# Automatic Calibration of Cameras with Special Motions

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## Abstract

*We consider the problem of auto-calibrating the intrinsic parameters of a camera moving with a special motion: the rotation axis of the camera being perpendicular to its translation direction. Our method for calibrating the camera is based on Kruppa's equation which in general requires solving a set of nonlinear equations. We prove in a theorem how to recover the true scale of the Kruppa's equation from the eigenvalues of a matrix formed using the fundamental matrix between two views.*

## 1. Introduction

The problem of camera calibration has been widely studied in the computer vision literature [12, 13, 14, 9, 1]. Such methods typically require the user to show a known object (calibration rig) to the camera and find the camera parameters by minimizing the reprojection error between the 2D projections of known 3D points in the object and their measured 2D projections in the image.

Another approach for calibration of cameras is to use auto-calibration methods [10, 11, 8, 2, 4, 1]. Such methods automatically match image points across several images and calibrate the cameras by solving nonlinear equations such as Kruppa's equations [3]. While very elegant, these methods suffer from the fact that solving Kruppa's equations requires solving nonlinear equations and are ill-conditioned. However, [5] shows that for special motions one can transform the nonlinear equations to linear ones using the fact that the unknown scale in the Kruppa's equation is related to the eigenstructure of a matrix formed using the fundamental matrix. As a specific example which is also relatively common, when the axis of rotation of a camera is perpendicular to the direction of translation, it has been shown [5] that one can recover the unknown scale of the Kruppa's equation as one of the eigenvalues of a suitable matrix. However, to the best of our knowledge, there is no work addressing the criterion to use in order to decide which eigenvalue corresponds to the true scale of the Kruppa's equation. In this paper, we address this problem by introducing a simple algorithm which we prove its correctness in a theroem.

In Section 2, we review the problem of self-calibration of cameras and restate the theoretical results on that for cameras undergoing special motions. In Section 3, we propose our main result on auto-calibration of a camera having a special motion: its rotation axis being perpendicular to the translation direction. We prove a theorem on how to recover the true scale of the Kruppa's equation from a matrix formed using the fundamental matrix between two views.

## 2. Self-Calibration of Cameras

Assume we have $N$ points $\{X^j \in \mathbb{R}^3\}_{j=1}^N$ in 3D space, where the vector $X^j$ contains the coordinates of point $j$ in the world reference frame. Let $(R,T) \in SO(3) \times \mathbb{R}^3$ be the pose of a camera, with respect to the world reference frame, where $R$ is a rotation matrix belonging to $SO(3) = \{R \in \mathbb{R}^{3\times 3} : R^\top R = I, \det(R) = +1\}$ and $T$ is the translation between the world and the camera frames. Then the projection of the point $X^j$ onto the image plane of this camera has homogeneous coordinates $\tilde{x}^j \in \mathbb{R}^3$ satisfying the following equation:

$$\lambda^j \tilde{x}^j = KRX^j + KT. \quad (1)$$

In this equation, $\lambda^j$ is the projective depth of a point $j$ and is equal to the third coordinate of $KRX^j + KT$. The matrix $K \in \mathbb{R}^{3\times 3}$ is called the *intrinsic parameter matrix* or *camera calibration matrix* and is of the following form:

$$K = \begin{bmatrix} fs_x & fs_\theta & o_x \\ 0 & fs_y & o_y \\ 0 & 0 & 1 \end{bmatrix}. \quad (2)$$

The calibration matrix is constructed using the intrinsic parameters of the camera, namely, the position of the optical center $(o_x, o_y)$, the size of the pixels $(s_x, s_y)$, the skew factor $s_\theta$ and the focal length $f$. The rotation $R$ and translation $T$ describe the relative position and orientation of the camera frame with respect to the world reference frame. They are also called the *extrinsic calibration parameters* of the camera. Note that $\tilde{x}^j$ in equation (1) describes the image point in the pixel coordinates, while $x^j = K^{-1}\tilde{x}^j$ is the image point in the metric coordinates. The SfM problem refers to the problem of inferring both extrinsic and intrinsic parameters of the camera.

