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THE LEARNING OF GENERAL MATHEMATICAL STRATEGIES

A developmental study of process attainments in mathematics,
including the construction and investigation of a process-
oriented curriculum for the first secondary year.

by

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ABSTRACT

Consideration of the relative place of content and process in the mathematics curriculum leads to the following questions:

1. What is the nature of the mathematical process and how does it relate to the content?
2. Does the process comprise learnable strategies; if so, what are feasible learning objectives for different ages?
3. Can content and process be learned simultaneously or are there incompatibilities between effective teaching methods?

A theoretical study shows that the content of mathematics - structures, symbol-systems and models - arises directly from the application of the basic processes of generalisation and abstraction, symbolisation and modelling, to the objects of experience.

Experimental studies based on (a) the development of a process-enriched curriculum for the early secondary years, and (b) age and ability cross-sectional studies of pupils' proof activity show that:

i. the awareness that proof requires consideration of *all cases* is generally weak among secondary pupils, but is relatively easily taught,
ii. with a process-enriched curriculum, 11 year olds can acquire strategies of experimenting, making generalisations and constructing complete (finite) sets but still have little sense of deducing one result from another,
iii. the main types of deficiency in proof-explanations are (a) fragmentary arguments, (b) non-explanatory re-statements of the data, (c) unawareness of suitable starting assumptions.

Strategies for improving proof activity are inferred from pupils' responses, and are shown to be effective in a sixth form teaching experiment.

An informal study shows that students entering university mathematics departments possess generalisation skills and logical awareness to a much higher degree than 15 year olds, but still have only vague ideas of the nature of axiom systems.

On question 3 the evidence suggests that there need be no substantial loss of content learning in the process-enriched curriculum, and both in this and in the teaching experiment an improvement in general understanding and involvement was observed.

CONTENTS

CHAPTER 1

THE MATHEMATICS CURRICULUM

GENERAL OBJECTIVES

It is not so long since mathematical education in England consisted, for the lower classes, of training to calculate accurately with large and complicated numbers, weights, measures and money, and, for the upper classes, of the rote learning of Euclid's books. No one would defend these practices today, but tradition dies hard, and beneath the superficially radical changes of the last fifteen years there is a continuing lack of a tough philosophy which unites current awareness of the nature of mathematics and of the process of learning into a sufficiently coherent and understandable framework to act as a guide for curriculum practice.

The first question to answer in constructing such a philosophy is what kind of knowledge of mathematics is appropriate for a general education. Some recent discussions (IMA, 1975) have emphasised the need for at least a certain segment of the secondary school leavers to be equipped with the basic numerical skills required for engineering and similar technical occupations. To get a correct perspective on this question it is necessary to distinguish between the secure and confident understanding of those number concepts and skills which are of wide application, and those which are needed at a high level of speed and accuracy only in certain occupations. The retentivity of different kinds of knowledge also has to be taken into account. But, in any case, a purely utilitarian approach to the curriculum would be sterile. It would imply that for pupils of average ability, destined for employment as clerks, secretaries, draughtsmen, technicians and the like, the curriculum would comprise the reading of instructions, the interpretation of maps and diagrams, writing for record-keeping, simple tabulation and tallying of quantities, and some practical science or craft. History, geography, the study of literature, expressive writing, art, music and theoretical science would have no place. An educational experience of this kind would do little or nothing to help its products to exercise judgements as citizens or as parents, and a society which provided no more than this would be failing to pass on its most highly regarded values and achievements.

If one accepts the obligation to transmit the skills and culture of civilisation as effectively as possible, the problem of selecting, from the vastness of knowledge, material for the brief years of schooling, is acute. One must seek the most general, the most pervasive, the most distinctive aspects of knowledge. Specifically, one must attempt to establish a structure of ideas which will facilitate the assimilation of further knowledge, and teach the actual skills, strategies and attitudes needed for the acquisition of knowledge. (Even for the apprentice to a trade, a distilled awareness of how to make effective use of the training experiences provided could be a most valuable acquisition.) For the teaching of these fundamental ideas and some of the component sub-skills for acquiring knowledge, structured - even programmed - learning experiences may be the most effective. But for appreciating the nature of different subjects, and for developing strategies and attitudes, participation in experiences which reflect without distortion the actual knowledge-getting methods of the different subjects is essential. This is essentially Bruner's assertion in The Process of Education (1960, see also 1959).

To apply these principles to mathematics one must decide what kind of subject it is. For most of those who are able to retain a positive attitude to it, it is, first, the means of gaining insights into some aspect of the environment. The form of the growth function of populations, the ways of turning a mattress, the concept of acceleration (and of the decrease in the rate of inflation) and the correct understanding of the statistical "law of averages", are everyday examples, and any given occupational or leisure situation will furnish many more. Secondly, the general attraction to puzzles and patterns, and the existence of a sprinkling of enthusiastic amateur number theorists, suggest that the capacity for appreciating mathematics as an art to enjoy is also present in many people, but is generally suppressed by distasteful school experiences. These two modes of interaction of people with mathematics represent the applied and the pure mathematical approaches, and they have been identifiable throughout history as the mainspring of mathematical activity.

The criterion of generality makes it desirable that valid general strategies of enquiry, discovery and verification should be identified, and their development promoted, through as many subjects as possible. The stages of question formulation, exploration, insight and verification (compare Poincaré, and Polanyi (1957)) may be experienced in mathematical work more easily than in most other subjects; the problem-solving strategies discussed by Polya (1954) in a mathematical context are seen by him as forming a general training in inductive reasoning in what is a particularly suitable field, since the performing of mathematical experiments requires no more elaborate observational method than reflection, and no more expensive equipment than pencil and paper.

Thus, in designing a curriculum to represent faithfully the nature of pure and applied mathematics, in content and process, many of the strategies to be learnt will contribute to general aims. Pupils will be learning how to conduct any enquiry, individually, and collectively, collecting data which bear on the problem, drawing conclusions and identifying new questions, exposing individual conclusions to discussion and argument and establishing public agreements.

CONTENT AND PROCESS - AN INTERNATIONAL SURVEY

On the international curriculum development scene, one of the most significant writings of the last two decades is Bruner's The Process of Education (1960), in which he argues for the involvement of pupils in activities as close as possible to those of researchers on the frontiers of knowledge. Bruner's book is based on a meeting of some thirty-five scientists, psychologists and curriculum developers at Woods Hole on Cape Cod in 1959; out of these discussions emerged some far-reaching principles for guiding curriculum reform. Starting from the hypothesis that any subject can be taught effectively in some intellectually honest form to any child at any stage of development, Bruner asserts that teaching should be designed, both globally and locally, to exhibit the fundamental structure of the subject, that is the basic ideas that lie at its heart. These basic ideas, and this structure, need to be realised by the learning materials in a form suitable to each age of pupil; so that they learn to use them in progressively more complex forms. This is a statement against 'teaching conclusions', and against the teaching of bits of a subject out of relation to the whole. "Intellectual activity is everywhere the same, whether at the frontier of knowledge or in a third-grade classroom....The school boy learning physics is a physicist...." (pp. 11-14)

Even more strongly, in Towards a Theory of Instruction (1959), Bruner writes:

> "Finally a theory of instruction seeks to take account of the fact that a curriculum reflects not only the nature of knowledge itself (the specific capabilities) but also the nature of the knower and of the knowledge-getting process. It is the enterprise *par excellence* where the line between the subject matter and the method grows necessarily indistinct. A body of knowledge, enshrined in a university faculty and embodied in a series of authoritative volumes, is the *result* of much prior intellectual activity. To instruct someone in these disciplines is not a matter of getting him to commit results to mind. Rather, it is to teach him to participate in the process that makes possible the establishment of knowledge. We teach a subject not to produce little living libraries on that subject, but rather to get a student to think mathematically for himself, to consider matters as a historian does, to take part in the process of knowledge-getting. Knowing is a process, not a product."

Alongside these we place contrasting quotations from Gagné (1970) and Ausubel (1968). These three pose sharply the question of the relative place of content and process, and represent viewpoints to which we shall want to refer later.

Gagné says:

> "Obviously, strategies are important for problem-solving, regardless of the content of the problem. The suggestion from some writers is that they are of overriding importance as a goal of education. After all, should not formal instruction in the school have the aim of teaching the student "how to think"? If strategies were deliberately taught, would not this produce people who could then bring to bear superior problem-solving capabilities to any new situation? Although no one would disagree with the aims expressed, it is exceedingly doubtful that they can be brought about solely by teaching students "strategies" or "styles" of thinking. Even if these can be taught (and it is likely that they can), they do not provide the individual with the basic firmament of thought, which is a set of externally-oriented intellectual skills. Strategies, after all, are rules which govern the individual's *approach* to listening, reading, storing information, retrieving information, or solving problems. If it is a mathematical problem the individual is engaged in solving, he may have acquired a strategy of applying relevant subordinate rules in a certain order - but he must also have available the mathematical rules themselves. If it is a problem in genetic inheritance, he may have learned a way of guessing at probabilities, before actually working them out - but he must also bring to bear the substantive rules pertaining to the dominant and recessive characteristics. Knowing strategies, then, is not all that is required for thinking; it is not even a substantial part of what is needed. To be an effective problem-solver, the individual must somehow have acquired masses of organised intellectual skills."

Ausubel says:

> ".....As far as the formal education of the individual is concerned, the educational agency largely transmits ready-made concepts, classifications, and propositions. In any case, discovery methods of teaching hardly constitute an efficient

> primary means of transmitting the content of an academic discipline.
>
> It may be argued with much justification, of course, that the school is also concerned with developing the student's ability to use acquired knowledge in solving particular problems, that is, with his ability to think systematically, independently, and critically in various fields of enquiry. But this function of the school, although constituting a legitimate objective of education in its own right, is less central than its related transmission-of-knowledge function in terms of the objectives of education in a democratic society, and in terms of what can be reasonably expected from most students....."

Of these, Ausubel's is unsupported assertion, except for the hint in the last phrase that some students are incapable of learning systematic, independent or critical thinking; and this in itself is by no means self-evident. It is arguable that even pupils who learn slowly will be best fitted to go on learning throughout their lives by an education which gives them some encouragement and orientation towards finding out and making their own judgements. Bruner's and Gagné's statements taken together form an acceptable rationale for a combination of content and process objectives.

Among those contributing to developments of the mathematical curriculum, Dienes, Papy and the Carbondale project may be seen as essentially content-oriented, Christiansen and Freudenthal as advocating mixed programmes, while Davis and Papert show a strong bias towards process.

Christiansen (1969) says:

> "The foremost goal of mathematics on scientific level is the *study of structures*.

> The most important means for the attainment of this goal is the *axiomatic method*.....but the relevant preparation of the use of the axiomatic method - on any level of school teaching - consists of an application of the inductive approach to a degree that goes far beyond what is at present customary."

And later,

> "In short, the inductive approach (in one of its forms) may be characterised in the following four steps: (1) Experimentation, (2) Observation, (3) Forming of a hypothesis, (4) Further Experimentation in order to test the hypothesis. The inductive approach forms a strong motivation for a subsequent use of (5) Deduction with regard to verification (or falsification) of the hypothesis (relative to some mathematical model)."

For Christiansen the reasons for the use of 'the inductive method' are (a) to give "joy and insight into the aesthetic values of mathematics" and (b) because it is "a special working method applicable by any human being trying to obtain cognition with regard to any field of knowledge." Thus process aspects of mathematics are to be developed both for the general experience of ways of gaining knowledge, for their attractiveness, and for the deeper insight which they give into the nature of the subject and the motives which lead people to pursue it. Freudenthal (1968) stresses particularly this last point, the need for active participation in order to appreciate the mathematising, systematising process. At the same time he asserts that this process began by dealing with everyday reality and only subsequently became turned in on itself.

> "Arithmetic and geometry have sprung from mathematising part of reality. But soon, at least from the Greek antiquity onwards, mathematics itself has become the object of mathematising. Arranging and rearranging the subject matter, turning definitions into theorems and theorems into definitions, looking for more general approaches from which all

> can be derived by specialisation, unifying several theories into one - this has been a most fruitful activity of the mathematician."

> "Systematisation is a great virtue of mathematics, and if possible, the student has to learn this virtue, too. But then I mean the activity of systematising, not its result. Its result is a system, a beautiful, closed system, closed, with no entrance and no exit. What humans have to learn is not mathematics as a closed system, but rather as an activity, the process of mathematising reality and if possible even that of mathematising mathematics."

Other speakers at the 1968 and 1969 international conferences at Utrecht and Lyons echoed the same theme, the need for school mathematics to relate strongly to real-life applications. Thus Pollak (1969) argued for the use of real applications in the classroom, and Engel (1969) devoted his lecture to the powerful applications in Operational Research of simple mathematical ideas such as those of graph theory and combinatorics, linear programming and game theory, and simulation methods.

In examining actual programmes of curriculum development, we begin with Dienes. For him, the goal of mathematical education appears to be the understanding of structural ideas, and the method, the exploration of structured apparatus or the playing of games designed to embody a particular structure.

> "By mathematics I understand *actual* structural relationships between concepts connected with numbers (pure mathematics), together with their applications to problems arising in the real world (applied mathematics). The learning of mathematics I shall take to mean the apprehension of such relationships together with their symbolisation, and the acquisition of the ability to apply the resulting concepts to real situations occurring in the world."

Thus Dienes' methods emphasise activities which fit in with children's natural modes of learning, as described by Piaget, and his best known contribution has been the provision of structured apparatus for the Primary School stage. However, in his recent booklet "The Six Stages in the Process of Learning Mathematics" (1973) Dienes appears to suggest that children can proceed along the full path of the mathematical process culminating in the construction of a fully formal axiom system, with theorems deduced by strict rules of proof. His six stages assume as a starting point that a desired structure is embodied in a variety of sets of material. Then follow:

1. Free play with the material
2. Prescribed structured games with the material
3. Abstraction of the structure by recognition of the common elements in the different games
4. Representation of the structure by some graphical method
5. Discovery of the properties of the structure and use of a (possibly symbolic) language for stating them
6. Choice of certain properties as axioms and others as rules of proof, and deduction of remainder as theorems.

These stages are illustrated by sequences using concrete material embodying the structures of (i) elementary logic, (ii) symmetries of the equilateral triangle and (iii) a total order relation. Dienes does not state whether a child is expected to work through all these stages in a continuous sequence, or whether he envisages a return after some years to a situation, previously explored informally, for the completion of stages (5) and (6). It is difficult to imagine stage (6) being anything other than a meaningless symbol-game to a pupil who needed stages (1) to (4). Other criticisms can be made of this scheme. Wheeler (1964) questions whether multiple embodiments are actually needed (or helpful) for the abstractive process; and the critical examination of the nature of proof offered in Chapter 6 of this work will suggest that most mathematical proof is mainly concerned with judgements relating to the substance of the concepts involved, and that strictly formal proof is relatively unimportant.

The present writer would claim that it is possible for pupils to have experiences much closer to those of the real mathematician through the solving, extension and generalisation of problems arising in the context of a normal syllabus. This is not to deny the value of structured materials as the starting points of mathematical investigation, but rather to assert that the motivation for the inquiry should be the desire to gain further insight. An inquiry which is forced along predetermined tracks is an imposed exercise, not an experience of the getting of knowledge. From the standpoint of the present discussion, Dienes is to be counted as one whose goals of instruction appear now to include both content and mathematical process, but whose recent proposals for process learning are a highly artificial distortion of normal mathematical experience, and also lack credibility from an educational standpoint.

More brief comments on other writers now follow. Papy's (1963) approach to content is the common one of "sets and structure"; regarding process, the exposition is strongly deductive throughout, with axioms stated and many "if-then" diagrams. Most of the exercises in Book 1 (for 12 year olds) simply practise the ideas of the chapter which precedes them, but a few, particularly in the section on number laws, require a deduction. In the later books such exercises occur more frequently, but there are none which invite the pupil to extend or generalise a problem for himself.

The CSMP at Carbondale, though describing itself as "a content-oriented approach" to the curriculum, devotes a considerable amount of time in its earlier stages (12-14 year olds) in developing the pupils' ability to construct formal mathematical proofs. This arises partly from its other emphasis on individualisation of the curriculum and of the material. (CSMP, 1972) This is discussed in Chapter 6. Davis (1967, 1970) explicitly affirms the centrality of process to mathematics. His characteristic mode of teaching is through "informal exploratory experiences" based on "paradigmatic situations". For example, starting with an unknown number of pebbles in a bag, two children in turn add to it or take from it a chosen number of pebbles and members of the class suggest by how many the total has changed. (Film, A Lesson with Second Graders, Madison Project). Another filmed lesson "Monotonic Sequences" shows a normal group of 13 year olds led to a sophisticated awareness of real numbers. Davis's curriculum is built up by a succession of such experiences with situations embodying all the key mathematical structures.

Papert (1972) embraces process more fully than any previously quoted curriculum developer, stating that the choice of content, particularly in the early years, should be made primarily in terms of the suitability for developing the awareness of the mathematical way of thinking. He describes the development by the pupil of simple computer programs to make a computer controlled toy turtle describe desired patterns. The concepts of sub-routines, iteration, de-bugging, partial solutions and so on thus acquire concrete representations. Papert emphasises the value of a long-term project in which the pupil develops from producing very simple patterns to ones of a self-set level of complication, as against the short classroom exercise suitable only for learning particular concepts and skills.

Thus, the importance of process as well as content has been asserted frequently on the international scene during the recent phase of curriculum development, but only a few curricula have achieved a satisfactory combination of these two aspects, and there has been little or no theoretical discussion of their relationship to each other.

Changes in the mathematics curriculum in England, as reflected in the most widely used series of texts, have introduced various types of discovery learning which may give modestly improved awareness of the process of making generalisations, but at the same time the presentation of material in a deductive framework and the demand to construct proofs have declined.

THE PRESENT INVESTIGATION - QUESTIONS AND RESULTS

The situation described above gives rise to the questions investigated in this thesis:

1. What is the nature of the mathematical process, how does it relate to thecontent of mathematics, and what is its importance as compared with the content?
2. Does the process comprise learnable strategies; and if so, what are the feasible learning objectives for different ages?
3. Can content and process be learned simultaneously, or are there incompatibilities between effective methods for the two aspects?

An initial theoretical study gives some answers to question 1. The experimental studies bearing on questions 2 and 3 fall into two parts. First, a process-enriched curriculum for the early secondary years (The South Nottinghamshire Project) has been developed in conjunction with two comprehensive schools and has provided the setting for classroom observations and written tests. Secondly, the process achievements of pupils in normal school settings have been studied in a sequence of experiments. One of these was an interactive study of small groups of pupils of different ages; two used written group tests in age and ability cross-sectional studies. There followed a sixth form experiment, investigating the improvement of process attainments by teaching.

The theoretical study shows (i) that the content of mathematics - structures, symbol-systems and models - arises directly from the application of the basic mathematical processes of generalisation and abstraction, symbolisation and modelling, to the objects of experience; there is thus a very close relationship between the two aspects.

On question 2, the experimental studies show:

(i) at 11, pupils in normal school situations can recognise, extend and describe patterns but do not attempt to explain, justify or deduce them; with a process-enriched curriculum, they can acquire strategies of experimenting, making generalisations, and constructing complete (finite) sets; and can give one-step explanations, but still have little sense of deducing one result from another.

(ii) that proof requires the consideration of *all cases* is not fully and spontaneously appreciated, even by sixth formers, but this awareness is relatively easily taught at the sixth form stage.

Other types of failure to give satisfactory proof-explanations are (a) disconnected, fragmentary arguments, (b) lack of insight into the situation leading to a non-explanatory restatement of the data, (c) lack of awareness of what are suitable assumptions or starting points for an argument.

(iii) Levels of proof-explanation reached in problems depend strongly on contextual factors, such as familiarity, complexity and whether or not the set is finite.

(iv) Improvement in sixth formers' proof activity, particularly in their awareness of "all cases" can be achieved by teaching based on methods derived from earlier studies reported here. Specifically, these consist of study of a fairly simple axiomatic system - Boolean algebra - with critical discussion of pupils' own proof arguments, and attention to strategic concepts of all cases, data and conclusion, agreed starting points, being systematic, classifying and exhausting cases.

(v) Students entering university mathematics departments possess generalisation skills and logical awareness to a much higher degree than 15 year olds, but still have only vague ideas of the nature of axiom systems.

On question 3 the evidence is not strong, but there appears to be no substantial loss of content learning in the process-enriched curriculum, and both in this and in the teaching experiment an improvement in general understanding and involvement.

CHAPTER 2

THE NATURE OF MATHEMATICS

THE FUSION OF CONTENT AND PROCESS

SYMBOLISATION

THE FUSION OF CONTENT AND PROCESS

To the Greeks, mathematics was the study of numbers, magnitude and figures; but even as early as this, deductive proof was equally well established as a characteristic of mathematical activity. Plato, in the Republic, says "Those who study geometry and arithmetic... assume the existence of odd and even numbers, and three kinds of angles: these things they take as known and consider that there is no need to justify them either to themselves or to others, because they are self-evident to everyone; and starting from them, they proceed consistently step to step to the propositions which they set out to examine."

Since "magnitudes" or measures, consist mainly of the application of numbers to geometrical figures - lines, surfaces and so on - together with other situations dealt with by analogy with these (weight, time), this view roots mathematics in the study of number and space. The explicit recognition of a wider subject matter can be attributed to Boole (1847, 1854) who, in his algebra of the Laws of Thought, used the letters x, y for propositions, . and + for "and" and "or" connectives, and 1 and 0 for truth and falsity. He said, "It is not the essence of mathematics to be conversant with the ideas of number and quantity" and "It is concerned with operations considered in themselves, independently of the various ways in which they may be applied." This was the culmination of a century or more of puzzlement about the nature of negative and imaginary numbers (e.g. d'Alembert), of infinitesimals, of "imaginary double points of infinity" (Stirling, 1717).

Thus, with Boole, the content of mathematics is being recognised as consisting essentially of the *relations* between objects, and not the objects themselves. At the same time it was becoming accepted that the starting points of the deductive mathematical system are not "self-evident truths" about given fundamental objects, but postulated relationships between undefined terms. This position is expounded later by Russell:

> "Pure mathematics consists entirely of such asseverations as that, if such and such a proposition is true of anything, then

> such and such another proposition is true of that thing. It is essential not to discuss whether the first proposition is really true, and not to mention what the anything is of which it is supposed to be true....If our hypothesis is about anything and not about some one or more particular things, then our deductions constitute mathematics. Thus mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true."

Thus the recognition that mathematics is not essentially about any particular kinds of object (numbers or space), but is characterised by its method, developed alongside greater clarity in the definition of that method. However, Russell's definition is too wide. Even he did not choose totally arbitrary sets of hypotheses to work with. The axiom systems which mathematicians actually study are those which have some significance in relation to the existing body of knowledge. Following Russell, work on the foundation of mathematics attempted to expose the assumptions of the deductive process itself, by which the consequences of axioms are deduced, and thus to bring logic inside the field defined as mathematics. One result of this is the notion of the formalised text (Bourbaki, p.7); more valuable is the Bourbakiste concept of classifying the whole of mathematics as the structures which stem from the notions of set and element. Thus relations are sets of ordered pairs, functions are kinds of relation, algebraic structures are sets with laws of composition (which are, themselves, functions), topologies are certain kinds of identified sets of subsets, and so on (Choquet, 1962).

In practice the content of modern mathematics still shows strong links with its origins, in number and space, though some accepted elementary theories, such as Boolean algebra, permutations of finite sets and graph theory are in principle more primitive. They have a natural place in the Bourbakiste scheme, and their inclusion in school curricula helps to demonstrate to pupils that mathematics has a wider context than simply number and space.

The mathematical process does not only consist of the exposition and demonstration of mathematical truth. The process of discovering the concepts and the generalisations has historically received less attention - in the tradition of Archimedes' elusive Method, and Fermat's undemonstrated Last Theorem, mathematicians have been considerably less articulate about the discovery process than about the exposition of their results. More recently, Poincaré (1956) Hadamard (1954), Weyl (1940), Polya (1945, 1954,1963), Lakatos (1963), Kilmister (1972), among others, have explored both the psychological aspects of the creative process and also the mathematical strategies themselves. The Bourbakiste analysis itself has emerged with important insights into the mathematical process. Choquet affirms that "the axiomatic method is analogous to an automatic production line; the mother-structures to the machine tools". These structures are "those associated with the equivalence relation, the order structures, the algebraic structures, the topological structures, etc. (Choquet, 1962).

Thus the way of generating new mathematics is to classify, to compare and order, to combine, to reverse, to transform, to recognise nearness, in the material one is studying. This applies both within mathematics and to non-mathematical material. Fielker (1973) has shown how a rich sequence of geometrical study for a primary school can be built up in practice by the application of the "mother-structures" to simple geometrical elements such as straight lines, circles and a set of wooden shapes, and Gattegno (1973) shows how an extensive mathematics curriculum can be developed by the use of the same basic structural operations on a set of Cuisenaire rods.

Consider how mathematics might develop in this way. Number and space provide the raw material. Each is a collection of concepts constructed by the mind out of its interaction with the world, numbers out of the experience of repetition, geometrical ideas from other perceptions of sameness in physical objects. In each of these fields further acts of classification take place in which sets are constructed of objects which are agreed to be the same in some way. Next, pairs of objects are compared with other pairs and sometimes the relationship is judged to be the same between the two pairs - it may be two pairs of numbers with a common difference, or such that the first is greater than the second, or pairs of objects of the same shape but different size. Considering the transformations which take the members of these pairs into each other leads to the identification, collection and classification of functions, for example: linear, square, reciprocal, enlargement, shear. Thus algebra arises as the set of structures which emerge from the study of number and space. At the same time, the set of functions becomes sufficiently large and variegated to constitute a third field of raw material in which the classifying, relating and transforming process can operate.

Wittmann(1975) has pointed out the similarity between the mother-structures, Piaget's groupings (1972) and the heuristic strategies of Polya (1954, 1962, 1965). Wittmann(1973) has also reconstructed the Piagetian groupings, giving formal mathematical definitions and showing them in operation in basic sorting, comparing and combining activities with objects. His list of the logical groupings is (1) inclusion, including the addition of a further subset to an initially separated one, or the removal of part of a set, as in the most primitive actions with sets of objects, (2) substitution, as when a set is separated into two or more subsets, the result being seen as equivalent to the original situation, (3) complete and (4) partial decomposition, as when a set of logic blocks is (3) fully separated into subsets with respect to two or more attributes, or, (4) decomposed with respect to one attribute, then *one* of the subsets further decomposed, and so on, (5) a combination of (1) and (2) involving enlarging or reducing one set or length to become equivalent to another, (6) generalising a relation, as when brother, cousin, having the same grandfather are recognised as all examples of having a common ancestor.

For each of these logical groupings, Wittmann describes a corresponding infra-logical grouping, in which the same operations exist but are tied more closely to physical operations with objects. If Piaget's claim that the groupings are the foundation of logical thinking is accepted, the provision of sets of objects of the types mentioned and the encouragement of activity with them plays a vital part in the intellectual development of young children. Such provision is, of course, normal practice, but there could be an extension of it and also intervention by a teacher or parent could promote the child's awareness of the operations.

Thus, to summarise, the content of mathematics is that body of knowledge which is generated by the application to experience of the fundamental classifying, relating and transforming processes. Peel (1971b) puts it thus: "Mathematics is the study of the properties of the operations by which man orders, organises and controls his environment. These operations constitute logico-mathematical structure." These fundamental operations underlie the processes both of abstraction and generalisation and of proof. But a further aspect of mathematics needs consideration along with these.

SYMBOLISATION

The representation of a situation by a diagram or a symbolic expression, or by a "model" in the abstract sense, is so central to mathematics that it is hard to realise that quite a high degree of algebraic sophistication was achieved by the Babylonians and the Greeks without any symbols apart from a crude number system. Dienes (1961) says

> "The structures now being considered by mathematicians are so complex that it would be quite impossible to dispense with symbolism. The symbols remind the mathematician of what it is that he is really supposed to be thinking about; but more than this, the mechanisms of some of the well learnt mathematical techniques make it possible for the mathematician to skip a great number of steps, or in other words he allows the mechanism to do a part of the thinking for him."

Weyl (1940) in a lecture entitled The Mathematical Way of Thinking argues that, for an adequate characterisation of mathematics, alongside the axiomatic method must be placed *symbolic construction,* as the method by which mathematics is distilled from the raw material of reality. (He cites as examples the capturing of the infinite set of integers by the positional notation, and of a point of a continuum by an infinite binary decimal; and an extension of this last process to topological schemes.) Incorporating symbolisation we arrive at the following definition of mathematics.

Mathematics consists of *structures, and their associated models and symbol systems*. By *structure* is meant *an inter-related system of relational concepts*. Examples of structures are the group S_3, the group in general, the rational numbers, the plane quadrilaterals, the functions $y = kx^2$. A *symbol system* is some set of physical objects, usually marks on paper, which has a set of transformation rules determining how these marks may be moved about, derived from the relationships among the denoted concepts. Thus the transformation $a = b/\sin 15^\circ \rightarrow a \sin 15^\circ = b$ is a physical movement of the marks (probably perceived as such when performed) which derives its validity from knowledge of the denoted concepts and their relationships. Symbols are visible and movable but concepts are invisible and abstract; hence the tendency to teach symbol-transformations by rote and thus to detach the symbols from their meanings.

Two systems are *models* each of the other if they are isomorphic in some respects, that is if there are correspondences between their elements, relations or compositions. The concept is a wide one. The isomorphism is generally analogical, not deductive. Thus a set of Cuisenaire rods may model the positive integers, pairs of them may (less fully) model the rationals, a group composition table is a model of the group, and so is its set of generators and relations.

The role of models in helping mathematicians to grasp elusive concepts may be seen at a number of points in history. The Greeks accepted natural numbers and geometrical figures as concrete objects for study; their existence was not in question. Nor was that of "magnitudes", which had concrete embodiment in lines and plane regions. Ratios they were not able to define explicitly, though they could define *equality* of ratios, and a ratio was no doubt thought of as embodied in the pair of lines or regions. Negative numbers were used from the sixteenth to the eighteenth centuries with considerable uncertainty as to their "existence"; eventually their representation in the four-quadrant plane of analytical geometry gave increased confidence in them. The plane representation of imaginary numbers had the same effect. It was not possible at that time to define these numbers satisfactorily, but the existence of a model in which the numbers and their operations could be interpreted consistently served instead. As well as geometrical models of elusive numbers, algebraic models of hypothetical geometries helped to increase confidence; for example, n-dimensional geometry could be regarded as a representation of the algebra of n-variables, which was a more familiar theory. (Bourbaki, 1968)

Peirce (1956) makes some comments, ostensibly about proof, but more particularly about the use of symbolic and diagrammatic representations. He draws a distinction between "corollarial reasoning", the kind by which a corollary is deduced from a general theorem - direct deduction - and "mathematical reasoning proper (which) is reasoning with specially constructed schemata". These schemata are the figures drawn in geometry, and the literal expressions transformed in algebra; and these have to be used not merely in the discovery process, but also in the proof. A figure drawn to prove, say, that the altitudes of a triangle are concurrent, must not look isosceles or it will mislead; a separate diagram must be drawn for an obtuse angled triangle and care must be taken to use the diagram as an *illustration*, checking that the assertions made are true for all the triangles which this particular one represents. The process is similar though generally less hazardous in algebra; a proof of a general solution for all cubic equations of form $az^3 + bz + c = 0$ would need to consider various combinations of positive, negative and zero values for a, b and c; if the learning of algebraic transformations had not already absorbed the problem of dealing with all possible values for the coefficients. Consider another illustration. "If a, b, c are elements of a group G, then $ab = ac \Rightarrow b = c$." It is almost impossible to imagine this phrased as a general enunciation "If, when an element of a group is combined (on the left) with each of two elements of the group, the results are equal, then the latter elements are themselves equal." The psychological value of the literal place-holder is apparent even here; the prospect of carrying through the whole proof without it is daunting. But the success of the proof-making depends on ensuring that the transformations made with these place-holders are precisely those permitted to all elements of a group, and special cases may need to be considered. In a sense, these procedures, both in geometry and in algebra, are using particular objects (symbols) to represent general concepts, given by definitions, and the proof is a "display for ease of refutation", the validity of which has to be judged intuitively.

CHAPTER 3

THE MATHEMATICAL PROCESS AND GENERAL STRATEGIES

INTRODUCTION

ABSTRACTION

GENERALISATION

SYMBOLISATION AND MODELLING

PROOF

<u>IN APPENDIX 3</u>

Pupils' investigations:

- Dress Mix-Ups
- Remainder Problem
- Filter Paper

INTRODUCTION

A more detailed analysis of the mathematical process in action will now be made, and general strategies identified. The best known set of strategies for mathematical work is Polya's set of general problem-solving strategies, expounded in How to Solve It (1945). These focus attention on Data, Conclusion and Conditions - thus presupposing a formulated, set problem; the methods include working forwards and backwards; drawing a diagram; designing a plan; studying other problems related to the given one, by logic or by analogy, and in method or in result; changing the conditions, adopting new viewpoints. Some research using these will be reviewed in Chapter 4, but these are not specifically concerned with the mathematical process; Polya himself affirms his concern with the improvement of problem-solving in everyday affairs in his introduction to Mathematics and Plausible Reasoning (1954). But in this and his subsequent two volumes, Mathematical Discovery (1962), he outlines a number of strategies specifically for *mathematical* investigation. Some of these are fundamental to the mathematical process - Generalisation/Specialisation, and Iteration (or Mathematical Induction) - but the remainder have more limited applications. (They comprise superposition, methods for maximum/minimum problems, the setting up of equations and the intersection of two loci.) In the same way, the methods in Klamkin's article on Transform Theory (1962) are rather specialised.

We shall discuss the mathematical process under the three headings (1) Generalisation and Abstraction, (2) Symbolisation and Representation and (3) Proof. It will be helpful to have a context in which to discuss these; the game of Frogs will be used.

Frogs

O O O x O O O

Some pegs of two colours are arranged in a row, separated by an empty hole. Red pegs move to the right, blue to the left; they may either *slide* into the hole, if adjacent to it, or *jump* into it over one peg of the opposite colour. The object is to interchange the colours in the least possible number of moves. Thus the first problem is to find rules to follow in making the moves so as to achieve the result. A little experiment soon shows that it is possible to become blocked, and that, whenever this happens, two pegs of the same colour are next to each other somewhere in the middle of the set. It is not so easy to see how this may be foreseen and avoided. A second problem is to relate the number of moves required to the number of pegs. Experiment establishes a table of values as follows: (the reader should experiment for himself using coins in a row of squares if pegs and pegboard are not available.)

Pegs (p)	Number of Moves (m)
3 of each	15
4 of each	24
2 of each	8

We might next try to predict the number of moves with 5 pegs of each colour: it might be good to include a value for 1 peg; this is 3 moves. Factorising the numbers gives

$$3 \rightarrow 15 = 3 \times 5$$
$$4 \rightarrow 24 = 4 \times 6$$
$$2 \rightarrow 8 = 2 \times 4$$
$$1 \rightarrow 3 = 1 \times 3$$

and a conjectured formula $m = p(p + 2)$; this may be verified with 5 pegs. The possibility of using different numbers of pegs of the different colours may be investigated; one may then conjecture perhaps that for p and q pegs, with $p > q$, $m = p(q + 2)$ or $m = q(p + 2)$. If one of these were correct it would imply an asymmetry between the smaller and larger of p and q, but this cannot at present be excluded as a possibility.

However, experiment shows that 3 red and 2 blue pegs require 11 moves, and that this is independent of which colour moves first. At some point a more analytical approach may be considered. In fact to have assurance regarding the numbers in the table above some means of recording the moves is necessary. Many ways are possible and several should be tried - this step is a crucial one in the application of mathematics to a situation. It will be found that it is sufficient to state which colour is moved each time, so that a whole game can be recorded as RBRRB, for example. (Slides and jumps provide an alternative coding.) Other games are (for 2 red, 2 blue) RBBRRBBR, (for 3 red, 2 blue) RBBRRRBBRRB, for (3,3) RBBRRRBBBRRRBBR, and a study of these could also lead to a conjectured generalisation regarding the number of moves required in all cases. However, the question of proof remains more difficult. Consider the following: to interchange p red and q blue pegs, the red ones must each move a total of q + 1 places, i.e. p(q + 1) places in all. Similarly, the blue pegs must move a total of q(p + 1) places, making altogether 2pq + p + q places. Some of these, however, are jumps. Each red peg must jump over each blue peg once, at some stage: thus pq of the moves will be jumps, and this will account for pq of the places required above. The minimum number of moves is thus pq + p + q which is symmetrical with respect to p and q and reduces, if p = q, to p(p + 2). This, however, assumes that the interchange can be effected always without blockages. A proof by induction can be given that with p = q, the minimal move game is playable, for all p, in p(p + 2) moves and that the rules given are unambiguous; the details of this are straightforward.

One possible proof of the general case goes as follows: Let a row of pegs of the same colour at either end of the row, terminated by either a peg of the other colour or by the hole be called a *stack*. Let a sequence of pegs of alternating colour, with a possible inclusion of the hole which may count as either colour, be called an *alternating pattern*.

Call any state in which there are two stacks separated by an alternating pattern a *successful state*. Then from every successful state it is possible to make a correct forward move into another successful state. The correct move is as follows: (i) if the hole is not adjacent to a stack, forward jumps can be made until it is. (ii) suppose it is then adjacent to the right hand stack. This end takes one of the four following forms, in each of which a correct forward move is indicated. (The hole is indicated by X)

(i).......B R B X R R R R .....
(ii)......B R B X B B B B .....
(iii).....R B R X R R R R .....
(iv)......R B R X B B B B .....

Hence a correct forward move is always possible, and the game is playable in the minimum number of moves.

The stages observable in this investigation include (1) experiment, leading to the formulation of questions, (2) the generation and systematic organisation of examples, leading to the making of a conjectured generalisation, (3) representations by diagram (as in the proof) and by symbols (when a *sequence of moves* is symbolised as RBR or SJS), (5) abstraction and definition of concepts (stack, alternating pattern, etc.). This example shows most of the important mathematical processes and will form the basis of the following discussion.

Three pieces of mathematical work by pupils appear in the Appendix 3: Dress Mix-Ups, The Remainder Problem and Filter Paper. Reference will also be made to these.

ABSTRACTION

The mathematical process has been described above as essentially that of classifying, comparing, transforming, combining, and so on, leading to the building up of structures. The way in which such actions lead to the building up of the hierarchy of abstractions which we recognise as mathematics will now be described in a little more detail. The game of Frogs furnishes one example. Abstraction is involved in identifying a relationship among the numbers in the table, and, before this, in recognising a functional relationship between the numbers of moves and pegs. Later, in the proof, a number of minor concepts were identified and defined - stack, alternating pattern, successful state. The recognition that move sequences could be represented either by a diagrammatic picture of the state of the pegs after each move, or simply by stating what colour peg was moved at each stage, is another act of abstraction. These are all made by observing the situation, by reflecting on it, and by becoming aware of the regularities in it. This ability and inclination to recognise similarities, is clearly a universal human urge. A small child given a set of Logic Blocks will spontaneously sort them. Szeminska (1965) reports how children aged 14 upwards, given a bag of miscellaneous objects and told "See what is in the bag" sorted them hierarchically. Peel (1971a)also reports an accelerating tendency during adolescence and through the years of higher education to move to higher levels of abstraction in thinking.

Three kinds of abstraction can be distinguished in mathematics. The first kind of abstraction, *concept recognition,* is that in which one or more known concepts are identified in a new situation; for example, given 23 × 64 = 46 × 32 one may recognise the reversal of the order of the four digits on the two sides, the remaining symbols being preserved. What has been recognised may be expressed symbolically ab × cd = dc × ba. This is often the first step in a cycle of mathematical activity leading to a statement of generalisation; in this case the obvious question is, "What is the class of numbers for which this is true?". (Dienes,(1961) calls this whole activity "primitive generalisation"). A second kind of abstraction, *concept extension* (Dienes calls it "mathematical generalisation") occurs when a new meaning is adopted for a concept which includes the old as a special case; examples are the extension of the concept of rotation of a figure to allow centres outside the figure as well as within it, or of the trigonometric functions from right-angled triangle definitions to those applying to angles of any magnitude. The third kind of abstraction, *concept creation*, occurs when one moves from consideration of a single object to the creation of a new class of which the object is a member. It is not possible to draw a clear line of demarcation between this and concept recognition, as often the "new" concept is a minor modification of existing concepts. Major levels of the number and geometry hierarchy are suggested in the following diagrams:

categories
group, field
natural numbers, addition
function + 5
3,23
chair, table, desk

transformations of the plane
(rotation, shear)
congruence, similarity, types of symmetry
triangle, parallel, line-symmetric
figures
physical objects

Progress in acquiring concepts at the higher levels here represents major intellectual achievement and gives significantly greater power of thought. A curriculum should therefore aim at helping pupils to reach the highest levels they can. These are the steps to which Skemp (1971) refers when he says that concepts of a higher order than those already possessed cannot be communicated by definition but require experience of a range of examples. (For example, to learn what a group is, one must either be given examples of actual sets of symmetries and number systems with, say, addition; or first learn, also by example, what are laws of composition, inverses and so on, and then be given the definition of the group in terms of these. Law of composition and inverse are concepts at the same level as group, and above that of natural number and addition.)

C.S. Peirce (1956) claims that this hierarchical abstractive process is of the essence of mathematics. He identifies it as the conversion of an attribute into an object which itself has attributes; examples he gives are the change from "honey is sweet" to "honey possesses sweetness" and subsequently the making of comparisons between different sweetnesses, that of honey and that of a honeymoon, and, later, from "a point moves" to "a point generates a line", line being thus defined as an entity which may then have attributes of its own; and another example is the transformation "magnetic attraction acts in certain directions → lines of force → tension in lines of force." In more mathematical terms, the movement is sometimes from considering objects or states which are being transformed to regarding the transformations as objects which themselves may be transformed. If we put this together with the object/class notion and say that concepts of higher order are those which are constructed by regarding the classes or transformations of the objects at one level as the objects to be classified or transformed at the next higher level, this explains most of the level allocations made in the table above.

STRATEGIES FOR ABSTRACTION

A variety of strategies may assist abstraction. Simply *describing the situation* requires the recognition of concepts in it. Other strategies are to *classify* - the question "Is this the whole set of such objects?" is often the key one which leads to the emergence of a quite important new concept. An obvious example is the glide reflection, needed to complete the set reflection, transation, rotation,.... of isometries of the plane. One easier to overlook is that of rotational symmetry, which arises similarly when the obviously regular (in some way) parallelogram, swastika or letter N fails to fit in the existing class of (line) symmetrical objects. A third example appears in the Remainder Problem (Appendix 3) where the types of pairs of numbers which arise include the easily recognised pairs, in which one divides the other, and some which "have no connection". In this case the pupil fails to identify the remaining case as that of numbers with a common divisor greater than 1; the strategy "Is this the whole set?" might have provoked the establishment of the concept. Sometimes the *representation of the situation* itself exposes new concepts, as in Dress Mix-Ups, where the concepts of transposition, 3-cycle and so on, as types of permutation are visible as soon as the arrow diagrams are drawn and compared. Often the *making of generalisations* leads to the creation of minor (and sometimes major) new concepts. This is shown in the Remainder Problem (Appendix 3); and in Lakatos' (1963) classic account of the history of Euler's theorem on the vertices, edges and faces of a polyhedron, in which the repeated efforts to refute the proof led to a long chain of increasingly radical revisions of the concept of a polyhedron. (See Chapter 2.) A similar but shorter example of a similar process can be realised by asking whether the theorem that the exterior angles sum to four right angles applies to non-convex polygons.

Another potentially powerful strategy for abstraction is *changing the model, embodiment or representation*. Dienes' multiple embodiment principle asserts that abstraction occurs when the common elements are identified in two or more perceptually different embodiments, but his experiments show that this method is not as effective as he expected. (Dienes, 1963, pp. 68-70) However, the *manipulation of a single concrete embodiment* such as number blocks, rods or sets of shapes, by classifying, transforming, combining and similar actions is undoubtedly a useful strategy for learning a concept and its associated network of relationships.

GENERALISATION

The making of generalisations is the ordinary bread-and-butter activity of mathematics. It is in the course of this activity that abstractions are made, some very minor ones (concept recognition) others more significant (concept creation). In

Frogs the initial motivation was, after first succeeding in playing one game, to form some general rules for playing that game, and, if possible, the more general game with any number of pegs. Later, the number of moves for a given game being found, again the urge was to generalise to find the number for any number of pegs. Later still, the generalisation from equal to different numbers of pegs was made. Abstractions were used in this process but the thrust of the activity was towards more general statements, thus enlarging the understanding *of the game*. On the more professional level of research, the relationship between the two activities is similar. Generalisations are what one directly seeks; new abstractions may emerge, and may well represent the more significant addition to knowledge in the long term. But to be shown as significant they must be used to establish some worthwhile generalisations.

STRATEGIES FOR GENERALISATION

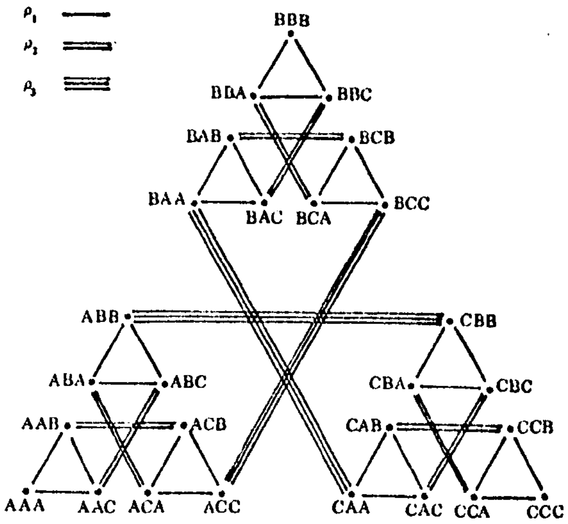
The following strategies appear from the above considerations to be relevant to generalisation.

Recognise relationship
Generate examples to test conjecture
Collect variety of examples: try big numbers
Organise examples systematically
Consider iteration: adding one
Make conjectures

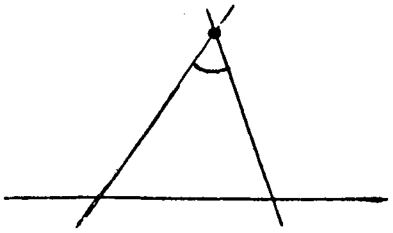
These will be considered further in subsequent studies.

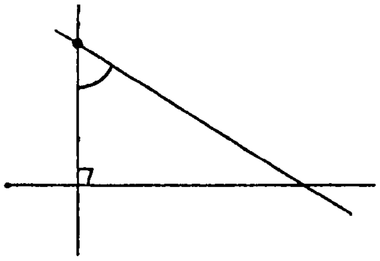
SYMBOLISATION AND MODELLING

This is closely connected with abstraction, in that the act of representation - the replacement of the situation by symbols or diagrams - implies the recognition of a correspondence between some aspects of the situation and the model. In Frogs we see this in the symbolic representation of the move sequence by strings of letters referring either to colours or to slides and jumps; and in the final proof, where the more primitive diagrammiatic representation of the game is required. Another instance is shown in the following diagram, which displays the entire set of possible states and moves in the Towers of Hanoi. The *pegs* and coded A, B, C and the *discs* are coded by position: the first letter denotes the peg on which the largest disc is placed, thenext that holding the next largest disc, and so on. (Jullien, 1972)



The difference between symbol systems for number and geometry needs clarification. It is clear that 1, 2, 3 are *symbols* and that the objects being combined and related are the underlying *numbers*. But is a drawing of an isosceles triangle, for example, a symbol? Or is it the actual object of consideration? It is, in fact, a representation of a *general* isosceles triangle, but, if accurately drawn, can be measured or folded to verify the equality of angles or sides; so it is a closer representation than a symbol. It may be called an *ikon* (following Bruner, 1959). The representation of natural numbers by strokes |, ||, ||| and so on is a comparable ikon system. The geometric *symbols* most closely corresponding to the numerals are, for example, t_{AB} for the translation sending A to B, and $A(50^{o})$ for the rotation through 50^{o} about A. (The symbol m for the reflection m is non-definitive without a specification of m). A geometric symbol system which has its own set of transformations is the representation of reflection by their mirrors, with the transformation of "swinging pairs". (See diagram).





This difference between numbers and geometry reflects the fact that geometry is an essentially more complex system, and that at the school level we work mainly with an ikonic representation of it, whereas in number, even young pupils work easily with the symbol system, with only occasional recourse to the ikons.

MODELLING IN APPLIED MATHEMATICS

Modelling is identified by Hall (1972) as the essence of applied mathematics.

> "The basis of modelling is the scientific method except that the emphasis is on finding a mathematical form for the scientific theory. The process starts from some given empirical situation which challenges us to explain its obvious regularities or discover its hidden laws. The first, and generally the most

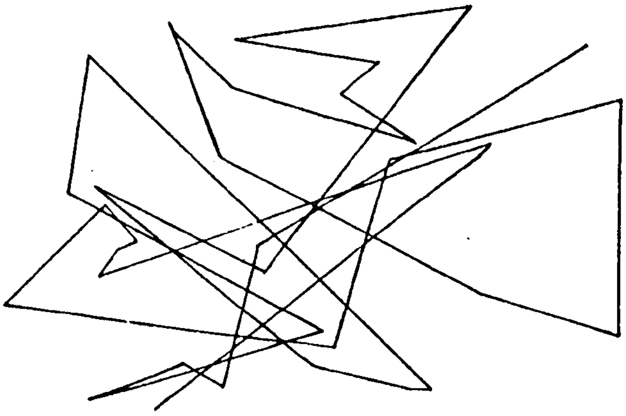
difficult step is to discover an appropriate mathematical formalism to describe the essential features of the situation.

Once the situation has been formulated mathematically, the model itself is set up. It will consist of postulated relations between the entities to describe their interdependence and their modifications as the situation changes. In particular, certain features may be selected as most significant and others as irrelevant. To be reliable a model must reproduce the major known results so the first check is to validate it in this way. A more searching test is to use the model to predict new results, especially ones in which the main effects are exhibited in some extreme form. The more unexpected the prediction the more convincing the test. In practical model building it may be necessary to iterate round the validation and prediction cycles several times before a model is obtained which is sufficiently accurate and easy to use. An important by-product of this process is the formulation of new mathematical problems and of new techniques for their solution. The most significant result is that the successful model involves the creation of new concepts. These provide the categories through which similar situations can be described and understood even when the details of the model are no longer applicable."

Hall gives the following example of the process:

" The figure below shows the record of the movements of a fox. The length of a straight segment is clearly one of the variables in the description and its distribution can be estimated by analysing this sample path in a histogram. Similarly, the variable angle between successive segments has a second histogram. The question arises whether these two variables describe the situation fully. We can simulate the path by selecting lengths and angles at random from those of the original paths so that the histograms must be identical. It soon becomes apparent, however, that the simulated path wanders over a much greater area than the original path so that some third factor must be operating.

The missing factor can be isolated by repeating the simulation with some additional constraints. By adding a boundary and selecting only those segments that lie inside the boundary, a path comparable with the original one is obtained. Thus the remaining factor is what a biologist would recognise as the fox's use of his home territory."



An example from school work of an "applied" problem, which illustrates problem-formulation and final reinterpretation in the practical situation, is Filter Paper (Appendix 3).

STRATEGIES FOR SYMBOLISATION

In most mathematical work the task is, as in Frogs, to select a suitable mode of representation from the established repertoire. This is extensive - Venn diagrams, tree diagrams, Cayley tables, arrow graphs, flow diagrams, Cartesian graphs, nomograms, scale plans, plane projections of three-dimensional objects, as well as letters, numerals in various positions (2^3, a_n) brackets, symbols for compositions and so on. Many of these are capable of adaptation to a variety of situations. It is possible that strategies of choosing and using these for new situations could be developed more consciously and more fully than they are in school mathematics, and that the resulting acquisitions would be of considerable general usefulness. One example of this may be seen in Dress Mix-Ups and the Remainder Problem also contains several types of symbolic and diagrammatic representation. (See Appendix 3)

PROOF

To the Greeks, the starting points of their mathematical system (the essential properties of numbers and angles) were self-evident truths. Nowadays, mathematics is seen to comprise closed axiomatic systems. Recent writers have gone further, and have attempted to describe the actual process of proof activity. That proof itself must be regarded as an act of communication rather than as a static statement is explained by René Thom (1973). Against the background of the adoption of a rigorous Bourbakiste mathematics for the curriculum for all pupils in the French secondary schools, he writes:

> "The real problem which confronts mathematics teaching is not that of rigour, but the problem of the development of "meaning" of the "existence" of mathematical objects;...."meaning" in mathematics is the fruit of constructive activity, of an apprenticeship...."

> "At best mathematicians base their universe on a kind of common stem made up of objects and theories which occur in standard teaching (for example, real and complex numbers, analytic and differentiable functions, manifolds, groups, vector spaces,...) and all proof, other than the more specialised, must proceed from this mathematical vernacular common to all. A proof of a theorem (T) is like a path which, setting out from propositions derived from the common stem (and thus intelligible to all), leads by successive steps to a psychological state of affairs in which (T) appears obvious. The rigour of the proof - in the usual, not the formalised sense - depends on the fact that each of the steps is perfectly clear to every reader, taking into account the extensions of meaning already effected in the previous stages. In mathematics, if one rejects a proof, it is more often because it is incomprehensible than because it is false. Generally this happens because the author, blinded in some way by the vision of his discovery, has made unduly optimistic assumptions about shared backgrounds. A little later his colleagues will make explicit that which the author had expressed implicitly, and by filling in the gaps will make the proof complete. Rigour, like the provision of supplies and support troops, always *follows* a breakthrough."

This insight, from a working mathematician, will be a useful guide when we consider in Chapter 6 what kind of proof activity to study in the school curriculum, and we observe the need to connect new conjectures by deductive steps with what is known already.

Lakatos (1963) contributes a view of the nature of mathematical proof as the display of a conjecture for ease of refutation as well as reinforcing Thom's view of it as the embedding of new results into the fabric of existing knowledge. Lakatos regarded the success of the formalists as having overshadowed the consideration of the more informal processes by which mathematics is generated. His long paper is a detailed case-study of the history of Euler's theorem on the vertices, edges and faces of polyhedra right up to its embedding in geometrical topology in the form of the Euler characteristic of a surface. He states his general aim thus:

> "Its modest aim is to elaborate the point that informal, quasi-empirical mathematics does not grow through a monotonous increase of the number of indubitably established theorems but through the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations."

Kilmister (1972) supports this view:

> "A theorem can be described as a conjecture: and the proof of the theorem is a search for counter-examples - which, because of the complex nature of mathematical concepts, is usually successful, requiring subsequent modifications either in the enunciation of the theorem or in the definitions of the mathematical objects entering it."

In the course of the dialogue which constitutes the paper, it is shown that a proof is not something which establishes the truth of a conjecture, but rather it is a decomposition of the original assertion into sub-conjectures; thus the conjecture and proof are both displayed for criticism.

Global counter examples, refuting the conjecture, and local counter examples, refuting some or all of the sub-conjectures, are distinguished; so are the different kinds of definition, including some designed for "monster-barring". The polyhedron comprising a small cube attached to the middle of a face of a large cube is one such apparently non-conforming object.

In the last part of his article, Lakatos distinguishes between naive guessing and deductive guessing. The former begins with a table of values of V, E, F for a number of solids and eventually finds the relation V - E + F = 2; the latter starts with a single point, then a polygon (V = E), and asks what would happen if additional polygons were attached to the first. Such "deductive guessing", the use of Choquet's mother-structures as generative tools, increases the content of the conjecture, in this case reaching a V, E, F formula for all normal n-spheroid polyhedra with multiply-connected faces and with cavities, this Euler characteristic providing one criterion for classifying such surfaces.

There are here many valuable pointers for the development of proof-centred classroom activities.

Some teachers have said that proof, for a pupil, is what brings him conviction. Although this is a valuable remark, in that it directs attention to the need for classroom explanations to have meaning for the pupil rather than be formal rituals, it is perhaps dangerous in that it avoids consideration of the real nature of proof. Conviction is normally reached by quite other means than that of following a logical proof. Lunzer (1973b) has suggested that productive thinking is more analogical than logical; and I would suggest that conviction arrives most frequently as the result of the mental scanning of a range of items which bear on the point in question, this resulting eventually in an integration of the ideas into a judgement. Proof is an essentially public activity which follows the reaching of conviction, though it may be conducted internally, against an imaginary potential doubter.

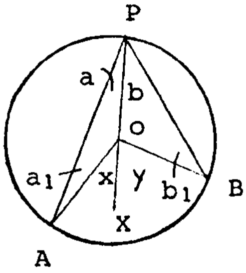
The mathematical meaning of proof carries three senses. The first is *verification or justification*, concerned with the truth of a proposition; the second is *illumination*, in that a good proof is expected to convey an insight into *why* the proposition is true; this does not affect the *validity* of a proof, but its presence in a proof is aesthetically pleasing. The third sense of proof is the most characteristically mathematical, that of *systematisation*, i.e. the organisation of results into a deductive system of axioms, major concepts and theorems, and minor results, derived from these. The classic example of this is the Bourbaki work; its objectives are both the increased assurance of correctness, and the great simplicity and flexibility obtained; thus it is regarded as a particularly good way of achieving verification, and it also contributes to illumination. (Psychologists such as Skemp (1971) and Ausubel (1968) would also comment that a well-connected system makes for ease of learning and retention).

Although we have asserted that proof is an essentially public activity, it grows out of the internal testing and acceptance or rejection which accompanies the development of a generalisation. One might hypothesise that this gradually becomes more externalised. First one tries one's generalisation on other people; conflict with their ideas often leads in younger children first to a reassertion, but eventually there is an appeal to evidence. Later, there may be the realisation of the need for a written statement of the proposition, for more effective attack by potential counter-examples and to avoid unconscious shifts of ground. The final stages are the awareness of the need to set out the argument in written form (as Lakatos shows), and the need for explicit starting assumptions or axioms. These developments took place in the early history of mathematics, and culminated in the acceptance by the mathematical community of the Euclidean model of proof as the best form for guaranteeing freedom from error.

It follows from the above analysis that pupils will not use formal proof with appreciation of its purpose until they are aware of the public status of knowledge and the value of public verification. The most potent accelerator towards achievement of this is likely to be cooperative, research-type activity by the class. In this, investigation of a situation would lead to different conjectures by different pupils and the resolution of conflicts by arguments and evidence.

The traditional classroom version of Euclidean proof brings out these points well. Consider the following proof:

Theorem: The angle at the centre of a circle is twice the angle at the circumference subtended by the same arc.



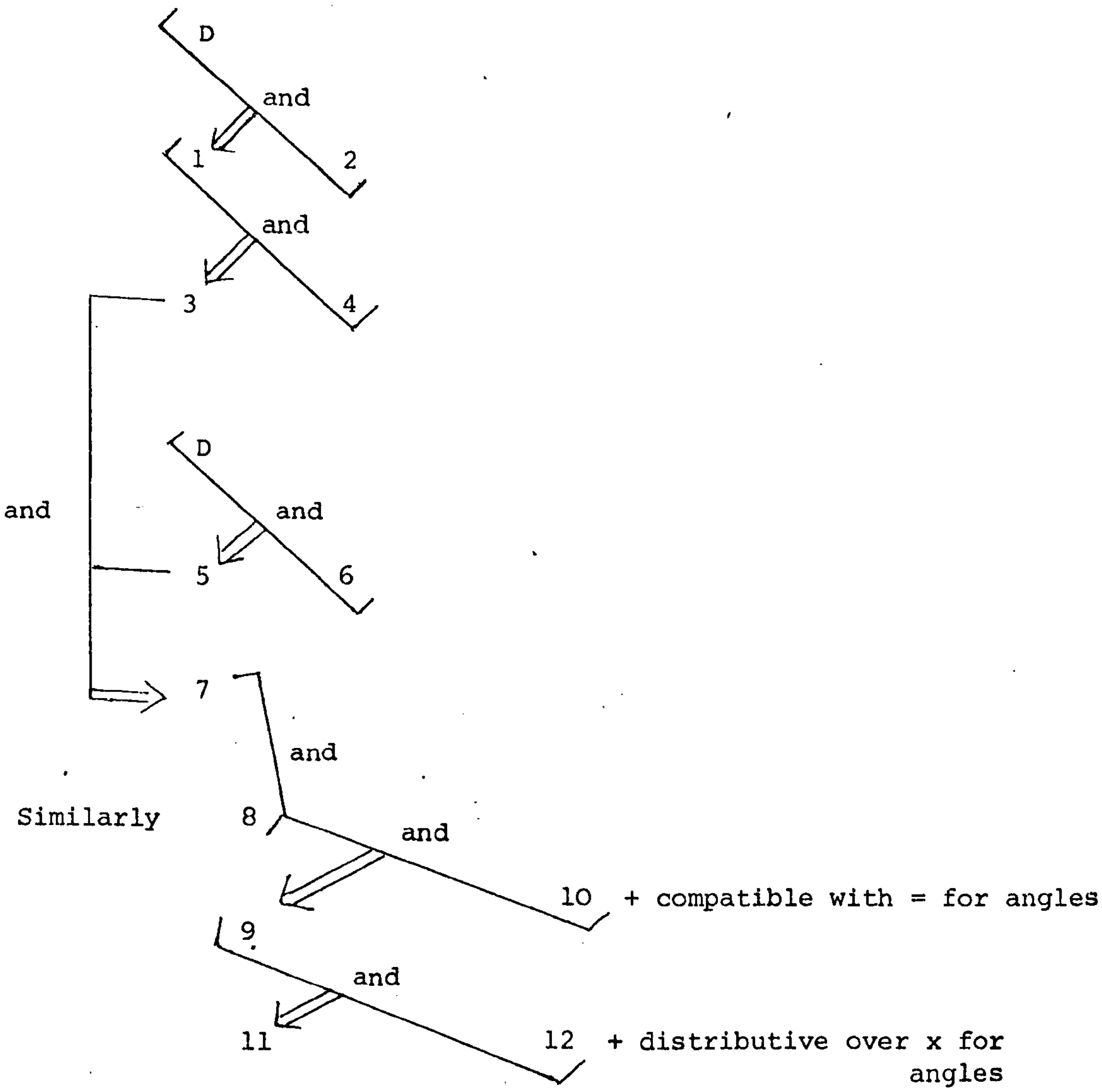
Given: Circle, Centre O, points A, P, B, on circumference (D)

Construction: Join PO and produce to any point X.

Let the angles a, b, a_1, b_1, x, y be as marked. To prove: $A\hat{O}B = 2A\hat{P}B$

Proof :	$OA = OP$	(1)	(2)	(radii of a circle)
	$\therefore\ a = a_1$	(3)	(4)	(base angles of an isosceles Δ)
	Also $x = a + a_1$	(5)	(6)	(exterior angle of a Δ)
	$\therefore\ x = 2a$	(7)		
	Similarly $y = 2b$	(8)		
	$x + y = 2a + 2b$	(9)	(10)	
	$= 2(a + b)$	(11)	(12)	
	i.e. $\underline{A\hat{O}B = 2\ A\hat{P}B}$			

This layout uses the left hand column for statements relating to the current proposition, and the right hand column for references to theorems being assumed as already known and established. The logical structure is shown in the diagram. (p 3.20)


D
and
1
2
and
3
4
D
and
5
6
and
7
and
Similarly
8
and
10
+ compatible with = for angles
9
and
11
12
+ distributive over x for angles

The proof structure revealed here helps to clarify and illustrate what we normally require of proof in the mathematical activity of the classroom (and perhaps of the research seminar).

A proof is a directed tree of statements, connected by implications, whose end point is the conclusion and whose starting points are either in the data or are generally agreed facts or principles.

Examples will be given below of pupils' attempts at proof which are intuitively recognisable as failures, and in which the failure can be identified as lack of one or more of the requirements in this definition. Difficulties arise mainly in two ways - in that there is no absolute criterion for the degree of explicitness required, nor for what theorems are "generally agreed". The Euclidean example quoted also illustrates the problem of "completeness", i.e. of giving a proof which applies to all cases implicitly included in the statement; quantifiers are often missing from Euclidean statements and the different classes of cases are often forgotten.

The Euclidean example shows what is needed for *verification*; and it contributes to *systematisation* in so far as the theorems used in the right hand column are chosen to be suitable for prior proof in a correct and satisfying deductive system. (It will be the more satisfying to the extent that they appear more fundamental than the current proposition). The relevance of this to the classroom is shown in the following example. In one test, pupils were asked why multiplying by ten could be effected by "adding a nought". Very few were able spontaneously to relate this to the movement of figures between columns of different place value; the majority appealed to the standard algorithm for long multiplication. This is an example of how proof activity in regard to well-known relationships requires agreements about which of them are to be regarded as the more fundamental ones from which others are to be derived.

Illumination is not particularly strong in this proof; the most striking fact is not the double angle property but the invariance of the $\hat{APB}$ as P moves, and this is better illuminated by other proof, for example that which translates the line pair PA, PB keeping P on the circle and $\hat{APB}$ constant, and shows that the arc AB remains of constant length.

The game of Frogs illustrates several of these points. The formula $m = p(p + 2)$ induced from the table of values is probably totally convincing; one's developed expectations of regularity in regard to the problem make one feel that the correct formula is probably of that order of complication. The RBR sequences give added conviction, though still no proof. The proof of the minimum number of moves, from considering the total number of slides and jumps needed, is illuminating as well as verificatory. The final proof is, on the other hand, non-illuminating; being an exhaustive check of possibilities; the induction proof for equal numbers of pegs (not detailed), also carries insight. The transition from the empirical approach of playing games and collecting examples to the deductive considerations is clearly seen; so is the way in which the empirical experiment builds up awareness of the concepts which must be used for the deduction.

The Remainder Problem shows how in some cases an explicit formula cannot be found, but only an algorithm. And even the earlier result, that the common difference in the number sequence is the product of the two given numbers, if coprime, is obtained only by fairly systematic empirical work, and is not symbolised or proved.

STRATEGIES FOR PROOF

These considerations suggest that the following concepts and strategies are relevant to proof activity.

Make exhaustive empirical check
Display conjecture for refutation - write it, discuss with others
Construct classes and deal with them separately
Identify data and conclusion
Connect data and conclusion logically

Embed in agreed existing knowledge
Identify and state assumptions and definitions
Recognise potential arbitrariness of assumptions and undefined terms

The occurrence of these in pupils' mathematical work and the feasibility of learning and using them at different stages will be studied in later chapters.

CHAPTER 4

CONTENT AND PROCESS IN EXISTING SCHOOL COURSES

INTRODUCTION

The analysis of the nature of mathematics made in Chapters 2-3 has implications for the secondary curriculum; these affect the nature of what is to be taught, what are appropriate methods, and the place of the learning of process. The South Nottinghamshire Project is to a large extent an attempt to design and establish a curriculum which takes account of these implications. It does this by basing its learning on abstraction from concrete situations, and by developing pupils' competence in the mathematical process - chiefly with regard to strategies of generalisation and proof. The departures from normal practice will be seen more clearly if we begin by reviewing some teaching material in current general use from the point of view of the analysis of mathematics made above.

THE CONTENT OF MATHEMATICS AND CURRENT TEACHING METHODS

The recognition that the content of mathematics is structures, that is, interrelated systems of relational concepts, with their associated symbol-systems and models, and that the mathematical process consists of generalising and abstracting, symbolising and modelling, and proof, has several implications for teaching.

The first is that the essential learning act is one of insight and the teacher's task is to prepare a situation in which greatest possible number and the highest possible quality of insights can be achieved. This is not to ignore the importance of insights in the learning of other subjects, but it follows because the actual subject matter of mathematics consists not of information, but of relationships. The most obviously effective means for achieving this is by a process of guided discovery; empirical research has indeed shown this to be the most effective method, if retention and transfer are the criteria. (Worthen (1968), Gagné and Brown (1961), Pigge (1965); and see Shulman (1970).

Different forms of guided discovery are observable in school texts, which may be distinguished as "deductive" and "empirical".

Guided deductive discovery is a method which instructs the pupil to perform a sequence of actions with material, or operations with numbers, which bring together the components of the required relationship, and asks questions the answers to which involve the recognition of the relationship. "Deductive" implies that the recognition is of the relationship and of the reason for it, or the necessity of it; *empirical discovery* is when only the *fact* of the relationship is discovered. The treatment in SMP Book C (p.4.3) of the relationships area of triangle = ½ area of parallelogram = ½ area of rectangle is of the first type. The pupil makes the figures on a geoboard, changing one into another. This is done four times with varying shapes of rectangle and with decreasing detail in the instructions, and in each case the questions "What is the area of the rectangle?/the parallelogram?/the triangle?" are asked (the answers are in numbers of geoboard unit squares.) The following page extends the applicability of the relationship to a new embodiment - the coordinate plane. The starting figures are now specified by the coordinates of their vertices, but the same transformations are to be drawn. The last questions require the finding of the areas of four parallelograms and three triangles given by coordinates, with the use of the relationship, now presumed to be learned. Further work discovers the same relationships in a second new embodiment, cut-out paper shapes, and this leads to the *calculation* of the areas of printed parallelograms and triangles by measurement of the bases and heights and use of the relationship area = length x breadth for the rectangle (this is assumed to be already familiar; it is not discussed). Thus this relationship is discovered deductively in three different embodiments, then *applied* to a fourth.

A rather different kind of guided deductive discovery is seen in MME Vol 1 (1967) in the treatment of the kite (a minor concept) (p.4.4). Instructions are given to fold and cut paper in a certain way, and various questions ask for *prediction of the results*, these to be checked by observation after cutting and unfolding. Whether the answer is "We shall get an isosceles triangle because a cut perpendicular to a fold will give a straight line", given before

Rectangles, parallelograms, triangles

Exercise A

1 Set up the rectangle in Figure 6 on a pinboard. Change it to the parallelogram and then to the triangle. What is the area of:

(*a*) the rectangle,
(*b*) the parallelogram,
(*c*) the triangle?

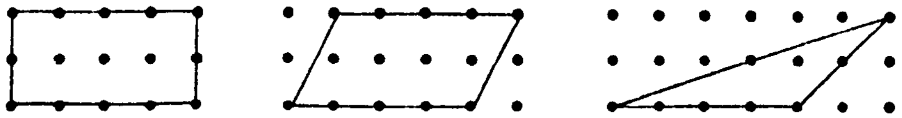


Fig. 6

2 Set up the triangle in Figure 7. Change it to the parallelogram and then to the rectangle. What is the area of:

(*a*) the rectangle,
(*b*) the parallelogram,
(*c*) the triangle?

Fig. 7

3 Using your pinboard, find the area of the triangle in Figure 8.

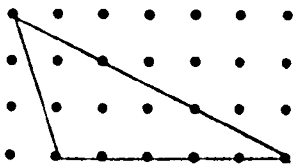


Fig. 8

4. (*a*) (See Fig. 4·05)

Fold a piece of paper into halves along AB, and make a second fold, OQ such that the angle AOB is acute. Cut along XPY. How many sides has this shape? What are its symmetries? What shape is symmetrical about one diagonal and has two pairs of equal sides? Open out your shape and check your conclusions. What are the properties of the kite?

(*b*) Fold another piece of paper in the same way as in (*a*). Cut along ZRY at right angles to the fold OB. The upper part of the cut, RY, is perpendicular to the fold OB. What does a cut perpendicular to a fold produce? What can you say about YPY′?

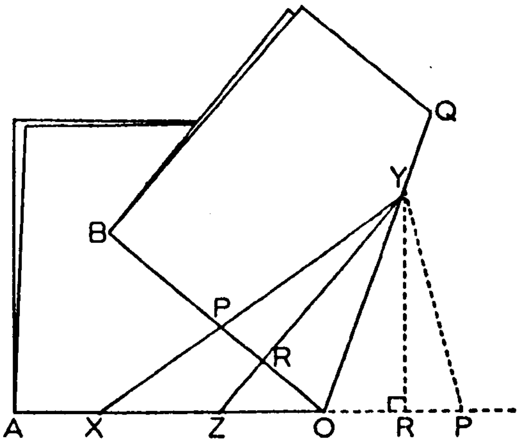


FIG. 4·05

What happens to the kite if YPY′ is a straight line? How many sides will the shape have? What can you say about the sides of the triangle? What do we call a triangle with two sides equal? Stick your isosceles triangle into your book and note its properties.

(*c*) Fold a piece of paper into halves as before, and make a second fold OQ, where angle AOB is a right angle (Fig. 4·06) Cut along XY. How many sides has this shape? What symmetries does it have? What can you say about the lines PY and PY′? What is the shape of the unfolded polygon? Open out and check your findings.

