0}^{\infty} R_k(X_k, Y_k)\Big]$$

for all $x \in E$, where the expectation is taken over the probability distribution $\mathbb{P}|_{X_0=x}$ induced by the policy $\pi \in \Pi_D$, and $Y_k = f_k(X_k)$. However, it will also be necessary to invoke randomized policies $\pi \in \Pi_R$ at certain points in our work.

We will generally refer to such an MDP as to be modelled by the 4-tuple

$$(E, A, Q_{\cdot}(\cdot,\cdot), R_{\cdot}(\cdot,\cdot)) .$$

We lastly note that in general, the theory of Markov decision processes is sufficiently rich as to be addressed without confronting such mathematical subtleties, and we will offer simplified notation whenever possible.

## 5.2 RESULTS FROM MDP THEORY

Results in this section refer to the theory of infinite horizon Markov decision processes and can be found in [12], [51] (Chapters 5 to 7) and [13] (Chapter 7). Most theory for MDPs deals with countable state-space models with bounded rewards; we will however, for future reference, summarize a set of results of interest that allow for the implementation of solution algorithms when the state space is not finite and rewards unbounded. All results will be stated without proof.

An important particularity of infinite horizon MDPs is that they are often simpler to solve than finite horizon models, and generally admit a *stationary* optimal policy. We say that a deterministic Markovian policy $\pi = (f_0, f_1, ...) \in \Pi_D$ is stationary, and denote it $(f)_{n\geq 0}$, if it uses a single decision rule at every *decision epoch* $n \geq 0$, i.e.

$$\pi = (f_n)_{n\geq 0} = (f)_{n\geq 0} = (f, f, ...) \text{ for some } f \in F_D \,,$$

and we note that randomized Markovian stationary policies are defined analogously. When certain structure assumptions are satisfied, an infinite horizon model can actually be seen as an approximation of its finite horizon equivalent, providing us with tools in the core of MDP theory, such as reward and value iteration, in order to attain approximations of the solution.

In the following we assume to be analysing a stationary problem, that is, the reward functions, transition probabilities and decision sets do not vary between epochs $n \geq 0$; also, we will assume that rewards are non-negative. We will refer to the *value of a policy* $\pi$ as the function $v^\pi : E \to \mathbb{R}$ defining the total expected reward

$$v^\pi(x) = \mathbb{E}_x^\pi\Big[\sum_{k=0}^{\infty} R(X_k, Y_k)\Big] \,,$$

for all $x \in E$; in addition, we call the *value of the MDP* the function $v : E \to \mathbb{R}$ defined as the optimal total expected rewards over policies $\pi \in \Pi_D$, given by

$$v(x) = \sup_{\pi\in\Pi_D} v^\pi(x) \,, \tag{5.2.1}$$

for all $x \in E$. We note that, even if a value $v$ exists for the MDP, it does not imply that an optimal policy in $\Pi_D$ will, since a different $\pi \in \Pi_D$ may attain the maximum for each $x \in E$. Our aim is to determine a policy $\pi^* \in \Pi_D$, if possible, that satisfies $v^{\pi^*}(x) \geq v(x)$ for all $x \in E$.

A key notion for characterizing the value function of an MDP is the *principle of optimality* or *Bellman equation*. This concept plays a central role in the theory of *contracting* MDPs, and presents a necessary condition for optimality, associated with the optimization method of dynamic programming in discrete-time decision making problems. If applicable, the value of the MDP at a certain state $x \in E$ may be expressed in terms of the pay-off $R$ from some initial action, plus the value of the remaining decision problem resulting from it, so that in a stationary problem $v$ satisfies

$$v(x) = (\mathcal{T}v)(x) , \tag{5.2.2}$$

where $\mathcal{T}$ denotes the *maximal reward operator*, given by

$$(\mathcal{T}v)(x) = \sup_{\alpha \in D(x)} \left\{ R(x,\alpha) + \int_E v(y) Q(\mathrm{d}y|x,\alpha) \right\} , \tag{5.2.3}$$

for all $x \in E$. The term within brackets in (5.2.3) is usually denoted

$$(\mathcal{L}v)(x|\alpha) = R(x,\alpha) + \int_E v(y) Q(\mathrm{d}y|x,\alpha) , \tag{5.2.4}$$

and referred to as the *reward operator*. Generally, we aim to construct a function space $\mathbb{V}$ so that, under reasonable conditions on states, actions, rewards and transitions to ensure that $\mathcal{T}v \in \mathbb{V}$, equation (5.2.2) has a unique solution and equals the value of the MDP. Another result of special relevance is Banach's fixed point theorem, which we reproduce here.

**Theorem 5.2.1** (Banach's Fixed Point Theorem)**.** *Let $(\mathbb{B}, d)$ be a complete metric space with a contraction mapping $T : \mathbb{B} \to \mathbb{B}$, i.e. such that there exists a constant $c \in [0,1)$ so that*

$$d(T(x), T(y)) \leq c d(x,y) ,$$

*for all $x, y \in \mathbb{B}$. Then, $T$ admits a unique fixed point $x^*$ in $\mathbb{B}$ (i.e. $T(x^*) = x^*$). Furthermore, $x^*$ can be found starting with an arbitrary element $x_0 \in \mathbb{B}$ and defining a sequence $(x_n)_{n \geq 0}$, with $x_n = T(x_{n-1})$ for $n \geq 1$, so that*

$$x_n \to x^* \quad as \quad n \to \infty .$$

Now, for any arbitrary measurable function $w : E \to \mathbb{R}$ satisfying $\inf_{x \in E} w(x) > 0$, define the *weighted supremum norm* $\| \cdot \|_w$ for functions $g : E \to \mathbb{R}$ by

$$\|g\|_w = \sup_{x \in E} \frac{|g(x)|}{w(x)} . \tag{5.2.5}$$

Also, let $\mathbb{M}_w(E)$ be the space of measurable real-valued functions given by

$$\mathbb{M}_w(E) = \{ g : E \to \mathbb{R} \; : \; g \text{ measurable and } \|g\|_w < \infty \} .$$

We note that convergence in $\mathbb{M}_w(E)$ implies pointwise convergence, since for each $x \in E$ and sequence of functions $(g_n)_{n \geq 0}$ with $g_n \in \mathbb{M}_w(E)$ for $n \geq 0$, it holds

$$|g_n(x) - g(x)| < \varepsilon w(x) \quad \text{if} \quad \|g_n - g\|_w < \varepsilon$$

for some $\varepsilon > 0$. Therefore, every Cauchy sequence of elements in $\mathbb{M}_w(E)$ converges to an element on its set, so that $\mathbb{M}_w(E)$ is a Banach space.

**Assumption 5.2.2.** *There exist constants $\mu, \kappa \in \mathbb{R}_+$ so that*

*i)* $\sup_{\alpha \in D(x)} R(x, \alpha) < \mu w(x)$ *for all* $x \in E$.

*ii)* $\int_E w(y) Q(\mathrm{d}y | x, \alpha) \leq \kappa w(x)$ *for all* $\alpha \in D(x)$ *and* $x \in E$.

In view of this assumption, function $w$ is sometimes referred to as a *bounding function* for the MDP. Moreover, the Markov decision process is called *contracting* if $\kappa < 1$, and guarantees that the optimization problem $v$ in (5.2.1) is well-defined, since for all $\pi \in \Pi_D$ it holds

$$\begin{aligned} v^\pi(x) &= R(x, \alpha_0) + \mathbb{E}_x^\pi \Big[ \sum_{k=1}^{\infty} R(X_k, Y_k) \Big] = R(x, \alpha_0) + \int_E v^\pi(y) Q(\mathrm{d}y | x, \alpha_0) \\ &\leq \mu w(x) + \mu \kappa w(x) + \mu \kappa^2 w(x) + \cdots = \frac{\mu}{1 - \kappa} w(x) < \infty , \end{aligned}$$

for all $x \in E$, and $\|v^\pi\| \leq \frac{\mu}{1-\kappa}$ so that $v \in \mathbb{M}_w(E)$. It is also guaranteed that the contracting infinite horizon MDP can be approximated by a finite horizon model, since it can be shown in a similar fashion that there exists a constant $c > 0$ such that

$$\lim_{n \to \infty} \mathbb{E}_x^\pi \Big[ \sum_{k=n}^{\infty} R(X_k, Y_k) \Big] \leq c \lim_{n \to \infty} \kappa^n w(x) = 0 .$$

**Lemma 5.2.3.** *Under Assumption 5.2.2, with $\kappa < 1$, it holds that*

$$\|\mathcal{T}g_1 - \mathcal{T}g_2\|_w < \|g_1 - g_2\|_w$$

*for any arbitrary $g_1, g_2 \in \mathbb{M}_w(E)$, where $\mathcal{T}$ is the maximal reward operator in* (5.2.3). *Furthermore, given a stationary policy $\pi = (f)_{n\geq 0} \in \Pi_D$ we have*

$$v^{\pi} = \lim_{n\to\infty}(\mathcal{T}_f \circ \overset{n}{\ldots} \circ \mathcal{T}_f)g = \lim_{n\to\infty}\mathcal{T}_f^n g\,, \tag{5.2.6}$$

*for all $g \in \mathbb{M}_w(E)$, and $v^{\pi}$ is the unique fixed point of $\mathcal{T}_f$ in $\mathbb{M}_w(E)$, with*

$$(\mathcal{T}_f g)(x) = (\mathcal{L}g)(x|f(x))\,, \tag{5.2.7}$$

*for all $g \in \mathbb{M}_w(E)$ and $x \in E$.*

Note that result (5.2.6) is a direct application of Banach's fixed point Theorem 5.2.1, when the metric is induced by the norm $\|\cdot\|_w$, since $\mathcal{T}_f$ is contracting. The result suggests that it is possible to determine the value of a stationary deterministic policy $\pi$ by repetitive application of $\mathcal{T}_f$ to an arbitrary initial function in $\mathbb{M}_w(E)$. Next, we reproduce a verification result that avoids general measurability problems; it states that candidates for the optimal solution to problem (5.2.1) are given by fixed points of the maximal reward operator.

**Theorem 5.2.4.** *Under Assumption 5.2.2, with $\kappa < 1$, let $g \in \mathbb{M}_w(E)$ be a fixed point of $\mathcal{T} : \mathbb{M}_w(E) \to \mathbb{M}_w(E)$. If there exists a decision rule $f \in F_D$ such that*

$$(\mathcal{T}g)(x) = (\mathcal{T}_f g)(x)\,,$$

*for all $x \in E$, with $\mathcal{T}_f$ as in* (5.2.7)*; then the value of the MDP is such that $v = g$ and $\pi = (f)_{n\geq 0} \in \Pi_D$ is an optimal deterministic stationary policy.*

In addition, the following structure theorem will provide us with means to determine the existence of optimal stationary policies to our problem in Chapter 6, along with proof for the usefulness of iterative methods, such as value iteration, in order to approximate optimal solutions.

**Theorem 5.2.5.** *Under Assumption 5.2.2, with $\kappa < 1$, let $\mathbb{C}_w(E) \subset \mathbb{M}_w(E)$ be a closed subset such that* *i)* $0 \in \mathbb{C}_w(E)$,

*ii)* $\mathcal{T} : \mathbb{C}_w(E) \to \mathbb{C}_w(E)$,

*iii)* *for all* $g \in \mathbb{C}_w(E)$ *there exists an* $f \in F_D$ *such that* $\mathcal{T}g = \mathcal{T}_f g$,

*with* $\mathcal{T}_f$ *as in* (5.2.7). *Then, it holds that*

*a)* $v \in \mathbb{C}_w(E)$ *and* $v = \mathcal{T}v$ *(value iteration).*

*b)* $v$ *is the unique fixed point of* $\mathcal{T}$ *in* $\mathbb{C}_w(E)$.

*c)* $v$ *is the smallest function* $g \in \mathbb{C}_w(E)$ *such that* $g \geq \mathcal{T}g$ *(superharmonic).*

*d)* *For all* $g \in \mathbb{C}_w(E)$

$$\|v - \mathcal{T}^n g\|_w \leq \frac{\kappa^n}{1-\kappa} \|\mathcal{T}g - g\|_w \, .$$

*e)* *There exists an optimal deterministic stationary policy* $\pi = (f)_{n\geq 0} \in \Pi_D$ *such that* $v = v^{\pi}$.

In view of Theorem 5.2.5, our efforts in Chapter 6 will be directed towards the construction of a complete metric space $(\mathbb{V}, d)$ satisfying conditions $(i) - (iii)$.

## 5.3 Piecewise Deterministic Models and Control Functions

Stochastic processes that evolve through random jumps at random time points and are governed by a deterministic flow in between jumps are referred to as *piecewise deterministic Markov processes* (PDMPs). When a continuous time optimization problem solely relates to piecewise deterministic stochastic processes, whose jump behaviour can be controlled, it can then be reduced to a discrete-time MDP and treated with previously introduced methods (see [1]). However, several mathematical complications arise as the action space becomes a function space; here, we reproduce some technical results found in [13] (Chapter 8) and [59] that ensure tractability of the problem.

If $(T_n)_{n\geq 0}$ denotes jump times in a PDMP, the evolution of the process up to time $T_{n+1}$ is known to the decision maker at time $T_n$, for all $n \geq 0$, so that he can fix a control action $\alpha(t)$ for all $T_n + t \leq T_{n+1}$. This is the basis for treating a continuous-time control problem as a discrete-time MDP, where an $\alpha \in \mathcal{A}$ is thought of as the action at time $T_n$ or epoch $n \geq 0$, and the action space $\mathcal{A}$ is given by a function space

$$\mathcal{A} = \{\alpha : \mathbb{R}^+ \to \mathcal{U} \ : \ \alpha \text{ measurable}\} ,$$

where $\mathcal{U}$ is the *control action space*, assumed to be a compact Borel subset of a Polish space, a separable and completely metrizable topological space. In addition, it is shown in [59] that for $\mathcal{A}$ to become a Borel space fitting previously presented theory, it can be endowed with the coarsest $\sigma$-algebra (with the fewest open sets) such that mappings

$$\alpha \mapsto \int_0^\infty e^{-t} w(t, \alpha_t) \mathrm{d}t$$

are measurable for all bounded and measurable functions $w : \mathbb{R}_+ \times \mathcal{U} \to \mathbb{R}$.

### Relaxed Controls and the Young Topology

In addition to deterministic controls, it is also necessary to consider the space of randomized controls. If allowed, a decision maker could choose to fix a randomized control action at a decision epoch $n \geq 0$ at time $T_n$. Doing so, he fixes a probability distribution $\rho(t) \in \mathbb{P}(\mathcal{U})$ for all $T_n + t \leq T_{n+1}$, where $\mathbb{P}(\mathcal{U})$ is the set of probability measures on the Borel subsets $\mathcal{B}(\mathcal{U})$. Then, we think of $\rho \in \mathcal{R}$ as an action at time $T_n$, where the function space $\mathcal{R}$ is given by

$$\mathcal{R} = \{\rho : \mathbb{R}^+ \to \mathbb{P}(\mathcal{U}) \ : \ \rho \text{ measurable}\} . \tag{5.3.1}$$

Trivially, we have $\mathcal{A} \subseteq \mathcal{R}$, since all deterministic controls are attainable in $\mathcal{R}$ through the adoption of measures with single mass points.

The set $\mathcal{R}$ is endowed with the *Young topology*, the coarsest such that for all $\rho \in \mathcal{R}$, the mapping

$$\rho \mapsto \int_0^\infty \int_{\mathcal{U}} g(t, u) \rho_t(\mathrm{d}u) \mathrm{d}t$$

is continuous for all functions $g : [0, \infty] \times \mathcal{U} \to \mathbb{R}$ which are measurable in the first argument and continuous in the second and satisfy

$$\int_0^\infty \max_{u \in \mathcal{U}} |g(t, u)| \mathrm{d}t < \infty . \tag{5.3.2}$$

With respect to this topology $\mathcal{R}$ is a separable, metric and compact Borel space.

Moreover, for a sequence of controls $(\rho_n)_{n\geq 1} \subset \mathcal{R}$ and fixed control $\rho \in \mathcal{R}$, $\lim_{n\to\infty} \rho_n = \rho$ if and only if

$$\lim_{n\to\infty} \int_0^\infty \int_{\mathcal{U}} g(t,u)\rho_{n,t}(\mathrm{d}u)\mathrm{d}t = \int_0^\infty \int_{\mathcal{U}} g(t,u)\rho_t(\mathrm{d}u)\mathrm{d}t\,, \tag{5.3.3}$$

for all functions $g$ satisfying (5.3.2).

## 5.4 Solution Algorithms and Value Iteration

There exists a whole family of dynamic programming algorithms commonly used to solve MDPs, notable variants of these include value iteration and Howard's policy improvement algorithm. These algorithms require storage for an indexed array of values $V$, along with an array of policies $\pi$. When concluded, optimal policies will be stored in $\pi$, and $V$ will contain the optimal sum of the rewards attained according to each policy in $\pi$.

Computational approximations of value functions and optimal policies in Chapter 6 have been obtained through the method of value iteration, combined with linear interpolation methods that approximate values over a discretized grid of the continuous state space. The method starts with an arbitrary value function $V_0$ meeting certain conditions of continuity and concavity, and uses equation (5.2.2) to update its values at a next stage, while storing within the array $\pi$ the optimal strategy for every point in the discretization of the state space. The contracting property of operator $\mathcal{T}$ guarantees the convergence of the method, which will stop according to a given tolerance for the value difference between steps.

Here, we present a simplified sequence diagram of the algorithm procedure. We let $\tilde{E} \subset E$ denote a discretization of the state space $E$, and we approximate values of $V$ at arbitrary points in $E$ through linear/spline interpolation and exponential/logarithmic transformations; in addition, we make use of Simpson's rule for numerical integrations. We denote by $\tilde{\mathcal{U}} \subset \mathcal{U}$ a discretization of the control action space and let $\delta$ be an arbitrary tolerance value, then:

- Define initial candidates $V(x)$ and $\pi(x)$ for all $x \in \tilde{E}$;
- **Repeat:**
  - For all $x \in \tilde{E}$ let $\pi(x) = \underset{u \in \tilde{U}}{\operatorname{argmax}} \{R(x,u) + \int_E V(y) Q(\mathrm{d}y|x,u)\}$
  - For all $x \in \tilde{E}$ let $V_1(x) = R(x, \pi(x)) + \int_E V(y) Q(\mathrm{d}y|x, \pi(x))$
  - **If**(max$(V_1(x) - V(x)) < \delta$) Bellman update $V = V_1$ & Break;
  - Bellman update $V = V_1$;
- Return $V$ and $\pi$.