Now, assume that we have two cameras observing the same scene. Without loss of generality, we assume the world reference frame is located at the center of the first camera, i.e., $(R_1, T_1) = (I, 0)$. The relation between the image points in the two cameras is then given as:

$$\lambda_2^j \tilde{x}_2^j = \lambda_1^j K_2 R K_1^{-1} \tilde{x}_1^j + K_2 T \quad (3)$$

where, for simplicity of notation, we used $(R, T)$ rather than $(R_2, T_2)$. One can eliminate the unknown scales $\lambda_1^j$ and $\lambda_2^j$ from (3) and get:

$$\tilde{x}_2^{j\top} F \tilde{x}_1^j = 0, \quad \text{where} \quad F = \widehat{K_2 T} K_2 R K_1^{-1}. \quad (4)$$

The matrix $F$ is called the *fundamental matrix* between the two cameras and incorporates intrinsic and extrinsic calibration parameters. Here, $\widehat{u} \in so(3)$ denotes the mapping of $u \in \mathbb{R}^3$ to the space of skew-symmetric matrices so that $\widehat{u}\, u = 0$.

Note that equation (4) is a bilinear equation in $\tilde{x}_1^j$ and $\tilde{x}_2^j$. As a result, having enough number of point correspondences (at least 8) $\{\tilde{x}_1^j, \tilde{x}_2^j\}_{j=1}^N$ between the two cameras, one can reconstruct the fundamental matrix $F$ up to a scale factor from the linear equation:

$$\begin{bmatrix} \tilde{x}_2^{1\top} \otimes \tilde{x}_1^{1\top} \\ \tilde{x}_2^{2\top} \otimes \tilde{x}_1^{2\top} \\ \vdots \\ \tilde{x}_2^{N\top} \otimes \tilde{x}_1^{N\top} \end{bmatrix} f = 0 \quad (5)$$

where $\otimes$ denotes the Kronecker product, and $f \in \mathbb{R}^9$ is obtained by stacking all rows of $F$ into a vector. So, one can recover the fundamental matrix $F_i$ from a set of point correspondences $\{\tilde{x}_1^j, \tilde{x}_2^j\}_{j=1}^N$ using the 7-point algorithm [1] and then recover the translation $T' = \gamma K T$ (up to a scale factor $\gamma$), since by equation (4), $T'$ is in the left nullspace of $F$.

Having $T_i'$ and $F$ recovered, one canonical reconstruction of the projective camera matrices is given by $(\widehat{T}'^{\top} F, T')$, where $\widehat{T}'^{\top} F = KRK^{-1} + T' v^\top$ for some $v \in \mathbb{R}^3$ [5]. If we let $Y = KK^\top$, it is easy to check that the following equation holds:

$$(\widehat{T}'^{\top} F - T' v^\top)\, Y\, (\widehat{T}'^{\top} F - T' v^\top)^\top = Y. \quad (6)$$

Multiplying both sides of the above equation on left by $\widehat{T}'$ and on right by $\widehat{T}'^{\top}$, we get:

$$(\widehat{T}' \widehat{T}'^{\top} F)\, Y\, (F^\top \widehat{T}' \widehat{T}'^{\top}) = \widehat{T}' Y \widehat{T}'^{\top}. \quad (7)$$

One can show that $\widehat{T}' \widehat{T}'^{\top} F = 1/\lambda F$ for some $\lambda \in \mathbb{R}$ and as a result equation (7) reduces to the well-known Kruppa's equation:

$$F\, Y\, F^\top = \lambda^2\, \widehat{T}' Y \widehat{T}'^{\top}. \quad (8)$$

# 3. Auto-Calibration of Cameras for Special Motions

In this section, we propose a method for auto-calibration of a camera when the rotation axis is perpendicular to the translation direction. [5] has shown that for this special motion, the true scale of the Kruppa's equation is one of the eigenvalues of $F^\top \widehat{T'}$. However, it is not clear whether the true scale corresponds to the smaller or the larger nonzero eigenvalues of $F^\top \widehat{T'}$ and more generally how to recover the true scale from the eigen-structur of $F^\top \widehat{T'}$. In this section, we first show that for such a special motion, the true scale of the Kruppa's equation can be either the larger or the smaller nonzero eigevalues of $F^\top \widehat{T'}$. Second, we propose a method to find the true scale based on the eigenvalue-eigenvector decomposition of the matrix $F^\top \widehat{T'}$. As will be shown later, this method would be extremely helpful for camera auto-calibration, since by finding the true scale we can estimate the calibration matrix by solving the linear Kruppa's equation.