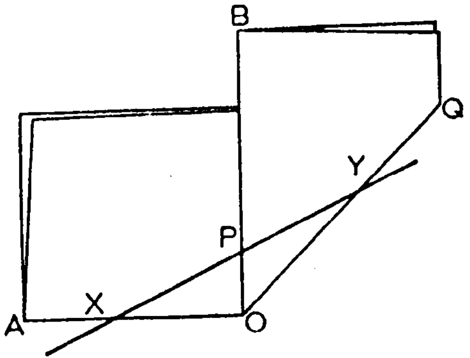


FIG. 4·06

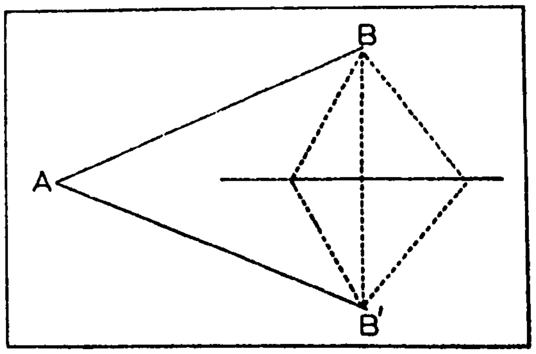


FIG. 4·07

(*d*) On a postcard, draw an angle BAB′ and bisect it. With a knife, cut along the bisector starting about 3 cm from A. Thread a piece of shirring elastic through pinholes at B and B′ and knot the ends at the back of the card (Fig. 4·07). Pass a loop of the elastic through the cut, and holding this at the back of the card, move it backward and forward along the cut. Notice how the kite changes through the isosceles triangle to the reflex quadrilateral. How does symmetry determine the shape of a quadrilateral? How does symmetry decide the angle properties of the quadrilateral?

RE-ENTRANT OR REFLEX KITE

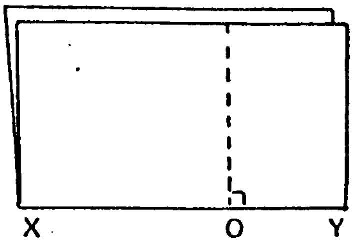
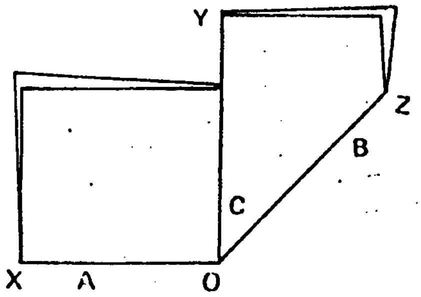


Fig. 4 07

1. Fold a piece of paper in half as before and label as the diagram, where O is anywhere between X and Y.
2. Using your set square or protractor draw the dotted line from O at 90 to the crease.



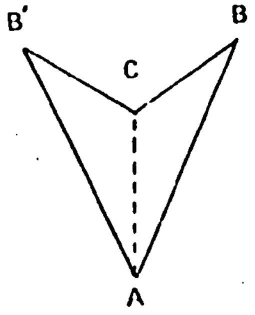


Fig. 4 09

3. Twist OY so that it lies along the dotted line, and make a second crease, OZ as in Fig. 4.08.
4. Draw any line AB across the corner O and label the point C.
5. Cut along AB.
6. Stick your quadrilateral into your notebook and make sure you have labelled it as in the diagram, and answer the following questions.
   - *(a)* Are the triangles ABC and AB'C congruent (identical)?
   - *(b)* Are the sides CB and CB' equal in length?
   - *(c)* Are the sides AB and AB' equal in length?
   - *(d)* Does the crease AC bisect the angles BCB' and BAB'?
   - *(e)* Is the line AC a perpendicular bisector of the line BB'?

As it has all the same properties as the kite this is also a kite called a cutting, or "We have got....because the cut....has given a straight line", given afterwards, the deductive element is emphasised by this procedure. Movement from the convex to the concave kite passing through the special case of the triangle is also a feature; we are making generalisations. The corresponding section in the revised version Maths Today Book 1 (1975) (p. 4.5) is now an example of *guided empirical discovery*. The pupil cuts out the figure but has only to *observe* the equalities in the figure; no prediction, no explanation and no generalisation is involved.

Of these, deductive discovery must be preferred, because the underlying relationship which is learnt is itself more general - it has more connections with other material; and also because the general strategy of seeking relationships may also be acquired from the activity.

The second implication of the relational/symbolic character of mathematics concerns the nature of the learning task. There is a need to distinguish clearly between the structural concepts and the symbols or models which represent them, and hence between the relationships themselves and the corresponding transformations of the symbol-system.

Several mistakes explainable in these terms have been made during the assimilation of "modern mathematics" into the curriculum. The ikonic representations of a set by braces or a Venn diagram have often been taught as if they themselves constituted the concept of set. The chapter on The Quadratic Function and its Graph in the Scottish series, Book 5 (1st edition) provides some illustrations of the relationship between a concept and its various models, (in the wide sense of p. 2.6). The relevant section of this begins by showing the graph of a linear function, which makes it possible to begin to make the discrimination, and to link a particular form of curve to a particular class of function. The quadratic function has several models - a table of pairs, an algebraic formula, an arrow graph (not very illuminating), a parabola-shaped

Cartesian graph. None of those *is* the quadratic function. A normal approach to learning this concept would involve developing the skills of transforming between these different models. To be able to do this one has to understand at least some facets of the concept itself. As genes become visible during mitosis, so the concept has to come to life to effect a transformation. If the transformations are taught by exposition, without the awareness of their rationale being transmitted, then the concept itself will not be acquired. This will become apparent if an application needs to be made. Even if the teaching does include the meaning, the same problem can arise if there are too many practice examples of the same type, since the residual learning may then consist of the transformation skills without the insights on which they were initially built. A suitable balance between repetition and variety is necessary. These principles can be seen operating in the earlier part of the chapter in question, where the above models for functions in general are established through exercises involving transforming from one into another.

Another teaching method can be described as *deductive exposition.* This has the superficial virtue that it suggests that mathematical principles can be deduced one from another, but the exercise in this case fails if the pupil cannot follow the deduction, and an impression that the workings of the subject are impenetrable may be given. The foregoing account of the nature of mathematics, the need shown in history for the support of models to aid the grasp of abstractions, the naturalness of the process of abstraction and the movement from lower to higher levels all suggest that structural relationships will be best learnt by abstraction from suitable concrete embodiments, rather than by deduction from other relationships. Other research also points in the same direction, (Collis (1975a), Lunzer, Bell and Shiu (1976)), and so do some of the results in this thesis. Examples of deductive exposition appear widely in the Scottish texts, and less frequently in the SMP books. The SMP sequence on the subtraction of negatives is an example. In this case it is a definition which is being justified deductively.

We shall now try to give a meaning to the 'subtraction' of shift numbers. We know that if we take away something from itself we are always left with nothing. It seems reasonable to expect that this is also true of shift numbers, so

$$^{+}3 - {}^{+}3 = 0.$$

But we know from Exercise D that

$$^{+}3 + {}^{-}3 = 0,$$

so subtracting $^{+}3$ gives the same answer as adding $^{-}3$.

This suggests that $-\,{}^{+}3$ means the same as $+\,{}^{-}3$.

We also have

$$^{-}3 - {}^{-}3 = 0$$

and

$$^{-}3 + {}^{+}3 = 0,$$

so perhaps $-\,{}^{-}3$ means the same as $+\,{}^{+}3$.

(*a*) What can you see from the following pairs of statements? The first one has been done for you.

(i) $^{+}5 - {}^{+}5 = 0,$

$^{+}5 + {}^{-}5 = 0;$

so $-\,{}^{+}5$ means the same as $+\,{}^{-}5$.

(ii) $^{+}7 - {}^{+}7 = 0,$

$^{+}7 + {}^{-}7 = 0;$

(iii) $^{-}4 - {}^{-}4 = 0,$

$^{-}4 + {}^{+}4 = 0;$

(iv) $^{+}2 - {}^{+}2 = 0,$

$^{+}2 + {}^{-}2 = 0;$

(v) $^{-}6 - {}^{-}6 = 0,$

$^{-}6 + {}^{+}6 = 0.$

We see that the idea that $-\,{}^{+}3$ means $+\,{}^{-}3$, and $-\,{}^{-}3$ means $+\,{}^{+}3$, works when we subtract a number from itself. Does it work when we subtract from other numbers? Let us consider some number patterns.

(*b*) Use the number pattern to copy and complete the following subtractions:

$$^{+}2 - {}^{+}2 = 0$$
$$^{+}3 - {}^{+}2 = {}^{+}1$$
$$^{+}4 - {}^{+}2 =$$
$$^{+}5 - {}^{+}2 =$$
$$^{+}6 - {}^{+}2 = {}^{+}4$$
$$^{+}7 - {}^{+}2 =$$
$$^{+}8 - {}^{+}2 = \quad .$$

Since the shift number from which we subtract $^{+}2$ is one unit larger each time, it seems reasonable to expect that the answers will also be one unit larger each time.

The proposed defining relationship is $x - y = x + (\text{inv } y)$.
The argument is: (compare p. 4.8)

$$y - y = 0$$

$$y + (\text{inv } y) = 0$$

for all shift numbers y, verified by 6 examples

Hence $\quad y - y = y + (\text{inv } y)$

Hence, plausibly, $\quad x - y = x + (\text{inv } y)$ for all x and all y.

The logical weakness of this argument is in the last step, but the psychological weakness lies in the lack of assimilation of the meaning of the first statement, $y - y = 0$ for all shift numbers y, before its employment in the comparison with the second. In Piagetian language, this is formal reasoning: making logical deductions from propositions whose content is unfamiliar, and for the majority of children this is not achieved until later in adolescence. This is the weakness of many worthy attempts at deductive exposition - worthy, because they stem from the teacher's attempt to display in his teaching the true nature of mathematics, not asserting rules without justification but showing them as necessary consequences of previous knowledge. Two mistakes are often combined. One is that illustrated above, of reasoning from insecure premises. The other is the assumption that a single sound chain of reasoning is convincing, whereas acceptance actually depends on the convergence of many implications, or rather the growing awareness of an interlocking scheme of ideas of which this particular idea forms a part. The initial mode of introduction of a concept such as this is probably less important than the subsequent provision of the varied experience needed to assimilate it.

The Scottish treatment of this topic also rests on the doubtfully acceptable deduction: since $x - y = z \Leftrightarrow y + z = x$ for positive numbers, so for negative numbers the result of subtracting y from x must be that number which, when added to y, gives x.

The approach of the South Nottinghamshire Project to this topic is that of "problem-situation embodying the concept". This aims to work persistently in a situation in which the study of the properties of plane figures - a pursuit of some interest in itself - is made in a context in which the addition and subtraction of negatives has to be incorporated into the system to enable the study to proceed, and in which therefore these operations acquire a meaning in concrete terms. Negative numbers arise first in the labelling of line-segments in various directions (this was begun as (L2, U5) etc. in a previous topic); this is then connected with the coordinates of points as $\overrightarrow{PQ} = Q - P$; this brings in, for example, $1 - 4 = \bar{3}$ and $5 + \bar{3} = 2$. Straying into other quadrants introduces the labelling of points by signed numbers, and the full set of additive and subtractive relations with them. Thus the consistency of the system can be gradually appreciated while the interest is maintained by the study of the properties of the geometrical figures.

This review of some current teaching material thus shows that the characterisation of mathematics as structures and symbol-systems explains the appropriateness of guided discovery and provides the conceptual framework in which the relation between "manipulation" and "understanding" can be more easily seen. At the same time, it has been argued that, of the methods used in current texts, deductive discovery is preferable to empirical, and deductive exposition has weaknesses which, it is suggested, may possibly be avoided by a more schematic form of learning.

PROCESS ASPECTS OF CURRENT SECONDARY COURSES

The O-Level Examination

The following question is taken from the SMP O-level Paper 2 for 1969. A study of this paper as a whole shows that the questions require, for the most part, the recognition of learnt concepts in familiar situations, except for the last parts, some of which are problems requiring the identifying and bringing together of two items of information rather than one.

1. $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix}$,

$\mathbf{D} = \begin{pmatrix} \frac{1}{2} & 0 \\ k & 1 \end{pmatrix}$.

(i) Evaluate **AB**.

(ii) Find the value of k which makes **CD** the unit matrix.

(iii) Simplify **CABD** with the value of k found above. What does this show about the inverse of **CA**?

(iv) What is the inverse of **AC**?

To see this, consider the following solution of Question 1.

(i) $AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(ii) $CD = \begin{pmatrix} 1 & 0 \\ k+2 & 1 \end{pmatrix}$ which = I if k = -2. (Requires manipulation including k)

(iii) CABD = C(AB)D
= CD
= I This shows $(CA)^{-1}$ = BD

(iv) ($(AC)^{-1}$ is not necessarily the same as $(CA)^{-1}$)

Since CD = I
ACD = A
So ACDB = AB
= I
Hence $(AC)^{-1}$ = DB

In this question part (iii) requires only the knowledge that the middle AB can be treated as a simple element and substituted, and of the meaning of the inverse. Part (iv) requires first the cautionary recognition of non-commutativity. The idea that A might be conveniently combined with CD, with a view to separating off AC, probably arrives only after some exploration of possibilities, and it is not confirmed as the desired route to a solution until the B is chosen to combine on the right. This is a characteristic two-step problem-solving process. Similar demands, possibly slightly less difficult, occur in the last parts of several other questions in the paper. No generalisations or proofs are asked for; this limited degree of problem-solving is the highest level strategy tested. A geometrical question asks for the plotting of a set of six given points (forming a rectangle) after each of two shears (given by invariant line and the image of one point), and requires the pupil to recognise the combined transformation as another shear; but there is no question of generalisation. O-level papers requiring open mathematical investigations have been set by special arrangement with schools; the AEB paper for Abbey Wood School in 1967 included such questions as "Investigate, algebraically rather than geometrically, matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a + b = c + d$." (ATM, 1968)

Secondary Texts - Generalisation

Only the lower levels of generalisation appear to any extent in the SMP or Scottish texts. Recognition of relationships, characterised as "Observe A and B: how are they related?" figures quite strongly in the SMP course: the guided discovery of the relationship "area of triangle = ½ parallelogram = ½ rectangle" is an activity of this type (see p. 4.3); so is the discovery of the position of the centre of rotation (Book C, p. 124-5). A somewhat similar example from the Scottish texts (1st edition, Book 4, p. 110) asks for four equiangular hexagons printed on a triangular tiling to be checked for similarity, i.e. are their sides in proportion, and asks also whether two equiangular triangles can be found on the tiling which are not similar. These examples might not appear to demand (or to develop) any very substantial strategy on the part of the pupil - he has only to do as instructed, to look at specified aspects of the situation and to say what he sees. However, comparison with any pre-1960 text shows that this kind of activity was almost non-existent then, and alerts one to the fact that the ability to perform such an act of controlled observation may need a degree of training. (It is shown in a later chapter that success at this activity is by no means universal, and that training in it does effect improvement). Nor should one underestimate the potential message in this activity that mathematical truth resides in objects, figures and their relationships and not in the authority of the teacher.

The next level of generalisation is represented by the question "Is G always true?" where G is a proposed generalisation (possibly the result of the previously discussed activity). To answer this, *examples must be generated* and tested against G. Questions of this type are comparatively rare in the texts, and almost always related to "bookwork". The question about equiangular triangles quoted a few lines above is of this type, and SMP Book Z asks of certain finite arithmetics, "Does every element have a unique inverse?"; and of course, in the Book Z section on proof we find many such questions. Opportunities for turning questions on, for example, the combination of transformations, from exercises on concepts to acts of generalisation are extensively missed. (e.g. Book X, p. 44 and Book Z, p. 142). Opportunities for developing the higher levels of generalisation, for example, the making of conjectures, do not appear in these texts.

Secondary Texts - Proof

The place of proof in the geometry of the SMP and Scottish courses is the subject of an already published article (Bell, 1974). To summarise, this shows that SMP Book 4 contains a chapter Conclusions from Data, which requires the proofs of various properties from assumptions stated in the questions; and that the Scottish texts contain some explicit development of deduction in geometry, starting from some concrete axioms about the rectangle. Pupils are given properties to prove, sometimes with guidance about starting assumptions. However, in both cases, this work has been omitted from later editions of the texts. Book Z (SMP) has a short chapter on Proof, with a 7-page discussion and a miscellaneous collection of examples to try.

It is suggested that these proof activities have been omitted because they were found to appear artificial to the pupils, and that a more suitable way for pupils to progress towards proof is through the giving of explanations in the course of their own investigations, and in defence of their conclusions when in conflict with those of others. This argument will be pursued in the next chapters, when proof will be analysed more fully. For the present, it is sufficient to note the continuing elimination of proof activites from the best-known texts.

CHAPTER 5

THE DEVELOPMENT AND EVALUATION OF A PROCESS-ENRICHED CURRICULUM FOR THE EARLY SECONDARY YEARS

IN APPENDIX 5

INTRODUCTION

In earlier chapters, some of the most important general mathematical strategies have been described and illustrated from individual pupils' work. Reference to current texts has supported the co-tention that the development of these strategies does not appear as an objective, except at the level of "following instructions to generate an example; observe certain aspects of it and state the relationship between them". The present chapter reports the results of a curriculum development project which has as one of its chief objectives the development of pupils' abilities to employ general strategies in mathematics, during the first few years of the secondary school course. Other chapters investigate which general strategies are acquired during normal courses; this one considers to what extent a special programme can achieve departures from the norm.

QUESTIONS

The particular questions at issue are (1) in what aspects of the mathematical process can pupils of this age participate, with understanding and involvement, (2) do they thus acquire actual transferable general strategies and skills, (3) are their attitudes to mathematics and their appreciation of it improved, (4) what is the effect on their learning of mathematical content, both immediately and in subsequent years. A fifth question, of considerable practical relevance, is (5) do process-oriented tasks provide suitable work for mixed ability classes.

HYPOTHESES IN RELATION TO THE QUESTIONS

These are expectations based on experience and previous research. Reynolds (1967), in his results relating to the performance of grammar school first-formers on a test of various aspects of proof, found a strong tendency to accept generalisations on the basis of a given small number of results, without further check; less than half could symbolise the function nth odd number = 2n - 1; about a third of them were unaware of the relevance of the set to which a given generalisation referred.

King (1974) succeeded in teaching some above-average 11 year olds to construct proofs by making minor modifications of proofs already taught. Lawson and Wollmann (1975) successfully taught 12 year olds a transferable strategy for controlling variables in simple scientific experiments. Thus the existing reported research gives rather little indication on the subject of question (1), that is, what process aspects of mathematics are accessible at the first year secondary level. On question (2), Lawson's and Wollmann's experiment gives grounds for expecting some successful learning of strategies, at least if these are explicitly taught. On question (3), relating to attitudes, gains in interest and enjoyment are likely to be strongly dependent on the teacher, and thus on whether the material provides him with a better means for promoting these positive attitudes. Gains in appreciation of the nature and purpose of mathematics, and of its being a meaningful and satisfying activity, depend more on the material but are very difficult to evaluate. This is a source of frustration to the teachers involved, since they generally regard improvements in their dimension as among the most important. On question (4), interaction between content and process learning, there are two studies of primary school mathematical attainment which are relevant. Biggs (1967) showed that in attainment on mechanical, problem and concept tests in arithmetic, the most successful primary schools were those whose teaching was essentially traditional, but included some use of structural apparatus or environmental activities; the mainly "motivational", i.e. environmental schools performed less well on the tests used; their more general objectives were not evaluated. Richards and Bolton (1971) obtained similar results, but also used tests of creativity, and on these the more activity-oriented school out-performed the others. These researches suggest that objectives which are consciously and skilfully embodied in the teaching are likely to be attained, but those which it is hoped to achieve incidentally are not. Question (5) relates to the suitability of process-oriented tasks for mixed-ability classes. Although previous research cannot be quoted, there are *a priori* grounds for adopting a positive hypothesis. Just as the same title set for an English composition can call forth responses at greatly different levels of abstraction and perception, and it is even possible to choose works of literature to which valid responses can be made at different depths, in the same way mathematical situations can be chosen in which some pupils will simply identify concepts, and observe and justify simple relationships, while others can operate at a much more abstract level, can find more general theorems and, quite naturally, use much more rigorous modes of proof. The seven strip patterns and some of the problems concerning divisibility of numbers provide examples of this, and more appear in the discussion of the South Nottinghamshire Project material below.

The South Nottinghamshire Project provides the setting for three studies which bear on these five questions. These are (I) a descriptive and analytical study of the teaching material, together with classroom anecdotes, which bears on questions (1) and (5), (II) two interschool comparisons, (IIA) of number attainments, bearing on questions (4) and (5), (IIB) of the attainment of general mathematical strategies, bearing on questions (1) and (2). These are preceded by a general description of the Project.

DESCRIPTION OF THE PROJECT

Like most curriculum development, the South Nottinghamshire Project has a number of interrelated aims. To quote from the Introduction:

> "It is a piece of development work operating in two South Nottinghamshire comprehensive schools, in association with the Shell Centre for Mathematical Education. It arose out of previous work on the preparation of suitable material for mixed ability classes of first year pupils in these schools. Its main focus was on exploring the use of practical materials, and the making of mathematical investigations at this stage.

> Thus as well as preparing material suitable for the mixed ability situation, we wished also to explore the feasibility of ideas we had about syllabus content and teaching methods. In particular we wanted to see how far pupils of this age could go in the investigation of mathematical situations by themselves, leading to the drawing of conclusions, the making of generalisations and the giving of explanations and proofs. We wanted to see whether this could be done in the context of the normal syllabus, and what effect it would have on all-round mathematical attainment, both immediately and in subsequent years. (Our hypothesis was that somewhat less material might be covered initially, but that performance in later years would be enhanced.) Regarding teaching method, we wished to explore the value at the early secondary stage of a range of simple concrete material - geoboards, pegboards, pattern shapes, matchsticks, number rods and blocks, point lattices and grids and so on - feeling that this would (a) enable well motivated problems to be posed, and (b) provide concrete "props" to aid the understanding of the mathematical ideas."

Thus the Project included both development and research activities; it was intended to improve the pupils' learning of mathematics, but there was also to be an evaluation to provide answers to some general questions currently being asked by those concerned with the teaching of mathematics.

The material so far published comprises fourteen units covering work for the first secondary year. The ground covered is generally similar to that of the SMP course, but the treatment is different throughout. Each unit comprises a general introduction for the teacher, a sequence of assignment cards, commentary for the teacher on the individual tasks, and a few examples of pupils' work, with commentary. The differences from the SMP course lie in the basing of the work of each 2½ week unit on a small number (from one to five) of investigations, rather than on a large number of short questions; and in the greater use of concrete materials as the setting for the investigations. Thus essentially there are larger scale tasks in which more of the direction of the activity is the pupils' responsibility. The notes for each unit begin with a statement of the content and process objectives embodied in it. A few examples of these statements of objectives are given below.

EXTRACTS FROM TEACHERS' NOTES

COORDINATES

Content Objectives:

(i) Given a point on a grid with axes and origin, state the coordinates of the point; and, conversely, given the coordinates, plot the point.

(ii) Relate the objective (i) to 6-figuremap references

(iii) State the relationship between the coordinates of points on a line (either symbolically or verbally), and give the coordinates of other points on it.

(iv) Given two points, state the coordinates of the mid-point

(v) Given a line segment, state its coordinate difference; and conversely plot a line segment with a given coordinate difference.

(vi) Make use of signed numbers, or some equivalent form, to specify the sense of a line segment.

Process Objectives:

(i) Employ and write about simple strategies involved in a game.

(ii) Seek relationships, and make generalisations, from sets of numbers and number pairs.

(iii) Write up descriptions of experiments and conclusions.

SHAPE

Content Objectives:

Recognise and name: triangle - isosceles and equilateral, quadrilateral, trapezium, kite, parallelogram, rectangle, square, rhombus.

Process Objectives:

(i) Modify part of a figure (keeping the rest fixed) to generate a new figure
(ii) Add to an existing figure to build new figures
(iii) For some pupils, develop a strategy for solving "How Many?" questions
(iv) Write an account of procedures and results
(v) Extend a problem and ask new questions.

SYMMETRY

Content Objectives:

(i) Recognise line and rotational symmetry
(ii) Be aware that "m is the mediator of PP'"
(iii) Construct the remainder of a part-given figure to have specified symmetry

Process Objectives:

(i) Experimenting, generating examples and recording results
(ii) Recognise the existence of a limited number of solutions to a problem, and employ a systematic procedure to find them.
(iii) Recognise the need for a definition of "sameness".
(iv) Write an account of procedure and results

SEQUENCES AND FUNCTIONS

Content Objectives:

(i) Interpolate and extrapolate terms in a sequence by considering patterns of differences between successive terms.

(ii) Recognise that in a linear sequence the differences between successive terms are constant, and in a quadratic sequence the differences are linear.

(iii) Induce an expression for the nth term of a linear sequence

(iv) Develop the beginning of function notation

(v) Possibly introduction to the concept of an inverse function.

Process Objectives:

(i) Inventing situations from which a number sequence can be derived.

(ii) Justifying, by reasoning from a given situation or diagram, a formula which has already been obtained by induction from a number pattern.

AREA

Content Objectives:

(i) Know that area of triangle = half area of rectangle; and area of rectangle = l x b.

(ii) Know how to find areas of straight-line figures by dissection and by subtraction

Process Objectives:

(i) Generating examples; predicting and verifying results

(ii) Tabulating results (for different numbers of pins)

(iii) Seeking relationships from tabulated results.

WHOLE NUMBER RELATIONSHIPS

Content Objectives:

(i) Understanding of the base 10 place value system

(ii) Recognition and use of the associative, commutative and distributive laws in performing calculations.

(iii) Understanding of the relationship between addition and subtraction, and between multiplication and division.

(iv) Knowledge of the algorithms for (a) subtracting numbers, up to a 3-digit number from a 3-digit number, (b) multiplying a 2- or 3-digit number by a 1-digit number

(v) Knowledge of addition and multiplication facts, and the number inter-relationships (up to 100) associated with divisibility and multiples.

Process Objectives:

(i) Extending an investigation for themselves (e.g. by choosing the next number to investigate in Sums of Divisors.)

(ii) Looking for relationships between numbers and predicting results in further cases; seeking generalisations.

(iii) Formulating rules and modifying them in the light of further evidence.

I. DESCRIPTIVE STUDY OF PROCESS ATTAINMENTS

The following strategies are identified and their mode of incorporation in the course discussed.

GENERALISATION

1. Generating examples - to satisfy given conditions or to test a given conjecture.
2. Classify and order systematically to obtain a complete set.
3. Recognise and (4) extend a pattern or relationships, numerical or spatial.
5. Express a relationship in general terms, algebraically or verbally ("Make a generalisation").

PROOF

1. Check all cases.
2. Establish sub-classes and check exhaustively.
3. Identify underlying general relationship ("key fact").
4. Connect data and conclusion.
5. Embed in agreed existing knowledge.

FORMULATING PROBLEMS

REPRESENTATION

1. Use of diagrammatic recording, graphs, tables
2. Use of algebraic symbolism.

ABSTRACTION

1. Actions with concrete embodiments
2. Abstractions resulting from generalisation.

GENERALISATION

(1) Generating examples - to satisfy given conditions or test a given conjecture.

(2) Classify and order systematically to obtain a complete set.

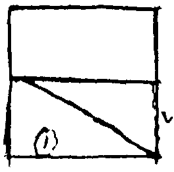
These are common features of the course. In the first unit, Shape, the tasks all require (1) and (2); for example, finding *all* the *different* triangles, or quadrilaterals, on a 9-pin geoboard, or all the 5-square shapes. It would be expected that (2) would be more difficult that (1); the question about (1) is whether it is a learnable general strategy, or whether it is simply a matter of (a) understanding the concepts in terms of which the conditions are being framed, and (b) having the necessary mental power to cope with the degree of complexity of the conditions. (This question is discussed further in Chapter 9; some evidence is also available from the test results reported in the second part of this chapter.) Direct evidence from the classroom makes it clear that for some pupils the activity is not an easy one. One pupil had counted the two triangles in Fig. 1 as the same, and even when her attention was drawn to them had considerable difficulty in recognising their difference. The example on p. 5.11 also shows the difficulties experienced by some pupils. What was it that was 'quite hard' for him in finding Nos. 4 and 5? Or that made him need his friend's help to find No. 6? But however the difficulty is categorised, it seems a reasonable hypothesis that the ability to generate a variety of examples to meet given conditions may be improved by experience of such situations; and it is a modest but necessary element of mathematical activity.

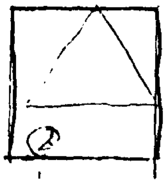
FIG.1.

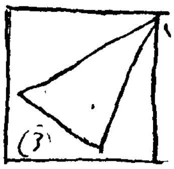
H

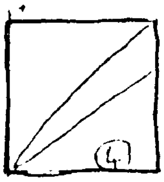
I have made one two three four ~~four~~ trangles on a 9 pin goabood. Number one ten two and there three were issay easy but Number 4 and 5 were quite hand. And Number 6 my frind tell me it. Number 7 ~~&~~ I made up my self. And number 8 my freuid

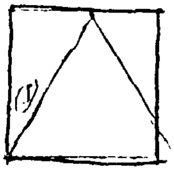
toded me as well / But Number 9 ~~we~~ I made up my selff. Number 10 I made by durling the loseeck- elastic band. Number 11 I made by etting the lasickbamd elastic go lose loose. Oh Number 12 I made to at wonch once. Number 13 teen and 14 I did by making two at woch. Number 15 I made my self. alltogther I made 8 differt ways alltogther

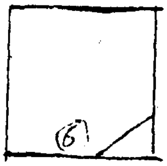




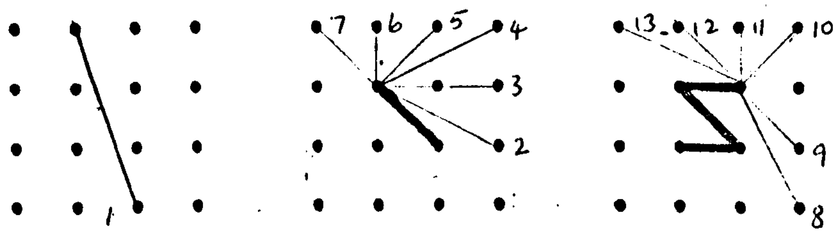






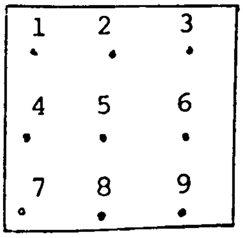


The use of systematic classification and ordering to obtain a complete set of figures was included as part of this activity with two objectives. The first was to make the pupils aware of the different possible types - right-angled, isosceles, right-angled *and* isosceles, obtuse-angled, for triangles, parallelogram, rectangle, trapezium and so on for quadrilaterals - and to provide the opportunity for supplying names for these, where necessary. Finding and classifying the *complete* set is a strategy which is likely to bring to a pupil's notice types of which he was not previously aware. The second objective was that of developing this strategy itself, for further use. As a check of the summaries given above will show, several such problems appear in the early parts of the course - Dividing the Board, in which all possible divisions of a 16-pin geoboard into congruent halves by a single elastic band are to be found; Line Segments - "How Many" different (in length and direction) segments on 9-pin boards, (in Coordinates) and towards the end of the year, in Fractions (Unit 13), classifying all the fractions which can be formed from rods of lengths 1 to 10 inclusive, and others. Finding a complete set was a difficult task for most of the first year pupils, partly, it seemed, because they were unused to this kind of definiteness and rather surprised that it could be obtained, partly because it involved a rather precise use of written recording, and partly because these problems required, for success, the keeping of one variable fixed while changing another. Thus, in Dividing the Board, it is necessary to identify the 2 different ways in which the band may cross the middle square, then to count the number of ways in which a halving of the middle square can be extended to the edge of the board, and within this set to separate straight from two-step extensions.



Even after discussion of this strategy with the class, only the ablest were capable of carrying it through for themselves. In the case of Triangles on the Geoboard, two methods of systematic counting were offered to the pupils (See Cards A3, A4). The second of these was generally useable by the pupils, the first they could follow as a demonstration, but found it hard to use themselves.

TRIANGLES - MAKING SURE Card A3



We have labelled the pins 1 to 9. To get a triangle we have to pick a 'triplet' - that is 3 pins. Here are some triplets: (1,2,3) (1,2,4), (1,2,5), (1,2,6), (1,2,7) (1,2,8), (1,2,9). Now try to work out a system to help you write down all the triplets possible.

Most of the triplets - like (1,2,4) in the diagram - make triangles, but some make straight lines - like (1,2,3). How many give straight lines?

Look at the 8 different triangles you found. Label each one with a letter.

Label each triplet with the same letter as the triangle it makes.

How many triplets are there for each triangle?

Are there any triplets without a letter?

TRIANGLES - CLASSIFYING Card A4

Label each triangle you found on the 9-pin board with a letter.

How many triangles have a right-angle?
Which ones are they?

An isosceles triangle has 2 sides the same.
How many isosceles triangles are there?
Which are they?

An equilateral triangle has all its sides the same?
Are there any equilateral triangles?

Which are the scalene triangles (ones with all their sides different)?

Compare results with your partner.

Write in your books what you have found, using the names of the triangles.

(3) Recognise and (4) extend a pattern or relationship, numerical or spatial.

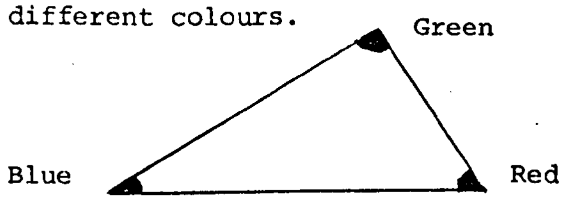
(5) Express relationship in general terms, algebraically or verbally ("Make a generalisation")

These activities involve both abstraction - the identification of common elements - and generalisation - the statement of the nature of the regularity, and of the extent of the class to which it applies.

Activities of this kind, with *spatial* relationships do occur in the SNP first year, but they are not common. In the units on Tessellations, the recognition and use of repetition by translation figures strongly, and may be verbalised; and the angle-sums of the triangle and the quadrilateral are to be recognised and stated from directed experiment on the tessellations (Card E7).

COLOURING ANGLES 1 Card E7

You require the tessellation you made with an irregular triangle. Choose one triangle in the pattern and colour its angles using three different colours.



Now colour blue all the angles, in the other triangles, which are the same as the blue one.

Do the same for the other colours (green and red) until all the angles are coloured.

1. Look at one point of the tessellation. Is there any pattern about how the colours go round the point?
2. If R = size of red angle, B = size of blue angle, C = size of green angle, what does R + B + C equal?

   Can you explain your answer?
3. What do the triangle's angles add up to? Does this work for other triangles?

In the Symmetry unit, though most of the activity consists of game-situations embodying the relevant concepts, the recognition and verbalisation of the relationship between the mirror and each pair of related points, PP^1 perpendicularly bisected by m, is the subject of one task.

Work in pairs. Card B6

Stretch 2 elastic bands across the middle of the board. They will be 2 lines of symmetry.

Now make a shape with another elastic band.

See if your partner can place another elastic band on the board so that both of the first elastic bands are lines of symmetry for the whole figure.

Now let your partner start and you complete the figure.

Record your interesting figures.

<u>SYMMETRY FOLDING</u> Card B2

Cut out a shape with line symmetry.

Fold it along its line of symmetry.

Choose a particular point on the shape and label it P.
Prick through P with a sharp point and open up the paper.
Label the other hole P^1. P^1 is called <u>the image of P</u>.

Join PP^1 and draw in the line of symmetry.

What do you notice?

Choose another point and repeat this......and another.

Write what you notice in your book.

Other activity involving these aspects of generalisation is concerned with numbers as coordinates. The objectives for Coordinates in the summary above mention the mid-point formula, and the use of coordinate-differences to specify line segments. The former gives an opportunity for generating examples of pairs of points, finding the mid-points geometrically and making a table of the coordinates of these, from which the relationships $\frac{1}{2}(x_1 + x_2)$, $\frac{1}{2}(y_1 + y_2)$ can be discovered. (An empirical generalisation, rather than a deductive one, in practice).

Other examples of recognising, extending and describing patterns, verbally and symbolically, in Coordinates, are illustrated in extracts J, K and L. The first of these (J) shows how a game situation is used to establish and practise the use of coordinates. Then the attempt to reconstruct the games from the records of the points played motivates the detection of the relationships among the coordinates which correspond to collinearity of the points. The second extract (K) shows the verbalisation of these relationships, and then their expression in symbolic form. The main difficulty here is the concentration on the relation between the pairs of coordinates of each point rather than on the way each coordinate changes as one moves along the line. The symbolisation of the relationship did not appear to present particular difficulty when approached in this way. The last of these examples (p. 5.19-20) shows how one boy, having worked out and understood the patterns which lead to the result 24 for the number of different line segments on a 9-pin board, extends them to find the results for boards of 16, 25 and 49 pins. He has made a verbal generalisation - the "number down" times its own number and take one - but the "number down" is not recognised as 2n - 1 for a board of n × n pins; nor is any explanation given, though the reasoning can be inferred from what he has written. This example shows how pattern-recognition, extension and verbalisation can take place at different levels; it is not possible to say that this strategy is possessed or not possessed by first year pupils in general, except in relation to a particular situation. However, it is still plausible that the habit of seeking patterns, re-opening problems by changing a parameter, and trying to verbalise and to explain the result can be developed by practice, particularly with discussion drawing attention to these aspects of the process. The extent to which *explanation* is possible for first year pupils is discussed under the next main heading below, Proof. Examples of Generalisation activities appear in almost all units of this course - see the Summary. Some of the unit on Sequences and Functions will be discussed under Symbolisation.

To summarise: of the five aspects of generalisation considered, (1), (3), (4) and (5) were attainable by most pupils, depending on the

J

FOUR In A Line

We played four in a line this game is simalar to battle ships. You mark out a chalk Sqaure and say eg. 2 along and 3 up then you put a peg on where it should be. The aim of the game is to get four in a line before your oponent.

Comments

There was a trap where someone has ~~a~~ three in a line with one end that can, be blocked.

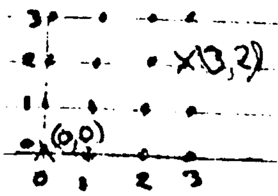
On my first win I had a diagonal line. The pattern was my first move 1,2 which equals 3 and so on in two's up to 4,5.

x = 4 for all the points and y goes up one at a time.

Coordinates – Fixing Position

In four-in-a-line we used ordered pair to fix positions.

eg.



The first number is called x, and gives the distance across. The ~~fir~~ second number is called y, This gives the distance up.

Four in a line.

This is a game and these are our moves.

Posicions	Gamos played	Pts.
(5,4)(6,3)(7,2)(4,5)	1st Game	1 won.
(5,4) (6,5)(6,6)	2nd Game	1 Lost.
(6,6)(7,5)(5,5)(5,7)(6,5) (8,5).	3rd Game	1 won.
(6)(3,2)(5,2)(6,1)	4th Gama	1 Lost.
(4,5)(4,4)(6,4)(4,3)(7,5) (5,2).	5th Game	1 Lost.
(6,6)(6,7)(6,8)(7,4)(7,5)	6th Gama	1 Lost.

K

23rd November

Coordinates

The 1st number is 2 less than the second number.

① (1,3) (2,4) (3,5) (4,6) (16,18) (28,30) (35,37)

② (7,1) (8,2) (9,3) (10,4) (12,6) (16,10) (21,15)

② The 1st number is six more than the 2nd number.

③ (10,0) (9,1) (8,2) (7,3) (5,5) (1,9) (3,7)

The second number (~~always~~) goes down in twos all along the line but everytime you get to the 2nd number its 8 less, 6, 4,

④ (1,2) (2,4) (3,6)

27th November

① (4,5) (5,4) (6,3) (7,2) (8,1) (9,0)

The first number + the second number add upto nine.

X+Y=9.

② (5,6) (5,7) (5,8) (5,9) (5,10) (5,20) (5,100)

The first number is always five the second number can be any.

X=5.

③ (3,0) (4,1) (5,2) (6,3) (7,4) (8,5) (9,6) (101,104)

~~(Number [illegible] to three less more than the first.)~~ (second

(~~Y~~ X

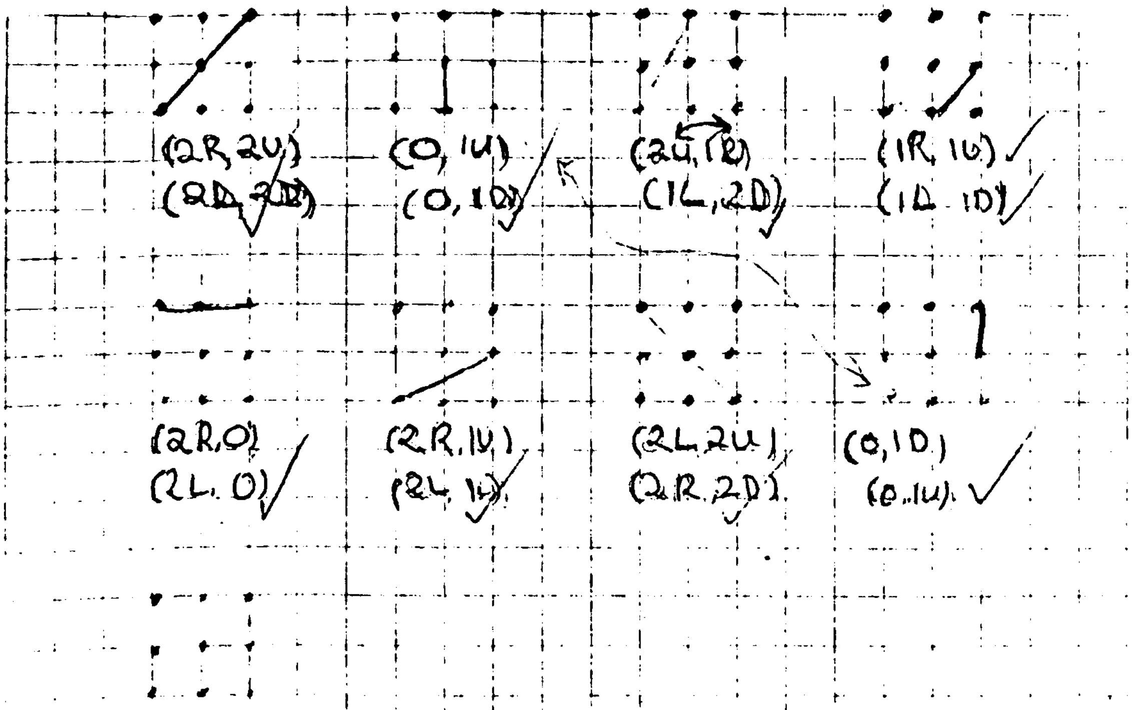
number two add three makes the first number.

Y+3=X.

④ (0,5) (1,4) (2,3) (3,2) (4,1) (5,0).

The first number and the second num = five.

L1



(2L, 1D)
('' 1U)
('' 2D)
('' 2U)
('' 0)

(2R, 1D)
('' 1U)
('' 2D)
('' 2U)
('' 0)

(1L, 1D)
('', 1U)
('', 2D)
('', 1U)
('', 0)

(1R 1D)
('' 1U)
('' 2D)
('' 2U)
('' 0)

(0, 1D)
('' 1U)
('' 2D)
('' 2U)

24.

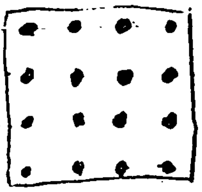
(3L, 1U)
('' 1D)
('' 2U)
('' 2D)
('' 0)
+
2

(3R, 1U)
('' 1D)
('' 2U)
('' 2D)
('' 0)
+
2

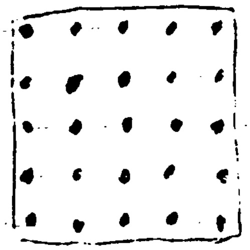
(2L, 1U)
('' 1D)
('' 2U)
('' 2D)
('' 0)
+
2

L2

(2R, 1U)	(1L, 1U)	(1R, 1U)	(0, 1U)
(" 1D)	(" ,1D)	(" 1D)	(" 1D)
(" 2U)	(" 2U)	(" , 2U)	(" 2U)
(" 2D)	(" 2D)	(" , 2D)	(" 2D)
(" 0)	(" 0)	(" , 0)	+2
+ 2 (70)	+ 2 (35)	+ 2 (63)	[illegible]



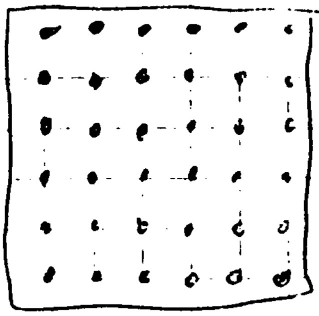
= 48



= 80

(4R, 1U)
(" , 1D)
(" , 2U)
(" , 2D)
(" , 3U)
(" , 3D)
(" , 4U)
(" , 4D)
(" , 0)

There will be more lists like this if I could carry on. With my others if you count the number down if you 'times' it by its own number and take one you get your.



= 120.

degree of sophistication of the patterns in question. (2) was less generally attainable. (This carries some attributes of Proof.)

PROOF AND EXPLANATION

(1) Check all cases
(2) Establish sub-classes and check exhaustively

Most of the situations which involve generalisation also contain the possibility of going on to proof or explanation, that is, to a statement of reasons why the relationship is true for all possible cases, and not just for those checked. But the first examples in the course of proof-type activity are those discussed in the previous section in which a *complete* set of examples has to be generated, where the proof involves constructing subclasses (possibly a hierarchy of them) and showing that these include all possible cases. This was described for Dividing the Board and illustrated for Line Segments. In the latter case the implicit argument is "every segment must have a first coordinate of 0, 1R or 1L, 2R or 2L - five possibilities; *each* of these can be paired with five similar possibilities for the second coordinate....." As well as providing fairly simple examples of proof, this construction of subclasses is often a vital element in more complicated proofs.

(3) Identify underlying general relationship ("key fact")
(4) Connect data and conclusion.

These types of proof appear first in the unit on Sequences and Functions, where the functional relationship may be inferred empirically from a constructed table of values, and proof depends on deducing the relationships directly from the generating conditions for the sequence; often something similar to mathematical induction comes in at this point. See Process Objectives (ii) for this unit (p. 5.7) and Card L2.

MATCHSTICK SHAPES 2 Card L2

Make these patterns of triangles in a row.
Make 2 more in the sequence.

Copy the table and fill in the gaps.

Number of triangles	1	2	3	4	5		10	20		43	100	n	
Number of sticks	3	5	7			17			61				m

When you have completed the table, answer the following questions:

(a) When you know the number of triangles, can you write down a rule to find the number of matches needed?

(b) By studying the matchstick patterns, can you explain <u>why</u> your rule works?

This card shows the teaching method adopted to direct attention to the n → f(n) relationship rather than the sequential one f(n)→ f(n + 1). When the formula n → 2n + 1 has been obtained from the table, it remains to refer back to the matchstick patterns and see that each added triangle requires 2 further sticks, and the first one has one extra stick as a start, so that 2n + 1 is indeed correct. Alternatively this reasoning may have taken place when making the jumps to 10 and 20 triangles in the table. We shall have occasion to refer to this point below; like all the other types of proof, it was not appreciated by most of the first year pupils. Once a pattern had been recognised in the course of constructing the table, it was difficult to make them entertain any serious doubt about it, and the reference back to the matchstick patterns was regarded as superfluous. For evidence of this, see extract M (p.5.23) where the only sequence for which a justification in terms of the stick patterns is given is n → 4n for squares of side n. Another type of proof is similar to this last but without the sequence aspect. A relation is inferred empirically from one or a few examples, and the question is whether it is true in general, and what justification can be offered. The Midpoint formula is an example; another is Arrow Diagrams 2. (Cards D4, N3)

M

Match stick Shapes

Size	1	2	3	4	5	6	7	8	9	10	11	▨	n
No of St	3	6	9	12	15	18	21	24	27	30	33	▨	3×n

All you have to do to get the no of sticks to the size of triangle is multiply by three.

Size	1	2	3	4	5	6	7	8	9
No of St	3	5	7	9	11	13	15	17	19

The pattern I have found is.
1+2=3 ✓
2+3=5 ✓ 2
3+4=7 ✓ 2
4+5=9 ✓ 2
5+6=11 ✓ 2
6+7=13 ✓ 2
7+8=15 ✓ 2
8+9=17 ✓ 2
9+10=19 ✓ 2

The differfence between the last number before the total, and the total goes up like this
1 3 5 7 9
2 4 6 8 10

size	20	30	43	100	n	M-1 √2
No of St	41	61	87	201	n×2	M

The pattern is ×2 and add 1. So when you get a bottom number you take 1 and divid by 2.

Match stick Shapes

Size	1	2	3	4	5	6	7	8	9	10
N of St	4	8	12	16	20	24	28	32	36	40

I think that the pattern will be ×s 4 to get the number of sticks. MY thery was right

The reason for this is that ~~the~~ a square is made up of 4 side

Size	1	2	3	4	5	6	7	8	9
N of St	4	7	10	13	16	19	22	25	28

The last bottom numbers go up in 3s 1 less than the number of sides.

To get the top number, ~~if you have the bottom number you do half the number and add whatever~~ the number in the line is. Wrong

To get the bottom number you ×3 and add one.

MIDPOINTS

Card D4

Work either on spotty paper or on pegboard.

Put two red pegs in the board (or plot two points).
Now put a blue peg in the exact middle of the line joining the two red pegs.
Record the coordinates of all three points.

Repeat this for other positions of the pegs, and make a table like this:

1st red peg	2nd red peg	middle blue peg

Study the results in your table.
Can you discover a rule for getting the coordinates of the midpoint from the coordinates or the end-points?

Describe your experiment, and write down your results and any rules discovered.

ARROW DIAGRAM 2

Card N3

Experiment with arrow diagrams to find out whether any of the following functions is the same as a <u>single</u> function:

1. $\xrightarrow{+1}\xrightarrow{+2}$
2. $\xrightarrow{-1}\xrightarrow{+3}$
3. $\xrightarrow{-2}\xrightarrow{-1}$
4. $\xrightarrow{\times 6}\xrightarrow{\div 2}$
5. $\xrightarrow{\div 3}\xrightarrow{\times 6}$
6. $\xrightarrow{\times 2}\xrightarrow{\times 3}$
7. $\xrightarrow{\div 2}\xrightarrow{\div 3}$
8. $\xrightarrow{\times 3}\xrightarrow{-2}$
9. $\xrightarrow{\div 2}\xrightarrow{+3}$
10. $\xrightarrow{\times 2}\xrightarrow{+0}$
11. $\xrightarrow{+2}\xrightarrow{\times 1}$

Write down your comments about any rules you discover.
Are there any special exceptions to the rules?

The former has been discussed under Generalisation. The latter leads to the empirical generalisation that pairs of functions which consist of just + and - functions or just × and ÷ functions, not mixing the two types, combine to form single functions. In both these cases the results are plausible, but proofs are beyond most pupils at this stage. A geometrical example occurs in the Tessellations unit, where, after tessellations of different triangles and quadrilaterals have been constructed, the question arises "Do *all* quadrilaterals tessellate?" The class being observed failed to respond to this, apparently because the concept of all possible quadrilaterals was too unfamiliar to them.

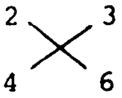
In addition to these examples, where a generalisation has first to be found, then tested empirically and proved if possible, there are some in which the generalisation is given; and others in which it is already known, and testing and proof are the only remaining tasks. One of the first type is Corner Numbers. Here an empirical check soon convinces the pupil of the truth of the generalisation; but, again, the proof offered by the teacher failed to evoke much responsefrom the pupils. This showed that

<u>CORNER NUMBERS</u> Card J18

You will need the 'train table' which you completed when doing the Trains problem.

1	2	3	4	5	6
2	4	6	8	10	12
3	6	9	12	15	18
4	8	12	16	20	24

You can see that we have drawn a square round a block of four numbers:



Notice that 2 x 6 = 12

and 4 x 3 = 12

so 2 x 6 = 4 x 3

Does this work for any other blocks of four numbers? Try some and see. Write them down and show any working out you do.

Can you find out why it works?

$$10 \times 18 = (2 \times 5) \times (3 \times 6) \text{ and } 15 \times 12 = (3 \times 5) \times (2 \times 6)$$

and that any opposite corner numbers such as 10 and 18 are obtained by multiplying the same four numbers as the other pair forming the rectangle.

(5) Embed in agreed existing knowledge

An example where the generalisation is well known occurred during the work on Decimals. A girl, asked why "adding a 0" multiplied a number by ten, performed the standard algorithm

$$\begin{array}{r} 43 \\ \underline{\times\ 10} \\ \underline{430} \end{array}$$

This is a reminder that explanation and proof involve, in the first place, the derivation of less fundamental results from more fundamental ones, but what is felt to be fundamental is a function of a person's own experience. To this girl, the multiplication algorithm was probably the most basic fact about multiplication - her working definition. Pupils of this age do not normally work from stated axioms. Generally they have not even begun to systematise knowledge, since this implies a degree of second-order reflection on the actual connections between different fragments of knowledge. They collect new relationships with some enthusiasm, but are not much interested in the economy of deducing minor results from major ones. They are not particularly impressed by the definiteness of proof, perhaps because they are used to adapting to an environment full of change. Also, looking at the situation more closely, it is natural and comparatively easy for these pupils to observe relationships among concrete (i.e. familiar) objects like particular numbers and shapes. The demand for an explanation then requires perception of another relationship of a higher level of generality, which is difficult. For example, in Matchstick Shapes it would be possible to try to derive the functions directly from the construction rules, without generating numerical values. This generally is too difficult, because the pupil needs the numerical values to help him grasp the function. Similarly, in Corner Numbers the underlying relationship is insufficiently tangible; the corner numbers themselves are needed, and then we are led inevitably to an empirical generalisation.

So, to summarise, none of these aspects of proof was generally accessible to the first year classes with whom the material was used.

The ones which came nearest to acceptance were the systematic enumeration of complete sets of examples with some classification, i.e. (1) and (2).

FORMULATING PROBLEMS

Some pupils were able to extend problems in response to the teacher's suggestion (see Line Segments, p. 5.19), but they were not in general able to do this spontaneously. (The second year SNP material develops this strategy.)

REPRESENTATION (INCLUDING SYMBOLISATION)

(1) Diagrammatic Recording, Graphs, Tables

Aspects of this which figure strongly in the SNP first year material are Recording, Algebraic Symbolism, Tables and Graphs (Arrow and Cartesian). The strategy involved is that of being able to use one or more of these forms of representation in appropriate situations. The geoboard and pegboard problems mentioned above all require the recording on spotty paper of figures made on the board. In the case of Triangles and Quadrilaterals this is necessary for eliminating repetitions and counting the number of different examples; for Dividing the Board it was for some pupils necessary to record on paper, turn the board through 180°, and check against the drawing to ensure a correct halving. The geometrical coordination required for this made it a non-trivial task for several pupils.
Recording is also used in the game Four in a Line (Coordinates) (see extract N) and recognition of winning lines from the record gives a lead into the equations of lines. (This strategy of recording, then reconstructing the game from the record can be a powerful way of developing abstractions; it is used frequently by Dienes). Tables of values for functions are used in Sequences and Functions, double entry tables in Whole Number Relationships and in Area (see Card F6) and both types of graph (arrow and Cartesian) in Linear Functions.

PATTERNS Card F6

Collect all your results together in a table like this:

		Number of pins inside					
		0	1	2	3	4	5
Number of pins on the boundary	3						
	4						
	5						
	6						
	⋮						

(The 1 is in the table because a (4,1) shape has an area of 1 square)

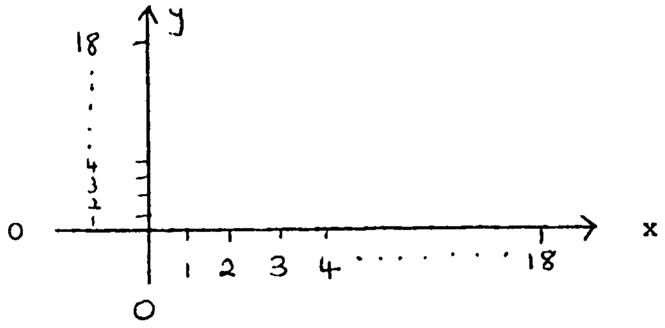
If there are gaps in the table try to predict what numbers go in them, and then check whether you were right or not.

Describe the patterns in the rows and in the columns of your table

In Cards N5, N6 the relation between position of graph and form of function is explored.

CARTESIAN GRAPHS 1 Card N5

You will need a sheet of squared paper and coloured pens



1. On your squared paper draw a set of axes, making sure that each scale will go from 0 up to about 18.

2. Graph the following functions on one set of axes, plotting five points in each case:

$\xrightarrow{+2}$, $\xrightarrow{-2}$, $\xrightarrow{+3}$, $\xrightarrow{-3}$

3 On another set of axes graph the functions

$\xrightarrow{\times 2}$, $\xrightarrow{\div 2}$, $\xrightarrow{\times 3}$, $\xrightarrow{\div 3}$

Write down comments about the positions of the graphs and the kinds of functions which they represent.

CARTESIAN GRAPHS 2 Card N6

On new sets of axes, graph the following combined functions.

$\xrightarrow{\times 2}\xrightarrow{+ 3}$ $\xrightarrow{x\ 2}\xrightarrow{+ 2}$ $\xrightarrow{- 2}\xrightarrow{\times 2}$ $\xrightarrow{\times 3}\xrightarrow{+ 2}$

$\xrightarrow{+ 1}\xrightarrow{\times 3}$ $\xrightarrow{\div 2}\xrightarrow{+ 3}$

Write down your comments about the positions of the graphs and the kinds of functions which they represent.

You will notice that each combined function is made up of a + or - function and a × or ÷ function. Investigate the effect of:

(a) keeping the × or ÷ function the same, and varying the + or -.

(b) keeping the + or 1 function the same, and varying the × or ÷.

Write down your comments on what happens.

In all these cases the strategy of suitable recording to preserve the results of an essentially evanescent experiment is being developed. The exercises were performed successfully, but these pupils were not placed in the position of having *spontaneously* to adopt a form of recording, so it is not possible to offer observations regarding their acquisition of this as a usable strategy.

(2) Algebraic Symbolism

Algebraic symbolism was developed in two units - Coordinates and Sequences and Functions. Relevant extracts from these (K and M) have already been referred to. In M (p.5.23) we see the beginning of the use of algebra in n^{x2}_{+1}, and there is also a case where the pupil has been feeling towards an explicit relationship but cannot express it, so he displays the pattern $1_2 3$, $2_3 5$, $3_4 7$. These examples suggest that the use of a symbolic language does not present great difficulty if just sufficient symbolism is given to enable observed relationships to be expressed. Problems arise when a symbolic expression is presented to a person who has not previously met that form, or has lost some part of its meaning. Such problems arose for a few pupils during number work; for example, 60 was not recognised as comprising six tens. These questions will

will be discussed below. The use of algebraic language in appropriate situations, such as the statement and investigation of number generalisations, is a relevant general strategy, but at this stage, the pupils are only just beginning to learn the language.

ABSTRACTION

(1) Actions with concrete embodiments

In the discussion of strategies for abstraction in Chapter 3 , it is shown that many of the other strategies, particularly those of Generalisation and Representation, both involve acts of abstraction and lead to further ones. The deliberate use of models, representations and concrete embodiments to provoke and facilitate abstraction is also discussed. This last strategy is used extensively in the South Nottinghamshire Project mainly as a *teaching* strategy, though it is also intended that the use of materials embodying particular schemas, such as base blocks, number rods and geometric shapes, should become a deliberate strategy for those pupils for whom it is helpful.' Some examples of this appear in the unit on Angle, where the wooden shapes acted for some pupils as a concrete memory, reminding them what 60° and 120° were like, and for others as measuring instruments with which to find other angles. For example, one girl picked up the equilateral triangle and decided its angle was 45°. She was asked to check that two of these made 90° by putting two triangles over the square. She could not see at first what was wrong, but on questioning decided that she was more sure about the square, and then by putting three triangles together arrived correctly at 60°. Other pupils worked out the angle of the octagon by fitting two of them and a square together to make 360°.

The following comments on the relative unsuitability of the abacus and the value of base blocks for helping pupils with the concepts of place value, are taken from the Report. They indicate some of the factors relevant to the use of structured material as an aid to abstraction.

"In general the use of the abacus did not seem helpful. The representation of 16 as  in base five rather than $\begin{array}{c|c} F & U \\ \hline 3 & 1 \end{array}$ differs only in using three rings instead of the written digit 3; the significance of the position as denoting fives is still a matter of decision - it is not displayed by the material, as it is if one uses three sticks each marked in 5 unit cubes, as in the multibase blocks. But the representation of three units by the numeral 3 is not a point of difficulty for these pupils, whereas the place-value idea is. Similarly the operations of addition and subtraction on the abacus involve rather special actions, e.g. adding 23 to 141 involves setting up 23 on the abacus; 141 standing beside it (unless a second abacus is available); and transferring all the rings onto the first abacus. Then when four rings are filling a peg and another two are to be added we take away five, put one of them on the next peg and leave the sixth one on the present peg. The four discarded ones have to be put right away; they have lost their value. This sequence of actions is a fairly complex skill which the abler pupils learned and used successfully; the weaker ones could not cope with it without many mistakes, particularly concerning the four discarded rings, which they felt ought to have a place somewhere. This preoccupation with manipulating the abacus correctly meant that the work gave no opportunity for learning about place value to those whose understanding of it was weak - rather the reverse.

Some work with base ten blocks with a group of weaker pupils began by asking them to do the following:

(1) 216 + 95 (2) 216 x 35 (3) 216 x 5

Three of them got (1) right; the fourth put

$$\begin{array}{r} 216 \\ \underline{95} \\ \underline{1166} \end{array}$$

None of the four could do (2). All tried (3) but no-one got it right. We decided that for this group the right material was base ten blocks; we felt that in attempting to improve their understanding of place value we needed to use all their existing knowledge of numbers, not abandon it, and in particular the words like six-ty, six hundred and so on, themselves linked with the base ten system. I worked with them and with this material for two subsequent sessions.

We started by one pair counting out 83 in the wood, 8 sticks and 3 units, and the other pair collected 203. I asked them to find twice 83, 203 - 83 and 5 x 83. The multiplication was done by addition so I asked whether they could do five 80s and five 3s and whether this would be the same. Their problem with this was not knowing what five 80s would be; they knew five 8s but could not connect the two. (The next 20 minutes of this work has been preserved on the audiotape.) We got on to three 80s which was agreed to be 24 sticks, i.e. 24 tens. "How many units?" 204, 208, 240, 304, 124 were all suggested by these four boys; mostly they had tried to count the total number of units on the 24 sticks which they each now had. "Could you find out which is right without counting?" "No." We tried 13 tens which they checked and agreed. I asked them to record "13 tens are 130, 24 tens are 240" (Jeremy: "How do we write it?....Is it add or times") We continued through 15, 18, 21, 27, 34 tens and more correct answers were coming more quickly, though there were still mistakes. By the end there was a feeling around that 15 → 150, 34 → 340 etc. but it was still an insecure feeling, and if there was a reason felt it was empirical rather than structural; it was what happened rather than what had to happen.

During the next lesson they did a number of written questions, using the blocks if they wished. The questions included 6 × 20, 3 × 40, 40 × 5, 20 × 6, 2 × 60, 3 × 4, 30 x 4. The connection between 6 × 20 (6 twenties) and 20 × 6 (twenty sixes) was recognised sometimes by some of them; but mainly for 20 × 6 they worked out twenty sixes.

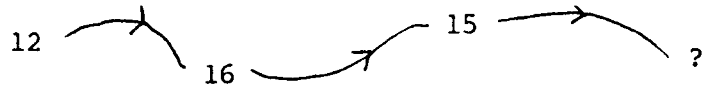
In spite of the previous day's work, although they mainly got 3 × 40 and 30 × 4 right, they still appeared not to see the connections between 3 × 4, 30 × 4, 3 × 40; these were all done independently, not using the result of one for the other.

Later we tried to take the step to 4 × 23, 5 × 23, 7 × 23. They got these right, generally, on the second attempt, and by counting sticks and cubes rather than as 4 × 20 and 4 × 3."

(2) Abstraction resulting from generalisation

While considering strategies for abstraction that of investigation should be included, even though it does not fit easily into the category. An example from the Number unit - Sums of Divisors, (Cards J20, J20a, J21) - shows how many quite important concepts and relationships can be learnt through such a piece of work, which also provides opportunities for generalisation, proof and representation, though with the disadvantage that the teacher cannot be sure that a given pupil will meet a given relationship during his investigation of the problem.

SUMS OF DIVISORS 1 Card J20



The numbers which divide exactly into 12 (not counting 12 itself)

1, 2, 3, 4, 6

The sum of these divisors is 1 + 2 + 3 + 4 + 6   16
So an arrow is drawn from 12 to 16.

Check that an arrow should be drawn from 16 to 15.

To which number should an arrow be drawn from 15?
Continue the chain.

Investigate chains starting with other numbers.

SUMS OF DIVISORS 1a Card J20a

Try the following starting numbers:

10, 18, 20, 26, 37, 28, 24

Complete a chain for each number.

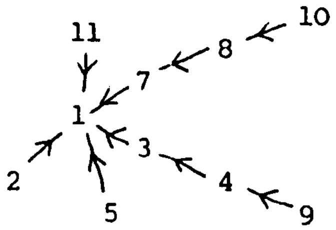
Try some more.

Write comments on anything you notice about any of your chains.

SUMS OF DIVISORS 2 Card J21

Use a whole page of your book.

Start by putting the number 1 in the middle of the page.
Build up your chains to form a 'tree'. (A few have been put in to show how it works.)



Continue to build up the tree.

Write comments on any discoveries you make.

Have any of the numbers got special names?

Such an activity contributes also to the appreciation of mathematics and the development of favourable attitudes. This particular activity was enjoyed by both classes observed. In the course of it, primes were characterised as numbers which "shot straight to 1"; and the following generalisations were found and used, (not all by all pupils):

(1) if the chain 8 → 7 → 1 has been found, and a later chain begins 14 → 10 → 8, it must continue 8 → 7 → 1 as in the previous chain.

(2) in testing for divisors, work systematically through 2, 3, 4....but note that 4 need not be tried if 2 does not divide, and so on.

(3) if 4 goes 5 times, then 5 is also a divisor.

(4) you need not go more than half way to the number in trying divisors - in fact, taking account of (3) you can stop as soon as the second factor becomes as big as the first.

(5) expressing the number in prime factors to begin with enables all combinations to be found.

GENERAL STRATEGIES OF ENQUIRY - WRITE-UPS

This is not strictly a *mathematical* strategy but it is intended to contribute to the awareness by pupils of the questions they are investigating, the experimental method and the generalisations which form the results. Thus it is seen as a means of promoting the development of the general strategies, and the appreciation of the mathematical method. Some examples already quoted show something of the development which took place during the trial year, extract H (p. 5.11) being an early example and J a somewhat later one. The earlier ones tended to be blow-by-blow accounts of what had been done; the later ones became better at identifying data and results. The activity appeared to be a significant and valuable one, but it is not possible to make more precise claims about results achieved.

MIXED ABILITY

The extract on page 5.11 shows how a pupil of low ability was able to make a meaningful activity out of Triangles on the Geoboard. The cards throughout the preceding section show how by starting from a concrete situation there are mathematically significant activities at the many levels which have been considered, ranging from generating examples up to explanations, proofs and extensions of the problems, and proceeding to higher levels of abstraction.

In concept-forming activities it was easy to provide for different ability levels - for example, in Dividing the Board, some pupils remained at the stage of constructing valid halvings, while others moved on to quarterings or to determining the total number. It was generally easy to suggest extensions for able pupils (see page 5.16) Less easy were activities involving number skills in which there was a wide spread of existing attainment among the pupils. In these cases different groups of pupils had to be given different assignments, as in Number Work, where a weak group worked with base blocks while the majority studied base 5 calculations (see pp. 5.30-31).

II. INTER-SCHOOL COMPARISONS

A. NUMBER ATTAINMENT

This study was designed to provide some evidence on questions (4) and (5) - about the compatibility of content and process attainments and the suitability of the material for mixed ability classes. Neither question can be answered definitively from the data, but on both questions some useful evidence was obtained. The activity also made progress in the construction of tests and the organisation of evaluations.

There are many hazards in the comparative evaluation of curricula; some of these have been discussed above. (Biggs, 1967; Richards and Bolton, 1971; see also Williams, 1971) The tests designed for these studies were both multi-faceted, that is, they yielded not only total scores but also scores on a number of subscales. This overcame some of the problems of compatibility by allowing the comparison of profiles of attainment between the different schools.

The number test used appears in Appendix 5. It was designed to cover seven facets of number knowledge: place value, tables (addition and subtraction), tables (multiplication and division), computation, estimates (approximate calculations), number relationships and applications (verbal problems). Only six of these were used in the comparison; results for tables (addition and subtraction) were too near the maximum to show any useful variance.

The test thus covers only the knowledge of concepts and skills relating to the positive integers, and is restricted to basic material; the "relationships" are those underlying calculation, such as $3 \times 40 = (3 \times 4) \times 10$, $361 \times 24 = (360 \times 24) + 24$.

The test was given in June 1975 to first year classes in one of the Project schools, and to four classes comprising the whole first year in a bilateral school. The Project classes contained the whole ability range; in the non-Project school there was a degree of streaming, as shown by the mean IQs of the classes. (Table 1, below)

The general emphasis of the SNP first year work with regard to number is on the understanding of place value and on informal, small-number calculation. The objectives for Whole Number Relationships (see p. 5.8) give an indication. The non-Project school followed a fairly traditional first year course, with rather more emphasis on calculation. The hypothesis was therefor that the Project classes would do better on place value and relationships than on computation, as compared with the non-Project school. Regarding overall performance, as between the two schools, it was hypothesised that any difference would be related to differences in mean IQ.

Sample

Seven classes are compared. Classes 1-3 are from a Project school (A), 4-7 are from the non-Project school (B). The mean non-verbal reasoning scores for each of the classes 1-7 are shown in Table 1.

	SCHOOL A			SCHOOL B			
Class	1	2	3	4	5	6	7
n	25	22	26	27	16	16	14
Mean NVQ	102	100	97	97	88	80	81
SD	11	14	14	8	11	12	12

TABLE 1

Thus, though class 4 of School B is similar to the School A classes, classes 5, 6 and 7 are of distinctly lower mean IQ. Class 4 is somewhat more homogeneous than those of School A.

Results

(i) Comparison of classes

The correlations between IQ and score for these seven classes as a whole, on each facet of the test, are shown in Table 2.

PV	T md	Comp	Est	Rel	Appn
0.62	0.44	0.60	0.57	0.63	0.39

TABLE 2

Since we wish to compare the effects of different teaching situations on these pupils we remove from the scores the differences due to differences in IQ, which, as these correlations show, are substantial. The linear regression program SMLR (Youngman, 1975) is used for this purpose. A correlational analysis (program CATT, Youngman, 1975) is applied to the residual scores to detect differences between classes. This program first applies Bartlett's variance test to ensure homogeneity of the variances, an overall F-test, and then a Scheffé test for significance of differences between each pair of means. The results of the preliminary tests are shown in Table 3, and the means and standard deviation for each group in Table 4.

	PV	T md	Comp	Est	Rel	App
Bartlett variance (should be >.05)	.009	.014	.94	.003	.90	.52
F-Test sig. level (should be <.05)	.004	.22	.0004	.77	.0005	.25
Indications	?,sig	?,n.s.	✓,sig	X,n.s.	✓,sig	✓,n.s.

TABLE 3

Thus computation and relationships and, less reliably, place value, show significant differences between the classes. The classes between which the Scheffé test indicates significant differences are shown in the bottom row of Table 4. (p. 5.39)

Class	M PV SD	M Tmd SD	M Comp SD	M Est SD	M Rel SD	M App SD
1	0.56 0·71	0.17 0·85	-0.07 1·09	0.21 0·44	0.14 2·85	-0.09 1·43
2	0.12 0·84	-0.28 1·25	-0.14 1·04	-0.03 0·55	-0.61 3·05	-0.31 1·16
3	0.00 1·05	-0.08 1·11	-0.31 1·00	-0.03 0·69	-0.10 2·70	0.05 1·42
4	0.18 1·11	0.32 0·80	0.85 0·88	0.02 0·87	2.41 2·82	0.53 1·27
5	-0.61 1·18	-0.09 1·14	-0.57 1·07	0.05 0·75	-0.84 3·07	0.17 1·07
6	-0.31 0·90	0.35 1·33	0.32 1·08	-0.11 1·07	-0.76 2·17	-0.36 1·19
7	-0.35 0·50	-0.48 1·64	-0.32 1·10	-0.17 0·82	-1.43 2·81	-0.18 0·91
Sig diff						
.05>	1 > 5	-	4 > 3,7	-	4 > 2	-
.01>>			4>> 5		4 >> 7	

TABLE 4

The first hypothesis, that classes 1-3 would do better on place value and relationships, and less well on computation and applications has some limited support in that classes 1-3 are generally better on place value than classes 4-7, and the non-significant results on applications show classes 1-3 lower in relation to 4-7 than on the other facets. But it is apparently refuted by the good performance of class 4 on relationships. However, this may be due to the fact that a number of the questions in the relationships section can be solved by computation; scrutiny of the scripts supports this, showing that class 4 have indeed used computation in this section. Taking the results as a whole, class 4 has performed above expectation and classes 5, 6, 7 below, with classes 1, 2 and 3 between. Thus the results show no differential effect between the schools attributable to the Project's emphases. The tendency for understanding of place value to be relatively better than computation in the Project schools is in the predicted direction.