CHAPTER 6

# MDP ALGORITHMS FOR WEALTH ALLOCATION PROBLEMS WITH DEFAULTABLE BONDS

Let $T > 0$ be a fixed time horizon and $(\Omega, \mathcal{G}, \mathbb{P})$ denote a complete probability space equipped with a filtration $\{\mathcal{G}_t\}_{t\geq 0}$. Here $\mathbb{P}$ refers to the real world (also called historical) probability measure and $\{\mathcal{G}_t\}_{t\geq 0}$ is the *enlargement* of a *reference filtration* $\{\mathcal{F}_t\}_{t\geq 0}$ denoted $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ and satisfying the usual assumptions of completeness and right continuity; $\mathcal{H}_t$ will be introduced later. We consider a frictionless financial market consisting of a risk-free bank account $B = (B_t)_{0\leq t\leq T}$, a pure-jump asset $S = (S_t)_{0\leq t\leq T}$ and a defaultable bond $P = (P_t)_{0\leq t\leq T}$. The dynamics of each of the components of the market are given as follows.

**Risk Free Bank Account.** Let $B_0 = 1$ and $r > 0$ denote the market fixed-interest rate, the deterministic dynamics of $B$ are given by

$$\mathrm{d}B_t = rB_t\mathrm{d}t\ .$$

**Pure Jump Asset.** Let $C = (C_t)_{0\leq t\leq T}$ be a compound Poisson process defined on $(\Omega, \mathcal{G}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, given by

$$C_t = \sum_{n=1}^{N_t} Y_n\ , \qquad (6.0.1)$$

where $N = (N_t)_{0\leq t\leq T}$ denotes a Poisson process with intensity $\nu > 0$ and $(Y_n)_{n\in\mathbb{N}}$ is a sequence of independent and identically distributed random variables, with $\mathbb{E}[Y_n] < \infty$, $Y_n \geq -1$ and distribution $\gamma(\mathrm{d}y)$. Here $\{\mathcal{F}_t\}_{t\geq 0}$ is a suitable complete and right-continuous filtration.

Asset $S$ is a piecewise deterministic Markov process (see Section 5.3) adapted to $\mathcal{F}_t$

and given by

$$\mathrm{d}S_t = S_{t^-}\left(\mu \mathrm{d}t + \mathrm{d}C_t\right),$$

where $\mu$ is the constant *appreciation rate* of the asset and $S_0 > 1$. Figure 6.1 illustrates some sample realisations of the process $S$.

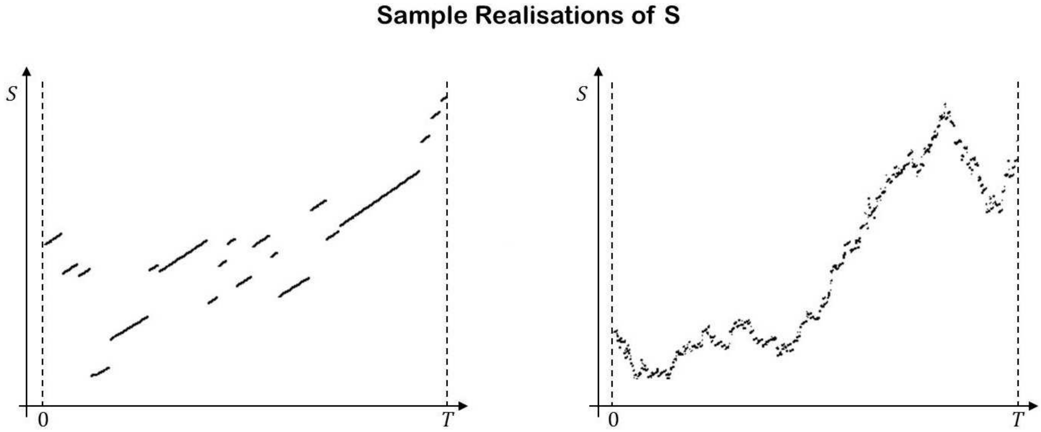


**Figure 6.1:** Sample realisations of the Piecewise Deterministic Markov Process $S$, with varying parameters. On the left hand side $\nu = 2$, $T = 10$, $\mu = 0.025$; on the right hand side $\nu = 40$, $T = 10$, $\mu = 0.05$. Jumps $Y$ follow truncated normal distributions.

**Defaultable Bond.** We consider a tradeable zero coupon bond with face value of one unit and recovery at default. Let $\tau > 0$ be an exponentially distributed random variable defined on $(\Omega, \mathcal{G}, \{\mathcal{H}_t\}_{t\geq 0}, \mathbb{P})$ with intensity $\lambda_{\mathbb{P}}$; we make use of the intensity-based approach for modelling Credit Risk (see [6]) and let the $\tau$ model the default time of the bond $P$. Here $\mathcal{H}_t = \sigma(H_s : s \leq t)$ is the filtration generated by the one-jump process $H_t = 1_{\{\tau \leq t\}}$, after completion and regularization on the right; $C_t$ and $H_t$ are assumed to be independent and $\lambda_{\mathbb{P}}$ is denoted the *hazard rate* of $\tau$, so that the compensated process

$$\mathrm{d}M_t = \mathrm{d}H_t - \lambda_{\mathbb{P}} \mathrm{d}(t \wedge \tau) \tag{6.0.2}$$

with $M_0 = 0$ is a $(\mathcal{G}_t, \mathbb{P})$-martingale, with $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$. Lastly, we denote by $Z = (Z_t)_{0\leq t\leq T}$ the $\mathcal{F}_t$-adapted *recovery process* of $P$, i.e. the process determining the wealth recovery upon default.

Then, the time-$t$ price of this defaultable bond $P$ with maturity at $T$ is given by

$$P_t = B_t \mathbb{E}_{\mathbb{Q}}\left[ B_T^{-1}(1 - H_T) + \int_t^T B_u^{-1} Z_u \mathrm{d}H_u \middle| \mathcal{G}_t \right], \tag{6.0.3}$$

where $\mathbb{Q}$ is a martingale measure equivalent to $\mathbb{P}$; intuitively, $P_t$ models the discounted $\mathbb{Q}$-expected value of the pay-off $(1 - H_T) + H_T Z_\tau$. The existence of such an equivalent measure on $(\Omega, \mathcal{G})$ follows from the results on change of measures presented in [6] (Chapter 4).

This Chapter will extract the real world dynamics of the defaultable bond (6.0.3) and set up a portfolio optimization problem $V$ of the form (1.2.2), allowing for investments on the three introduced financial instruments. We will present the conversion of the problem into a discrete-time Markov decision process (MDP), so that the value function is characterized as the unique fixed point to a dynamic programming operator. Then, optimal wealth allocations will be numerically approximated through the method of value iteration and the dependence of optimal portfolio selections will be explored in terms of the *risk premium* and different parameters describing the system. Our results suggest significantly different allocation procedures to those in [5, 8, 16, 37] under an exponential family of utilities, and extends the work to more general families of logarithmic and exponential utility functions.

The rest of the Chapter is organised as follows. Sections 6.1 and 6.2 derive the $\mathbb{P}$-infinitesimal dynamics of the financial products and set up an allocation problem by means of characterizing the dynamics of a joint wealth process. Section 6.3 follows a procedure in order to introduce an equivalent MDP to our optimization problem, and presents the main technical results in the Chapter. Sections 6.4 and 6.5 will provide proof of our results and justify the use of value iteration techniques in order to approximate optimal solutions. Finally, sections 6.6 and 6.7 present a numerical analysis and make comments on optimal portfolio strategies, drawing comparisons with previous results that lead to the key contributions of this work; in addition possible extensions of the model and drawbacks of this approach are discussed.

## 6.1 THE $\mathbb{P}$-DYNAMICS OF THE DEFAULTABLE BOND

Following results in [6] (Section 4.4) and [33] (Section 8.6), let $\eta = \eta(\tau) = \phi e^{-\lambda_{\mathbb{P}}(\phi-1)\tau}$ be a random variable satisfying $\eta > 0$ and $\mathbb{E}_{\mathbb{P}}[\eta] = 1$, where $\phi$ is a strictly positive constant. Then, the change of measure with *Radon-Nikodým density process*

$$\eta_t = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{G}_t} = \mathbb{E}_{\mathbb{P}}[\eta(\tau)|\mathcal{G}_t] = \mathbb{E}_{\mathbb{P}}[\eta(\tau)|\mathcal{H}_t]\,, \tag{6.1.1}$$

is such that $\tau$ is an exponentially distributed random variable under $\mathbb{Q}$, with intensity $\lambda_{\mathbb{Q}} = \phi\lambda_{\mathbb{P}}$; this is observed by noting that

$$\mathrm{d}\mathbb{Q}(\tau \le t) = \phi e^{-\lambda_{\mathbb{P}}(\phi-1)t}\mathrm{d}\mathbb{P}(\tau \le t) = \phi\lambda_{\mathbb{P}}e^{-\phi\lambda_{\mathbb{P}}t}\mathrm{d}t\,.$$

In practice, default intensities are independently estimated, using credit ratings and company data for the real world intensity $\lambda_{\mathbb{P}}$ and derivatives prices (including CDS and Options) for $\lambda_{\mathbb{Q}}$; their underlying ratio $\phi$ is named the 'Risk Premium' and represents the reward investors claim for bearing the risk of default in $P$.

**Proposition 6.1.1.** *The stochastic process $\eta_t$ defined by* (6.1.1) *is a $(\mathcal{G}_t, \mathbb{P})$-martingale with $\eta_0 = 1$ and*

$$\mathrm{d}\eta_t = \eta_{t^-}(\phi - 1)\mathrm{d}M_t\,,$$

*where $M_t$ is defined by* (6.0.2).

*Proof.* Expanding the conditional expectation in (6.1.1) we get

$$\begin{aligned}\eta_t &= \mathbb{E}_{\mathbb{P}}[\eta(\tau)|\mathcal{H}_t] = H_t\phi e^{-\lambda_{\mathbb{P}}(\phi-1)\tau} + (1-H_t)\int_t^\infty \phi e^{-\lambda_{\mathbb{P}}(\phi-1)x}\lambda_{\mathbb{P}}e^{-\lambda_{\mathbb{P}}(x-t)}\mathrm{d}x \\ &= H_t\phi e^{-\lambda_{\mathbb{P}}(\phi-1)\tau} + (1-H_t)e^{-\lambda_{\mathbb{P}}(\phi-1)t} \;=\; \phi^{H_t}e^{-\lambda_{\mathbb{P}}(\phi-1)(\tau\wedge t)}\,.\end{aligned}$$

Then, direct application of Itô's formula for non-continuous semi-martingales to $\eta_t$ yields

$$\begin{aligned}\mathrm{d}\eta_t &= \eta_{t^-}(\phi-1)[\mathrm{d}H_t - (1-H_t)\lambda_{\mathbb{P}}\mathrm{d}t] \\ &= \eta_{t^-}(\phi-1)[\mathrm{d}H_t - \lambda_{\mathbb{P}}\mathrm{d}(t\wedge\tau)]\end{aligned}$$

proving the result. □

In order to obtain the $\mathbb{P}$-dynamics of $P$ defined by (6.0.3) we make use of the models for valuation of contingent claims subject to default risk in [23]. We first define the concept of a *gain process*; we denote by $G = (G_t)_{0\le t\le T}$ the wealth gain process resulting from holding one defaultable bond $P$, given by

$$\mathrm{d}G_t = \mathrm{d}P_t + Z_t\mathrm{d}H_t\,, \tag{6.1.2}$$

with $G_0 = P_0$. Note that $P$ and $G$ differ in the sense that $G$ incorporates the wealth recovered in case of default in $P$, so that $G_t = Z_\tau$ for $t \geq \tau$. In addition, we make the following assumption.

**Assumption 6.1.2** (Recovery of Market)**.** *The wealth recovery upon default in $P$ is given by a fraction of its current market value, i.e. $Z_t = (1-L)P_{t^-}$ for all $t < T$, with $0 \leq L \leq 1$ constant.*

**Lemma 6.1.3.** *The price dynamics of the defaultable bond $P$ in* (6.0.3)*, under the Recovery of Market assumption and real world probability measure $\mathbb{P}$, are given by*

$$\mathrm{d}P_t = P_{t^-}[(r + \phi\lambda_{\mathbb{P}} L)\mathrm{d}t - \mathrm{d}H_t] \quad \textit{for} \quad t \leq T \wedge \tau, \quad \textit{and} \tag{6.1.3}$$

$$\mathrm{d}P_t = 0 \quad \textit{for} \quad \tau < t \leq T\,, \tag{6.1.4}$$

*with $P_0 = e^{-(r+\phi\lambda_{\mathbb{P}} L)T}$.*

*Proof.* The derivation of these equations follows from the application of Theorem 1 in [23]. We use arbitrage-free arguments to obtain a pricing expression for $P_t$; the key observation is that its future expected gain $G$ in (6.1.2), up to time $\tau \wedge T$, must match the attainable risk-less reward under measure $\mathbb{Q}$; that is, the discounted gain $e^{-rt}G_t$ given by

$$e^{-rt}G_t = e^{-rt}P_t + (1-L)\int_0^t e^{-rs}P_{s^-}\mathrm{d}H_s \tag{6.1.5}$$

for $t \in [0, \tau \wedge T]$, must be a $\mathbb{Q}$-martingale. Noting that $\mathbb{P}(\tau = T) = 0$ a.s., we may assume that default does not occur at maturity time. Recall from (6.0.3) that $P$ is discontinuous only at the default time and that $P_t = 0$ for $t \geq \tau$, we may denote $P_t = (1 - H_t)U_t$, where $U_t$ is a continuous process. Plugging this expression for $P$ into (6.1.5) above and applying Itô's formula we obtain

$$\mathrm{d}(e^{-rt}G_t) = e^{-rt}\Big[(1 - H_{t^-})\mathrm{d}U_t - r(1 - H_{t^-})U_{t^-}\mathrm{d}t - LU_{t^-}\mathrm{d}H_t\Big]$$

for $t \in [0, \tau \wedge T]$. It is possible to rewrite the above equation in terms of a compensated jump process, through the inclusion and subsequent subtraction of a compensator in the jump differential term $\mathrm{d}H_t$, so that

$$\mathrm{d}(e^{-rt}G_t) = e^{-rt}\Big[(1 - H_{t^-})(\mathrm{d}U_t - (r + \lambda_{\mathbb{Q}} L)U_{t^-}\mathrm{d}t) - LU_{t^-}\mathrm{d}M_t^{\mathbb{Q}}\Big]\,,$$

where

$$\mathrm{d}M_t^{\mathbb{Q}} = \mathrm{d}H_t - \lambda_{\mathbb{Q}}\mathrm{d}(t \wedge \tau)$$

with $M_0^{\mathbb{Q}} = 0$ is a $(\mathcal{G}_t, \mathbb{Q})$-martingale. Therefore, for $e^{-rt}G_t$ to be a $\mathbb{Q}$-martingale the following must hold

$$\mathrm{d}U_t = (r + \phi\lambda_{\mathbb{P}}L)U_t\mathrm{d}t\ ,$$

since we recall that $\lambda_{\mathbb{Q}} = \phi\lambda_{\mathbb{P}}$. Finally, note that $\mathrm{d}P_t = \mathrm{d}U_t - U_{t^-}\mathrm{d}H_t$ and $P_t = U_t$ for $t < \tau$, the result follows. □

## 6.2 Wealth Dynamics and the Allocation Problem

Consider an investor wishing to invest in this market. Denote by $\pi_t^B$ the percentage of total wealth at time $t$ invested on the risk-less bond; analogously $\pi_t^S$ and $\pi_t^P$ denote the time-$t$ proportions on the asset and defaultable bond. The *portfolio process* $\pi = (\pi_t^B, \pi_t^S, \pi_t^P)_{0\le t\le T}$ is a $\mathcal{G}_t$-predictable process taking values in

$$\mathcal{U} = \{(u_1, u_2, u_3) \in \mathbb{R}_+^3 : \sum_{i=1}^{3} u_i = 1\}\ , \tag{6.2.1}$$

so that short selling is not allowed and wealth is fully invested at all times and remains positive. Note that $\pi_t^P = 0$ for $t > \tau$; furthermore, although the price of $P$ drops to zero at default we must account for the gain derived from its recovery value, i.e. we consider the $\mathbb{P}$-dynamics of the gain process $G = (G_t)_{0\le t\le T}$ in (6.1.2) with regards to portfolio optimization purposes. From (6.1.3) and (6.1.4), the dynamics of $G$ are determined by

$$\mathrm{d}G_t = G_{t^-}[(r + \phi\lambda_{\mathbb{P}}L)\mathrm{d}t - L\mathrm{d}H_t] \quad \text{for} \quad t \le T \wedge \tau, \quad \text{and}$$
$$\mathrm{d}G_t = 0 \quad \text{for} \quad \tau < t \le T\ ,$$

with $G_0 = P_0$.

Denote by $X^\pi = (X_t^\pi)_{0\le t\le T}$ the wealth process associated to a strategy $\pi \in \mathcal{U}$. Then, its time-$t$ infinitesimal gain is given by

$$\mathrm{d}X_t^\pi = X_{t^-}^\pi \cdot \left[(1 - \pi_t^P - \pi_t^S)\frac{\mathrm{d}B_t}{B_t} + \pi_t^S\frac{\mathrm{d}S_t}{S_{t^-}} + \pi_t^P\frac{\mathrm{d}G_t}{G_{t^-}}\right] .$$

Denote by $X_t^{\pi,c}$ the purely continuous component of $X_t^{\pi}$; the explicit form of $X$ is derived using Itô calculus and noting that

$$\begin{aligned} \mathrm{d}\log(X_t^{\pi}) &= \frac{\mathrm{d}X_t^{\pi,c}}{X_{t^-}^{\pi}} + \log(1-\pi_t^P L\mathrm{d}H_t) + \log(1+\pi_t^S \mathrm{d}C_t) \\ &= [r+\pi_t^S(\mu-r)+\pi_t^P\phi\lambda_{\mathbb{P}}L]\mathrm{d}t + \log(1-\pi_t^P L)\mathrm{d}H_t + \log(1+\pi_t^S\mathrm{d}C_t)\,, \end{aligned}$$

so that

$$\begin{aligned} \log(X_t^{\pi}) &= \log(X_0) + \int_0^t [r+\pi_s^S(\mu-r)+\pi_s^P\phi\lambda_{\mathbb{P}}L]\mathrm{d}s \\ &\quad + \log(1-\pi_\tau^P L)H_t + \sum_{n=1}^{N_t}\log(1+\pi_{T_n}^S Y_n)\,, \end{aligned} \tag{6.2.2}$$

which follows noting that $H_t$ is a single jump process with jump size 1, at time $\tau$; and random jumps $Y_i$ in $C_t$ occur at random times $T_i$ in $N$. Hence, exponentiating equation (6.2.2) we have

$$X_t^{\pi} = X_0 e^{\int_0^t (r+\pi_s^S(\mu-r)+\pi_s^P\phi\lambda_{\mathbb{P}}L)\mathrm{d}s}(1-\pi_\tau^P L)^{H_t}\prod_{n=1}^{N_t}(1+\pi_{T_n}^S Y_n)\,, \tag{6.2.3}$$

where $X_0$ stands for the initial wealth.