Before beginning our analysis, we state Theorem 6.15 from [5].

**Theorem 1** *Given an unnormalized fundamental matrix $F = \lambda \widehat{T'} KRK^{-1}$ with $||\widehat{T'}|| = 1$, if $T = K^{-1}\widehat{T'}$ is parallel to $\omega$ (the axis of $R$), then $\lambda^2 = ||F^\top \widehat{T'} F||$ and if $T$ is perpendicular to $\omega$, then $\lambda$ is one of the two nonzero eigenvalues of $F^\top \widehat{T'}$. More precisely, for $v \in \mathbb{R}^3$ satisfying $\omega = \widehat{T} K^{-1} v$, the eigenvector of $F^\top \widehat{T'}$ corresponding to the true scale of the fundamental matrix is given by $\widehat{T'} v$,* i.e.*,*

$$(F^\top \widehat{T'})\, \widehat{T'} v = \lambda\, \widehat{T'} v. \quad (9)$$

Now, the question is if there is a way of deciding whether the smaller or the larger nonzero eigenvalue of $F^\top \widehat{T'}$ corresponds to the true scale, $\lambda$. Although Remark 6.19 of [5] says there is no way to tell which eigenvalue is the correct one, we show in this paper that there is actually a way of finding the true one.

First, we show that depending on the motion and calibration parameters of the camera, $\lambda$ can be either the larger or the smaller nonzero eigenvalue of $F^\top \widehat{T'}$. Using the fact that $F = \lambda \widehat{T'} KRK^{-1}$ we can write:

$$F^\top \widehat{T'} = \lambda (KRK^{-1})^\top \widehat{T'}^\top \widehat{T'} = \lambda (KRK^{-1})^\top (I - T' T'^\top). \quad (10)$$

Then by taking the trace from both sides we get:

$$\begin{aligned} \text{trace}(F^\top \widehat{T'}) &= \lambda(1 + 2\cos(\theta)) \\ &\quad - \lambda \text{trace}((KRK^{-1})^\top T' T'^\top), \end{aligned} \quad (11)$$

where $\theta$ is the angle of rotation along $\omega$. On the other hand, using the fact that one of the eigenvalues of $F^\top \widehat{T'}$ is zero, and one is equal to $\lambda$, we have:

$$\text{trace}(F^\top \widehat{T'}) = 0 + \lambda + \gamma, \tag{12}$$

where $\gamma$ is the third eigenvalue of $F^\top \widehat{T'}$. Thus, by equality of (11) and (12) we get:

$$\begin{aligned}\gamma &= [2\cos(\theta) - \text{trace}(T'^\top (KRK^{-1})^\top T')]\lambda \\ &= (\,2\cos(\theta) - T^\top R^\top S^{-1} T\,)\lambda.\end{aligned} \tag{13}$$

where $S^{-1} = K^\top K$.

If for a calibration matrix with $S^{-1} = I$ we let $T = 1/2\, e_1$ and $R = \exp(\widehat{e}_3\theta)$, then the relation between the two nonzero eigenvalues of $F^\top \widehat{T'}$ is as follows:

$$\gamma = 7/4 \cos(\theta)\lambda.$$

As a result, for $\theta = \pi/6$, we get $\gamma = 7\sqrt{3}\lambda/8 > \lambda$ and for $\theta = \pi/3$ we get $\gamma = 7\lambda/8 < \lambda$. This shows that depending on the motion and calibration parameters, the relation between $\lambda$ and $\gamma$ varies and we can not find the true scale based on ordering of the eigenvalues of $F^\top \widehat{T'}$. So, the question is: can we still determine which eigenvalue corresponds to the true scale of $F$? We show in the following that the answer to this question is positive.