There are several significant differences between class 4 and classes 5, 6, 7. Since the correlation between individual IQs and scores on each facet has been removed, this indicates differences arising from different teaching or other experiences of the classes. It is possible that these are related to streaming, class 4 being the top stream of the year.

The residual scores are also subjected to a discriminant function analysis (DSFN, Youngman, 1975). This confirms the previous results. The two significant functions account for 50% and 28% of the variance respectively; the first loads on computation and relationships (0.84 and 0.78 respectively, with no other loadings >.35) and the second on place value (0.87, no other >.25). The means of the seven classes with respect to these functions are shown on Graph 1 (p. 5.41)

Thus the greatest differences between the seven classes are on computation and on relationships, and on these class 4 (D on the graph) is clearly superior to the rest, the three Project classes, (A, B and C) lying among the remaining non-Project classes. On place value the differences are smaller and in this case the three Project classes score above those from the non-Project school.

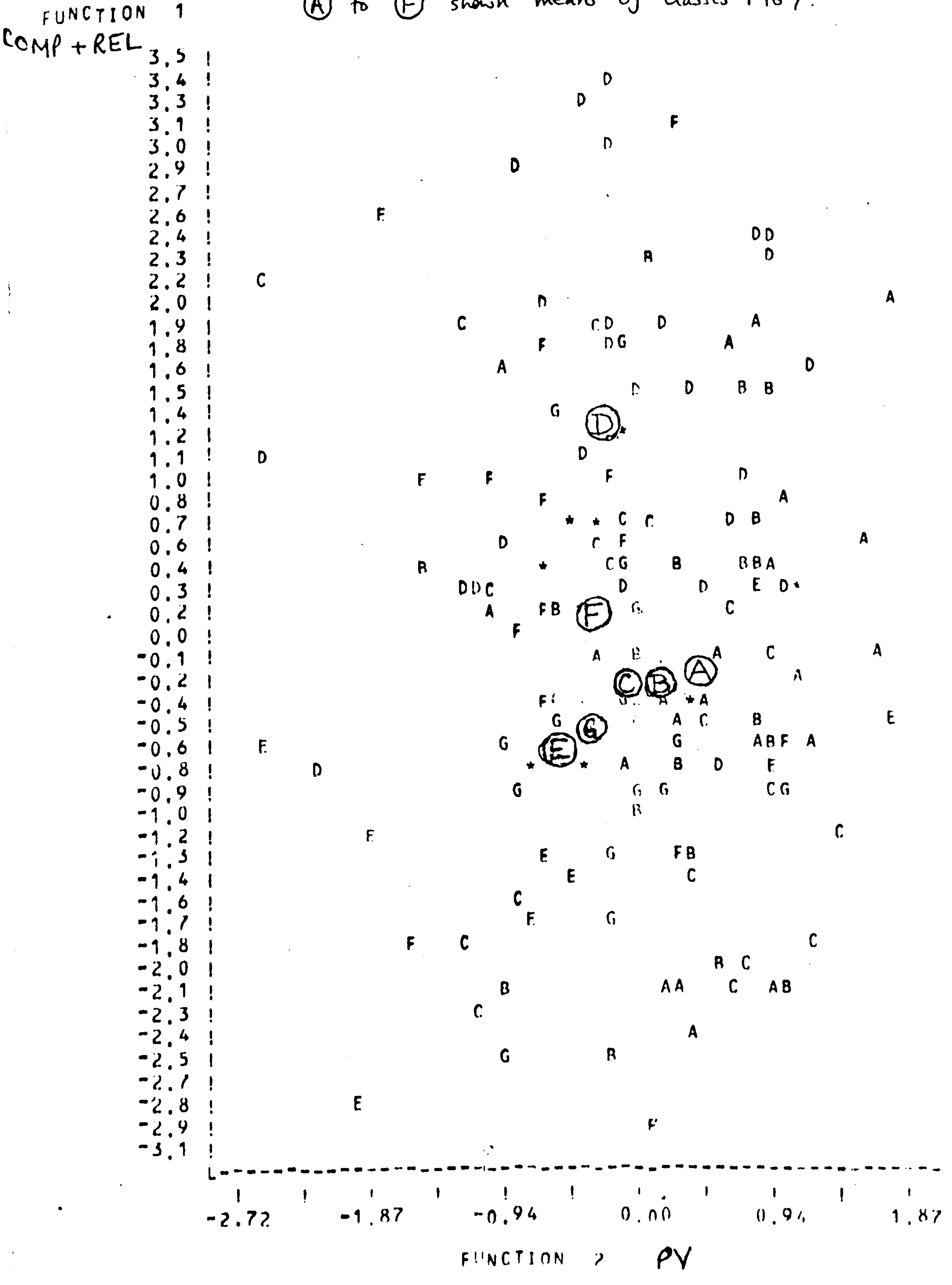
An evaluation somewhat comparable to this one, though a a larger scale, was made by Edinburgh University for the Fife Mathematics Project (Crawford, 1975). This involved some 20 schools which had devoted various proportions of their first year time, from 0 to 50%, to enrichment material of somewhat similiar character to the South Nottinghamshire Project material. The test was on the standard Scottish first year syllabus, to which the Project work contributed only indirectly. The results (residuals after removing IQ components) showed great variations between schools, but these were quite independent of their degree of involvement in the Project. The conclusion was drawn there, as here, that any benefits of a different kind which might be accruing from the project were not at the expense of achievements on the standard syllabus.

(ii) Comparison by Ability Levels

In the previous analysis the correlation between IQ and score for each variable was removed by the use of residuals, but classes

FUNCTION 1 AGAINST FUNCTION 2 WITH GROUP MEMBERSHIP LABELLED

Ⓐ to Ⓕ shown means of classes 1 to 7.



GRAPH 1

were kept intact. These classes showed distinctive performances which could be related to differences in their teaching and school situations, the apparent inference being that the top stream over-performed and lower streams under-performed, even after the IQ correlation was eliminated. An alternative method of analysis compares, not classes, but subsets within each school of pupils, in six different IQ ranges, under 80, 80-90, and so on.

Hypotheses

A common assumption is that, in a mixed ability class, it is more difficult to ensure that all pupils make satisfactory progress, and that those at the extremes of the ability range are those more likely to suffer. It would follow from this that the relationship between IQ and test scores would be non-linear, with the middle range of pupils scoring above the line and the extremes below. See Fig. 3.

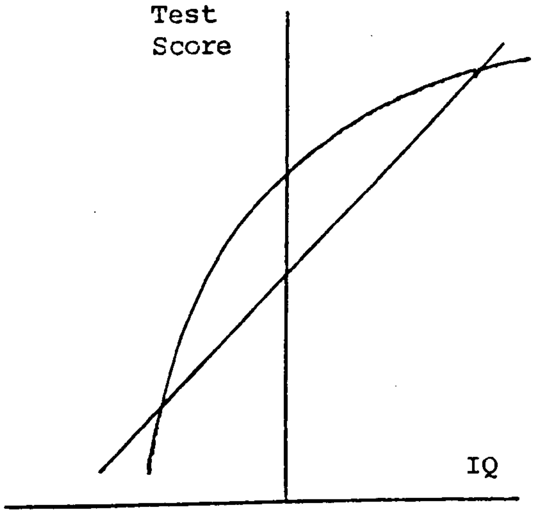


FIG 3

However, this makes the unwarranted assumption that there is a "normal state of affairs", and that, in this, the relationship is linear. All we can do is to compare different schools, and look for *differences* in this tendency. It is of course possible that different forms of organisation might lead to different gradients for the line. This also would show on the comparative graphs.
The hypothesis, then, is that in a "mixed ability" school the tendency for the test score/IQ graph to bulge upwards will be more marked than in a streamed school.

Sample

The same test results as were analysed in the previous section were used from School A, Project, mixed ability, and School B, non-Project, partially streamed. (See Table, p.5.37)

Results and Discussion

Figure 4 shows the table of results by IQ subsets, and the corresponding graph, for the total scores on all six facets of the number test. There are no significant differences in the scores of the subsets between the two schools. Both curves shows some evidence of the kind of bulge discussed above, the project school slightly more so than the other, but the upper part of this is clearly due to a ceiling effect in the test. The difference in the lower subsets is more than half the standard deviation but is not significant on account of the small size of the School A group. Figures 5, 6 and 7 show the results on three separate facets of the test. None of the differences are significant, but it is interesting to note that the superiority of School A on place value and of School B on computation, remarked upon above, show throughout the ability range. It is also worth observing the strength of the dependence of results on IQ in comparison with inter-school differences.

MEAN SCORES ON A NUMBER TEST OF PUPILS IN DIFFERENT IQ RANGES AT TWO SCHOOLS

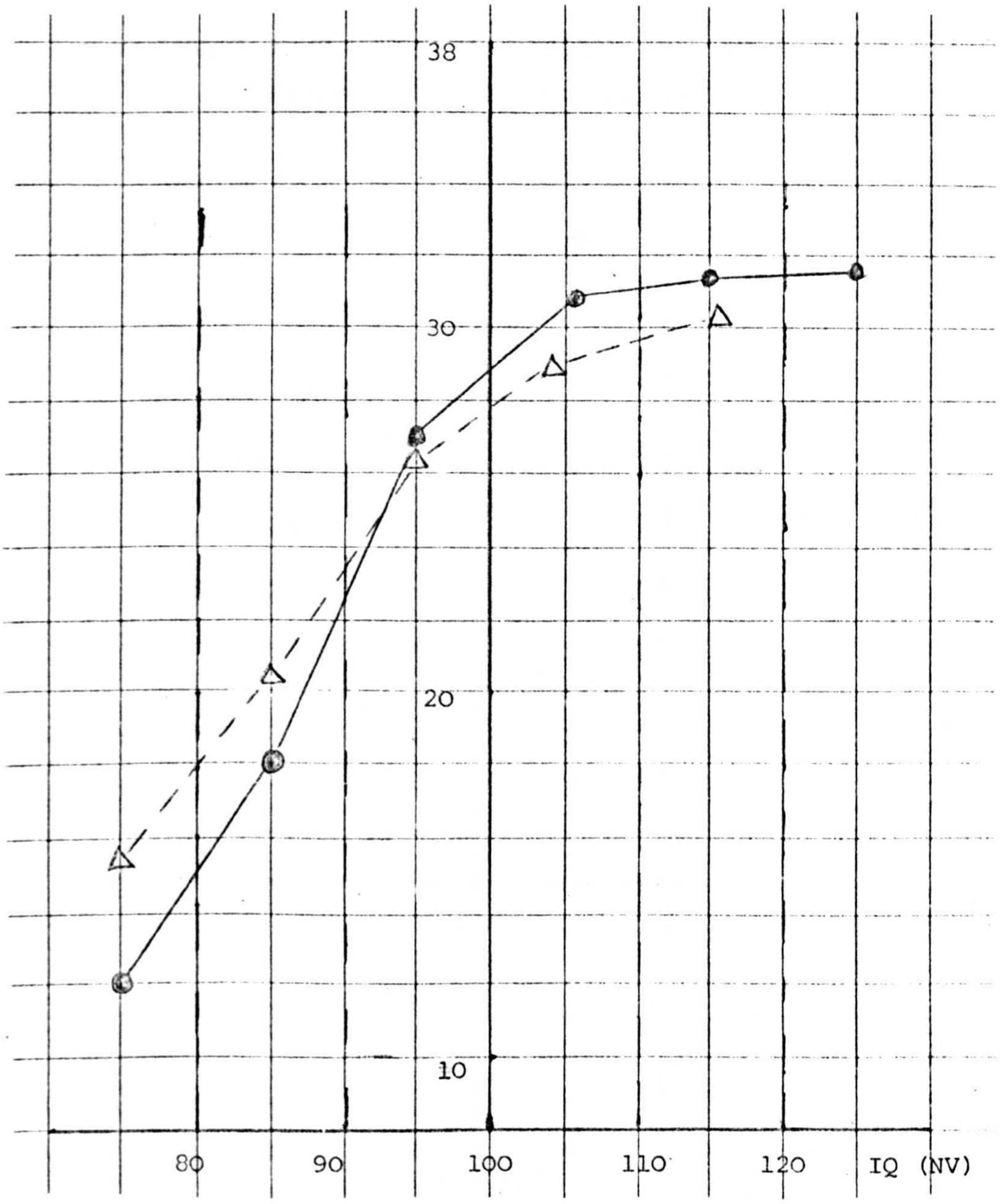


FIG 4

of pupils	5	11	17	14	11	5	
MEAN	12.0	18·1	26·9	30·8	31·6	31·8	SCHOOL A
SD	5·4	6·5	5·6	1·6	5·11	4·5	●——●
of pupils	24	20	20	12	2	0	
MEAN	15.7	20·5	26·3	29·0	31·0		SCHOOL B
SD	4·2	7·6	6·8	5·3	3·0		△——△
	NO ADJACENT DIFFERENCES REACH SIGNIFICANCE						
	80	90	100	110	120		IQ (NV)

MEAN SCORES ON SUB-TESTS OF A NUMBER TEST BY PUPILS OF DIFFERENT IQ RANGES IN TWO SCHOOLS

FIG 5

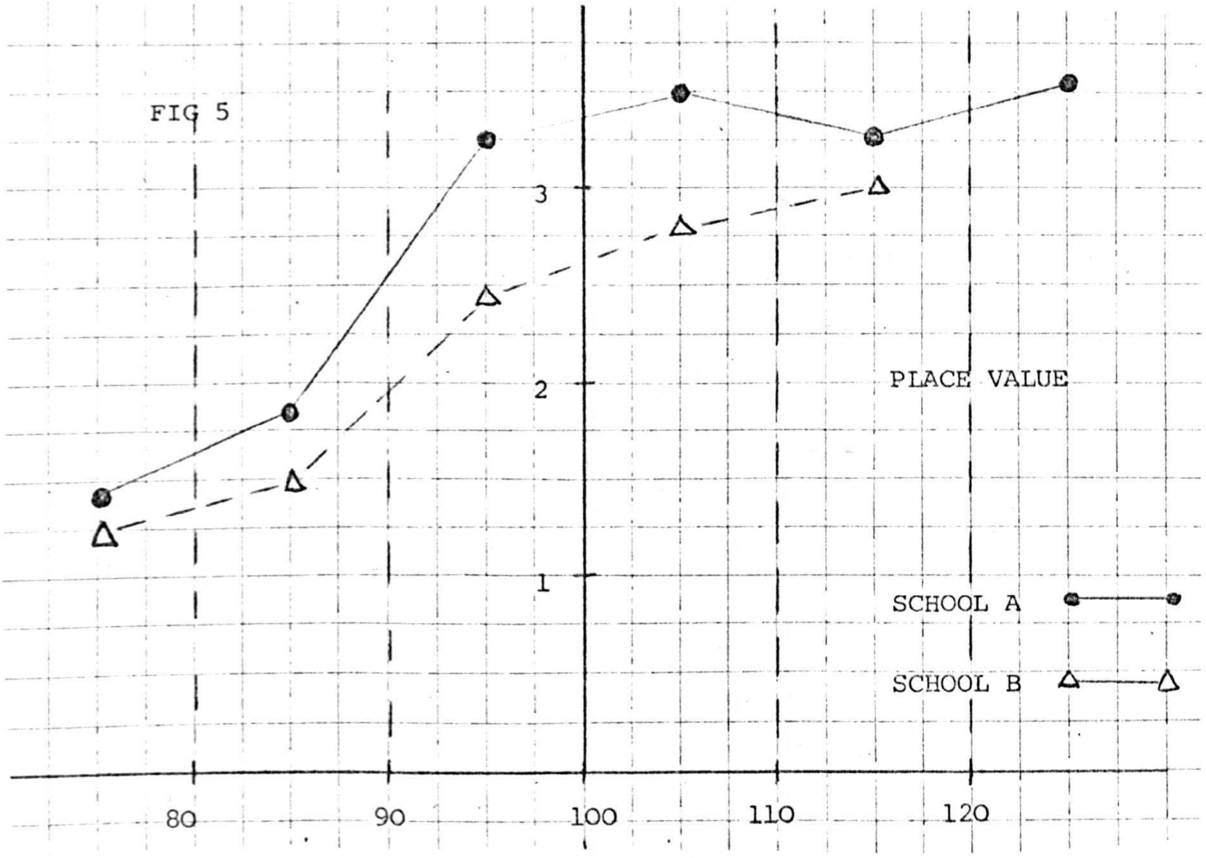
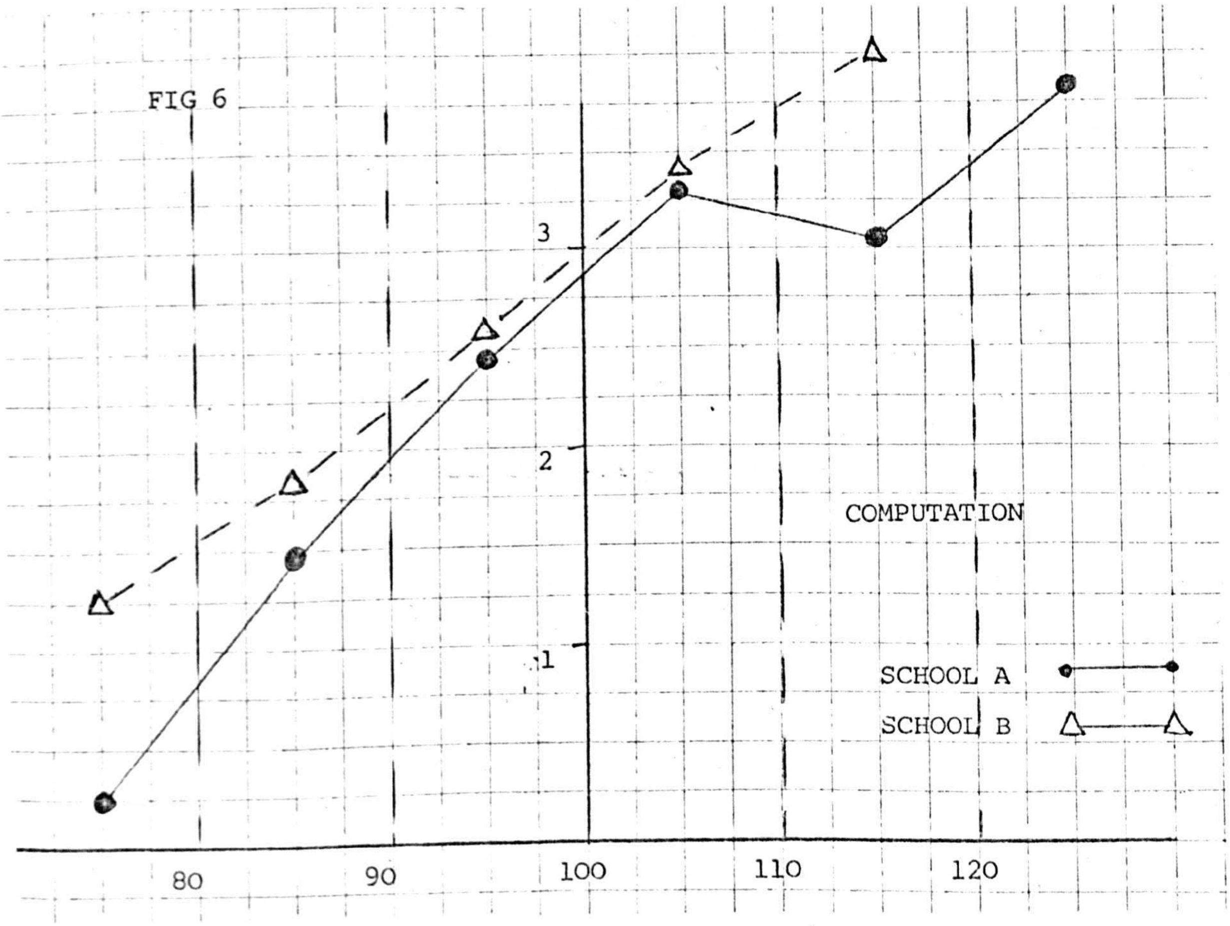
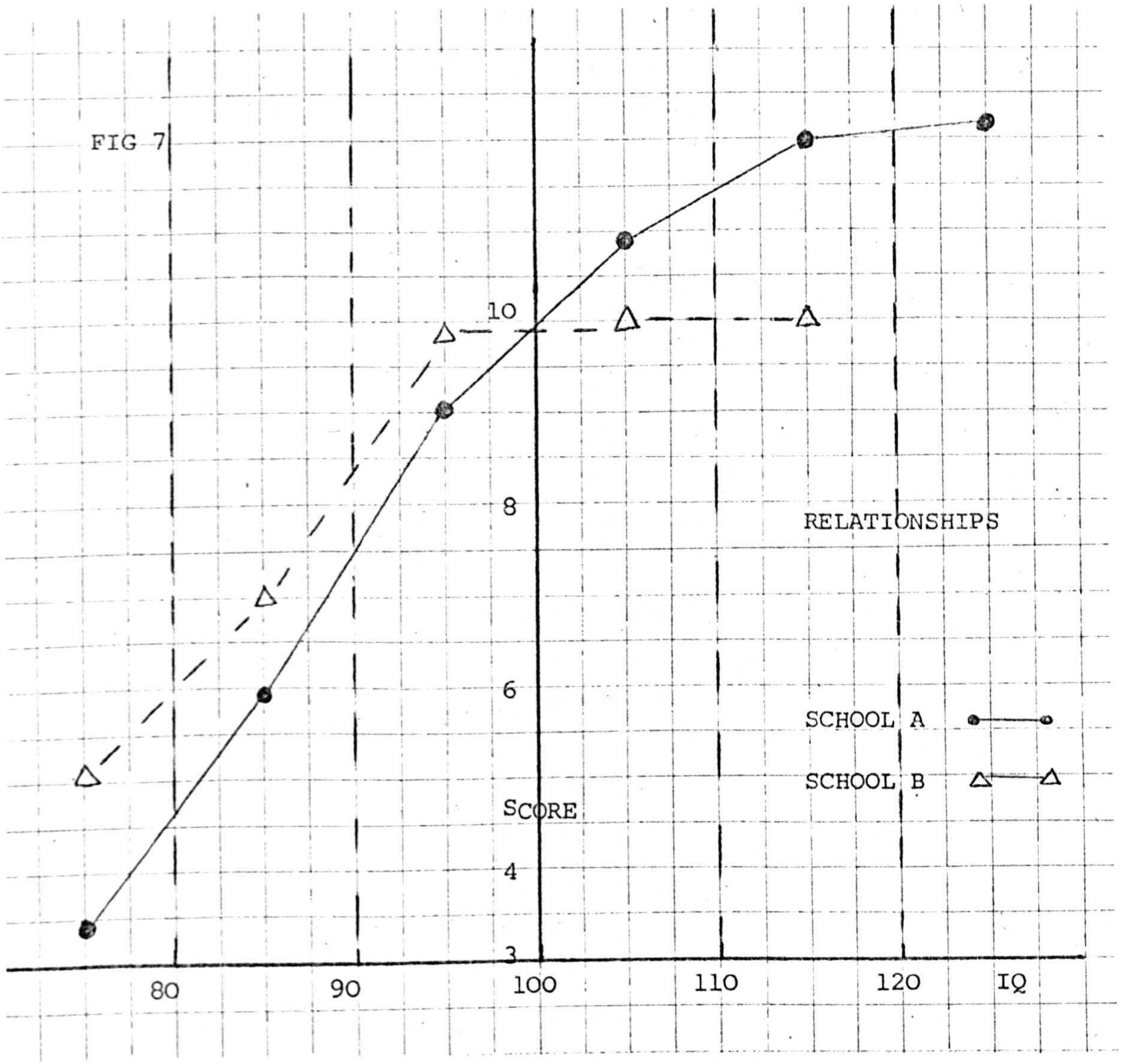


FIG 6



MEAN SCORES ON SUB-TESTS OF A NUMBER TEST BY PUPILS OF DIFFERENT IQ RANGES IN TWO SCHOOLS

FIG 7



B. ATTAINMENT OF GENERAL STRATEGIES

This study is aimed at question (2) of those posed at the beginning of this chapter - regarding the attainment of transferable general strategies. The test has been designed to evaluate the strategies described in the first part of this chapter as being within the capabilities of the first year pupils. It was given, early in the second half term of the course, to two classes from Project schools and two classes from a non-Project school.

The aims of this testing were (a) to check that the test was capable of revealing the effects of the teaching of strategies, and to develop a suitable marking scheme, (b) to note any differences between schools at this stage after the first half term of the course (c) to establish a base line from which changes over the year could subsequently be measured. It was also intended that the development and use of the test should contribute to the clarification of the teaching objectives and the subsequent improvement in effectiveness of the teaching material.

The general style of test questions was to set up a situation involving the generation of examples to test a generalisation, or leading to the making of a generalisation, and then to ask for an explanation of the reason for the truth of the generalisation. A mixture of number and geometrical questions was included; some questions required the generation of a complete set, and some involved proofs of impossibility. The content of particular mathematical ideas was intended to be sufficiently elementary to allow the general strategies to be the dominant factor. Parts of questions 3 and 4 illustrate these points; a fuller analysis is given below.

Question 3

This is like question 1.
This time choose a number *bigger* than ten. Write it here...........
Add it to ten and write the answer...............
Take ten away from it, and write down what is left...............
Add the two last answers...........

Try this with other numbers. Is there any pattern in the results?
If so, describe it. Explain why it happens.

Question 4

Suppose you have a lot of stamps of value 6p and 15p but no others.
You can make up various amounts of pastage from these.
If you want to, you can make 27p as 15 + 6 + 6.
Can you make 29p? Use the space below for your trials.

Test Development

Marks for the different items were initially allocated to three scales - M, G and E, for "mathematical", generalisation and explanation. The hypothesis was that on items requiring the last two strategies the Project schools would eventually show some superiority, but on items requiring only simple mathematical operations and no learnt strategy there would be little difference between Project and non-Project schools. One of the main aspects of the testing and analysis procedures was to be the observation of differential results between different facets of the tests, so as to reduce the effects of other unknown differences between classes, teachers and schools. An item analysis was performed, and also factor analyses seeking 3, 5 and 7 factors from the results of the 22 separate test-items, and tests of difference between the means for each item between the two groups, Project and non-Project. From these results, four modified scales were established, item-analysed and used for tests of difference between groups. This process will now be described.

The factor analyses are dominated by the high inter-correlations between parts of the same question. This is inevitable with this type of test. Independent items cannot be used without altering the nature of the activity, since generalising from examples produced oneself is a different activity from doing so from a set offered in the item; similarly, an explanation of a generalisation found for onself is different from one of a generalisation proposed. The factor analysis can offer guidance regarding how far the characteristics which the tester supposes he has built into the item are reflected in differential performance characteristics of the pupils. But it is clear that in many cases items which seem clearly different in nature correlate highly because, for example, it is impossible to score for an explanation if one has failed to make the generalisation. The main points of interest in the factor analysis are therefore those where items from different questions are brought together in a factor, or items from the same question are separated. In the final decision regarding scale allocation, the prima facie nature of the item, its occurrence in factors and its actual intercorrelations with the other items were all considered. The four scales adopted are:

1. Generating examples: to meet given criteria, stating how or why given examples fail to qualify, classifying examples, finding complete sets. Items 1, 5, 6, 7, 8, 12, 14, 15.

2. Recognising relationships and patterns, extending patterns; expressing relationships verbally. Items 10, 17, 19, 20, 21.

3. Giving explanations or proofs. Items 4, 11, 13, 16, 18, 22

4. Following verbal instructions to produce data. Items 2, 3, 9

The item analysis using these scales is shown in Table 5. A copy of the test and detailed notes on the test development are contained in Appendix 5.

```
SCALE     ITEMS     MEAN       SIGMA      ALPHA

   1        8.      7,982      4.574     0,8183
   2        5.      5,204      3.160     0,7468
   3        6.      2,265      2.614     0,6731
   4        3.      3,770      1,824     0,7142
   5       22.     19,221     10.310     0,8969

ITEM  SCALE   MEAN   SIGMA   R(TOTAL)   R(SCALE)
  1     1     0,60   0,490   0.3200     0,4119
  2     4     1,40   0,723   0.5194     0,8746
  3     4     1,22   0,909   0.5808     0,8948
  4     3     0,44   0,728   0.6144     0,7010
  5     1     1,37   0,854   0.6363     0,7517
  6     1     1,13   0,945   0.6209     0,7802
  7     1     1,29   0,947   0.5533     0,6406
  8     1     0,67   0,910   0.5370     0,6236
  9     4     1,15   0,627   0.3660     0,6027
 10     2     0,68   0,895   0.5848     0,6049
 11     3     0,39   0,746   0.4427     0,5552
 12     1     1,19   0,804   0.6577     0,6480
 13     3     0,57   0,786   0.5235     0,5947
 14     1     0,94   0,905   0.7124     0,7351
 15     1     0,79   0,936   0.6655     0,6713
 16     3     0,25   0,603   0.3947     0,5761
 17     2     0,82   0,952   0.6816     0,7031
 18     3     0,57   0,743   0.5851     0,6737
 19     2     1,03   0,694   0.4786     0,6360
 20     2     1,10   0,921   0.6323     0,7806
 21     2     0,97   0,991   0.6782     0,7875
 22     3     0,25   0,617   0.4152     0,6064
```

TABLE 5

HYPOTHESES FOR TEST

Reference to the description of the SNP course in the previous chapter leads to the following hypotheses in relation to the present test.

Generating examples to meet given conditions, and constructing complete sets are strongly encouraged during the early part of the SNP course. During the first seven weeks, most classes will have covered the Shape, Statistics and Symmetry units, this including several investigations in a geometrical setting. The work on Number Patterns and the occasions for giving explanations arise later in the year. Hence on Scale 1 (Generating Examples) Project classes should be superior to non-Project classes at the time of testing.

On Scales 2 and 3 little or no difference is to be expected at this stage.

On Scale 4 there should be no differences.

SAMPLE

Two classes (54 pupils), one from each Project school, and two classes (59 pupils) from a non-Project school in a similar area took the test. The mean IQs of the classes on the NFER DH (Calvert) test, were 99, 103 (Project) and 98, 102 (non-Project).

RESULTS

The results of tests for significance of the differences between means are given below, with significance levels for the F-ratio.

	1 Generating Examples to Criterion	2 Recognition of Relationships	3 Explanations	4 Following Instructions
Max	15	10	12	6
Project Mean n=54 S.D.	9.50 3.8	5.98 2.8	2.87 2.6	3.57 1.8
Non-Project Mean n=59 S.D.	6.59 4.8	4.49 3.3	1.71 2.5	3.95 1.8
Sig. level	.001	.012	.018	N.S.

TABLE 6

The first hypothesis is confirmed, substantial differences being recorded. (Significance 0.001) On Scale 4 there is no significant difference, as predicted. However, there are unexpected differences on Scales 2 and 3 significant at the .05 level.

DISCUSSION

The superiority of the Project classes on scales 2 and 3 seems to indicate that even after only half a term, the orientation towards recognising relationships and giving explanations is having a measurable effect. That following instructions shows a non-significant difference in the other direction may perhaps be accounted for by the fact that the non-Project classes had made extensive use of SMP work-cards. These involved considerably more following of written instructions than the mainly orally-introduced activities of the Project. Although the two groups are well matched with respect to non-verbal reasoning, the possible effects of different earlier (primary school) experience should perhaps not be ruled out. Future plans include the comparison of gains over the year on this test; this will provide valuable confirmation or otherwise of the present results.

Finally, it is worth noting also the relative levels of attainment on the different scales. For the Project classes, the mean score on scales 1, 2 and 4 are around 60%, on scale 3 about 25%. The greater difficulty of explanation items is evident.

CONCLUSIONS AND SUGGESTIONS FOR FURTHER INVESTIGATION

The results of the test of general mathematical strategies, and the classroom observations, make it reasonable to assume that the curriculum of the South Nottinghamshire Project does result in improved learning of these strategies. They also show that the strategies of experimenting and generalising are the more easily improved, while those of explanation and proof are more difficult at this stage. The results of the inter-school comparison on the number test are less conclusive. Although they show no significant differences overall, the two schools compared are sufficiently different in character to make it impossible to make confident inferences about what differences would exist between the Project pupils' number attainments under a more content-oriented curriculum, and their actual attainments under SNP. It seems reasonable to deduce that there are not substantial losses, but to say more than this is probably unjustified. (In the revision of the first year material being made for the 1976/7 year, a "skills booklet", to be used individually by pupils during part of the time devoted to number has been included. This is to provide for the practice stage of the basic computational skills, which follows the stage of understanding of principles.) On the positive side, evidence from the SNP schools suggests that the Project materials do succeed in involving the pupils in more genuinely mathematical activity than the standard courses, and that there are noticeable effects on the pupils' general understanding of the nature of mathematics and in their confidence and know-how in approaching it subsequently.

On the suitability for mixed ability classes, the provision of extensions of problems for abler pupils was easy and natural. Less able pupils in general did useful and satisfying work at the lower levels of abstraction, but the extension of the project into the third year and beyond would present increasing problems in the choice of common starting points; some differentiation would probably be necessary.

Some remarks on the styles of task which have evolved during the classroom trials may be appropriate. The chief difference from orthodox courses is in the extensive investigation of a few situations, as opposed to the working of a larger number of short exercises. Generally, the concrete materials - geoboards, pegboards, tessellations, matchsticks and so on - provide the situations; these embody the concepts of shape, symmetry, pattern, angle relationships, number relationships, sequences and functions. The "how many" question leads to classification and hence to new concepts, and also to proof by exhaustion (see pages 5.10-5.13, 5.19-5.20). The "find a rule" question leads to generalisation; and "does it always work?", "why does it work?" lead to proof (see pages 5.14-5.16, 5.24-5.25). Inter-pupil discussion, arising from the reaching of different conclusions, both removes some errors and sharpens the construction of proofs and explanations. These types of activity clearly do not feature in a major way in most standard courses; equally clearly, they are essential to the achievement of the improvement in process attainments in the SNP. For comparison, the SMP chapter on line and rotational symmetry concludes with an exercise in which the symmetry in nine different situations is to be recognised. (Book A, pages 81-3); the SNP explores just three situations, paper folding, geoboard and pegboard, but in greater depth and including the construction of figures with a variety of symmetry. Similarly, the SMP Ratio chapter (Book D) finishes with an exercise of 13 questions such as:

"4. There is four times as much nitrogen as oxygen in air. How much oxygen is there is 25 litres of air?"

The SNP unit on Fractions and Ratios represents ratios by pairs of number rods, and generates all the pairs of numbers corresponding to each ratio, and all the different ratios embodied in a given pair of rods - $\frac{1}{4}, \frac{4}{1}, \frac{1}{5}, \frac{5}{1}, \frac{4}{5}, \frac{5}{4}$. These interrelationships in the rod situation are studied, and just one or two others drawn from "real life".

These differences are based on the theory that what needs to be learnt are not simply particular key techniques, such as the Unitary Method, (though these have their place), but interrelated systems of concepts. Thus a situation which can be investigated and manipulated so as to expose many different relationships is of more value than one in which the sole task is to identify the aspects of a concept and apply an appropriate method. To use Skemp's (1971) phrase, such learning is more *schematic*. In a ratio question, if two quantities x and y are given, the preferred method would automatically consider x/y, y/x, x + y, x/(x + y), y/(x + y) and so on before deciding which of these needed to be used. The learning of ratio would emphasise the interrelationships among these quantities. That this style of task makes for more effective learning is suggested by Skemp's theory, by the results of reflection on Dienes' multiple embodiment principle (e.g. Wheeler, 1964, and see p. 3.8 above), and by the recent finding (Lunzer, Bell and Shiu, 1976) that structural factors are the greatest determinant of difficulty in mathematical problems, as well as by the present work. In view of the persistence of conflicting methods in standard texts, it is highly desirable that it should be tested in a specifically designed experiment.

The other most significant feature of the SNP style of task relates to algebra. The principle of using symbolism to express and transform relationships which have acquired some meaning for the pupil is discussed on pages 5.16 and 5.29. The aim is to use symbols easily and naturally, accepting pupils' own choices or suggesting agreed ones, as an extension of and improvement on ordinary language or existing symbols (as R3, U2 for right and up displacements). This principle is extended in the second year SNP material, where letters are used to express number generalisations and to denote unknown numbers which are then found, but still within the context of a problem situation. Observations indicate that this is markedly more successful than more formal approaches to algebra, and this, too, seems worthy of experimental investigation.

CHAPTER 6

GENERALISATION AND PROOF - PREVIOUS RESEARCH

<u>IN APPENDIX 6</u>

INTRODUCTION

Viewed internationally, the proof aspect of mathematics is probably the one which shows the widest variation in approaches. The present French syllabus adopts an axiomatic treatment of geometry from the third secondary school year (age 14), (though examination questions do not demand such knowledge (Bell, 1975)) and early American developments based primary school number work on the laws of algebra. In England, proofs of geometrical theorems have been steadily disappearing from O-level syllabuses for thirty years, and "it continues to be the policy of the SMP to argue the likelihood of a general result from particular cases". (Preface to Book 5).

The 1967 Report by the Mathematical Association "Suggestions for 6th Form Work in Pure Mathematics" discusses a number of criticisms by university teachers of the school preparation of prospective mathematics students, and suggests some remedies. The first section is on Methods of Proof, and mentions students' lack of "clear ideas of what constitutes proof", and that they "readily confuse a theorem and its converse". Later, under "necessary and sufficient conditions", students rarely use these terms correctly", and again, referring to mathematical induction "this popular method of proof is often applied with more enthusiasm than understanding". The remedies suggested include mainly a wide range of examples - of theorems whose converses are untrue, examples of induction where the first few values of n are untypical, and so on.

The results in this thesis imply that much more serious and continuous attention to the development of proof activity throughout the secondary school is needed to bring students to a stage at which they can progress with reasonable ease to the deductive expositions normally offered at universities. Another Mathematical Assocation Report on The Use of the Axiomatic Method in Secondary Teaching (1966) suggests that axiomatics, in the form of Groups and Boolean Algebra, might come into the course after the age of 15 or 16, after some "much earlier" preparatory work involving deductive method. This earlier work is described thus: "From a large number of stated intuitive assumptions a coherent edifice of results can be built up deductively, and checked stage by stage against the concrete situation that gave rise to those assumptions. From time to time individual assumptions can be taken up, to see how they are related to other assumptions." In that this implies an interest in the deductive structure itself on the part of 13-14 year olds, it is considerably more ambitious than the traditional school treatments of Euclid, the deductive aspects of which have gradually disappeared from the O-level examinations because they were found impossible to teach, with meaning, to most pupils. The experimental work reported in this thesis will show what concepts of proof secondary pupils are currently able to use, and also indicate ways in which their powers in this field of activity might be developed.

The opening paragraph of this chapter indicated a wide divergence in the views of proof held by different writers, and different development projects. The Comprehensive School Mathematics Project at Carbondale, Illinois represents one end of the spectrum. In the course for pupils of grammar school ability (CSMP, 1972, Braunfeld, 1973) pupils of 12-14 years follow two independent courses at the same time, one less and one more formal. In the formal course they learn to write proofs based explicitly on the axioms of propositional logic. The first page shown (Appendix 6) is from a 12 year old; this is purely an exercise in logic. The other, from a 13 year old, is a proof that $-(x + y) = -x + -y$; this uses the associative, commutative and cancellation laws and the definition of the additive inverse for integers.

The CSMP view is that by beginning thus, with detailed chains of inference using the stated laws of logic, pupils can acquire a firm foundational knowledge of what a proof is without having to induce this knowledge from the ordinary proofs they see presented by the teacher. In the course of the three years (12-14), the mathematical content will increase and the detail of the logic diminish. Whole sequences of such deductions will be referred to in single lines, but the pupil will be aware of what logical consequences he is taking for granted (CSMP, 1972).

The question which arises here is whether pupils of this age would recognise the need for the detailed logical proofs as a foundation for mathematical proof, or whether they would see them as a somewhat foolish whim of the mathematics teacher. Since such methods have in fact been reached historically as a result of an eventual awareness of the unsatisfactory nature of less rigorous methods, it would seem sensible, not necessarily to follow the precise course of history, but at least to ensure that the pupil can feel the need for the axiomatic approach before he is required to follow it. Moreover, since most actual mathematics is done in a more informal way and at a more concrete level, this could hardly be described as bringing real mathematics into the classroom. However, the main object of quoting this work is to illustrate the differing views of proof in relation to the curriculum. Dienes (1973) appears to suggest that the final level of proof in school mathematics is a purely formal system in which strings of symbols are transformed according to stated rules; after sketching a study of totally ordered sets, the following proof that 2 comes after 4 is given:

Rule 1: Rxy ⇒ NRyx
Rule 2: (Rxy and Ryz) ⇒ Rxz

Theorem: NR SSO SSSSO
Proof: 1. R.See (Axiom)
2. R SSSO SSO (e = SSO)
3. R SSSSO SSSO (e = SSSO)
4. R SSSSO SSO (3,2, Rule 2)
5. NR SSO SSSSO (4,Rule 1)

Lester (1975), following Suppes, uses a similar but simpler system, as a step towards examining "the development of the ability to write a correct mathematical proof" in pupils aged 9 to 17.

The chief fault of these views is that they assume that mathematical proof is purely concerned with *verification*, whereas it has normally been expected also to convey *illumination*. But the preceding comments also apply.

PROOF AND LOGIC

Logic underlies mathematical proof in two apparently different ways. It comprises the *basic relationships and transformations* involved at every step of an argument - such as $[(\text{all } P \text{ are } Q) \text{ and } x \in P] \Rightarrow x \varepsilon Q$, $[P \Rightarrow Q] \Leftrightarrow [\sim Q \Rightarrow \sim P]$, $\sim(\forall x\ \exists y,\ y < x) \Leftrightarrow \exists x\ \forall y,\ y \geqslant x$ - and also the *recognised methods of proof,* such as reductio ad absurdum, disproof by counter example, identification and exhaustion of all possibilities; and mathematical induction. Most of the latter consist of the same basic relationships used globally, as a logical structure for a whole proof. A considerable amount of research exists on the understanding and use of the simpler basic relationships (mainly implication) in a variety of contexts. What is most relevant to mathematics will be quoted here. It is mostly above the level at which mistakes are commonly made in school mathematics - whether a rectangle can be a square, for example, and whether one has assumed the equivalent of what one is trying to prove, and whether what is being used is actually a case of the theorem being quoted.

Thus Henle (1962) shows that a high proportion of apparent logical errors consist of (a) confusing the truth of the conclusion with the validity of the reasoning, (b) the omission of a premise or the inadvertent assumption of a non-existent premise, or (c) a misreading of the meaning of a premise, rather than actual errors of logical inference. This study used ordinary verbal material.

Wason (1968) shows how even intelligent subjects tend to adhere tenaciously to their hypotheses, if confirming evidence has been found, and fail to consider alternative hypotheses. Lunzer (1973b) says "problems of logical inference constitute a special class and should not be taken as a touchstone for the quality of thinking in general" , and "productive thinking is more often analogical than logical". The factors affecting performance in logical problems are (a) structure, e.g. modus ponens, contrapositive, converse or inverse, with or without quantifiers and (b) context. Ennis (1965), Hill (1961), and Varga (1972) show that with the easier structures in familiar concrete settings children aged 6-9 can make correct inferences. Wason and Shapiro (1971) and Abbott (1974) also show the relevance of context. O'Brien shows that at ages 14 to 17, structure remains dominant, and context is unimportant; it would appear that his contextual variations are less significant. (1971, 1972, 1974)

More details of these studies of implication; O'Brien (1972) used items of the type "if the car is shiny, it is fast; the car is fast; is it shiny?", in four contexts - causal, class inclusion, nonsense and random - and in four forms, modus ponens, contrapositive, converse and inverse. (Thus the item quoted is of class inclusion and in the converse form). The subjects were girls aged 14 to 17 (grades 9-12), of mean IQ about 110. As the graph (from O'Brien's article) shows, the greatest differences were between the forms. The overall percentage successes were, for modus ponens 95%, contrapositive 63%. inverse 32% and converse 11%. The differences between ages was relatively small, and consisted entirely of gains from age 14 to 15 on inverse from 22% to around 35%, and on converse from 6% to 13%. Differences between contexts were minimal. (O'Brien, 1972) The bulk of these errors were due to what O'Brien calls Child's Logic, that is the assumption that a statement implies its converse and inverse, when the correct reponse would be to say "can't tell". A previous study by O'Brien et al (1971) covering ages from 7 to 17 showed that the percentage of pupils *consistently* using correct "Math Logic" was below 4% up to the age of 13, reaching 10% at 15 and 18% at 17, while those using "Child Logic" consistently declined slowly from 70% at 7 years to 50% at 13, 30% at 15 and 24% at 17 years. An even more striking result is that, in another study, O'Brien (1973) found that students who had followed a year's course in logic performed very similarly to the subjects of the previous studies. Still more recently (1974), O'Brien rejects the concept of Child's Logic in favour of a more detailed analysis of the relative difficulty of the different logical forms. The work of Lunzer (1973a) and his collaborators with English children confirms the high level of difficulty of logical items right up to university student age.

The results of a short set of logical problems given by the present writer to small groups of pupils aged 11-15 (Appendix 6) agreed with these findings and showed the inability of the 11-13 year olds to

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tolerate uncertain results. Where the implication was "cannot tell", they substituted, often, the untrue converse.

All these questions involve selecting two or three statements from a slightly larger number and combining them, usually by a logical "and". Some aspects of them might be susceptible to improvement by teaching; for example, the use of a family tree diagram or a Venn diagram would be helpful in some questions; but in general we seem to be observing phenomena closely related to general intellectual development. Thus, to summarise, we note in these results that 11 year olds make logical deductions involving combining 2 or 3 statements, as long as they have definite outcomes, and are in situations concretely given or familiar to the pupils. By the age of about 15 indefinite outcomes and more hypothetical situations are handled, but confusion of statement and converse and similar logical errors may persist much longer.

Evidence of similar difficulties at an advanced level is provided by the results of part of an examination paper set to 101 university honours mathematics students at the end of their first year by J.A. Anderson in 1971 (unpublished). This extended a range of items concerning necessary and sufficient conditions and converses used by P.R. Buckland (1969) with a group of 17 postgraduate teacher training students having degrees containing mathematics as a major component. The general results of this were similar. Anderson's examination items and results are reproduced in Appendix 6. Of this sample, 11% and 15% respectively fail to interpret correctly the meaning of "necessary" and "sufficient" conditions, taking the converse of the correct statement; and, further, there are 20% who, without making this error, still fail to recognise the truth of the contrapositive. Some other items show the difficulty of statements containing both necessary or sufficient and the negation of an "or". Not relevant to our immediate point but very relevant to the question of the concept of proof are the results of an item where a total of 24% of students regard a relation satisfied by infinitely many integers as true for all integers. Again the question arises of whether these difficulties could be eliminated by more persistent teaching. Although this might well be the case, and might produce better results on items explicitly involving these points, the errors are likely to remain when the points arise without warning in the course of normal mathematical work.

If logic is so hazardous, can school pupils be expected to make any worthwhile progress with proof? Everyday thinking has its own safeguards - many supporting strands to the argument rather than one, more frequent checks for correspondence between the results of deduction and known facts, and a distrust of over-long or intricate chains of reasoning, which signals caution. The more mathematical, abstract, symbolic the material becomes, and the farther one builds out along a single line of deduction from the known, the greater self-protection against error is needed. This is no doubt why the tradition has developed a well defined form for proof which both provides some protection by its form, and also is a public display, for ease of checking by others. For pupils, there is no *need* for proofs to comprise lengthy chains or to use sophisticated logical steps. It may be that teachers have sometimes overestimated pupils' ability to recognise the validity of an argument. Later work in this chapter will show that logical complexity is by no means the greatest obstacle to pupils' development in proof; more severe limitations are the lack of general concepts and skills of proof, such as the need to consider all cases, to identify data and conclusion, to connect them logically and to embed the result in existing knowledge.

EXISTING RESEARCH ON PROOF

Reynolds' (1967) study covered parts of all the aspects of proof distinguished above. He gave two 20-item tests to a large sample of pupils in each of the forms first, third, fifth, non-mathematical sixth and mathematical sixth in a number of grammar schools. The results of 22 of the items were analysed, these items being classified under the headings Generalisation, Symbols, Assumptions, Converses, Reductio ad Absurdum and Deduction. The items are brief and, in many cases, ask the pupil a question about some aspect of proof rather than giving him a piece of mathematics to do. However, some do involve continuation of a deduction and two of the 22 analysed require the solution of a mathematical problem.

In Reynolds' general conclusions he compares his results with the expectations from Piagetian theory, which were that the responses should fall "into two broad categories according to the degree of completion of their cognitive structures", the first and third forms exhibiting the "acquisition of formal thought" (Piaget's stage IIIA), the fifths and sixth the "full use of formal thought" (IIIB). (Reynolds (1967, p. 188). He found that even the sixth formers showed evidence of formal thought in this sense only occasionally, and that the general picture was one of a steady improvement with age, with a substantial amount of concrete thought at all levels, including the mathematical sixths.

Reynolds' questions are designed to test the possession of an axiomatic concept of proof; he shows that this is largely absent among school pupils. But since he does not define intermediate stages, his detailed results are difficult to interpret. Two questions invited false generalisations from examples; one suggested considering which was the larger of 2^n, $2n + 1$ for $n = 1$, $n = 2$; and what deduction could be made. The other gave sixteen examples of pairs of even numbers expressed as the sum of two primes (Goldbach's conjecture), and asked whether these facts showed the truth of the conjecture for *all* even numbers. Over 15+, a majority avoided the former trap, but 75% of fifth formers and 20% of mathematical sixths accepted the latter.

In terms of the stages defined in the next chapter, this shows a minority of first year pupils still at Stage 1 - abstraction of relationship without sense of explanation, deduction or verification - and large numbers throughout the age range at Stage 2 - Generalisation, with check. Stage 3 - Proof, all cases - cannot be identified from Reynolds' questions. The greatest improvement over the ages 11-18 was shown in the use of Reductio ad Absurdum; about 60% of mathematical sixth formers used it successfully, but only 17% of first years. (It will be seen later that the present research also shows the relatively easier learnability of particular nameable skills as compared with more general concepts of explanatoriness.)

Thus, although Reynolds' study serves to confirm the difficulty of logical aspects of axiomatic proof, it does not provide much information about what kinds of proof activity are accessible to pupils.

King (1973) reports the development and testing of a unit of instruction on proof for able 11 year olds. The subject matter consisted of six theorems of the kind suggested by the Cambridge (Mass) Conference on School Mathematics (1963), for experiment with pupils of this age:

Thm 1. If $N|A$ and $N|B$, then $N|(A + B)$
Thm 2. If $N|A$ and $N|B$, then $N|(A - B)$
Thm 3. If $N|A$ and $N|B$ and $N|C$, then $N|(A + B + C)$
Thm 4. If $N|A$ and $N\nmid B$, then $N\nmid (A + B)$
Thm 5. If $N|A$ and $N\nmid B$, then $N\nmid (A - B)$
Thm 6. There is no largest prime number.

It is necessary to distinguish immediately between the content of a theorem and its formal statement. Many 11 year olds would be aware of the truth of the theorems 1-5 but few would be able to understand them and still fewer to prove them, in this form. In fact, we could describe four forms of statement for any theorem:

(a) implicit, unverbalised awareness
(b) informal statement
(c) formal statement
(d) symbolic statement.

A formal statement of theorem 1 might be "If two numbers A and B are both multiples of a third number (N), then their sum (A + B) is also a multiple of the third number (N)". The bracketed letters may be left out; either way we would call this form (c). An informal statement might be any pupil's rendering which was essentially equivalent to this. Much of our work with pupils of the ages in question is conducted with informal statements. (There is doubt about whether it is best to leave pupils at level (a) or take them to level (c)). In King's study, it would appear that pupils were required to reproduce and understand the proofs in symbolic form e.g. for theorem 1, something of the form

$$N|A,\ N|B \Rightarrow A = KN,\ B = LN \text{ for some } K,L.$$
$$A + B = KN + LN$$
$$= (K + L)N$$
$$\text{Hence } N|A + B.$$

The meaning of the "divides" symbol, practice in transforming between $N|A$ and $A = KN$ (numerically and symbolically) and in the distributive law similarly, were all included in the teaching programme; the general proof was *derived* from numerical examples, but then done symbolically. The criterion for understanding was to be able to (i) generate numerical examples of the theorem (ii) apply it to given numerical data (iii) prove theorems 2, 3, 5 on their own, having been taught the others (iv) explain and defend each step. Of these, (i) and (ii) are concerned with the content of the theorem, not its proof, but (iii) and (iv) could provide indications of the pupils' understanding of the proofs.

The ten pupils (mean IQ 117) studied the material for 17 days, long enough being allowed for 80% of them to achieve 80% success on each aspect of the work and tests. Thus it appears that 11 year old pupils can be taught to understand and construct proofs, given sufficient intensive teaching. However, the learning of the *content* of these theorems and their application to numerical examples would not be expected to cause any difficulty to bright 11 year olds; and the theorem-proving criterion test required minimal transfer from the learning situation - from A + B to A - B and to A + B + C. One might conjecture that familiarisation with the symbolic notation and the acquisition of some sense of the generality of the set of numbers referred to formed major parts of the learning. That some difficulties of this kind were experienced by the pupils is evident from the account of the stages in the development of the instructional unit for Theorem 6. At first it was shown that if $P_1, P_2 \ldots\ldots P_n$ was the entire set of primes, the integer $Q = (P_1.P_2.P_3 \ldots .P_n) + 1$ was not divisible by any of the assumed complete set of primes, and so was a prime, contradicting the assumption. $P_1.P_2.P_3 \ldots\ldots P_n$ was subsequently replaced by $2 \times 3 \times 5 \times \ldots\ldots \times P$, P being the assumed largest prime, and then the argument was taken inductively thus: if 2 is the only prime, consider 2 + 1: this is not divisible by 2, so is another prime - contradiction. Now if 2,3 are the only primes, consider $(2 \times 3) + 1$: and so on. A desk computer was used to compute products such as $2 \times 3 \times 5 \times 7 \times 11 \times 13$. This provides an example of pupils' inability to conceive first a hypothetical set of all primes, or a hypothetical (in fact, non-existent) largest prime, and their need to have the multiple products computed before they became sufficiently concrete to be worked with. This is all predictable from Piaget's theory in general, and from Collis's (1975a) results in particular. The method of proof by contradiction also caused difficulty and was made the subject of a cartoon story in the teaching programme, as well as being illustrated by many concrete examples. Thus although King's report of his study omits some important aspects of children's performance, it does show that the "all cases" aspect of proof was not readily appreciated.

Thus King's study essentially shows that 11 year olds can be taught to express given short, simple arguments in a symbolic and systematic form. It does not imply their ability to construct arguments of this type for situations in which they have not been trained, whether familiar or not.

An experiment by Collis (1973) on pupils' ability to work in a defined mathematical system, shows difficulties similar to those experienced by a pupil who is attempting to work from stated assumptions or axioms. Subjects were required to work in a novel arithmetic system with an operation * such that $a * b = a + (2 \times b)$. (Numerical examples were given) Three parallel tests were given, one involving letters, one small numbers, and one big numbers. Each test contained 5 items; we give the first and the third:

	Test 1	Test 2	Test 3
1.	$a * b = b * a$	$4 * 6 = 6 * 4$	$4728 * 8976 = 8976 * 4728$
3.	$a * x = a$	$4 * 5 = 4$	$4932 * 8742 = 4932$

In Test 1, subjects were asked when the statements would be true; in the others, whether they were true or false. "Can't tell" was an option in each case. The test was given to 30 pupils of each age from 7 to 17. The 8-12 year olds almost universally ignored the definition of * and replaced it by the familiar + or ×. Subjects of about 13 years began in this way but typically stopped at item 3, where + or × did not make sense, and went back to the beginning, trying to interpret * properly, but lost control of the situation. No subjects below the age of 16 succeeded in working correctly within the defined system; only 26% at age 16 and 63% at 17 achieved this.

What seems to be happening here is that since combinations by * are neither already memorised nor the subject of a known algorithm, the pupils are forced to work from the rule each time. This implies tolerating a lack of closure, in a similar way to that required when asking, of a theorem needed at a point in a proof, "Is this theorem a previous one in the deductive sequence? Am I allowed to assume it?"

The main part of the axiomatic concept of proof: an awareness that deduction must proceed from identified starting points, and an ability to avoid making unrecognised assumptions in the course of the argument. The implication of Collis's experiment is that this concept of proof would not be possessed by these pupils in general below the age of 17; of course, the possibility of younger pupils being *taught* this awareness is not excluded. This agrees with Reynolds' results. King and Reynolds also confirm each other in showing that the concept of "all cases" is not readily appreciated by younger secondary pupils, nor even held securely by older ones.

CHAPTER 7

PILOT STUDY OF PROOF ACTIVITY OF SECONDARY SCHOOL PUPILS IN AN INTERACTIVE SETTING

IN APPENDIX 7

Pupils' work (Networks)

INTRODUCTION

The aim of this research as a whole is to probe more deeply the levels of generalisation and proof activity at which pupils at the secondary level are able to work with understanding. The hypothesis which one would draw from existing English practice - the disappearance of geometrical proof from O-level syllabuses, and the SMP policy of "arguing the likelihood of a general result from particular cases", on the assumption that recent trends in English practice are based on experience of what pupils can understand - is (1) that the normally attainable level is that of recognition and description of a relationship, without any sense of explanation or deduction, or any consciousness that a generalisation is essentially an assertion about all members of a class of cases, (so that unless some appropriate method of verification is used, the assertion remains a conjecture rather than an established result). A second part of this hypothesis would be (2) that the axiomatic concept of proof, as requiring explicit starting assumptions and stated definitions, is even farther from attainability.

The informal pilot work to be described here aimed at exploring how far the concept of proof as (1) covering all cases and (2) needing starting axioms and definitions, was present in pupils of secondary age. Observations of the possession of corresponding skills, and of the teaching conditions favouring the development of these concepts and skills, were also intended. The results of this pilot work were used to formulate stage descriptions which are the basis of the systematic age-cross-sectional study reported in the next chapter. It is therefore unnecessary to report on age and stage aspects of the pilot work here. However, a number of relevant concepts emerged from these interactive discussions. These will be described and illustrated.

PROCEDURE

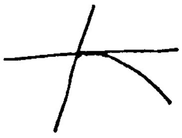
This pilot study is based on sessions conducted during the summer term, 1974, with seven groups, of six pupils each, at local comprehensive schools. These were from one first form, one second form, three fourths and two sixths; the pupils were of average ability relative to their year. The problems were first discussed informally with the pupils and they were then left to continue on their own, working in pairs, recording their work and writing results and conclusions. This work was interrupted occasionally for discussions designed to elicit the proof stages at which the pupils were working. Five situations were used. The group investigation of Networks will be described in detail; the discussion of Axiom Systems more briefly; for the others, conclusions will be reported.

GROUP INVESTIGATION OF NETWORKS

A group of six 15 year olds (from the lowest GCE stream in a 7 or 8 stream comprehensive school) investigated the possibility of drawing networks with given numbers of junctions of given order. The theorem in the background was that it is impossible to draw a network having an odd number of odd junctions, but this particular theorem was never formulated. We started by trying to draw some given networks unicursally and from this it appeared that the order of the junctions was a relevant factor. Eventually, we restricted ourselves to networks whose junctions were of orders 3, 4 and 5 only, and recorded the number of junctions of each of these orders. The table of these was as follows:

Order	3	4	5
1st figure	2	0	0
2nd figure	2	2	0
3rd figure	2	3	2

I then asked the group to draw (in any number of strokes) a network having 2 junctions of order 3,1 of order 4, and 1 of order 5. (We called such a network a (2,1,1) network subsequently.) "I don't think so" one boy responded immediately, "because the even junctions have to add up to counter-balance the odd junctions." The pupils carried on with work on this, in pairs. After a few minutes, in which the experiments seemed to indicate that (2,1,1) was impossible, it was suggested that they should broaden the investigation and try to formulate rules for what networks were possible and what impossible; starting by compiling a table which extended the one shown. One raw conjecture which emerged soon was "(1) even junctions outnumbering odd ones (2) with more complicated networks the bigger the proportion of small junctions, i.e. three's (3) any number of 3s and 4s. This boy's sheet of working is reproduced to give some impression of the kind of network which he and his partner had been considering. (Appendix 7) Another pair, determined to find a (2,1,1) network, had bent the rules to allow a line to approach a point tangentially to another line as in the diagram, making this a 5-junction. The third pair had a table including (2,1,1) (3,1,1) (2,1,2) (2,3,1) (1,2,1) (3,2,1), all of which they claimed to have drawn. I asked them to check their (2,1,1), which they found to be a (1,1,1).



At this point I decided to help them all to formulate a precise conjecture which they could hope to test and confirm or refute definitely. The first-mentioned group's statement, even in the form produced when I asked them to write it as a definite statement, was still too vague to refute (App.7 JS $_2$, at the top). Eventually they agreed to test the proposition (1) "It is impossible to have one 6-junction and one 10-junction without at least four 3-junctions."

The third pair had still not found a (2,1,1) network and said "I think it's impossible, but I don't know why....we can get a (1,1,1) ....," so I suggested they should test (3) "It is impossible to obtain a (2,1,1) network from a (1,1,1)".

The second pair had on their own formulated (2) "Even sets (2,2,2), (4,4,4), etc.) are all possible, odd sets are impossible." This has arisen because they had invented standard ways of producing 3-junctions, 4-junctions and 5-junctions (see Appendix 7).

These three conjectures were written up on the blackboard at this point and the pupils invited to try to prove their own conjecture and to disprove the others. It was immediately pointed out that (2) was wrong because we had a (1,1,1) network; so that was amended to read "odd sets, except (1,1,1), are impossible". At this stage, all three groups were fully convinced of the truth of their own conjectures, and that further efforts to prove them were superfluous; and they were not much interested in other people's conjectures. However, they agreed to try.

A few minutes later I had drawn a (3,3,3) network, and at the same time the second pair had produced a network refuting conjecture (1) - which they displayed with enthusiasm. Two of the conjectures had now been destroyed. A little while later the third pair claimed to have a proof of their conjecture and this was explained to the class. There was some discussion about whether this was a proof. They volunteered that "this is only one (1,1,1) network; to prove it you would have to draw all the (1,1,1) networks that you can." This was readily agreed to be the case by the rest of the group. "How many can you?" I asked; "You could never draw all the (1,1,1) networks there are....could you state the argument in such a way that it would clearly apply to all (1,1,1) networks which could be drawn?" The session had to end without the chance to pursue this point.

OTHER PROBLEMS

Odd and Even was a collection of questions about whether the results of adding or multiplying two even numbers, two odds, or an even and an odd, are always even, always odd, or sometimes even, sometimes odd; together with some similar questions about consecutive numbers; and about multiples of three. The pupils were asked whether they were sure their conclusions were always true, including for big numbers.

Coins A was the problem: Given three coins all showing heads, by a succession of moves each turning over two coins, obtain three tails. The extension to four coins, turn three at a time was used with some groups. Coins B used 3 coins in a row, and the two permitted moves were P: "turn over the left hand coin, interchange the positions of the other two," and Q, similarly turning over the right hand coin. The task was, Given TTT, obtain HTH.

Diagonals of a Polygon concerned the relation d = s - 3 between the numbers of sides and diagonals of polygons. After the initial investigation, a proof by considering the diagonals radiating from a single point was shown, and questions of the validity of this, and of applicability to non-convex polygons were raised. (A fuller discussion of this problem and of Coins A may be found in Chapter 9).

AXIOM SYSTEMS

One group of six 18 year olds also took part in a general discussion on proof intended to probe for any evidence of awareness of the need for explicit "axioms", and of their arbitrary nature. This was not found. The first question raised was about the subtraction of negative numbers - they could not formulate a general statement for this rule, nor had they any idea how one might prove such a statement, nor that definitions would need to be made.

Next they were asked to prove the exterior angle theorem for a triangle. In the course of discussion they showed an appreciation of the invalidity of a circular argument, and their response to a question about starting points was, "You have to go back to where everybody's knowledge is basic, to things that everyone knows or assumes." But they could not suggest what were the most basic things in geometry.

Thus although there was an awareness that a deductive sequence must start somewhere, there was certainly no distinction made at that point between self-evident truths and explicit axioms chosen with hindsight.

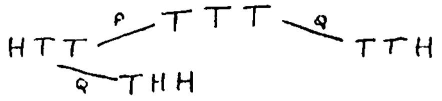
RESULTS

Dependence of level of proof on familiarity with the concepts involved in the situation. In *Odd and Even*, proofs of the results for dvisibility by three consisted of trials with one or two small numbers, and sometimes (for the fourth formers, not the second formers) a trial with big numbers. The results for odd and even numbers were proved by general arguments including the step that only the last digit is relevant.

Acceptance of best level of proof attainable. This is another aspect of the above episode. Also, in *Diagonals of a Polygon*, the sixth formers found, or accepted when shown, the proof for radiating lines, and realised that it did not cover non-radiating diagonalisations. They were, however, fully convinced that the *result* applied to these cases, their conviction resting on the check of a single example. They were not prepared to withhold judgement, nor did they recognise that the proof of the more general case might reveal new aspects of the situation.

Global approach to a situation. Most of the second formers who tried the extension of Coins A to "four coins, turn three", decided that either both versions were possible, or both impossible, because of their similarity. A similar tendency to view a situation globally, and not to separate out different aspects of it, was identified by Lunzer (1973b) in the responses of pupils of similar age to the problem of the variation of the area and perimeter of a rectangle.

Symbolisation; the adoption and use of a diagrammatic representation for a problem may present difficulties comparable with those of solving the problem without it. After three pairs of fourth formers had attempted Coins B, a representation of states and moves as in the diagram was shown to them. Only the pair who had already solved the problem were able to extend this diagram and thus confirm their result.



The possible difficulties in adopting a useful representation were also shown in some first formers' attempts at Coins A. Invited to record the game in their way, one girl wrote "Turn 10 and 2, turn 2 and 5....." (They were using 10, 5 and 2 penny coins); others wrote

TTT	and TTT
H H	HTH
T H	THH
...	

All these can be interpreted as a *tendency for the actions, rather than the states, to dominate the thinking.*

The remaining sequence of observations centres on the *degree of detachment from the concrete* which pupils have achieved, as shown by expectation of rationality in the situation, their recognition that they are working with a *rule governed situation*, their awareness of the need for *written statements of generalisations and definitions* which must be treated literally.

The *expectation of rationality* is what appeared to be lacking when, on Diagonals of a Polygon, pupils in various groups were prepared to accept and leave unchecked, non-confirming entries in their list of numbers of sides and diagonals. The non-awareness of its being a *rule governed situation* was shown when, on Coins A, some first formers asked whether it was absolutely impossible or just too difficult for them, suggested that *I* might find a way of doing it, or perhaps a computer would succeed; some of the fourth formers attempting Coins B were similarly not clearly aware of the deterministic character of the situation. Another possibly significant factor was the incidence of cheat moves. In both coin problems, in first, second and fourth year groups there were some pupils who tried, for example, standing a coin on edge, or flicking one over surreptitiously, or other infringements of the rules. They did this, knowing that it would be rejected, but nevertheless could not resist it. Similarly, in

Networks, there was the attempt by one pair to get 5 lines out of an even junction by making two of them tangential at the point. In Networks, we also saw the necessity for *written statement of a conjecture* and the refinement of it into testable form. Finally, in the non-convex cases of Diagonals of a Polygon, the sixth formers showed a lack of ready awareness that the result would depend on the *definition* of diagonal adopted; they felt that there was a correct definition which they could argue about but which the teacher would identify for them.

DISCUSSION

The above observations include some which help to give substance to a hierarchy of levels of proof, and others which suggest caution in the application of such a hierarchy. Relevant to levels of proof are (1) the expectation of rationality, related to the appreciation of the need for, and value of, checking apparently non-conforming cases; (2) the recognition of the rule-governed nature of a situation; of the need for written statement of a conjecture if it is to be defended publicly; and of the value of recording states, not just actions. These concepts seem to correspond approximately to the level of recognising that *all cases* need checking to establish a generalisation; some, but not all, fourth formers showed awareness of them, but younger pupils generally did not.

The recognition of a situation as rule-governed, and the identification of the rules, are equivalent to Dienes' (1963) essential step of mathematical abstraction, which, he considers, occurs when, and only when, the common characteristics of two or more embodiments of a structure are identified. (In fact, as his protocols show, this abstraction from several embodiments is a very difficult mental act, and does not seem to be necessary for the acquisition of a concept, though it may represent a powerful extension of it.) Higher levels of proof activity, such as the recognition of the need for definitions, and of identified starting-points for arguments, were observed only in the sixth formers, and appeared to be not very highly developed.

These all represent steps along the road from the conception of knowledge as existing in a ready-made form in the outside world, waiting to be hacked off bit by bit and absorbed by the learning child, to the sense of mathematical knowledge as comprising closed systems of propositions, logically related to each other, and having contact with the outside world at a number of points, but not having a *deductive* relationship to it. This development requires what Piaget identifies as the chief new characteristic of adolescent, as opposed to child, thought: "a reversal of the direction of thinking between reality and possibility in the subjects' method of approach." (Inhelder & Piaget, 1958)

Two indications for teaching may perhaps be inferred from these observations. First, the value of the public classroom situation for promoting the development towards higher levels of activity; and second, the probable value of identifying and naming the concepts discussed in these paragraphs, as they arise from such activity.

The two cautionary observations are the dependence of proof level on familiarity, and the acceptance of the best level obtainable. These suggest that the optimal development of proof strategies will not be achieved in the context of only partly-assimilated ideas, hence that there is an incompatibility between content and process learning. They also provide a reminder that pupils will not necessarily operate in every given situation at the highest level of which they are capable; in attempting to establish stages this will need to be borne in mind.

Little comment has been made about the observation regarding symbolisation. The immediate inference is that more experience with a wider range of modes of symbolisation might be beneficial. This is potentially a very important aspect of the mathematical process and deserves a substantial study of its own.

Finally, the broad stages of development towards proof which emerge from this study are:

Stage 1	ABSTRACTION Recognition, extension, description of pattern or relationship.
Stage 2	CHECK Empirical check or attempt at deduction.
Stage 3	PROOF Awareness of need to consider *all possible cases* and to state conditions for truth.
Stage 4	AXIOMS Awareness of need for explicit statements of starting points of arguments, and of definitions used.

CHAPTER 8

A STUDY OF STAGES IN THE GENERALISING AND PROVING ACTIVITY OF PUPILS AGED 11 TO 17

<u>IN APPENDIX 8</u>

INTRODUCTION

In spite of the dangers of determinism which are inherent in stage concepts, especially if related too closely to ages, the Piagetian designation of stages of concrete and formal thought has provided robust concepts which aid the understanding of child and adolescent modes of thought and assist in the design of effective teaching materials. Piaget and his collaborators have gone further, and assigned the learning of number and its operations, and of measures of length, area and other quantities to particular stages. More helpfully for teachers, perhaps, these experiments have identified stages *within* these concepts; the steps from the concept of area to its measurement by area units, and subsequently by calculation are an example (Piaget, Inhelder & Szeminska, 1960). Other workers have established stages in other mathematical concepts, for example that of function (Thomas, 1975; Orton, 1970). It is plausible that in the understanding of proof, well-marked stages should exist; in view of its close connection with formal reasoning. However, of the relevant researches reviewed in the last chapter, Collis (1975a) shows an inability to work from definitions before age 17, and Reynolds (1967) shows the non-attainment of the axiomatic stage by school pupils, and considerable over-generalisation in the early years. The pilot study reported above led to the tentative proposal(p.7.10) of the four stages (1) Abstraction, (2) Check, (3) Proof, (all cases) and (4) Axioms. Using the pilot work as a guide, the present study seeks to establish such stages, if possible, and to describe them in terms of a range of pupil behaviours. The method is to give the same test problems to a class of pupils of each year group throughout the secondary school age range.

HYPOTHESIS

The pilot work suggests that the second stage will predominate, but that there will be a substantial amount of first stage performance in the first year or two and a fair amount of work at the third stage at age 15 and above.