Let $\Pi$ denote the family of all measurable portfolio processes $\pi$ taking values in $\mathcal{U}$. For a given increasing and concave utility function $U:(0,\infty)\to\mathbb{R}^+$, let

$$V_\pi(t,x,h) = \mathbb{E}_{t,x,h}[U(X_T^{\pi})]$$

denote the expected terminal reward associated to a portfolio strategy $\pi\in\Pi$, for current state $(t,x,h)\in[0,T]\times\mathbb{R}^+\times\{0,1\}$. Here, $\mathbb{E}_{t,x,h}$ denotes the expectation under the conditional probability measure $\mathbb{P}|_{(X_t^{\pi}=x,H_t=h)}$. The optimal policy $\pi^*\in\Pi$ is the one that maximizes the reward, so that

$$V_{\pi^*}(t,x,h) = \sup_{\pi\in\Pi} V_\pi(t,x,h)\,, \tag{6.2.4}$$

for all $(t,x,h)\in[0,T]\times\mathbb{R}^+\times\{0,1\}$. As mentioned before, we aim to numerically approximate the policy $\pi^*$, so as to explore the dependence of optimal portfolio selections on the *risk premium*, utility of choice, and additional parameters defining the model. Note that $V_{\pi^*}(T,x,h)=U(x)$ for all $(x,h)\in\mathbb{R}^+\times\{0,1\}$ and problem $V_{\pi^*}$ is tractable since $\mathbb{E}[Y_n]<\infty$.

## 6.3 An Equivalent Discrete-Time Markov Decision Process

We follow a similar approach to that in [11] and [12] in order to reduce problem (6.2.4) to a discrete-time MDP, allowing for $V_{\pi^*}$ to be computationally identified as the unique fixed point to a maximal reward operator.

Let $\Psi = (\Psi_n)_{n\geq 0}$ denote the increasing sequence of joint jump times in $N$ and $H$, given by

$$\Psi_n = T_n 1_{\{T_n<\tau\}} + \tau 1_{\{T_{n-1}<\tau<T_n\}} + T_{n-1} 1_{\{\tau<T_{n-1}\}} \, , \tag{6.3.1}$$

with $\Psi_0 = 0$. Intuitively, $\Psi$ represents an ordered discrete counting process incorporating default time $\tau$ to jump times $(T_n)_{n\geq 0}$ in asset $S$; in addition, we refer to the counting steps $n \geq 0$ of $\Psi$ as *decision epochs*. We define the MDP composed by the following 4-tuple $(E, \mathcal{A}, Q, R)$, an explanatory diagram is presented in Figure 6.2.

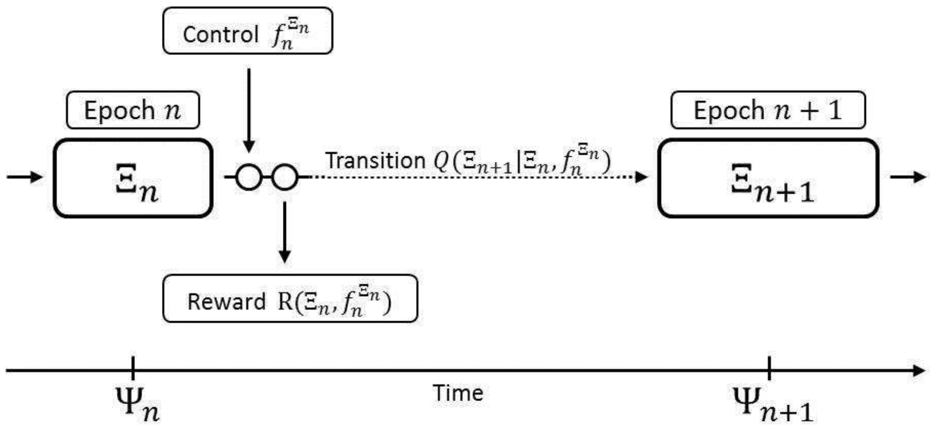


**Figure 6.2:** Explanatory diagram of the structure of the MDP $(E, \mathcal{A}, Q, R)$; variables $\Xi_n$ and $\Xi_{n+1}$ refer to the states of the system at epochs $n$ and $n+1$ subsequently. We observe that each decision epoch $n$ takes place at time $\Psi_n$.

**The State Space $E$** is given by $E = [0, T] \times \mathbb{R}^+ \times \{0, 1\}$ and supports times $\Psi_n$, with associated wealth $X_{\Psi_n}$ and states of default process $H_{\Psi_n}$, right after each jump. We use the notation $\Xi_n$ to denote the $n$-th *state of the system*, given by

$$\Xi_n = \begin{cases} (\Psi_n, X_{\Psi_n}, H_{\Psi_n}) \in E & \text{if } \Psi_n \leq T \, , \\ \Delta & \text{otherwise} \, , \end{cases}$$

for $n \geq 0$. $\Delta \notin E$ is an external *absorption state* and allows for us to set up an *infinite horizon* optimization problem as described in Chapter 5.

**The Action Space $\mathcal{A}$** stands for the set of deterministic *control* actions

$$\mathcal{A} = \{\alpha : \mathbb{R}^+ \to \mathcal{U} \text{ measurable}\} , \tag{6.3.2}$$

where $\mathcal{U}$ is given in (6.2.1). A control $\alpha \in \mathcal{A}$ is a function of time and $\alpha(t) \in \mathcal{U}$ determines the allocation of wealth at time $t$ after a jump in $\Psi$. We note that for a given state $\Xi_n \in E \cup \{\Delta\}$ only a subclass of actions $D_n(\Xi_n) \subseteq \mathcal{A}$ may be admissible (for example, if bond $P$ defaulted).

In addition to $\mathcal{A}$, we denote by $F$ the set of all deterministic *policies* or *decision rules* given by

$$F = \{f : E \cup \{\Delta\} \to \mathcal{A} \text{ measurable}\} . \tag{6.3.3}$$

At any decision epoch $n$, a policy $f_n \in F$ maps a state $\Xi_n$ to an admissible control action in $D_n(\Xi_n)$; we denote the resulting control by $f_n^{\Xi_n}$. The policy determines, as a function of the system state, the control chosen at epoch $n$; this results in a function $f_n^{\Xi_n} : \mathbb{R}^+ \to \mathcal{U}$ that models the time evolving allocation of wealth in our portfolio $\pi$, so that

$$\pi_t = f_n^{\Xi_n}(t - \Psi_n) \quad \text{for} \quad t \in [\Psi_n, \Psi_{n+1}) . \tag{6.3.4}$$

A portfolio process $\pi \in \Pi$ is called a *Markovian portfolio strategy* if it is defined by a *Markov policy*, i.e. a sequence of functions $(f_n)_{n\geq 0}$ with $f_n \in F$ (see Section 5.1). We recall that if policies $f_n \equiv f$ for all $n \geq 0$, the Markov policy is called *stationary*, implying that decisions are independent of the epoch number and only dependent on the system state. Figure 6.3 illustrates the characterization of a Markovian portfolio strategy in a diagram. It is key to note that for a specified Markov policy, the controls to take at each epoch are random, since they depend on the system states to be observed.

**The Transition Probability $Q$.** For current state $\Xi_n \in E$ and control $f_n^{\Xi_n} \in D_n(\Xi_n)$, the transition density describes the probability for the system to adopt a specific state in epoch $n+1$ (or time $\Psi_{n+1}$). Let $f_n^{\Xi_n}(t) = (\alpha_t^B, \alpha_t^S, \alpha_t^P) \in \mathcal{U}$ denote the proportions of wealth allocated to each financial instrument at $t \geq 0$ time units after jump time

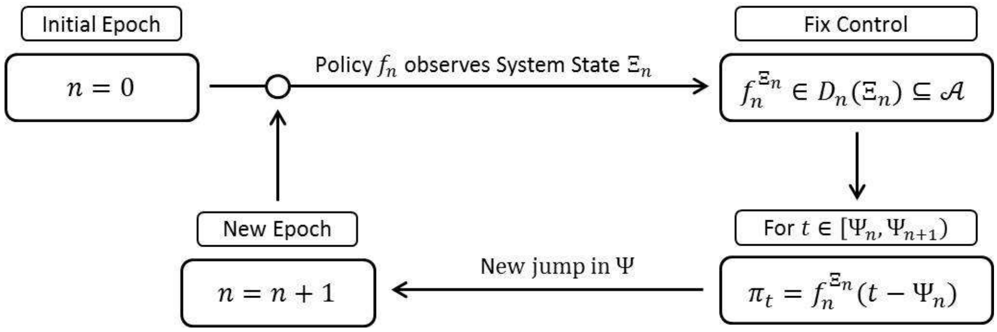


**Figure 6.3:** Characterization of a Markovian portfolio strategy $\pi \in \Pi$ defined by a Markov policy $(f_n)_{n\geq 0}$, with $f_n \in F$.

$\Psi_n$, according to control $f_n^{\Xi_n}$; we note from (6.3.4) that this is equivalent to the global portfolio wealth allocation $\pi_{t+\Psi_n}$ at time $t + \Psi_n$. Analogously, let $\Gamma_t^{f_n^{\Xi_n}}$ denote the associated wealth $t \geq 0$ time units after $\Psi_n$; this is equivalent to the global wealth $X^{\pi}_{t+\Psi_n}$ at time $t + \Psi_n$. Note from (6.2.3) that $\Gamma_t^{f_n^{\Xi_n}}$ is a deterministic function of the last system state, given by

$$\Gamma_t^{f_n^{\Xi_n}}(X_{\Psi_n}, H_{\Psi_n}) = X_{\Psi_n} e^{\int_0^t (r+\alpha_s^S(\mu-r))\mathrm{d}s}\big[H_{\Psi_n} + (1-H_{\Psi_n})e^{\int_0^t \alpha_s^P \lambda_{\mathbb{P}} L\phi \mathrm{d}s}\big] \,. \tag{6.3.5}$$

For an arbitrary $\Xi_n = (t', x, h)$, Lemma B.1.1 in Appendix B shows that the transition density kernel $Q$ is given by

$$\begin{aligned} Q(B|\Xi_n, f_n^{\Xi_n}) &= \mathbb{P}(\Xi_{n+1} \in B|\mathcal{G}_{\Psi_n}, f_n^{\Xi_n}) = \\ &= \nu \int_0^{T-t'} e^{-(\nu+(1-h)\lambda_{\mathbb{P}})s} \int_{-1}^{\infty} 1_B(t'+s, \Gamma_s^{f_n^{\Xi_n}}(x,h)(1+\alpha_s^S y), h)\gamma(\mathrm{d}y)\mathrm{d}s \\ &+ (1-h)\lambda_{\mathbb{P}} \int_0^{T-t'} e^{-(\nu+\lambda_{\mathbb{P}})s} 1_B(t'+s, \Gamma_s^{f_n^{\Xi_n}}(x,0)(1-\alpha_s^P L), 1)\mathrm{d}s \,, \end{aligned} \tag{6.3.6}$$

for $B \subseteq E$; in addition

$$Q(\{\Delta\}|\Xi_n, f_n^{\Xi_n}) = 1 - Q(E|\Xi_n, f_n^{\Xi_n}) \,.$$

Since $\Delta$ is an absorbing state we define $Q(\{\Delta\}|\Delta, \alpha) = 1$ for all controls $\alpha \in \mathcal{A}$. Intuitively, formula (6.3.6) gives the probability for the system state at epoch $n+1$ to fall within a subset $B$ of the state space, given all information in $\mathcal{G}_{\Psi_n}$.

**The Reward Function $R$** is a function $R : E \times \mathcal{A} \to \mathbb{R}$ given by

$$R(t, x, h, \alpha) = e^{-(\nu+(1-h)\lambda_{\mathbb{P}})(T-t)} U(\Gamma^{\alpha}_{T-t}(x,h)) \,. \tag{6.3.7}$$

The adoption of such a non-negative reward function ensures the reducibility of optimization problem (6.2.4) to an infinite horizon discrete-time Markov decision process, as it will be shown in Lemma 6.3.1 below. We note that the term $e^{-(\nu+(1-h)\lambda_{\mathbb{P}})(T-t)}$ defines the likelihood of no jumps in a Poisson process with rate $\nu+(1-h)\lambda_{\mathbb{P}}$ over a period of time $T-t$, this will be a key observation in the proof of Lemma 6.3.1. In addition, we define $R(\Delta,\alpha)=0$ for all $\alpha\in\mathcal{A}$.

For an arbitrary state $(t,x,h)\in E$, we let $v(t,x,h)$ denote the *optimal total expected reward* over all Markov policies $(f_n)_{n\geq 0}$ with $f_n\in F$, given by

$$v(t,x,h)=\sup_{(f_n)}\mathbb{E}_{t,x,h}\Big[\sum_{k=0}^{\infty}R(\Xi_k,f_k^{\Xi_k})\Big]\,, \tag{6.3.8}$$

where $\mathbb{E}_{t,x,h}$ denotes the expectation under the probability measure $\mathbb{P}|_{(X_t=x,H_t=h)}$. We now present an equivalency result between the portfolio optimization problem (6.2.4) and the MDP $(E,\mathcal{A},Q,R)$.

**Lemma 6.3.1.** *For any $(t,x,h)\in E$, we have $V_{\pi^*}(t,x,h)=v(t,x,h)$.*

***Proof.*** We treat the case $t=0$, arbitrary time points can be proved similarly upon redefinition of terminal time $T'=T-t$ and adjustment of notation (see [13], Chapter 8). Denote by $\Pi_M$ the set of all Markovian portfolio strategies and note that $\Pi_M\subseteq\Pi$. Due to the Markovian structure of the state process the optimal strategy in (6.2.4) must be Markovian (see [4]), so that

$$V_{\pi^*}(0,x,h)=\sup_{\pi\in\Pi}V_\pi(0,x,h)=\sup_{\pi\in\Pi_{\mathrm{M}}}\mathbb{E}_{x,h}[U(X_T^\pi)]\,, \tag{6.3.9}$$

i.e. the supremum is attained in the set $\Pi_M$. Any $\pi\in\Pi_M$ is defined by sequence of decision rules $f_n\in F$ forming a Markov policy $(f_n)_{n\geq 0}$ as described in (6.3.4); therefore, for such a policy we need to show that

$$\mathbb{E}_{x,h}[U(X_T^\pi)]=\mathbb{E}_{x,h}[\sum_{k=0}^{\infty}R(\Xi_k,f_k^{\Xi_k})]\,.$$

The proof is conceptually similar to that in [11] (Theorem 3.1). Note that

$$\begin{aligned}\mathbb{E}_{x,h}[U(X_T^\pi)] &= \mathbb{E}_{x,h}\Big[\sum_{k=0}^{\infty}U(X_T^\pi)1_{\{\Psi_k\leq T<\Psi_{k+1}\}}\Big]\\ &= \sum_{k=0}^{\infty}\mathbb{E}_{x,h}\Big[\mathbb{E}_{x,h}\Big[U(X_T^\pi)1_{\{\Psi_k\leq T<\Psi_{k+1}\}}\Big|\mathcal{G}_{\Psi_k}\Big]\Big]\,,\end{aligned}$$

where $\Psi$ is the increasing counting process in (6.3.1) incorporating default time in $H_t$ to jump times in $N_t$; we recall these are $\mathcal{G}_t$-adapted processes with exponentially distributed jumps and intensities $\lambda_{\mathbb{P}}$ and $\nu$. In view of (6.3.4) and (6.3.5) we note that wealth $X^\pi$ can be expressed as a deterministic function of the previous system state, i.e.

$$X_t^\pi = \Gamma_{t-\Psi_k}^{f_k^{\Xi_k}}(X_{\Psi_k}, H_{\Psi_k}) \,,$$

for $t \in [\Psi_k, \Psi_{k+1})$, with $X_0^\pi = x$. Therefore

$$\begin{aligned}
\mathbb{E}_{x,h}[U(X_T^\pi)] &= \sum_{k=0}^{\infty} \mathbb{E}_{x,h}\Big[\mathbb{E}_{x,h}\Big[U(\Gamma_{T-\Psi_k}^{f_k^{\Xi_k}}(X_{\Psi_k}, H_{\Psi_k}))1_{\{\Psi_k \le T < \Psi_{k+1}\}}\Big|\mathcal{G}_{\Psi_k}\Big]\Big] \\
&= \sum_{k=0}^{\infty} \mathbb{E}_{x,h}\Big[U(\Gamma_{T-\Psi_k}^{f_k^{\Xi_k}}(X_{\Psi_k}, H_{\Psi_k}))\mathbb{P}(\Psi_{k+1} > T \ge \Psi_k|\mathcal{G}_{\Psi_k})\Big] \,.
\end{aligned}$$

In addition, we note that

$$\begin{aligned}
\mathbb{P}(\Psi_{k+1} > T \ge \Psi_k|\mathcal{G}_{\Psi_k}) &= 1_{\{T \ge \Psi_k\}}\mathbb{P}(\Psi_{k+1} > T|\mathcal{G}_{\Psi_k}) \\
&= 1_{\{T \ge \Psi_k\}} e^{-(\nu+(1-H_{\Psi_k})\lambda_{\mathbb{P}})(T-\Psi_k)} \,.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbb{E}_{x,h}[U(X_T^\pi)] &= \sum_{k=0}^{\infty} \mathbb{E}_{x,h}\Big[1_{\{T \ge \Psi_k\}} e^{-(\nu+(1-H_{\Psi_k})\lambda_{\mathbb{P}})(T-\Psi_k)} U(\Gamma_{T-\Psi_k}^{f_k^{\Xi_k}}(X_{\Psi_k}, H_{\Psi_k}))\Big] \\
&= \sum_{k=0}^{\infty} \mathbb{E}_{x,h}\Big[R(\Xi_k, f_k^{\Xi_k})\Big] \,,
\end{aligned}$$

completing the proof. □

### 6.3.1 Main Results

It has been shown that value function $V_{\pi^*}$ in (6.2.4) can be derived as the sum of expected rewards $v$ in (6.3.8); in what follows, we make use of the theory exposed in Section 5.2 and present results confirming the usefulness of iterative methods in order to approximate optimal portfolio strategies for our problem. The efforts are directed towards the construction of a complete metric space satisfying conditions $(i) - (iii)$ in Theorem 5.2.5, so that $V_{\pi^*}$ is identified as the fixed point to a reward operator. Proof of the results is postponed to the next Sections.