We already know that $T'$ and $\widehat{T'}v$ with $v$ satisfying $\omega = \widehat{T}K^{-1}v$ are two eigenvectors of $F^\top \widehat{T'}$ corresponding to the eigenvalues 0 and $\lambda$, respectively. Obviously, $\widehat{T'}\widehat{T'}v$ would be orthogonal to these two eigenvectors, thus $\{T', \widehat{T'}v, \widehat{T'}\widehat{T'}v\}$ forms an orthogonal basis for $\mathbb{R}^3$. As a result, the third eigenvector corresponding to the eigenvalue $\gamma$ can be written as $\widehat{T'}\widehat{T'}v + \alpha T' + \beta \widehat{T'}v$, for some $\alpha, \beta \in \mathbb{R}$, so we have:

$$F^\top \widehat{T'}(\widehat{T'}\widehat{T'}v + \alpha T' + \beta \widehat{T'}v) = \gamma(\widehat{T'}\widehat{T'}v + \alpha T' + \beta \widehat{T'}v). \tag{14}$$

Using the fact that $T'$ and $\widehat{T'}v$ are eigenvectors of $F^\top \widehat{T'}$, and $\widehat{T'}\widehat{T'}\widehat{T'} = -\widehat{T'}$, equation (14) can be written as:

$$-F^\top \widehat{T'}v + \beta\lambda \widehat{T'}v = \gamma(\widehat{T'}\widehat{T'}v + \alpha T' + \beta \widehat{T'}v). \tag{15}$$

By left multiplying the equation (15) by $T'^\top$, we get

$$\alpha = -T'^\top F^\top \widehat{T'}v \,/\, \gamma \tag{16}$$

also by rearranging the terms in equation (15), we get

$$(F^\top + \gamma\widehat{T'})\widehat{T'}v = \beta(\lambda - \gamma)\widehat{T'}v - \alpha\gamma T'. \tag{17}$$

Finally, by left multiplying this equation by $(\widehat{T'}v)^\top$, and using the fact that $\widehat{T'}v$ is orthogonal to $T'$ we obtain

$$\beta = \widehat{T'}^\top v^\top (F^\top + \gamma\widehat{T'})\,\widehat{T'}v \,/\, [(\lambda - \gamma)\,||\widehat{T'}v||^2]. \tag{18}$$

Now, looking more closely to the structure of the eigenvectors of $F^\top \widehat{T'}$ corresponding to $\lambda$ and $\gamma$ we can see that the eigenvector corresponding to $\lambda$ lives in the plane orthogonal to $T'$ while the third eigenvector $\widehat{T'}\widehat{T'}v + \alpha T' + \beta\widehat{T'}v$ would not be in this plane as long as $\alpha \neq 0$. Thus, if $\alpha \neq 0$, the only eigenvector being orthogonal to $T'$ is the one corresponding to the true scale of the fundamental matrix (since the first eigenvector corresponding to the zero eigenvalue is parallel to $T'$). As a result, we must investigate the conditions under which $\alpha$ is not zero as well as the structure of the eigenvalues for the case of $\alpha = 0$.

Substituting $F = \lambda\widehat{T'}KRK^{-1}$ in equation (16) we get:

$$\begin{aligned}\alpha &= -\lambda \,/\, \gamma\; T'^\top (\widehat{T'}KRK^{-1})^\top \widehat{T'}v \\ &= -\lambda \,/\, \gamma\; T'^\top (KRK^{-1})^\top \widehat{T}'^\top \widehat{T'}v\end{aligned} \tag{19}$$

which is zero only when $T'^\top (KRK^{-1})^\top$ is parallel to $T'^\top$. This means that there exists a constant $\eta \in \mathbb{R}$ such that $T'^\top (KRK^{-1})^\top = \eta T'^\top$, which using the fact that $\widehat{T}' = K\,T$ can be simplified to $RT = \eta T$. This holds only for two cases: (1) $T$ is parallel to the axis of rotation which is obviously in contradiction with our original assumption, or (2) $R = I_3$ with $\eta = 1$.