SAMPLE AND PROCEDURE

Two problems were used, one concerning numbers, one from geometry. The first required the making of a generalisation and giving of reasons for it, the second offered a generalisation and asked for it to be tested.

The problems were given in written form to one class of girls of each age from 11 to 17 inclusive at a girls' grammar school mainly serving a large council housing estate. The 11-15 year classes were chosen from the middle of the ability range for the school; the lower sixth formers were a smaller than average group of girls who were available at the required time - none of the latter were studying mathematics for A-level. In each class, half the girls did the number problem and half the geometrical one; they were given as much of the 40 minute period as they needed to complete the work. In all, about 80 pupils did each problem.

STAGES

A preliminary scrutiny of the scripts led to the adoption of the following elaborated stage descriptions for the classification of the responses. Stage 4 did not appear, and Stage 0 was required for unsuccessful responses.

Stage 0: Non-recognition of relationship, regularity or pattern. This includes non-expectation of regularity in the given situation and also the inability to work in the situation with sufficient accuracy or consistency to observe the regularity existing in it.

Stage 1: Abstraction: Recognises pattern or relationship in given data; can extend verbally or symbolically. Does not seek to explain it or deduce it. If reasons are requested, they may be given but they are regarded as concomitant facts, not as justifications of the statement of relationship.

If a relationship is proposed, checks it in just one or two cases; regards it as true or false as an indivisible entity, not as comprising a class of cases each needing independent consideration.

Stage 2: Check. Recognises that a statement of relationship applies to a class of cases, so that either a variety of cases must be checked and a probable inference made that other cases conform, or a deductive argument or global insight that covers implicitly a class of cases must be supplied.

At Stage 2.1 the variety of cases checked is neither great nor systematic; there is little awareness that the extrapolation is only probable, and deductive arguments consist of fragments not firmly linked to data or conclusion or to each other.

At Stage 2.2 there is greater variety, more systematic choice, more cautious extrapolation, connected though incomplete deductive chains and the use of 1-1 correspondence and of iterative arguments.

Stage 3: Proof, all cases. In this stage, there is full awareness of the need to deal with all cases (except possibly special or extreme ones like 0 and 1), so if empirical methods are used it is with explicit acknowledgement of their limitations; deductive chains are complete (or recognised not to be) and apply to the whole class of cases of which the subject is aware.

PROBLEMS AND RESPONSES

Double and Add the Next

> "Start with 6; double it; 12; add the next number.
> 12 + 13 = 25   Start 6, finish 25.
>
> Starting with 12, double it; 24; add the next number.
> 24 + 25 = 49   Start 12, finish 49.
>
> Do two more like this and note the starting and finishing numbers.
>
> Now we shall decide what finishing number we want and try to find what starting number we need.
>
> Can you finish on 13? What starting number do you need?
>
> Can you finish on 21?
>
> Can you finish on 14?
>
> Find some rules about what numbers you can and can't get as finishing numbers.
>
> Find also a rule for finding the starting number for a given finishing number.
>
> For each rule, say whether it is always true, or only sometimes, and give reasons."

The most obvious rule regarding finishing numbers is that even numbers are impossible and most subjects found this. The majority were also able to give a reason for it, though the reasons varied in explicitness. Only three subjects out of the 80 who did this problem observed that the possible finishing numbers were not *all* odd numbers, but only those of the form 4n + 1. (The even and odd dichotomy seemed too strong to break out of; many subjects, asked subsequently to see whether 15 was a possible finishing number, said "Yes, start with 3½" without recognising that the introduction of fractions removed the basis for the rejection of 14.) The obtaining or not of an explicit reversal rule - e.g. subtract 1 and divide by 4 - was the aspect of the problem which showed the strongest relation to age through the range considered.

Stage 0 comprises pupils who fail to make a generalisation whether about the finishing numbers or the reversal rule. There were two subjects in the first year and three in the second in this category.

> 1AH is inconsistent, sometimes using the function $x + (x + 1) = y$ and sometimes the correct one. Of rules for finishing numbers, she says "There is not any"; for the reversal rule she repeats the forward process $6 \rightarrow 12$, $12 + 13$. This is a clear case of low expectation of regularity and of insufficient skill to extract the generalisation.
>
> 1LH* says "You cannot have 14 but you can have numbers like 53, 13, 21:" she fails to extract the even/odd distinction.
>
> 2RB says that you cannot get "numbers like 14 which you subtract down to 7...."
>
> 2NH gives no rule for finishing numbers and for a reversal rule says "Do the sum again."
>
> 2JM$_2$ has a wrong rule $(4x + y)$ i.e. adding on any number for the "next number". She also has mistakes in her working, and infers "You can't finish on an odd number" which is consistent with the cases she has generated, but not with those given on the sheet (25,49)

Stage 1 was the predominant stage for the 11 and 12 year old groups; and persists in smaller numbers through to the 16+ group. These pupils obtain the even/odd relationship but offer no reasons (except possibly restatements of the data or irrelevant comments.)

> 1KH$_2$: "You cannot get even numbers as finishing numbers, only odd ones". (No further comment or reason)
>
> 2HK says "From 13 it is always the odd one." No reason.
>
> 6TQ: "You can finish with numbers 13 and 21 because these are numbers in which you can go in twice and have one number left over....." with three further rules, one of which is wrong. In spite of the word "because" this merely states that odd numbers are possible, and gives no reason.

Stage 2 comprises pupils who supply relevant reasons. It is perhaps surprising that empirical checks of the even/odd rule, going beyond the cases obtained in response to the questions on the sheet, do not appear in the scripts. The differences are between those whose reasons form a connected argument linking data and conclusion, and

* Scripts whose numbers are starred appear in Appendix 8.

those whose reasons, though relevant, leave gaps in the deductive chain. In the case of the reversal rule, there are similar differences between fully explicit rules and those which leave a gap to be filled by trial and error.

A full proof of the even/odd relation involves stating

a) doubling the starting number gives an even number
b) the next number to this even number will be odd
c) the finishing number is therefore even + odd, which is odd.

Stage 2 responses omitted one or two of these points. Some extracts from Stage 2 responses are:

> 1JI: "You cannot finish on an even number because if you double an uneven number the answer will come to an even number. e.g. 7 → 14 → 29."
>
> 3CY: "You are able to finish only on an odd number because an odd and even number also make an odd number."

A fully explicit reversal rule was:

> 4JC: "To find the starting number take one away and divide by four."

A typical partial rule would give an explicit rule for part of the process and leave the rest to trial and error.

> 1BM: "Find consecutive numbers which add to the finishing number, then halve the lower one."

Stage 3 requires a full proof of the even/odd relation and an explicit reversal rule. Another characteristic of some Stage 3 scripts was the full and explicit realisation of the relation between the exclusion of fractions and the validity of the rule for finishing numbers. 6AW is typical of several 16 and 17 year old responses.

> 6AW: "To finish on 14 you would need $3\frac{1}{4}$": then "A final number must not be an even number as the number you start with will not be a whole number." She has an explicit reversal rule, and also points out that it will not be true for numbers below 5.

This displays a very clear awareness of the link between the data and the conclusion which is equivalent to a "complete deductive chain."

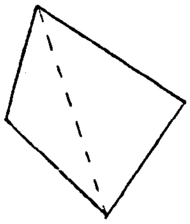
One of the few pupils below the sixth form who found the 4n + 1 rule for finishing numbers was 3AT: she used a systematic empirical approach:

> 3AT: Shows 3 x 4 + 1 = 13 and 5 x 4 + 1 = 21; finds finishing numbers for 1, 2, 3, 4, 5 and says "You cannot get 1, 2, 3, 4, 6, 7, 8, 10 and so on as finishing numbers because they do not come into the pattern of adding 4 each time." Checks 101, 105 and says "This rule is always true so far but you would have to go through this procedure for every number to certify this."

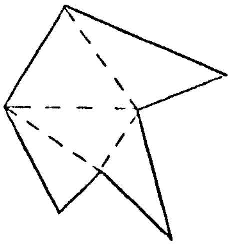
Diagonals of a Polygon

A. Draw a polygon. Draw as many diagonals in it as you can, without any of them crossing. How many are there?

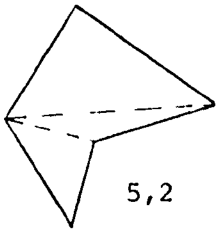
B.



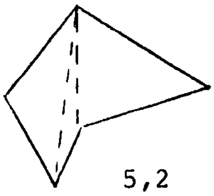
4,1



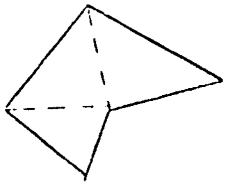
7,4



5,2



5,2



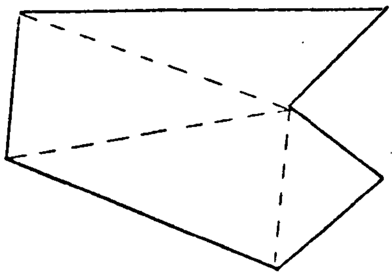
5,2

Some diagrams have been drawn here. It seems that "The greatest number of non-crossing diagonals which can be drawn in a polygon is three less than the number of sides."

Is this statement true for all polygons?

Investigate this fully; then state your conclusions and your reasons.

The most notable feature of the responses to this problem was the interplay of empirical and deductive work. The youngest pupils drew their conclusions almost entirely on empirical grounds made firm assertions that the statement was "true for all polygons" based on the inspection of only the cases presented on the sheet. Middle school pupils generated a conscious variety of cases to check, and tended to make more cautious assertions. Deductive arguments became steadily more relevant, more coherent and more complete; only a handful of pupils gave complete proofs of the relationship, and these were all based on the radial lines figure, and failed to recognise that this way of drawing diagonals was not possible in all polygons.



No pupil (except possibly one third year) appreciated that the maximum number of diagonals might depend on where they were drawn. A full deductive proof at this level would state that if diagonals were drawn radiating from a single point, they go to all but three of the vertices - these being the point itself and the two adjacent vertices. Thus the number of diagonals is three less than the number of vertices, which is the same as three less than the number of sides.

Stage 0: Five pupils in the first two years failed to recognise the potential regularity in the situation, or had insufficient knowledge or skill to achieve a result. There was also one such response in the 7th year. (5,2 refers to a polygon of 5 sides and 2 diagonals, and similarly.)

> 1JG: 5,2 shown. "Not true because polygons can have any number of sides and diagonals."
>
> 1DG*: 17,11 shown. "Not true because some polygons can be bigger and some smaller."(Diagonals go to points other than vertices.)
>
> 7JA: 4,1 shown. "All the polygons drawn in the diagrams have different numbers of sides, from 4 to 7, therefore the diagonals.......must differ."

Stage 1: These assert "true for all" from a small number of confirming instances or else "not true" if one case fails. If reasons are given they do not go beyond restatement of the data.

> 3SW*: One figure (8,5) shown. "Yes, it is probably true for all polygons."

* Scripts whose numbers are starred appear in Appendix 8.

Stage 2: These responses are distinguished from those of Stage 1 either by generation of a number of figures to check, or by the attempt to provide a deductive argument.

Stage 3 is marked by the achievement of a complete deductive chain, or, if this is not attained, by the empirical check of a variety of cases and explicit recognition of the limitation of the empirical method. Thus Stage 2 comprises a wide range of responses, differing in the degree of conscious variety and of systematic ordering in the cases checked empirically, and in the relevance and coherence of the deductive arguments.

Stage 2.1:

> 2AH: 1 figure. "Yes, this is true....you are dividing your polygon into triangles and if you add up all the degrees of the triangles it will come to the polygons degrees."
>
> 5SW*: Shows 3 different ways of putting diagonals into a pentagon. Says "true for all" and adds: square has 1, triangle, 0.

This gives no deductive reasons but does check carefully on different possibilities.

Stage 2.2:

> 6LP: 7 figures, 3-9 sides, in order: "From this evidence it seems that....."
>
> 2KJ: 4 figures, all radial. "True for all....2 nearest points cannot be joined......it forms a triangle in which no diagonals can be drawn.....so subtract 3!"
>
> 5JC: 8 figures, mixed, sizes 4-10. "For each of the above.... true....one side added.....then another non-crossing diagonal can be found....."

The first two of these are empirical responses with check of a variety of cases; the others are incomplete deductive arguments.

Stage 3:

5CB : 4 figures mixed. "True for all.....cannot be joined to next points or to itself....."

2KP: 1 figure. "Yes,.....can't go back to where it started.... or to the 2 on either side."

These are complete deductive arguments (expressed informally).

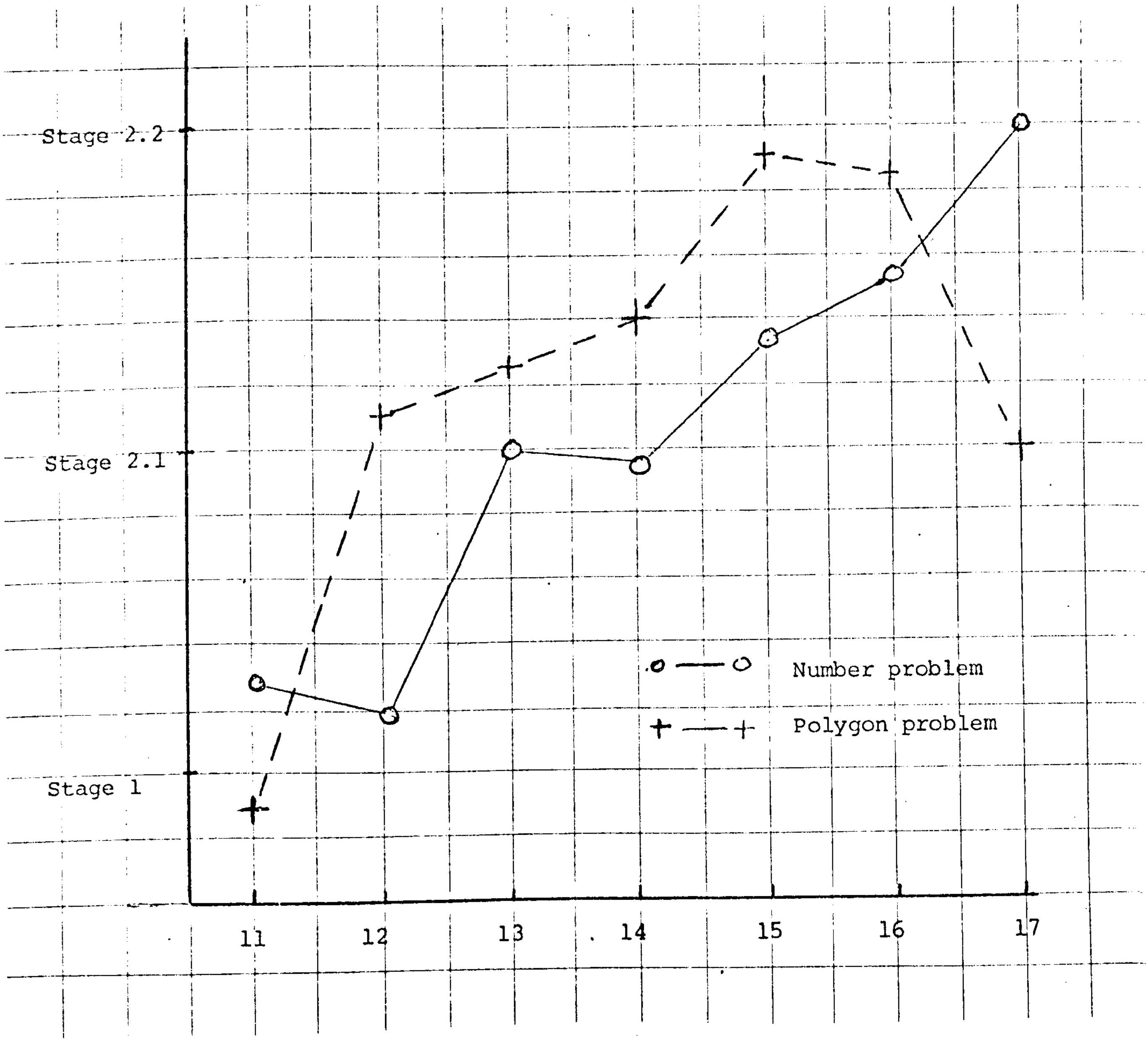
RESULTS

These are shown in the table and the graph below.

	NUMBER PROBLEM					POLYGON PROBLEM				
	STAGE					STAGE				
AGE	0	1	2.1	2.2	3	0	1	2.1	2.2	3
11+	2	7	3	1	0	4	7	2	0	0
12+	2	9	0	2	0	1	4	4	3	2
13+	0	3	5	1	1	0	4	2	5	1
14+	0	5	3	4	0	0	3	2	6	1
15+	0	2	3	3	1	0	0	2	5	1
16+	0	2	3	4	2	0	1	1	10	1
17+	0	0	1	1	4	1	1	2	1	1

NUMBER OF PUPILS AT EACH STAGE IN EACH YEAR GROUP

MEAN STAGE OF GROUPS OF DIFFERENT AGES ON TWO PROBLEMS OF GENERALISATION AND PROOF



DISCUSSION

Although the class means show a fairly steady improvement with age (except for one exceptional result), there is a wide variation within classes, with almost all stages appearing at almost all ages.

The non-conforming result of the 17 year old group on the polygon problem can probably be explained by the smallness and untypical nature of this group; their lack of contact with mathematical studies for 18 months has affected their approach to the unfamiliar nature of the polygon problem, though not to the number problem.

The hypotheses are generally confirmed but shown to be optimistic. Stage 1 is the mean at age 11, and there is a sharp rise between 11 and 13, but there is less evidence of Stage 3 among the upper forms than predicted. We must ask whether the stage reached depends on the particular problem, or whether there is a *general* capacity or preference for deductive proof. Neither of these two problems demands knowledge of concepts which would not be possessed by virtually all 13 year olds (though some younger pupils failed to understand "diagonal" and some had not the concept of even and odd numbers). However, the degree of complexity of the problem is relevant - the length of the deductive chain required, and the extent to which the concepts to be linked are exposed or hidden in the situation. It was thought that it might be possible to minimise variation from this factor and so to observe more clearly pupils' progress towards the *concept* of deductive proof, as distinct from *skill* in constructing proof, by asking them to select from a number of offered responses. The sheet containing three proposed proofs (Polygons II, See Appendix 8,) was given to a number of first and second year pupils when they had finished the first sheet; none of them selected the deductive proof. Later informal work suggested that though Stage 1 pupils select the empirical proof, most Stage 2 pupils choose the deductive one.

FOLLOW-UP

Response to Counter-Examples

Pupils in the first, second and seventh forms who completed their problem before the end of the period were given a follow-up sheet, designed to see whether they could modify their original generalisation or proof when faced with a counter-example. (See Appendix 8) In the case of the number problem, this asked them to consider whether 15 was a possible finishing number; the question was whether they would be able to extend the rule "even numbers are impossible" into the full 4n + 1 rule. Of the six first formers, six second formers and three seventh formers to which this applied, one in each of the first two groups (none in the third) was able so to modify her rule. Most of the rest responded "Yes, you must start with 3½" without noticing that the introduction of fractions removed the basis for excluding even numbers, like 14, as finishing numbers.

The counter-example offered following the polygon problem was designed for those who had adopted the radial lines proof; it showed a polygon in which there was no point from which radial lines could be drawn. This only applied to three seventh formers and none of these recognised the significance of the counter-example.

It would be of interest to explore this further, to see whether the ability to respond to a counter-example by modifying a generalisation or proof can be allocated definitely to one of the stages.

IMPLICATIONS FOR TEACHING

1. The critical question for the curriculum is how higher levels of deductive thinking may be encouraged. One prerequisite is clearly that the concepts being dealt with should be familiar;

learning new concepts is incompatible with rigorous establishment of relationships involving them. Thus there is a conflict for the teacher between teaching his pupils more advanced concepts and developing their deductive skills. This choice would be made easier if it were demonstrable that an emphasis on deductive skills conveys the power to learn new material more quickly and effectively. I think this is probably true, but the demonstration of it is a complicated task.

2. One of the steps towards deduction may be the acquisition of a taste for certainty;this may be acquired through problems based on small finite sets of possibilities where exhaustion is a feasible strategy. Some such problems have been used with 10 year olds. Whether certainty from exhaustive check or global insight without check (which seem to be the alternatives at this age) is the more satisfying to pupils would be interesting to study.

3. There are suggestions among the results that the strategies of reversal, systematic classification and ordering are capable of development, possibly helped by some explicit teaching.

   For example, on the polygon, what was required was either (a) the radial lines proof, consisting of the recognition of the value of this systematic way of drawing diagonals from a single point, and the making of a one-one correspondence between vertices and diagonals, with a fixed number of vertices not being involved; or (b) an analysis of how, if a polygon is built up, side by side, starting with a triangle, at each step a new diagonal can be drawn. (In the latter case, the difficulty is to be satisfied that the ways of adding a side which have been considered comprise all the possible ones. In fact, the most satisfactory way of making this proof apply to all polygons is to reverse it, that is to show that every polygon (with any given diagonalisation) can be reduced step by step to a triangle, and that at each stage this can be

> done so as to reduce the number of diagonals by one. An investigation of the awareness of this with older school pupils and university mathematics students would be interesting.

Thus, the methods of proof contain certain general strategies (a) *ordering* (in the drawing of the diagonals from a single point) (b) making a *one-one correspondence*, (c) *iterating* - building up step by step and attending to what is the *same* about each step. The question arises whether the development of awareness of these strategies is a feasible way of improving pupils' capacity for proof. Also necessary in this proof is an awareness of the class of possible cases which the actual diagram represents. For example, does a particular non-convex polygon represent all polygons better than a particular convex or regular one? Does a particular way of drawing diagonals adequately represent all possible ways? It might be said that this is the essence of mathematics - the art of dealing with the general by working with the particular. C.S. Peirce (1956) makes this point in relation to the geometry of Euclid and to algebraic proofs.

CHAPTER 9

A STUDY AND ANALYSIS OF PUPILS' GENERALISING AND PROVING ACTIVITY

INTRODUCTION

This experiment is designed (a) to provide a substantial body of evidence concerning pupils' abilities and achievements in relation to generalisation and proof, in the form of open responses to a variety of problem-situations, and (b) to test a number of hypotheses which emerge from the two-problem, age cross-sectional study described in Chapter 8, and from previous work. It was preceded by oral work with a number of individual pupils, and some written trials of questions; some additional hypotheses for testing came from this work.

HYPOTHESES

(1) In the age cross-sectional study, four stages were recognised in the responses of the pupils, showing a fairly strong relationship to age. The main points of interest were the development in connectedness of the deductive arguments, and the developing awareness of the meaning of generalisation over "all cases". However, it is clear that the two problems used, though very different, were a very small sample of the range of mathematical situations met by the secondary school pupil, and that the next task was to experiment with a wider range of problems. Another limitation of this experiment was that each pupil attempted only one problem. It was decided to design the present experiment so as to observe to what extent a particular pupil would perform at the same stage on different problems. For this purpose, it was necessary to modify the stage descriptions so as to emphasise criteria referring to observable aspects of the pupil's written response, rather than criteria relating to his stage of thinking, such as "regards a relationship as true or false as an independent entity, not as comprising a class of cases". These make it easier to link with Piagetian stages, but harder to achieve reliable stage-allocation. It was clear that the more satisfactory and rigorous approach was first to seek categories which would permit significant sorting and description of performance on the wider range of problems; then to consider aspects of performance which might be characteristic of

particular stages. Thus in the present experiment ten diverse problems are used, and each pupil attempts two. More problems from each pupil would have been desirable, but this was judged to be too great an incursion into teaching time. The hypothesis is that pupils' performances on their two problems will fall into the same or adjacent categories.

(2) From the pre-testing of problems (as well as from earlier work) a number of other hypotheses may be stated. These will not be formal hypotheses to be tested statistically since each refers to only a small number of problems which have other diverse characteristics; but informal comments relating to them will be made. With regard to context, it appears that generalisation arising from familiar non-mathematical activities can lead to high-level proof-explanations, or at least to high motivation to explain. The Coin-Turning problem and Noughts and Crosses are such situations included among the ten problems chosen. The degree of familiarity with the situation and of all the different relationships within it seem to be the operative factors; this is stronger for Noughts and Crosses. On the other hand, the geometrical situations pretested proved more difficult to explain than the number ones, because whereas number generalisations can be checked by trying the calculation with particular numbers, and this process often leads directly to a general insight, in geometrical problems the corresponding process is less well-defined - it involves not just choosing some numbers, but trying to make figures according to given conditions. Also, the basic properties and algorithms for number are well known and felt to be fundamental, whereas it is less clear what is fundamental in geometry, perhaps particularly so since the Euclidean system has disappeared from the curriculum. Hence we hypothesise that, in relation to context, proof levels will be lowest in geometry and highest in familiar non-mathematical contexts, with number situations coming between them.

(3) The next hypothesis relates to the set over which generalisation takes place. The strong tendency of pupils to generalise from a few examples, which was evident in the age cross-sectional study with

regard to the even/odd rule and the ready acceptance of the generalisation about the diagonals of the polygon, is a quite normal way of learning which we all use; we generalise first, then look for confirmation or refutation. How thorough we are in seeking evidence depends on what degree of certainty is desired in the particular instance - and what is possible. It appeared in pilot work that pupils were more interested in insight than in certainty; it also appeared that they were interested in that level of certainty which was easily attainable. Hence some situations were introduced based on finite sets ('Choose any number between 1 and 10') to see whether the method of checking all cases would be adopted, by those pupils who were aware of the 'all cases' aspect of generalisation, in preference to seeking a general insight. The response to these in the pilot interviews was that the checking could be done and would give the definite answer but was of no interest; it was 'roundabout', as one pupil put it. In the main experiment two number situations are set up in this way. The hypothesis is that the full check of the finite set will not be adopted as a method of justifying the generalisation.

(4) Another factor observed as promoting high motivation was the impossibility situation, but this was a motivation towards solving the problem rather than towards presenting a rigorous proof. The Coin-Turning problem and Stamps both produced this reaction in pretests. Pupils became intrigued when solution began to seem impossible and showed considerable pleasure and relief when they reached conviction that it was indeed impossible. However, their written explanations of their conclusions varied widely from their oral report. It seemed that a mental scanning of possibilities had led to a decision, but the organisation of this process and its presentation on paper was a boring task; so the giving of a complete and cogent argument was neglected. The hypothesis is therefore that the impossibility situations will show high levels of solution but some incompleteness of explanation.

(5) A further feature of the design of this experiment concerns the use of an arrow diagram to record the moves, in Coin-Turning. In previous work with similar problems using actual coins with groups of 12 and 13 year old pupils, it became clear to the pupils that it was necessary to record the moves of a game in order to substantiate a claim to have solved the problem - and sometimes even to be sure for oneself that one had not made a mistake. However, it appeared to be easy to record a mistaken move, and difficult to spot the mistake from a record (for example, 4 coins, turn 3, TTTT, THHH, HTHT, THHT...). It was decided to include in the present experiment the recording of allowable moves in Coin-Turning, to observe whether this enabled pupils to give a more cogent argument of the impossibility of solving the problem as given. The hypothesis for testing is that this will not be helpful, that is, that the handling of the recording will be as difficult as solving the problem without it.

(6) In two of the number problems, Add and Take and One and the Next, it is appropriate to use some simple algebra. This amounts in the first case to expressing the process as (10 + x) + (10 - x) and showing that this always equals 20, and in the second case to expressing numbers in terms of multiples of 3 as, 3k, 3k + 1, 3k + 2, or as M(3), M(3) + 1, M(3) + 2. It is expected that these representations of multiples of three will probably be used if and only if they have been taught, but the expression (10 + x) + (10 - x) should be capable of construction by any pupil who has learnt some algebra. This should be a majority of the sample, since it covers 15 year olds of all abilities. The hypothesis is that a majority of pupils will use algebra for Add and Take, and a small number will do so for One and the Next.

(7) a) Factors determining performance, b) strategies for teaching. There are some further questions on which it is hoped that the experiment will provide information but on which it does not seem appropriate to formulate hypotheses. These concern pupils' development in attainment with regard to generalisation and proof, and are

(a) what are the determining factors of performance? and (b) what kinds of learning experiences bring about improvements? The second question cannot be answered directly from the present experiment, although the recognition of developmental sequences, large or small, may be helpful. Evidence on the first question may be forthcoming if the reasons for pupils' breakdowns on the problems can be observed or inferred. The three types of factor recognised and discussed below (p.9.57) are (i) knowledge of relevant facts, principles and skills, (ii) general reasoning ability, including ability to deal with logical complexity, and (iii) possession of relevant general strategies. (Evans (1968) obtained three such factors in a major longitudinal study of pupils aged 14 to 17; his third was a Problem Solving factor.) Each of these leads to its own conclusion regarding possibly useful teaching.

THE PROBLEMS

The problems used in this study all require the provision of an explanation and justification of a generalisation; in some cases the generalisation has to be found first, in other cases one is offered and has to be tested. The aim is to offer a representative piece of mathematical activity, subject to the limitations of the test situation; and the task is left as open as possible in order to obtain an accurate reflection of the pupil's thinking. The mathematical concepts involved are intended to be very well known, so that they do not present difficulties which interfere with the generalisation and proof strategies which we wish to observe. In all, ten problems were used. The ten problems are chosen to display variety in a number of ways, as suggested in the hypotheses. These ways are (1) Context - numerical, geometrical or from outside mathematics (as in Coin Turning and Noughts and Crosses) (2) whether the set of cases over which the generalisation is made is finite or infinite and (3) whether it is given or needs to be constructed; for example, Add and Take ("Choose any number between 1 and 10") concerns a given finite set, Stamps concerns a finite set, needing construction, of possible linear combinations of 8 and 20 around 70, Midpoints concerns a given, infinite set of numbers on the line, and so on; (4) whether the generalisation is stated, for test and explanation, or whether it has to be found first (for example, One and the Next and Quads, respectively); (5) whether it is a positive generalisation or statement of impossibility, (as in Stamps and Coin Turning) or the construction of a complete set of examples under given conditions as in Triangles; (6) whether the generalisation is a well known one, as in Adding a Nought, or a new one.

These characteristics of the problems are shown in Table 1.

TABLE 1

TEST	CONTEXT (1)	SET		GENERALISATION		
		fin/inf (2)	given/ const (3)	(4)	(5)	(6)
Add and Take	num	fin	given	not stated but immed	pos	new
Coin Turning	outside	fin	const	stated	imp	new
Adding a Nought	num	inf,N	given	stated	pos	well known
Diags of Polygon	num/geom	inf	const	stated	pos	new
Midpoints	num/geom	inf,NxN	given	stated	pos	fairly familiar
Noughts & Crosses	outside	fin	const	stated	yes/no	new
Stamps	num	fin	const	stated	imp	new
Quads	geom	inf	const	not stated	pos	new
One & the Next	num	fin	given	stated	pos	new
Triangles	geom	inf	const	not stated	find set	new

SAMPLE AND PROCEDURE

The problems were made up in pairs into five half-hour tests. Each test was worked by about 40 pupils, 13-15 from each of three schools (one grammar, two comprehensive) in different parts of the country. The comprehensive schools were asked to choose their sample of pupils to that together with the grammar school sample the whole ability range was covered as well as possible. All pupils were fourth formers, thus aged between 14.8 and 15.8 at the time of the test. Pupils sitting next to each other worked different tests, to minimise the risk of copying. The instructions for all the tests were the same.

> "These questions are about finding rules and giving reasons. You will have plenty of time - two problems to do in half an hour - so experiment fully, think about all the possibilities and be sure as you can before giving your answers."

A selection of responses to each problem appears in Appendix 9.

CATEGORY SYSTEM

Preliminary sorting of the responses, using the previous stage descriptions as a guide, led to the adopting of the following general categories for describing the characteristics of generalisation, explanation and proof activity in a way which would apply to the whole range of problems.

Previous work suggested the importance of the distinction between the empirical and deductive types of response - that is, between those in which the basis of the argument was empirical, and those in which though examples were generated, their function in the proof argument was as illustrations rather than as evidence. The distinction was in most cases quite clear, though there were some doubts at the borderline between X and Rgr.

The O category was eliminated as far as possible but there were some failures which left too little material for satisfactory categorisation.

The Category Ø has not appeared in earlier work. Some responses, although they generated at least one correct example, did not use it to test the generalisation. They acted on the instructions given, but showed no awareness that the result obtained was *dependent* on the detailed nature of the process, that the results obtained from that process might have some feature which the results of another process might not have. This is perhaps the same phenomenon as was observed in the pilot experiment and described as a lack of expectation of rationality. It was not particularly easy to distinguish in practice and was not in fact very much used, but it seemed of sufficient psychological importance to retain as a category with a view to its further investigation.

In the Stages study, variety and systematic organisation of examples was observed as significant. A V category - variety of example - was considered for this study but rejected since although conscious selection of examples with the right kind of variety is a significant skill in empirical inference, it is often not possible to decide from a script how carefully a given set has been chosen.

Quantity of examples may indicate either lack of insight or the seeking of variety. On the other hand, an S category proved easier to recognise and indispensable for sorting the responses to Triangles. In this problem a well organised set of examples of more than one type represented a solution as competent as the explanatory ones in other problems.

The F category is self-explanatory. The various levels of deductive response represent an attempt to describe by significant details the observed gradation in quality. The three aspects of relevance, connectedness and reference to agreed facts as starting points are the dominant factors.

EMPIRICAL

O : Misinterpretation, failure to generate correct examples or to comply with given conditions. (As far as possible, mistakes and misinterpretations are ignored and categories allotted accepting the pupils' interpretations of the problem).

Ø : Non-dependence: One or more examples correctly worked, but not used to test the general statement; lack of awareness of connection between conclusion and details of the data.

X : Extrapolation from empirical check. Truth of general statement inferred from an incomplete set of particular cases; any apparent 'reasons' are assertions that the conditions have been complied with, or descriptions of the working out of particular cases. The basis of the inference is clearly empirical.

(If there is a general restatement of the process and its conclusion, category Rgr is the right one; if there are remarks added by way of explanation but which are actually irrelevant, use category D)

S : Systematic: Finds at least some complete subsets of cases, is clearly attempting to find all.

F : Check of full finite set of cases.

DEDUCTIVE

This subset of the categories includes all responses where a deductive element contributes to the drawing of the conclusion.

D : Dependence: Attempts to make a deductive link between data and conclusion, but fails to achieve any higher category.

Rgr : Relevant, general restatement: Makes no analysis of the situation, mentions no relevant aspects beyond what are actually in the data, but re-presents the situation as a whole, in general terms, as if aware that a deductive connection exists but unable to expose it.

Rcd : Relevant, collateral details: Makes some analysis of the situation, mentions relevant aspects which could form part of a proof, possibly identifies different subclasses but fails to build them into a connected argument; is fragmentary.

Einc : Connected, incomplete: Has a connected argument with explanatory quality, but is incomplete.

E comp : Complete Explanation: Derives the conclusion by a connected argument from the data and from generally agreed facts or principles.

ANALYSIS OF RESPONSES AND CATEGORISATION FOR EACH PROBLEM

ADD AND TAKE

Choose any number between 1 and 10. Add it to 10 and write down the answer. Take the first number away from 10 and write down the answer. Add your two answers.

1. What result do you get?
2. Try starting with other numbers. Do you get the same result?
3. Will the result be the same for all starting numbers?
4. Explain why your answer is right.

DESCRIPTION OF RESPONSES AND NOTES ON ALLOCATION OF CATEGORIES

A typical response chooses 3, obtains 13, 7, 20; then tries 5, obtaining 15, 5, 20; and answers "Yes" to question 3. The crux of an explanation is the recognition that the same chosen number is added and subtracted in obtaining the result, so that it cancels, leaving the two tens. (This implies an awareness of commutativity and associativity). The display of the process in algebraic language, $(10 + x) + (10 - x)$, contains the essence of this insight, again depending on the implicit awareness of the two laws. A valid justification of the result can also be given by checking all ten cases.

Allocation of the responses to the categories is straightforward. Some scripts make mistakes in the given process; the most common is, at the second step, to take the first number away from the previous answer instead of from ten, giving as a final result $20 + x$. Such mistakes are ignored as far as possible in the allocation to categories; a valid explanation of the pupil's own generalisation is accepted. This is done since we are evaluating the explanatory qualities of the response, not the correctness of the mathematical operations.

Category 0 contains two blank scripts, and one response which works one example but makes no generalisation.

Category X can be subdivided. In X_1 the response shows no worked examples, but simply answers "Yes" to the questions. Category X_2 comprises two types; some give "explanations" which are simply accounts of the calculation in a particular case, or assert that they have followed the instructions.

No. 10: I choose a number between 1 and 10 and I choose 2. I add to 10 it made 12 and then I took 2 away from 12 and it came to 10 then I add both numbers 10 + 12 which made the answer 22.

No. 18: My answer is right because I have done what the paper tells me.

The other X_2 type offers an "explanation" which has a degree of generality but does not reach the level required for category Rgr: "represents the situation as a whole, in general terms, as if aware that a deductive connection exists but unable to expose it."

No. 14: I think my answers are right because I took a number, added to 10 then took it away. And then the two answers I got I added them together. The answer is always 20.

No. 2 is similar. These are in category X_2. The category X_3 response shows a number of examples worked out, says that the answer is always 20, and gives an empirically-based explanation, for example:

No. 26: because which ever starting number I put in it adds up to twenty.

No. 5 is similar. Two pupils test examples outside the prescribed range: No. 11 includes fractions among the chosen numbers, and one (No. 40) uses numbers outside the range 1-10 (12, 11, 19, 64 and 0) This is category X_4.

Examples of Rgr are:

No. 4: ...whatever number you add to ten and then take the same away when you add the two answers together they both add up to the same.

Nos. 7, 23, 33 are similar. One with more feeling of inevitability is:

No. 22:...If you choose a number between 1 and 10 and add it to ten, then if you take the first number away from ten then it will be whatever is needed to make 20....

No. 36:....Yes, the answer will be the same for all the starting numbers, e.g.

$$\begin{array}{r} 10 \\ \underline{4}\;+ \\ 14 \end{array} \qquad \begin{array}{r} 10 \\ \underline{4}\;- \\ 6 \end{array}$$

both added together = 20, and will always add to if you start off with a number like 10, 20, 30, 40, etc.

These all carry some feeling that the result is an inevitable consequence of the nature of the given process. It seems as if the pupil has some insight into the process, which convinces him, but he is unable to articulate the connections.

No category F responses are found.

The possible complete explanations are referred to above. Responses which contain all the necessary points but not clearly expressed are put into category E inc; those which make some analysis of the situation but only have part of a correct explanation go into category Rcd.

Two responses use algebra:

No. 20: If the starting number is called x, then the equation being done is (10 + x) + (10 - x) added together, this always comes to twenty.

No. 19: If you let x be 10 and y be the number you get $x + y$
$x - y$
and when you add the equation together you are left with 2x which was 10 so you have 2 x 10 which will always be 20.

These are both E comp.

Other examples are:

E comp: No. 28:....because....one number is going to be a certain number above 10, when added. When subtracted, the number is going to be the same amount of units BELOW 10, and so when they are added together, they are bound to add up to 20, as it is just the same as adding 10 + 10.

Einc: No. 8:...because if you take any number from ten it will leave you with the number that when added to the original number will make 10, e.g. 8 from 10 = 2, 8 + 2 = 10 which is the number it is all centred round so when two lots of 10 are used, the answer is bound to be 20.

Rcd: No. 25:...because the number which is first added to 10 is then taken away from the amount which is got from adding it on in the first place.

RESULTS

O : blank or no generalisation
Nos. 6, 31, 34

X_1 : no additional examples; answer "Yes" and no more
Nos. 37, 38, 39

X_2 : with "explanation" recounting the calculation in a particular case or action-based general re-statement.
Nos. 1, 2, 10, 14, 17*, 18

X_3 : with genuine empirical justification
Nos. 5*, 26

X_4 : including examples outside the given range
Nos. 11, 40*

Rgr : general re-statement of data and conclusion, with a sense of necessary connection
Nos. 3, 4, 7, 22, 23, 30, 33, 36, 42

Rcd : some analysis, but only partial explanation
Nos. 9, 15, 43

Einc : contain all necessary points for E comp but not clearly put together.
Nos. 8*, 12*, 13, 21, 25,27

E comp : explain cancelling of the chosen number, leaving two tens
Nos. 19, 20, 24, 28, 29*, 35, 41

* Scripts whose numbers are starred appear in the Appendix

COIN TURNING

This is a coin turning game but played with pencil and paper.

1. The first is about 3 coins and a move consists of turning over any two.

   Using as many such moves as you wish, get from 3 tails to 3 heads.

   Make your moves like this:
   T T T
   H H T
   T H H
   .... and so on.

   If you can do it, show your list of moves. If you think it is impossible, explain why.

2. The diagrams below show all the possible ways of putting down three coins. An arrow has been drawn from THT to HTT to show that this is a possible single move. TTT to HHH is not a possible single move so these will not be joined.

   Complete one of these diagrams by drawing arrows to show all the possible single moves. (The spares are for use if you make mistakes on the first one.)

   TTH HHH THT
   HHT TTT HTT
   HTH THH

   TTH HHH THT
   HHT TTT HTT
   HTH THH

   TTH HHH THT
   HHT TTT HTT
   HTH THH

   TTH HHH THT
   HHT TTT HTT
   HTH THH

3. Now explain again why your answer to No. 1 is right.

RESPONSES AND CATEGORIES

A typical good response would make a number of trials, become convinced of the impossibility, then explain it by showing that the various arrangements of 2 heads, 1 tail can change into each other or into 3 tails, but not into any other combination; these two form a closed system. The arrow diagrams are intended to help pupils to see this if they have not done so already.

A considerable number of pupils (about 40% of the sample) fail to make any useful progress on their problem. Four do not attempt it. Fourteen make mistakes and either obtain 3 heads, or reach no conclusion. Three make correct trials, not reaching 3 heads, but draw no conclusion. All these are placed in category O. One tries and says it is impossible, but adds nothing further. (Category X) One other gives an 'explanation' not good enough for Rcd, so goes into category D.

No. 2: It's impossible because when you turn a tail you end up with more heads and when you turn a head you don't have enough.

There are three 'general restatements':

No. 28:....impossible if two coins have to be turned over in one move. There are not enough coins. 3 heads can easily be gained by moving one coin at a time, but it is impossible by moving two.

We do not classify this as Rcd; it is closer to "mentions no relevant aspects beyond what are actually in the data" (Rgr); than to "makes some analysis of the situation, mentions relevant aspects which could form part of a proof." (Rcd). No. 30 is similar, but adds "If there were 4 coins, this could be carried out" which is closer to Rcd but still not there. No. 21 is a classic restatement.

The criteria adopted for allocation to the remaining three categories in this problem are these. For E comp we require an analysis of how 2 heads, 1 tail either reproduces 2 heads, 1 tail or goes to 3 tails. For E inc we require the statement that it remains always 2 heads, 1 tail, or 3 tails. Examples are

E comp No. 24:....there will always be one tail. This is because the first move was to turn two tails over, to make two heads but we are left with one tail and this tail is turned over in the second move to make a head but one more coin has to be turned over and this coin is a head which will then become a tail and this goes on without getting rid of a tail.

Closer to the borderline is:

E comp No. 11:...the only possible moves left would leave you with two heads. The only way to get out of it would be to return to three tails. Yet even then this would be impossible as you'd change from three tails to two heads and a tail then would have to change two more so again you would have two heads and a tail. This would carry on for ever.

Einc: No. 20:....because moving two coins at once, and starting with 3 tails, causes there always to be either 2 or 0 heads showing. One coin is always left out.

Four E comp and three E inc scripts are found. The remaining scripts, classified Rcd, all make some analysis of the situation. Many of the statements are inaccurate and in some cases even the sets of trials are faulty. A common phrase is "there is always an odd one". A few examples are given:

No. 40:...because you are turning them over in 2s and no matter how hard you try you will still end up with 2 of the same, and one odd one which you haven't turned.

No. 25:...because 2T should be showing on the table of moves to enable the player to get the 3H in a row.

USE OF THE ARROW DIAGRAMS

Of the pupils in category O, X and D who tried to use the diagrams none had a correct set, nor any new explanation. For those in categories E inc and E comp the possibility of improving their explanation by using the diagram did not exist. Of those in Rgr and Rcd, five had a correct arrow diagram but the only improvement in explanation was given by No. 13 who repeated the previous explanation "because there are two moves and 3 coins and two doesn't go into three" with the addition "and HHH doesn't join with TTT".

One other pupil (No. 28, Rgr) regarded the arrow diagram as an illustration and repeated the previous non-explanation.

Rgr: No. 28:...as you can see, there is no arrow going from 3 tails to 3 heads. This is because the move is impossible.....

Thus these diagrams do not enable those who cannot solve the problem, or explain it, to improve their performance. It seems that making up a correct diagram and recognising the significance of the closed systems of states which it shows are new problems, at least as difficult as the original ones. (This result relates to the findings of Kilpatrick and Lucas that the use of diagrams in problem solving is apparently uncorrelated with success.)

RESULTS

O_1 : blank
Nos. 3, 18, 38, 39

O_1 : mistakes, obtain 3H or no conclusion
Nos. 1, 4, 7, 14, 16, 17, 22, 23, 33, 34, 36, 37, 41, 42

O_2 : correct trials, no conclusion
Nos. 5, 10, 31

X : correct trials, statement of impossibility, no explanation
No. 12

D : as X but with imprecise explanation
No. 2*

Rgr : apparent "explanation" merely restates data
Nos. 21, 28*, 30

Rcd : some analysis, possibly with inaccurate statements or faulty trials
Nos. 8, 13, 19, 25, 26, 27, 35, 40*, 43

E inc : analysis, including statement always 2H, 1T (or 3T)
Nos. 9, 15, 20*

E comp : includes analysis of how 2H, 1T and 3T remain a closed set
Nos. 6, 11*, 24, 29

* Scripts whose numbers are starred appear in the Appendix

ADDING A NOUGHT

If you want to multiply by ten, you can add a nought; for example, 243 x 10 = 2430.

1. Is this true for all whole numbers?

2. Explain why your answer is right.

RESPONSES AND CATEGORIES

This is a well known number generalisation, which pupils use frequently. The aim of the problem is to see whether they can give an explanation of it. An adequate explanation needs to appeal to the place value system, and to state that the effect of adding a nought is to shift each digit of the given number into a place whose value is ten times that of its original place; thus each part of the number is multiplied by ten. One main point of interest is to see what more fundamental principles the pupils appeal to in their answers.

The majority of the responses to this problem fall into Category Rgr, since the pupils are unable to give a true explanation - they may have been given one when they first learnt the principle but have long since forgotten it - and they can only reassert the principle and give other examples. This category is subdivided for description purposes. However, there are first three category O responses, two giving no explanation at all, and one garbled memory relating to the multiplication algorithm.

No. 22.....because I was taught in Junior School and because you are adding one unit so you have to move all the units up.

17 responses add examples in support of their reassertions of the principle. Seven of these appear to use the examples as justifications of the principle, (implicitly ignoring the circularity of the argument), while others are clearly offering them as illustrations. The first seven are classed as category X (inference from a check of particular

cases, and the latter as Rgr. Three of the X responses give contrasting examples of multiplying by numbers other than ten, emphasising the fact that the principle applies to ten and not to other multiples. These are classed as X_2, the remainder as X_1.

No. 28:...I think it is right because:

20 x 10 = 200
554 x 10 = 5540
775816 x 10 = 7758160

but if you times it by 6, 7 or 8 this does not happen

2 x 6 = 12
2 x 7 = 14
2 x 8 = 16......

In the Rgr category, three responses give lengthy description of the multiplication algorithm; short extracts are quoted. These are labelled Rgrl.

No. 20:

$$\begin{array}{r} 208 \\ \underline{10} \\ 000 \\ \underline{2080} \\ 2080 \end{array}$$

First we do 0x8....but the second line we are multiplying by ten. So the first column we have to put in a nought. Because 10 x any number won't be less than 10.....

No. 41:....instead of starting directly underneath the 0, you start the next answer one place to the left....

Two are restatements of the generalisation, without examples (Rgr2)

No. 38:...you've got to put a 0 on the end otherwise you will only get the answer of your number multiplied by one. So the nought makes it into the number you want....

Two of the responses give 10 x 10 = 100 as their first example; but these look more like the one-example check than a step in an explanation; no special category is made for these.

No. 12:...the answer is correct because if 10 is multiplied by 10 (10 x 10) it equals 100, so in short it moves one place.

Seven responses mention either multiplying by 100 as needing two noughts, or multiplying decimals as requiring moving the point, thus supporting their statements by mentioning its extensions. These are labelled Rgr4.

No. 4: If it was a fraction, you would have to move the decimal point, e.g. 0.05 x 10 = 0.5. As there is no decimal point, i.e. a whole number, you just add a nought.

No. 33: Because the number ten has one nought on it and when multiplied with another number you put a nought on it.....if you multiply by 100 which has two noughts on it, you add the noughts to the answer, for example 243 x 10 = 24300.....

Seven responses attain the Rcd category; two of these draw attention to the classification of numbers as between 1 and 10, 10 and 100 and so on, and to the fact that multiplication by ten moves them up a category.

No. 21: Numbers fall into categories of 1000 s, 100s, 10s, units.
By adding a nought you increase it by one power of ten.
e.g. 2 x 20 = 40
2 x 200 = 400
2 x 2000 = 4000
2 x 20000 = 40000
All you have to do is multiply the two integers and then add the number of noughts in the sum, e.g. 2 x 2000 = 2 x 2 = 4, add 3 noughts = 2 x 2000 = 4000

This is actually statement of the associative law rather than the principle in question, but it does "analyse the situation and mention relevant aspects which could form part of a proof." The other six Rcd responses refer generally to the movement of figures between columns of different value, but do not explain this in terms of what it does to each figure. The latter is the requirement for E comp. One response separates the effect of multiplying by 10 on the 200, the 40, and the 3 but does not explain how 200 x 10 = 2000; this is classed E inc.

Rcd: No. 27: shows 240, 2400, 24000 in columns labelled HTU etc. but says only "this diagram shows how this works."

Rcd: No. 11 shows a similar tabular arrangement but uses it as an illustration, not an explanation. "All you do is move the numbers up one so the nought can fit in."

E comp: No. 35:...putting a nought before the point moves all the numbers up the thousands, hundreds, tens and units scale, making each of the numbers ten times bigger than they were previously.

RESULTS

O : no explanation
Nos. 26, 40

O : unintelligible
No. 22

X_1 : examples used as if to justify principle
Nos. 2, 18, 25*, 36

X_2 : as X_1 but with some contrasting examples, multiplying not by 10.
Nos. 3, 23, 28

Rgr1 : action-description of the algorithm
Nos. 1, 20*, 41

Rgr2 : restatements of the principle, without examples
Nos. 5, 38

Rgr3 : restatements, with examples as illustrations
Nos. 8, 12, 13, 14, 15*, 16, 19, 29

Rgr4 : as Rgr3 but including extension to decimals or to multiplying by 100.
Nos. 4, 6*, 9, 24, 32, 33, 37

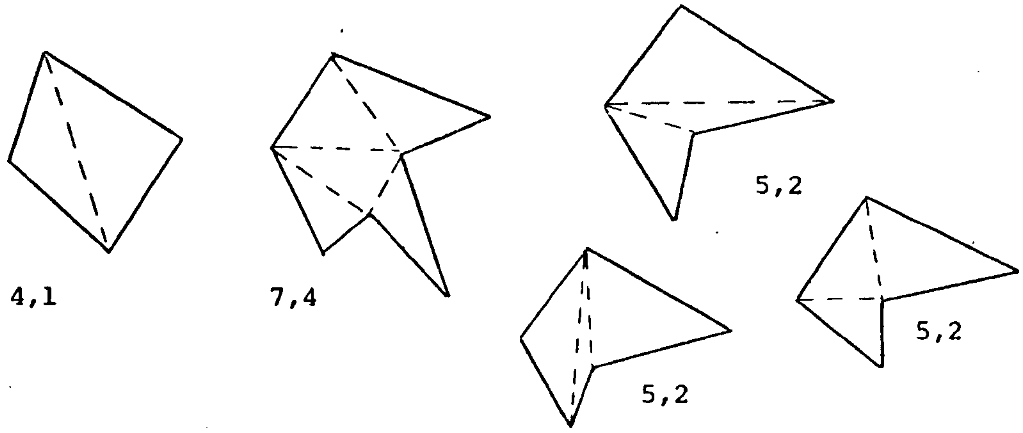
Rcd : refer to movement of figures between columns of different value, or to the classification of numbers as between 1 and 10, 10 and 100 and so on.
Nos. 7, 11, 17*, 21, 27, 31, 34, 39*

Einc : separates effect of x 10 on 200, 40 and 3 but does not explain this.
No. 10.

Ecomp : separates digits and explains effect of x 10 on each as shifting figures between columns of different place value.
Nos. 30, 35*

* Scripts whose numbers are starred appear in the Appendix

DIAGONALS OF A POLYGON



Some diagrams have been drawn here. It seems that "the greatest number of non-crossing diagonals which can be drawn in a polygon is three less than the number of sides."

Is this statement true for all polygons?

Investigate this fully; then state your conclusions and your reasons.

RESPONSES AND CATEGORIES

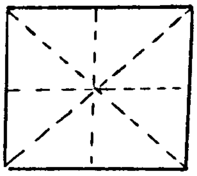
A good response to this problem might begin by checking the truth of the statement for a wider class of polygons than that shown, and for more different ways of drawing the diagonals within a given polygon. The data shows that three diagonalisations of the five sided polygon all have the same number of diagonals. The only deductive proof observed at this stage relates the set of diagonals radiating from a single vertex one to one with the vertices to which they go - that is, one to each vertex of the polygon, except that from which they radiate and its two adjacent ones; hence d = s -3. Some sense of the incompleteness of this proof, in not applying to non-radiating diagonalisations of a given polygon and not applying at all to polygons in which radiating sets are impossible, is shown by the two pupils in this sample who achieve this proof. There are some signs of what could form the beginning of other proofs - of an inductive type, starting with the triangle with no diagonals and seeing that each additional side adds one diagonal, or by dealing with the number of triangles formed rather than with the diagonals. But many pupils remain entirely at the empirical level of checking a variety of cases.

Three scripts are in category O - no response or some diagrams but no written conclusion. Fourteen make empirical trials and draw a conclusion, but contain mistakes, some recognised but ignored, others apparently not noticed. These are categorised as X^{o}.

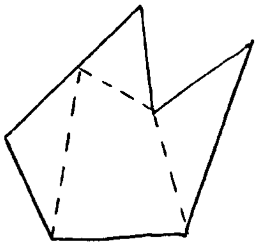
No. 11: 8 polygons shown, various shapes and sizes. These polygons show that there can be more or less than three diagonals on a polygon because there may be 15 points that 16 diagonals can reach or 12 points that 9 diagonals can reach.

In this cases some diagonals had been missed. Two pupils did not understand the term diagonal at all;

No. 28 shows:



and



Ten responses draw conclusions empirically from correct experiments and this forms category X. (One of these misinterprets the statement.) Six are classic examples of check generally of a good variety of examples, and a positive assertion. Three make a more cautious assertion, but in only one of these (No. 36) is there a full awareness that it may be always true but that a proof of this would require check of all the possibilities. In the other cases, there is no sign of thinking of the generalisation beyond the range of examples tried.

No. 36:...I can't see a way of proving this statement is true except by drawing and working out all the possibilities but as it does work for these numbers, it should for others.

No. 31: 5 diagrams: It is true for them that I've drawn.

There are two cases of D; one shows a quadrilateral and states that when one diagonal has been drawn the other cannot be drawn without crossing; the other mentions that the polygons are divided into triangles but does not consider their number. One response is categorised Rgr. This draws attention to the pattern 4,1; 5,2; 6,3 and so on, without adding any analysis.

Ten responses are in category Rcd; they make some analysis of the situation and mention relevant aspects which could form part of a proof. Six of these mention the relationship between the number of triangles and either the number of sides or the diagonals, or both (Rcdl)

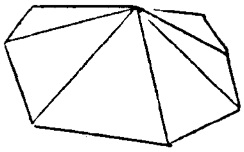
No. 41: The diagonals split the polygons up into triangles. These are always two triangles less than the amount of sides...

Three analyse the drawing of the diagonals, and either try to explain how the non-crossing condition limits the number, or to consider how many vertices the diagonals can go to. (Rcd2)

No. 23:...because say you have a shape with 4 sides, you start out at point 1 and that only leaves you with 3 other points to go to (or does it?) e.g. you start at one point which will go across the middle so no more can be done without crossing.

Two responses were graded E inc. Both use radiating diagonals and both show an awareness that this arrangement is not always possible, but neither extends the proof.

No. 16:



If in an 8,5 polygon you go from one point, you must have only 5 other points to go to, as you can't go along any of the edges of the shape, that leaves you with six points to go to, as you must be at the joint of 2 lines, and you can't go to yourself. Therefore there are only 5 points left. In the case of 7, 4 and 6,3 and 9,6 you have to go from 2 points.

RESULTS

O : Nothing or no conclusion
Nos. 7, 15, 18

X^o : mistakes or misunderstandings leading to conclusion "no" or false "yes"
Nos. 2, 3, 6*,8, 11, 12, 13, 14, 25, 26, 28, 37, 38, 40

X : correct trials leading to "yes"
Nos. 4, 15, 10, 19, 20, 22, 27, 31, 36*,39

D : attempts at explanation but not relevant
Nos. 24, 33*

Rgr : states pattern 4,1; 5,2; 6,3
No 1*

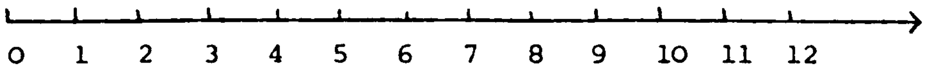
Rcd1 : relates number of sides to number of triangles
Nos. 17, 21, 32, 34, 35*, 41

Rcd2 : analyses drawing of diagonals
Nos. 9*, 23, 29

Einc : uses radiating diagonals, relates 1-1 to vertices
Nos. 16*, 30

* Scripts whose numbers are starred appear in the Appendix

MIDPOINTS



A and B can be any two whole number points on the number line. M is the point half way between them.

1. If A is at 2 and B is at 8, at what number is M?
2. Add A's number to B's number and halve the result. Do you get M's number?
3. Will the rule in No. 2 work for every possible position of A and B on the line, including bigger numbers?
4. Explain why your answer is true.

RESPONSES AND CATEGORIES

The main difficulty with this problem is in recognising that the result is not obvious. To help this as much as possible the question is set up so as to require the pupil first to find the mid point geometrically, and then to calculate (A + B)/2 and verify that the two results are the same. Many pupils feel that such a calculation is bound to give the mid point, hence the Rgr category is large. The best explanations from this sample only reach the Einc category. One of these translates AB to bring A to O, the other type argues that if M is the mid point, A and B are reached by going the same distance from M, to the left and to the right respectively, so A + B will be the same as 2M. No algebra is used by any pupil; nothing approaching $a + \frac{b-a}{2}$ is seen. The categories will now be discussed in order.

Category O contains two blank sheets, and three in which the given example has been worked out, but no response is made to the question about further positions. Thus no generalisation is made.

The Ø category is needed for four responses which respond to question 4 by commenting on the fact that the calculation can always be done:

No. 8: My answer is true because it is easy to find any half of a whole number if A and B can be two whole number points.

No. 9 is similar, and Nos. 5, 6 refer to using halfway points if you have odd numbers. These fail to show awareness of the need to connect data and conclusion; they do not use their examples to test the rule. Hence they are categorised Ø.

The X category is divided; X_1 comprises responses which show nothing beyond the example given, and "Yes" to question 3. X_2 comprises those which either show further examples checked, or state that they have checked others.

In three of these cases (Nos. 2, 21, 22) it is clear that the check is genuine.

No. 2: If A is at 4 and B is at 8, the number M is at 6.
4 + 8 = 12. Half of 12 is 6. (Four similar examples given)

Another is less clear (No. 40)

No. 40:....this is true because 15 + 17 = 32, 32 ÷ 2 = 16

The remaining four in category X_2 simply state that they have made further checks (Nos. 11, 24, 31, 35)

No. 24: Because I tried it out and it does.

In category D are placed two responses which are attempting a global explanation and not an empirical check, but who do not say sufficient to be categorised as Rgr or Rcd.

No. 39: Because M is the middle of two numbers and if you half two numbers you half.

Category Rgr contains three types of response. There are nine (Rgr1) which simply make a general restatement of the rule without showing or claiming any further examples. These are the ones who apparently feel the result is obvious.

No. 30: Because when you add two numbers together and half them, you get the same answer as when you find half way between them.

No. 1: This is true and works in every case because any two numbers added together and then divided by 2 have an answer that is halfway between the two numbers because all the process of dividing by two is - is halving.

A further four responses (Rgr2) both make general restatements and either support or illustrate them with examples. In three of these cases the check appears to be genuine, i.e. M has been found geometrically and numerically and the results compared (Nos. 23, 32, 38) while in the other case this is not clear (No. 3)

No. 38: Six examples, the last being:....A = 30, B = 40. 30 + 40 = 70, ½ of 70 = 35, M = 35 on the number line. If two numbers are added together, e.g. 2 and 8 and then halved giving 4 (sic) the middle point will be the same because you are finding the distance between those two numbers.

The next two responses are graded Rcd. These state or show a relevant aspect of the situation which could form part of a proof but are not connected enough for E inc.

No. 12: 15 examples, apparently genuine checks: If M is halfway between the two numbers then we have to count up the numbers in between and divide them by 2 to find a point halfway between them. Therefore the sum of the two numbers divided by 2 = M which is exactly halfway between them.

The reference to finding and halving the distance between A and B puts this in a category above all previously mentioned responses, though it is clearly not explanatory.

There are three E inc responses of which we quote two as they represent different types.

No 10:...because the number before M, (A) is always the same distance away from M as B is, e.g.

$$\begin{array}{ccccc} 35 & & 36 & & 37 \\ A & \underset{1}{\longleftrightarrow} & M & \underset{1}{\longleftrightarrow} & B \end{array}$$

e.g. 2 $14 \underset{3}{\longleftrightarrow} 17 \underset{3}{\longleftrightarrow} 20$

Therefore the two numbers when added together must be twice the amount of M. Also see e.g. 2, 20 - 3 = 17 add this 3 to 14 and you are given 17. Therefore if you add these it equals 34 which when halved equals 17. This works for all cases.

No. 36: Because the number in the middle has the same number going to each side, e.g. A = 2, B = 8 so there is 6 numbers in between. The middle number <u>5</u> has 3 squares each side. So if we moved the numbers to the beginning, so we start at 0, but still have 6 numbers inbetween, e.g. A = 0, B = 6. We know that the middle must be half of 6 and this is 3. So if this works, then we could just add the two together. 6 + 0 = 6. Half this and it would be 3. So if this can work at the beginning it should also work anywhere else on the line.

RESULTS

O : blank or no comment on generalisation
Nos. 7, 14, 18, 17, 33

Ø : no awareness shown of dependence of $\frac{A+B}{2}$ on M being midpoint; comment on details of the calculation instead.
Nos. 5, 6*, 8*, 9

X_1 : "Yes" to generalisation but no evidence, examples or statements.
Nos. 13, 16, 19, 25, 26

X_2 : Show or claim further examples checked.
Nos. 2*,11, 22, 24, 31, 35, 40*

D : attempt explanation but irrelevant or very fragmentary
Nos. 21, 39

Rgrl: general restatement, but no examples beyond the one given
Nos. 1, 15, 20, 27, 28, 29, 30, 34, 37

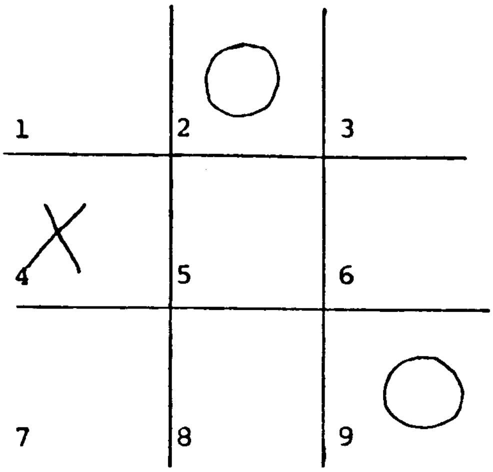
Rgr2: with further examples
Nos. 3, 23, 32, 38*

Rcd : relevant aspect displayed or stated
Nos. 4, 12.

Einc : show the essential step but not fully connected
Nos. 10*, 36, 41

* Scripts whose numbers are starred appear in the Appendix

NOUGHTS AND CROSSES



You are X and it is your turn. You are thinking of going to square 6.

1. Is this a good move?

2. Explain fully why you think so.

RESPONSES AND CATEGORIES

This problem is set in a non-mathematical context, but one familiar to the pupils. It is of a somewhat different character from most of the other problems of this set. The implicit generalisation for test is whether the set of possible consequences of the proposed move all lead to a win for X, or all to a win for O, or neither of these. The explanation consists of tracing the consequent moves through to show how the game must end. A high proportion of the sample succeeded in this. There are two other main categories of response - apart from O, which contained four blank scripts. The higher of these two categories is for responses which follow the game through partially, but not to the end, and the lower for those who do not pursue any definite follow through, but make imprecise judgements about what will happen.

No. 33: It is a good move because they can not get a horizontal line and they have two more goes before they can get a diagonal line and by that time you may have won.

No. 24: This is not a good move because if I did move to square six the game would end in a draw.

This set contains eight responses (Nos. 7, 9, 13, 19, 24, 28, 33, 38); we put these in category D, since there is the attempt to make a deductive link (granted that the players play predictably) between the given situation and the outcome. These are also similar responses to those categorised as D on other problems.

Category Rcd contains five responses where the follow through is partial.

No. 27: I think that if I was to move into square 6 it would not be a very good move because my opponent would probably go in square 5 and then I would have wasted my turn, because I could not get 2 crosses joining on to it. If I was to move into square 8 I would be able to stop my opponent from getting two possible lines and so I would not have wasted my move.

Here the pupil has not pursued the consequences of putting X into 6 beyond noting that the horizontal line 4 5 6 would probably not materialise. As for the suggested alternative, though, as stated, it stops one winning move on the opponent's part, it allows another O into 3. Of the other scripts classed as Rcd, three are full follow throughs but with mistakes (Nos. 2, 15, 39). One (No. 31) compares the value of a move into 6 with a move into 7 in terms of the number of possibilities of winning opened up by each, but without precise following through.

The remaining responses all have full follow through and so are categorised as E. However, this is divided into two sub-categories. E comp is reserved for those responses which mention explicitly in some way the winning character of the situation where a player has two marks in each of two lines, so that the opponent can block one or other of these but not both. The E inc responses are those which give instructions which, though effective, mention only one of the choices open in these situations. The distinction is not great, but on the other hand, it does discriminate non-trivially between responses. Two typical examples follow:

Einc: No. 17: No, it isn't a good move because if I go in square 6 she will go in square 5 to stop me from getting the middle line then I go to square 7 to stop her from getting the line going down then she goes in square 1 to get the line going diagonal.

Ecomp: No. 41: No, this is not a good move because O would go in square 5 to prevent me getting a line and will also have two alternatives for getting a line next go. I will only be able to stop one of these lines and so O will win the game.

RESULTS

O : blank
Nos. 14, 18, 26, 40

D : imprecise extrapolation, no definite follow through
Nos. 7, 9*, 13, 19, 24, 28, 33, 38*

Rcd : definite follow through but not to end of game or with mistakes
Nos. 2*, 15, 27, 31*, 39

Einc : follow through to end but alternatives not mentioned
Nos. 3, 4, 8, 11, 17, 22, 25, 34, 35

Ecomp: with mention of alternatives
Nos. 1, 5, 6, 10*, 12, 16, 20, 21, 23, 29, 30, 32, 36, 37, 41.

* Scripts whose numbers are starred appear in the Appendix

STAMPS

1. Anne has plenty of 8p and 20p stamps, but no others. She has a parcel to post costing 70p. Can she put on the correct amount exactly?

2. Explain why your answer is right.

RESPONSES AND CATEGORIES

A full response to this problem following most pupils' approach, requires the establishment of the set of possible combinations of 8 and 20 which must be checked to see whether 70 can be obtained. Most economically this means recognising that only 0, 1, 2 or 3 twenties are possible, and trying different multiples of 8 with each, showing that 70 is straddled and cannot be obtained exactly. Alternatively those aware of the properties of the highest common factor can see that only multiples of 4 are obtainable. This approach was not used by any pupil in the sample. However, several used similar arguments in part of their solution, for example recognising that any useful multiple of 8 would have to be an integral number of tens, so that 40 is the only possibility. Others regarded 70 as an odd number and so unobtainable from two evens.

Category 0 is needed for three scripts, one blank and two misreadings. Scripts of the next level of attainment are difficult to categorise, both X and Rgr being prima facie correct. These state that there is no combination of 8 and 20 which makes 70; some add that the nearest is 72p. The basis of the inference is clearly empirical, and the deductive aspects of this problem reside in the establishment of the boundaries of the class of possibilities. Hence we allocate these nine responses to category X. We next have a group of responses which make some analysis of the situation, though still none of them say definitely there cannot be more than three 20s, or anything similar. The nearest to this is No. 23, which states first that four 20s make 80p, nine 8s make 72p and eight 8s 64p, then shows 3 x 20 + 1 x 8, 3 x 20 + 2 x 8, 5 x 8 + 1 X 20. This is the most systematic of this group. At the other end of the scale:

No. 27: 4 x 8 = 32 + 2 x 20 = 40 = 32 + 4 = 36
5 x 8 = 40 + 10 x 20 = 20 = 40 + 20 = 60

Some of these are given in the spirit of an argument, while others simply report the results of trials. These are classed as Rcd since they make some analysis of the situation. They also involve the selection of pertinent examples, whereas less particular choice is required in the examples checked for a category X response in other problems. The Explanatory category has to include the three, referred to above, which regard 70 as odd, so unobtainable from two evens. These are classed as E^O. The genuine explanations are easy to recognise. Six of them make an exhaustive systematic check.

No. 13 first shows 8s or 20s alone are impossible. Using a
mixture: 1 x 8p + 1 x 20p = 28p worth, 42p remaining
4 x 8p + 1 x 20p = 52p worth, 18p remaining
1 x 8p + 2 x 20p = 48p worth, 22p remaining
1 x 8p + 3 x 20p = 68p worth, 2p remaining
Notice that the amounts remaining each time could not be made up correctly using either a) 8p stamps, or b) 20p stamps only, c) a mixture.

Five use the fact that any useful multiple of 8 must be a whole number of tens.

No. 11: Because 8 does not go into an odd number so you can't add 20 to it to form 70. The only multiple of 10 8 goes into is 40 and so you would need 30p more and 20 does not go into 30.

Note in the first line that "odd number of tens" is presumably what is in mind.

RESULTS

O : nothing or misinterpretation
Nos. 16, 18, 19

X : states conclusion, may give a few examples, but no analysis
Nos. 2, 6, 8, 9, 12, 25*,26, 29, 36

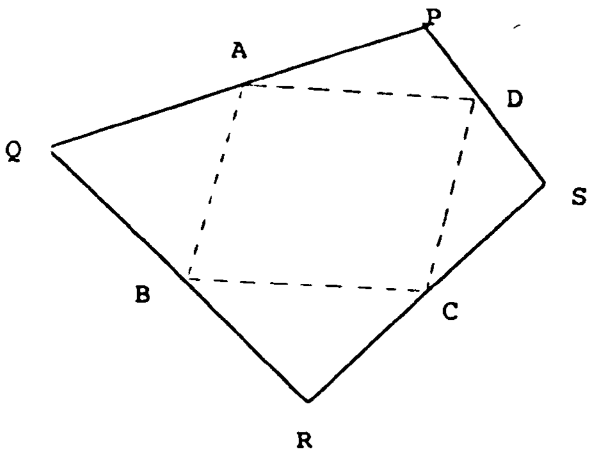
E^{O} : cannot make an odd number (70) from two evens
Nos. 28*, 32, 33

Rcd : some analysis, selected examples given as illustrations
Nos. 5, 7*, 10, 15, 20, 22*, 23, 27, 31, 35

E comp : exhaustive check of all relevant combinations
Nos. 1, 3, 4, 11, 13, 14, 17, 21*, 24*, 30, 34

* Scripts whose numbers are starred appear in the Appendix

QUADS



A, B, C, D, are the midpoints of the sides of the quad PQRS. In some quads the midpoint figure ABCD is a rectangle.