Let $\mathbb{M}(E)$ define the set of measurable functions mapping the state space $E$ into the positive subset of the real line, given by

$$\mathbb{M}(E) = \{g : E \to \mathbb{R}^+ : g \text{ measurable}\} .$$

We recall from (5.2.3) that the *maximal reward operator* $\mathcal{T}$ for the MDP $(E, \mathcal{A}, Q, R)$ is a *dynamic programming* operator acting on $\mathbb{M}(E)$, such that

$$(\mathcal{T}g)(t,x,h) = \sup_{\alpha\in\mathcal{A}} \left\{ R(t,x,h,\alpha) + \sum_k \int g(s,y,k) Q(\mathrm{d}s, \mathrm{d}y, k|t,x,h,\alpha) \right\} ,$$

for all $g \in \mathbb{M}(E)$ and $(t,x,h) \in E$. Additionally, the term within brackets is denoted

$$(\mathcal{L}g)(t,x,h|\alpha) = R(t,x,h,\alpha) + \sum_k \int g(s,y,k) Q(\mathrm{d}s, \mathrm{d}y, k|t,x,h,\alpha) , \tag{6.3.10}$$

and referred to as the *reward operator*, so that

$$(\mathcal{T}g)(t,x,h) = \sup_{\alpha\in\mathcal{A}} (\mathcal{L}g)(t,x,h|\alpha) . \tag{6.3.11}$$

Now, let $\mathbb{C}_\vartheta(E)$ be the function space defined by

$$\mathbb{C}_\vartheta(E) = \{g \in \mathbb{M}(E) : g \text{ continuous and concave in } x \text{ and } \|g\|_\vartheta < \infty\} , \tag{6.3.12}$$

where

$$\|g\|_\vartheta = \sup_{(t,x,h)\in E} \frac{g(t,x,h)}{(1+x)e^{\vartheta(T-t)}} , \tag{6.3.13}$$

for fixed $\vartheta \geq 0$ satisfying conditions in Lemma 6.4.1.

**Theorem 6.3.2.** *Operator $\mathcal{T}$ is a contraction mapping on the metric space $(\mathbb{C}_\vartheta(E), \|\cdot\|_\vartheta)$.*

**Theorem 6.3.3.** *There exists an optimal stationary portfolio strategy $\pi^* \in \Pi$, defined by a Markov policy $(f)_{n\geq 0}$ with $f \in F$ as shown in* (6.3.4)*, so that $V_{\pi^*}$ in* (6.2.4) *is the unique fixed point of $\mathcal{T}$ in $\mathbb{C}_\vartheta(E)$.*

Theorem 6.3.3 implies that a single decision rule $f : \Xi_n \to \mathcal{A}$ is optimal for all epochs $n \geq 0$, and the control chosen after each jump in $\Psi$ is only dependent on the state of the system $\Xi$; we note that this incorporates information on time left to deadline, current wealth and event of default in $P$. Moreover, since $V_{\pi^*}$ is characterized as a unique fixed point to a dynamic programming operator the use of computational approaches to approximate its value is justified.

## 6.4 Proof of Theorem 6.3.2

We begin with the presentation of a contraction result for later use. Let $\mathbb{M}_\vartheta(E)$ define the function space given by

$$\mathbb{M}_\vartheta(E) = \{g \in \mathbb{M}(E) : \ \|g\|_\vartheta < \infty\} ,$$

where the norm $\|\cdot\|_\vartheta$ is as in (6.3.13). In view of (5.2.5), $\|\cdot\|_\vartheta$ is a *weighted supremum norm* and $\mathbb{M}_w(E)$ is a Banach space, since every Cauchy sequence of elements converges to an element in the set.

**Lemma 6.4.1.** *For sufficiently large $\vartheta \in \mathbb{R}_+$ it holds $\|\mathcal{T}g_1 - \mathcal{T}g_2\|_\vartheta < \|g_1 - g_2\|_\vartheta$, for all $g_1, g_2 \in \mathbb{M}_\vartheta(E)$ .*

***Proof.*** For all $g_1, g_2 \in \mathbb{M}_\vartheta(E)$, it holds

$$\begin{aligned}
(\mathcal{T}g_1 - \mathcal{T}g_2)(t,x,h) &\leq \sup_{\alpha\in\mathcal{A}}\{(\mathcal{L}g_1)(t,x,h|\alpha) - (\mathcal{L}g_2)(t,x,h|\alpha)\} \\
&= \sup_{\alpha\in\mathcal{A}}\Big\{\sum_k \int (g_1 - g_2)(s,y,k)Q(\mathrm{d}s,\mathrm{d}y,k|t,x,h,\alpha)\Big\} \\
&\leq \|g_1 - g_2\|_\vartheta \sup_{\alpha\in\mathcal{A}}\Big\{\sum_k \int (1+y)e^{\vartheta(T-s)}Q(\mathrm{d}s,\mathrm{d}y,k|t,x,h,\alpha)\Big\} .
\end{aligned}$$

Denote by $I$ the expression within brackets on the right hand side. In view of (6.3.6), it reads

$$\begin{aligned}
I &= \nu\int_0^{T-t} e^{-(\nu+(1-h)\lambda_{\mathbb{P}})s}\int_{-1}^{\infty}(1+\Gamma_s^\alpha(x,h)(1+\alpha^S y))e^{\vartheta(T-t-s)}\gamma(\mathrm{d}y)\mathrm{d}s \\
&+ (1-h)\lambda_{\mathbb{P}}\int_0^{T-t} e^{-(\nu+\lambda_{\mathbb{P}})s}(1+\Gamma_s^\alpha(x,0)(1-\alpha^P L))e^{\vartheta(T-t-s)}\mathrm{d}s .
\end{aligned}$$

Note that for all $(t,x,h,\alpha) \in E \times \mathcal{A}$ we have

$$1 + \Gamma_s^\alpha(x,h) < 1 + xe^{(2r+\mu)t+\lambda_{\mathbb{P}}L\phi} \leq k(1+x) ,$$

for some $k \in \mathbb{R}^+$, therefore there exists a constant $c \in \mathbb{R}^+$ so that

$$1 + \Gamma_s^\alpha(x,h)(1-\alpha^P L) \leq c(1+x) ,$$

and

$$\int_{-1}^{\infty}(1+\Gamma_s^\alpha(x,0)(1+\alpha^S y))\gamma(\mathrm{d}y) = 1 + \Gamma_s^\alpha(x,0)(1+\alpha^S\bar{y}) \leq c(1+x) ,$$

for all $x \in \mathbb{R}^+$, since $\bar{y} = \mathbb{E}[Y] < \infty$. Thus, the following holds for $I$

$$\begin{aligned} I &\leq c(1+x)e^{\vartheta(T-t)} \cdot \Big\{\nu \int_0^{T-t} e^{-(\nu+(1-h)\lambda_{\mathbb{P}}+\vartheta)s}\mathrm{d}s + (1-h)\lambda_{\mathbb{P}} \int_0^{T-t} e^{-(\nu+\lambda_{\mathbb{P}}+\vartheta)s}\mathrm{d}s\Big\} \\ &\leq c(1+x)e^{\vartheta(T-t)}(1-e^{-(\vartheta+\nu+\lambda_{\mathbb{P}})(T-t)})\Big(\frac{\nu}{\nu+\vartheta} + \frac{\lambda_{\mathbb{P}}}{\nu+\lambda_{\mathbb{P}}+\vartheta}\Big) \,. \end{aligned}$$

There trivially exists a constant $\vartheta \in \mathbb{R}_+$ big enough so that

$$c_\vartheta = c(1-e^{-(\vartheta+\nu+\lambda_{\mathbb{P}})(T-t)})\Big(\frac{\nu}{\nu+\vartheta} + \frac{\lambda_{\mathbb{P}}}{\nu+\lambda_{\mathbb{P}}+\vartheta}\Big) < 1 \,.$$

Thus,

$$\|\mathcal{T}g_1 - \mathcal{T}g_2\|_\vartheta = \sup_{(t,x,h)\in E} \frac{(\mathcal{T}g_1 - \mathcal{T}g_2)(t,x,h)}{(1+x)e^{\vartheta(T-t)}} \leq \|g_1 - g_2\|_\vartheta c_\vartheta < \|g_1 - g_2\|_\vartheta \,,$$

completing the proof. □

We note from the proof of the Lemma that part (ii) in Assumption 5.2.2 is satisfied with $\kappa < 1$. Upon noting that for all $(t,x,h,\alpha) \in E \times \mathcal{A}$ it holds that

$$R(t,x,h,\alpha) \leq \mu(1+x)e^{\vartheta(T-t)} \quad \text{for some } \mu > 0,$$

we conclude from the results in Section 5.2 that the MDP $(E, \mathcal{A}, Q, R)$ is contracting, and therefore problem $v$ in (6.3.8) is well-defined.

Since $\mathbb{C}_\vartheta(E)$ in (6.3.12) is a closed subset of $\mathbb{M}_\vartheta(E)$, the contracting property of $\mathcal{T}$ in Theorem 6.3.2 follows. However, we must provide proof for the concavity of the mapping $x \mapsto (\mathcal{T}g)(t,x,h)$, along with the continuity of $(t,x,h) \mapsto (\mathcal{T}g)(t,x,h)$; here, we do so separately.

### 6.4.1 The Proof of Concavity

**Lemma 6.4.2.** *For all $g \in \mathbb{C}_\vartheta(E)$, the mapping $x \mapsto (\mathcal{T}g)(t,x,h)$ is concave.*

***Proof.*** We begin introducing the concept of *invested amounts*. In view of (6.3.5), at $t$ time units after a last decision epoch in $E$ with wealth $x$ and default state $h$, a control action $\alpha \in \mathcal{A}$ with fractions $\alpha(t) \in \mathcal{U}$ for all $t \geq 0$ yields the wealth amounts $a(t) =$

$\alpha(t)\Gamma_t^\alpha(x,h)$. It is therefore possible to define an alternative convex action space of invested amounts, given by

$$\mathbb{A}_{x,h} = \{a : \mathbb{R}_+ \to \mathbb{R}_+^3 \ : \ \sum_{i=1}^{3} a_i(t) = \Gamma_t^\alpha(x,h) \text{ for some } \alpha \in \mathcal{A}\} \ .$$

We denote by $\Gamma_t^a(x,h)$ the deterministic wealth evolution in time for a control $a \in \mathbb{A}_{x,h}$; in addition, we refer to controls $a$ and $\alpha$ as being *equivalent* if $\Gamma_t^a(x,h) = \Gamma_t^\alpha(x,h)$.

The dynamics of $\Gamma_t^a(x,h)$ are expressed in terms of invested amounts and given by

$$\frac{\mathrm{d}\Gamma_t^a(x,h)}{\mathrm{d}t} = \Gamma_t^a(x,h)r + a_t^S(\mu - r) + (1-h)a_t^P \lambda_{\mathbb{P}} L\phi \ ,$$

which is a first order linear differential equation, its general form solution is given by

$$\Gamma_t^a(x,h) = \frac{\int_0^t \phi(s)[a_t^S(\mu - r) + (1-h)a_t^P \lambda_{\mathbb{P}} L\phi]\mathrm{d}t + C}{\phi(t)} \ ,$$

with $\phi(t) = e^{-\int r\mathrm{d}t}$. Therefore

$$\Gamma_t^a(x,h) = e^{rt}\Big(x + \int_0^t e^{-rs}[a_t^S(\mu - r) + (1-h)a_t^P \lambda_{\mathbb{P}} L\phi]\mathrm{d}t\Big) \ ,$$

which is a linear function on $(x,a)$. For an arbitrary fixed $t' \geq 0$ and $h \in \{0,1\}$, fix wealths $x_1, x_2 \geq 0$ with $x_1 \neq x_2$ and set controls $\alpha_1, \alpha_2 \in \mathcal{A}$ so that

$$\begin{aligned}(\mathcal{T}g)(t', x_1, h) &= (\mathcal{L}g)(t', x_1, h|\alpha_1) \ , \text{ and} \\ (\mathcal{T}g)(t', x_2, h) &= (\mathcal{L}g)(t', x_2, h|\alpha_2) \ ,\end{aligned}$$

where operators $\mathcal{L}$ and $\mathcal{T}$ are given (6.3.10) and (6.3.11) by respectively. Now, choose equivalent controls $a_1 \in \mathbb{A}_{x_1,h}$ and $a_2 \in \mathbb{A}_{x_2,h}$ so that

$$a_1(t) = \alpha_1(t)\Gamma_t^{\alpha_1}(x_1,h) \quad \text{and} \quad a_2(t) = \alpha_2(t)\Gamma_t^{\alpha_2}(x_2,h) \ ,$$

for $t \geq 0$. Fix $\kappa \in (0,1)$ and let

$$\begin{aligned}x_3 &= \kappa x_1 + (1-\kappa)x_2 \ , \text{ and} \\ a_3 &= \kappa a_1 + (1-\kappa)a_2 \ .\end{aligned}$$

Note that $a_3 \in \mathbb{A}_{x_3,h}$ since $\sum_{i=1}^3 a_{3,i}(0) = x_3$. Hence,

$$(\mathcal{T}g)(t', x_3, h) = \sup_{\alpha \in \mathcal{A}} (\mathcal{L}g)(t, x_3, h|\alpha) = \sup_{a \in \mathbb{A}} (\mathcal{L}g)(t, x_3, h|a) \geq (\mathcal{L}g)(t, x_3, h|a_3) \ ,$$

with,

$$\begin{aligned}(\mathcal{L}g)(t',x_3,h|a_3) &= e^{-(\nu+\lambda_{\mathbb{P}}(1-h))(T-t')}U(\Gamma^{a_3}_{T-t'}(x_3,h)) \\ &+ (1-h)\lambda_{\mathbb{P}}\int_0^{T-t'} e^{-(\nu+\lambda_{\mathbb{P}})s}g(t+s,\Gamma^{a_3}_{t'}(x_3,h)-La^P_{3,s},1)\mathrm{d}s \\ &+ \nu\int_0^{T-t'} e^{-(\nu+(1-h)\lambda_{\mathbb{P}})s}\int_{-1}^{\infty} g(t+s,\Gamma^{a_3}_{t'}(x_3,h)+ya^S_{3,s},1)\gamma(\mathrm{d}y)\mathrm{d}s\,,\end{aligned}$$

where $a^P_{3,s}$ and $a^S_{3,s}$ denote the wealth amounts invested in the defaultable bond $P$ and stock $S$ respectively $s \geq 0$ time units after $t'$, according to control $a_3 \in \mathbb{A}_{x_3,h}$. We recall that $(x,a) \mapsto \Gamma^a_t(x,h)$ is a linear mapping, utility $U$ is a concave function and $g$ is concave on its second argument, so that

$$\begin{aligned}(\mathcal{T}g)(t',x_3,h) &\geq \kappa(\mathcal{L}g)(t',x_1,h|a_1)+(1-\kappa)(\mathcal{L}g)(t',x_2,h|a_2) \\ &= \kappa(\mathcal{T}g)(t',x_1,h)+(1-\kappa)(\mathcal{T}g)(t',x_2,h),\end{aligned}$$

completing the proof. □

### 6.4.2 Enlargement of the Action Space

In order to settle the continuity of the mapping $(t,x,h) \mapsto (\mathcal{T}g)(t,x,h)$, we will naturally make use of the enlargement of the action space $\mathcal{A}$ in (6.3.2) to the set of *randomized controls*. We recall from (5.3.1) that this is given by

$$\mathcal{R} = \{\rho : \mathbb{R}^+ \to \mathbb{P}(\mathcal{U}) \text{ measurable}\}\,,$$

where $\mathbb{P}(\mathcal{U})$ defines the set of probability measures on the Borel subsets $\mathcal{B}(\mathcal{U})$ of the compact set $\mathcal{U}$ in (6.2.1). Such an enlargement of the action space is common in these circumstances (see [51], [4], [13]) and will provide us with tools to settle the desired result. We recall that $\mathcal{A} \subseteq \mathcal{R}$, since all deterministic controls are attainable in $\mathcal{R}$ through the adoption of measures with single mass points. Also, the set $\mathcal{R}$ is endowed with the *Young Topology* as explained in Section 5.3, so that $\mathcal{R}$ is a separable, metric and compact Borel space. Then, for a sequence of controls $(\rho_n)_{n\geq 1} \subset \mathcal{R}$ and fixed control $\rho \in \mathcal{R}$, $\lim_{n\to\infty}\rho_n = \rho$ if and only if

$$\lim_{n\to\infty}\int_0^T\int_{\mathcal{U}} g(t,u)\rho_{n,t}(\mathrm{d}u)\mathrm{d}t = \int_0^T\int_{\mathcal{U}} g(t,u)\rho_t(\mathrm{d}u)\mathrm{d}t\,, \tag{6.4.1}$$

for all functions $g : [0, \infty] \times \mathcal{U} \to \mathbb{R}$ which are measurable in the first argument and continuous in the second and satisfy

$$\int_0^\infty \max_{u \in \mathcal{U}} |g(t,u)| \mathrm{d}t < \infty \, . \tag{6.4.2}$$

As a standard procedure, previous functions (6.3.5) and (6.3.6) defined on the set of deterministic Markovian controls $\mathcal{A}$ need to be extended to $\mathcal{R}$. For $\rho \in \mathcal{R}$, we define the wealth dynamics between jump times in (6.3.5) as

$$\mathrm{d}\Gamma_t^\rho(x,h) = \int_{\mathcal{U}} \Gamma_t^\rho(x,h)[r + u^S(\mu - r) + (1-h)u^P \lambda_{\mathbb{P}} L\phi]\rho_t(\mathrm{d}u)\mathrm{d}t \, ,$$

for all $(x,h) \in \mathbb{R}^+ \times \{0,1\}$, so that

$$\Gamma_t^\rho(x,h) = \Gamma_t^{\bar{\rho}}(x,h)$$

is deterministic, with $\bar{\rho} \in \mathcal{A}$ defined by $\bar{\rho}_t = \int_{\mathcal{U}} u\rho_t(\mathrm{d}u)$. On the other hand the transition density $Q$ in (6.3.6) extends to

$$\begin{aligned} Q(B|t,x,h,\rho) \; = \; & \\ \nu \int_0^{T-t} & e^{-(\nu+(1-h)\lambda_{\mathbb{P}})s} \int_{-1}^{\infty} \int_{\mathcal{U}} 1_B(t+s, \Gamma_s^\rho(x,h)(1+u^S y), h)\rho_s(\mathrm{d}u)\gamma(\mathrm{d}y)\mathrm{d}s \\ + \; (1-h)\lambda_{\mathbb{P}} & \int_0^{T-t} e^{-(\nu+\lambda_{\mathbb{P}})s} \int_{\mathcal{U}} 1_B(t+s, \Gamma_s^\rho(x,0)(1-u^P L), 1)\rho_s(\mathrm{d}u)\mathrm{d}s \, , \end{aligned} \tag{6.4.3}$$

where we recall $\gamma(\cdot)$ defines the density distribution of jumps $Y$ in asset $S$. We note that, by definition, deterministic controls can perform no better than relaxed ones. Here, we introduce a result showing that, in fact, deterministic controls in $\mathcal{A}$ do perform as well as randomized ones in $\mathcal{R}$.