As a result, $\alpha$ is always nonzero as long as the rotation is not the identity. When the camera motion is only translational, both eigenvectors of $F^\top \widehat{T'}$ corresponding to the nonzero eigenvalues, lie in the same plane orthogonal to $T'$. However, when this is the case, resulting from the rotation being the identity, we can assume that the rotation axis is parallel to the translation direction (but with zero degrees of rotation!). So, by the results of [5], we can get the true scale by $\lambda = ||F^\top \widehat{T'}F||$.

However, we can find a stronger result from our previous analysis. When the rotation is the identity, we have $\theta = 0$ and $R = I_3$. So, from equation (13) which gives the relation between the nonzero eigenvalues we have:

$$\gamma = (2 - T^\top S^{-1} T)\lambda = (2 - T'^\top T')\lambda = \lambda\,, \tag{20}$$

since $||T'|| = 1$. This means that when the motion of the camera is only translational then both eigenvalues of $F^\top \widehat{T'}$ are equal to each other and thus equal to the true scale of the fundamental matrix. Otherwise, when the rotation is not the identity, we can find the true scale as the eigenvalue of $F^\top \widehat{T'}$ corresponding to the eigenvector being orthogonal to $T'$ which is unique from the above analysis.

We can summarize our result in the following theorem.

**Theorem 2** *Given an unnormalized fundamental matrix $F = \lambda\widehat{T'}KRK^{-1}$ with $||\widehat{T'}|| = 1$, if $T = K^{-1}\widehat{T'}$ is perpendicular to $\omega$ (axis of $R$), then the eigenvector corresponding to the true scale is orthogonal to $\widehat{T'}$. If both* *eigenvectors corresponding to the two nonzero eigenvalues are orthogonal to $\widehat{T'}$, then both eigenvalues are equal and give the true scale.*

As a result, the algorithm for finding the unknown scale of the Kruppa's equation is as follows:

- Find the eigenvalue-eigenvector decomposition of the matrix $F^\top \widehat{T'}$. Let $\{v_1, v_2, T'\}$ be the eigenvectors corresponding to eigenvalues $\{\lambda_1, \lambda_2, 0\}$.
- Find which eigenvector $v_i$ is orthogonal to $T'$ i.e. $v_{i^*}^\top T' = 0$.
- If $v_{i^*}$ is the only eigenvector orthogonal to $T'$, then the true scale of the fundamental matrix is equal to the corresponding eigenvalue i.e. $\lambda = \lambda_{i^*}$.
- If both eigenvectors are orthogonal to $T'$, then the true scale is equal to either of the nonzero eigenvalues of $F^\top \widehat{T'}$ which are equal i.e. $\lambda = \lambda_1 = \lambda_2$.

**Remark 1** *When the data are noisy, we can find the true scale by searching for the eigenvector which has the minimum inner product with the eigenvector corresponding to the smallest eigenvalue of $F^\top \widehat{T'}$.*

**Example 1** *Let $T = \begin{bmatrix}1 & 2 & 1\end{bmatrix}^\top$, $\omega = \begin{bmatrix}2 & -1 & 0\end{bmatrix}^\top$ and $K = \begin{bmatrix}0.5 & 1 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}$ and let the true scale of the fundamental matrix to be $\lambda = 5$. Then, for the matrix $F^\top \widehat{T'}$ we get $\lambda_1 = 5.00$ and $\lambda_2 = -1.78$. We have $v_1^\top T' = 0$ while $v_2^\top T' = -0.68$. Thus, we the true scale is given by $\lambda_1 = 5$. Now, if we have a pure translation of $T = \begin{bmatrix}1 & 2 & 1\end{bmatrix}^\top$, then we get two nonzero eigenvalues for $F^\top \widehat{T'}$ which are equal i.e. $\lambda_1 = \lambda_2 = 5$ while $v_1^\top T' = v_2^\top T' = 0$. In this case, the true scale is simply the nonzero eigenvalue $\lambda = \lambda_1 = \lambda_2 = 5$.*

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