1. Find out what has to be special about the quad PQRS for ABCD to be a rectangle.

2. Give reasons to justify your answers. Use the plain or spotty side of the paper provided for your trial drawings.

RESPONSES AND CATEGORIES

Any quadrilateral with perpendicular diagonals has a rectangle for its midpoint figure. On joining PR and QS, it follows either from the midpoint theorem or from the enlargement from centre Q taking QAB to QPR that AB||PR; similarly DC||PR, and AD||BC||QS. Hence ABCD is a rectangle if, and only if, $PR \perp QS$. When setting this problem it was thought that traditionally taught 15 year olds would use the midpoint theorem and those who had followed a modern syllabus would use enlargements. In the event, no pupils used either of these approaches effectively, and only a few showed any fragments of either method. It was recognised that a proof would be difficult, but it was expected that many pupils might check empirically various known types of quadrilateral for PQRS and observe which had rectangles as midpoint figures. This would probably lead to naming the kite as the required outer figure . A fair amount of such work was done, but a surprisingly large number of pupils confused the meanings of rectangle, parallelogram and quadrilateral.

The great majority of pupils simply reached generalisations (mainly wrong ones) or added some fragmentary explanations. Only three responses were explanatory.

Category O contains two cases where the problem of relating the shapes of the inner and outer figures was not accepted. These both comment on the sizes of the figures.

No. 29: The figure ABCD is half the size of the figure PQRS only if the points are in middle or thereabout.

A further five category O cases make trials but draw no conclusions nor make any comment. Category X contains those which reach a conclusion from empirical trials. Four of these deal in named shapes - two decide PQRS must be a diamond, one a diamond or kite, one a parallelogram.

No. 10: I think that the quad will have to be diamond for the midpoints to be a rectangle.

A fifth (No. 21) says "PQRS must have a line of symmetry...down the middle"and shows a kite. Another specifies "two sides the same length, and the other two also the same length but not as long." (No. 15) In all there are seven of these responses, which successfully make trials (five with correct conclusions, two with wrong ones), maintaining their hold on the testing process by using different shapes which they know. We call these X_2.

No. 6:.....lines QS and PR would have to cross to give four segments with 90^o angles....(correct diagram)....to obtain a rectangle from any quad it is necessary that QP and RS should be parallel and also that QR and PS should be parallel.

A less successful group (X_1) are only able to extract less definite conclusions.

No. 14: The quad has got to have sides with exactly 90^o and the sides of the quad have got to be straight....

No. 4: The angles AQB and DSC have to be smaller than the other two angles because AB and DC are shorter.......

Two others specify four sides all different, and nothing special about PQRS.

Three responses are categorised D. These are all based on the one or other of the interpretations already described; all make trials and derive a conclusion, but add reasons which are, however, irrelevant.

No. 34: ABCD is always a rectangle as when you join the midpoints of the quad you cut off triangles.

We next have a group which treats ABCD in the given diagram as a rectangle. These pupils conclude that every quad gives a rectangle because 4 midpoints implies a four-sided inner figure, perhaps adding that more sides in the outer figure would not give a rectangle inside.

No. 22: With the quad having 4 sides, no matter how long...they have a midpoint and therefore....a rectangle is produced. This particular quad has 4 sides so therefore there are 4 midpoints, by which a rectangle is produced.

No.s 3, 19, 31, 28 are similar. These all have analyses of the situation so are categorised $R^{o}cd$, the o indicating misinterpretation.

The best responses to this problem only reach category Rcd. There are six of these:

No. 35: The quad must have 2 long sides and 2 short sides, because if the four sides were the same length, when joined up, the midpoints make a square.

No. 8: PQRS must have at least two lines parallel or two sides of the same length for ABCD to be a rectangle. If two sides of PQRS are the same length they will both have the same centre. Therefore two of the four points ABCD will be the same length along their respective lines. For some reason they will then be joined to the other two points by a parallel line.....

This last is probably the best of the sample. Another of the better ones makes a systematic but non-deductive analysis:

No. 18:...for ABCD to be a rectangle the quad must have 2 pairs of equal sides and one right angle. Because
a) no right angle - lines not parallel
b) only one pair equal lines - not parallel, nor of even length
no right angles
c) no equal lines - lines are not parallel.

RESULTS

O_1 : misinterpretation
Nos. 1, 29

O_2 : trials, no conclusion
Nos. 7, 13, 20, 33, 36

X_1 : trials, indefinite conclusion
Nos. 2, 4, 14, 16, 25*,27, 32

X_2 : trials of different known shapes, definite conclusion
Nos. 6, 10, 11*,12, 15, 21, 24

D : trials and conclusion with added reason but irrelevant
Nos. 9, 23*,34

R^ocd : some analysis but of the simplified situation produced by confusing meanings of rectangle/quad/parallelogram
Nos. 3*,19, 22, 28, 31

Rcd : reasonable and precise conclusions (not necessarily correct) with some deductive analysis
Nos. 8, 17, 18, 26*,30*, 35

* Scripts whose numbers are starred appear in the Appendix

ONE AND THE NEXT

Write down any number up to fifteen. Write down the next number and add it to the first. Write down your answer. You have now written down three numbers.

Gail says that one, and only one of these numbers is in this list.

3, 6, 9, 12, 15, 18, 21, 24, 27, 30

1. Is she right?
2. Will she always be right?
3. Explain why.

RESPONSES AND CATEGORIES

This is based on the result that, of two consecutive numbers and their sum, just one is a multiple of three. Proving this requires separation of the three cases in which (a) the first number, (b) the second number, (c) neither of these is a multiple of three. (In the first case, $3x + (3x + 1) \neq M(3)$; the second case is similar; in the third case $(3x + 1) + (3x + 2) = 6x + 3 = M(3)$). The question is limited to $x \leqslant 15$, thus admitting a full check of all cases as a valid proof. No pupil used algebra.

A comparatively large number of pupils misinterpreted the data. The most serious misinterpretation was a failure to observe the "next number" condition, adding any two numbers whatever, or any two from the list. These seven responses form Category 0.

No. 33: 5,7 = 12....No....because it depends on what number the person picks e.g. 5 + 7 = 12, 12 is the only one there but if you have 6 + 9 = 15 all three are there.....

A further nine make misinterpretations which permit reclassification of the response, accepting the pupil's interpretation. These read the question as requiring that the third number, the <u>sum</u> of the two consecutive numbers is in the list, not one and only one of the three numbers. This is easily refuted by a counter-example; six responses which do this are categorised X_1. A further three add that the sum of two consecutive numbers is always odd, so that only the odd numbers in the list can be produced in this way. These are included in Category Einc, since they comprise the fullest explanation possible of this case. Straightforward X type responses which assert the conclusion on the basis of a number of trials, are categorised X_2; there are three of these. Six responses list the full set of cases - category F. Responses which make a few trials then attempt an explanation go in cateogry D if their attempts are largely irrelevant, or Rgr if they have the style of a general restatement of the data and conclusion. Examples of D are:

No. 47: Ten examples checked and crossed out. Because it is like a table and there is only 3 numbers between each other number and so it is impossible to do. If you can only pick a number up to 14 or over because the table only goes up to 30.

No. 31: One example 7 + 8 = 15: Yes, she will always be right. Because 3 goes into all the numbers so many times and if you pick a number like 7 and one like 8, 3 does not go into them but they add up to 15 which 3 does go into.

These appear here, rather than in category X, because they place reliance on the deduction and not simply on the empirical work. No. 31. fails to reach category R because the factor 3 is part of the data, and the rest is a description of the process with one example, not a general restatement.

Examples of Rgr are:

No. 7: (4 + 5 = 9) Because as there are 3 numbers to be written down and all the numbers in the list are multiples of 3 one number out of 3 will be a multiple of 3.

No. 4(7 + 8 = 15) Yes, because the numbers go up in 3s. If you use 2 numbers whichare not in the list, and under 15, they will equal a number which is in the list. All the numbers in the list can be divided by 3. To.use numbers not in the list, they can not be divided by 3 but added together they become numbers which can be divided by 3, therefore they are in the list, so Gail will be right.

Responses more successful at explanation but still very limited are those which identify the three different subclasses of cases - where the multiple of three is the first number, the second number or the sum. These make "some analysis" of the problem and are categorised as Rcd. An example is:

No. 6: Yes she will be right all the time because the numbers given above will either add or come into the sum, e.g. take any of the numbers from 1-15 12+13 = 25. The number 12 has appeared. She is going up in threes and this is important because every sum you make will have a number which will divide by 3, e.g. $\underline{6} + 7 = 13$, $8 + \underline{9} = 17$, and these numbers of course appear in the list given, thus proving that Gail is right all the time. As long as you have a number which can be divided by three it will always work.

She identifies the three cases and states the importance of "going up in threes" and of multiples of three, but does not connect these together or relate them to properties of multiples of three. The best explanation found in this sample is still very incomplete. It covers the three cases in its numerical examples, and manages just to state how one of the cases is true.

No. 1: If you choose a number which is in the 3 x timetable, naturally the next number won't be as it is the next number up and numbers in the 3 x timetable only occur every third number. The sum of your two numbers added together minus your second number always gives your first number.

RESULTS

O : "next number" condition not observed
Nos. 15, 24, 25, 26, 28, 29, 33

X_2 : several trials, conclusion "yes"
Nos. 20, 43*,45

X_1 : misinterpretations (mainly sum is M(3)); counter-example leads to "no".
Nos. 5, 13, 14, 21, 27, 46*

F : full check of all cases
Nos. 2, 3, 8, 9, 10, 11

D : few examples, attempts at explanation
Nos. 31, 32, 41, 47

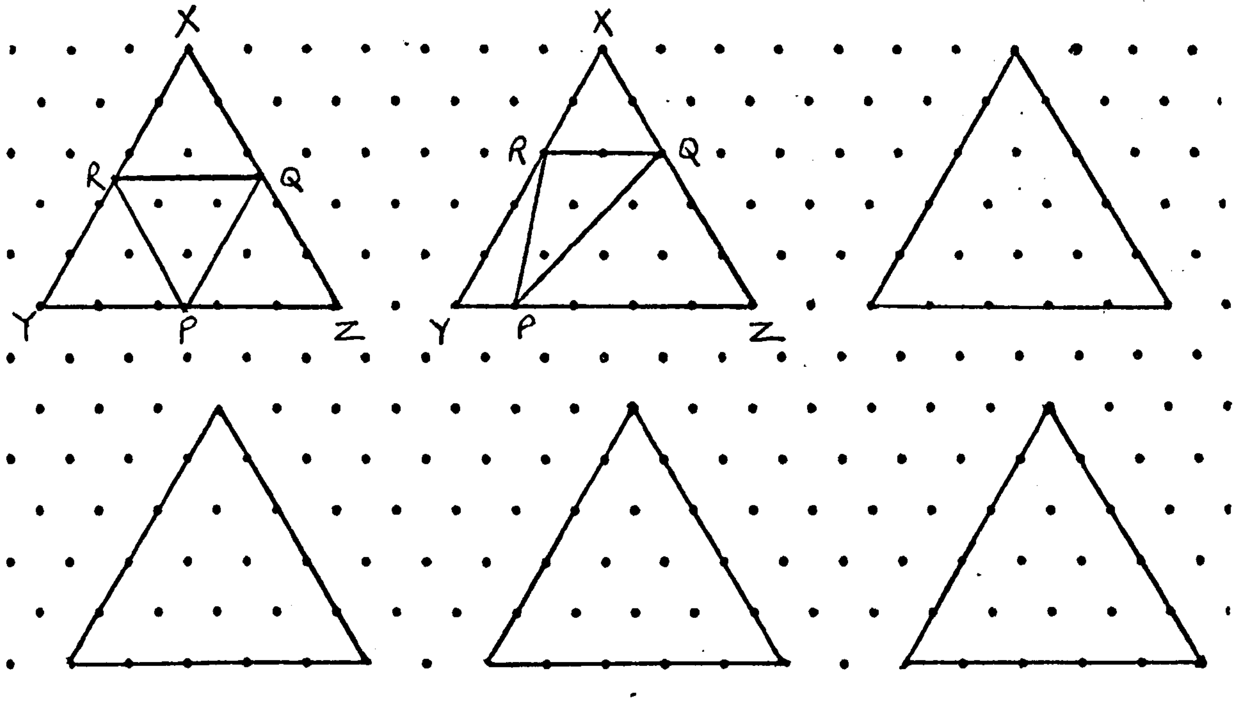
Rgr : restatement of generalisation
Nos. 4, 7*

Rcd : some analysis, identifies subclasses
Nos. 6, 12*,42

Einc : explains result for at least one subclass
Nos. 1, 22, 23*,44

* Scripts whose numbers are starred appear in the Appendix

TRIANGLES



The points Q, P and R can be anywhere on the sides of the big triangle.

In the first triangle above PQR is equilateral.

1. Can PQR be equilateral with P, Q and R in any other positions? If so, what are all the other possibilities?

2. Explain why your answer is right.

RESPONSES AND CATEGORIES

Triangles requires the generation of sets of possibilities, with arguments to justify the completeness of the results. The sets are (a) the set of "tilts" with P,Q,R on YZ, ZX, XY respectively and YP = ZQ = XR, (b) the set of "parallels" with, say, P at X and Q, R on XY, XZ with XQ = XR, and similarly with P at Y or Z, (c) the set of "re-namings" which generates six triangles from each previous one by permuting the vertices P, Q, R. Proofs of completeness in cases (b) and (c) are straightforward; in case (a) the best approach is probably to consider the consequences of rotating about the centre of the big triangle if that of the small triangle does not coincide with it.

A considerable number of pupils are unable to keep to the conditions of this problem, and draw triangles PQR with vertices not on the sides of XYZ. See, for example, Nos. 24, 47. These nine are placed in category O. A further three misinterpreted 'equilateral' to mean 'congruent', but these are categorised elsewhere, accepting their interpretation.

For most pupils the main task in this problem is finding as many new triangles as possible. There is a range of responses of different quality in this empirical phase. Those which simply find some new triangles, ignoring the requirement to find all, are categorised X. Those which show some degree of system, some awareness of the need to find <u>all</u> triangles, are categorised S_1. Those which are clearly attempting to find all, and succeed in finding at least some complete subsets, are categorised S_2. Some examples follows:

X: No. 23 shows 4 correct parallel-type examples and 2 incorrect 'tilts' and says because all the triangles are equilateral and points P,Q,R touch the sides.

S_2: No. 2: Set of 33 possibilities (parallel and renaming types; no tilts). If the triangle found with its numbered points can be used a further two times these seem to be 33 possibilities.†

S_1: No. 13: 2 mirror-image tilts shown. Yes the others are 3 and 4 (the two shown) There are only 3 combinations because the triangle has 3 equal sides.

† This has worked through two types of variation (missing some cases) before closing the search.

Two cases of Rgr appear. These are a little different from the standard type as for example Add and Take; they are assertions that there are no more triangles, unsupported except by the empirical work.

No. 12: (shows 6 trials, no successes) No. PQR cannot be equilateral with P, Q and R in any other positions because the triangle XYZ is an equilateral triangle and only if the points QPR are in the middle of each of the 3 sides of the large triangle can the smaller triangle be equilateral.

The Rcd category scripts go beyond this by making some analysis of how different possibilities are generated, but not connectedly.

No. 7: (shows minimal tilts) PQR shown in the first diagram can be placed in three different positions. As there are three sides to this triangle then each side is able to be moved around three different ways still making PQR equilateral.

This makes some analysis of the situation and relevantly mentions rotating 3 different ways.

Category Einc is used for those three responses which analyse connectedly the conditions for obtaining new triangles, either by moving the vertices of the given one or ab initio.

No. 1: In the first triangle where PQR is equilateral the points P, Q and R are all exactly halfway between the lines YZ, ZX and XY respectively. The triangle YXZ is split into 3 equilateral triangles, all exactly the same size. When either R, Q or P are moved so they are not in the centre of their particular line, the triangle is not split into 3 triangles with the same dimensions etc. so the triangle PQR will not be equilateral. By moving any one of the points up or down their line you are making it either longer or shorter. If you make it longer another line will be made shorter, so the resulting triangle will not be an equilateral as all equilateral triangles have sides of the same length.

This fails because the analysis considers only one of P, Q, R moving at a time, and so misses the possible tilts - as well as missing the other possible types of variation. But it has the approach needed for an explanation of this problem.

RESULTS

O : blank, misinterpretation or vertices not on sides of XYZ
Nos. 21, 24*, 25, 26, 28, 29, 43, 44, 45, 46, 47*

X : finds some cases, ignores request for "all"
Nos. 5, 10, 11, 20, 22, 23, 31*, 42

S_1 : some awareness of requirement of all, some systematic subsets
Nos. 6, 13*, 15, 33

S_2 : some complete subsets; seeking all
Nos. 2*, 27, 32

Rgr : restates data
Nos. 12, 41

Rcd : some analysis of possibilities
Nos. 3, 4*, 7, 8

Einc : connected analysis of how all of some subset of possibilities obtained
Nos. 1, 9*, 14

* Scripts whose numbers are starred appear in the Appendix

COMBINED RESULTS

The establishment of reliable allocation to general categories of the responses to this varied set of problems requires the formulation of local criteria for the different categories for each problem, then the allocation of scripts, using the general definitions of the categories supplemented by the particular local criteria for each problem. The arguments for the validity of the local criteria are contained in the sections discussing each problem. The reliability of the judgements made in allocating scripts was checked by having them re-allocated by a second marker. The second marker was provided, for each problem, with a list of the categories, in which a total of six scripts had been entered by the experimenter, these being spread over the different categories. If necessary, the basis of allocation of these six scripts to their categories was discussed. The second marker then allocated the remainder of the scripts according to his own judgement. The proportion of *these remaining scripts* which were allocated to the same category by the experimenter and the second marker is shown as the agreement coefficient in Table 1; the range of this coefficient is from 0.69 to 0.97. The disagreements were all between adjacent categories or sub-categories (e.g. X_0/X), except for the following. In Add and Take, on one script there was disagreement between Rgr and Einc, and on one script between Rcd and E comp; these were in the same direction as 8 of the 9 adjacent-category disagreements; that is, there was a consistent tendency for the second marker to grade slightly lower than the first. In Adding A Nought, there were three X/Rgr disagreements, arising from the difficulty of judging whether examples were being used as *evidence* on which to base the conclusion (X), or as *illustrations* to a deductively-based conclusion (Rgr). On Diagonals of a Polygon, there were three X/Rcd2 disagreements arising from a similar difficulty of judgement. On Noughts and Crosses there were two Rcd/Einc disagreements; in this problem these categories are non-adjacent, being separated by Emist.

A less rigorous procedure was adopted for One and the Next and Triangles, since these were the first problems categorised, and the second marker was unfamiliar with the material. In this case the second marker was informed of the experimenter's categorisation of each script, and asked simply to identify cases of disagreement. There was disagreement over one script in the set for each of the two problems. The final allocations to categories, for all problems, were made after the reliability check had been conducted and disagreements had been discussed with the second marker.

Table 2 shows the number of responses to each problem in each of the major categories. For this table, certain combinations of categories have been made. The category F responses, which occur only in One and the Next, have been entered in the E comp column; in Triangles, the S_1 and S_2 categories have been entered under D and Einc respectively; and the E_o responses to Stamps are entered in the Einc column. In each case these are shown separately from any other entries in the same cell.

TABLE 2

TEST	O,Ø	X	D	Rgr	Rcd	Einc	Ecomp	Total	A.C.*
Add and Take	3	13		9	3	6	7	41	.69
Coin Turning	20	1	1	3	9	3	4	41	.94
Adding a Nought	3	7		20	8	1	2	41	.77
Diags of Polygon	3	24	2	1	9	2		41	.86
Midpoints	9	12	2	13	2	3		41	.74
Noughts & Crosses	4		8		5	9	15	41	.80
Stamps	3	9			10	3Eo	11	36	.97
Quads	7	15	3		11			36	.77
One & the Next	7	9	4	2	3	4	6F	35	-
Triangles	11	8	$4S_1$	2	4	3 $+3S_2$		35	-
Totals	70	98	24	50	64	37	45	388	
%	18%	25%	6%	13%	16%	9%	12%		

* Agreement Coefficient

Tables 3 to 7 show to what extent each pupil performed at the same level in the two problems constituting his test. Each of the 41 pupils who worked the first two problems is denoted in Table 3 by his serial number. Thus the table shows that pupil No. 38 was classed as X_1 on Add and Take, and as 0 on Coin Turning. The (different) 41 pupils who worked the third and fourth problems are similarly represented in Table 4 by their serial numbers; and so on.

ADD AND TAKE

COIN TURNING

	O	X_1	X_2	X_3	X_4	Rgr	Rcd	Einc	Ecomp	Total
O		38 39	18			3				4
O_1	34	37	1 14 17			42 4 7 22 23 33 36			41	13
O_2	31		10	5						3
X								12		1
D			2							1
Rgr						30		21	28	3
Rcd				26	40		43	8 13 25 27	19 35	9
Einc							9 15		20	3
Ecomp	①				11				24 29	4
Total	3	3	6	2	2	9	3	6	7	41

TABLE 3

DIAGONALS OF A POLYGON

ADDING A NOUGHT

	O	X_o	X	D	Rgr	Rcd_1	Rcd_2	Einc	Total
O		26 40	22						3
X_1	18	2 25 26							4
X_2		3 28					23		3
Rgr_1			20		1	41			3
Rgr_2		38	5						2
Rgr_3	15	12 8 13 14	19				29	16	8
Rgr_4		37 6	4	24 33		32	9		7
Rcd	7	11	39 27 31			34 17 21			8
Einc			10						1
Ecomp						35		30	2
Total	3	15	9	2	1	6	3	2	41

TABLE 4

NOUGHTS AND CROSSES

MIDPOINTS

	O	D	Rcd	Einc	Ecomp	Total
O	14 18	7 33		17		5
Ø		9		8	5 6	4
X_1	26	13 19		25	16	5
X_2	40	24	2 31	35 11 22		7
D			39		21	2
Rgr_1		28	15 27	34	1 20 29 30 37	9
Rgr_2		38		3	23 32	4
Rcd				4	12	2
Einc					10 36 41	3
Total	4	8	5	9	15	41

TABLE 5

STAMPS

QUADS

	O	X	E_o	Rcd	Ecomp	Total
O_1		29			1	2
O_2		36	33	7 20	13	5
X_1	16	2 25	32	27	4 14	7
X_2		12 6		15 5 10	21 11 24	8
D		9		23	34	3
$\overset{o}{\text{Rcd}}$	19		28	22 31	3	5
Rcd	18	8 26		35	17 30	6
Total	3	9	3	10	11	36

TABLE 6

TRIANGLES

ONE AND THE NEXT

	O	X	S_1	S_2	Rgr	Rcd	Einc	Total
O	24 25 26 28 29		15 33					7
X_1	21 46	5	13	27			14	6
X_2	43 45	20						3
F		10 11		2		3 8	9	6
D	47	31		32	41			4
Rgr						4 7		2
Rcd		42	6		12			3
Einc	44	22 23					1	4
Total	11	8	4	3	2	4	3	35

TABLE 7

DISCUSSION

(1) Categories and Stages

The tables showing pupils' performances compared on the two problems which each attempted show that the hypothesis that they will make similar responses to the two problems is untenable in general. One table, Add and Take/Coin Turning, shows a fair degree of association, and if Rcd, Einc and Ecomp responses are taken together, and compared with the remaining categories, on each problem, the resulting 2 x 2 table shows a significant association at the 0.001 level.

	O,X,Rgr	Rcd,E
O,X,D Rgr	21	4
Rcd,E	4	12

$\chi^2 = 10.6$, $p = 0.001$

Table 8

None of the other tables shows significant association. There appear to be two reasons for this. One is that the ten problems are very diverse indeed, both in their nature and in the levels of difficulty they presented to the pupils, and hence in the responses. This is good for the descriptive aspects of this study, in that it has produced a wide range of examples of pupils' mathematical activity; but it has made it difficult to establish coherences across problems - at least by direct statistical methods. Some common characteristics do exist and they will be described below. The second proposed reason for lack of association is more fundamental. It is that a pupil's level of explanation in a particular problem depends directly on his familiarity with the concepts and relationships involved, and his degree of insight into the problem. He seems willing to accept whatever level of insight he can reach in the problem as his explanation, and not to be aware that this is incomplete. This contrasts with the situation of the trained mathematician, who will know whether or not he has explained a result satisfactorily. The evidence for this explanation of the results is as follows:

(1) The remark "I can't explain it" or the equivalent appears scarcely at all in these 388 scripts.

(2) The prevalence of Rgr responses, the general "re-statements" which give the impression that the writer feels he has settled the question, but which add nothing to what is in the data.

(3) The general tone of most of the responses is as if a satisfactory explanation has been given. The ones quoted in the previous sections provide a liberal supply of examples of this.

The most striking contrast in levels of explanation on a pair of problems is between Noughts and Crosses and Midpoints. Here 24 pupils trace the moves of the game situation through several steps to its conclusion, but only 3 of them give as much as an incomplete explanation of Midpoints. The scripts of the remaining pupils on Midpoints were carefully studied by the experimenter and the second marker, without being able to detect any sense that the pupils were aware of the inadequacy of their explanation. (It was not possible on this occasion to conduct follow-up interviews, but this is a suggestion for future research).

(2) Context

This states that familiar non-mathematical situations, number and geometrical situations will be in this order of increasing difficulty with regard to the provision of explanations. The number of Einc and E comp responses combined are, for the non-mathematical situations; Noughts and Crosses, 24; Coin Turning, 7; Number: Add and Take, 13, Stamps 14, Adding a Nought 3; Geometrical; Quads 0, Triangles 3 + $3S_2$. The numerical evidence appears to support the hypothesis, but the problems are by no means equal in all respects other than context. However, the good performance on Noughts and Crosses does appear to be attributable to familiarity, and the poor performance on Quads partly to lack of agreed basic principles. Adding a Nought shows that what is familiar as a process may be difficult to explain. In Coin Turning the generally poor performance was due partly to inability to perform and record the manipulations without mistakes, and partly to the factor mentioned under (4), that is, the tendency not to present full explanations in impossibility situations. This factor is affecting Stamps too, but this is a particularly easy problem in which to reach a conclusion, so that even after this loss the number of full explanations is high.

(3) Finite Sets

The method of exhaustive check is adopted by noone in Add and Take, and by 6 pupils in One and the Next. It seems that in Add and Take, conviction of the truth of the generalisation comes to most pupils fairly quickly and even those pupils who do not actually give an explanatory response think they have done so. In One and the Next,a much harder problem to explain, there are pupils who cannot achieve an explanation but, being aware of this, adopt the next best alternative for justifying the result. Or it may be that the less obvious nature of this generalisation makes extensive empirical check appear more reasonable. Taking the two problems together, it is clear that a large number of pupils could have given more cogent justification of the results than they actually did, if they had employed such a check, but that they did not do so. This supports the hypothesis that pupils are more concerned with illumination than with certainty.

(4) Impossibility Situations

These are Coin Turning and Stamps. Coin Turning failed to be particularly attractive from this point of view because many pupils made mistakes which led to the false assumption that the problem was soluble. From those who did reach the correct conclusion, the explanations were, as predicted, incomplete. Stamps did produce the expected good performance, and with an unexpectedly large number of responses presenting a fully argued case.

(5) Use of Record (Coin Turning)

This has been discussed under the particular problem. The result was as predicted; pupils who could not solve the problem could not, in general, construct and use correct records.

(6) Use of Algebra

Two pupils used algebra fully, and one partially, for Add and Take; none did so for One and the Next. The hypothesis is not confirmed. It would appear that the teaching of algebra to these pupils has not made them aware of its value in expressing general number properties and relationships. Perhaps more experience of problems of the kind used in the test would help.

(7a) Factors determining performance

The three proposed factors were (i) specific knowledge required, (ii) logical complexity, and (iii) possession of general strategies. The first has been commented on already; it was clearly very important. There was also plenty of evidence of the operation of the second factor. For example, in Triangles, some subjects drew triangles some of whose vertices were not on the sides of the big triangle, (e.g. No. 43) and a few interpreted "equilateral" to mean "congruent"; in One and the Next, some failed to observe the "next number" condition, while others assumed that the last number of the three had to be a multiple of three; in the second step of Add and Take some took the first number from the second number instead of from ten; in Diagonals some drew lines in the figure which did not terminate at vertices, and others drew too few diagonals or miscounted the ones they had drawn. In some of these cases the lack of knowledge of a particular concept is the reason for failure, but in most the breakdown appears to be an inability to coordinate all the data - the information-receiving apparatus seems to be overloaded. In some cases this complexity factor interacts with the knowledge factor and the grasp of a concept which is not well understood is lost. The failures to attach P, Q, R to the sides of XYZ in Triangles must be attributed to the complexity factor, as the meaning of the individual terms is quite clear; whereas to read "congruent" for "equilateral" is an equally clear knowledge failure. Only a few mistakes in Diagonals are attributable to lack of knowledge of the concept; missing diagonals and miscounting are failures of coordination. The misreading of Add and Take, which was comparatively common, must be attributed similarly. It seems to arise as follows:

Subjects have written a chosen number (say 3), and have added it to ten (13). Now they are asked to "take the first number away from 10." What they do is to take the first number away not from 10 but from 13, which they have just written down; it must be that their own written 13 is much more immediate to them than the 10 which only appears on the printed page. This is not unlike other documented phenomena related to the perceptual field - ten-year-olds will often write, in $15 \div 3 = \Delta \div 3$, the answer 5. In One and the Next the significance of the term "next number" may conceivably be missed because of unfamiliarity with it in this type of context; but both this and the reading of "one and only one of these numbers" and "the last of these numbers" are more plausibly attributed to the complexity factor. This factor is well recognised in the literature of psychology and mathematical education (Collis 1975b, p. 76) and it is related to the "acceptance of lack of closure" (Lunzer, 1973b). It is also implicitly recognised by the mathematics teacher who knows that when a pupil says "I don't understand this problem", often he needs only to be asked to read it aloud to achieve comprehension, his previous sketchy reading having failed to collect all the data.

In Midpoints, another type of logical failure was prevalent, that of not distinguishing the finding of the midpoint geometrically from finding its coordinate from the formula; "it's the average" seemed to be a statement which linked the two aspects and made them difficult to separate. This last failure might be aided by the development of a strategy of stating data and conclusion; on the other hand, deciding what is data and what conclusion presents the same difficult problem of separating the two aspects; naming them may or may not help. This is similar to the once-familiar problems of understanding formal Euclidean proof.

Thus logical complexity is clearly an operative factor. Whether the possession of general strategies is helpful is a question which cannot be answered in this experiment. What can be done is to identify where possible the points of breakdown in the problem and to formulate general strategies whose value might be tested in a further experiment. This is part of the task of the next section.

(7b) Strategies for Improvement

There are first a number of what are not so much strategies as *strategic concepts*, which might plausibly be taught to pupils and which might improve performance. (1) One is simply the concept of *all cases*. Pupils aware of this should be able to replace X responses by F, in problems such as Add and Take, and, where not possible, as in Diagonals of a Polygon, to acknowledge the logical gap and to take steps to seek a general insight. (2) The concepts of *data and conclusion* might help, as suggested above. (3) Several strategies which might be described as *being systematic* could be very helpful. The notion of *identifying different types of case* and dealing with them successively is required for One and the Next (the three possible types of starting number, M(3), M(3)+1, M(3)+2,) Triangles (the possible different *kinds* of new triangle), Stamps (0, 1, 2 or 3 twenties together with appropriate numbers of 8s) and could be useful in Adding a Nought (one-digit numbers, 2 digit and so on.) The use of *mathematical induction* (under a less sophisticated name as "adding one") could be helpful in Diagonals of a Polygon in teaching towards the one-to-one relation between radiating diagonals and their terminal vertices.

The facilitation of progress from Rgr and Rcd stages to E is not easy, on account of the pupils' apparent satisfaction with their existing performance and the difficulty of showing them convincingly what it lacks. The best strategy is probably to *vary the situation*, in any identifiable way. In Add and Take, one could ask how the process could be altered to give something other than twenty; or could vary the chosen number by one at a time and see what happens. In Midpoints, varying by one was used by one of the few pupils who achieved Einc (No. 10). He wrote $35 \underset{1}{\rightarrow} 36 \underset{1}{\leftarrow} 37$, then $14 \underset{3}{\rightarrow} 17 \underset{3}{\leftarrow} 20$. In the same problem, *trying big numbers* was suggested in the questions, and seems to have helped in a few cases by forcing attention to the processes being performed. Trying special numbers loses the generality but may be a step towards a fuller insight; it helped another pupil on Midpoints (taking A at O, and general B: No. 36). These may all be feasible strategies or concepts for improving explanations (and insights generally), but the first requirement is a context in which they can be developed, that is sufficient experience of solving problems and giving explanations. The present experiment has shown that such problems need to be in contexts which are already familiar. The development of such work thus may be competing for course time with the teaching of new concepts. It may be that at present the balance allows too little time for the development of these process abilities as distinct from new content. The second requirement for the development of improved explanations is the ability to distinguish an explanation from a lower level statement, in particular from a restatement. This recognition may be developed by discussing different proposed written explanations of a given result, with a group of pupils. This ability to consider and evaluate explanations is probably a very important step towards being able to improve one's own explanations. An experiment to test the efficacy of these methods in improving sixth formers' understanding of proof forms the subject of the next chapter.

A different and complementary approach to the teaching of the *axiomatic* concept of proof would be to see it embodied in a small-scale system. Possible systems are the seven point geometry, Steiner's (1968, 1975) voting systems, the school version of Euclidean geometry - and Boolean algebra. Euclid's Elements itself formed a model from which many generations of thinkers derived their notions of deduction and of axiom systems, perhaps in a way not essentially different from that in which multiple classification is embodied in logic blocks or the place value system in multibase blocks. We are here discussing a study in which one of these systems is actually built up with reflection on its deductive structure, so that the concept of axiom-system is abstracted from it. This would be for advanced and/or able pupils, since both experience and the present results show that most school pupils are far from such ideas at present. An outline of such a study of Boolean algebra formed a partial guide for the teaching programme of the experiment of Chapter 10; it appears in Appendix 10.

CHAPTER 10

THE IMPROVEMENT OF GENERAL MATHEMATICAL STRATEGIES BY TEACHING: A SIXTH FORM EXPERIMENT

IN APPENDIX 10

INTRODUCTION

Underlying all the research reported in this thesis has been the question "To what extent can general mathematical strategies be learnt?". In the age cross-sectional experiment of Chapter developments with age were noted in the coherence of deductive arguments and in the recognition of the non-validity of empirical extrapolation from a number of cases; but whether these developments could be accelerated by suitable teaching could not be determined from the data of that experiment. The ten-problem experiment of Chapter 9 exposed more fully both these and other strands of development with regard to generalisation and proof, such as the construction of classes of cases and the sense of what are acceptable as agreed assumptions on which arguments may be based, but, again, this did not study the effects of teaching. The General Mathematics Test used with the South Nottinghamshire Project's first year secondary classes showed their superiority over non-project classes in the generation of examples to test generalisations or to meet criteria, and to a smaller extent in the giving of explanations. This does suggest the susceptibility of certain strategies to educational influence, but is based on the comparative performance of two groups on a single occasion so that the differences cannot unquestionably be attributed to differences in their secondary school experience rather than to earlier differences in experience. The experiment of the present chapter studies the effect, on the generalisation and proof attainments of a sixth form group, of specially-directed teaching over a period of about six weeks. It also involves the development of a more objectively-markable test, based on the responses obtained in the ten-problem study. This is used as both pre- and post-test, and the results of the trained group compared with those of a control group. The experiment was conducted in collaboration with Mrs. B.C. Edmonds, who taught both the classes as part of her normal teaching duties in the sixth form college used. The present writer was responsible for developing and providing the criterion test, and for consultation regarding the teaching programme. The detailed teaching was the responsibility of Mrs. Edmonds.

PREVIOUS RESEARCH

Several researches exist which bear on the learning of general strategies. The first two to be quoted concern general heuristic strategies, the remainder, mathematical strategies, except for one which is about an aspect of scientific method.

Covington and Crutchfield (1965) developed a General Problem Solving Program based on strategies such as planning one's attack, searching for uncommon ideas, transforming the problem and using analogies. The setting is not mathematical, but consists of stories of how two children solve a number of puzzles and mysteries with the help of their uncle, and high school science teacher and part-time detective. Groups of 10 and 11 year old pupils studied this program with their teachers and were tested for problem-solving ability, creative thinking and attitude. They showed considerable gains on all of these, and the gain in problem-solving ability was still significant five months later. Other workers followed up this work though with less successful results (Kilpatrick, 1969).

Lucas (1974) studied the effect of heuristically-oriented teaching of a university calculus course on the students' ability to solve problems. An experimental group and a control group each contained about 15 students; both groups received "enquiry-style" instruction except that with the experimental group, the same style was adopted during problem-solving sessions, and "the problems were discussed more thoroughly for the sake of problem-solving", whereas the control group had "an expository treatment of problem solution". Also, the experimental group received 12 papers outlining and demonstrating various heuristic strategies, and their problem solutions were graded to reward heuristic usage. This programme lasted for eight weeks. Pre- and post-test interviews were administered: during these, each subject talked through the solution of seven problems. Significant differences on the post-test, favouring the experimental group, were found on total score for the problems, on plan and approach, and on the strategies of using a well-chosen mnemonic notation, using the method or result of a related problem, and of separating and summarising data. There was no difference in the frequency of use of diagrams, but there was a lower incidence of incorrect diagrams in the experimental group. Among the results, the first was significant at the .005 level, the others at levels of .025 or .05, which suggests that the simple, well-defined strategy of choosing a mnemonic notation is more susceptible to teaching than the others.

Brian (1966) analysed the mathematical process into (1) constructing mathematical models, (2) conjecturing, (3) settling conjectures as true or false, and (4) using known or given axioms, theorems or algorithms on problems where they clearly apply. A short course (about two weeks) designed to help students acquire these processes was given to a group of 17 college students. This resulted in a significant improvement on the third process, the settling of conjectures, but not on the others. The fourth process is described by Brian as the primary aim of most present mathematics teaching. In terms of the strategies we have defined in Chapter 4, Brian's first process is formulating questions and making representations, and his second and third processes appear as higher and lower levels of generalisation. Thus Brian's result suggests that testing generalisations may be easier to learn than making generalisations, or formulating questions, if we may assume equal emphasis on the different processes in the teaching he provided.

Wills (1967), constructed a programmed unit to teach the following problem-solving procedure; (a) a difficult problem is given, requiring a certain generalisation, (b) similar, but simpler problems are presented, (c) the results of these are tabulated so as to reveal a pattern, (d) the generalisation suggested by the pattern is applied to the initial problem, and the result checked. The subject matter was recursive definitions and figurate numbers; the age of the students is not stated, nor is the length of the instructional period. The pre- and post-test comprised 60 problems on a wide variety of topics which could be solved by steps (b) and (c), that is, by the generation and tabulation of examples, from which a generalisation can be made and applied to the problem. 561 students took part, in three groups; the first used the programmed unit only, the second also had back-up instruction from teachers, while the third was a control. Both of the first two groups made highly significant gains compared with the control, and there was no significant difference between these two. In terms of the set of strategies discussed in this thesis, Wills' experiment shows successful teaching of one well-defined strategy for generalisation, including the generation of relevant examples and the making of generalisations. Wilson (1967) attempted to improve performance on theorem-proving tasks by either (a) task-specific heuristics, or (b) identifying data and conclusion and seeking to make a connection, or (c) planning a solution in general terms. The task-specific heuristics did not improve performance, even on the tasks at which they were directed; of the general heuristics, planning a solution was successful in only a few of the tasks. Thus this attempt to teach strategies for theorem-proving tasks was largely unsuccessful.

Post (1967) had ten classes of 12 year olds given a six week period of instruction and practice in the processes of problem-solving, but obtained no significant differences in comparison with a control group.

Lawson and Wollman, (1975) trained classes of fifth and seventh grade (10-11 and 12-13 year olds) in controlling variables, in the Piagetian task with bending rods. Transfer was investigated (a) to another task involving controlling variables (the pendulum), (b) to one (the beam balance) involving a different aspect of formal reasoning, in this case proportional reasoning, and (c) to other tasks, e.g. Peel's passages for showing imaginative judgements. Transfer was obtained to the task (a) involving the same strategy but not to (b) or (c). The training consisted of evoking the subjects' intuitive judgements about "fair tests", of clarifying and exposing their judgements to them, supplying verbal forms focussing the experience, e.g. fair test, variables, all factors the same except the one being tested, and getting the pupils to describe their actions and the rationale for them.

To summarise the results of these researches, it is clear that it is possible to teach at least some of the strategies which comprise the mathematical process, but that some are easier to teach than others. The *testing* of conjectures (Brian), the adoption of a mnemonic notation (Lucas), the finding of a generalisation by generating and tabulating examples (Wills) and the learning of one strategy for scientific experiment (Lawson & Wollman) are shown here as the most susceptible to teaching. Wilson's failure may perhaps be attributed to a less careful identification of the strategies actually required for his tasks. But, as in all teaching experiments, fully consistent results cannot be expected because of the difficulty of specifying the teaching in sufficient detail to identify the significant aspects.

HYPOTHESIS

The reports of these researches say so little about the methods used to teach the strategies in question that it is difficult to make useful deductions regarding what methods are most likely to be successful.

Gagné (1970) points out that strategies are essentially higher-order rules and that they are normally learned in the course of performing the activity to which they relate. An appropriate method is therefore experience of the activity, preferably with verbal guidance. Thus, to be specific, the suggested method for learning strategies of generalisation and proof is to provide problems involving them, to discuss the solution of the problems and to identify and name the concepts which have emerged as important in the present studies, such as "all cases", extreme values, iteration, "re-statement", connected, data, conclusion, agreed starting points. This can be done to some degree in any mathematical context, but as the Ten Problem Study (Chapter 9) shows, it is best done in the context of already-familiar ideas. In designing the present experiment, which has to take place under normal school conditions, about two thirds of the teaching time will need to contribute to the learning of syllabus content; in this, process aspects will be emphasised as much as possible. The remaining third of the available time will be given to material on familiar content, chosen particularly to illuminate the concepts listed above. The hypothesis is that *such teaching will lead to improvements in pupils' concepts and strategies of proof.*

CRITERION TEST

The behaviours which it was aimed to improve in this experiment were essentially those studied in the Ten Problem Study - the making of generalisations and the giving of explanations and proofs - but a more accurate method of measuring them was required. This implied more and shorter questions, and the same set of questions for each pupil. The solution adopted was to take some of the Ten Problems, to present them along with correct and incorrect responses, and to ask the sixth formers to distinguish these and to give reasons. (This is a different activity, and a more concentrated one, than the spontaneous provision of generalisation or proof but it is highly relevant to proficiency in these strategies.) For example, the first questions are based on the problem Add and Take. This is given in essentially its original form and the question continues:

> Susan:
>
> The result will always be 20. If you chose a number between 1 and 10 and add it to 10, then if you take the first number away from 10 it will be whatever is needed to make 20.
>
> Yvonne:
>
> Always 20. Whatever you add you always take it away so it cancels out. But as you add 10 and take the number from 10, you get double 10 which is 20.
>
> Have these pupils proved their answers?
>
> Susan's: Yes/No . Yvonne's: Yes/No
>
> Give your reasons:
>
> ____
>
> ____

A question based on One and the Next, and contributing to the factor X, is

Amanda: 1 + 2 = 3, 2 + 3 = 5, 3 + 4 = 7. So John is right.

Bob: 4 + 5 = 9, 7 + 8 = 15, 1 + 2 = 3, 10 + 11 = 21
So John is right.

Have they proved their answers? Amanda: Yes/No

Bob: Yes/No

1. Say why you think so:

2. Say which you think has the better set of examples.

Amanda/Bob

Give your reasons:

The full test and notes on marking are included in Appendix 10 together with notes on the development of the test, and item and factor analyses.

The factors included in the test are:

F/X: The recognition of the conclusiveness of a full check of a finite set of cases, and the non-validity of a conclusion based on a subset only.

C: The construction of an identified set of subsets of the relevant cases, and dealing with these exhaustively.

E: The recognition of a genuine explanation as distinct from a re-statement of data or an irrelevant statement.

Eg: E in relation to geometrical situations, in particular, the polygon problem.

EE: Ability to state explicitly the nature of the distinction between an explanatory and a non-explanatory response.

SAMPLE

The experimental class consisted of 12 first-year sixth formers (5 boys, 7 girls) who were studying Boolean algebra as one topic of their A-level mathematics course. The first control class comprised 11 pupils (all boys) in a parallel group, studying Mechanics. A second control group consisted of 3 pupils of the same age group, taking English A-level with no mathematics. Strict matching of the two groups was not possible; the O-level mathematics grades of the experimental group were slightly higher than those of the first control. The criterion test was administered to all three groups immediately before and after the teaching programme.

TEACHING PROGRAMME

The first two classes were both taught by Mrs. Edmonds for two hours per week and studied mathematics with other teachers for a further three hours per week. The experimental teaching programme occupied a total of 20 hours. During this time, the first control group studied Mechanics for 20 hours; the second control group studied no Mathematics at all. The design of the experimental teaching programme was to some extent limited by the fact that it had to serve as part of the normal A-level course for these pupils; a fact of which the teacher, being their normal teacher, was very much aware. Hence, of the 20 hours available, 13 hours were occupied by the teaching of the Boolean algebra topic with special emphases intended to develop the awareness of aspects of proof within the topic, and the remaining 7 hours by work and discussion based on some of the material from the Ten Problem study (Chapter 9) which had not been used in constructing the criterion test. (One problem was discussed which was also in the test; this is referred to below.)

The first of the special emphases given to the Boolean algebra course was aimed not directly at improving performance on the criterion test, but at developing the pupils' confidence in evaluating a mathematical situation for themselves, at using symbolic forms to model a given situation and the studying of the relationships among the concepts extracted, considering whether they can be simplified and reduced to a smaller set. Thus the laws of Boolean algebra were not given initially, as rules to be practised, but derived by examination of the properties extracted intuitively from a body of logically connected information.

Similarly, in subsequent lessons, starting with some experiment with a "switch board" containing switches and lamps which could be connected in various ways by plugging in wires, some laws of switching circuits were observed, codified and compared with those derived from logical situations. Broadly, the aims of this approach were (a) the development of the pupils' understanding of the mathematical process, and (b) to develop their reliance on their own powers of reasoning rather than simply on received instruction. (See article in Appendix 10) The second special emphasis of the Boolean algebra course was that pupils were encouraged to discuss and criticise their colleagues' proposed results and reasons. This emphasis became the basis of the last 7 hours of the 20 hour programme, when the validity of proof-arguments was examined more closely.

The first of these sessions discussed the validity of checks by examples to settle conjectures such as "If $a * b = a + b - ab$, with a, b integers, is * a commutative operation"; and similarly, if $a \otimes b = a/b^2$. The contrast between disproof by a single counter-example and the need for a general argument for proof of the positive statement was exposed. In subsequent sessions the problems Stamps, Quads, Coin Turning, Midpoints and Triangles were discussed. On some occasions pupils were asked to write proofs, and these were then duplicated and circulated to the whole class for criticism. (For further details of this teaching see Edmonds (1976)).

RESULTS

It was expected (i) that the teaching programme would improve the pupils' performance substantially on all aspects of the criterion test; (ii) that the first control group would make small gains arising from their normal study of mathematics; and (iii) that all groups would show small gains due to previous experience of the test.

The mean scores of the experimental group and the two control groups on each of the six scales, and on the total, are shown in Table 1.

Group	N	Scale	X	F	C	Eg	E	EE	Total
		Max	3	3	2	2	6	4	20
E	12	Pre	1.25	2.33	0.75	1.00	3.08	1.08	9.58
		Post	2.50**	2.50	1.25	1.25	3.75	1.33	12.58*
C_1	11	Pre	2.00	1.09	1.00	1.27	2.36	1.09	8.82
		Post	2.00	1.27	1.09	1.27	3.18	1.36	10.18
C_2	3	Pre	0.67	1.67	0.67	0.67	1.00	0.00	4.67
		Post	0.33	2.00	1.00	0.67	1.33	0.33	5.67

** Gain significant at .01 level in comparison with combined control groups;
* At .05 level.

TABLE 1

The estimates of significance of the gains in Table 1 were obtained by first calculating gain scores for each individual, using a short specially-written Fortran program and then applying a correlational analysis, which compared gains within and between the two groups (experimental and combined control) using Tukey's Q test. (Youngman, 1975)

DISCUSSION

The results offer partial but not total confirmation of the hypothesis stated above. Both control groups show small gains, though that made by the first control is not significantly different from that made by the small second control group. The overall gain by the experimental group is significant, but this arises mainly from Scale X. The experimental group gains more than the main control on C and Eg but not significantly so, and scales F, E and EE show no differences between the groups. The main conclusion to be drawn is probably that the distinguishing of valid from invalid informal proofs, as required by the criterion test, requires judgements of relevance and logical completeness which need more time or more intensity of teaching - or more general mathematical maturity - than they received here. On the other hand, the recognition of invalid inference from a limited number of examples requires only fairly superficial observation, together with a general sensitivity to the matter, and this was quite easily learned.

Edmonds (1976), commenting on these results, suggests that there were improvements in the experimental group's performance which did not appear in the test results. Their more critical attitude led to the rejection of some proposed proofs which the mark scheme defined as acceptable. For example, on Add and Take, Ann's explanation that the addition of a positive and a negative number gave zero was judged to require proof; similarly, her omission of an explicit statement that the 20 arose from adding the remaining two tens was criticised. These comments highlight the fact that the completeness of a proof is a matter of judgement of what is crucial to this particular result, (so needs mentioning); and what can be regarded as already agreed among those reading the proof.

The results of the second control group, of pupils taking A-level English, are of some interest. Their particular weeknesses, compared with the mathematics students, lie on scales X, E and EE. On X it was noticeable from their comments that they all regarded the check of a sufficiently varied set of examples as fully convincing; they were insensitive to the need for *all* cases to be conforming. However, their F results show them equally capable of recognising that a full empirical check is valid. For example, "Jayne shows a complicated polygon....she can say definitely that the statement is true." Regarding their performance on E and EE, it has been observed by Backhouse (1967) that performance by sixth formers on Valentine's Reasoning Test showed significant differences between pupils studying different subjects both before and after the full sixth form course, but no significant differences in the gains made by the different groups. Mathematics students scored highest, English students among the lowest on the Valentine test.

Most of the failures on F were due to assuming that "a correct proof" implied an argument with some explanatory quality, so that a complete check of all cases was not regarded as a full proof. One pupil said "Tessa has only showed it, not proved why."

The C scale requires some comment. The two items concerned are from Stamps; James and Richard. James's proof (that 70 cannot be made up from 8s and 20s) is a muddled but complete check of the cases 0, 1, 2 and 3 x 20; whereas Richard's is a systematically arranged check, but omits consideration of 0 x 20. These items showed poor reliability; and although in the pre-test nearly half the pupils were right on one of these, only about 15% had both right. The conflict between the superficial degree of organisation of the answers and the actual completeness of the check has proved a difficulty. Although a mean gain from 0.75/2 to 1.25/2 is recorded for the experimental group, it should be remembered that this actual question was discussed during the teaching programme, which makes the post-test score look very low.

In conclusion, if we consider the general level of the results in relation to the content of the questions, even bearing in mind the limitations of this experiment, we are led to the suggestion that there are a number of straightforward concepts and skills related to the empirical aspects of proof which could be improved quite substantially by suitable emphases in teaching; and others, mainly related to deductive aspects, which involve judgements of relevance and of explanatoriness, and of whether the assumptions on which the arguments are based are sufficiently fundamental, which develop much more gradually. If they are considered important it would seem necessary to devote greater attention to activities involving judgement and the construction and criticism of arguments than is normal in mathematics courses.

CHAPTER 11

CONCEPTS AND STRATEGIES OF GENERALISATION AND PROOF POSSESSED BY ENTERING UNIVERSITY MATHEMATICS STUDENTS: AN INFORMAL STUDY

INTRODUCTION

The previous studies have shown how wide a gap exists between the general mathematical strategies possessed by school pupils, even including sixth formers studying mathematics and the fully developed mathematician. As a first attempt at filling this gap in the pattern of development, an informal study was made of 52 students in October 1975, at the beginning of their first year in the mathematics department of Nottingham University.

HYPOTHESIS

It was expected that the performance of these students would be superior to that of the pupils previously tested in the following respects:-

A. Strategies for Generalisation

1. Interplay of empirical and deductive work - use of empirical work to suggest generalisations, and to test proposed generalisations.
2. Construction of classes of case to be dealt with exhaustively.
3. Use of 1-1 correspondence and iterative arguments.
4. Recognition of symmetries and isomorphism.
5. Spontaneous extension and generalisation of problems.

B. Concepts of Proof

1. Clear awareness of the invalidity of a partial empirical check and the validity of a complete check.
2. Distinction between implication and equivalence.
3. Precise literal treatment of statements of propositions and definitions.
4. Relevance to proof of (a) starting assumptions
   (b) definition of terms
5. Nature of axioms as (a) basic statements of relationship among the undefined terms of a theory, (b) logically arbitrary, but in practice chosen with hindsight.

Of these, A1-3 and B1 have been exhibited to a limited extent among the better responses from school pupils; considerable improvements are expected on these. On B2 and B3 deficiencies have been evident in the pupils' work. Little or no opportunity has been given for showing B4 and B5 in earlier work; they appear well beyond the capabilities of pupils in general so were not incorporated in the test material.

SAMPLE AND PROCEDURE

As this was a pilot study, two forms of test were prepared, to allow a larger number of questions to be tried. Each form contained three questions. 52 first year students in a university mathematics department worked at them for about 40 minutes during their first meeting with their tutor of the academic year. The tutors were free to decide whether or not they wished their tutor group to take part in the survey; in the event, 27 of form A were returned for analysis, and 25 of form B.

The six questions used included three adapted from earlier studies, Coin Turning, Diagonals of a Polygon and Quads, to facilitate comparisons; and three new questions aimed at sampling understanding on B2 and B5 above. B3 and B4 would be exposed, it was hoped, in Diagonals of a Polygon, and the reformulation of the question was intended to assist in this.

QUESTIONS AND RESPONSES

MEANS

Theorem: The arithmetic mean of any two positive real numbers is not less than their geometric mean.

Proof: $$\frac{a+b}{2} \geqslant \sqrt{ab}$$

Hence $$a + b \geqslant 2\sqrt{ab}$$

Squaring to remove square roots,

$$(a + b)^2 \geqslant 4ab$$

Hence $$a^2 + b^2 - 2ab \geqslant 0$$

So $$(a - b)^2 \geqslant 0$$

This is true for all real numbers a&b, hence the theorem follows.

Is this theorem proved? If not, say what is wrong and give a correct treatment of the situation.

RESULTS

Of the 25 responses to this question, 17 correctly stated that the argument needed to be reversed, 5 accepted it, and 3 said it would be 'better' to reverse it.

Other observations: One student argued by contradiction: "assumed $\frac{a+b}{2} \ngeqslant \sqrt{ab} \ldots\ldots \Rightarrow (a - b)^2 \ngeqslant 0$' One tried to check some numerical cases. Of the 17 responses which were correct, 8 also mentioned the need to ensure that the positive square root was taken. Thus, although a majority (68%) of this sample correctly identified the logical error, there are still 20% who failed to reject a proof of form Q ⇒ P as a substitute for P ⇒ Q, and a further 12% who regarded it as a matter of preference rather than necessity.

COINS

Take 3 coins showing all tails.

A move consists of turning over any two coins.

1. Using as many such moves as you wish, obtain all heads. Prove or disprove that this is possible.
2. Extend this problem to 4 coins, turning 3 at a time. Prove your results.
3. Generalise your results as far as possible.

This question was included particularly to observe improvements on Strategies A4 and A5; it produced additionally an observation relating to A1.

RESULTS

3 coins, turn 2:	25 correct, 1 wrong
4 coins, turn 3:	19 correct, 7 wrong
n coins, turn n-1:	13 attempts, 7 correct statements, 6 wrong
	3 attempted proof/explanations, 1 correct, 2 wrong

No (n,m) generalisations.

Other observations: The student who was wrong on the 3,2 problem was attempting an over-sophisticated approach involving coding HHT as the binary number 110, and stating that turning two coins involves changing the number by a multiple of 11, i.e. of 3. (He apparently missed 101). He applied the same method to the 4, turn 3, problem and deduced that this too is impossible.

On the 4,3 problem, four students, working empirically, on reaching repetitions of previous states assumed that, as in the 3,2 problem, not all states were possible.

The proof explanations involved an application of parity arguments to the situation; the errors arose from vagueness and failure to check that the generalisations observed were actually true.

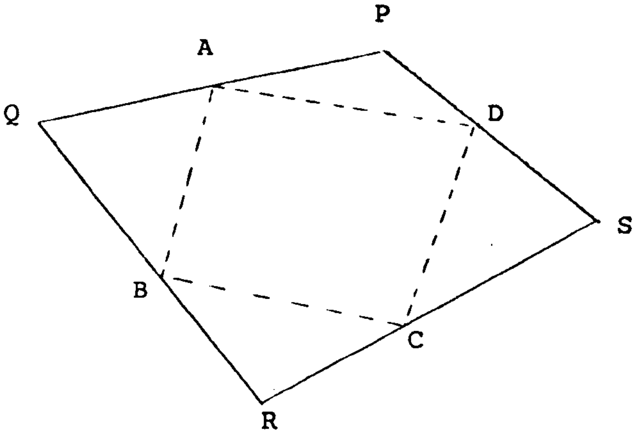
A typical response with a correct result but a wrong imprecise reason:

"4 moves are required with 4 coins. The system works with 4 coins since an odd number of coins is turned over, so at each move effectively one coin is turned over, the other 2 moves having no overall effect."

Compared with school pupils to whom this problem was given, the students were of course much more competent in making and recording their trials. Their mistakes arose less from initial empirical errors than from hasty generalisations with inadequate checks. Most of them saw the problems as a class of problems embodying some general principles of parity; for them the task was to find out just which of these principles applied, and how, and their manipulation of the system with pencil and paper was limited to what they needed for checking these points. No one observed that one could use symmetry; if one can get from TTTT to THTH, then one must be able to get from there to HHHH.

The two questions, one on form A, one on form B, on axioms will be considered later. The two remaining questions on form B are described next.

QUADS



A, B, C, D are the midpoints of the sides of the quad PQRS.
In some quads the midpoint figure ABCD is a rectangle.

1. Find out what has to be special about the quad PQRS for ABCD to be a rectangle and prove your results.

2. It is suggested that in order to obtain a rectangle as the midpoint figure ABCD, the quadrilateral PQRS must be a rhombus. Check this and prove the correct result.

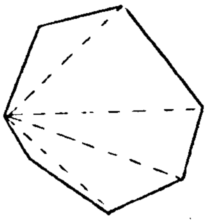
RESULTS

The aims of this question were to observe general differences in approach between school pupils and the students, and in particular to see how far students were aware of the distinction between rhombus outside ⇒ rectangle inside, which is true, and rectangle inside ⇒ rhombus outside, which is not true, since the kite, and in fact, any quadrilateral with perpendicular diagonals, has a rectangle for its midpoint figure.

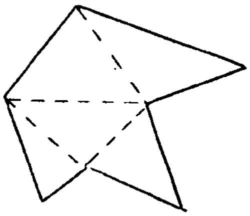
The general difference was quite striking. Although most of the school pupils who knewing the meanins of the terms well enough to attempt the question as intended had treated the question globally and empirically, trying various types of inside and outside figure, only two students did so. The remainder all treated it analytically, using known geometrical theorems and proving lines parallel to each other. 18 students of the 25 obtained PQ $\perp$ QS or something similar. (Lunzer (1973b) observed similarly that, whereas for 11 year olds a rectangle either got bigger or smaller as a whole, older children were able to separate its area and perimeter as distinct attributes which could vary independently of each other.)

Question 2 was the attempt to detect the confusion between implication and equivalence in this context. On only 8 of these scripts was it possible to tell whether or not it had been committed; the error appeared to have been made in 3 of these 8 scripts.

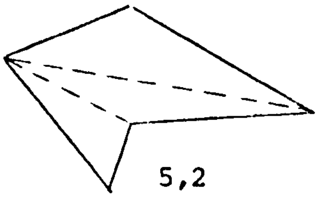
DIAGONALS OF A POLYGON



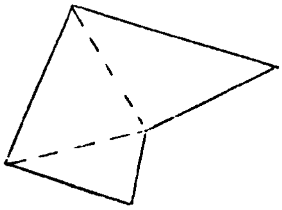
7,4



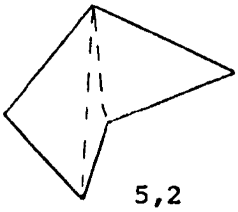
7,4



5,2



5,2



5,2

Theorem: The greatest number of non-crossing diagonals which can be drawn in a polygon is three less than the number of sides.

Proof: In the left hand diagram, it is clear that diagonals can be drawn from one vertex to each other vertex, except three; these being the first vertex itself and the two adjacent ones to it. Similar radiating sets of diagonals can be drawn in any polygon; hence the theorem is true for all polygons.

Is this theorem proved? If not, say what is wrong and give a correct treatment of the situation.

RESULTS

A correct treatment is probably best obtained by abandonning the radiating sets idea and considering what happens when each new diagonal is added, thus:

Without any diagonals the polygon contains 1 region. Each added diagonal increases the number of regions by 1, so after adding D_n

diagonals the polygon contains $D_n + 1$ regions. If no more diagonals can be added, every region is a triangle, needing $3(D_n + 1)$ sides. n of these are sides of the polygon, and the diagonals provide $2D_n$.

$$\text{Hence} \quad 3(D_n + 1) = n + 2\,D_n\,,$$
$$\text{Whence} \quad \underline{D_n = n - 3}$$

Necessary definitions are: an n-sided polygon is a sequence A_1, $A_2 \ldots . A_n$ of distinct points in the plane, together with the n line-segments A_1, A_2, $A_2A_3 \ldots A_{n-1}A_n$; such that the plane is divided into a single finite inside region and an infinite outside region. A diagonal of such a polygon is any join $A_p\ A_q$ of two of its points which is not a side and which lies inside the polygon.

The two points of the proof which it was intended that students should query were (a) 'similar radiating sets of diagonals' cannot be drawn in all polygons; (b) no proof is offered that a radiating set gives the greatest possible number of non-crossing diagonals, as compared with the other possible diagonalisations, e.g. of the type shown in the other 7, 4 polygon. It was also expected that some would comment on the dependence on implicit definitions of polygon and diagonal.

Of the 25 responses obtained, 3 accepted the proof as correct, thus missing points (a) and (b). 10 rejected the proof because it 'relies on intuition from looking at the diagrams', or because it does not deal explicitly with an n-sided polygon. These could be said to be reacting to the superficial aspects of the proof rather than its substance. The remaining 12 criticised the proof on more substantial grounds:

7 said it did not apply to non-radiating diagonalisations;

1 said also that some polygons had no radiating sets;

4 (including the previous 1) said that the question assumed that diagonals were internal.

Of these, none mentioned that a definition of diagonal was needed to clarify the matter, but one noted the omission of an argument that the radiating sets gaves the maximum number of diagonals for a given polygon.

There were four attempts to give a correct proof but none was successful. All made assumptions as substantial as the theorem viz. (i) that the sum of the interior angles of an n-sided polygon was (n - 2) x 180°, or (ii) that the number of triangles formed was n - 2.

Hence although we see here some improvement in comparison with the school pupils, on B3 and B4 - the precise literal treatment of a statement and awareness of the importance of starting assumptions and definitions - there is clearly still a considerable gap between the students' performance and the approach of the mature mathematician.

The two items relating to axioms asked the same questions, one in relation to a geometrical theorem, the other in relation to a number theorem. Four examples are inserted here to give the spirit of the responses.

QUESTIONS OF PROOF G

Consider the following statement: "The sum of the interior angles of a triangle is 180°".

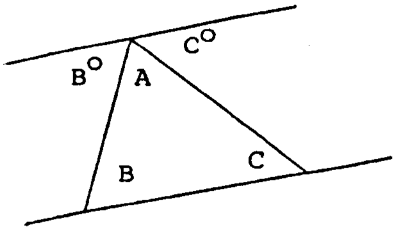
A. It can be justified by experiment with objects in the physical world.

B. It has nothing to do with the physical world, but it is the only possible way of making a geometric system work.

C. It cannot be said to be true in any absolute sense, but it can appear as a theorem in a system based on suitable axioms.

Say which of A, B, C is closest to your own point of view, and sketch the steps of the argument you would use to justify this theorem.

Example 1



$A + B + C = 180°$

An axiom - the angles at a point on a straight line add up to 180°.

Angles B and B^O are the same, alternate angles and also C and C^O.

As angles at a point on a straight line add up to 180° the sum of interior angles of a triangle is 180°.

Example 2

"Mathematics has evolved because of the world. This case is a case of two dimensions which is easily seen to everyone. Thus the fact can lead to the theory.

To say it is unreal but makes a geometric system work is untrue because it is real and we see triangles about us constantly. However, this basis of mathematical knowledge within our own world leads us to less real situations of greater than four dimensions, complex numbers, etc.

It is true to say that it is a theorem based on a number of axioms. But the axioms must be true in an absolute sense or theory would break down somewhere."

QUESTIONS OF PROOF N

Why does (-1) (-1) = +1 ?

A. It can be justified by experiment with objects in the physical world.

B. It has nothing to do with the physical world, but is the only possible way of making a number system work.

C. It cannot be said to be true in any absolute sense, but it can appear as a theorem in a system based on suitable axioms.

Say which of A, B, C is closest to your own point of view, and sketch the steps of the argument you would use to justify this law.

Example 1

"B is the closest to my point of view. To have a number system which works the answers to 3 calculations must be known

i) 1×1

ii) 1×-1

iii) -1×-1

Since 1 is the identity element of multiplication of real numbers, $1 \times 1 = 1$. Again using this fact, $1 \times -1 = -1$. Resolution of -1×-1 may occur by

$$-1 \times -1 = -(1 \times -1) = -(-1)$$

Since the - sign means that the figure is at the opposite side of the origin, $-(-1)$ means that the answer figure is at the opposite side of the origin to -1 and of magnitude 1. The answer to -1×-1 must therefore be $-1 \times -1 = 1$."

Example 2

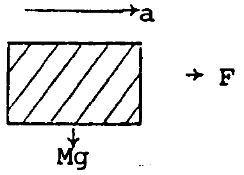
"C is closest to my own point of view.

It certainly cannot be justified by experiments with objects in the natural world as a negative number is merely an extension of the number system below zero and cannot be represented by a number of objects.

However, it is wrong to say that is nothing to do with the physical world. It is possible to have a negative velocity for example, but only in the sense that this velocity is contrary to the direction in which one is measuring.

Everyone is taught that (+1) × (+1) = (+1). It is relatively easy to see that (-1) × (=1) = (-1).

e.g.



'F = ma'
In this system

F = -ma. Therefore F is -ve too.

So really (-1) × (-1) = (+1) must be true. If the force in the above system is in the opposite direction we can take it as -F in the direction we are measuring. The mass is +ve, hence we have the produce of two -ves being +ve."

RESULTS ON AXIOMS QUESTIONS

On the 26 responses to the geometrical version, 18 chose C, 6 chose A, one A and C, one B.

Of those choosing A, one suggested a experiment with beams and wires, one a more sophisticated version of walking round the triangle and adding the changes in direction, one measuring the angles.

Of those choosing C, only one stated that the properties of parallels were "always taken from granted"; one other took as an axiom that the angle between parallel lines was 0^{o}; the only other axioms suggested were that the angles formed on a straight line made 180^{o} (2 students) or that the circle was 360^{o} (2 students).

One student said that "the angle sum of a triangle = 180°" was itself an axiom but used without comment the properties of angles on parallel lines.

Of the 23 responses to the number version, 16 chose C, 4 chose B, 2 chose A. 1 chose B/C (2 omitted the question).

These did not always correspond to the type of explanation subsequently offered. Four students (ABBC) justified (-1) × (-1) = + 1 by appeal to some other system involving negation, e.g. "taking away a hole", - means "going to the other side of the origin", negative = "a contradiction." One justified it by continuing the number pattern, -1 × 2, -1 × 1, 1 × 0, -1 × -1.

One said it was "too great a part of the number system not to work", another that "if (-1) × (-1), then complex numbers would be unnecessary."

In all, six students stated or implied that number properties could be deduced from a suitable set of axioms, definitions and/or rules, but only one, and a more doubtful second, recognised that these were in a certain sense arbitrary, in that "other number systems are possible", to quote the clearest statement made.

DISCUSSION

On strategy A1 - interplay of empirical and deductive work - the responses to Coins showed almost a reversal of the school pupils' approach. The students moved quickly into deductive work with parity relationships and often failed to make adequate use of empirical checks. On strategies A2 and A3 no definite observations could be made. On A4, the useful symmetry in Coins was not recognised; and on A5, in the same problem, only the most obvious generalisation was made.

The proof-concept B1 appeared (for example, in Diagonals of a Polygon) to be well established; but there was a fair amount of confusion between implication and equivalence (B2) in Means and Quads. The Polygon problem showed most students to be treating propositions literally (B3) but non explicitly showed awareness of the need for definitions (B4b) and some gave potentially circular arguments (B4a). On axiom systems (B5), it appears that although a majority of the students have some awareness of the nature of an axiomatic system, the idea is extremely vague. It amounts generally to knowing that mathematical theories can be deduced from definite starting points, but considers these to be basic truths about the real world rather than logically arbitrary. The notion of a logically self-contained system built on relationships between undefined terms is almost certainly not yet present. The greatest opportunities for improvement would seem to be on A1 and A5, extension and generalisation of problems with skilled interplay between empirical and deductive work, and in the concepts of proof B2-B5.

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APPENDICES

The appendices are numbered to correspond to the chapter to which they belong. There is an appendix for each chapter, except Chapters 1, 2 and 4.

APPENDIX TO CHAPTER 3

Pupils' investigations:

- Dress Mix-ups
- The Remainder Problem
- Filter Paper

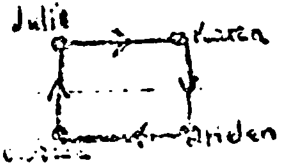
TEXT
BOUND INTO THE
SPINE

DRESS MIX-UPS

Dress mix-ups.

The girls could get their clothes mixed up 2 ways in the first type.

Like this.



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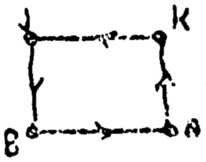

The dresses could be passed to the wrong girl six ways in the second type.

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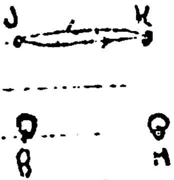

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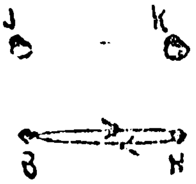

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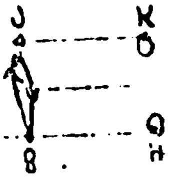

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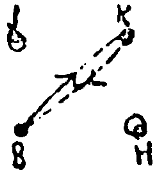

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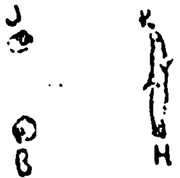

The girls could get their dresses mixed up 2 ways in the third type.

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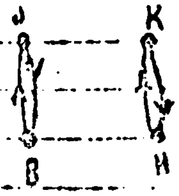

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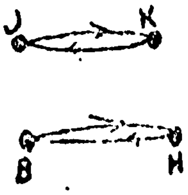

There could be 8 mix-ups in the 4th type.

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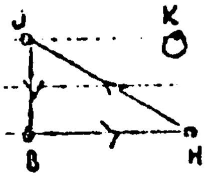

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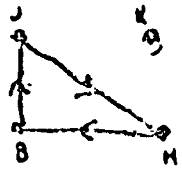

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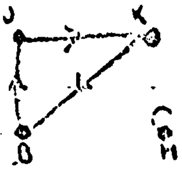

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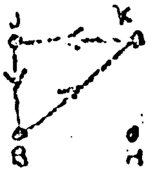

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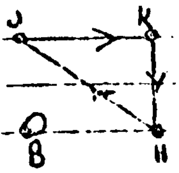

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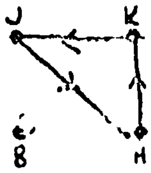

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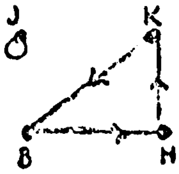

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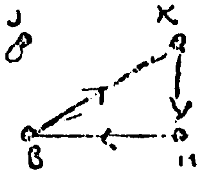

There could be 4 mix-ups in the 5th type.

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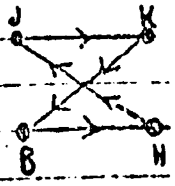

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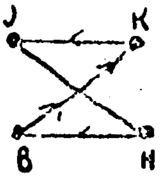

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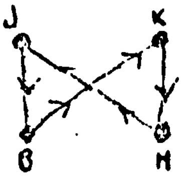

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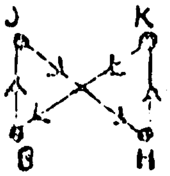
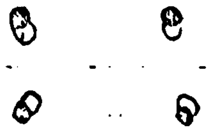


The girls could get their dresses mixed up one way in the 6th type and one way in the 7th type.