**Lemma 6.4.3.** *For all $g \in \mathbb{C}_\vartheta(E)$ it holds*

$$(\mathcal{T}g)(t,x,h) = \sup_{\alpha \in \mathcal{A}} (\mathcal{L}g)(t,x,h|\alpha) = \sup_{\rho \in \mathcal{R}} (\mathcal{L}g)(t,x,h|\rho),$$

*for all $(t,x,h) \in E$.*

***Proof.*** We recall that $\mathcal{A} \subseteq \mathcal{R}$, so that for all $g \in \mathbb{C}_\vartheta(E)$ it holds

$$\sup_{\alpha \in \mathcal{A}} (\mathcal{L}g)(t,x,h|\alpha) \leq \sup_{\rho \in \mathcal{R}} (\mathcal{L}g)(t,x,h|\rho) \, ,$$

for all $(t,x,h) \in E$. In addition, recall that for all $\rho \in \mathcal{R}$ we have $\bar{\rho} \in \mathcal{A}$; so that the result will follow from

$$(\mathcal{L}g)(t,x,h|\rho) \leq (\mathcal{L}g)(t,x,h|\bar{\rho}) ,$$

for all $\rho \in \mathcal{R}$. Now, note from the function (6.3.7) that $R(t,x,h,\rho) = R(t,x,h,\bar{\rho})$, since $\Gamma_t^{\rho}(x,h) = \Gamma_t^{\bar{\rho}}(x,h)$ by definition. In addition, any function $g \in \mathbb{C}_\vartheta$ is concave on its second argument, so that by Jensen's inequality we have

$$\int_{\mathcal{U}} g(t+s,\Gamma_s^{\rho}(x,h)(1+u^S y),h)\rho_s(\mathrm{d}u) \leq g(t+s,\Gamma_s^{\rho}(x,h)(1+\bar{\rho}^S y),h) ,$$

and

$$\int_{\mathcal{U}} g(t+s,\Gamma_s^{\rho}(x,0)(1-u^P L),1)\rho_s(\mathrm{d}u) \leq g(t+s,\Gamma_s^{\rho}(x,0)(1-\bar{\rho}^P L),1) ,$$

for all $(t,x,h) \in E$. Hence, it holds that

$$\begin{aligned}(\mathcal{L}g)(t,x,h|\rho) &= R(t,x,h,\rho) + \sum_k \int g(s,y,k)Q(\mathrm{d}s,\mathrm{d}y,\mathrm{d}u,k|t,x,h,\rho) \\ &\leq R(t,x,h,\bar{\rho}) + \sum_k \int g(s,y,k)Q(\mathrm{d}s,\mathrm{d}y,k|t,x,h,\bar{\rho}) \\ &= (\mathcal{L}g)(t,x,h|\bar{\rho}) ,\end{aligned}$$

completing the proof. □

### 6.4.3 The Proof of Continuity

**Lemma 6.4.4.** *The mapping $(t,x,h) \mapsto (\mathcal{T}g)(t,x,h)$ is continuous, for all $g \in \mathbb{C}_\vartheta(E)$.*

***Proof.*** Note that all sets in $\{0,1\}$ are open and therefore it suffices to prove that $(t,x) \mapsto (\mathcal{T}g)(t,x,h)$ is continuous. In view of Lemma 6.4.3, we note we can make use of relaxed controls within $\mathcal{R}$, since

$$(\mathcal{T}g)(t,x,h) = \sup_{\rho \in \mathcal{R}} (\mathcal{L}g)(t,x,h|\rho).$$

We recall that $\mathcal{R}$ is a compact Borel space with respect to the *Young topology*, therefore, in view of the definition of $\mathcal{L}$ in (6.3.10) the proof would follow from the continuity of the mappings $E \times \mathcal{R} \to \mathbb{R}$ given by

$$(t,x,\rho) \mapsto e^{-(\nu+(1-h)\lambda_{\mathbb{P}})(T-t)} U(\Gamma_{T-t}^{\rho}(x,h)) , \tag{6.4.4}$$

and

$$(t,x,\rho) \mapsto \sum_k \int g(s,y,k) Q(\mathrm{d}s, \mathrm{d}y, k|t,x,h,\rho) , \tag{6.4.5}$$

for fixed $h \in \{0,1\}$. Since utility $U$ is a continuous function and the exponential term in (6.4.4) is continuous on time, continuity of mapping (6.4.4) reduces to showing that

$$(t,x,\rho) \mapsto \Gamma^{\rho}_{T-t}(x,h)$$

is continuous. From the definition of $\Gamma$ in (6.3.5), this is equivalent to showing that

$$\int_0^t \int_{\mathcal{U}} u^S(\mu - r)\rho_s(\mathrm{d}u)\mathrm{d}s \quad \text{and} \quad \int_0^t \int_{\mathcal{U}} u^P \lambda_{\mathbb{P}} L\phi \rho_s(\mathrm{d}u)\mathrm{d}s \tag{6.4.6}$$

are continuous in $(t,\rho)$. Following the approach in [11] (Prop. 4.3) we provide proof for the first integral expression in (6.4.6), the second is proved in a similar fashion. Let $(t_n,\rho_n)_{n\geq 1} \subset [0,T] \times \mathcal{R}$ be a sequence with $(t_n,\rho_n) \to (t,\rho)$, in order to ease notation let $\epsilon_{n,s}$ and $\epsilon_s$ denote

$$\epsilon_{n,s} = \int_{\mathcal{U}} u^S(\mu - r)\rho_{n,s}(\mathrm{d}u) \quad \text{and} \quad \epsilon_s = \int_{\mathcal{U}} u^S(\mu - r)\rho_s(\mathrm{d}u) \, .$$

Then,

$$\begin{aligned} \Big| \int_0^{t_n} \epsilon_{n,s}\mathrm{d}s - \int_0^t \epsilon_s \mathrm{d}s \Big| &\leq \Big| \int_0^{t_n} \epsilon_{n,s}\mathrm{d}s - \int_0^t \epsilon_{n,s}\mathrm{d}s \Big| + \Big| \int_0^t \epsilon_{n,s}\mathrm{d}s - \int_0^t \epsilon_s\mathrm{d}s \Big| \\ &\leq (\mu - r)|t_n - t| + \Big| \int_0^t \epsilon_{n,s}\mathrm{d}s - \int_0^t \epsilon_s\mathrm{d}s \Big| \, . \end{aligned}$$

Noting that function $u \mapsto g(t,u) = g(u) = u^S(\mu - r)$ is such that satisfies (6.4.2), it follows from the characterization of convergence in $\mathcal{R}$ in (6.4.1) that

$$(\mu - r)|t_n - t| + \Big| \int_0^t \epsilon_{n,s}\mathrm{d}s - \int_0^t \epsilon_s\mathrm{d}s \Big| \xrightarrow{n\to\infty} 0 \, .$$

We now turn our attention to the mapping (6.4.5), we note from the definition of the kernel $Q$ in (6.4.3) that continuity follows from that of functions

$$W_1(t,x,\rho) = \int_0^{T-t} e^{-(\nu+(1-h)\lambda_{\mathbb{P}})s} \int_{-1}^{\infty} \int_{\mathcal{U}} g(t+s, \Gamma^{\rho}_s(x,h)(1+u^S y), h)\rho_s(\mathrm{d}u)\gamma(\mathrm{d}y)\mathrm{d}s \tag{6.4.7}$$

and

$$W_2(t,x,\rho) = \int_0^{T-t} e^{-(\nu+\lambda_{\mathbb{P}})s} \int_{\mathcal{U}} g(t+s, \Gamma^{\rho}_s(x,0)(1-u^P L), 1)\rho_s(\mathrm{d}u)\mathrm{d}s \, , \tag{6.4.8}$$

for a fixed $h \in \{0,1\}$. We follow a procedure in [12] (Lemma 1) to prove the continuity of equation (6.4.7), that of (6.4.8) is proved in a similar fashion. We begin assuming that $g \in \mathbb{C}_\vartheta(E)$ is a bounded function and we let $(t_n, x_n, \rho_n)_{n\geq 0} \subset [0,T] \times \mathbb{R}^+ \times \mathcal{R}$ be a sequence with $(t_n, x_n, \rho_n) \to (t, x, \rho)$; in order to ease notation let $g'_n$ and $g'$ denote functions given by

$$g'_n(s,u) = g(t_n + s, \Gamma_s^{\rho_n}(x_n, h)(1 + u^S y), h) \quad \text{and} \quad g'(s,u) = g(t + s, \Gamma_s^{\rho}(x, h)(1 + u^S y), h)\,.$$

Then

$$\begin{aligned}
&|W_1(t_n, x_n, \rho_n) - W_1(t, x, \rho)| \leq \\
&\qquad \leq \Big| \int_{T-t}^{T-t_n} e^{-(\nu + (1-h)\lambda_{\mathbb{P}})s} \int_{-1}^{\infty} \int_{\mathcal{U}} g'_n(s,u) \rho_{n,s}(\mathrm{d}u) \gamma(\mathrm{d}y) \mathrm{d}s \Big| \\
&\qquad + \int_0^{T-t} e^{-(\nu + (1-h)\lambda_{\mathbb{P}})s} \int_{-1}^{\infty} \int_{\mathcal{U}} |g'_n(s,u) - g'(s,u)| \rho_{n,s}(\mathrm{d}u) \gamma(\mathrm{d}y) \mathrm{d}s \\
&\qquad + \Big| \int_0^{T-t} e^{-(\nu + (1-h)\lambda_{\mathbb{P}})s} \int_{-1}^{\infty} \int_{\mathcal{U}} g'(s,u) (\rho_{n,s}(\mathrm{d}u) - \rho_s(\mathrm{d}u)) \gamma(\mathrm{d}y) \mathrm{d}s \Big|\,.
\end{aligned}$$

Since $g$ is a bounded function, the first term converges to 0 as $n \to \infty$. Due to dominated convergence and the continuity of $\Gamma$ and $g$ the second term does also converge to 0 as $n \to \infty$. Finally, the third term converges to 0 due to the characterization of convergence in $\mathcal{R}$ in (6.4.1).

Now, we recall from (6.3.13) that for all $g \in \mathbb{C}_\vartheta$ there exists some constant $c_g \in \mathbb{R}^+$ satisfying $g(t,x,h) \leq c_g(1+x)e^{\vartheta(T-t)}$. Let $w(t,x,h) = g(t,x,h) - c_g(1+x)e^{\vartheta(T-t)}$ define a negative and continuous function; then, there exists (cf. [4], Lemma 7.14) a decreasing sequence of bounded functions $(w_n)_{n\geq 1}$ with $w_n \to w$ pointwise, therefore

$$W'_n(t,x,\rho) = \int_0^{T-t} e^{-(\nu + (1-h)\lambda_{\mathbb{P}})s} \int_{-1}^{\infty} \int_{\mathcal{U}} w_n(t+s, \Gamma_s^{\rho}(x,h)(1 + u^S y), h) \rho_s(\mathrm{d}u) \gamma(\mathrm{d}y) \mathrm{d}s$$

defines a bounded and decreasing sequence of continuous functions with

$$\begin{aligned}
&W'_n(t,x,\rho) \to \\
&\quad W_1(t,x,\rho) - c_g \int_0^{T-t} e^{-(\nu + (1-h)\lambda_{\mathbb{P}})s} \int_{-1}^{\infty} (1 + \Gamma_s^{\rho}(x,h)(1 + \bar{\rho}_s^S y)) e^{\vartheta(T-t)} \gamma(\mathrm{d}y) \mathrm{d}s\,.
\end{aligned} \tag{6.4.9}$$

as $n \to \infty$. Since the pointwise limit of non-increasing sequences of continuous functions is upper semicontinuous, it follows that the right hand side function in (6.4.9) is upper semicontinuous. In addition, the term

$$c_g \int_0^{T-t} e^{-(\nu + (1-h)\lambda_{\mathbb{P}})s} \int_{-1}^{\infty} (1 + \Gamma_s^{\rho}(x,h)(1 + \bar{\rho}_s^S y)) e^{\vartheta(T-t)} \gamma(\mathrm{d}y) \mathrm{d}s$$

is continuous, therefore $W_1$ is upper semicontinuous. Taking $w(t,x,h) = -g(t,x,h) + c_g(1+x)e^{\vartheta(T-t)}$ lower semicontinuity of $W_1$ is achieved, proving the result on continuity for $W_1$ and completing the proof. $\square$

## 6.5 Proof of Theorem 6.3.3

We recall from Lemma 6.3.1 that the value of the original portfolio optimization problem (6.2.4) can be derived as the sum of expected rewards $v$ in (6.3.8). Theorem 6.3.3 implies that, in addition, the value function $V_{\pi^*}$ is characterized as a unique fixed point to a dynamic programming operator, so that the use of computational methods to approximate its value is justified. The main line of the proof is directed towards the use of Theorem 5.2.5 in the introductory results section.

***Proof of Theorem 6.3.3.*** We recall that the MDP $(E, \mathcal{A}, Q, R)$ is such that $v = V_{\pi^*}$ and Assumption 5.2.2 is satisfied, with $\kappa < 1$. In addition, the Banach space $\mathbb{C}_\vartheta(E)$ is a closed subset of $\mathbb{M}_\vartheta(E)$ satisfying

i) $0 \in \mathbb{C}_\vartheta(E)$,

ii) $\mathcal{T} : \mathbb{C}_\vartheta(E) \to \mathbb{C}_\vartheta(E)$.

Thus, according to Theorem 5.2.5 the proof would follow from the existence, for all $g \in \mathbb{C}_\vartheta(E)$, of a deterministic policy $f \in F$ such that $\mathcal{T}g = \mathcal{T}_f g$, with

$$(\mathcal{T}_f g)(t,x,h) = R(t,x,h,f^{(t,x,h)}) + \sum_k \int g(s,y,k) Q(\mathrm{d}s, \mathrm{d}y, k | t,x,h,f^{(t,x,h)}) \,,$$

for all $(t,x,h) \in E$.

It follows from a well-known result in [4] (Chapter 7) that there exists a randomized policy $f : E \to \mathcal{R}$ such that $\mathcal{T}g = \mathcal{T}_f g$ for all functions $g \in \mathbb{C}_\vartheta(E)$. However, we note from Lemma 6.4.3 that the deterministic policy $\bar{f} : E \mapsto \mathcal{A}$ given by

$$\bar{f}_s^{(t,x,h)} = \int_{\mathcal{U}} u f_s^{(t,x,h)}(\mathrm{d}u) \in \mathcal{U}$$

for all $(t,x,h) \in E$, is measurable and such that $\mathcal{T}g = \mathcal{T}_{\bar{f}} g$, therefore completing the proof. $\square$

## 6.6 Numerical Analysis

In what follows we present and analyse computational results to our discrete-time infinite-horizon optimization problem $(E, \mathcal{A}, Q, R)$ defined in (6.3.1)-(6.3.8), for different measures of risk aversion. Numerical approximations of optimal allocation strategies $\pi^* \in \Pi$, along with optimal values $V_{\pi^*}$, are obtained through the method of value iteration as introduced in Section 5.4 and justified by the results in Theorem 6.3.3. For this matter, we have made use of an homogeneous space discretization as introduced in [11] (Section 5.3).

We recall that the equivalency result in Lemma 6.3.1 warrants the optimality of these strategies in the original portfolio optimization problem (6.2.4), where alterations on wealth allocations are only decided at times of jumps in the market (a jump in asset $S$ or a default in $P$) and span as time-dependent allocation functions until the next market jump; these jumps are referred to as *epochs* within the context of the MDP. Thus, we take advantage of the flexibility of the method regarding the choice of utility function and, in view of the original problem, determine distinctions on optimal wealth allocation strategies under different families of utilities, as well as the impact of generalizing utilities towards risky investments. Additionally, we assess the influence on allocation strategies of the different parameters defining the model and, more importantly, the effect of the short selling restriction imposed on the original definition of the problem.

Results in this section allow for us to complement and draw comparisons with the work in [5, 8, 16, 37], expanding its scope as discussed in Section 6.7. The focus is on popular power, logarithmic and exponential utility measures of risk aversion. The *constant relative risk aversion* (CRRA) family of power utility functions is given by

$$U(x) = \frac{x^{1-c}}{1-c} \quad \text{for } 0 < c < 1, \tag{6.6.1}$$

so that the level of *relative risk aversion* is constant and given by $R(x) = -\frac{xU''(x)}{U'(x)} = c$, where $U'$ and $U''$ denote the first and second order derivatives of $U$ respectively. The logarithmic family of utility functions is on the other hand given by

$$U(x) = \log(x + c) \quad \text{for } c \in \mathbb{R}_+, \tag{6.6.2}$$

and its level of risk aversion is $R(x) = \frac{x}{x+c}$, so that it is a CRRA utility measure only if $c = 0$; if $c > 0$ this is an *increasing relative risk aversion* (IRRA) measure. Finally, the exponential family of measures is a popular *constant absolute risk aversion* (CARA) family given by

$$U(x) = 1 - \frac{e^{-cx}}{c} \quad \text{for } c \in \mathbb{R}_+, \tag{6.6.3}$$

so that the *absolute risk aversion* level is constant and given by $A(x) = -\frac{U''(x)}{U'(x)} = c$.

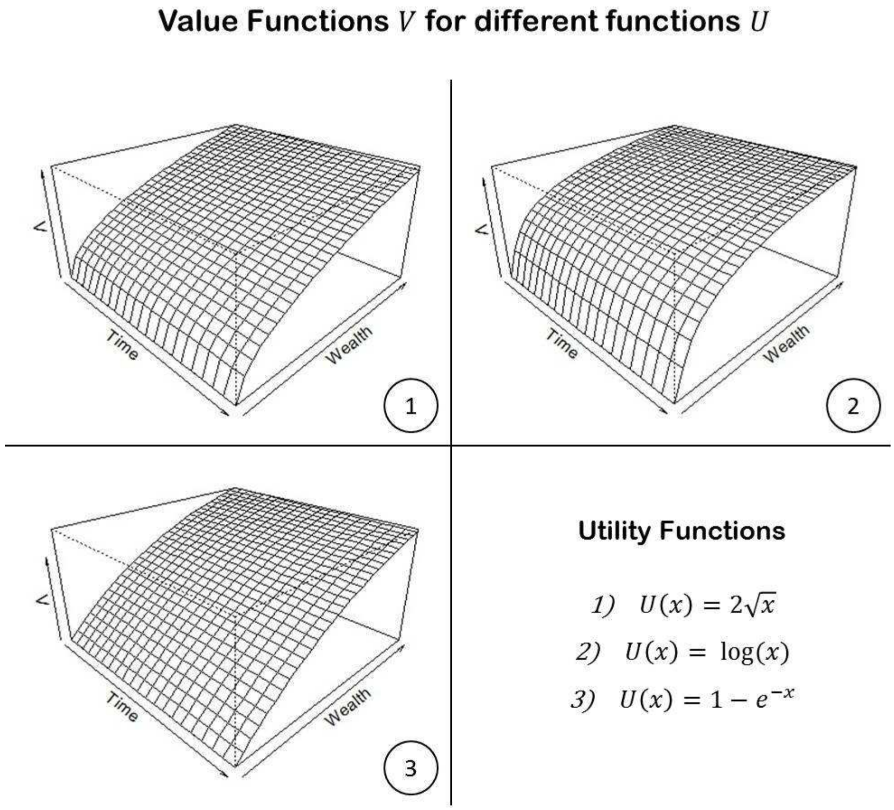


**Figure 6.4:** Approximation of pre-default $V$ for different utility functions $U$. Results obtained through the method of value iteration with convergence in 10 iterations. $T = 1$, $r = \mu = 0.05$, $\lambda = 0.25$ $\phi = 1.3$, $L = 0.5$ and $\nu = 10$.