6th type

7th type



The Remainder Problem

The original problem was:-

When a boy counted his sweets in fours he had 2 left over, when he counted them in fives he had one left over. How many sweets did he have?

I thought about this problem and wrote it down like this

4r2
5r1 6,26,46,66,86,...

The first number of sweets he could have had was 6 but after that there were many more numbers which continued to go up in 20s. I noticed that

5x4 = 20

and the numbers went up in 20s. To check this rule I tried a couple more examples

3r2
4r2 2,14,26,38,50,62
3x4 = 12 the numbers go up in 12s

8r4
9r1 28,100,172...
8x9 = 72 It goes up in 72s

We looked at the problems
1. How do you find what the starting number is going to be?

2. How many do they go up in each time?

3. What happens with three or more counts.

First I tried to find what the starting number is going to be. But at first it didn't seem a very easy problem.

There were exceptionally cases when 4r2 and 5r1 would add up to 6 the first number but this rarely worked.

For a while I left this and went on to the problem of what the numbers went up in. I had found that the first numbers when multiplied together made the amount the numbers went up in.

But then I found these two cases

4r2
6r4 22,34,46....
4r3
6r1 19,31,43....

This made up think that the numbers always go up ~~by~~ either by the multiples of the two numbers or half that amount.

When this example came up

2r0
4r2 6,10,14,18,22,

I thought that when all the first numbers are prime numbers the numbers go up in those numbers multiplied together but when those numbers are not all prime numbers the numbers go up in half this multiple.
But another case cropped up and put me off the trail

3r1
6r4 10,16,22,28,34,40,46

this time the numebrs should go up in a third of 3x6

3x6 = 18 1/3 of 18 = 6

To test this rule we tried some more examples

4r1
8r5
13,21,29...

4r2
8r6
14,22,38.....

5r2
10r7
17,27,37,....

The rule seemed to be correct. If the larger number is a multiple of the smaller number then the numbers always go up in the multiples of the two numbers divided by the smallest number
e.g.
3r1
9r4 13,22,31,40,49,58.

e.g. $\left.\begin{matrix}3r1\\9r4\end{matrix}\right\}$ 13,22,31,40,49,58

Nine is a multiple of three

3x9 = 27   27÷3 = 9

To multiply two numbers together and divide by the smallest always leaves you with the largest number.

I ended up with -

If the numbers have no connection they go up in the multiples of the two numbers and if one number is a multiple of the second the numbers go up in the largest number.

* * * *

Next I tried using three or four counts. Here are a few examples of them

$\left.\begin{matrix}3r2\\4r1\\5r3\end{matrix}\right\}$ 53,113,173,233....

3,4 and 5 have no connection so the numbers go up in 60s - 3x4x5

$\left.\begin{matrix}2r0\\3r1\\4r2\\5r3\end{matrix}\right\}$ 58,118,178,238....

As 4 is a multiple of 2 the numbers go up in 2x3x4x5÷2

In this next example the numbers have no connection so go up in 2x3x5x7 = 210

$\left.\begin{matrix}2r1\\3r2\\7r4\\5r0\end{matrix}\right\}$ 95,305,515

I found out that in some cases you can predict what the last digit is going to be. In the last example 2r1 means it must be an odd number and 5r0 means to divide by 5 exactly the numbers must all end in 5 or 0. As 0 isn't an odd number the numbers must all end in 5

This doesn't help every time. But I did find that there was a pattern in the tables of first numbers

	3r0	3r1	3r1
5r0	0	10	5
5r1	6	2	11
5r2	12	7	2
5r3	3	13	8
5r4	9	4	14

	4r0	4r1	4r2	4r3
5r0	0	5	10	15
5r1	16	1	6	11
5r2	12	17	2	7
5r3	8	13	18	3
5r4	4	9	14	19

The numbers go from 0 to (in this case) 19 in order in a pattern.
The numbers go from 0 diagonally down from the top corner

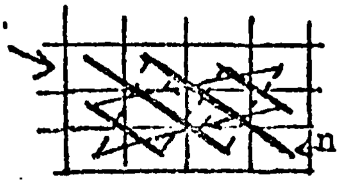
	2r0	2r1	
3r0	0		
3r1		1	
3r2	2		☐

Then you want the next number down at ☐ but as it goes off the table you have to look horizontally across for the 2. Then the 3 goes off the table so you look vertically upwards from where the 3 should be. The 4 also goes off the edge of the table so you must look horizontally across to find it and then as usually diagonally down till you reach the bottom and the table is finished.

	2r0	2r1	
3r0	0	3	
3r1		1	←4
3r2	2		

	2r0	2r1
3r0	0	3
3r1	4	1
3r2	2	5

It always works in the same pattern



This kind of table works for any numbers

	11r0	11r1	11r2	11r3	11r4	11r5	11r6	11r7	11r8	11r9	11r10
4r0	0	12	24	36	4	16	28	40	8	20	32
4r1	33	1	13	25	37	5	17	29	41	9	21
4r2	22	34	2	14	26	38	6	18	30	42	10
4r3	11	23	35	3	15	27	39	7	19	31	43

As we know how the table works we can predict what the first number is going to be by workingit out from the table but not actually writing all the numbers in.If the numbers are large it would take a long time.

We also found a few cases that didn't work at all
e.g. 5r0 / 10r1 3r2 / 6r4 4r3 / 8r1
but didn't really have time to go into this.

Conclusion

I couldn't find a really efficient way of predicting what the first number is going to be,only by the method of the tables. What I found about the way the numbers go up has been mentioned earlier.When I tried using more than 2 counts the results seemed to be the same and fitted in with any patterns or rules about two counts.

One of the things I didn't cover was when it was impossible to find any whole numbers above 0 which would fit.This happened when there was two numbers of which one was a multiple of the other with a completely different remainder.As there are no numbers there is not much you can do with them.

Jane Ockenden / 38

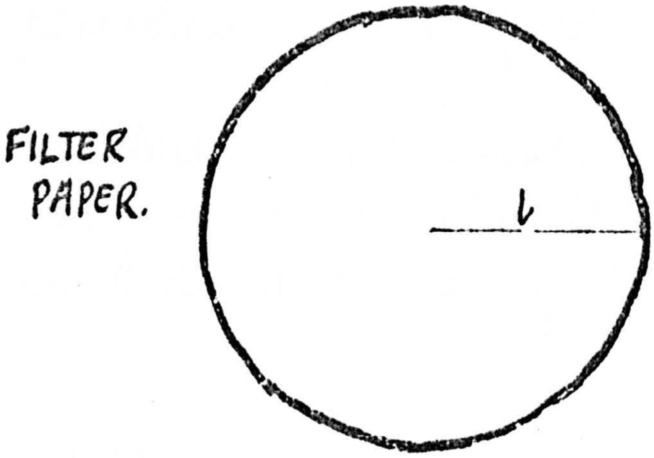
W. Staden

FILTER PAPER

PROBLEM:

What is the cone of greatest volume that can be made from a piece of filter paper of any given radius?

A piece of filter paper is a perfect circle, of radius, let us say, l.



If this piece of filter paper is folded into a cone, l becomes the slant height of the cone and the centre of the circle becomes the apex of the cone.

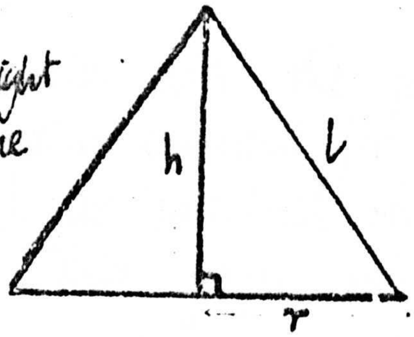
The cones which can be made from that sheet of filter paper an vary a lot in dimension, but the slant height will always be onstant and equal to the radius of the paper.

For a typical cone,

t h = perpendicular height

r = radius of base of cone

l = slant height.



Now by Pythagoras, who stated that in a right angled triangle (as in the one above) the squares of the hypotenuse was equal to the sum of the quares on the other two sides:

$$l^2 = h^2 + r^2$$

Whence $r^2 = l^2 - h^2$

and $h^2 = l^2 - r^2$

formula for the volume of a cone $= \frac{1}{3}.\pi r^2 h$

where r = radius of base of cone

h = perpendicular height of cone.

and h are variables in the equation, but for a constant height r may be expressed in terms of h and vice-versa then we may substitute in volume $= \frac{1}{3}.\pi r^2.h$ and hence end up

one variable.

Thus substituting

$r^2 = l^2 - h^2$ in vol. $= \frac{1}{3}.\pi.r^2.h$

we get

$\frac{1}{3}.\pi(l^2-h^2).h$ X

i.e. vol, $V = \frac{\pi}{3}.h.l^2 - \frac{\pi}{3}.h^3$

graph of $V = \frac{\pi}{3}.hl^2 - \frac{\pi}{3}.h^3$ were drawn, ie a graph

owing how volume varies with the perpendicular height of cone, (l always remains constant for that piece of filter paper), cubic graph would be obtained since there is a term in h^3 ent in the equation. This graph will be of the form

$$y = ax^3 + bx^2 + cx + d$$

we have $V = -\frac{\pi}{3}h^3 + \frac{\pi}{3}l^2.h$

i.e. $y = -ax^3 + cx$

here the graph $V = -\frac{\pi}{3}h^3 + \frac{\pi}{3}l^2 h$ crosses the x axis

0 : letting $+\frac{\pi}{3} = a$, and $+\frac{\pi l^2}{3} = b$

we get:

$$0 = -ah^3 + bh$$

dinding by h we obtain:

$$0 = -ah^2 + b$$

$$\therefore -b = -ah^2$$

$$\therefore \frac{-b}{-a} = h^2$$

multiplying both sides by $\frac{-1}{-1}$ we obtain

$$\frac{b}{a} = h^2$$

$$\therefore h = \pm\sqrt{\frac{b}{a}}$$

which means that h has real roots

Also in $V = -\frac{\pi}{3}h^3 + \frac{\pi}{3}l^2 h$

when $h = 0$, $V = 0$

$\therefore$ h has three real roots, 0 being the third.

On this graph of $V = -\frac{\pi}{3}h^3 + \frac{\pi}{3}l^2 h$

there are thus a turning points on the graph, hence a maximum point and a minimum point.

Where there is a maximum point, here the V coordinate is the maximum volume attainable from the filter paper and the h coordinate the corresponding height of that cone.

∴ differentiate $V = \frac{\pi}{3} . l^2 h - \frac{\pi}{3} h^3$ with respect to h

remembering that l is a constant for that piece of paper,

$$\frac{dV}{dh} = \frac{\pi}{3} l^2 - \frac{\cancel{3}\pi}{\cancel{3}} h^2$$

$$= \frac{\pi}{3}\left(l^2 - 3h^2\right)$$

thus there are turning points when

$$\frac{\pi}{3}\left(l^2 - 3h^2\right) = 0$$

multiplying both sides by $\frac{3}{\pi}$ we obtain

$$l^2 - 3h^2 = 0$$

$$\therefore l^2 = 3h^2$$

$$\therefore \frac{l^2}{3} = h^2$$

$$\therefore h = \pm\sqrt{\frac{l^2}{3}}$$

i.e. $h = +\sqrt{\frac{l^2}{3}}$ or $-\sqrt{\frac{l^2}{3}}$

Ⓐ Ⓑ

to find out whether Ⓐ is a maximum point or not we find $\frac{d^2V}{dh^2}$ of $\frac{\pi}{3} l^2 . h - \frac{\pi}{3} h^3$

$$\frac{dV}{dh} = \frac{\pi l^2}{3} - \pi h^2$$

$$\therefore \frac{d^2V}{dh^2} = -2\pi h \quad \text{when } h = +\sqrt{\frac{l^2}{3}} \quad \text{this} = -2\pi\sqrt{\frac{l^2}{3}}$$

since $\frac{d^2V}{dh^2}$ is a negative quantity,

when $h = +\sqrt{\frac{l^2}{3}}$ there is the maximum point, hence the maximum volume.

to check, substitute h for $-\sqrt{\frac{l^2}{3}}$

$$\therefore \frac{d^2V}{dh^2} = -2\pi . -\sqrt{\frac{l^2}{3}}$$

$$= +2\pi\sqrt{\frac{l^2}{3}}$$

$\therefore$ when $h = -\sqrt{\frac{l^2}{3}}$ there is the minimum volume, obtained from the cone of this height

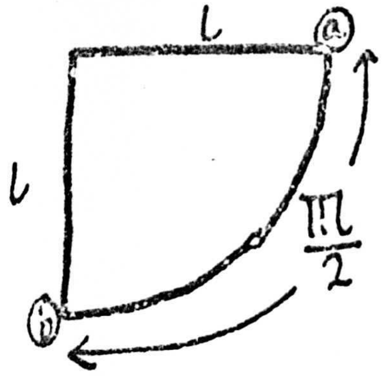
Now compare the volume of the cone formed by lding laboratory filter paper in the normal way, to the volume the cone that could be made by ~~by~~ folding it in such a ay that the height of the cone was $\sqrt{\frac{l^2}{3}}$, where l equals he radius of the filter paper.

In the laboratory, the filter paper is taken and folded in half, it is then folded in half again so that a quarter Segment of a circle is produced:
if the radius of the filter paper is l, as we have Supposed all along, the circumference

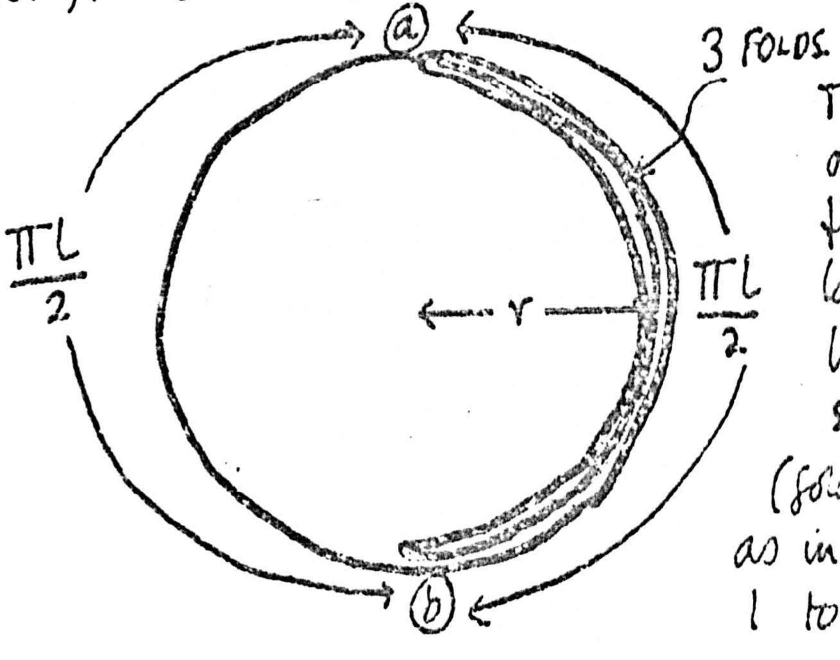
$$= 2\pi l$$

∴ the length of the quarter segment of the circle

$= \frac{1}{4} . 2\pi l \cdot \frac{\pi l}{2}$



long ⓐ ⓑ there are 4 folds of paper, and when the cone produced from this paper, 3 folds are made to form one side 4th fold, the other thus:



This is bottom view of the cone, ie. from the other diagram, looking down on the lines along ⓐ ⓑ and separating 3 lines (folds of paper) to the right as in the diagram, and 1 to the left

Thus the perimeter of this circle at the base of the cone

$= \frac{\pi l}{2} + \frac{\pi l}{2} = \frac{2\pi l}{2}$

$= \pi l$

Let the radius of the base of the cone $= r$

circumference of this circle $= 2\pi r$

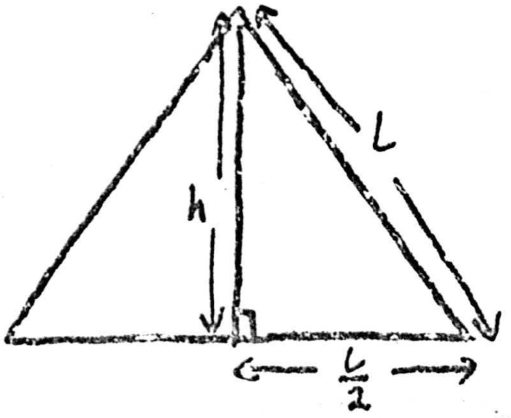
but circumference also $= \pi l$

$\therefore \; 2\pi r = \pi l$

$\therefore \; r = \frac{\pi l}{\pi . 2}$

$= \frac{l}{2}$

Now if the radius of the base of the cone = $\frac{L}{2}$, and the slant height is L, pythagoras's theorem may be applied to find the height of the cone, h.



by pythagoras:

$$\left(\frac{L}{2}\right)^2 + h^2 = L^2$$

$$\therefore h^2 = \frac{4 \cdot L^2}{4} - \frac{L^2}{4}$$

$$\therefore h^2 = \frac{3L^2}{4}$$

$$\therefore h = \frac{L}{2} \cdot \sqrt{3}$$

Now volume of cone is given by

$$V = \frac{1}{3} \pi r^2 h$$

$\therefore$ Volume (given that $r = \frac{L}{2}$, $h = \frac{L}{2}\sqrt{3}$)

$$= \frac{1}{3} \cdot \pi \left(\frac{L}{2}\right)^2 \cdot \frac{L}{2}\sqrt{3}$$

$$= \frac{\pi L^3}{8} \cdot \frac{\sqrt{3}}{3}$$

$$\frac{\sqrt{3}}{3} = 3^{\frac{1}{2}}.3^{-1} = 3^{-\frac{1}{2}}$$

$$\therefore \text{Vol} = \frac{\pi l^3}{8.\sqrt{3}}$$

$$\sqrt{3} = 1{\cdot}732$$

$\therefore$ Vol in terms of π and l

$$= \frac{\pi l^3}{13{\cdot}856} \text{ cu. units.}$$

If the paper is folded so that the height $= \sqrt{\frac{l^2}{3}}$
Referring back to page 2 of the investigation:
from equation X

$$\text{Vol.} = \frac{1}{3}\pi(l^2 - h^2).h$$

substituting $\sqrt{\frac{l^2}{3}}$ for h we get,

$$\text{Vol} = \frac{\pi}{3}\left(l^2 - \left(\sqrt{\frac{l^2}{3}}\right)^2\right).\sqrt{\frac{l^2}{3}}$$

$$\therefore \text{Vol} = \frac{\pi}{3}.\sqrt{\frac{l^2}{3}}.l^2 - \frac{\pi}{3}.\sqrt{\frac{l^2}{3}}\left(\sqrt{\frac{l^2}{3}}\right)^2$$

$$= \frac{\pi l^3}{3.\sqrt{3}} - \frac{\pi l^3}{3.3.\sqrt{3}}$$

$$= \frac{3\pi l^3 - \pi l^3}{3.3.\sqrt{3}}$$

$$= \frac{\cancel{2}\pi l^3}{9 \times \underset{0{\cdot}866}{\cancel{1{\cdot}732}}}$$

$$= \frac{\pi l^3}{7{\cdot}794} \text{ cu. units.}$$

$$\text{Since } \frac{\pi l^3}{13{\cdot}856} < \frac{\pi l^3}{7{\cdot}794}$$

filter papers in the laboratory, as regards obtaining the maximum ume from them when folded into a cone, if they were folded so at the height of the cone was $\sqrt{\frac{l^2}{3}}$, could be made

hold $\frac{(13.856 - 7.794)}{7.794} \times 100$ % more $\simeq$ 80% more.

However the difficulty lies in folding the paper that the height $= \sqrt{\frac{l^2}{3}}$, where l is the radius of the er, that is why, for simplicity's sake, the paper is folded in the lab. how it is.

APPENDIX TO CHAPTER 5

Number test

General Mathematics Test (Strategies)

Further notes on development of the sub-scales

NAME.....................Form.........

NUMBER DIAGNOSTIC SHEET 6

Place Value	Tables A,S	Tables M,D	Computation -, x	Estimates	Relationships	Applications
4	6	7	4	3	15	5

Fill in the answers in the spaces provided. Do any working at the side.

P1. What number is seventeen less than 2010? ________

P2. Write these numbers in the boxes in order of size, starting with the smallest and ending with the largest.

496 946

649 469

☐ ☐ ☐ ☐

P3. Put a ring round the letter at the end of the line which goes up in tens

310	311	312	313	A
420	520	620	720	B
352	362	372	382	C

P4. Write the number which consists of six hundreds, four tens and thirteen units. ________

P4

Fill the boxes

T1. 8 + 5 = ☐ **T2.** 5 + ☐ = 13

T3. 6 + 7 = ☐ **T4.** ☐ + 4 = 18

T5. 9 + 9 = ☐ **T6.** 14 - 9 = ☐

Tas

6

T7. 8 x 5 = ☐ **T11.** 3 x ☐ = 21

T8. 6 x 4 = ☐ **T12.** ☐ x 8 = 48

T9. 3 x 8 = ☐ **T13.** 36 ÷ 9 = ☐

T10. 7 x 9 = ☐

Tmd

7

C1. Subtract
$$\begin{array}{r}325\\ -\ \ 72\\ \hline \\ \hline\end{array}$$

C2. Subtract:

481 - 39 ________

C3. Multiply
$$\begin{array}{r}519\\ \times\ \ \ \ 8\\ \hline \\ \hline\end{array}$$

C4. Multiply:

235 by 40 ________

$\mathrm{C}\frac{\quad}{4}$

E1. Ring the number nearest to 58.

48 49 54 60 68

E2. Ring the number nearest to 5 x 340

850 1000 1250 1500 1750

E3. Ring the number nearest to the total

36 + 5 + 1 + 27 + 92

58 100 130 150 190 3000

$\mathrm{E}\frac{\quad}{3}$

R1. Fill the boxes with the correct signs, + or -

37 + 68 - 24 = 68 ☐ 24 ☐ 37

(2)

R2. Given that 36 + 58 = 94, do the following without working out.

36 + 68 = ☐

37 + 57 = ☐

94 - 36 = ☐

(3)

R3. Given that 18 x 35 = 630, do the following without working out.

19 x 35 = ☐

18 x 34 = 630 - ☐

(2)

R4. Mark each statement ✓ if it is right; X if wrong; ? if you are not sure.

5 x 60 =
(6 x 50
(60 x 50
(5 x 10 x 6

(3)

R5. Mark each statement ✓ if it is right; X if wrong; ? if you are not sure.

40 x 80 =
(320
(3200
(12,000

(3)

R6. Mark each statement ✓ if it is right; X if wrong: ? if you are not sure.

44 - 19 = 19 - 44

44 x 19 = 19 x 44

R $\overline{15}$

(2)

A1. Harry gets paid 10p for his paper round each day except Sundays. Andrew does a Sunday round only, and gets 35p for it. How much more does Harry get than Andrew in a week?

__________p

A2. 180 pupils are to have a medical check-up. The doctor can see 12 pupils in half an hour. How long will he take to see all the pupils?

__________hours

A3. John buys 3 bags of sweets. He pays 45 pence. Each bag contains 20 sweets. How much does each bag cost?

__________pence

A4. There is going to be a school outing to the country which will cost 40p per child. 115 children want to go and a bus holds 30 children.

How many buses will be needed?

__________buses

A 5

How much money will the children pay altogether?

____

Name ....

Form ....

GENERAL MATHEMATICS TEST

FINDING AND PROVING RULES

Read the questions carefully.

Do all your working in the spaces left on the paper.

A pencil and rubber will be needed for questions 2 and 5.

Time allowed: 1 hour.

1. Choose a number, less than 10, and write it down here..............

Add the number to 10, and write the answer ....

Take your first number away from 10, and write what is left ....

Add the two last answers; write the result here..................

Choose another number and do all the same things.

Item 2: Max 2 for 2 correct examples.
1 for 1 correct example or two consistent examples from a different interpretation.

What happens?

Will the same thing always happen? Try further numbers if you wish.

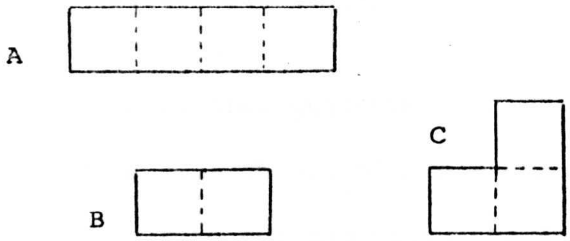
Item 3: Max 2 for 'always 20' or a correct generalisation from the actual example, obtained.

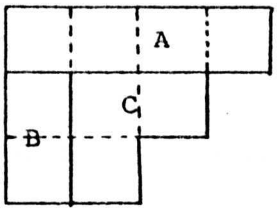
Answer : Yes/No.

Explain why this happens.

Item 4: Max 2 for an explanation including how the added and subtracted numbers cancel.
1 for a general restatement or partial explanation.

2.





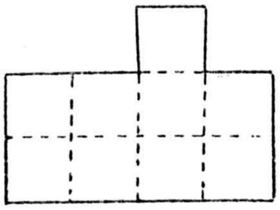
The pieces A, B and C can be used to make other shapes.

Here is one example

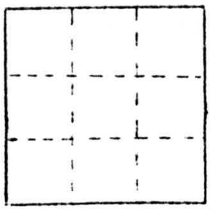
Try to make the shapes below by putting together the three pieces A, B and C each time.

If you can, say YES, and mark the shape to show how you would do it.

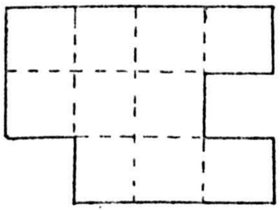
If you say NO, use this space to explain why it can't be done.



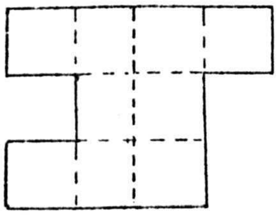
YES/~~NO~~



~~YES~~/NO



~~YES~~/NO



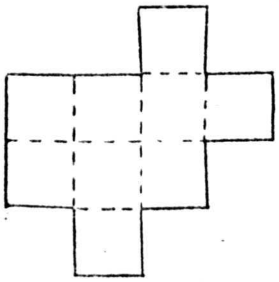
YES/~~NO~~

em 5
for all correct;
1 for each wrong
swer down to 0.

Items 6, 7, 8: Each Max 2 for correct identification of reason eg (6) A is too long, (7) there are 10 squares, (8) when A is in place there is an odd square.

0 for 'they won't fit!

1 for intermediate cases.



~~YES~~/NO

3. This is like question 1.

This time choose a number bigger than ten. Write it here............

Add it to ten and write the answer. ....

Take ten away from it, and write down what is left..............

Add the two last answers..............

Try this with other numbers.

Marking as for Q 1.

Item 9 : Max 2.

Is there any pattern in the results? If so, describe it.

Item 10 : Max 2.

Explain why it happens.

Item 11 : Max 2.

4. Suppose you have a lot of stamps of value 6p and 15p but no others.

You can make up various amounts of postage from these.

If you want to, you can make 27p as 15 + 6 + 6.

Can you make 29p? Use the space below for your trials.

Item 12:
1 for correct answer unless there is positive evidence of misunderstanding,
2 with evidence of experimentation.

Answer: YES/NO

Explain why.

Item 13: 2 for some explanation of choice of 'near' combinations for checking.

Find all the amounts you can make up from 6's and 15's which are above 30 and below 40.

Answer: I can make............33, 36, 39.................... Item 14

but not...[illegible]. 31 32 34 35 37 38 (40)... Item 15

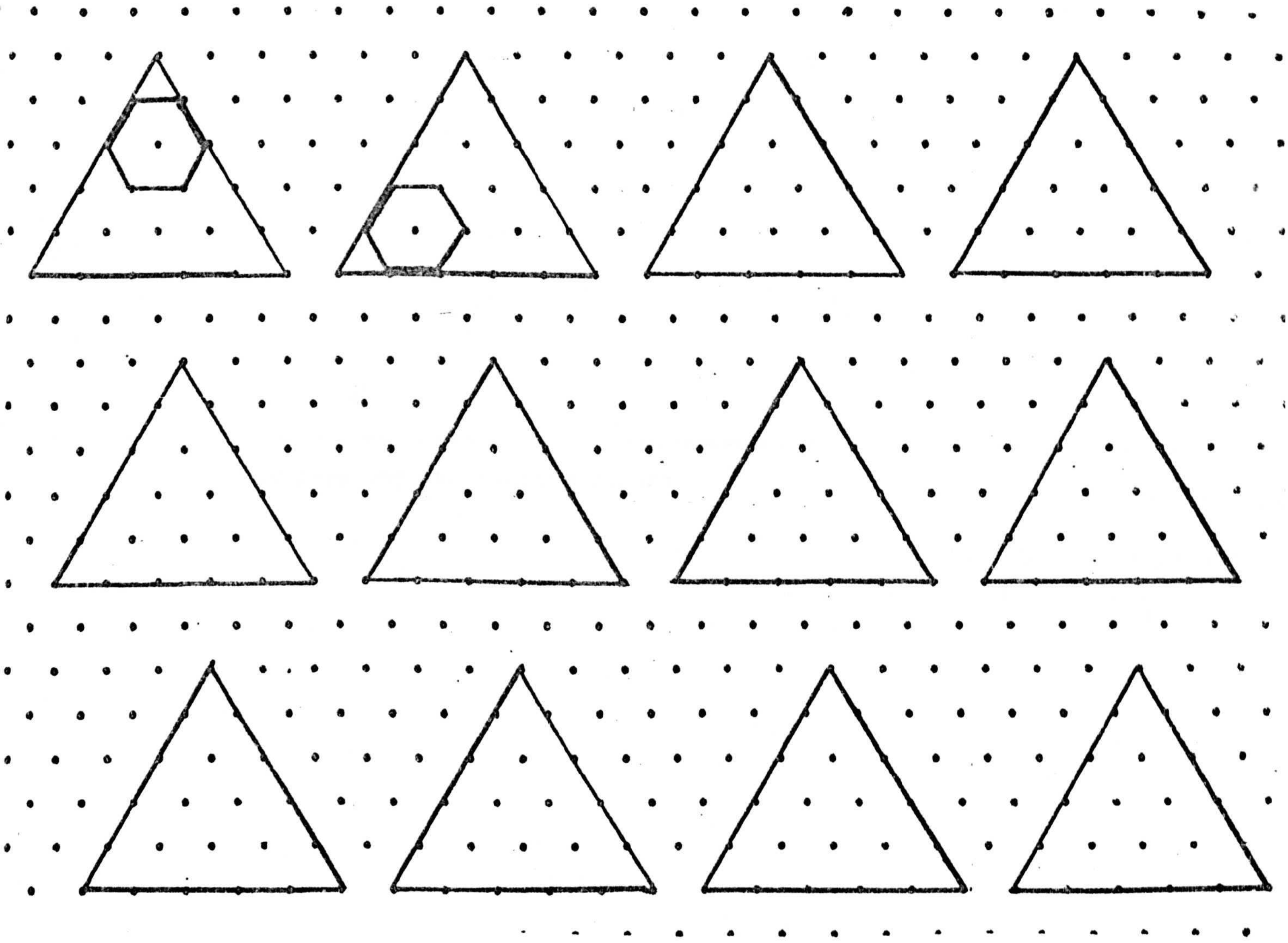
In each case, 2 for correct set, −1 for each wrong entry. Ignore 40.

5. In the first two triangles below, a figure has been drawn in two different places.

In how many different places can you put this figure in these triangles? (It must fit on the dots).

Then write down your answer below.

Try some drawings.



Answer________________________different places.

Are you sure you have got exactly all the different places?
Not too many nor too few?

Say why you are sure.

Item 1: 1 for correct set.

Item 16: 2 for a proper explanation indicating systematic trial eg. 3 on bottom, 2 on sides, 1 at top.

1 for an imprecise indication

0 for 'I have tried all the different places,

6. Look at your last answer to Question 4.

Describe any pattern in the numbers you have found.

Item 17: 2 for any
correct description of pattern
in pupils set of numbers.

Explain why the pattern occurs.

Item 18: 2 for brief explanation
eg. 'because 6 and 15 are
both multiples of 3'.

2 + 3 = 5

4 + 6 = 10

6 + 9 = 15

8 + 12 = 20

....

....

Write two more number sentences to show how the pattern continues

Item 18 : Max 2
for 2 correct ess

Item 19 : Max 2.

16 + ... = ... + ... = 60

Here are two number sentences taken from the same pattern. Can you finish them?

---

29 + ... = ...

Item 20 : Max 2
for eg. 'cannot start with odd no.'

Jim says that this number sentence is taken from the same pattern and he asks Mandy to finish it.

Mandy says that it is impossible.

Who is right? Explain your answer carefully.

Paul says that he thinks that the last number is always something times five.

Is he right? Can you explain it?

Item 22 : Max 2 for
mentioning correctly 2s + 3s.
0 for 'they all end in five'

FURTHER NOTES ON THE DEVELOPMENT OF THE SUB-SCALES

The factor analyses (Tables 1, 2 below) are dominated by the high inter-correlation between parts of the same question. This is inevitable with this type of test. Independent items cannot be used without altering the nature of the activity, since generalising from examples produced oneself is a different activity from doing so from a set offered in the item; similarly, an explanation of a generalisation found for oneself is different from one of a generalisation proposed. The factor analysiscan offer guidance regarding how far the characteristics which the tester supposes he has built into the item are reflected in differential performance characteristics of the pupils. But it is clear that in many cases items which seem clearly different in nature correlate highly because, for example, it is impossible to score for an explanation if one has failed to make the generalisation. The main points of interest in the factor analysis are therefore those where items from different questions are brought together in a factor, or items from the same question are separated. In the final decision regarding scale allocation, the prima facie nature of the item, its occurrence in factors and its actual intercorrelations with the other items were all considered. Some notes on particular items are given. The four scales adopted are:

1. Generating examples: to meet given criteria, stating how or why given examples fail to qualify, classifying examples, finding complete sets. Items 1, 5, 6, 7, 8, 12, 14, 15.

2. Recognising relationships and patterns, extending patterns; expressing relationships verbally. Items 10, 17, 19, 20, 21.

3. Giving explanations or proofs. Items 4, 11, 13, 16, 18, 22

4. Following verbal instructions to produce data. Items 2, 3 and 9

The separation of scales 1 and 4 was made in response to the factor analysis. This brings together items 2, 3 and 9 into a factor separate from items 1, 5, 12; the difference is that expressed in the scale titles. In Scale 1 items, trials have to be made and examples selected which meet certain criteria e.g. multiples of 6 and 15 around 29, hexagons which fit into a triangle and have not been found already; in items 2 and 9 one simply chooses a starting number and follows instructions. Item 3 correlates highly (0.75) with item 2, and although it was originally considered as a 'making generalisation' item, since it requires only the assertion that 'the result is always 20' following the process defined in item 2, it seems reasonable to include it with that item.

Of the other original 'making generalisation' items, No. 10 emerged as highly correlated (0.66) with its explanation item, No. 11; in contrast to items 3 and 4, where the 'generalisation' is easy and the explanation hard, (means 1.22 and 0.44 out of 2), here the recognition of the pattern is much less obvious and the step from there to giving an explanation not so big (means 0.68, 0.39 out of 2). For example, possible sequences of answers in Item 9 are 12-22-2-24, 25-35-15-50 with the first to last doubling explainable as two lots of the first number, one with 10 added, one with 10 taken away. However, we have kept item 11 in the explanation scale, and item 10 in the modified generalisation scale, now called Recognising relationships.

This new scale 2 collects with items 10 and 17, the first three of the four parts of question 7. Here the general pattern has to be abstracted from the four examples 2 + 3 = 5, 4 + 6 = 10, 6 + 9 = 15, 8 + 12 = 20; it does not have to be verbalised but has to be extended by the provision of more examples first simply following one, then meeting other criteria, viz. starting with 16 then ending with 60 then starting with 29 (impossible). These three items cohere strongly in all the factor analyses, but the following item 22, requires insight into and verbal explanation of the 2n + 3n = 5n pattern and the 7 factor tables associate this with another explanation item (No. 4). (See table 2).

Item 17 asks for detection of the pattern in the results of question 4 i.e. in 33, 36, 39; it thus falls clearly into scale 2, though statistically it sticks on all analyses with the other items in Question 4, since it depends on reasonably correct results in Question 4. In this case we resist the statistical pull and retain it in Scale 2.

Scale 1 contains mainly items from Questions 2 and 4. The latter involve generating numerical examples to meet certain conditions - e.g. multiples of 6 and 15 between 30 and 40 - while in question 2, item 5 requires experiment with geometrical shapes to determine whether they fit in certain frames. Items 6, 7, 8 require 'explanations' of why certain frames cannot accommodate the pieces. These explanations are of a somewhat different character from the verbal/numerical/insight explanations of Items 4, 11 and 22. The present ones require little more than a statement of why the attempt to fit the shapes broke down. Item 8, for completeness, requires a two step statement that shape A must go across the long row, and then there is an isolated square, impossible to fill; but this still is not comparable with items 4, 11 and 22. It seems reasonable here to accept the statistical suggestion that associates these items 5-8 all together, as a scale consisting of "Generating Examples to meet given conditions, and stating how or why certain examples fail to quality." Item 1 appears also in this scale. This mark is obtained for giving a <u>correct complete set</u> of hexagons placed in different positions within a triangle (Question 5). Although this represents one of the generalisation strategies developed during the SNP course, the mark was dropped initially from the scoring because, the Project schools' scripts being marked first, the item appeared too easy. However, the non-Project classes scored less well and it is therefore an important item for comparison purposes. It does not correlate particularly highly with any other variables, but fits reasonably statistically and naturally on grounds of content into Scale 1.

Scale 3 (Explanation) has by now been discussed fairly fully. Items 4 and 11 define it and are brought together in the 5-factor analysis. Item 22 is clearly of the same type, item 16 (why the set of hexagons is complete) is close in type of thinking though different in context; item 18 is also a genuine explanation item, though it is capable of being answered correctly at the more superficial level which might suggest scale 2 rather than scale 3. (Why are 33, 36, 39 all multiples of 3? Because 6 and 15 are both multiples of 3!) The 7-factor analysis brings together items 4 and 22, and items 13, 16 and 18, so in this scale the apparent nature of the items and the statistical analyses are in good agreement. The item analysis following these scales is shown in Table 5.

ACTOR PATTERN MATRIX (CONVENTIONALLY SCALED). SALIENTS MARKED WITH ASTE

	1	2	3	4	5
1	0,16	0,31	0,12	-0,03	-0,23
2	-0,04	0,01	0,80*	-0,01	-0,05
3	-0,01	-0,04	0,80*	0,04	0,01
4	0,17	-0,02	0,22	0,11	0,30*
5	0,05	0,59*	-0,10	0,20	-0,08
6	0,05	0,64*	-0,15	0,12	0,00
7	-0,08	0,51*	0,14	0,03	0,01
8	-0,23	0,73*	-0,05	-0,05	0,22
9	-0,00	0,15	0,27	-0,22	0,29
10	0,01	0,08	0,09	-0,05	0,66*
11	-0,01	-0,03	-0,13	0,03	0,74*
12	0,29	0,32*	0,12	0,17	-0,17
13	0,16	0,68*	0,09	-0,23	-0,13
14	0,89*	0,01	-0,07	0,01	0,01
15	0,90*	-0,03	0,09	-0,11	-0,02
16	0,25	0,10	0,07	-0,07	0,11
17	0,61*	-0,08	-0,06	0,23	0,09
18	0,23	0,05	0,06	0,18	0,15
19	0,03	-0,05	-0,09	0,62*	-0,06
20	-0,08	-0,05	0,02	0,78*	-0,00
21	-0,10	0,09	0,11	0,65*	-0,01
22	0,05	-0,08	-0,01	0,37*	0,09

TABLE 1

5-factor analysis (oblique) of the 22 items

FACTOR PATTERN MATRIX (CONVENTIONALLY SCALED). SALIENTS MARKED WITH ASTERIS

	1	2	3	4	5	6	7
	Sc 1	Sc 4					
1	0.18	0,15	0,00	0.14	0,19	-0.20	-0,09
2	0.03	0,80*	0,04	-0.03	0,02	0.01	-0,11
3	-0.03	0,74*	0,00	0.04	-0,05	-0.02	0,14
4	0.13	0,14	-0,08	-0.11	0,11	0.05	0,56*
5	0.10	-0,07	0,21	0,17	0,45*	-0,07	-0,03
6	0.21	-0,08	0,14	-0.02	0,61*	0.00	-0,09
7	0.01	0,17	0,03	0.06	0,44*	0.00	-0,03
8	-0.14	-0.04	-0,13	0.09	0,65*	0.10	0,13
9	-0.01	0.27	-0,13	0.13	0,03	0.39*	-0,19
10	0.00	0.09	0,00	0,01	0,04	0.71*	-0,07
11	-0.03	-0,14	0,03	-0.09	0,01	0.71*	0,08
12	0.08	0,06	0,26	0.56*	-0,08	-0.03	-0,14
13	-0.11	0,00	-0,20	0.31*	0,12	-0.05	-0,00
14	0.89*	-0,06	0,03	-0.02	0,02	0.01	-0,00
15	0.90*	0.10	-0,10	-0.02	-0,02	-0.03	0,02
16	0.04	-0,01	-0,07	0.38*	-0,15	0.12	0,11
17	0.44*	-0,11	0,24	0.22	-0,22*	0.11	0,07
18	-0.04	-0.04	0,15	0.42*	-0,21	0.14	0,20
19	0.01	-0.08	0,66*	-0.01	-0,06	0.01	-0,08
20	-0.09	0.05	0,79*	-0.06	-0,05	0.03	-0,01
21	-0.05	0,12	0,63*	-0.11	0,15	-0.04	0,08
22	-0.14	-0,15	0,17	0.12	-0,09	-0.14	0,60*

TABLE 2

7-factor (oblique) analysis of the 22 items

APPENDIX TO CHAPTER 6

Examples of pupils' deductions - CSMP

Logical problems test, with results

Logical questions from University first year examination, with results (Anderson)

Prove:

(a) $P \Rightarrow T,\ T \Rightarrow (Q \Rightarrow {\sim}S),\ {\sim}S \Rightarrow R \vdash (P \wedge Q) \Rightarrow R$

Demonstration	Analysis
1 $P \Rightarrow T$	1 Assump
2 $T \Rightarrow (Q \Rightarrow {\sim}S)$	2 Assump
3 ${\sim}S \Rightarrow R$	3 Assump
4 ${\sim}[(P \wedge Q) \Rightarrow R]$	4 Assump
5 $(P \wedge Q) \wedge {\sim}R$	5 Sub In (4)
6 ${\sim}R$	6 Conj. Simp. (5)
7 $P \wedge Q$	7 Conj. Simp. (5)
8 P	8 Conj. Simp. (7)
9 Q	9 Conj. Simp. (7)
10 T	10 Mod. Pon. (1, 8)
11 $Q \Rightarrow {\sim}S$	11 Mod. Pon. (2, 10)
12 S	12 Mod. Toll. (3, 6)
13 ${\sim}S$	13 Mod. Pon. (9, 11
14 $S \wedge {\sim}S$	14 Conj. Int. (12, 13)
15 $(P \wedge Q) \Rightarrow R$	15 Indirect Inference (4-14) (Discharging ${\sim}[(P \wedge Q) \Rightarrow R]$

Prove: $(\forall x, y)[x, y \in \alpha \Rightarrow {}^{-}(x \oplus y) = {}^{-}x \oplus {}^{-}y]$

Demonstration	Analysis
1. $x, y \in \alpha$	1. assumption
2. ${}^{-}(x \oplus y) \oplus (x \oplus y) = 0$	2. Lemma 20 (I)
3. $({}^{-}x \oplus {}^{-}y) \oplus (x \oplus y) = {}^{-}x \oplus ({}^{-}y \oplus y) \oplus x$	3. assoc. and comm.
4. $= {}^{-}x \oplus x$	4. Lemma 20 (I)
5. $= 0$	5. Lemma 20 (I)
6. ${}^{-}(x \oplus y) \oplus (x \oplus y) = {}^{-}x \oplus {}^{-}y \oplus x \oplus y$	6. SAE
7. ${}^{-}(x \oplus y) = {}^{-}x \oplus {}^{-}y$	7. Theorem 16
8. $x, y \in \alpha \Rightarrow {}^{-}(x \oplus y) = {}^{-}x \oplus {}^{-}y$	8. Ded. Thm. (discharging 1)
9. $(\forall x, y)[x, y \in \alpha \Rightarrow {}^{-}(x \oplus y) = {}^{-}x \oplus {}^{-}y]$	9. UG

P1. Andrew is Sharon's brother.

Tina is John's sister.

Sharon is Tina's cousin.

Gary is John's cousin.

Tick any statement which you are sure must be correct.

Cross any statement which you are sure must be false.

			Ages			
			11	12	13	15
X	a)	Tina is Gary's sister.	*	64	100	100
✓	b)	Tina is Gary's cousin.	55	82	100	100
X	c)	Andrew is John's brother.	*	73	82	67
			*45 both parts correct			
	d)	Gary is Andrew's brother.	08	0	9	67
	e)	Sharon is Gary's sister.		9	9	50

P2. There were seven children at Susan's party.

Six of them each brought a present, and four of the presents were toys.

Three of the children won prizes.

Tick(✓) any statement below which you are sure must be correct.

Cross(x) any statement below which you are sure must be false.

X	a)	All of the children who brought presents brought toys.	66	100	82	100
✓	b)	Some of the children who brought presents also won prizes.	50	73	73	100
	c)	Some of the children who brought toys won prizes.	*	9	36	83
			*0 both parts correct			
	d)	All of the children who won prizes had brought presents.	*	9	36	67

P4. In the Smith family, all those who cook also help with washing up.

Peter Smith can cook. Jane Smith washes up.

Tick the sentence below which you are sure must be correct.

Cross any statement which you are sure must be false.

a) All the Smith family both cook and wash up. * 20 18 33

b) Peter and Jane Smith both cook and wash up. * 0) 0

c) All the Smith family either cook or wash up. * 20 18 50

✓ d) Peter Smith cooks and washes up. 62 50 64 10

e) Jane Smith cooks and washes up. * 0 9 17

* 01 all four parts co

P5. The paperboy delivers 15 newspapers to 12 houses.

Tick the statements which must be true.

a) One house must have no papers. * 100 91 100

b) Each house must have at least one paper. * ——————

✓ c) At least two houses must have the same number of papers. * 70 36 100

✓ d) At least one house must have more than one paper. * 91 100 100

* 62 all parts correct

Distribution of incorrect answers to P4

(a)	✓	40	18	17
	X	40	64	50
(b)	✓	50	82	83
	X	50	18	17
(c)	✓	20	36	17
	X	60	45	33
(d)		—		
(e)	✓	50	55	67
	X	50	36	17

TEST

(*a*) A necessary condition that the statement P be true is that $x < 5$. Which one of the following statements must be true?

(i) If $x < 5$, then P is true.
(ii) If $x < 5$, then P is false.
(iii) If $x \geqslant 5$, then P is true.
(iv) If $x \geqslant 5$, then P is false.
(v) None of these.

(i) $\boxed{11}$ (ii) $\boxed{-}$ (iii) $\boxed{-}$ (iv) [✓ 88] (v) $\boxed{2}$ (101)

(*b*) Let X be the set of all integers between 0 and 9 inclusive, and let P be the statement '$\forall x \in X, \exists y \in X$ such that $3x-2y = 5$'.

Which one of the following statements is correct?

(i) P is true.
(ii) $x = 5, y = 5$.
(iii) x must be odd.
(iv) $x = 2, y = 4$ is a counterexample.
(v) None of these.

(i) $\boxed{-}$ (ii) $\boxed{21}$ (iii) $\boxed{10}$ (iv) $\boxed{19}$ (v) [✓ 50] (100)

(*d*) A sufficient condition that the statement Q be true is that $y > 0$. Which one of the following must be true?

(i) If Q is true, then $y \leqslant 0$.
(ii) If Q is false, then $y \leqslant 0$.
(iii) If Q is true, then $y > 0$.
(iv) If Q is false, then $y > 0$.
(v) None of these.

(i) $\boxed{-}$ (ii) [✓ 64] (iii) $\boxed{15}$ (iv) $\boxed{-}$ (v) $\boxed{20}$ (99)

(*e*) It has been proved that infinitely many integers satisfy a relation R. Consider the statement S: 'all integers satisfy R'. Which one of the following is true?

(i) S is clearly true.
(ii) S is true, but requires proof.
(iii) S is more likely to be true than false.
(iv) S is false.
(v) None of these.

(i) $\boxed{11}$ (ii) $\boxed{13}$ (iii) $\boxed{22}$ (iv) $\boxed{17}$ (v) [✓ 37] (100)

(*g*) A necessary and sufficient condition that P be true is that A or B be true. Which one of the following is correct?

(i) If A is false or B is false, then P is false.
(ii) P is true if and only if A is true and B is true.
(iii) If P is false, then A is false or B is false.
(iv) P is false if and only if A is false.
(v) None of these.

(i) $\boxed{7}$ (ii) $\boxed{3}$ (iii) [✓ 35] (iv) $\boxed{2}$ (v) $\boxed{52}$ (99)

(*i*) A necessary condition that P be true is that either A or B, but not both, be false. Which one of the following is true?

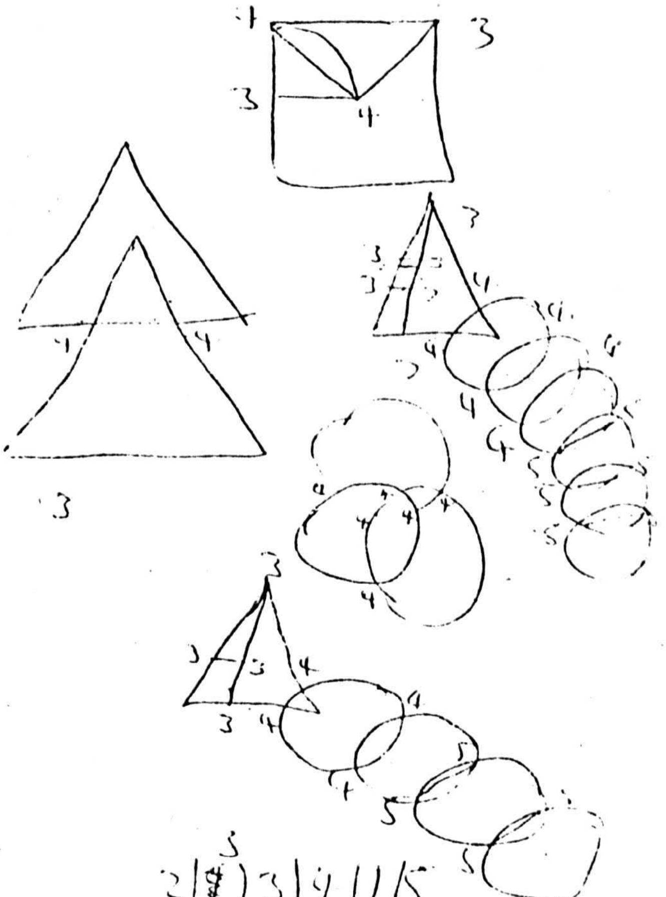
(i) If P is false, at least one of A and B is true.
(ii) If P is false, both A and B are true.
(iii) If P is true, (A or B) is false.
(iv) If P is true, precisely one of A and B is true.
(v) None of these.

(i) $\boxed{4}$ (ii) $\boxed{4}$ (iii) $\boxed{16}$ (iv) [✓ 71] (v) $\boxed{4}$ (99)

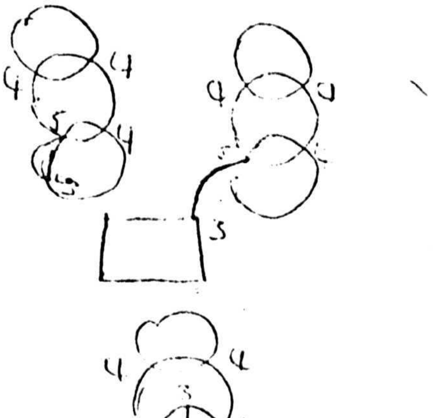
APPENDIX TO CHAPTER 7

Pupils' written trials (Networks)

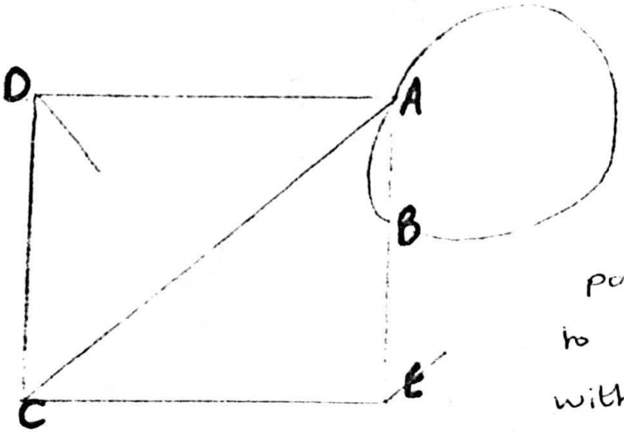
MF



5
2|~~4~~|3|4||5



3|3|1

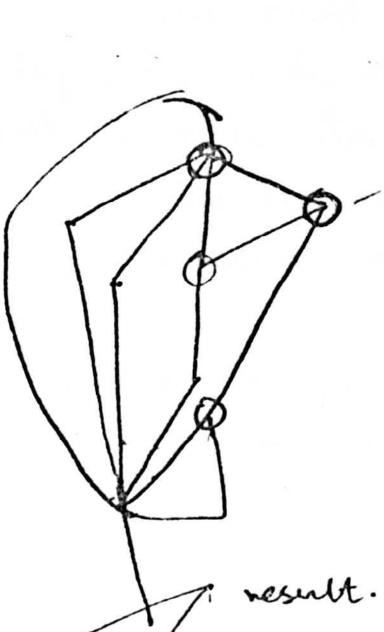


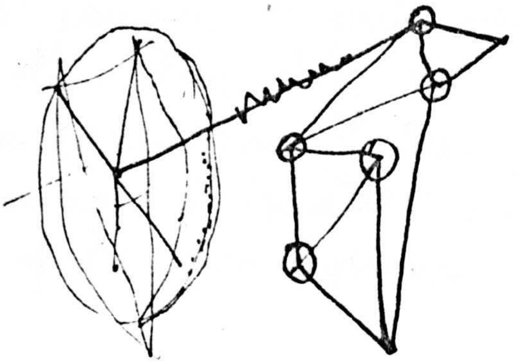
CD

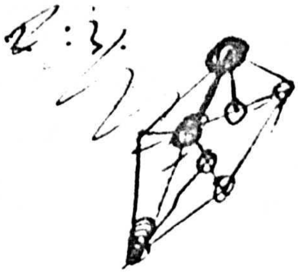
Starting from a (1, 1, 1,) it is impossible to get a (2, 1, 1) e.g. drawing a line at point D or E it is impossible to finnish it with joining up with another junction

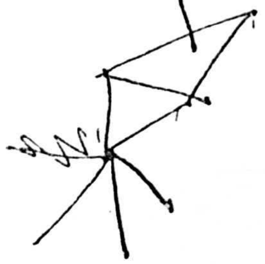
∴ it is impossible to get a (2, 1, 1,) netw

JS1









result.

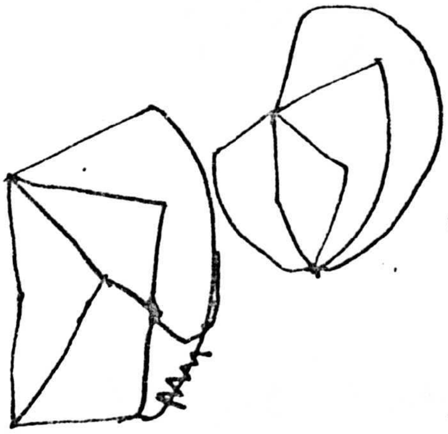
conditions =

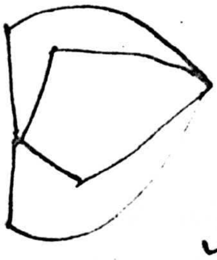
① even junctions outnumbering odd ones.

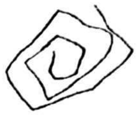
② with none complicated networks ~~[illegible]~~ the bigger the proportion of smaller junctions i.e. three's.

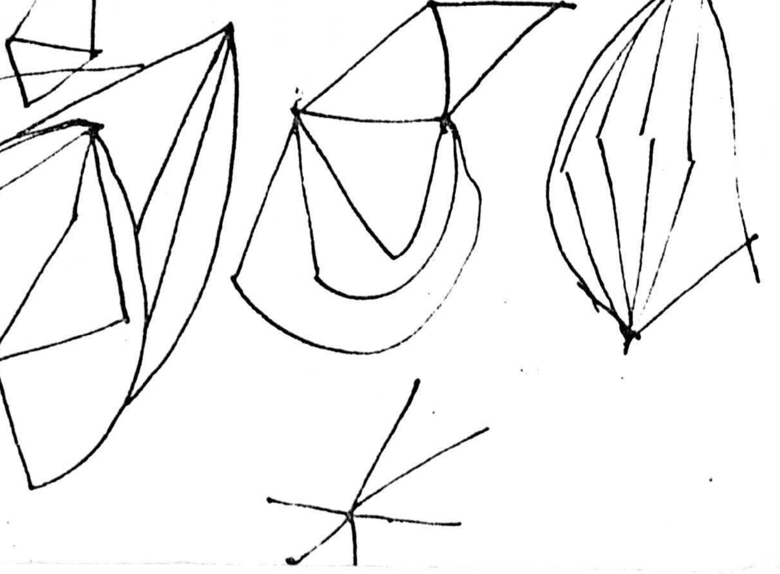
③ any number of 3's or fours.

orders.	3	4	5	6	7	8	9	10	11.
network. ✓	11	11	0	0	0	0	0	0	0
✓		11	0	0	0	0	0	0	0
✓	11	0	0	0	0	0	0	0	-
✓	1		1	1					
✓	5		(4	1					
✓	4	1		11					
✓	8	11		11					









JS2

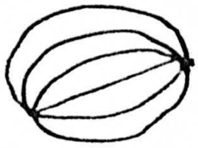
When adds junctions to each other unless they are of parity with each other then they will require junctions of smaller orders to allow the completion of the figure.

The amount of smaller junctions is ascertainable by subtraction of the two figures.

example

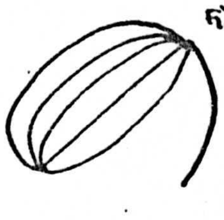
an order 6 junction. +
an order 6 junction :-

Fig ①



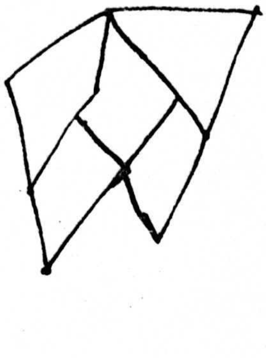
In fig 1. it is easily recognisable that they have only junction of order 6.

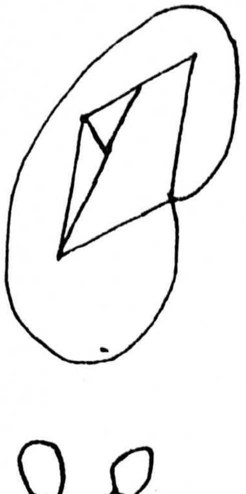
Fig ② Fig ③

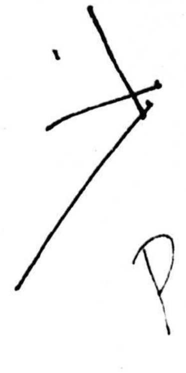


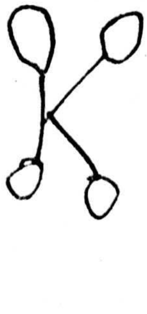
but in fig ②. an order 5j & an order 6j are added together & a junction is vacant. to anchor it to the 5j fig. one requires a 3j fig. (fig ③)

This rule holds to no matter how low the figures digress :-
3j + 3j = 0j ; 3j + 3j + 3j = 3j









APPENDIX TO CHAPTER 8

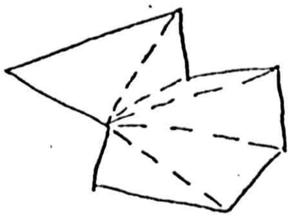
Follow-up sheets

Selected responses

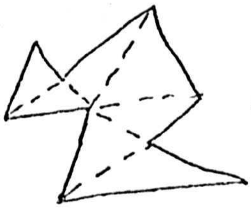
Three people, Jane, Philip and Karen did this question. Read their answers and decide which of them you agree with most.

Jane: said "It is true for all polygons. The diagrams on the sheet prove it to be true."

Philip said "If you draw diagonals from one point, they go to all the corners except three; the three being the point itself and the two ~~to~~ next to it. You can do this in any polygon, so the number of diagonals is always 3 less than the number of corners".



Karen said "I have drawn polygons of all shapes and sizes, including big and complicated ones and they ~~statement~~ all work out right. So I think the statement is probably true for all polygons." ✓

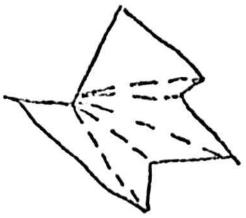


9, 6.

I think that ~~Karen~~ I agree with Karen.

DIAGONALS OF POLYGONS II

1. Did you use a diagram something like this, with diagonals all going from one point, to prove the statement on the previous sheet?

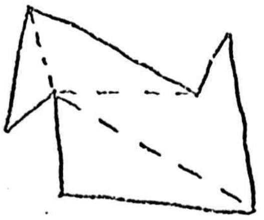


2A. If not, do you consider that this does prove the statement? Say why.

2B. If you did, go on to question 3.

~~3. [illegible]~~

3. Consider this diagram:



7, 3

Does this conflict with your previous conclusion?

Does it conflict with your previous reasons?

If so, give your new conclusion and your new reasons.

DOUBLE AND ADD THE NEXT II

1. Can you finish on 15?

2. Can you change your previous rule (if necessary) to give a rule which tells you, for any given number, whether it is a possible finishing number?

3. Say why you are sure your rule ~~applies to~~ covers all cases.

K . 22/2/63

J.K.

26 + 27 = 51
42 + 43 = 85
28 + 29 = 57

you can do it with an even number ~~not~~ ~~a~~ or can odd one
It ~~will~~ can be only half of it.
Sometimes it can be true
It can not be true

JLH

Double and add the next
Paper 1
JLH 11.7.63

tart 13 ~~Finish~~ Finish 53
Start 3 Finish 13
Start 10 Finish 21
you cannot do this sum
~~you cannot do this sum. Start 3 Finish 14~~
you cannot have 14 as a finishing number but you can have numbers like 53, 13, 21. That it seems better to have a number between 1 to 20 to double than to have a ~~[illegible]~~ number higher than 30.

Double and add the next

5 SAM

S.A.L.L.

8.7.59.

Double and Add the Next

① Start with 3. finish with 13
② start with 25 finish with 101
③ To finish with 13 start with 3
④ To finish with 21 start with 5
⑤ To finish with 14 is [redacted] impossible using whole numbers
⑥ Using whole numbers you cannot finnish on an even number.
⑦ Subtract 1 and divide by 4.
⑧ These rules are always true. ⑦ is using the reverse process and ⑥ is impossible.

3KC

Double and Add the next. K.C [redacted] 4.3.61

Start with 24, double it 48; add the next number.
48 + 49 = 97 Start 24, Finish 97.

Start with 5, double it 10 add the next number
10 + 11 = 21 Start 5 Finish 21.

You can Finish on 13. Start number will be 3.
You can Finish on 21. Start number will be 5.
You cannot Finish on 14.
You always have to finish on odd numbers
If you start with a number If you times it by 4 and add one you will get the finishing number. So if you have a given finishing number you will divide it by 4 and then subtract by 1..
If you double any number you will always get a even number then you add the next number witch will always give you an odd number.

6 DK

D.B. 2.11.57 Double and Add the Next

Start 5; doubled; 10; add next no
10+11=21 Start 5 finish 21
Start 9, doubled; 18; add next no
18+19=37 Start 9 finish 37.

To finish on 13 Starting number is 3.
" " " 21 starting number is 5
You cannot finish on 14.

All finishing numbers must be odd numbers. This is always true because when an odd and an even no are added the answer is always odd
All finishing numbers must be

Find 2 nos (one odd, one even) which add up to finishing no then halve the even no to find the starting no. This is always true

---

6AW

29.5.58. DOUBLE AND ADD THE NEXT.

Start 7, double it 14, add the next number 14+15=29
Start 7, finish 29.

Start 10, double it 20, add the next number 20+21=41
Start 10, finish 41

3) You would need 3 start 3 double 6+7=13

4) You would need 5. ~~Take a~~ start 5 double 10+11=21

5) You would need ~~6~~3¼ double 6½+7½=14

6) The final number must not be an even number, as the number you start with will not be a whole number.

7) Take one away from the finishing number, then ~~half it~~ divide by 4 then, this finds the starting number.

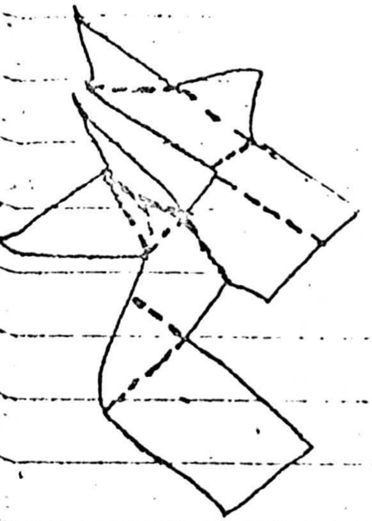
21 - 1 = 20 ÷ 4 = ~~10~~ 5 starting number

27 - 1 = 26 ÷ 4 = 6½ starting number.

These first rules should always be true, but the second will only be sometimes true. If a small number eg below 5 is taken in the ~~start~~ finishing number, it does not divide properly into four.

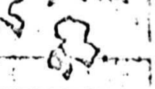
1 DG

DIAGONALS OF POLYGONS I



There are eleven diagonals in This polygon

This statement is not true because some polygons can be bigger and some a be smaller. There is also in the difference of the shape. e.g. and re were more diagonals for ⓐ than for ⓑ.



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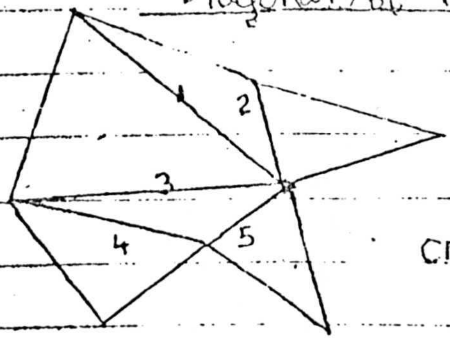
3 SW

S.W.

28/2/61

Diagonals of Polygons

A.



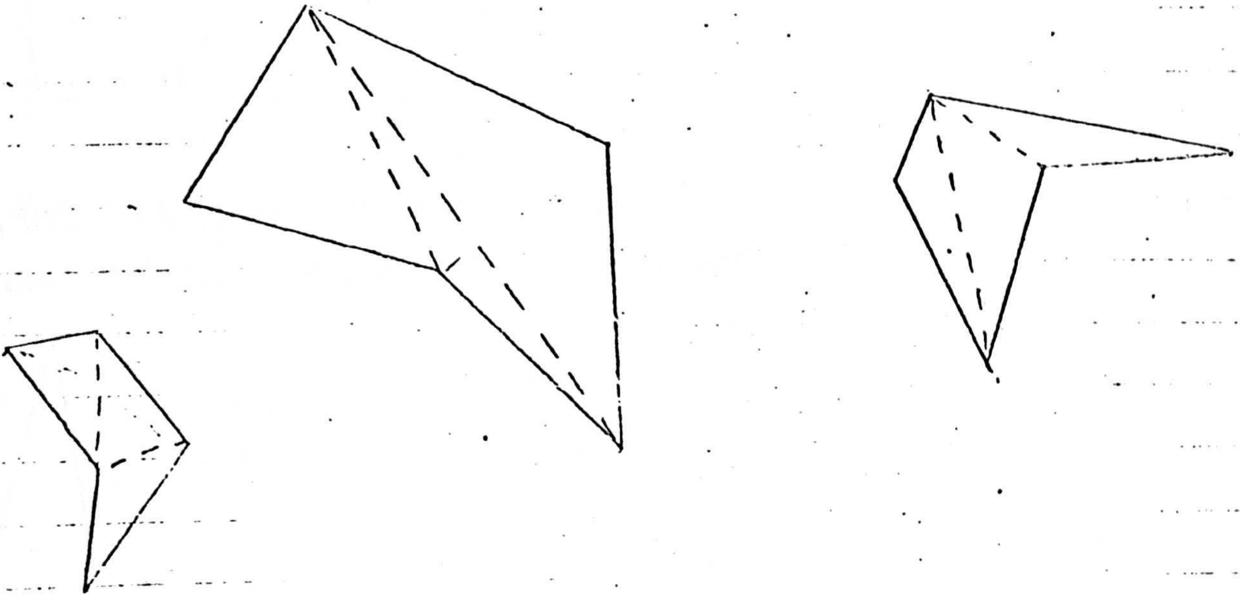
There are 5 diagonals + Sides which do not cross one another

B. Yes it is probably true for all polygons, that the greatest number of non crossing diagonals which can be drawn in a polygon is three less than the number of sides.

5

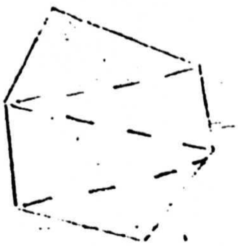
S.W
2/2/59.

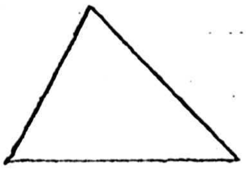
A.



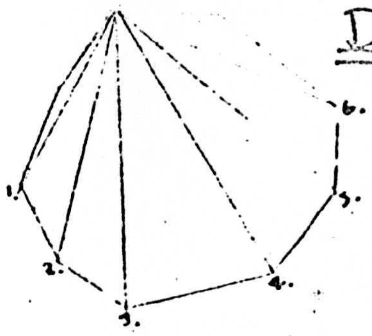
there are 2 diagonals in a polygon without other diagonals crossing them.

B.





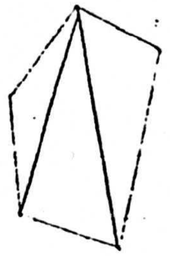
"The greatest number of non-crossing diagonals which can be drawn in a polygon is three less than the number of sides" is true for all polygons. This is because in a pentagon 3 diagonals can be drawn which is 3 less than the number of sides. In a square only one diagonal can be drawn which is 3 less than the number of sides and in a triangle no diagonals can be drawn as there are only 3 sides in a triangle.



Diagonals of polygons 14.1.61 C.S

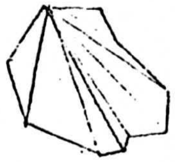
There are 6 diagonals, in a polygon

Yes, the statement is true about the no. of diagonals in a polygon is 3 less than the number of sides.



5 sides

2 diagonals



10 sides

7 diag

It is always true that whatever shape you do, and draw in the diagonals it will always be 3 less than the no. of sides, because each line goes to a corner, and two corners are really on the same line & there is the one which they are coming from.

APPENDIX TO CHAPTER 9

Selected responses

ADD AND TAKE

Choose any number between 1 and 10. Add it to 10 and write down the answer. Take the first number away from 10 and write down the answer. Add your two answers.

1. What result do you get?
2. Try starting with other numbers. Do you get the same result?
3. Will the result be the same for all starting numbers?
4. Explain why your answer is right.

14 → 6 → 20

) 20

9 → 19 → 1 ⇒ 20 , 1, 11, 9 ⇒ 20

10 → 20 → 0 → 20

Yes. I always get 20.

) Yes. The result should be the same for all starting numbers.

This is because each time you are taking 10 and adding to it any number from 1 to 10 e.g. 10+4, then taking the supplement and adding it to 10+4.

e.g.

10+4+6 = 20.

Each time you would get 20 as the resu

Adding the supplement is just another way of aying take the first number from 10 and ad

t to your other answer

0+4=14 14+6=20

ADD AND TAKE

Choose any number between 1 and 10. Add it to 10 and write down the answer. Take the first number away from 10 and write down r the answer. Add your two answers.

1. What result do you get?
2. Try starting with other numbers. Do you get the same result?
3. Will the result be the same for all starting numbers?
4. Explain why your answer is right.

→ 14 → 6 → 20

1) 20

2) 9 → 19 → 1 → 20 , 1, 11, 9 → 20

10 → 20 → 0 → 20

Yes. I always get 20.

3) Yes. The result should be the same for all starting numbers.

4). This is because each time you are taking 10 and adding to it any number from 1 to 10 e.g 10+4, then taking the supplement and adding it to 10+4.

e.g.

10 + 4 + 6 = 20.