Figure 6.4 presents pre-default value functions under different choices of measures; we note that these are increasing in wealth and decreasing in time. In these cases, optimal allocation strategies correspond to varying fractional distributions of wealth between the defaultable bond and the bank account; and convergence in the grid has been in all cases achieved under 10 iterations, using an initial candidate $V$ according to the strategy of investing all wealth in $B$.

### 6.6.1 Discussion of Parameter Choices

In view of the questions to address within this section, numerical simulations are undertaken with a set interest rate of $r = 0.05$. In addition, values such as jump intensities $\lambda$ and $\nu$, risk premium $\phi$, loss at default $L$ and appreciation rate of the stock $\mu$ are, unless otherwise stated, fixed to sensible positive values within a financial context. This is done using parameter choices for numerical simulations in [5] and [16] as a reference, therefore allowing for direct comparisons of our results with recent work on portfolio management with defaultable bonds, and establishing general properties on optimal strategies with respect to variations on utility functions and time, wealth and default state values.

### 6.6.2 Performance Analysis of Utility Functions

Optimal allocations under different utilities vary on time, wealth values and level of aversion towards risky investments. Under an exponential measure of constant absolute risk aversion, the level of optimal risky investments is highly dependent on wealth values; in this case, both $\pi^P$ and $\pi^S$ are decreasing functions of wealth for $x > \kappa$, with $\kappa \in \mathbb{R}_+$ small as observed in the case of a defaultable bond in Figure 6.5. In addition, a slight decrease on the aversion towards investing in $P$ is noticed as

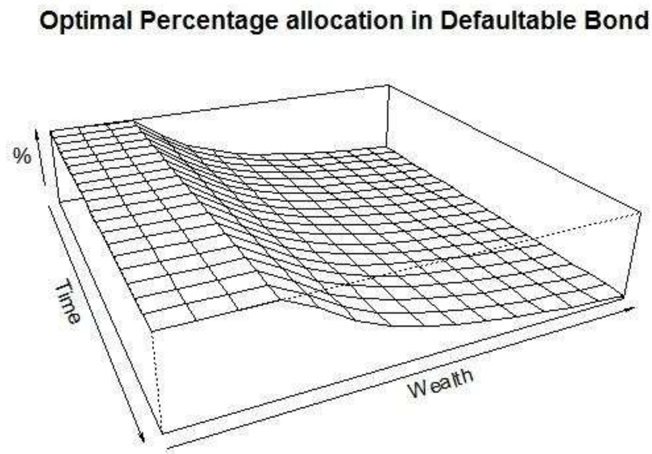


**Figure 6.5:** Optimal $\pi^P$, for $U(x) = 1 - e^{-x}$ and varying values of $t \in [0, T]$ and $x \geq 0$. Parameters $r = 0.05$, $\nu = 10$ and $\lambda = 0.25$.

time approaches deadline. On the contrary, the optimal wealth distribution remains invariant with regards to changes in wealth for both power and logarithmic utilities, however, there exists a mild increase of aversion towards the exposure to risky bonds as time approaches deadline, while remaining nearly time-invariant when the planning horizon is large; this is specially noticeable within logarithmic utilities and has been previously reported as discussed in Section 6.7. Certainly, as time approaches deadline (and maturity in $P$ under definition (6.0.3)) there exists an increase on the value of $P$ and a decrease on the likelihood of default, implying that the defaultable bond gets relatively cheap only when the planning horizon is large.

Additionally, stock investments remain time-invariant under both these measures; a previously reported result that is discussed in Section 6.7. Figure 6.6 below presents varying levels of the optimal percentage allocation $\pi^S$ for varying values of the difference between the appreciation rate of the stock $\mu$ and the interest rate $r$ under power utility functions $U(x) = \frac{x^{1-c}}{1-c}$, showing that this is a linearly increasing function on $\mu - r$ and a decreasing function on the level of constant relative risk aversion $R(x) = c$. However, the short-selling restriction imposed to the portfolio optimization

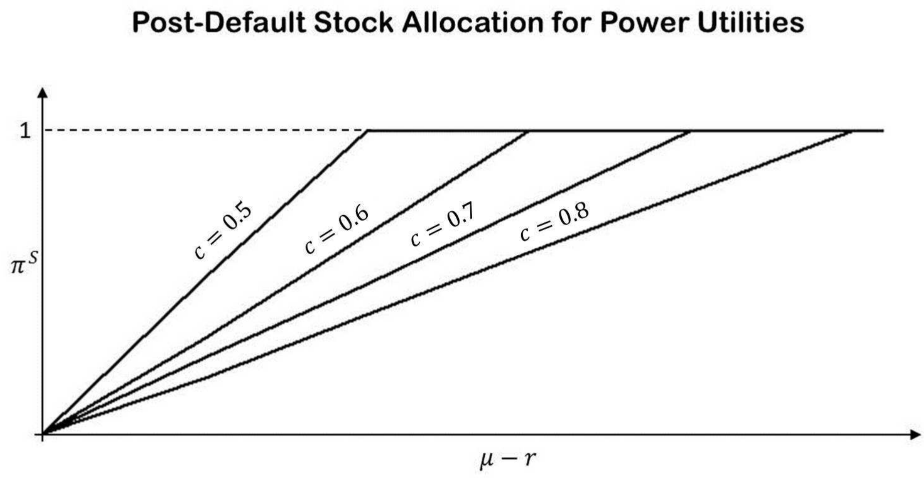


**Figure 6.6:** Optimal $\pi^S$ after default, as a function of the distance between the appreciation and interest rate and for different power utility measures $U(x) = \frac{x^{1-c}}{1-c}$. Maximum allocation equals 1, since no short-selling is allowed. Here, $\lambda = 0.25$ $\phi = 1.3$, $L = 0.5$ and $\nu = 10$.

problem causes allocations $\pi^S$ to remain invariant to a default event only if pre-default bond allocations $\pi^B$ are strictly positive; if $\pi^B = 0$ at default time, both bond and stock percentage investments may increase following a default event in $P$.

Moreover, we note in Figure 6.7 that for fixed $t \in [0, T]$ and wealth $x \in \mathbb{R}_+$, the value function $V$ is such that $V(t, x, 0) \geq V(t, x, 1)$ for all $(t, x) \in [0, t] \times \mathbb{R}_+$. In

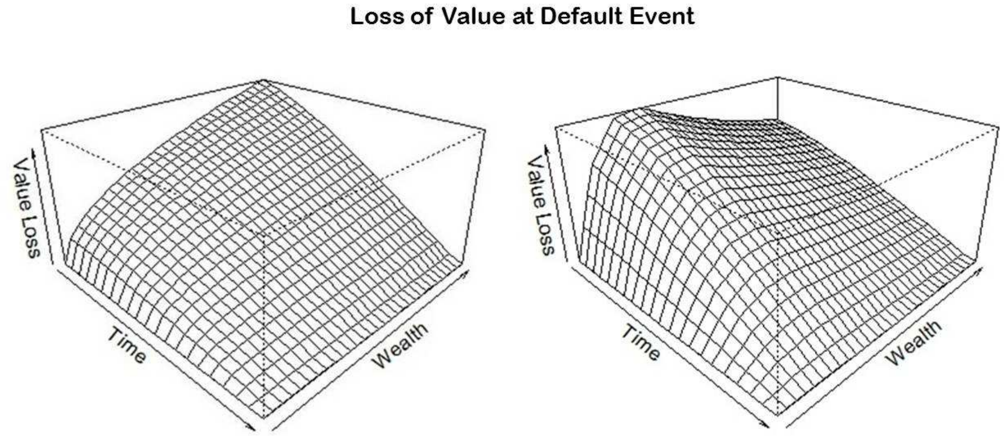


**Figure 6.7:** Approximation of the loss in $V$ at default. Here $T = 1$, $r = \mu = 0.05$, $\lambda = 0.25$ $\phi = 1.3$, $L = 0.5$ and $\nu = 10$. On the left hand side $U(x) = \frac{\sqrt{x}}{2}$, on the right hand side $U(x) = 1 - e^{-x}$.

addition, $V(t, x, 0) - V(t, x, 1)$ is decreasing in time and equal to 0 at $t = T$, a common feature under all utilities. Certainly, a default event decreases the dimensionality of the problem through a reduction in the choices of investment opportunities. Under exponential utilities and for $x > \kappa$, the loss in value is a decreasing function on wealth.

Finally, utilities analysed present common properties with regards to alterations on the values of several parameters defining the model. Optimal allocations $\pi^P$ are increasing functions of the risk premium $\phi$ and decreasing functions of the loss value $L$ at default, as illustrated in Figure 6.8 for a given pre-default state $(t, x, 0) \in E$ and utility $U(x) = 2\sqrt{x}$ in a two-Bond market. A higher incentive for bearing risk in $P$ motivates a higher investment; on the contrary, the opposite effect is caused by decreasing the return on recovery, despite the fact that it increases the yield on the bond. It is also never optimal to invest in a defaultable bond provided $\phi \leq 1$. In addition, optimal

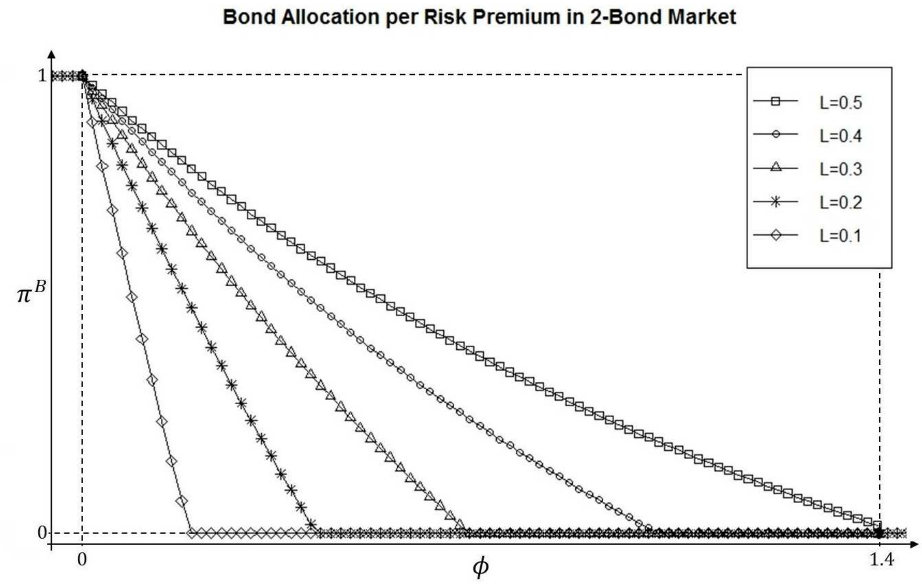


**Figure 6.8:** Approximation of pre-default $\pi^B$ in a two-Bond market, for different risk premium $\phi$ and loss on default $L$. Parameters $r = 0.05$, $\nu = 10$, $\lambda = 0.25$ and utility $U(x) = 2\sqrt{x}$.

risky investments present a similar dependency on the level of aversion under different utilities; these are decreasing functions of the level of relative/absolute risk aversion, as observed in Figure 6.9 for a defaultable bond under power and exponential utilities.

## 6.7 Discussion

This final Chapter has studied an extension of the work in [11–13] to the context of a defaultable market, in order to present a numerical technique for the analysis of optimal wealth allocation strategies for risk adverse investors, allowing for the use of broad families of utility functions. The original continuous-time portfolio optimization problem has been transformed into a discrete-time Markov Decision Process and its value function has been characterized as the unique fixed point to a dynamic programming operator, justifying the use of value iteration algorithms to provide the approximations of results of our interest.

The chapter has analysed the dependence of optimal portfolio selections on the risk

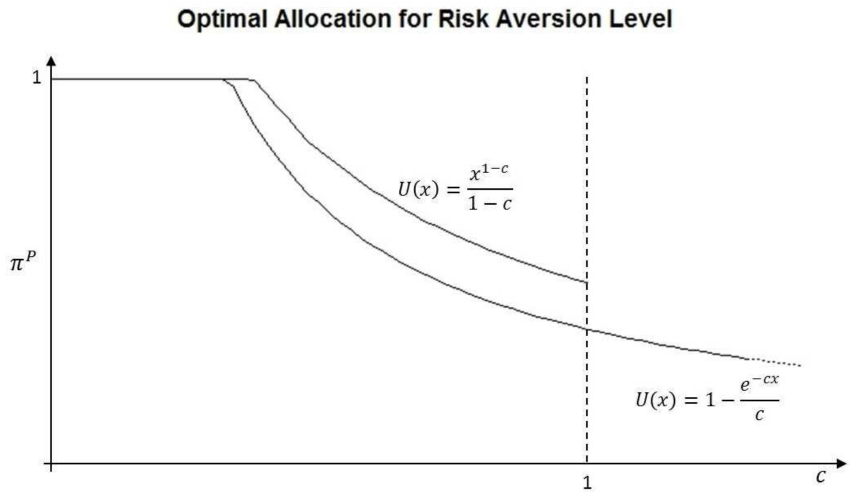


**Figure 6.9:** Optimal allocation $\pi^P$ for utilities $U(x) = \frac{x^{1-c}}{1-c}$, $U(x) = 1 - \frac{e^{-cx}}{c}$ and varying values of $c \geq 0$ in a two-Bond market with fixed $(x, t, 0) \in E$. Parameters $r = 0.05$, $\nu = 10$ and $\lambda = 0.25$

premium, recovery of market value and several other parameters defining the model, and has extended the scope of the results in [5, 8, 16, 37] to broader families of utility functions, highlighting relevant divergences on optimal strategies with respect to variations and generalizations in choices of utilities. In addition, the work as examined the impact of a short selling restriction within the market, identifying a dependency on optimal stock allocations with respect to default event on a corporate bond.

We recall that the work in [5, 8, 16, 37] covers continuous markets primarily driven by Brownian components and focuses on power utility functions and a restrictive choice of logarithmic utility. The analysis in Section 6.6 suggests that, similarly to [5, 8, 16], investments on defaultable bonds are only justified when the associated risk is correctly priced, measured in terms of risk premium coefficients $\phi$. Also, similar monotonicity properties on optimal defaultable bond allocations have been identified in comparison to those presented in [5] and [16], under power and logarithmic utilities, so that these are decreasing on $\phi$, increasing on $L$ and there exists a reduction of the risk aversion as time approaches maturity; this work suggests that such properties extend to generalizations of logarithmic utility functions defined in (6.6.2). On the contrary, under exponential measures in (6.6.3), there exists a slight increase in the risk aversion towards $P$ in time, and optimal defaultable bond allocations are highly dependent on the wealth value and decreasing for $x > \kappa$, for some small $\kappa \in \mathbb{R}_+$. Additionally, we observed that in this case $V(t, x, 0) - V(t, x, 1)$ is decreasing on $x$ for $x \geq \kappa$.

Furthermore, it has been shown that the investment in the risky bond and stock is always prioritised as the levels of constant relative or absolute risk aversion are diminished. Also, optimal stock investments have been identified as linear functions of the appreciation rate of the stock and interest rate, similarly to [41]; however, unlike results reported in [5] and [16], a short-selling restriction has been identified to trigger a dependency on the allocation with respect to default event in $P$.

### 6.7.1 Extensions and Limitations of the Model

This Chapter has treated a portfolio optimization problem involving one bank account, a pure jump asset and a defaultable bond. The problem of considering a diversified portfolio involving multiple assets and defaultable bonds is a natural extension to this work, not addressed in here in order to avoid technicalities part of extensive models.

Other natural extensions of the model under the reduction to an MDP approach were pointed out by Bäuerle and Rieder (see [11]). These include the introduction of regime switching markets, where the different economical regimes are modelled by a continuous-time Markov chain $(I_t)_{t \geq 0}$ in a similar manner to [16], so that parameters and coefficients defining the bank account, asset and defaultable bond vary according to the different states of $I$. In this scenario, the state space within the formulation of the MDP gains a degree of dimensionality, but the embedding procedure remains similar. In addition, models with partial information can be considered upon assuming that $I$ is a *hidden* process and making use of filtering theory.

Finally, we note that this work has made rather strong assumptions regarding most parameters defining the model. The interest rate, stock appreciation rate, default intensities and loss on default rate have all considered constant. An extension to Brownian models for such parameters would not be tractable under the approach presented in this Chapter; however, the inclusion of different economical regimes as discussed above could present a more realistic case of study.

# Appendices

APPENDIX A

# APPENDIX TO CHAPTER 3

In this Appendix, we include some lengthy analytical calculations leading to minor results that are part of Chapter 3 in this thesis.

## A.1 CLOSED FORM EXPRESSION OF FUNCTIONS $G$ AND $H$

**Lemma A.1.1.** *Function $G$ in* (3.1.4) *is always positive and can be expressed as*

$$\begin{aligned} G(t,x) &= 1-\Big(2\alpha e^{-\sigma x}-\alpha^2 e^{-2\sigma x}\Big)\Phi\Big(\frac{x-\lambda(T-t)}{\sqrt{T-t}}\Big) \\ &\quad +\Big(\frac{\alpha^2\sigma}{\lambda-\sigma}e^{2(\lambda-\sigma)x}-\frac{2\alpha\sigma}{2\lambda-\sigma}e^{(2\lambda-\sigma)x}\Big)\Phi\Big(\frac{-x-\lambda(T-t)}{\sqrt{T-t}}\Big) \\ &\quad +\frac{4\alpha(\sigma-\lambda)}{2\lambda-\sigma}e^{\frac{\sigma}{2}(\sigma-2\lambda)(T-t)}\Phi\Big(\frac{-x+(\lambda-\sigma)(T-t)}{\sqrt{T-t}}\Big) \\ &\quad +\frac{\alpha^2(\lambda-2\sigma)}{\lambda-\sigma}e^{2\sigma(\sigma-\lambda)(T-t)}\Phi\Big(\frac{-x+(\lambda-2\sigma)(T-t)}{\sqrt{T-t}}\Big) \end{aligned} \tag{A.1.1}$$

*for all $(t,x)\in[0,T]\times\mathbb{R}_+$ and $\lambda\in\mathbb{R}/\{\frac{\sigma}{2},\sigma\}$. If $\lambda=\frac{\sigma}{2}$ it is given by*

$$\begin{aligned} G(t,x) &= 1+2\sigma\alpha\sqrt{\frac{T-t}{2\pi}}e^{-\frac{(x+\frac{\sigma}{2}(T-t))^2}{2(T-t)}}-\Big(2\alpha e^{-\sigma x}-\alpha^2 e^{-2\sigma x}\Big)\Phi\Big(\frac{x-\frac{\sigma}{2}(T-t)}{\sqrt{T-t}}\Big) \\ &\quad -\Big(2\alpha^2 e^{-\sigma x}+2\alpha(1+\sigma x)+\sigma^2\alpha(T-t)\Big)\Phi\Big(\frac{-x-\frac{\sigma}{2}(T-t)}{\sqrt{T-t}}\Big) \\ &\quad +3\alpha^2 e^{\sigma^2(T-t)}\Phi\Big(\frac{-x-\frac{3\sigma}{2}(T-t)}{\sqrt{T-t}}\Big) \end{aligned} \tag{A.1.2}$$

*for all $(t,x)\in[0,T]\times\mathbb{R}_+$. Finally, if $\lambda=\sigma$, it is given by*

$$\begin{aligned} G(t,x) &= 1-2\sigma\alpha^2\sqrt{\frac{T-t}{2\pi}}e^{-\frac{(x+\sigma(T-t))^2}{2(T-t)}}-\Big(2\alpha e^{-\sigma x}-\alpha^2 e^{-2\sigma x}\Big)\Phi\Big(\frac{x-\sigma(T-t)}{\sqrt{T-t}}\Big) \\ &\quad -\Big(2\alpha e^{\sigma x}-\alpha^2(1+2\sigma x)-2\alpha^2\sigma^2(T-t)\Big)\Phi\Big(\frac{-x-\sigma(T-t)}{\sqrt{T-t}}\Big) \end{aligned} \tag{A.1.3}$$

*for all $(t,x)\in[0,T]\times\mathbb{R}_+$.*

***Proof.*** We recall from (3.1.4) that function $G$ is given by

$$G(t,x) = (1-\alpha e^{-\sigma x})^2 + 2\sigma\alpha \int_x^\infty (e^{-\sigma z} - \alpha e^{-2\sigma z})(1 - F_{S^\lambda_{T-t}}(z))\, \mathrm{d}z$$

for all $(t,x) \in [0,T] \times \mathbb{R}_+$. It is trivial that $G$ is always positive; now, we note from (3.1.3) that the distribution of $S^\lambda_{T-t}$ is given by

$$F_{S^\lambda_{T-t}}(x) = \mathbb{P}(S^\lambda_{T-t} \le x) = \Phi\Big(\frac{x-\lambda(T-t)}{\sqrt{T-t}}\Big) - e^{2\lambda x}\Phi\Big(\frac{-x-\lambda(T-t)}{\sqrt{T-t}}\Big) ,$$

for all $(t,x) \in [0,T] \times \mathbb{R}_+$, so that function $G$ is split up as

$$G(t,x) = (1-\alpha e^{-\sigma x})^2 + 2\sigma\alpha\big(A(x) + B(t,x) + C(t,x)\big) , \tag{A.1.4}$$

with

$$A(x) = \int_x^\infty (e^{-\sigma z} - \alpha e^{-2\sigma z})\, \mathrm{d}z ,$$
$$B(t,x) = \int_x^\infty (\alpha e^{-2\sigma z} - e^{-\sigma z})\Phi\Big(\frac{z-\lambda(T-t)}{\sqrt{T-t}}\Big)\, \mathrm{d}z ,$$

and

$$C(t,x) = \int_x^\infty (e^{(2\lambda-\sigma)z} - \alpha e^{2(\lambda-\sigma)z})\Phi\Big(\frac{-z-\lambda(T-t)}{\sqrt{T-t}}\Big)\, \mathrm{d}z .$$

Equation $A$ is easily derived to be

$$A(x) = \frac{1}{\sigma}e^{-\sigma x} - \frac{\alpha}{2\sigma}e^{-2\sigma x} ;$$

on the other hand, we make use of integration by parts in order to derive $B$, so that

$$\begin{aligned} B(t,x) &= \Big(\frac{1}{\sigma}e^{-\sigma z} - \frac{\alpha}{2\sigma}e^{-2\sigma z}\Big)\Phi\Big(\frac{z-\lambda(T-t)}{\sqrt{T-t}}\Big)\Big|_x^\infty \\ &- \int_x^\infty \Big(\frac{1}{\sigma}e^{-\sigma z} - \frac{\alpha}{2\sigma}e^{-2\sigma z}\Big)\phi\Big(\frac{z-\lambda(T-t)}{\sqrt{T-t}}\Big)\frac{\mathrm{d}z}{\sqrt{T-t}} , \end{aligned}$$

with $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$. Careful evaluation of the integral expression above yields

$$\begin{aligned} B(t,x) &= \Big(\frac{\alpha}{2\sigma}e^{-2\sigma x} - \frac{1}{\sigma}e^{-\sigma x}\Big)\Phi\Big(\frac{x-\lambda(T-t)}{\sqrt{T-t}}\Big) \\ &+ \frac{\alpha}{2\sigma}e^{2\sigma(\sigma-\lambda)(T-t)}\Phi\Big(\frac{-x+(\lambda-2\sigma)(T-t)}{\sqrt{T-t}}\Big) \\ &- \frac{1}{\sigma}e^{\frac{\sigma}{2}(\sigma-2\lambda)(T-t)}\Phi\Big(\frac{-x+(\lambda-\sigma)(T-t)}{\sqrt{T-t}}\Big) , \end{aligned}$$

for all $\lambda \in \mathbb{R}$.

Equation $C$ needs to be treated independently depending on the value of the parameter $\lambda$. If $\lambda \neq \frac{\sigma}{2}, \sigma$, integration by parts gives

$$\begin{aligned} C(t,x) &= \Big(\frac{1}{2\lambda-\sigma}e^{(2\lambda-\sigma)z} - \frac{\alpha}{2(\lambda-\sigma)}e^{2(\lambda-\sigma)z}\Big)\Phi\Big(\frac{-z-\lambda(T-t)}{\sqrt{T-t}}\Big)\Big|_x^{\infty} \\ &+ \int_x^{\infty}\Big(\frac{1}{2\lambda-\sigma}e^{(2\lambda-\sigma)z} - \frac{\alpha}{2(\lambda-\sigma)}e^{2(\lambda-\sigma)z}\Big)\phi\Big(\frac{-z-\lambda(T-t)}{\sqrt{T-t}}\Big)\frac{\mathrm{d}z}{\sqrt{T-t}}, \end{aligned}$$

so that

$$\begin{aligned} C(t,x) &= \Big(\frac{\alpha}{2(\lambda-\sigma)}e^{2(\lambda-\sigma)x} - \frac{1}{2\lambda-\sigma}e^{(2\lambda-\sigma)x}\Big)\Phi\Big(\frac{-x-\lambda(T-t)}{\sqrt{T-t}}\Big) \\ &+\frac{1}{2\lambda-\sigma}e^{\frac{\sigma}{2}(\sigma-2\lambda)(T-t)}\Phi\Big(\frac{-x+(\lambda-\sigma)(T-t)}{\sqrt{T-t}}\Big) \\ &-\frac{\alpha}{2(\lambda-\sigma)}e^{2\sigma(\sigma-\lambda)(T-t)}\Phi\Big(\frac{-x+(\lambda-2\sigma)(T-t)}{\sqrt{T-t}}\Big). \end{aligned}$$

If $\lambda = \frac{\sigma}{2}$ we have

$$\begin{aligned} C(t,x) &= \Big(z+\frac{\alpha}{\sigma}e^{-\sigma z}\Big)\Phi\Big(\frac{-z-\frac{\sigma}{2}(T-t)}{\sqrt{T-t}}\Big)\Big|_x^{\infty} \\ &+ \int_x^{\infty}\Big(z+\frac{\alpha}{\sigma}e^{-\sigma z}\Big)\phi\Big(\frac{-z-\frac{\sigma}{2}(T-t)}{\sqrt{T-t}}\Big)\frac{\mathrm{d}z}{\sqrt{T-t}}, \end{aligned}$$

so that,

$$\begin{aligned} C(t,x) &= \frac{\alpha}{\sigma}e^{\sigma^2(T-t)}\Phi\Big(\frac{-x-\frac{3\sigma}{2}(T-t)}{\sqrt{T-t}}\Big) - \Big(x+\frac{\alpha}{\sigma}e^{-\sigma x}\Big)\Phi\Big(\frac{-x-\frac{\sigma}{2}(T-t)}{\sqrt{T-t}}\Big) \\ &-\frac{\sigma}{2}(T-t)\Phi\Big(\frac{-x+\frac{\sigma}{2}(T-t)}{\sqrt{T-t}}\Big) + \sqrt{\frac{T-t}{2\pi}}e^{-\frac{(x+\frac{\sigma}{2}(T-t))^2}{2(T-t)}}. \end{aligned}$$

Finally, if $\lambda = \sigma$, function C is given by

$$\begin{aligned} C(t,x) &= \Big(\frac{e^{\sigma z}}{\sigma} - \alpha z\Big)\Phi\Big(\frac{-z-\sigma(T-t)}{\sqrt{T-t}}\Big)\Big|_x^{\infty} \\ &+ \int_x^{\infty}\Big(\frac{e^{\sigma z}}{\sigma} - \alpha z\Big)\phi\Big(\frac{-z-\sigma(T-t)}{\sqrt{T-t}}\Big)\frac{\mathrm{d}z}{\sqrt{T-t}}, \end{aligned}$$

so that,

$$\begin{aligned} C(t,x) &= \frac{1}{\sigma}e^{-\frac{\sigma^2}{2}(T-t)}\Phi\Big(\frac{-x}{\sqrt{T-t}}\Big) + \Big(\alpha x - \frac{1}{\sigma}e^{\sigma x}\Big)\Phi\Big(\frac{-x-\sigma(T-t)}{\sqrt{T-t}}\Big) \\ &+\alpha\sigma(T-t)\Phi\Big(\frac{-x-\sigma(T-t)}{\sqrt{T-t}}\Big) - \alpha\sqrt{\frac{T-t}{2\pi}}e^{-\frac{(x+\sigma(T-t))^2}{2(T-t)}}. \end{aligned}$$

Results (A.1.1)-(A.1.3) follow by plugging in expression (A.1.4) equations $A$ and $B$ along with the corresponding choice of $C$, according to the choice of parameter $\lambda$. $\square$

**Lemma A.1.2.** *Function $H$ in* (3.4.4) *is given by*

$$
\begin{aligned}
H(t,x) \;\; &= \;\; \sigma\alpha \cdot \Bigg\{ \Big[ 2\alpha(\sigma+\lambda)e^{-2\sigma x} - (\sigma+2\lambda)e^{-\sigma x} \Big] \Phi\Big( \frac{x-\lambda(T-t)}{\sqrt{(T-t)}} \Big) \\
&\quad + \Big[ \sigma e^{(2\lambda-\sigma)x} - 2\alpha\sigma e^{2(\lambda-\sigma)x} \Big] \Phi\Big( \frac{-x-\lambda(T-t)}{\sqrt{(T-t)}} \Big) \\
&\quad + 2\alpha(\lambda-2\sigma)e^{-2\sigma(\lambda-\sigma)(T-t)} \Phi\Big( \frac{-x+(\lambda-2\sigma)(T-t)}{\sqrt{(T-t)}} \Big) \\
&\quad - 2(\lambda-\sigma)e^{-\frac{\sigma}{2}(2\lambda-\sigma)(T-t)} \Phi\Big( \frac{-x+(\lambda-\sigma)(T-t)}{\sqrt{(T-t)}} \Big) \Bigg\}
\end{aligned}
$$

*for all* $(t,x) \in [0,T] \times \mathbb{R}_+$.

***Proof.*** We note from (3.4.1) and (3.4.4) that function $H$ is given by

$$
H(t,x) = G_t(t,x) - \lambda G_x(t,x) + \frac{1}{2} G_{xx}(t,x) \,. \tag{A.1.5}
$$

From the expression for equation $G$ in (3.1.4), we note that

$$
\begin{aligned}
G_x(t,x) &= 2\alpha\sigma(e^{-\sigma x} - \alpha e^{-2\sigma x}) F_{S^\lambda_{T-t}}(x) \,, \\
G_{xx}(t,x) &= 2\alpha\sigma^2(2\alpha e^{-2\sigma x} - e^{-\sigma x}) F_{S^\lambda_t}(x) + 2\alpha\sigma(e^{-\sigma x} - \alpha e^{-2\sigma x}) \frac{\mathrm{d}}{\mathrm{d}x} F_{S^\lambda_{T-t}}(x) \,,
\end{aligned}
$$

and

$$
G_t(t,x) \;\; = \;\; -2\sigma\alpha \int_x^\infty (e^{-\sigma z} - \alpha e^{-2\sigma z}) \frac{\mathrm{d}}{\mathrm{d}t} F_{S^\lambda_{T-t}}(z) \mathrm{d}z \,.
$$

We recall from (3.1.3) that $F_{S^\lambda_{T-t}}$ is given by

$$
F_{S^\lambda_{T-t}}(x) = \mathbb{P}(S^\lambda_{T-t} \le x) = \Phi\Big( \frac{x-\lambda(T-t)}{\sqrt{T-t}} \Big) - e^{2\lambda x} \Phi\Big( \frac{-x-\lambda(T-t)}{\sqrt{T-t}} \Big) \,,
$$

for all $(t,x) \in [0,T] \times \mathbb{R}_+$. Hence, we have

$$
\begin{aligned}
G_x(t,x) \;\; &= \;\; 2\alpha\sigma\big(e^{-\sigma x} - \alpha e^{-2\sigma x}\big) \Phi\Big( \frac{x-\lambda(T-t)}{\sqrt{(T-t)}} \Big) \\
&\quad - 2\alpha\sigma(e^{(2\lambda-\sigma)x} - \alpha e^{2(\lambda-\sigma)x}) \Phi\Big( \frac{-x-\lambda(T-t)}{\sqrt{(T-t)}} \Big) \,,
\end{aligned} \tag{A.1.6}
$$

for all $(t,x) \in [0,T] \times \mathbb{R}_+$; in addition, noting that

$$
\frac{\mathrm{d}}{\mathrm{d}x} F_{S^\lambda_{T-t}}(x) = \frac{2}{\sqrt{2\pi(T-t)}} e^{-\frac{(x-\lambda(T-t))^2}{2(T-t)}} - 2\lambda e^{2\lambda x} \Phi\Big( \frac{-x-\lambda(T-t)}{\sqrt{T-t}} \Big) \,,
$$

we get

$$G_{xx}(t,x) = 2\alpha\sigma^2\big(2\alpha e^{-2\sigma x} - e^{-\sigma x}\big)\Phi\Big(\frac{x-\lambda(T-t)}{\sqrt{(T-t)}}\Big)$$
$$+ 2\alpha\sigma\big(2\alpha(\lambda-\sigma)e^{2(\lambda-\sigma)x} - (2\lambda-\sigma)e^{(2\lambda-\sigma)x}\big)\Phi\Big(\frac{-x-\lambda(T-t)}{\sqrt{(T-t)}}\Big)$$
$$+ 2\alpha\sigma\sqrt{\frac{2}{\pi(T-t)}}\big(e^{(\lambda-\sigma)x} - \alpha e^{(\lambda-2\sigma)x}\big)e^{-\frac{x^2+\lambda^2(T-t)^2}{2(T-t)}}, \tag{A.1.7}$$

for all $(t,x) \in [0,T] \times \mathbb{R}_+$.

Similarly, in view of

$$\frac{\mathrm{d}}{\mathrm{d}t}F_{S^\lambda_{T-t}}(x) = \frac{x}{(T-t)^{\frac{3}{2}}\sqrt{2\pi}}e^{-\frac{(x-\lambda(T-t))^2}{2(T-t)}}$$

we have

$$G_t(t,x) = -\frac{2\sigma\alpha}{(T-t)^{\frac{3}{2}}\sqrt{2\pi}}\int_x^\infty (e^{-\sigma z} - \alpha e^{-2\sigma z})ze^{-\frac{(z-\lambda(T-t))^2}{2(T-t)}}\mathrm{d}z\,;$$

so that careful evaluation of the integral term yields

$$G_t(t,x) = 2\sigma\alpha^2(\lambda-2\sigma)e^{-2\sigma(\lambda-\sigma)(T-t)}\Phi\Big(\frac{-x+(\lambda-2\sigma)(T-t)}{\sqrt{(T-t)}}\Big)$$
$$- 2\sigma\alpha(\lambda-\sigma)e^{-\frac{\sigma}{2}(2\lambda-\sigma)(T-t)}\Phi\Big(\frac{-x+(\lambda-\sigma)(T-t)}{\sqrt{(T-t)}}\Big)$$
$$+ \alpha\sigma\sqrt{\frac{2}{\pi(T-t)}}\big(\alpha e^{(\lambda-2\sigma)x} - e^{(\lambda-\sigma)x}\big)e^{-\frac{x^2+\lambda^2(T-t)^2}{2(T-t)}}, \tag{A.1.8}$$

for all $(t,x) \in [0,T] \times \mathbb{R}_+$. Finally, plugging expressions (A.1.6)-(A.1.8) into (A.1.5) completes the proof. $\square$

## A.2 Moment Generating Function of $X^x$

**Lemma A.2.1.** *The moment-generating function of $X^x_t = x \vee S^\lambda_t - B^\lambda_t$ is given by*

$$M_{X^x_t}(s) = \mathbb{E}[e^{sX^x_t}] = e^{s(x+\sigma t(\frac{s}{2}\sigma-\lambda))}\cdot\Phi\Big(\frac{x-\sigma t(\lambda-\sigma s)}{\sigma\sqrt{t}}\Big)$$
$$+ \frac{\sigma s}{\sigma s-2\lambda}e^{(\frac{2x\lambda}{\sigma}+\frac{\sigma^2 s^2 t}{2}-(\lambda\sigma t+x)s)}\cdot\Phi\Big(-\frac{x+\sigma t(\lambda-\sigma s)}{\sigma\sqrt{t}}\Big)$$
$$- \frac{2\lambda}{\sigma s-2\lambda}\cdot\Phi\Big(-\frac{x-\lambda\sigma t}{\sigma\sqrt{t}}\Big) \tag{A.2.1}$$

*for $s \in (-\infty,0)\cup(0,\frac{2\lambda}{\sigma})$; in addition, $M_{X^x_t}(0) = 1$.*

***Proof.*** Result $M_{X_t^x}(0) = 1$ is rather obvious. For $s \in (-\infty, 0) \cup (0, \frac{2\lambda}{\sigma})$, we recover a result in [54] (Appendix D) stating that the cumulative density function of $X_t^x$ is given by

$$F_{X_t^x}(y) = P(X_t^x \leq y) = \Phi\Big( - \frac{x - y - \lambda\sigma t}{\sigma\sqrt{t}}\Big) - e^{-\frac{2y\lambda}{\sigma}} \Phi\Big( - \frac{x + y - \lambda\sigma t}{\sigma\sqrt{t}}\Big) ,$$

for all $(t, y) \in [0, T] \times \mathbb{R}_+$. Noting that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}y} F_{X_t^x}(y) &= \frac{1}{\sigma\sqrt{2\pi t}} e^{-\frac{(x-y-\lambda\sigma t)^2}{2\sigma^2 t}} + \frac{1}{\sigma\sqrt{2\pi t}} e^{-\frac{2\lambda}{\sigma} y} e^{-\frac{(x+y-\lambda\sigma t)^2}{2\sigma^2 t}} \\ &+ \frac{2\lambda}{\sigma} e^{-\frac{2\lambda}{\sigma} y} \Phi\Big( - \frac{x + y - \lambda\sigma t}{\sigma\sqrt{t}}\Big) , \end{aligned}$$

we expand $\mathbb{E}[e^{sX_t^x}]$, so that

$$\begin{aligned} \mathbb{E}[e^{sX_t^x}] &= \frac{1}{\sigma\sqrt{2\pi t}} \int_0^\infty e^{sy} e^{-\frac{(x-y-\lambda\sigma t)^2}{2\sigma^2 t}} \mathrm{d}y + \frac{1}{\sigma\sqrt{2\pi t}} \int_0^\infty e^{(s - \frac{2\lambda}{\sigma})y} e^{-\frac{(x+y-\lambda\sigma t)^2}{2\sigma^2 t}} \mathrm{d}y \\ &+ \frac{2\lambda}{\sigma} \int_0^\infty e^{(s-\frac{2\lambda}{\sigma})y} \Phi\Big( - \frac{x + y - \lambda\sigma t}{\sigma\sqrt{t}}\Big) \mathrm{d}y . \end{aligned}$$

Now, the third term on the right hand side may be integrated by parts as long as $s < \frac{2\lambda}{\sigma}$, yielding

$$\begin{aligned} \mathbb{E}[e^{sX_t^x}] &= \frac{1}{\sigma\sqrt{2\pi t}} \int_0^\infty e^{sy} e^{-\frac{(x-y-\lambda\sigma t)^2}{2\sigma^2 t}} \mathrm{d}y + \frac{\sigma s}{\sigma s - 2\lambda} \frac{1}{\sigma\sqrt{2\pi t}} \int_0^\infty e^{(s-\frac{2\lambda}{\sigma})y} e^{-\frac{(x+y-\lambda\sigma t)^2}{2\sigma^2 t}} \mathrm{d}y \\ &- \frac{2\lambda}{\sigma s - 2\lambda} \Phi\Big( - \frac{x - \lambda\sigma t}{\sigma\sqrt{t}}\Big) . \end{aligned}$$

Evaluation of the above integrals yields the result. □

APPENDIX B

# APPENDIX TO CHAPTER 6

In this Appendix, we include some lengthy analytical calculations leading to minor results that are part of Chapter 6 in this thesis.