Each time you would get 20 as the resul

Adding the supplement is just another way of saying taking the first number from 10 and ad it to your other answer

g. 10+4=14

10−4 = 6

14+6 = 20

ADD AND TAKE

Choose any number between 1 and 10. Add it to 10 and write down the answer. Take the first number away from 10 and write down r the answer. Add your two answers.

1. What result do you get?
2. Try starting with other numbers. Do you get the same result?
3. Will the result be the same for all starting numbers?
4. Explain why your answer is right.

⑥	④	⑤	③	⑨
16	14	15	13	19
4	6	5	7	1
20	20	20	20	20

The result I get is 20

whatever your number is the answer is always 20

The result is the same for any starting number from 1-10

The result is always the same because because ~~starting number~~ ~~minus 10~~ 10 minus starting number

= X X + starting number + 10 = 20

ADD AND TAKE

Choose any number between 1 and 10. Add it to 10 and write down the answer. Take the first number away from 10 and write down r the answer. Add your two answers.

1. What result do you get?
2. Try starting with other numbers. Do you get the same result?
3. Will the result be the same for all starting numbers?
4. Explain why your answer is right.

12
10
22

10
10
20
10
20

1
11
1
22

4
14
4
24

Result is 22

My answer is right because I started with 2 added it to 10 then took my first number away which leaves me with 10 add 10+12 to which adds up to 22.

5
15
10
25

8
18
10
28

9
18
10
29

29

ADD AND TAKE

Choose any number between 1 and 10. Add it to 10 and write down the answer. Take the first number away from 10 and write down r the answer. Add your two answers.

1. What result do you get?
2. Try starting with other numbers. Do you get the same result?
3. Will the result be the same for all starting numbers?
4. Explain why your answer is right.

$\underline{\underline{2}}$

$2 + 10 = 12,$

$10 - 2 = 8$

$8 + 12 = 20.$

$\underline{\underline{3}}$

a) $3 + 10 = 13$

$10 - 3 = 7$

$13 + 7 = 20$

$1\frac{1}{2}$

c) $1\frac{1}{2} + 10 = 11\frac{1}{2}$

$10 - 1\frac{1}{2} = 8\frac{1}{2}$

$11\frac{1}{2} + 8\frac{1}{2} = 20.$

The result is 20.

Starting with any number, even improper fractions, between 1 and 10 the result will always be the same.

This is right because 10 is the number which you must always have ∴ if you add a number to it (10) or take the same number from it the answers will compliment each other (to add to 20) because 10 and 10 is 20 This is the same as saying $10 - 2 + 10 + 2 = 20$ or $10 - 7\frac{3}{4} + 10 + 7\frac{3}{4} = 20$ This is proved because if you add a positive number to its own negative the result will be nought — $-2 + 2 + 10 + 10 = 2$ it is just like saying $10 + 10 = 20.$

It is just like saying $2 + 10 = 20 - 8$

or $3 + 10 = 20 - 7$

$1\frac{1}{2} + 10 = 20 - 8\frac{1}{2}$

ADD AND TAKE

Choose any number between 1 and 10. Add it to 10 and write down the answer. Take the first number away from 10 and write down the answer. Add your two answers.

1. What result do you get?
2. Try starting with other numbers. Do you get the same result?
3. Will the result be the same for all starting numbers?
4. Explain why your answer is right.

number chosen 9
+ 10 = 19
− 9 = 1 ×
added answers = 20

number chosen 4
+ 10 = 14
− 4 = 10 ×
add answers = 24

number chosen 7
+ 10 = 17
− first no = 3
ans = 20

number chosen 5
+ 10 = 15
− first no = 5
ans = 20

number chosen 9
+10 = 19
− first number = 1
ans = 20

1). The results on numbers below 10 are always 20

numbers chosen 12
+ 10 = 22
− 12 = ⁻12
+ = 9'

2). The answers still revert to 20 with no's over 10.

3). Yes.

No. chosen. 11
+ 10 = 21
10-11 = -1
+ = 20

no chosen 19
+ 10 = 29
10-19 = -9
20-9 = 20

No chosen 64
+ 10 = 74
10-64 = ⁻54
74-54 = 20

No chosen 0
+ 10 = 10
-0 10 = 10
+ = 20

Mandy Florrer
2 Ned

COIN TURNING

This is a coin turning game but played with pencil and paper.

1. The first is about 3 coins, and a move consists of turning over any two.

   Using as many such moves as you wish, get from 3 tails to 3 heads.

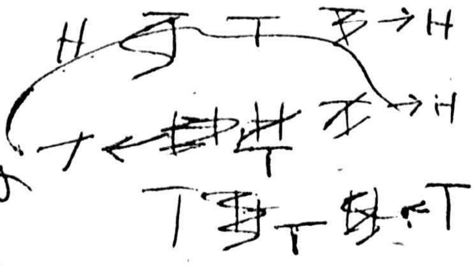
   Make your moves like this:

   T T T
   H H T
   T H H
   .... and so on.

   If you can do it, show your list of moves. If you think it is impossible, ~~explain why.~~

~~Don't Understand what you mean.~~
~~Because you've already got four heads~~
~~and you only have to turn one over~~
~~to get 3. So it's diff.~~

Its impossible because
when you turn a tale
you end up with more heads
and When you turn a head
you don't have enough



11

COIN TURNING

This is a coin turning game but played with pencil and paper.

1. The first is about 3 coins, and a move consists of turning over any two.

   Using as many such moves as you wish, get from 3 tails to 3 heads.

   Make your moves like this:

   T T T
   H H T
   T H H
   .... and so on.

   If you can do it, show your list of moves. If you think it is impossible, explain why.

I think this is not possible as the only possible moves left would ave you with ~~[illegible] least~~ two heads. The only way to get out of it uld ~~to~~ be to return to three tails yet even then this would impossible as you'd change from three tails to two heads d a tail then would have to change two more so again u would have two heads and a tail. This would carry on for er.

COIN TURNING

This is a coin turning game but played with pencil and paper.

1. The first is about 3 coins, and a move consists of turning over any two.

   Using as many such moves as you wish, get from 3 tails to 3 heads.

   Make your moves like this:

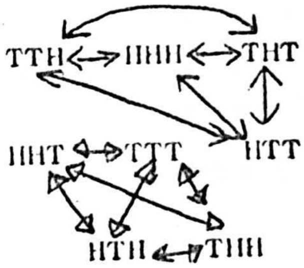
   T T T
   H H T
   T H H
   .... and so on.

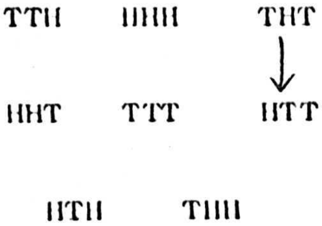
   If you can do it, show your list of moves. If you think it is impossible, explain why.

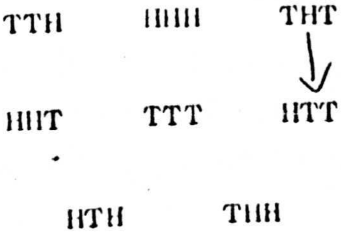
I think this is impossible, because by only moving two coins, I cannot get more than two heads showing at once. Because ~~only~~ moving two coins at once, and starting with 3 tails, causes there always to be either 2 or no heads showing. One coin is always left out.

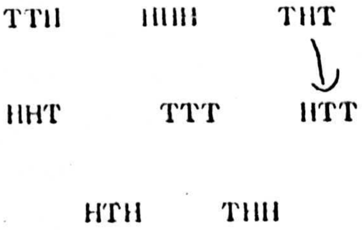
2. The diagrams below show all the possible ways of putting down three coins. An arrow has been drawn from THT to HTT to show that this is a possible single move. TTT to HHH is not a possible single move so these will not be joined.

Complete one of these diagrams by drawing arrows to show all the possible single moves. (The spares are for use if you make mistakes on the first one.)









3. Now explain again why your answer to No 1 is right.

My answer is right because by using all possible single moves it is impossible to get from TTT to HH H. Each of these has its own 'circle' of moves, and [redacted] from each starting point their are 3 moves to choose from. The trouble is, in neither 'circle' of moves ~~is~~ any one combination of coins the same.

COIN TURNING

This is a coin turning game but played with pencil and paper.

1. The first is about 3 coins, and a move consists of turning over any two.

   Using as many such moves as you wish, get from 3 tails to 3 heads.

   Make your moves like this:

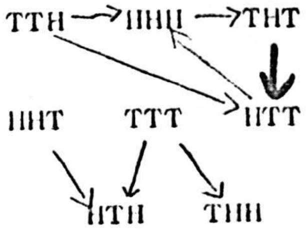
   T T T
   H H T
   T H H
   .... and so on.

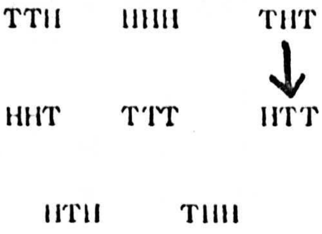
   If you can do it, show your list of moves. If you think it is impossible, explain why.

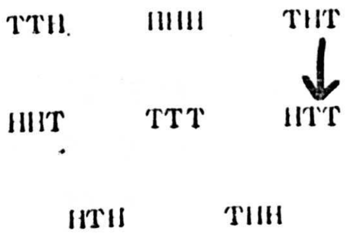
think that it is impossible, if two coins have to be turned over,
n one move. There are not enough coins. 3 heads can easily
e gained by moving one coin at a time, but it is impossible
y moving two.

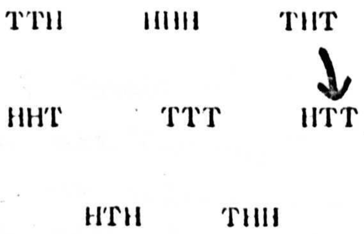
2. The diagrams below show all the possible ways of putting down three coins. An arrow has been drawn from THT to HTT to show that this is a possible single move. TTT to HHH is not a possible single move so these will not be joined.

Complete one of these diagrams by drawing arrows to show all the possible single moves. (The spares are for use if you make mistakes on the first one.)









3. Now explain again why your answer to No 1 is right.

As you can see, there is no arrow going from 3 tails to 3 heads. This is because the move is impossible, There are many moves which are possible in between.

COIN TURNING

This is a coin turning game but played with pencil and paper.

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   Using as many such moves as you wish, get from 3 tails to 3 heads.

   Make your moves like this:

   T T T
   H H T
   T H H
   .... and so on.

   If you can do it, show your list of moves. If you think it is impossible, explain why.

T	T	T
H	H	T
H	H	T
T	H	H
T	H	H
T	T	H
H	H	T
H	H	T
T	T	T
H	H	T

H	H	H
T	T	H
T	T	H
T	T	H
H	T	T
T	T	H
H	H	H
T	H	T

yes i think its' impossible because you are turning them over in 2's and no matter how hard you try you will still end up with 2 of the same, and one odd one which you havent turned.

ADDING A NOUGHT

If you want to multiply by ten, you can add a nought; for examply, 243 x 10 = 2430.

1. Is this true for all whole mumbers?

2. Explain why your answer is right.

① This is true for all whole numbers

ⓐ When multiplying by 10 the decimal point moves one Place to the right. Thus leaving a space inbetween the first number and the decimal Point, which the nought fills.!, 220×10=2200

15

ADDING A NOUGHT

If you want to multiply by ten, you can add a nought; for example, 243 x 10 = 2430.

1. Is this true for all whole mumbers?

2. Explain why your answer is right.

1. Yes this is true for all whole numbers

2. my answer is right because each time you multiply a number by 10 you are making it ten times bigger eg 10 × 10 = 100, 100 × 10 = 1000, 20 × 10 = 200, 30 × 100 = 3000 so if you just add a nought to the end of the number you are making it ten times bigger without even having to multiply the number by ten.

ADDING A NOUGHT

If you want to multiply by ten, you can add a nought; for examply, 243 x 10 = 2430.

1. Is this true for all whole mumbers?

2. Explain why your answer is right.

495·5 x10 = 4955 1·0056 x 10 = 10·056

95·4 x 10 = 10954. 7·4325 x 10 = 74·325

1095 4
10
10954
495 5
10
4955

s this is true for all whole numbers.

ly answer is right because decimal places, and ces in front of the decimal point are graded in e or more multiple of 10, g. after 1, 10, 100, 1000. and so instead of working t the sum you need just add the appropriate number noughts. The same is, if you multiply by a 100, ept you add two noughts because a hundred is 10 hes 10.

ADDING A NOUGHT

If you want to multiply by ten, you can add a nought; for examply, 243 x 10 = 2430.

1. Is this true for all whole mumbers?

2. Explain why your answer is right.

1, Yes this is true for all whole numbers.

~~#~~ An example for my answer is:

f you multiply this ~~some~~ sum by 1

08x1=208

hen multiply it by 10. You've got to make the answer

0 times bigger. I will show how we do this

$$\begin{array}{r} 208 \\ 10 \times \\ \hline 000 \\ 2080 \\ \hline 2080 \end{array}$$

First we do 0x8, then 0x0, then 0x2. That gives us a total of 0. But the second line we are multiply-ing by ten. So the first ~~row~~ column we have to put in a nought. Because 10x any number won't be less than 10. Even

0x0=0. Then we do 1x8 (Not 10x8 because we have put 0 in the first column to show we are multiplying by 10)

hen we do 1x0, then 1x2. We plot are awnsers in

e appriate columns, then add all the number up.

hich equal's 2080. In the first place, it would be

icker to add a 0. Instead of doing 0x8, 0x0 etc.

ADDING A NOUGHT

If you want to multiply by ten, you can add a nought; for example, 243 x 10 = 2430.

1. Is this true for all whole numbers?

2. Explain why your answer is right.

no it is not true of all ~~[illegible]~~ whole numbers
for example 243 X 4 does not make 2430
it makes 972.
because when you multiply by 10 you
add a nought as it is just 10 X 243
o if you got 243 and wrote 243 down
10 times it would come out as 2430

ADDING A NOUGHT

If you want to multiply by ten, you can add a nought; for examply, 243 x 10 = 2430.

1. Is this true for all whole mumbers?

2. Explain why your answer is right.

① yes it is true

② because ~~Wotch~~ What ever Whole number you × by 10 you just add A 0

eg ① 100146 × 10 = 1001460

② 4766429 × 10 = 47664290

③ 276428 × 10 = 2764280

That is Why i think I am right.

ADDING A NOUGHT

If you want to multiply by ten, you can add a nought; for example, 243 x 10 = 2430.

1. Is this true for all whole mumbers?

2. Explain why your answer is right.

This is true for all whole numbers .

this is right because by adding nought you are in Fact moving the decimal point one place to the right which n itself is multiplying by ten. This only works for whole numbers which are before the decimal point if you Add a nought after the decimal point you are making no difference what so ever but putting a nought before the point moves all the numbers up the thousands, hundre tens and units scales, making each of the numbers ten times bigger than they were previously

ADDING A NOUGHT

If you want to multiply by ten, you can add a nought; for example, 243 x 10 = 2430.

1. Is this true for all whole mumbers?

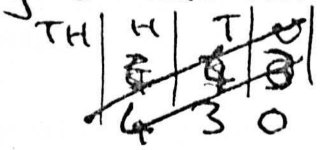
2. Explain why your answer is right.

① If you Times A whole number by 10 All you have to do is ADD A 0 E.g

$$\begin{array}{r} 43 \\ \times 1\,0 \\ \hline 0\;0 \\ 4\,3\;0 \\ \hline 4\,3\;0 \end{array}$$

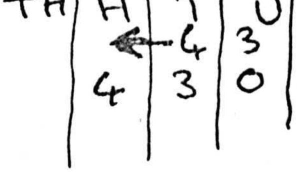
make it easyer by ADDing 0 like this

43 x 10 = 43$\underline{0}$



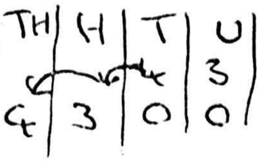
② If you Times A whole number by ten you Are making it ten times bigger So there-fore if you ADD 0 you make it Ten Times bigger E.g "Say if you had 1, And you ~~ADD~~ put 0, Next To it iT there fore makes 1, Ten x bigger 10,"

The same thing would happen if you wanted to x by 100, All you have To do is ADD 2, 0s There by making it 100 times bigger. Everytime you multyply by ten you move your number to the left therfore there is A Space which you fill in with a 0. E.g



43 x 10

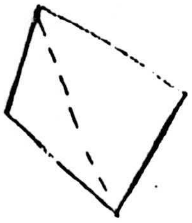
43x 100

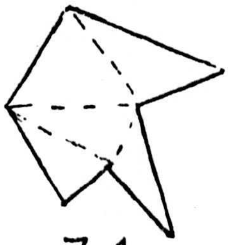


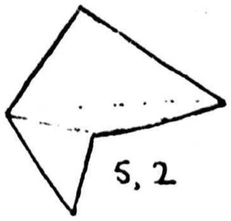
DIAGONALS OF POLYGONS

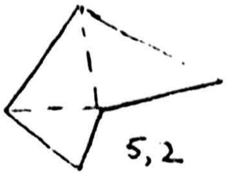
B.

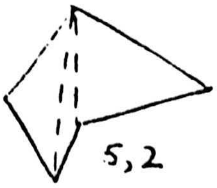

96.


4, 1


7, 4


5, 2


5, 2


5, 2


6, 2


6, 3


8, 5


9, 6

Some diagrams have been drawn here. It seems that "The greatest number of non-crossing diagonals which can be drawn in a polygon is three less than the number of sides."

Is this statement true for all polygons?

Investigate this fully; then state your conclusions and your reasons.

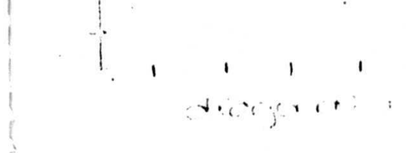
The statement is true for all polygons.

sides	diagonals
4	1
5	2
6	3
7	4
8	5
9	6

There does [illegible] seem to be a pattern or relationship between these sets of numbers ie the sides of the polygon and the number of diagonals.

If you have ~~an~~ 6 sided shape there are 3 non-crossing diagonals.
If you have an eight sided shape there are 5 non-crossing diagonals

As the number of sides go up 1 so do the number of non-crossing diagonals

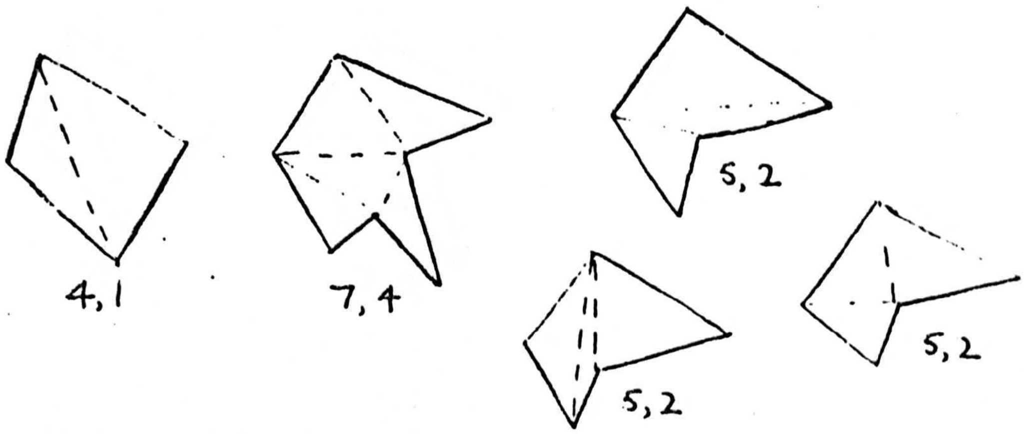


Take the numbers and put them on a graph to find the line that they form.

AR

DIAGONALS OF POLYGONS

B.

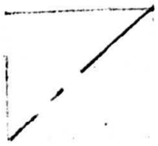


Some diagrams have been drawn here. It seems that "The greatest number of non-crossing diagonals which can be drawn in a polygon is three less than the number of sides."

Is this statement true for all polygons?

Investigate this fully; then state your conclusions and your reasons.

1) This Statement is true for all polygons



square
4 sides
1 diagonal

An ocragon
8 sides
5 diagonals

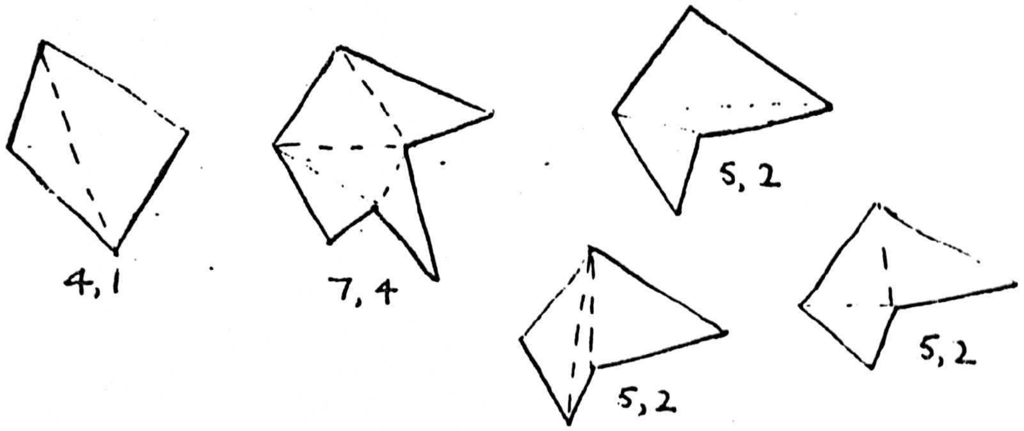
Pentagon
5 sides
3 diagonals

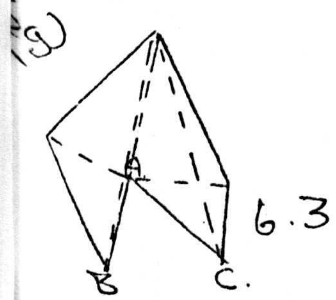
Starting at 4 sides the number of diagonals which can be drawn is 1. for a 5 sided polygon the nuber of diagonals increases by 2 to 3 diagonals. An 8 sided figure has 5 diagonals.

Conclussion

The number of diagonals which can be drawn in a polygon, is three less than the number of sides!.

DIAGONALS OF POLYGONS



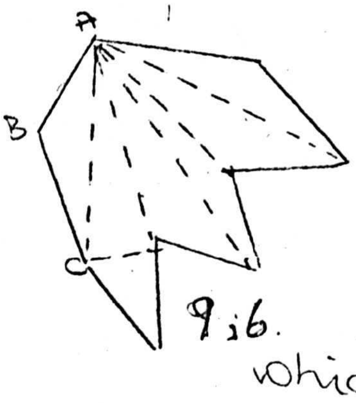


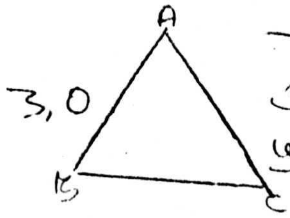
Some diagrams have been drawn here. It seems that "The greatest number of non-crossing diagonals which can be drawn in a polygon is three less than the number of sides."

Is this statement true for all polygons? Yes

Investigate this fully; then state your conclusions and your reasons.

This is because if you start at a point on the polygon (eg at A). and draw the number of diagonals from that point A each time. Then start again at B or C the lines cross those draw from the Point A.





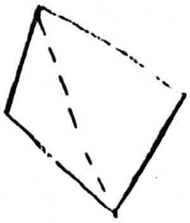
This is also true because the smallest number of lines you can have for any shape is three and a triangle has no diagonals at all A quadrilat[eral] which has four sides has only one [dia]gonal that can be drawn without it being crossed.

[T]he triangle has 3 sides 0 diagonal - difference 3

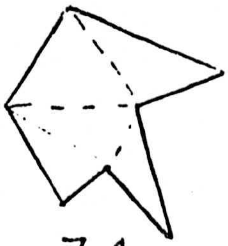
[T]he quadrilateral has 4 " 1 " — " "

DIAGONALS OF POLYGONS

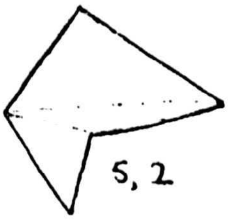
B.



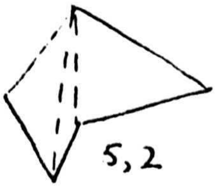
4, 1



7, 4



5, 2



5, 2

5, 2

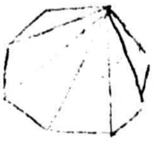
Some diagrams have been drawn here. It seems that "The greatest number of non-crossing diagonals which can be drawn in a polygon is three less than the number of sides."

Is this statement true for all polygons? Yes

Investigate this fully; then state your conclusions and your reasons.

I came to this conclusion with experimenting with other polygons.

6, 3



8, 5

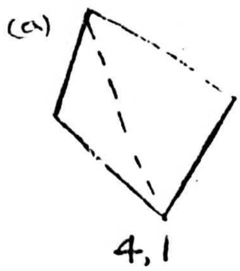
If in an 8,5 polygon you go from one point, you must have ~~at~~ only 5 other points to go to, as you can't go along any of the edges of the shape, ~~be~~ that leaves you with six points to go to, as you must be at the joint of 2 lines, and you can't go to yourself Therefore there are only 5 points left.

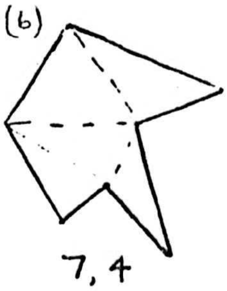
In the case of 7,4 and 6,3, 9,6 you have to go ~~to~~ from 2 points. I think then that if the second number is less than 3 you can go from 1 point only and if it is smaller than 7 you have to go from 4 points, such as in the polygon 10, 7.

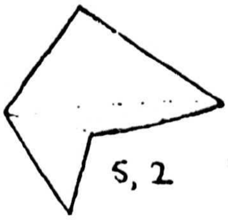
7,6

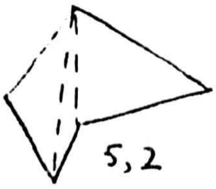
DIAGONALS OF POLYGONS

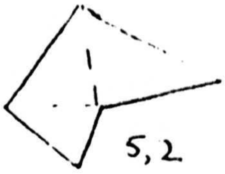
B.











Some diagrams have been drawn here. It seems that "The greatest number of non-crossing diagonals which can be drawn in a polygon is three less than the number of sides."

Is this statement true for all polygons?

Investigate this fully; then state your conclusions and your reasons.

Yes this is true for all polygons.

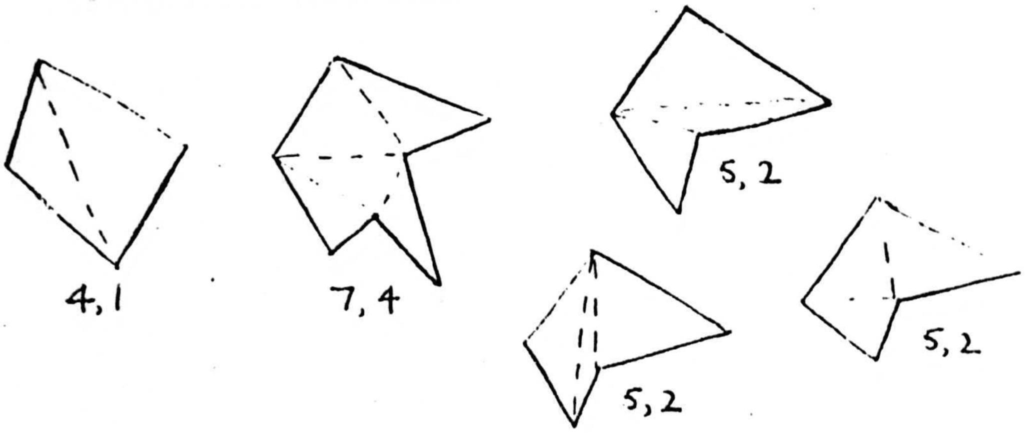
This is because each non-crossing diagonal ~~goes~~ meets two points on the polygon and so as for diagram (a) –  there is no way that those two points can be used again, and the only other diagonal which could be made would cross the other.

But, as in diagram (b) some of the points are used twice but this is possible because the diagonals can go in different directions without crossing each other. In this 7 sided figure it is only possible for ~~four~~ three of the ~~sides~~ points to be used twice and so only 4 diagonals can be made.

In a 5 sided figure only 3 1

DIAGONALS OF POLYGONS

B.

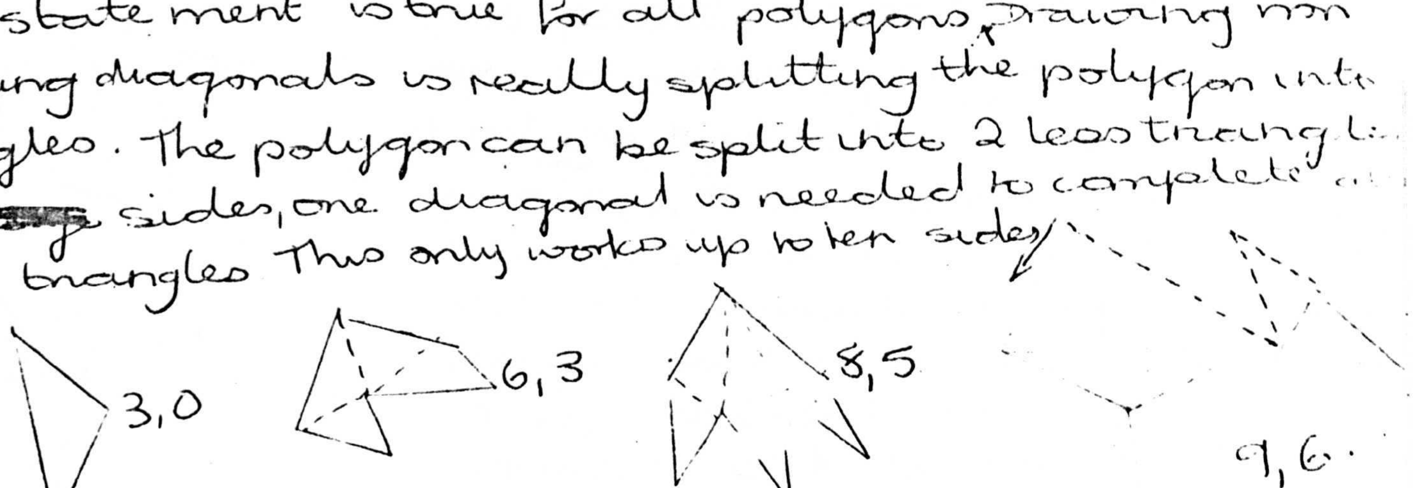


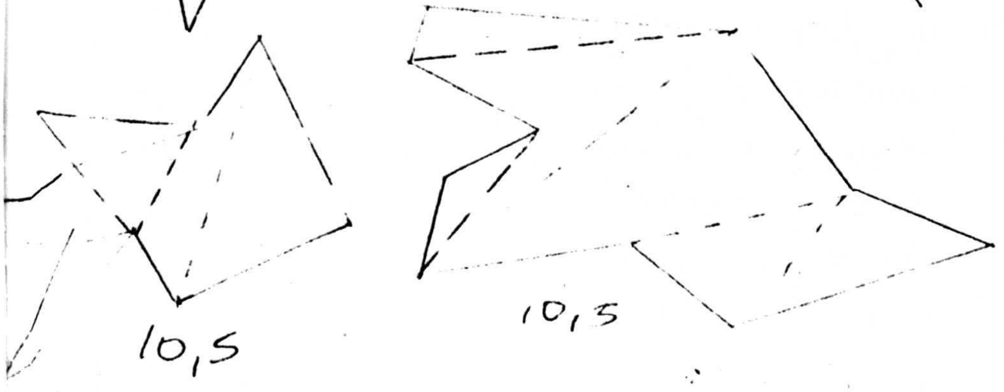
Some diagrams have been drawn here. It seems that "The greatest number of non-crossing diagonals which can be drawn in a polygon is three less than the number of sides."

Is this statement true for all polygons?

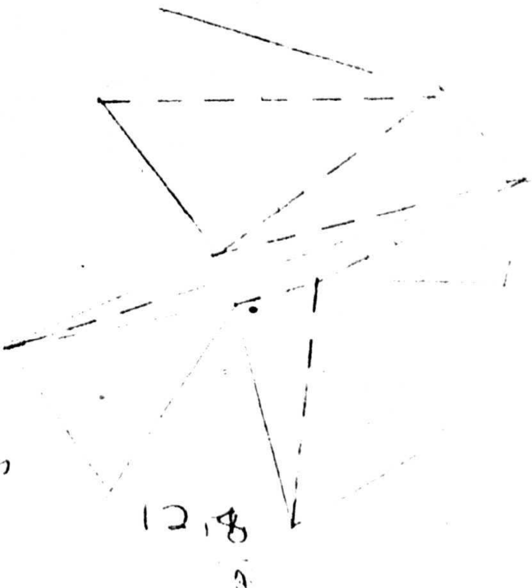
Investigate this fully; then state your conclusions and your reasons.

This statement is true for all polygons under 10 sides, drawing non crossing diagonals is really splitting the polygon into rangles. The polygon can be split into 2 less triangles han sides, one diagonal is needed to complete a hese triangles This only works up to ten sides



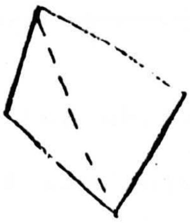


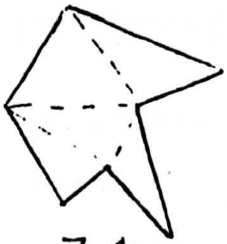
After ten sides the number of diagonals becomes irregular to the number of sides

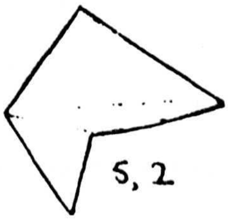


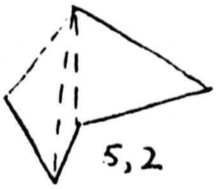
DIAGONALS OF POLYGONS

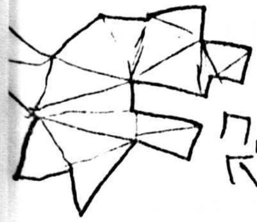
B.









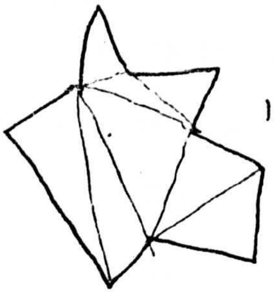


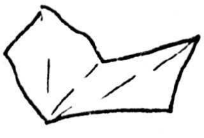
Some diagrams have been drawn here. It seems that "The greatest number of non-crossing diagonals which can be drawn in a polygon is three less than the number of sides."

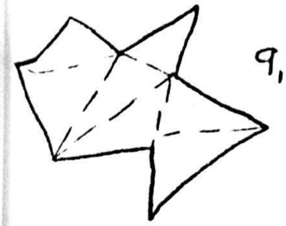
Is this statement true for all polygons? YES

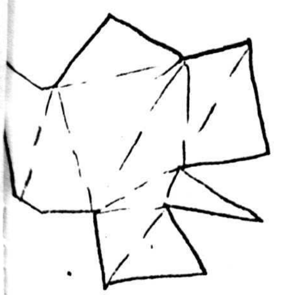
Investigate this fully; then state your conclusions and your reasons.

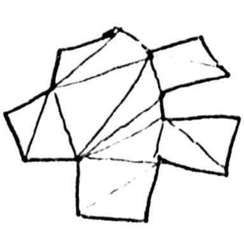
It is ~~possible~~ impossible to get more ~~than~~ non-crossing diagonals drawn in a polygon than 3 less than the no. of sides. It is proved so in the diagrams and works for all of them and it is impossible to try every single polygon ~~so~~ and as it works for these I shouldn't see why it wouldn't work for higher no's of sides in polygons. I can't see a way of proving this statement is true except by drawing and working out all the possibilties but as it does work for these no's it should for others

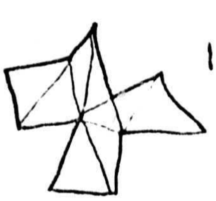


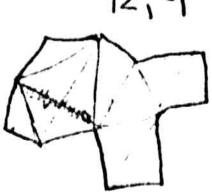












MIDPOINTS

A and B can be any two whole number points on the number line.

M is the point half way between them.

1. If A is at 2 and B is at 8, at what number is M?

2. Add A's number to B's number and halve the result. Do you get M's number?

3. Will the rule in No. 2 work for every possible position of A and B on the line, including bigger numbers.

4. Explain why your answer is true.

① M is 5 because it is the midpoint between 2 and 8.

② Yes. 2+8=10 Half of 10 is 5.

③ Yes the rule does work for every possible position.

④. If A is at 4 and B is at 8. The number m is at is 6. 4+8 = 12. Half of 12 is 6.

If A is at 0 and B at 9. The number m is at is 4½ 0+9=9. Half of 9 is 4½.

If A is at 3 and B at 10. The number m is at is 6½ 3+10=13. Half of 13 is 6½.

If A is at 1 and B is at 9. The number m is at is 5. 1+9 is 10. Half of 10 is 5.

If A is at 10 and B is at 12. The number m is at is 11. 10+12=22 Half of 22 = 11.

MIDPOINTS

A and B can be any two whole number points on the number line.

M is the point half way between them.

1. If A is at 2 and B is at 8, at what number is M? 5.

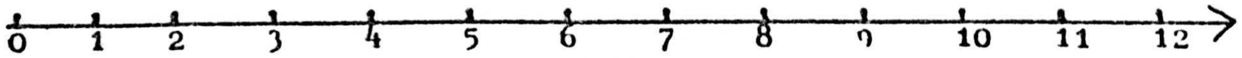
2. Add A's number to B's number and halve the result. Do you get M's number? Yes,

3. Will the rule in No. 2 work for every possible position of A and B on the line, including bigger numbers. Yes.

4. Explain why your answer is true.
Because the Fust Number is even and the last number is even, and So it can be halved.

8

MIDPOINTS



A and B can be any two whole number points on the number line.

M is the point half way between them.

1. If A is at 2 and B is at 8, at what number is M? 5

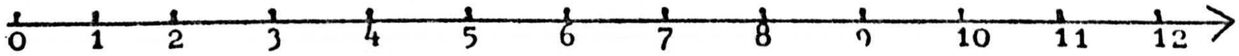
2. Add A's number to B's number and halve the result. Do you get M's number? yes

3. Will the rule in No. 2 work for every possible position of A and B on the line, including bigger numbers. 2 and 10 half of that equals 6

4. Explain why your answer is true.

My answer is true because it is easy to find any half of a whole number ~~with~~ if A and B, can be two whole number Points.

MIDPOINTS



A and B can be any two whole number points on the number line.

M is the point half way between them.

1. If A is at 2 and B is at 8, at what number is M? 5

2. Add A's number to B's number and halve the result. Do you get M's number? yes.

3. Will the rule in No. 2 work for every possible position of A and B on the line, including bigger numbers. yes

4. Explain why your answer is true.

Because the number before M, (A) is Alway the same distance away from M as B is e.g.

35 36 37

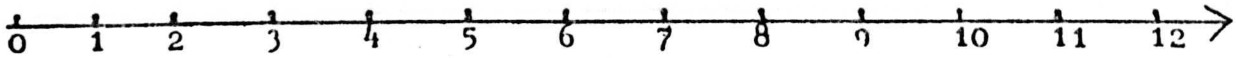
$A \xleftrightarrow[1]{} M \xleftrightarrow[1]{} B$

eg. 2. $14 \xleftrightarrow[3]{} 17 \xleftrightarrow[3]{} 20$

∴ the two numbers when added together must e twice the amount of M.

Also see eg. 2. ~~[illegible] xx/x~~ 20 - 3 = 17 add his 3 to 14 and you are given 17 ∴ if you add hese it equals 34 which when halved equals 17. This works for all casses

MIDPOINTS



A and B can be any two whole number points on the number line.

M is the point half way between them.

1. If A is at 2 and B is at 8, at what number is M?

2. Add A's number to B's number and halve the result. Do you get M's number?

3. Will the rule in No. 2 work for every possible position of A and B on the line, including bigger numbers.

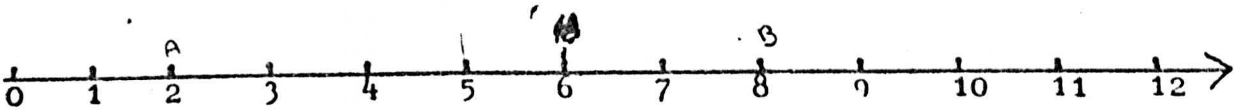
4. Explain why your answer is true.

1. If A is at 2 and B is at 8, M = 5.
2 If A is 2 and B is 8 = 10
halve 10 = 5
M = 5 on the number line
This works if A is 3 and B = 7 = 10
½ of 10 = 5
M = 5. on the number line
If A = 3 and B = 9 = 12
½ of 12 = 6
M = 6 on the number line.
3. If A = 12 and B = 20. = 32. ½ of 32 = 16
M will = 16
This rule will work for bigger numbers
eg A = 12 + B = 15, = 27
halve 27 = 13½
The half way point is 13½.

A = 30 and B = 40
30 + 40 = 70
½ of 70 = 35
M = 35 on the number line.

4. If 2 numbers are added together eg 2 and 6 and then halved giving (4) The middle point will be the same because you are finding the distance between those two numbers.

MIDPOINTS



A and B can be any two whole number points on the number line.

M is the point half way between them.

1. If A is at 2 and B is at 8, at what number is M?

2. Add A's number to B's number and halve the result. Do you get M's number?

3. Will the rule in No. 2 work for every possible position of A and B on the line, including bigger numbers.

4. Explain why your answer is true.

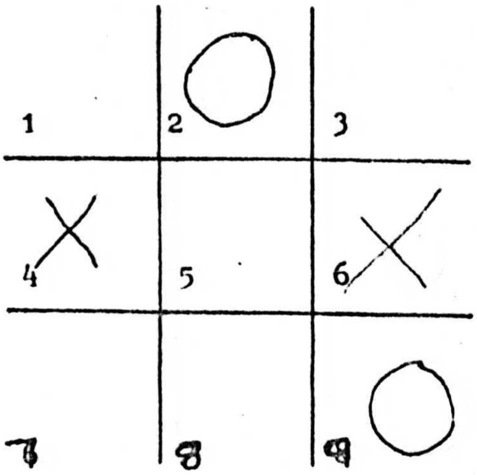
1. m = 5

2. 8 + 2 = 10   10 ÷ 2 = 5   Yes

3. 3 + 5 = 8   8 ÷ 2 = 4   Yes the rule will be the same.

4. This is true because 15 + 17 = 32   32 ÷ 2 = 16

NOUGHTS AND CROSSES



You are X and it is your turn.

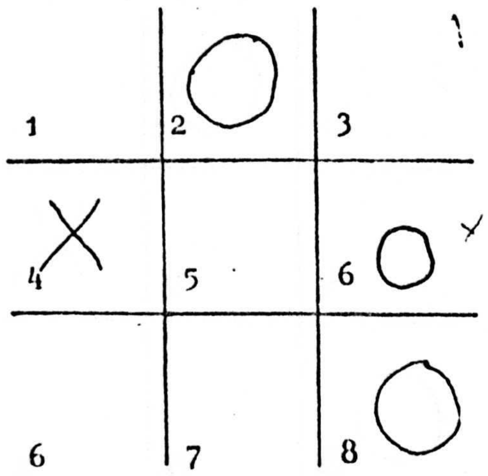
You are thinking of going in square 6.

1. Is this a good move?

2. Explain fully why you think so.

Yes. It is a fairly good move because if the person who is O decides to move into square 3. She could have got two chances of winning with one more go. Because she could have got a row of 1.2.3 or a row 3.6.9, but with the X in 6 she can be prevented to have ~~[illegible]~~ row 3.6.9 ~~and~~ ∴ can be stopped to getting row 1.2.3 by X moving to square 1. It is also, a good move because I can get a row of squares 4.5.6

NOUGHTS AND CROSSES



You are X and it is your turn.

You are thinking of going in square 6.

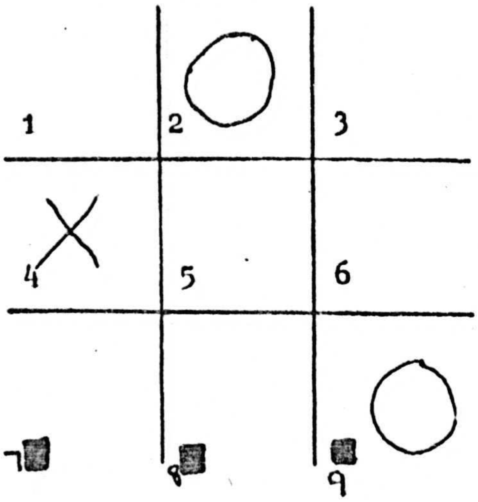
1. Is this a good move?

2. Explain fully why you think so.

1) I think that it is a good move.

2) I think this is a good move because you can put X in the 5 and if you did nat put O in 6 he would have got it.

NOUGHTS AND CROSSES



You are X and it is your turn.

You are thinking of going in square 6.

1. Is this a good move?

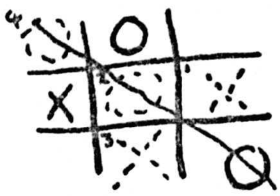
2. Explain fully why you think so.

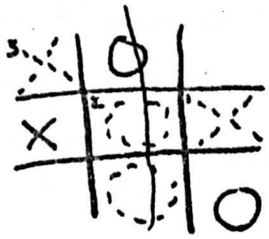
NO

BECAUSE IF I PUT A CROSS in no 6 then "O" will easily SEE hat if she does not put a 0 in square 5 I will win

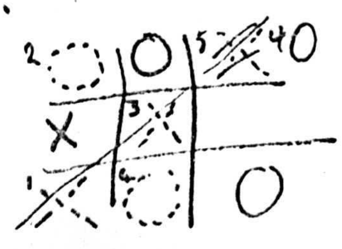
∴ she has two chances of winning; if I go in square 7 she will go in square 1 and vice verser

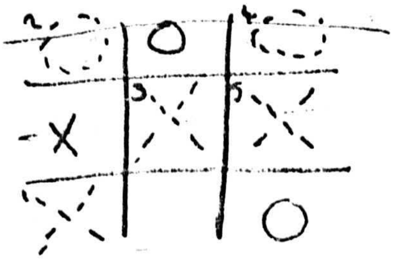
∴ O will win the game if I go in square 6



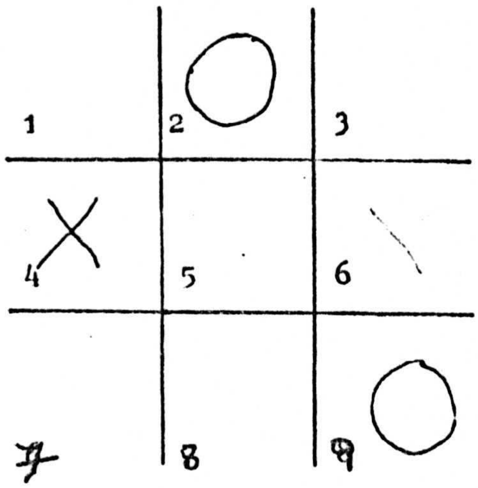


IF I HAD THE CHOICE of a MOVE I would choose square 7.





NOUGHTS AND CROSSES



You are X and it is your turn.

You are thinking of going in square 6.

1. Is this a good move?

2. Explain fully why you think so.

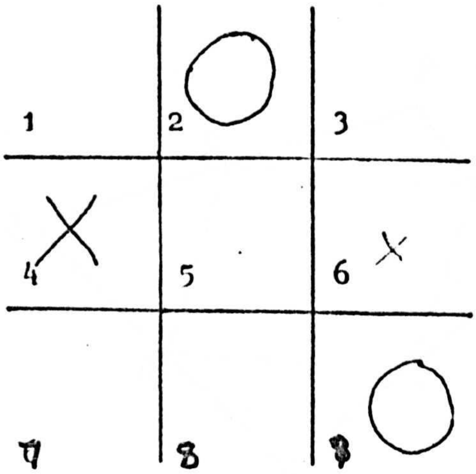
1). I do not think that it's the best possible move.

2). The reasons why, are that it would be better to move to 7 because, it would mean that you could perhaps make the row 1,4,7 or the diagonal 3.5.7. where, as if you move to 6 you have only got the one possible row which is row 4.5.6.

If you moved to space 6 it would prevent the opposition from making the line, 3.6.9 ~~also~~, but it would not prevent any other rows or lines.

If you moved to square number 7 it would prevent the opposition from getting the diagonal 3.5.7, also the row, 7.8,9, of which it occupies square 9. Also of course they will not be able to have the row 1.4.7. because both squares 4 & 7 would be occupied

NOUGHTS AND CROSSES



You are X and it is your turn.

You are thinking of going in square 6.

1. Is this a good move?

2. Explain fully why you think so.

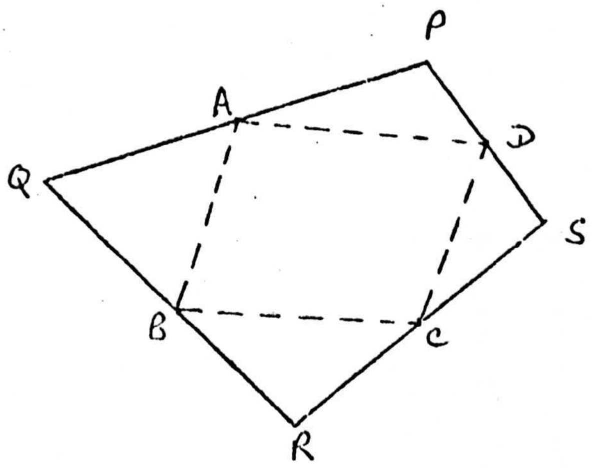
1. Yes.

2. If it was my turn and I put my x in square 6 that would stop my partner from getting a point ~~because~~ in one way but yet in another, I could lose out on a point.
This question depends on how both you and your partner play. If I put my x in square 6 she would put hers in 5

3

see back

QUADS



A, B, C, D, are the midpoints of the sides of the quad PQRS.

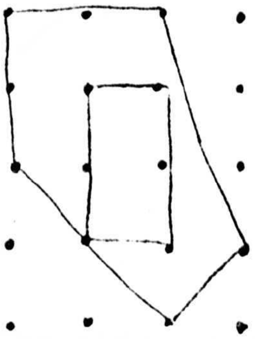
In some quads the midpoint figure ABCD is a rectangle.

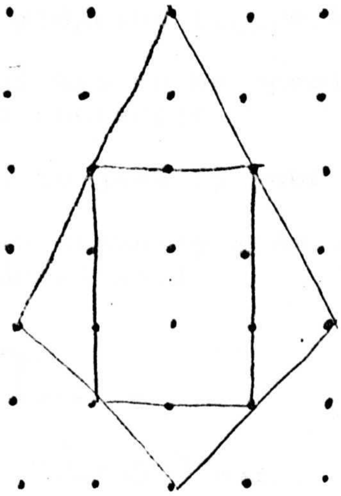
1. Find out what has to be special about the quad PQRS for ABCD to be a rectangle.

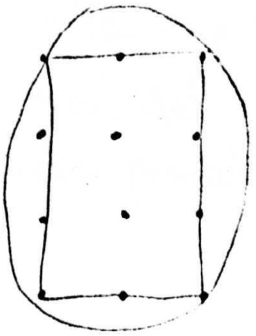
2. Give reasons to justify your answers.

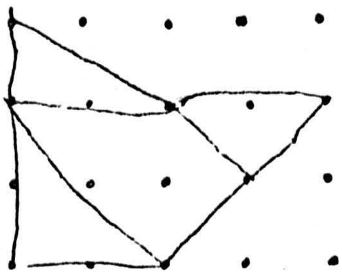
Use the plain orspotty side of the paper provided for your trial drawings.)

The Quad PQRS has to have 4 sides. You can't ave a midpoint figure of a rectangle if there are ore than 4 sides.

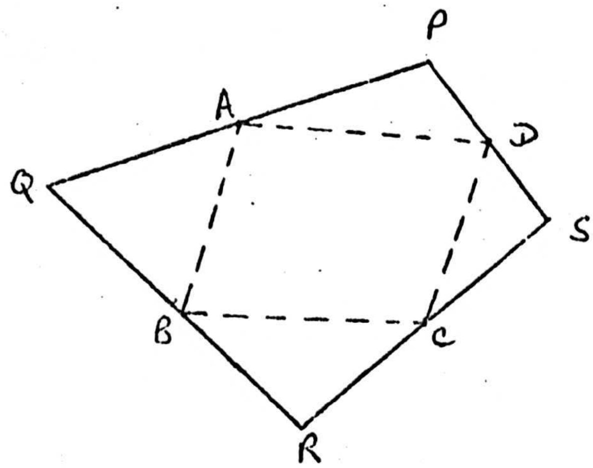








QUADS



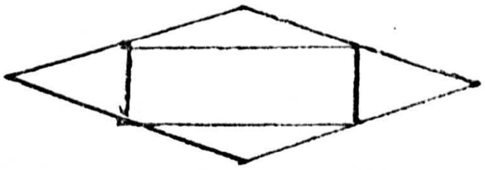
A, B, C, D, are the midpoints of the sides of the quad PQRS.

In some quads the midpoint figure ABCD is a rectangle.

1. Find out what has to be special about the quad PQRS for ABCD to be a rectangle.

2. Give reasons to justify your answers.

Use the plain orspotty side of the paper provided for your trial drawings.).

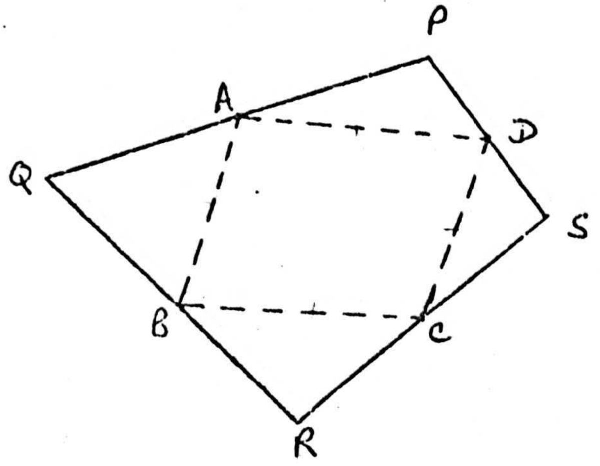


The Same is also True of a parallelogram which has The Top and bottom parallel and The Side parallel

For ABCD To be a rectangle The quad PQRS must be a diamond Shaped quad which is either longer across The width Than The length or Visa-Versa. The lengths of Side must be equal.

P.T.O.

QUADS



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1. Find out what has to be special about the quad PQRS for ABCD to be a rectangle.

2. Give reasons to justify your answers.

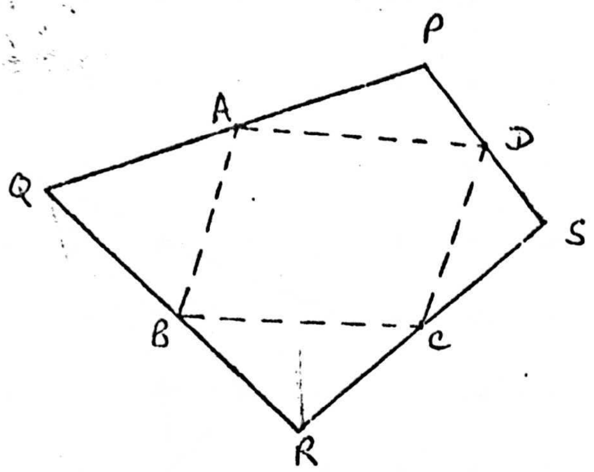
   Use the plain orspotty side of the paper provided for your trial drawings.)

1. [illegible] ~~then be a rectangle PQRS would have to have the opposite sides parallel to each other~~

The quad PQRS would have to be a parrallelagram for ABCD to be a rectangle

Because then the lines AB + DC would not be sloping.

QUADS



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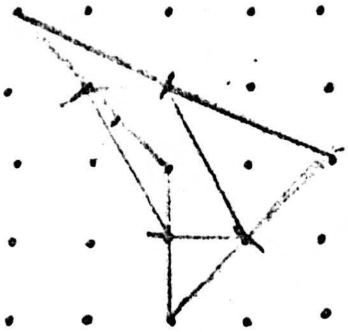
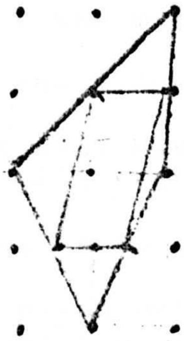
1. Find out what has to be special about the quad PQRS for ABCD to be a rectangle.

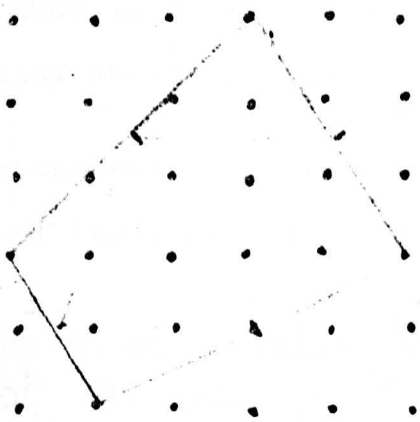
2. Give reasons to justify your answers.

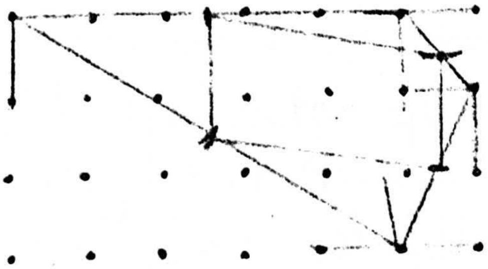
Use the plain orspotty side of the paper provided for your trial drawings.)

For ABCD to be a rectangle the quad has to have at least 2 angles of 45° and the half way marks have to be opposite and the sides have to be parallel

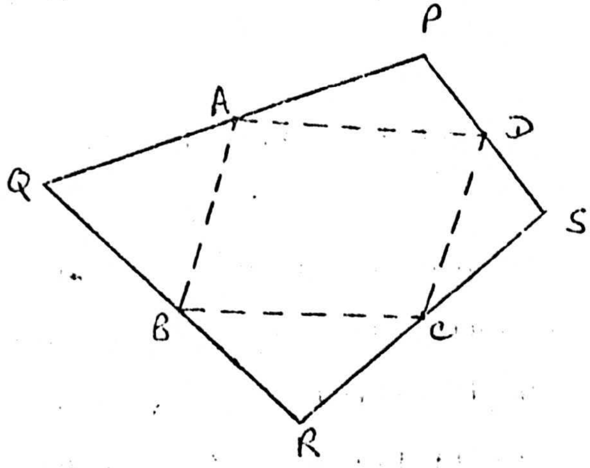








QUADS



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In some quads the midpoint figure ABCD is a rectangle.

1. Find out what has to be special about the quad PQRS for ABCD to be a rectangle.

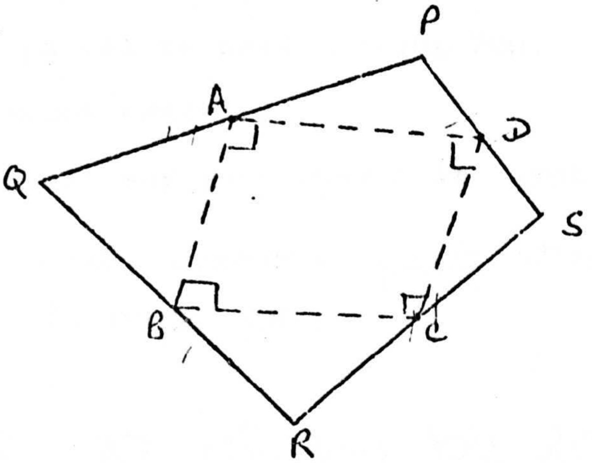
2. Give reasons to justify your answers.

Use the plain orspotty side of the paper provided for your trial drawings.)

① If the inner shape is to be a rectangle the outer quad must be one with 2 of the sides one length and the other two another length and the 2 same lengths must touch each other.

② This has to be because if the lengths of the outer quad are all different the lengths of the sides of the inner quad are also all different. If the two sides of the same length are not touching this makes the inner quads sides that are equal together and form a parallelagram

QUADS



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In some quads the midpoint figure ABCD is a rectangle.

1. Find out what has to be special about the quad PQRS for ABCD to be a rectangle.

2. Give reasons to justify your answers.

Use the plain orspotty side of the paper provided for your trial drawings.)

The sides of the quadrilateral PQRS all have to be equal in length and the angles between the sides of the quadrilateral have to be 90° if A.BCD is to be a rectangle. Thus, PQRS must be a square. This is because a rectangle must have corners of 90° and if the ~~PQ~~ midpoint(A) on PQ is to join the midpoint (B) of the line QR the lines adjoing AB must be be ~~of equal distance to that of PQ, QR~~ at 90°, therefore the only possibility of this is for R.S to = QR, QP

STAMPS

1. Anne has plenty of 8p and 20p stamps, but no others. She has a parcel to post costing 70p. Can she put on the correct amount exactly?

2. Explain why your answer is right.

No she cannot put the correct amount of stamps on.

~~20~~ 70 divided by 20 is 3 with 10 left over if you divide the remainder by 8 the answer would be 1 remainder 2.

So the girl could put 3 x 20p + 1 x 8p stamps on and by some more, or put and extra 8p stamp on and pay 6pence extra.

70 divided by 8 = 8 remainder 6. The girl could put 9 x 8p stamps on the parcel which is 72p. Then she would only be paying 2p extra.

STAMPS

1. Anne has plenty of 8p and 20p stamps, but no others. She has a parcel to post costing 70p. Can she put on the correct amount exactly?

2. Explain why your answer is right.

1. No she cannot put the correct amount on.

2. If 70p is to be made from 8p and 20p stamps 70p must have factors of 8 or 20 or a combination of the two to make 70. But which ever way you try you cannot make 70 from 8's and 20's.

9 × 8 = 72 Too ~~little~~ much
8 × 8 = 64 Too little
3 × 20 = 60 Too little
4 × 20 = 80 Too much
(1× 20)+ (6×8) = 68 too little
(1×20)+(7×8) = 76 too much
(2×20) +(4×8) = 72 too much
(2×20) +(3×8) = 64 too little
(3×20) +(1×8) = 68 too little
(3×20) +(2×8) = 76 too much

These are all the possibilities any where near 70.

22

STAMPS

1. Anne has plenty of 8p and 20p stamps, but no others.

She has a parcel to post costing 70p. Can she put on the correct amount exactly? ~~You~~ She cannot come to the exact 70p for her parcel to be posted.

2. Explain why your answer is right.

My answer is right because. both these numbers are even. these two numbers added together make an even number and the amount which is aimed for is an odd number 70p, add ~~three~~ twenties and 2 ~~8~~ eights you have too much 76p add 3 ~~twenties~~ twenties 1 eight. it is 68p too little. and the other way round has the same effect. none of these numbers By themselves add up to 70p so when they are combined it is hopeless to say that they would come to that amount. 70p.

STAMPS

1. Anne has plenty of 8p and 20p stamps, but no others. She has a parcel to post costing 70p. Can she put on the correct amount exactly?

2. Explain why your answer is right.

1. Anne cannot put on the correct amount exactly.

2. This is because multiples of 20p always make an even amount, eg. 20p. 40p. 60p. 80p. etc. 70p is odd. The next thing to try is multiples of 8p which give an even number, added to a multiple of 20p. (The even number has to be a multiple of ten)
eg. The first multiple of 8p which is also a multiple of ten is 8×5 = 40. and 40p + 20p = 60p but we need 70p and so it will not work.

25

STAMPS

1. Anne has plenty of 8p and 20p stamps, but no others. She has a parcel to post costing 70p. Can she put on the correct amount exactly? **NO**

2. Explain why your answer is right.

There is no combination of 8p and 20p stamps that will give an exact total of 70p.

②

STAMPS

1. Anne has plenty of 8p and 20p stamps, but no others. She has a parcel to post costing 70p. Can she put on the correct amount exactly? No

2. Explain why your answer is right.

8 and 20 are even numbers and 70 cannot be divided by an even number as the seven is an odd number So any multiples of 8 or 20 or both will not go into 70.

7

ONE AND THE NEXT

Write down any number up to fifteen. Write down the next number and add it to the first. Write down your answer. You have now written down three numbers.

Gail says that one, and only one, of these numbers is in this list:

3, 6, 9, 12, 15, 18, 21, 24, 27, 30

1. Is she right?

2. Will she always be right?

3. Explain why.

4 + 5 = 9

① Yes she is right 9 appears in the list

② ~~No not always~~ Yes

③ ~~Because the numbers Gail has written down are all multiples of 3 so for eg. 5+6=11 and 11 does not appear in the list. It is chance that~~

~~Because some~~ numbers for eg 14

Because ~~whatever number is chosen from 1 up to 15~~ as there are 3 numbers to be written down and all the numbers ~~are~~ in the list are multiples of 3 one number out of three will be a multiple of 3.

12

ONE AND THE NEXT

Write down any number up to fifteen. Write down the next number and add it to the first. Write down your answer. You have now written down three numbers.

Gail says that one, and only one, of these numbers is in this list:

3, 6, 9, 12, 15, 18, 21, 24, 27, 30

1. Is she right?
2. Will she always be right?
3. Explain why.

6 + 7 = 13

1. Yes, Gail is right

2. Yes.

3. Because if you have two numbers that 3 does not divide into, and they follow each other, eg. 7 and 8. You will add them up and it will add up to a number that three will divide into. In this case, the number is 15 and as the list of numbers are numbers that can be divided by three, 15 is in this list of numbers.

If you already start with a number that will be divided by 3 such as 3 + 4, it will always add up to a number that will not be divided by 3. In this case the number is 7 and is therefore not on the list. So you can only end up with one number that

23

ONE AND THE NEXT

Write down any number up to fifteen. Write down the next number and add it to the first. Write down your answer. You have now written down three numbers.

Gail says that one, and only one, of these numbers is in this list:

3, 6, 9, 12, 15, 18, 21, 24, 27, 30

1. Is she right?
2. Will she always be right?
3. Explain why.

5+6=11

1) NO

2) NO

3) Because the only numbers which are right in the list are the odd numbers, all the others cannot be made by to numbers next to each other

43

ONE AND THE NEXT

Write down any number up to fifteen. Write down the next number and add it to the first. Write down your answer. You have now written down three numbers.

Gail says that one, and only one, of these numbers is in this list:

3, 6, 9, 12, 15, 18, 21, 24, 27, 30

1. Is she right? yes

2. Will she always be right? yes

3. Explain why. She will all way be right because ~~one number out of~~ three will allways go in to one of the number's

5 6 11

7 8 15

13 14 27

3 4 7

14 15 29

10 11 21

11 12 23

9 10 19

~~4, 8, 12, 16, [illegible] 24, 28, 32, 36, 40~~

~~7 8 9 24~~

~~7 8 15~~

~~6 7 15~~

46

ONE AND THE NEXT

Write down any number up to fifteen. Write down the next number and add it to the first. Write down your answer. You have now written down three numbers.

Gail says that one, and only one, of these numbers is in this list:

3, 6, 9, 12, 15, 18, 21, 24, 27, 30

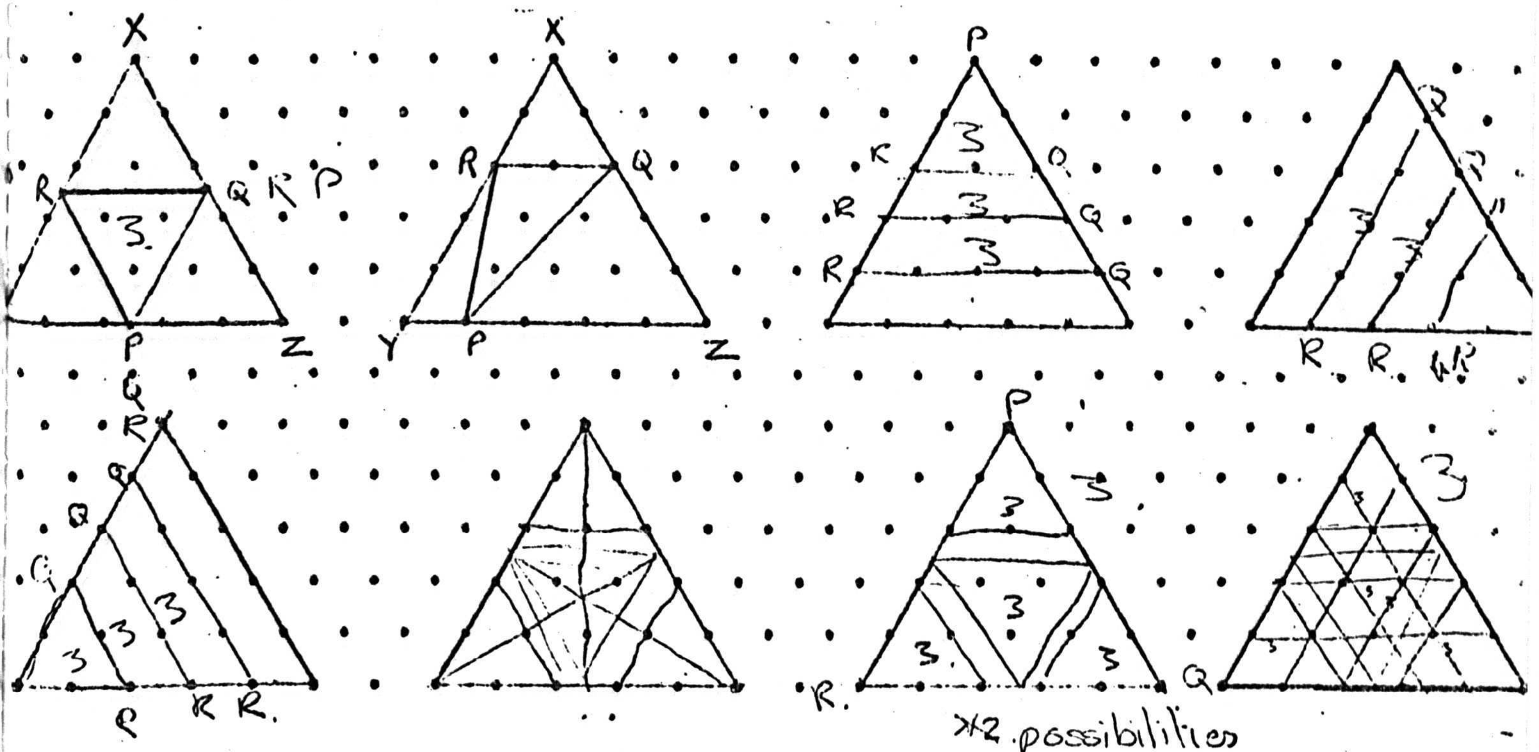
1. Is she right? YES

2. Will she always be right? ~~YES~~. NO.

3. Explain why.
She will not always be right because for example. 2 + 3 = 5 she does not have this in her answers.

2

TRIANGLES



~~×2~~ possibilities
~~×2~~ "
11×3 ".
33 possibilities.

The points Q, P, and R can be anywhere on the sides of the big triangle.

In the first triangle above PQR is equilateral.

1. Can PQR be equilateral with P, Q and R in any other positions? If so, what are all the other possibilities?

2. Explain why your answer is right.

If the triangle found within its unnumbered points can be used a further two times there seem to be. 33 possibilities

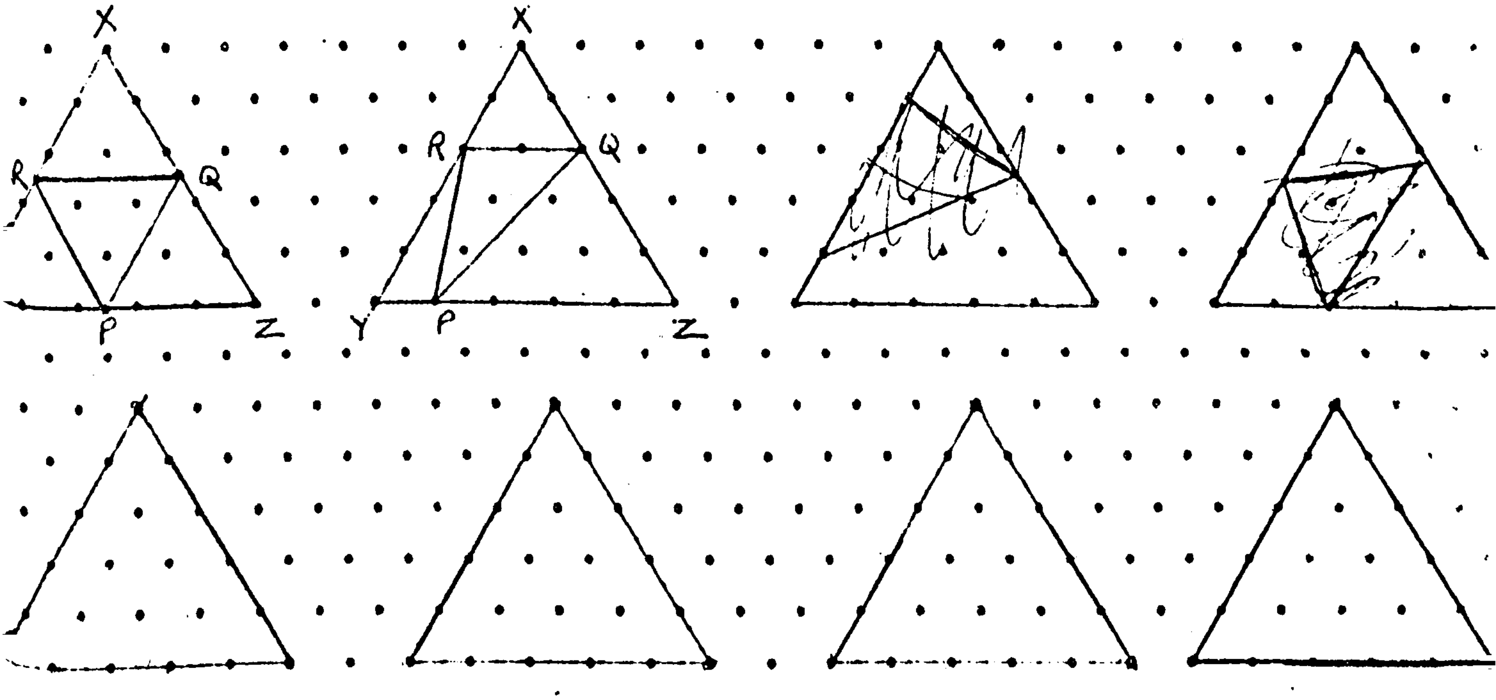
3
3
3
3.
3
3
3
3
3
3.
3.

[illegible]

5 4 4 1

4

TRIANGLES



The points Q, P, and R can be anywhere on the sides of the big triangle.

In the first triangle above PQR is equilateral.

1. Can PQR be equilateral with P, Q and R in any other positions? If so, what are all the other possibilities?

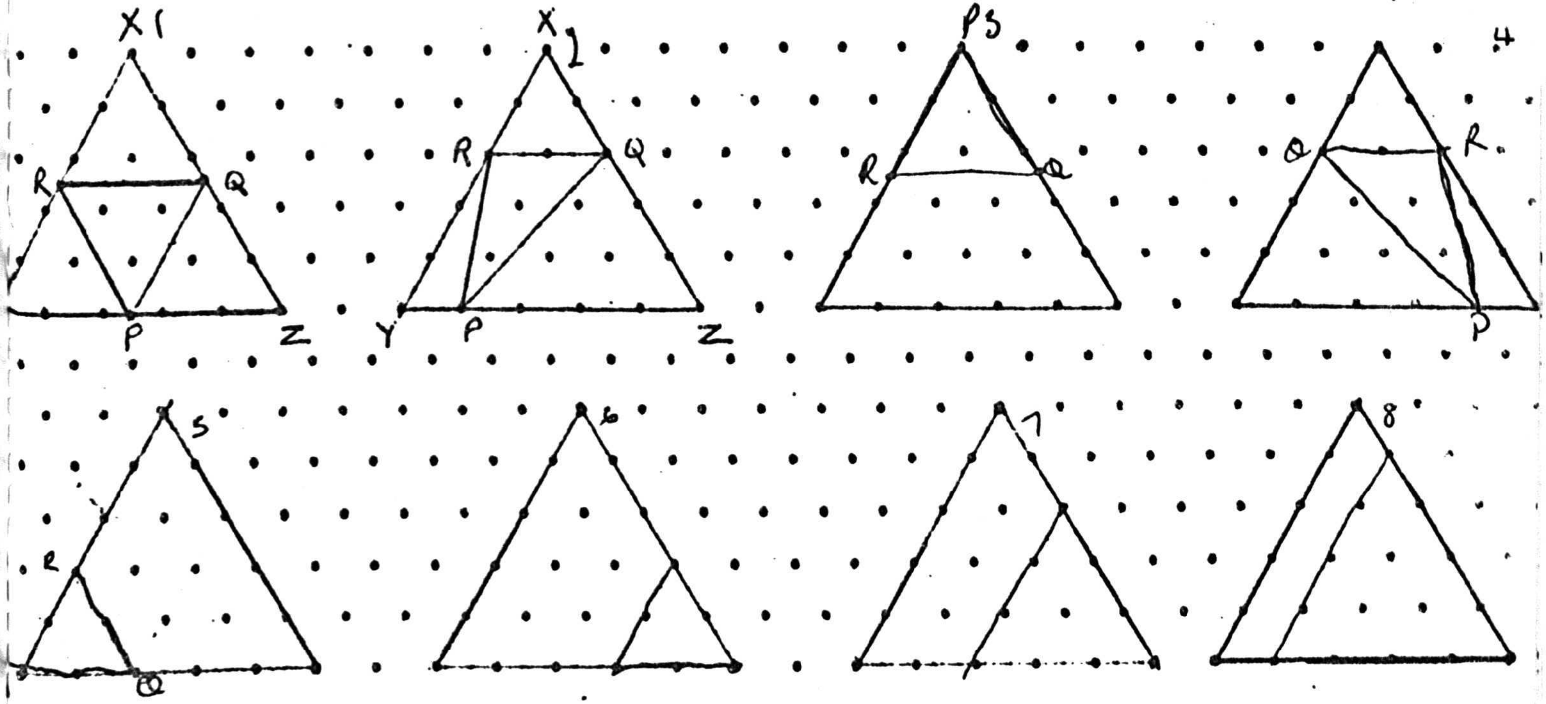
2. Explain why your answer is right.

1. No

2. Because for an equilateral triangle all the sides must be equal and the angles must be equal. There is only one position where ΔPQR can be equilateral. To do it in other positions the corners have to be used. If they are no the triangle becomes a right angled triangle or an isosceles. ΔPQR is half the length (its sides) of ΔXYZ. To move position and make the sides equal the points would have to be inside the triangle or in the corners.

TRIANGLES

9



The points Q, P, and R can be anywhere on the sides of the big triangle.

In the first triangle above PQR is equilateral.

1. Can PQR be equilateral with P, Q and R in any other positions? If so, what are all the other possibilities?

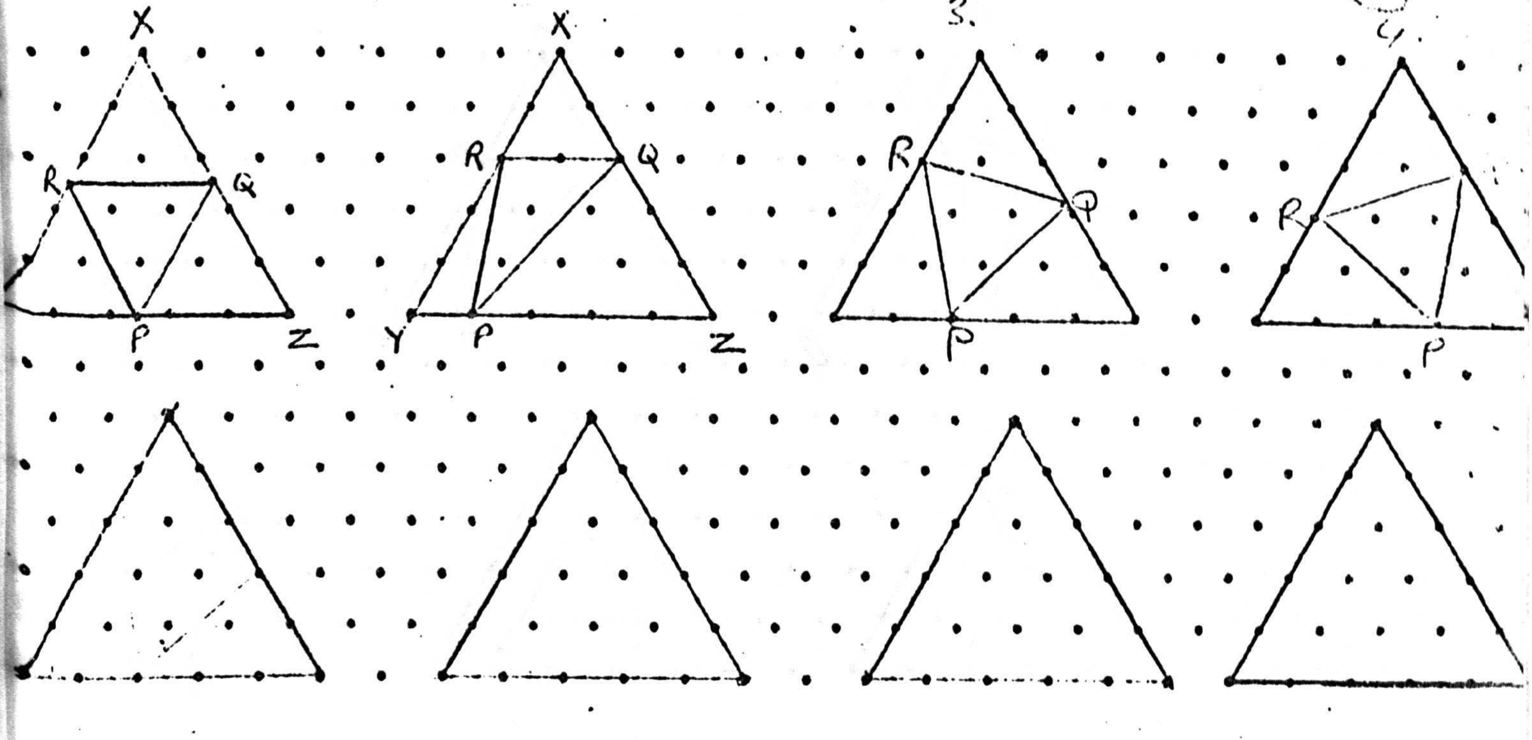
2. Explain why your answer is right.

PQR can be equilateral in another posistion if the triangle is turned upside down so that p is now x and if R Q are in the same posistion. In triangles 2 and four the inner triangles are equilateral to each other but are not equilateral triangles. In triangle 5 there is an equilateral triangle In triangle 6 there is an equilateral triangle In triangles 6 and 7 there are equlateral triangle

There will always be an equlateral triangle of PQR is the space are equal to the spaces up the side of the big triangle. Altogethere there are 14 triangles that can be made by this method (not including larger triangle)

13

TRIANGLES



The points Q, P, and R can be anywhere on the sides of the big triangle.

In the first triangle above PQR is equilateral.

1. Can PQR be equilateral with P, Q and R in any other positions? If so, what are all the other possibilities?

2. Explain why your answer is right.

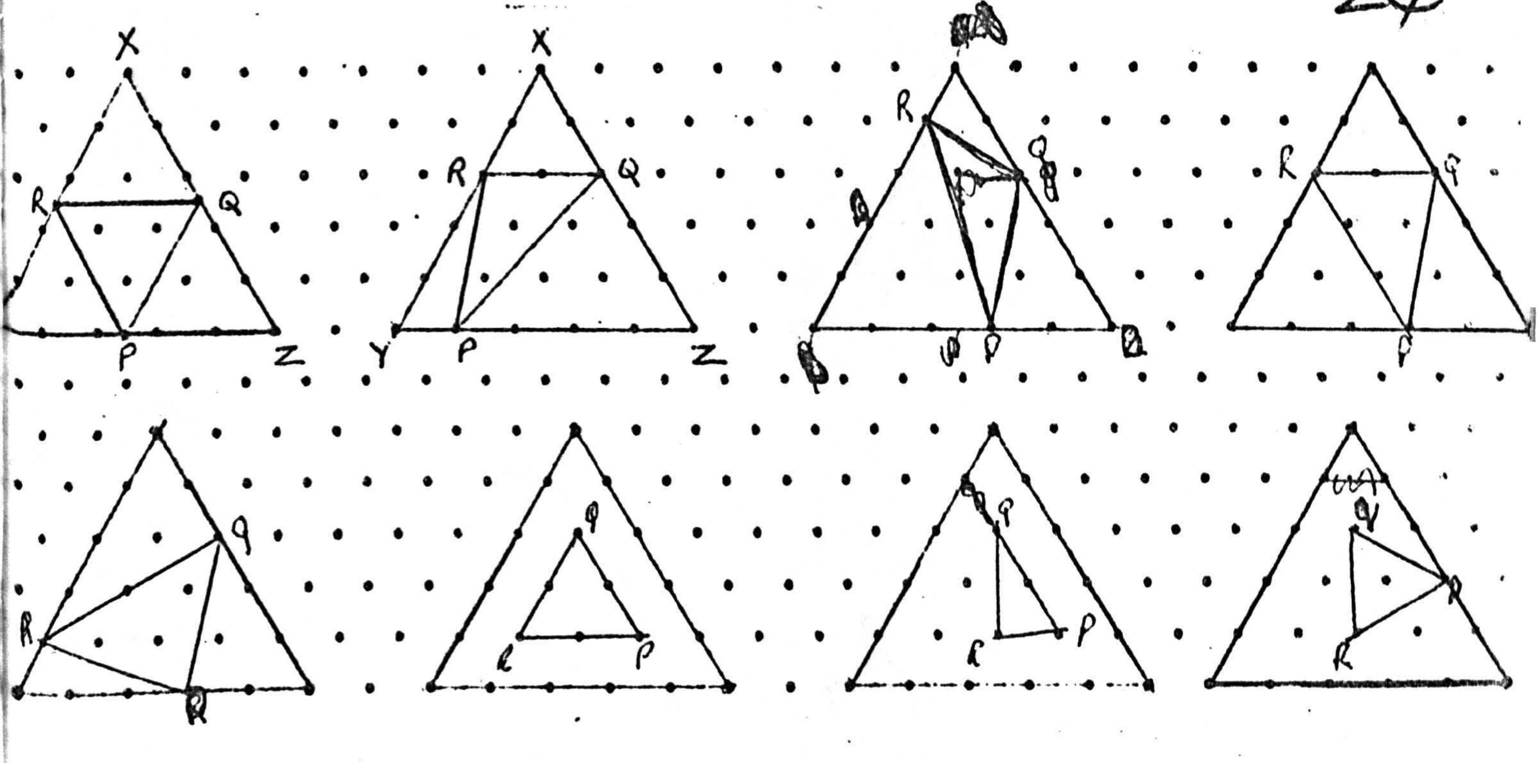
1 Yes, the others are 3 and 4.

2. There are only 3 combinations because the triangle has 3 equal sides.

Correct ~~register~~ but too-limited variations. Close, too soon.

TRIANGLES

24



The points Q, P, and R can be anywhere on the sides of the big triangle.

In the first triangle above PQR is equilateral.

1. Can PQR be equilateral with P, Q and R in any other positions? If so, what are all the other possibilities?

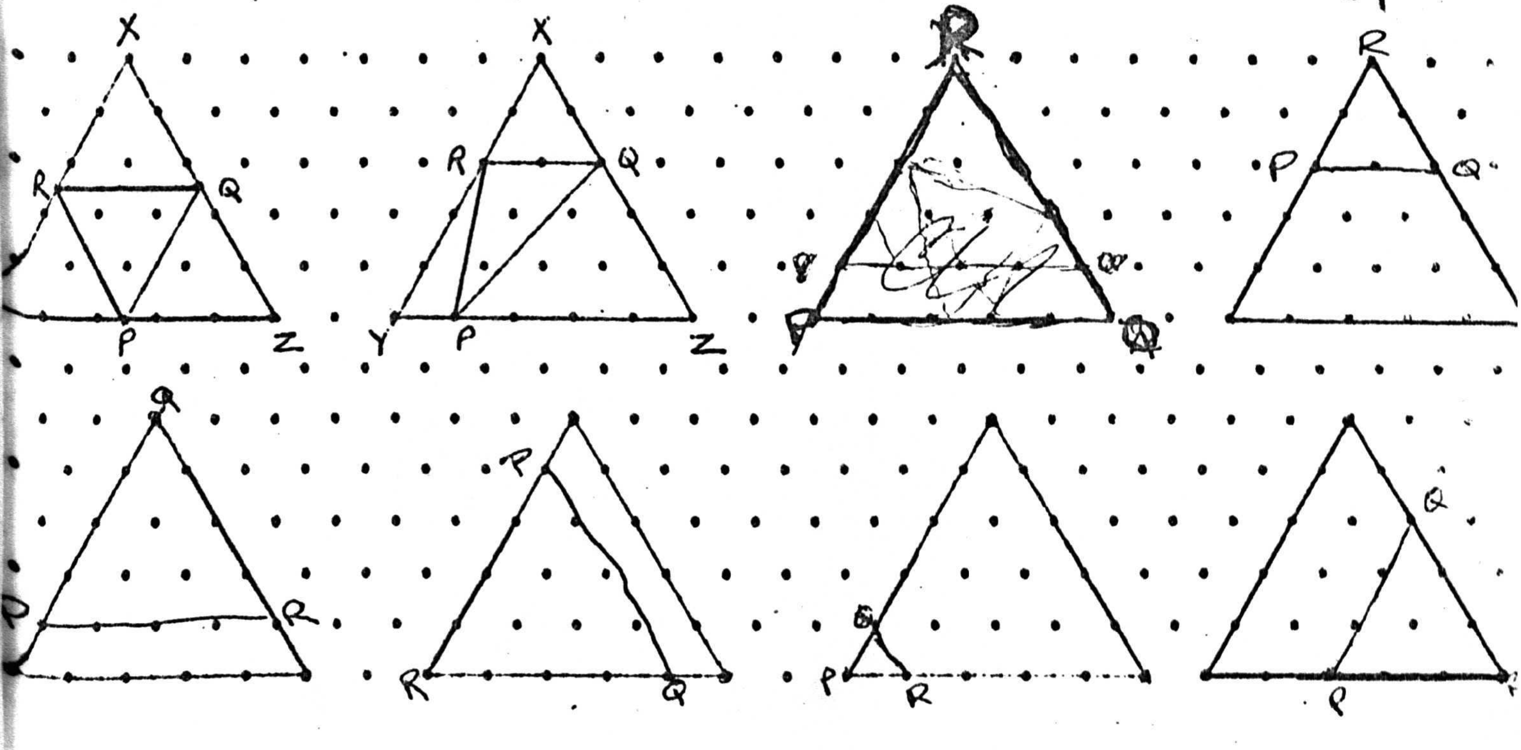
2. Explain why your answer is right.

1. yes PQR can be equilaterical because it could be in any Position as it likes.

2. Because all my sides are equal with the Big triangle.

TRIANGLES

T 31



The points Q, P, and R can be anywhere on the sides of the big triangle.

In the first triangle above PQR is equilateral.

1. Can PQR be equilateral with P, Q and R in any other positions? If so, what are all the other possibilities?

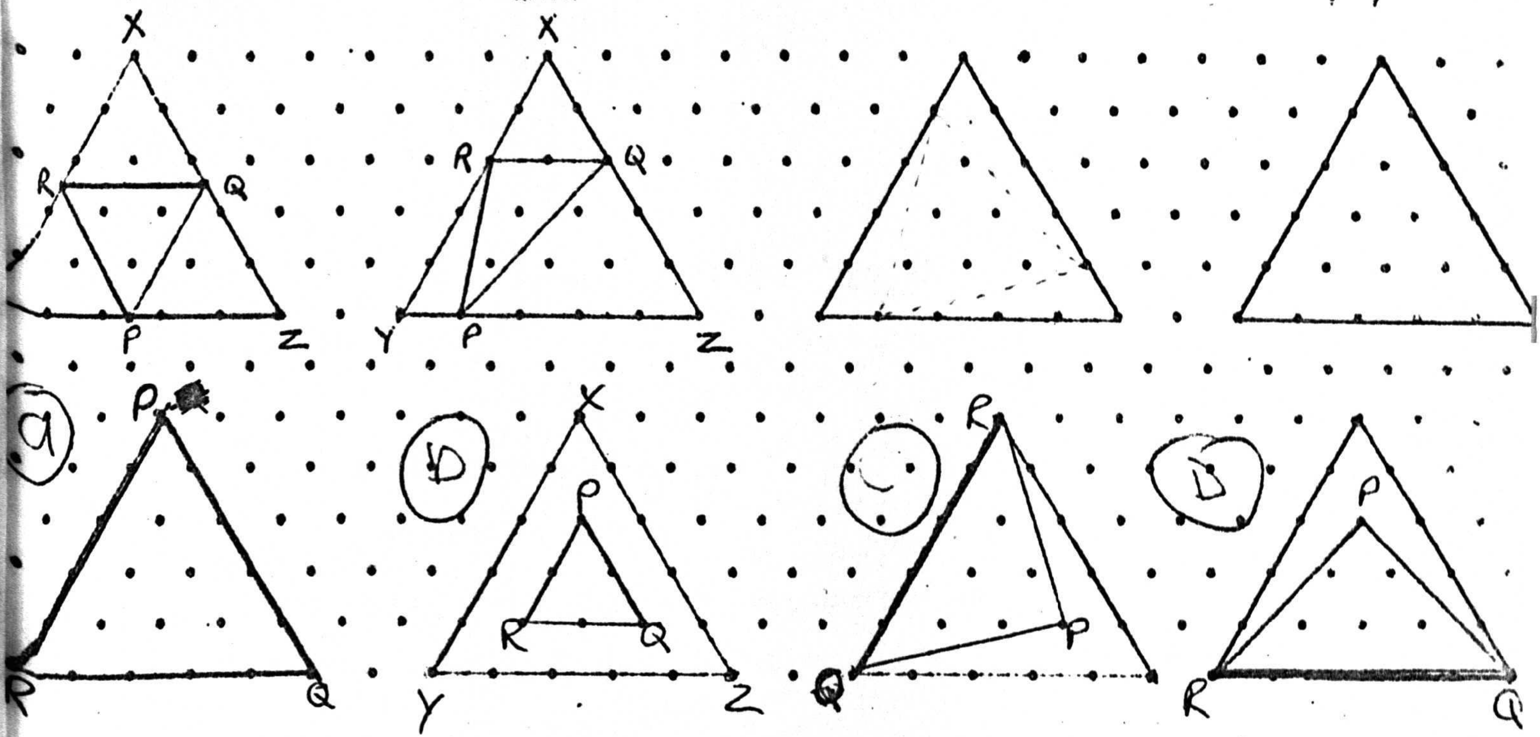
2. Explain why your answer is right.

The triangles above are right because all the sides are the same length. Some are bigger then others but the sides on the triangles are the same length they are not the same length as all the other triangles.

e.g. the Last one the sides are the same length from P. to Q from Q to R and from R to P.

TRIANGLES

47



The points Q, P, and R can be anywhere on the sides of the big triangle.

In the first triangle above PQR is equilateral.

1. Can PQR be equilateral with P, Q and R in any other positions? If so, what are all the other possibilities?

2. Explain why your answer is right.

Well if you put your anower in the triangle you have two triangles but there is always more than one triangle in side itself. one ~~triangle~~ triangle is only its full size. These are all equilateral triangles except ⓓ because the bottom ~~it~~ is longer than its sides.

APPENDIX TO CHAPTER 10

GENERALISING AND PROVING

This test is about making generalisations and proving them. Here is an example:

"Squares are bigger?

Choose some whole numbers, less than ten, and square them. Do they get bigger? Does this always happen? Prove your answer."

Investigate this for a few minutes, then study the following answers given by different pupils, and give your comments on them.

Philip: $3 \times 3 = 9$ $8 \times 8 = 64$ Yes, the numbers always get bigger.

Adrian: They always get bigger because you multiply them.

Patricia: n^2 is $n \times n$ which is greater than n.

Wendy: A number x the number = the number + the number + the number, and so on, a certain number of times. This is bound to be more than the number itself.

Have the pupils proved their answers?

Correct answers to this question would be:

Philip: No, he has only tried two cases.

Adrian: Has not proved it unless he also proves that multiplying makes numbers bigger.

Patricia:has written it in letters but has not proved anything.

Wendy: would have proved it but she has forgotten 1; this is a whole number for which it is not true.

The questions in the test are similar to this.

ADD AND TAKE

Problem: " Choose any number between 1 and 10 inclusive. Add it to 10 and write down the answers.Take the first number away from 10 and write down the answer. Add your two answers.

Try this starting with different numbers. Will the result be the same for all starting numbers?

Prove your answer."

Investigate this for a few minutes and consider the following answers.

Michael and Jenny both started by checking examples.

Michael:	10 + 2 = 12	10 + 8 = 18	10 + 5 = 15
	10 - 2 = 8	10 - 8 = 2	10 - 5 = 5
	20	20	20
Jenny:	10 + 6 = 16	10 + 3 = 13	10 + 10 = 20
	10 - 6 = 4	10 - 3 = 7	10 - 10 = 0
	20	20	20

Which do you think is the better set of examples?

Michael's/Jenny's

Give your reasons: ① X

Susan:

The result will always be 20. If you choose a number between 1 and 10 and add it to 10, then if you take the first number away from 10 it will be whatever is needed to make 20.

Yvonne:

Always 20. Whatever you add you always take it away so it cancels out. But as you add 10 and take the number from 10, you get double 10 which is 20.

Have these pupils proved their answers?

Susan's : Yes/No Yvonne's : Yes/No ② E ④ E

Give your reasons: ③ EE ⑤ EE

<u>Wendy</u>:

Always 20; because $(10 + x) + (10 - x) = 20$

<u>Kevin</u>:

$$\begin{array}{rrrrr}
10 + 1 = 11 & 10 + 2 = 12 & 10 + 3 = 13 & 4 \rightarrow 14 & 5 \rightarrow 15 \\
10 - 1 = \underline{9} & 10 - 2 = \underline{8} & 10 - 3 = \underline{7} & \underline{6} & \underline{5} \\
\underline{20} & \underline{20} & \underline{20} & \underline{20} & \underline{20}
\end{array}$$

$$\begin{array}{rrrrr}
6 \rightarrow 16 & 7 \rightarrow 17 & 8 \rightarrow 18 & 9 \rightarrow 19 & 10 \rightarrow 20 \\
\underline{4} & \underline{3} & \underline{2} & \underline{1} & \underline{0} \\
\underline{20} & \underline{20} & \underline{20} & \underline{20} & \underline{20}
\end{array}$$

Always 20

Are these correct proofs? Wendy's: Yes/No —

Kevin's: Yes/No ⑥ F

Give your reasons:

____

____ ⑦ F

____

<u>Ann</u>:

Always 20;it is the same as saying $10 - 2 + 10 + 2 = 20$.
If you add a positive number to its own negative the result will be nought.

<u>Hazel</u>:

Always 20; this is because if you take 2 from 10 you get 8, and 8 from 10 you get 2. The two numbers always make up ten and you cannot have any other number coming in.

Are these correct proofs? Ann's: Yes/No ⑧ E ⑩ E

Hazel's:Yes/No ⑨ EE ⑪ EE

Give your reasons:

____

____

____

STAMPS

Problem:" Anne has plenty of 8p and 20p stamps, but no others. She has a parcel to post costing 70p. Can she put on the correct amount exactly?

Prove your answer."

Investigate this for a few minutes, then comment on the following three answers.

James:

Three 20's cannot be made up to 70 with 8's. 6 x 8 + one 20 = 68 and with 7 x 8 you get 76. All 8's gives 64 or 72. Two 20's + 4 x 8 = 72. This accounts for all the numbers of 20's which are small enough. So 70 is impossible.

Lesley:

8 and 20 are both multiples of 4, and 70 is not, so you cannot get 70 by combining 8's and 20's.

Richard:

With one 20 you must add 50 which cannot be done with 8s.
With two 20's you need 30 which cannot be done with 8's.
With three 20's you need 10 which cannot be done with 8's.
Four 20's are too much. Thus 70 is impossible.

Have these pupils proved their answers?

James Yes/No ⑫ C

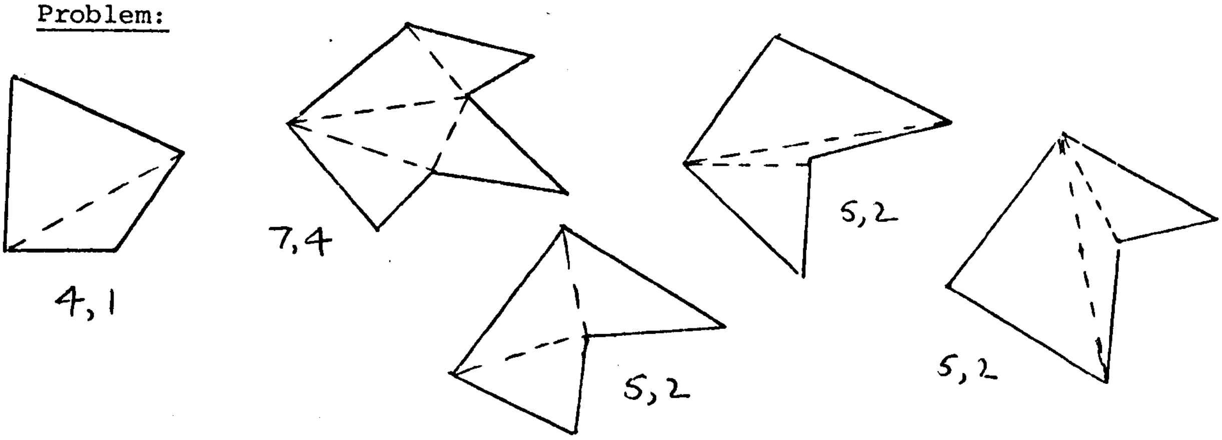
Lesley Yes/No

Richard Yes/No ⑬ C

Explain your answers and add comments about good and bad points in these three pupils' proofs.

DIAGONALS OF A POLYGON

Problem:



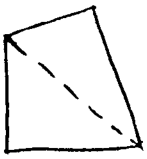
Some diagrams have been drawn here. It seems that "the greatest number of non-crossing diagonals which can be drawn in a polygon is three less than the number of sides."

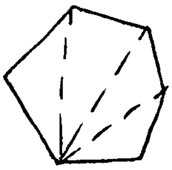
Is this statement true for all polygons?

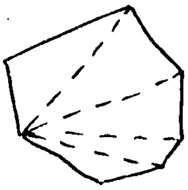
Investigate this fully; then state your conclusions and your reasons.

Two pupils drew some more examples of polygons and wrote their conclusions.

Linda:

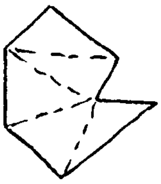

4,1

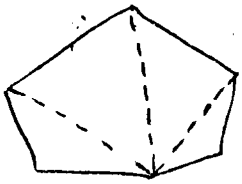

6,3

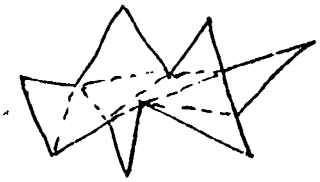

7,4

The statement is probably true for all polygons.

Jayne:


7,4


6,3


12,9

The statement is true for all polygons.

Study each of these, and give your comments on whether the set of examples proves the conclusion in each case.

Linda's: ____

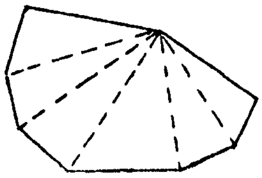
Jayne's: ____

⑭ X

Three pupils decided that the statement was true for all polygons, and gave the following reasons:

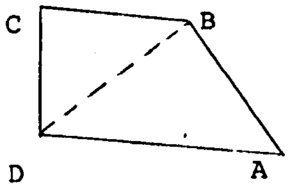
Julie: Conclusion: the statement is true for all polygons.

Reason: If in an 8,5 polygon you go from one point, you have only 5 other points to go to, as you cannot go along edges of the shape and you cannot go to yourself. Other sizes of polygon work the same way.



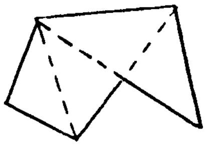
Susan: Conclusion: The statement is true for all polygons.

Reason: You can't join every point on a polygon to a single point without crossing another line e.g. in the diagram you cannot join CA or the lines will cross.



Valerie: Conclusion: The statement is true for all polygons.

Reason: the diagonals split the shape into triangles and the number of triangles is always 2 less than the number of sides. Also there is one less diagonal than the number of triangles.



For each of these, say whether you think the reasons prove the statement, and add your comments.

Julie's: Yes/No ________________________________

____

____

Susan's: Yes/No ________________________________ (15) Eg

____

____

Valerie's: Yes/No__________________________________ (16) Eg

____

ONE AND THE NEXT

Problem: "Write down two consecutive whole numbers, both less than twelve. Add them and write down the answer.

John says that one, and one only, of these three numbers will be in the set of multiples of three.

Multiples of three   3   6   9   12   15   18   21

Example: 3 + 4 = 7   3 is a multiple of 3, 4 and 7 are not."

Question: Is this always true?

Investigate this for 5 minutes and then consider the following answers given by different people.

Amanda: $1 + 2 = \underline{3}$, $2 + \underline{3} = 5$, $\underline{3} + 4 = 7$. So John is right.

Bob: $4 + 5 = \underline{9}$, $7 + 8 = \underline{15}$, $\underline{1} + 2 = \underline{3}$, $10 + 11 = \underline{21}$.
So John is right.

Have they proved their answers?   Amanda: Yes/No

Bob: Yes/No

1. Say why you think so:

______________________________ ⑰ X

2. Say which you think has the better set of examples?

Amanda/Bob

Give your reasons:

⑱ E

____

____

____

Tessa and Stephen answered as follows:

Tessa: $1 + 2 = \underline{3}$ $4 + 5 = \underline{9}$ $7 + 8 = \underline{15}$ $10 + 11 = \underline{21}$

$2 + \underline{3} = 5$ $5 + \underline{6} = 11$ $8 + \underline{9} = 17$

$\underline{3} + 4 = 7$ $\underline{6} + 7 = 13$ $\underline{9} + 10 = 19$ So John is right.

Stephen: Suppose A + B = C

If A is a multiple of 3, then B cannot be since A and B are consecutive number.

Then C cannot be a multiple of 3 because A is and B is not. If B is a multiple of 3, A cannot be and so C cannot be.

So John is right.

Have they proved their answers?

3. Tessa: Yes/No ⑲ F Stephen: Yes/No

4. Which do you think is the <u>better</u> answer? Tessa/Stephen

Give reasons for your answers to 3 and 4:

---

5. Have these pupils proved their conclusions? Explain your answers.

Paul: John will not always be right. If when you add your two numbers three does not go into it it won't be in the list.

Karen: He will always be right. If you start with a number in the list you have got one already. If you start with any other number the answer will be in the list. If you start with a number like 8 then the following number is in the list.

Paul: ⑳ E

Karen:

MAGIC SQUARES

The following square is to be filled with the numbers 2 to 10 inclusive, so that the sum of each row, each column and each diagonal is the same. A start has been made.

Is this a good start?

Explain your reasons.

4		
	6	
		8

NOTES AND MARK SCHEMES

ADD AND TAKE

Michael and Jenny

The significant difference is that Jenny has considered the extreme value 10, which might conceivably have been a special case.
Item 1: X: Marks: Jenny with reason mentioning 10 or extreme value - 1 mark on scale X. (This was originally allocated to a scale V(variety) but the item analysis suggested it should be included in X)

Susan and Yvonne

Items 2, 4E, 3, 5EE. Susan's is a restatement of the given process. Yvonne's is explanatory.
1 mark on scale E for each correct Yes/No answer, and 1 on EE for each correct reason. But the first mark is withheld if the reason given shows that the choice has been made on invalid grounds; and the second mark is given only if the reason shows a clear awareness of the explanatory status of the response, e.g. "Susan only says what happens, but Yvonne sayswhy" gains 2 + 2. In contrast "Susan No, Yvonne, Yes;""Susan's explanation is not clear as she refers to 'it' and does not explain this," is accepted as indicating that the correct response "No" has been validly selected, i.e. the non-explanatory nature of the response is recognised, but the subject cannot articulate the nature of the difference. Hence it scores only 1E + 1E.

Wendy and Kevin

Wendy's response is ambivalent, and proved impossible to mark reliably, so was excluded. It is a step towards explanation in that it expresses the relation in general, algebraic terms and exposes it to view, but is not strictly an explanation in iteself without observing also that the rearrangement, so that +x and -x cancel, is permissible.

Items 6, 7F: Kevin's is a correct full check of the finite set of possibilities. The mark for item 6 is given for recognising this. The item 7 mark is for a correct explicit statement of the reason, or if the question is taken in include rationals, for recognising that this is an incorrect extrapolation.

Ann and Hazel

Items 8, 10E, 9, 11EE: Ann's is explanatory. Hazel's does not relate directly to the process given, since 10 - 2 = 8 and 10 - 8 = 2 do not both occur in one example - what relates to 10 - 2 is 10 + 2.

Examples:

Ann:	"Yes": 1E
	"Not correct - she should add that 10 + 10 = 20" 1E
	"She explains how the first number cancels" 1E + 1EE
Hazel:	"Not enough examples" 0
	"As her answer reads the answer is ten, not twenty" 1E
	"She does not explain how the added 2 and the subtracted 2 cancel." 1E + 1EE

STAMPS

James checks all possibilities in a disorganised way. Richard is systematic but omits consideration of 0 x 20 so his proof is invalid. Lesley's is correct.

Items 12, 13C: Marks: Lesley's mark was excluded because of poor showing with item analysis. 1 mark on C for each of the others. No marks for the comments unless they show that selection was on faulty grounds, in which case the mark is lost.

Examples: "Richard has used examples, carefully planned out. James has muddled it up a bit, but his point is still there." (with Yes/Yes) This loses the C1 mark for James's "Yes", as there is no sense that the check needed to be complete.

Similarly, "Richard concentrates only on 20's when a combination of both is needed."

Similarly, "Richard's answer is better than James's as it is laid out more clearly."

DIAGONALS OF A POLYGON

Linda and Jayne

Item 14 X: The main point here is that the assertion "true for all" (Jayne) is certainly not justified by the check of 3 cases. Detection of this scores 1 on scale X (caution in extrapolation). Linda's assertion "probably true" is more acceptable, but is based on convex polygons with radiating diagonals only. A comment to this effect originally scored 1 on scale V (variety of relevant examples) but this proved an unreliable mark and was excluded.

Julie/Susan/Valerie

Items 15, 16. Susan's comment is fragmentary and Valerie's is a side-step, i.e. it relates the number of diagonals to the number of triangles but proves nothing about either. Correct identification of these (with a not invalid reason) scores 1 + 1 on Scale Eg. A separate scale was established for these, since the thinking involved differed somewhat from that required for the numerical situations; for example, Susan's proof could be rejected because it refers to a particular polygon, so is not general. Julie's proof is valid for convex polygons with radiating diagonals: statement that it is valid scored 1 on E, that it is not because it fails to deal with other types scored 1 on C on the original marking, but this was subsequently excluded as an unreliable item.

ONE AND THE NEXT

Amanda and Bob

Item 17X. Neither Amanda nor Bob have proved their answers; both have extrapolated from a few examples. Score 1 on scale X if this is recognised. The better set of examples is Amanda's since she has covered all three different types of situation, where the multiple of three is the first, second or third of the three numbers appearing in the equation. This originally scored on scale C as it involves recognising the importance the classification into types but in the statistical analysis it showed poor correlation with the other C items but high correlation with scale E so it was transferred as it can quite legitimately be regarded as an aspect of explanation.

Tessa and Stephen

Item 19F. Tessa has a full check of the finite set of cases - score 1F. Stephen's proof is insightful but incomplete in that only two of the three types of situation are covered. This was other poor item statistically and was excluded.

Paul and Karen

Paul's second statement is correct, but does not prove his first statement, since it does not consider the given process. The mark is given here (on E) if this irrelevance is recognised. The mark for Karen's response was not used.

MAGIC SQUARES

Less than half the pupils had time for this question. Marks are allotted as follows: the highest allowable of these being given. (These are not included in the analysis).

Not done	0
Something done	1
Any definite detailed reason	2
A correct detailed reason	3

Examples:

"Yes, since each column row or diagonal must contain these numbers." - score 1

"This is a poor start because it adds up to 18 which will be hard to keep up throughout the square." - score 2

"This is not a good start because by simple checking the 10 will not work anywhere." - score 3.

TEST STATISTICS AND NOTES ON DEVELOPMENT

Reliability of the Measures

In an experiment of this character, where the samples are small, statistical analyses cannot be expected to produce uniform results. It was, however, considered worth while to apply a number of checks even though the results might have to be treated with caution. The first of these was an item analysis, giving the correlations of each item with its own scale total and with the total for the whole test, and giving Cronbach's alpha for each subscale (Youngman, 1975). For this purpose, experimental and control groups were combined, giving a total of 26 cases.

CALE	ITEMS	MEAN	SIGMA	ALPHA	MS	MEAN	SIGMA	ALPHA
✓ 1	2.	0,846	0,769	0,4400	2.	1,000	0.620	-0,0462 X
E 2	15.	6,462	2,835	0,6671	15.	7,308	5.360	0,7620
X 3	2.	1,000	0,784	0,4615	2.	1,308	0,773	0,4851
F 4	3.	1,731	1,129	0,6825	3.	1,923	1.141	0,7201
C 5	4.	1,192	0,833	-0,1194 ?	4.	1,462	0,970	0,2725
total 0	26.	11,231	3,662	0,6113	26.	13,000	4.315	0,7312
		PRE TEST				POST TEST		

TEM	SCALE	MEAN	SIGMA	R(TOTAL)	R(SCALE)	MEAN	SIGMA	R(TOTAL)	R(SCALE)
1	✓ 1	0,54	0,499	0,2269	0,8178	0,73	0,444	0,4622	0,6991 } to scale X
2	E 2	0,65	0,476	0,3770	0,4806	0,73	0,444	0,6431	0,6749
3	E 2	0,35	0,476	0,2412	0,3599	0,35	0,476	0,4497	0,3905
4	E 2	0,65	0,476	0,5757	0,6207	0,69	0,462	0,5408	0,7059
5	E 2	0,31	0,462	0,3449	0,4132	0,35	0,476	0,4497	0,5589
6	E 2	0,50	0,500	0,6092	0,5865	0,54	0,499	0.4470	0,4292 X
7	F 4	0,62	0,487	-0,0149	0,8622	0,65	0,476	0.1312	0,8721
8	F 4	0,42	0,494	0,3287	0,8231	0,54	0,499	0.3576	0,8843
9	E 2	0,58	0,494	0,3941	0,4607	0,69	0,462	0,3283	0,5323
10	E 2	0,15	0,361	0,2642	0,3751	0,27	0,444	0,4019	0,4605
11	E 2	0,42	0,494	0,4563	0,5375	0,69	0,462	0.6181	0,6315
12	E 2	0,15	0.361	0.3510	0,4400	0,27	0,444	0.3618	0,3573
13	C 5	0,42	0,494	0,0523	0,5500	0,58	0,494	0,4150	0,8088
14	E 2	0,50	0,509	-0,1260	-0,0267	0,46	0,499	0.2146	0,2596 X
15	C 5	0,42	0,494	0,2649	0,5500	0,58	0,494	0.2887	0,7285
16	✓ 1	0,31	0,462	0,0263	0,7833	0,27	0,444	-0.0804	0,6991 X
17	X 3	0,38	0,487	0,6195	0,8062	0,65	0,476	0,4310	0,8125
18	E 2	0,73	0,444	0,1803	0,3676	0,58	0,494	0,5413	0,5186 X
19	E 2	0,54	0,499	0,4797	0,4690	0,69	0,462	0,5601	0,5075
20	E 2	0,54	0,499	0,4797	0,4690	0,50	0,500	0,5883	0,6635
21	X 3	0,62	0,487	0,1160	0,8062	0,65	0,476	0,1686	0,8125
22	C 5	0,08	0,266	0.3360	0,4532	0,15	0,361	0,1977	0,3466
23	F 4	0,69	0,462	0,2468	0,6532	0,73	0,444	0,3216	0,6431
24	C 5	0,27	0,444	0,4354	0,3804	0,15	0,361	0,1977	0,2367 X
25	E 2	0,15	0,361	0,6136	0,6337	0,27	0,444	0,2814	0,3315
26	E 2	0,23	0,421	0,1899	0,1655	0,23	0,421	-0.0635	0,1672 X

TABLE 1

X = excluded.

Examination of these results, combined with consideration of the nature of the items, led to the exclusion of six of the 26 items from further analysis, the elimination of scale V and the transfer of its one remaining item to scale X, and the separation of scale E into Eg, E and EE. These decisions are discussed above, in relation to the Mark Scheme. The characteristics of the test with this revised marking scheme are shown in Table 2. (Pre- and post-test results are here combined, giving 52 cases.) These item statistics were then considered satisfactory in relation to the numbers involved.

SCALE	ITEMS	MEAN	SIGMA	ALPHA
1 X	3.	1.788	0.987	0.4019
2 F	3.	1.827	1.139	0.7057
3 C	2.	1.000	0.784	0.3750
4 Eg	2.	1.135	0.833	0.5973
5 E	6.	2.885	1.660	0.6860
6 EE	4.	1.096	1.096	0.4684
7 Total	20.	9.731	3.701	0.7186

ITEM	SCALE	REV	KEY	MEAN	SIGMA	R(TOTAL)	R(SCALE)
1	1	0	0	0.63	0.482	0.3764	0.6061
2	5	0	0	0.69	0.462	0.5595	0.6564
3	6	0	0	0.35	0.476	0.3697	0.6366
4	5	0	0	0.67	0.469	0.5253	0.7418
5	6	0	0	0.33	0.469	0.4052	0.7240
6	2	0	0	0.63	0.482	0.1282	0.8666
7	2	0	0	0.48	0.500	0.4340	0.8560
8	5	0	0	0.63	0.482	0.4088	0.7171
9	6	0	0	0.21	0.408	0.3685	0.6846
10	5	0	0	0.56	0.497	0.5630	0.7078
11	6	0	0	0.21	0.408	0.3812	0.4270
12	3	0	0	0.50	0.500	0.2182	0.7845
13	3	0	0	0.50	0.500	0.3326	0.7845
14	1	0	0	0.52	0.500	0.5956	0.7686
15	4	0	0	0.62	0.487	0.5299	0.8398
16	4	0	0	0.52	0.500	0.4708	0.8489
17	1	0	0	0.63	0.482	0.2577	0.6465
18	5	0	0	0.12	0.319	0.2865	0.4239
19	2	0	0	0.71	0.453	0.3092	0.6487
20	5	0	0	0.21	0.408	0.4067	0.4331

TABLE 2

Pre- and post-test reliability was also calculated, for the experimental group (12 cases) and the two control groups combined (14 cases). The results appear in Table 3.

Scale	X	F	C	Eg	E	EE	Total	N
No. of items	3	3	2	2	6	4	20	
Pre/Post: Exptl.	.68	-.22	-.04	.33	.40	.58	.67	12
Control.	.77	.28	-.11	.66	.82	.53	.92	14

TABLE 3

The generally lower values for the experimental group are to be expected; their performance has presumably been influenced non-uniformly by the teaching. The values for F are affected by the fact that on one question the entire experimental group were fully successful both on pre- and post-tests. The values for C probably indicate an inherent lack of reliability in these two questions; this is discussed in the context of the actual results (p.10.12).

THE MATHEMATICAL PROCESS AS ILLUSTRATED BY BOOLEAN ALGEBRA

Boolean algebra provides a small-scale system in which the whole mathematical process, from real-life, raw material through general laws to an axiom system, can be seen; moreover, the system has distinct realisations and it can be seen how the axioms link them together. All the necessary material can be found in Boolean Systems, by D. Kaye.

It is possible to start either from switching problems or from logic. Sixth formers are generally more strongly motivated by the former. A good starting problem is the 'landing light' problem: the state of a light is to be changed by every change ineitherof two switches. With or without some preliminary consideration of the effect of putting two circuit elements, which may be closed (conducting) or open, in series and in parallel, a solution to this problem can be reached by trial. Denoting open circuits by 0, closed by 1, and circuit elements (such as switches or combinations of switches) which may be either state ,by variables, the laws of combination of such elements can be described by the Table 1 which gives the state of xsy and xpy for every possible combination of values of x and y.

x	y	xsy
0	0	0
0	1	0
1	0	0
1	1	1

x	y	xpy
0	0	0
0	1	1
1	0	1
1	1	1

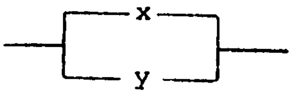


TABLE 1

A circuit element which is necessarily in the opposite state to x is denoted by x^1 (called the complement of x) Table 2 gives conditions for the landing light.

x	y	ℓ
0	0	1
0	1	0
1	0	0
1	1	1

Table 2

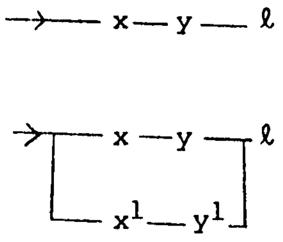


Fig. 1

The last three lines of this are satisfied by the upper circuit of Fig. 1; the first line is satisfied by the addition of the loop in the lower circuit, and it can be verified that this does not affect the other lines in the table. The expression for this circuit is $(xsy)p(x^1sy^1)$. Inspection of Table 1 shows that xsy behaves just like the arithmetical product xy, and xpy like x + y except that 1 + 1 = 1. It is common (though not universal) to use these familiar symbols, and the landing light function then becomes $xy + x^1y^1$. (A useful exercise is to show that $xy^1 + x^1y$ provides an alternative solution). In practice x and x^1, and y and y^1, each form a single changeover switch. It is fairly easy to show that the expression corresponding to a given table can be derived by choosing from the full expression $xy + x^1y + xy^1 + x^1y^1$ the one term which contributes the value 1 for each appropriate row of the table. Thus, in this case, for the first row, x = 0, y = 0, so 1 is given only by x^1y^1; and the last row needs xy. Thus a variety of practical problems can be solved by (a) writing the table corresponding to the required conditions (b) deriving the function (c) drawing the circuit corresponding to the function.

But the algebra provides an additional benefit - by suitable manipulations the expression can be brought into a form which minimises the number of different switches required. It is clear from the circuits of Table 1 that xy = yx and x + y = y + x, and it follows immediately from the appropriate circuits that the associative laws hold, and also <u>two</u> distributive laws, x(y + z) = xy + xz and x + yz = (x + y)(x + z).

The second of these is the <u>dual</u> of the first, that is, obtained from it by interchange of + and . (and of 0 and 1 if they appear). A collection of problems which yield to these methods is given by Kaye in the above mentioned book, pages 15 and 22, and by Giles (1970), pages 22, 25, 31. One of the more striking examples of the power of the method is in the design of binary adders (Kaye p 23).

In the course of this work a number of possible general laws other than those checked already will arise - some may be used inadvertently at first, others may be spotted and checked. These may range from 0 + x = x, 0 . x = 0, through the (probably missed) 1 + x = 1 to de Morgan's laws for complementing sums and products: $(x + y)^1 = x^1y^1$ and its dual. The following extensive list is given by Kaye (p. 19).

S1. Commutative Laws
(a) x + y = y + x   (b) x.y = y.x

S2. Associative Laws
(a) x + (y + z) = (x + y) + z   (b) x.(y.z) = (x.y).z

S3. Distributive Laws
(a) x.(y + z) = x.y + x.z   (b) x + y.z = (x + y)(x + z)

S4. Laws of Tautology
(a) $x + x = x$ (b) $x.x = x$

S5. Laws of Complementation
(a) $x + x^1 = 1$ (b) $x.x^1 = 0$ (c) $(x^1)^1 = x$

S6. Laws of Absorption
(a) $x + x.y = x$ (b) $x.(x + y) = x$

S7. De Morgan's Laws
(a) $(x + y)^1 = x^1.y^1$ (b) $(x.y)^1 = x^1 + y^1$

Laws with 0 and 1
S8. (a) $0 + x = x$ (b) $1.x = x$

S9. (a) $1 + x = 1$ (b) $0.x = 0$

S10. (a) $1^1 = 0$ (b) $0^1 = 1$

However, this list is neither minimal nor maximal. For example, the law of absorption $x + xy = x$ can be derived from the others in the list as follows:

$$\begin{aligned} x + xy &= x.1 + x.y \\ &= x(1 + y) \\ &= x.1 \\ &= x; \end{aligned}$$

while there are other useful laws, such as $(xy^1 + x^1y)^1 = xy + x^1y^1$ which could be added - in fact, these last two expressions occur so often that an additional composition is sometimes defined, $x \Delta y = xy^1 + x^1y$. The best reflection in the classroom of the process of mathematical inquiry would consist of collecting the actual list of possible general laws which emerged from the work of the class. These could then be examined and reduced to an agreed minimal list of 'axioms' by trying to prove as many as possible from others. It would probably be best not to press this process too far at this stage, but to regard it just as getting a rather long list down to a more comfortable number. Further efforts might be better motivated after the next phase of the work, which will be a study of logic similar to that of switching circuits. The task then will be to see how far the two systems agree, and it will be important to know that the sets of laws being checked against each other fully characterise the systems.

The development of switching algebra could be broken off at this point and the study of logic taken up. The source of interest in this subject is the insight which it can give into the logical structure of ordinary textual material, but it is typical of the mathematical process that it should begin by crystallising out some problem material which displays sharply and vividly the relationships which exist in normal material. Aristotle studied syllogisms, and Lewis Carroll invented puzzles; for our purpose the following problem forms a suitable illustration.

What conclusion can be formed from the following statements?

(a) A dishonourable man is never perfect.

(b) An honourable man never lies.

(c) A man is not perfect unless he is always tactful.

(d) Every tactful man tells an occasional lie.

These statements refer to sets of people; the first says that the complement of the set of honourable men has no intersection with the set of perfect men; in symbols, $H^1 \cap P = \emptyset$

The full set of statements may be expressed as:

$$\begin{aligned} H^1 \cap P &= \emptyset \\ H \cap L &= \emptyset \\ PCT & \\ TCL & \end{aligned}$$

The third and fourth of these may be combined to give

PCL.

This is an example of the general law most frequently needed in these problems - $\left\{ XCY \text{ and } YCZ \right\} \Rightarrow XCZ$ (1)

The fact that PCL contradicts the first two statements can be seen from Venn diagrams: see Fig. 2.

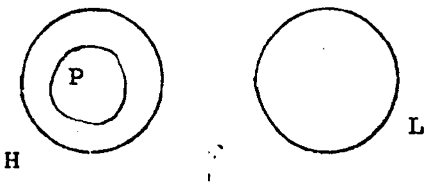


Fig. 2.

Thus the set P, of perfect men, is empty. But the diagram also makes it clear that PCH and $H^1 \cap P = \emptyset$ are equivalent; and in general

$$XCY \Leftrightarrow X \cap Y^1 = \emptyset \Leftrightarrow Y^1 C X^1 \qquad (2)$$

Thus the first two statements of the problem data are equivalent to

PCH
HCL^1

whence PCL^1 follows by the transitivity of inclusion, and the contradiction with PCL is evident. Experience with a few more problems suggests that here is a general method; transform all statements into inclusion form and transitivity will then expose all possible consequences.

Four further characteristics of mathematical process have now emerged. They are (a) the extraction of the relational aspects of a situation and the elimination of its particular content - this is evident from the expressions like PCH, which focus attention away from the particular nature of the sets denoted by P and H and toward their relationship, (b) the use of diagrams and symbols as models of the situation, (c) the derivation of rules for manipulation of the symbol system (this involves consideration of the meaning of the symbols), and the subsequent <u>use</u> of these manipulative rules without further reference to the meaning of the symbols, (d) the search for a general method which will apply to all problems of a type.

On working through a number of such problems a collection of general laws is built up. These may each be verified, as they arise, using the Venn diagram model. One short sequence of work produced the following in addition to the two noted above:

$$XCY \Rightarrow X \cap Z \; C \; Y \cap Z \qquad (3)$$

$$(X \cup Y)^1 = X^1 \cap Y^1 \qquad (4)$$

$$(XCY \text{ and } XCZ) \Rightarrow X \subset (Y \cap Z) \qquad (5)$$

As with circuits, the list of necessary laws could be shortened by proving some from others.

The set of laws arising thus has some aspects in common with those of switching algebra (e.g. complements, commutativity) but also some clear differences, e.g. the absence of inclusion in switching algebra, and the predominance of the associative and distributive laws there. How close is the correspondence? A reasonable approach would be to begin by checking all the found (and non-redundant) laws in each system to see whether they apply to the other. It will be a help to harmonise the notations by replacing $\cup$ and $\cap$ by + and . in set algebra.

Next, can we define inclusion in terms transferable into switching algebra - that is, in terms of +, ., ^{1}.? Law (2) above shows that this is easy: we define XCY as $XY^1 = 0$. The rest of law (2) is then an application of the commutative law, $XY^1 = Y^1X$. The remainder of laws (1) to (5) can then be proved by the following methods.

(1) Suppose XCY and YCZ; i.e. $XY^1 = 0$ and $YZ^1 = 0$.

$$\begin{aligned} \text{Then } XZ^{} &= X.1.Z^1 && \text{by S8 and S2}\\ &= X(Y + Y^1)Z && \text{by S5}\\ &= XYZ^1 + XY^1Z^1 && \text{by S3 and S2}\\ &= X.0 + 0.Z^1 && \text{given}\\ &= 0 && \text{by S9 and S8} \end{aligned}$$

Hence XCZ.

(3) Requires consideration of $XZ(YZ)^1$.

$$\begin{aligned} XZ(YZ)^1 &= XZ(Y^1 + Z^1) && \text{by S7}\\ &= XZY^1 + X(ZZ^1) && \text{by S3 and S2}\\ &= XY^1Z + 0 && \text{by S1, S2, S5}\\ &= 0.Z + 0 && \text{given}\\ &= 0 && \text{by S9 and S8} \end{aligned}$$

Thus all the laws of set algebra which have emerged from our work can be derived from the Kaye list of laws for switching algebra. To know whether all possible laws of set algebra can be so derived requires that we check whether sets, with their defined compositions of $\cup$, $\cap$ and ^{1}, satisfy all the same laws as switching circuits. At this point it becomes useful to eliminate any redundant laws. At first this is easy, but as the list gets shorter more ingenuity is required, and more care to avoid mistakes. A few examples follow. (We shall only deal with one of each dual pair; the other needs only the dual of every step in the proof.)

S6 has been proved above.

$$\begin{aligned} \text{For S9:}\quad 0 &= xx^1 && \text{by S5}\\ &= x(x^1 + 0) && \text{by S8}\\ &= xx^1 + x.0 && \text{by S3}\\ &= 0 + x.0 && \text{by S5}\\ &= x.0 && \text{by S8} \end{aligned}$$

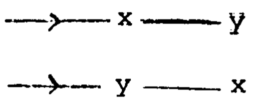
For S7 we show that $(x + y)^1 = x^1y^1$ by showing that x^1y^1 satisfies the conditions in S5 for being the complement of $(x + y)$ (we shall take S5 as the definition of the complement); but we shall either have to assume, or prove that the complement so defined is unique.

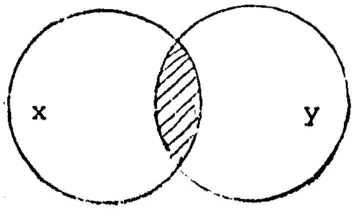
First $x^1y^1(x + y)$	$= x^1y^1x + x^1y^1y$	by S3
	$= xx^1y^1 + yy^1x^1$	by S2 and S1
	$= 0 + 0$	by S5 and S9
	$= 0$	by S8
Also $x^1y^1 + (x + y)$	$= (x^1 + (x + y))(y^1 + (x + y))$	by S3b
	$= (1 + y)(1 + x)$	by S2 and S5
	$= 1.1$	by S9
	$= \underline{1}$	by S8
Hence, assuming uniqueness $x^1y^1 = (x + y)^1$		by S5

It is also fairly easy to prove S4; and S10 follows from S5 if unique complements are assumed. Thus the set is reduced to S1, S2, S3, S5 and S8. It is possible but harder, to prove the associative laws S2 from the other four, and one can decide whether or not to take this further step (Kaye gives the proof on p. 97)

It is now a more straightforward matter to verify that both circuits and sets satisfy these laws and hence any laws derived from them. We give diagrammatic proofs.

S1

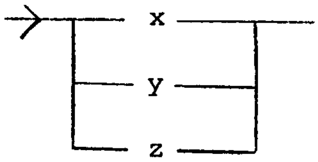


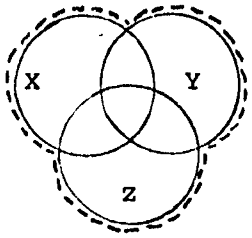


xy = yx

x + y = y + x similarly

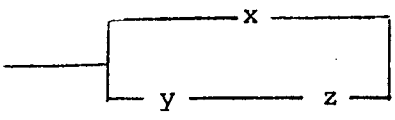
S2



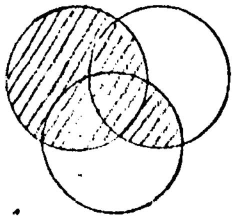


x + (y + z) = (x + y) + z

x(yz) = (xy)z similarly

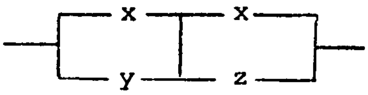


x + yz

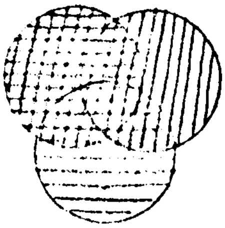


x + yz

S3



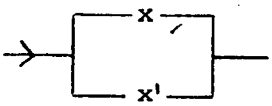
$(x + y)(x + z)$

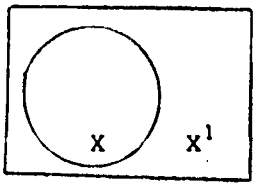


circuits have the same state for all combinations of 0, 1 for x, y z.

$x + y$ |||| $x + z$ ≡

S5





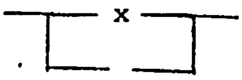
$x + x' = 1$: since one is closed

$X + X' = 1$

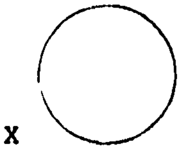
$XX' = 0$

$xx' = 0$ since one is open

S8



$x + 0 = x$



$X + 0 = X$

References

Kaye, D. (1968) Boolean Systems. Mathematical Topics (Longmans)

Giles, G (1970) Switchboards and Logical Paths, Mathematics Teaching Pamphlet No. 15 (ATM, Nelson, Lancs.)

APPENDIX TO CHAPTER 11

Selected responses

FORM A

MEANS

Theorem: The arithmetic mean of any two positive real numbers is not less than their geometric mean.

Proof:
$$\frac{a+b}{2} \geqslant \sqrt{ab}$$

Hence $a + b \geqslant 2\sqrt{ab}$

Squaring to remove square roots,

$$(a+b)^2 \geqslant 4ab$$

Hence $a^2 + b^2 - 2ab \geqslant 0$

So $(a-b)^2 \geqslant 0$

This is true for all real numbers a&b, hence the theorem follows.

Is this theorem proved? If not, say what is wrong and give a correct treatment of the situation. No

Should prove first statement. Also the conclusion does not lead to the theorem. Should prove as follows.

[illegible]. ~~arith mean.~~

Assume $\frac{a+b}{2} \ngeq \sqrt{ab}$.

$\therefore a+b \ngeq 2\sqrt{ab}$.

$\therefore (a+b)^2 \ngeq 4ab$

$\therefore a^2+b^2-2ab \ngeq 0$

$\therefore (a-b)^2 \ngeq 0$ which is false $\therefore$ orig assumption was false.

$\therefore \frac{a+b}{2} \geq \sqrt{ab}$. $\therefore$ arithmetic mean is always greater than or equal $\therefore$ is

COINS

Take 3 coins showing all tails.

A move consists of turning over any two coins.

1. Using as many such moves as you wish, obtain all heads. Prove or disprove that this is possible.

2. Extend this problem to 4 coins, turning 3 at a time. Prove your results.

3. Generalise your results as far as possible.

For 3 coins. one move turns 2 over. Another move will turn one of theo back. ∴ There must always be one with T showing

e.g TTT
HHT
HTH
HHT etc.

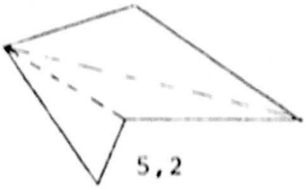
. The same applies here. The most that will be on head is 3 as in the next move two will come to heads.

TTTT ①
HHHT ②
HTTH③ + now at least one must go back to tails
THHH. ie stage 2

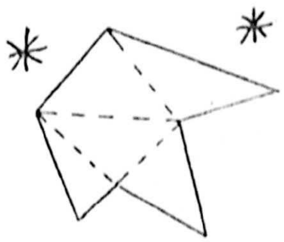
Any number of coins n. cannot be all turned over at one time if n-1 coins are uned Each move.

Exception: n=2 as n-1 = 1 + 2 is divisible by 1

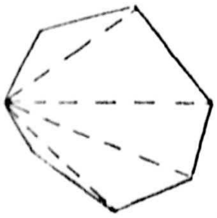
DIAGONALS OF POLYGONS



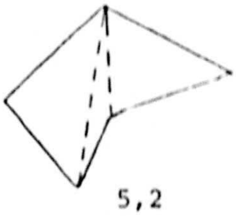
5,2



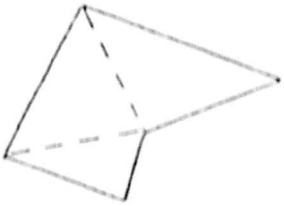
7,4



7,4



5,2



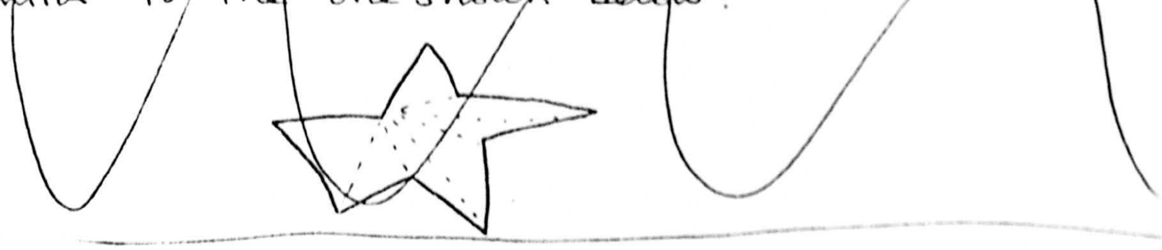
5,2

Theorem: The greatest number of non-crossing diagonals which can be drawn in a polygon is three less than the number of sides.

Proof: In the left hand diagram, it is clear that diagonals can be drawn from one vertex to each other vertex, except three: these being the first vertex itself and the two adjacent to it. Similar radiating sets of diagonals can be drawn in any polygon; hence the theorem is true for all polygons.

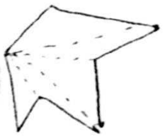
Is this theorem proved? If not, say what is wrong and give a correct treatment of the situation.

The theorem is not proved. It is possible to have a shape similar to the one shown below.



The theorem is not proved. In the diagram asterisked above the diagonals do not all radiate from one vertex, which defeats the reasoning of the proof.
For greatest number of non-crossing diagonals all vertices must be connected by one single line

The theorem is not proved, as in the left hand diagram the diagonals are all drawn from one vertex. in the second polygon (7, 4), 3 diagonals re drawn from one vertex, while one is drawn from another. The other three gures have diagonals drawn in a similar way ~~from~~ as the first i.e. as radiating om one vertex. As the second polygon does not have a similar treatement to e rest, a similar arguement to the proof above cannot be applied to it



If (7, 4) is drawn in this way, the theorem will apply to it.

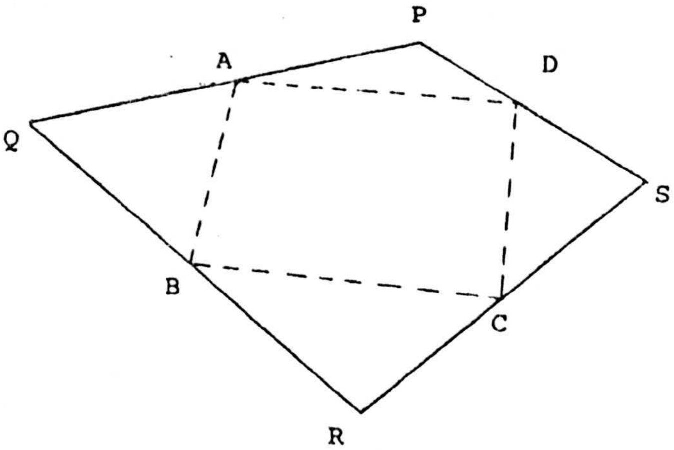
The theorem has been demonstrated; not proved; a proof must be general. For an n-sided figure, $(n>3)$, once it has been completely devided into triangles with atleast one side of the figure forming a side of the triangle, no more non-crossing diagonals can be drawn. The total no of degrees in an n-sided polygon is $180(n-2)$ Hence the total no. of Δs is $n-2$. The total no of sides in all the Δs $\therefore$ is $3(n-2)$, $\therefore$ The total no. of non-edge sides is $3(n-2)-n$, and the total different no of non-edge sides is $\frac{3(n-2)-n}{2}$ Since each Δ shares a side in common with the one next to it. i.e, total no. of sides of Δs = diagonals of polygon

$$= \frac{3(n-2)-n}{2} = n-3$$

TRADITIONAL 'O' AND 'A'-LEVEL

FORM B

QUADS



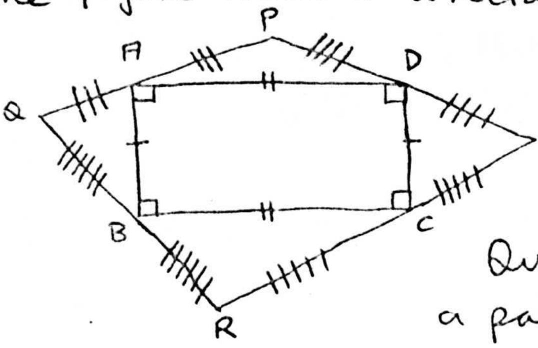
A, B, C, D, are the midpoints of the sides of the quad PQRS.

In some quads the midpoint figure ABCD is a rectangle.

1. Find out what has to be special about the quad PQRS for ABCD to be a rectangle and prove your results.

2. It is suggested that in order to obtain a rectangle as the midpoint figure ABCD, the quadrilateral PQRS must be a rhombus. Check this and prove the correct result.

(1) Assume figure ABCD is a rectangle.



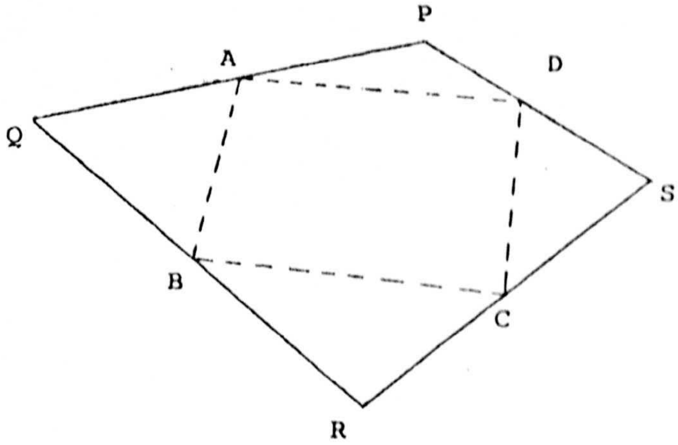
~~Quad PQRS must also be a rectangle~~

Quad PQRS must be a parallelogram for ABCD to be a rectangle.

J.C.P

FORM B

QUADS

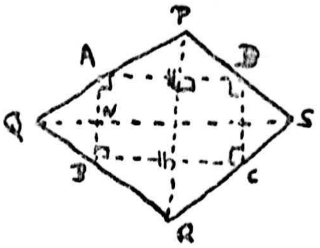


A, B, C, D, are the midpoints of the sides of the quad PQRS.

In some quads the midpoint figure ABCD is a rectangle.

1. Find out what has to be special about the quad PQRS for ABCD to be a rectangle and prove your results.

2. It is suggested that in order to obtain a rectangle as the midpoint figure ABCD, the quadrilateral PQRS must be a rhombus. Check this and prove the correct result.



PA = AQ ∴ AD is parallel to BC to QS
QB = BR and AB is parallel to DC to PR
RC = CS ∴ ABCD is always a parallelogram.
SD = DP
For PHD = 90° and PHA = 90° QP = QR and PS = RS

2. If PQRS is a Rhombus

QPR is similar to QAB ∴ PR is parallel to AB
similarly
PR is parallel to DC
QS is parallel to AD
QS is parallel to BC
∴ ABCD is a parallelogram
∴ To be a rectangle PHA = PHD = ANQ = QNB = 90°
∴ QP = QR = RS = SP ∴ the quad must be a Rhombus