## B.1 TRANSITION DENSITY KERNEL $Q$

Let $f_n^{\Xi_n}(t) = (\alpha_t^B, \alpha_t^S, \alpha_t^P) \in \mathcal{U}$ denote the proportions of wealth allocated to each financial instrument in the introduction of Chapter 6 at $t \geq 0$ time units after jump time $\Psi_n$, according to control $f_n^{\Xi_n}$. Analogously, let $\Gamma_t^{f_n^{\Xi_n}}$ in (6.2.3) denote the associated wealth $t \geq 0$ time units after $\Psi_n$.

**Lemma B.1.1.** *For an arbitrary $\Xi_n = (t', x, h)$, the transition density kernel $Q$ in the MDP $(E, \mathcal{A}, Q, R)$ in Section 6.3 is given by*

$$\begin{aligned} Q(B|\Xi_n, f_n^{\Xi_n}) &= \mathbb{P}(\Xi_{n+1} \in B|\mathcal{G}_{\Psi_n}, f_n^{\Xi_n}) \\ &= \nu \int_0^{T-t'} e^{-(\nu+(1-h)\lambda_{\mathbb{P}})s} \int_{-1}^{\infty} 1_B(t'+s, \Gamma_s^{f_n^{\Xi_n}}(x,h)(1+\alpha_s^S y), h)\gamma(\mathrm{d}y)\mathrm{d}s \\ &+ (1-h)\lambda_{\mathbb{P}} \int_0^{T-t'} e^{-(\nu+\lambda_{\mathbb{P}})s} 1_B(t'+s, \Gamma_s^{f_n^{\Xi_n}}(x,0)(1-\alpha_s^P L), 1)\mathrm{d}s\,, \end{aligned}$$

*for $B \subseteq E$; in addition*

$$Q(\{\Delta\}|\Xi_n, f_n^{\Xi_n}) = 1 - Q(E|\Xi_n, f_n^{\Xi_n})\,.$$

***Proof.*** For an arbitrary $\Xi_n = (t', x, h)$ at epoch $n$, the transition probability to a new state $\Xi_{n+1} \in E \cup \{\Delta\}$ at epoch $n+1$ is given by

$$Q(B|\Xi_n, f_n^{\Xi_n}) = \mathbb{P}(\Xi_{n+1} \in B|\mathcal{G}_{\Psi_n}, f_n^{\Xi_n}) = \mathbb{P}(\Xi_{n+1} \in B|\mathcal{G}_{t'}, f_n^{\Xi_n})\,, \tag{B.1.1}$$

where $\mathcal{G}_{t'}$ intuitively denotes all the information in the system up to time $t'$. For $B \subseteq E$, the next epoch comes at the time of the first jump in either the asset $S$ or the default process $H$ (and always before deadline $T$); in addition, we note that cases $h = 0$ and $h = 1$ must be treated separately since in the latter there are no more jumps in $H$. Due to the Markovian structure of the problem, we rewrite (B.1.1) as

$$\begin{aligned} Q(B|\Xi_n, f_n^{\Xi_n}) &= \mathbb{P}(\Xi_{n+1} \in B|\Xi_n = (t', x, 1), f_n^{\Xi_n}) \cdot h \\ &+ \mathbb{P}(\Xi_{n+1} \in B|\Xi_n = (t', x, 0), f_n^{\Xi_n}) \cdot (1-h) , \end{aligned} \tag{B.1.2}$$

The first term in the right hand side of (B.1.2) is derived upon noting that the intensity of the poisson jump process $N$ in (6.0.1) is $\nu$, and that the distribution of the jumps $Y \geq -1$ is given by $\gamma(\mathrm{d}y)$. Under control $f_n^{\Xi_n}$, the percentage of wealth invested in asset $S$ at any time $t \geq 0$ after $t'$ is given by $\alpha_t^S$, analogously, the total wealth is given by $\Gamma_t^{f_n^{\Xi_n}}(x, 1)$, so that

$$\begin{aligned} &\mathbb{P}(\Xi_{n+1} \in B|\Xi_n = (t', x, 1), f_n^{\Xi_n}) = \\ &\int_0^{T-t'} \nu e^{-\nu s} \int_{-1}^{\infty} 1_B(t'+s, \Gamma_s^{f_n^{\Xi_n}}(x,1)(1+\alpha_s^S y), 1)\gamma(\mathrm{d}y)\mathrm{d}s \, . \end{aligned} \tag{B.1.3}$$

For the second term in (B.1.2) we must consider the events

- $\mathcal{C}_1 =$*"Next jump in Asset S arrives before jump in Default process H"* , and
- $\mathcal{C}_2 =$*"Jump in Default process H arrives before next jump in Asset S"* ,

so that we can extend the above expression according to the laws of conditional probabilities, yielding

$$\begin{aligned} \mathbb{P}(\Xi_{n+1} \in B|\Xi_n = (t', x, 0), f_n^{\Xi_n}) &= \mathbb{P}(\Xi_{n+1} \in B|\Xi_n = (t', x, 0), f_n^{\Xi_n}, \mathcal{C}_1)\mathbb{P}(\mathcal{C}_1) \\ &+ \mathbb{P}(\Xi_{n+1} \in B|\Xi_n = (t', x, 0), f_n^{\Xi_n}, \mathcal{C}_2)\mathbb{P}(\mathcal{C}_2) \, . \end{aligned}$$

The jump intensity of $H$ is given by $\lambda_{\mathbb{P}}$; thus,

$$\mathbb{P}(\mathcal{C}_1) = \int_0^{\infty} \nu e^{-\nu s} \int_s^{\infty} \lambda_{\mathbb{P}} e^{-\lambda_{\mathbb{P}} r} \mathrm{d}r \mathrm{d}s = \frac{\nu}{\nu + \lambda_{\mathbb{P}}} ,$$

and analogously $\mathbb{P}(\mathcal{C}_2) = \frac{\lambda_{\mathbb{P}}}{\nu+\lambda_{\mathbb{P}}}$. In addition, we denote $\phi_S$ and $\phi_H$ the next jump times of $S$ and $H$ respectively, so that their conditional probability density functions

$f_{\phi_S|\mathcal{C}_1}(\cdot|\mathcal{C}_1)$ and $f_{\phi_H|\mathcal{C}_1}(\cdot|\mathcal{C}_1)$ are given by

$$f_{\phi_S|\mathcal{C}_1}(\cdot|\mathcal{C}_1) = \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathbb{P}(S \leq s, \mathcal{C}_1)}{\mathbb{P}(\mathcal{C}_1)} = (\lambda_{\mathbb{P}} + \nu)e^{-(\nu+\lambda_{\mathbb{P}})s} = f_{\phi_H|\mathcal{C}_1}(\cdot|\mathcal{C}_1) \ .$$

Then, in a similar manner to B.1.3, we have

$$\begin{aligned}&\mathbb{P}(\Xi_{n+1} \in B|\Xi_n = (t', x, 0), f_n^{\Xi_n}, \mathcal{C}_1)\mathbb{P}(\mathcal{C}_1) = \\ &\int_0^{T-t'} \nu e^{-(\nu+\lambda_{\mathbb{P}})s} \int_{-1}^{\infty} 1_B(t' + s, \Gamma_s^{f_n^{\Xi_n}}(x, 0)(1 + \alpha_s^S y), 0)\gamma(\mathrm{d}y)\mathrm{d}s \ ,\end{aligned} \tag{B.1.4}$$

and

$$\begin{aligned}&\mathbb{P}(\Xi_{n+1} \in B|\Xi_n = (t', x, 0), f_n^{\Xi_n}, \mathcal{C}_2)\mathbb{P}(\mathcal{C}_2) = \\ &\int_0^{T-t'} \lambda_{\mathbb{P}} e^{-(\nu+\lambda_{\mathbb{P}})s} 1_B(t' + s, \Gamma_s^{f_n^{\Xi_n}}(x, 0)(1 - \alpha_s^P L), 0)\mathrm{d}s \ .\end{aligned} \tag{B.1.5}$$

Finally, plugging equations (B.1.3), (B.1.4) and (B.1.5) in expression (B.1.2) completes the first part of the proof. The additional result

$$Q(\{\Delta\}|\Xi_n, f_n^{\Xi_n}) = 1 - Q(E|\Xi_n, f_n^{\Xi_n})$$

is trivial. □

# REFERENCES


[1] ALMUDEVAR A. A dynamic programming algorithm for the optimal control of piecewise deterministic Markov processes. *SIAM J. Control Optim. 40* (2001), 525–539.

[2] AVRAM F., CHAN T. AND USABEL M. On the valuation of constant barrier options under spectrally one-sided exponential Lévy models and Carr's approximation for American puts. *Stoch. Proc. Appl. 100* (2002), 75–107.

[3] BELLMAN R. E. *Dynamic Programming*. Princeton University Press, 1957.

[4] BERTSEKAS D. P. AND SHREVE S. *Stochastic Optimal Control*. Academic Press, 1978.

[5] BIELECKI T. R. AND JANG I. Portfolio optimization with a defaultable security. *Asia-Pac. Finan. Markets 13* (2006), 113–127.

[6] BIELECKI T. R. AND RUTKOWSKI M. *Credit Risk: Modelling, Valuation and Hedging*. Springer, 2002.

[7] BLACKWELL D. Discounted dynamic programming. *Ann. Math. Stat. 36* (1965), 226–235.

[8] BO L., WANG Y. AND YANG X. An optimal portfolio problem in a defaultable market. *Adv. Appl. Probab. 42* (2010), 689–705.

[9] BRUSS F. T. A unified approach to a class of best choice problems with an unknown number of options. *Ann. Probab. 12* (1984), 882–891.

[10] BÄUERLE N. AND RIEDER U. Portfolio optimization with Markov-modulated stock prices and interest rates. *IEEE T. Automat. Contr. 49* (2004), 442–447.

[11] Bäuerle N. and Rieder U. MDP algorithms for portfolio optimization problems in pure jump markets. *Financ. Stoch. 13* (2009), 591–611.

[12] Bäuerle N. and Rieder U. Optimal control of piecewise deterministic Markov processes with finite time horizon. *Modern Trends of Controlled Stochastic Processes: Theory and Applications* (2010), 123–143.

[13] Bäuerle N. and Rieder U. *Markov Decision Processes with Applications to Finance*. Springer, 2011.

[14] Bäuerle N. and Rieder U. Control improvement for jump-diffusion processes with applications to finance. *Appl. Math. Opt. 65* (2012), 1–14.

[15] Caffarelli L. A. The obstacle problem revisited. *J. Fourier Anal. Appl. 4* (1998), 383–402.

[16] Capponi A. and Figueroa-Lopez J. E. Dynamic portfolio optimization with a defaultable security and regime switching. *http://arxiv.org/abs/1105.0042* (2011).

[17] Carr P. Randomization and the American put. *Rev. Financ. Stud. 11* (1998), 597–626.

[18] Du Toit J. and Peskir G. The trap of complacency in predicting the maximum. *Ann. Probab. 35* (2007), 340–365.

[19] Du Toit J. and Peskir G. Predicting the last zero of Brownian motion with drift. *Stochastics 80* (2008), 229–245.

[20] Du Toit J. and Peskir G. Predicting the time of the ultimate maximum for Brownian motion with drift. *Math. Contr. Theo. Finan.* (2008), 95–112.

[21] Du Toit J. and Peskir G. Selling a stock at the ultimate maximum. *Ann. Appl. Probab. 19* (2009), 983–1014.

[22] Dubins L. E. and Savage L. J. *How to Gamble if You Must: Inequalities for Stochastic Processes*. McGraw Hill Series in Probability and Statistics, 1965.

[23] DUFFIE D. AND SINGLETON K. J. Modelling term structures of defaultable bonds. *Rev. Financ. Stud. 12* (1999), 687–720.

[24] EKSTROM E. AND PESKIR G. Optimal stopping games for Markov processes. *SIAM J. Control Optim. 47* (2008), 684–702.

[25] ELIE R. AND ESPINOSA G. E. Optimal selling rules for monetary invariant criteria: tracking the maximum of a portfolio with negative drift. *Math. Finan.* (2013), n/a.

[26] ESPINOSA G. AND TOUZI N. Detecting the maximum of a scalar diffusion with negative drift. *SIAM J. Control Optim. 50* (2012), 2543–2572.

[27] ETHERIDGE A. *A course in Financial Calculus*. Cambridge University Press, 2002.

[28] FLEMING W. H. AND PANG T. An application of stochastic control theory to financial economics. *SIAM J. Control Optim. 43* (2004), 502–531.

[29] GRAVERSEN S. E., PESKIR G. AND SHIRYAEV A. N. Stopping Brownian motion without anticipation as close as possible to its ultimate maximum. *Theor. Probab. Appl. 45* (2001), 41–50.

[30] GRAVERSEN S. E. AND SHIRYAEV A. N. An extension of P. Lévy's distributional properties to the case of a Brownian motion with drift. *Bernoulli 6* (2000), 615–620.

[31] GUO X. AND ZHANG Q. Closed-form solutions for perpetual American put options with regime switching. *SIAM J. Appl. Math. 64* (2004), 2034–2049.

[32] HOWARD R. A. *Dynamic Programming and Markov Processes*. The M.I.T. Press, 1960.

[33] JEANBLANC M., YOR M. AND CHESNEY M. *Mathematical Methods for Financial Markets*. Springer, 2009.

[34] KARATZAS I. AND SHREVE S. *Methods of Mathematical Finance*. Springer, 1998.

[35] KIRCH M. AND RUNGGALDIER W. J. Efficient hedging when asset prices follow a geometric Poisson process with unknown intensities. *SIAM J. Control Optim. 43* (2004), 1174–1195.

[36] Kyprianou A. E. and Pistorius M. R. Perpetual options and canadization through fluctuation theory. *Ann. Appl. Probab. 13* (2003), 1077–1098.

[37] Lakner P. and Liang W. Optimal investment in a defaultable bond. *Math. Finan. Econ. 3* (2008), 283–310.

[38] Le H. and Wang C. A finite time horizon optimal stopping problem with regime switching. *SIAM J. Control Optim. 48* (2010), 5193–5213.

[39] Malmquist S. On certain confidence contours for distribution functions. *Ann. Math. Stat. 25* (1954), 523–533.

[40] McKean H. P. A free boundary problem for the heat equation arising from a problem in mathematical economics. *Ind. Manag. Rev. 6* (1965), 32–39.

[41] Merton R. C. Lifetime portfolio selection under uncertainty: the continuous case. *Rev. Econom. Statist. 51* (1969), 247–257.

[42] Mikhalevich V. S. Sequential bayes solutions and optimal methods of statistical acceptance control. *Theor. Prob. Appl. 1* (1956), 395–421.

[43] Mordecki E. Optimal stopping for a compound Poisson process with exponential jumps. *Biblioteca Especializada del Banco Central de Uruguay, http://www.bvrie.gub.uy/* (1998).

[44] Pedersen J. L. Optimal prediction of the ultimate maximum of Brownian motion. *Stoch. Rep. 75* (2003), 205–219.

[45] Pedersen J. L. Optimal stopping problems for time-homogeneous diffusions: a review. *Rec. Adv. Appl. Probab. 00* (2005), 427–454.

[46] Peskir G. On the American option problem. *Math. Financ. 15* (2005), 169–181.

[47] Peskir G. Principle of smooth fit and diffusions with angles. *Stochastics 79* (2007), 293–302.

[48] Peskir G. and Shiryaev A. N. Solving the Poisson disorder problem. *Research Report No. 419, Dept. Theoret. Statist. Aarhus* (2000).

[49] PESKIR G. AND SHIRYAEV A. N. *Optimal Stopping and Free Boundary Problems*. ETH Zürich, Birkhäuser, 2006.

[50] PHAM H. Smooth solutions to optimal investment methods with stochastic volatilities and portfolio constraints. *Appl. Math. Opt. 46* (2002), 55–78.

[51] PUTERMAN M. L. *Markov Decision Processes: Discrete Stochastic Dynamic Programming*. Wiley, 2005.

[52] SCHÄL M. Control of ruin probabilities by discrete-time investments. *Math. Meth. Oper. Res. 62* (2005), 141–158.

[53] SHIRYAEV A. N. *Optimal Stopping Rules*. Springer, 1978.

[54] SHIRYAEV A. N., XU Z. AND ZHOU X. Y. Thou shalt buy and hold. *Quant. Finan. 8* (2008), 765–776.

[55] SNELL J. L. Applications of martingale system theorems. *Trans. Amer. Math Soc 73* (1952), 293–312.

[56] VARADHAN S. R. S. *Stochastic Processes*. American Mathematical Society, 2007.

[57] VLADIMIR I. A. *Ordinary Differential Equations*. Springer, 2006.

[58] VUIK C. Some historical notes about the Stefan problem. *Nieuw Archief voor Wiskunde 11* (1993), 157–167.

[59] YUSHKEVICH A. A. On reducing a jump controllable Markov model to a model with discrete time. *Theo. Probab. Appl. 25* (1980), 58–68.

[60] ZHOU X., DAI M., JIN H. AND ZHONG Y. Buy low and sell high. *Cont. Quant. Finan.* (2010), 317–334.