Topics in $2 + 1$ gravity and conformal field theory

Richard Gunn Johnson, MSci.

Thesis submitted to The University of Nottingham
for the degree of Doctor of Philosophy

February 2009

Abstract

We study the Hamiltonian dynamics for a system of two colliding point particles coupled to $(2+1)$-dimensional gravity with a negative cosmological constant by anchoring the dynamics of the system to its spatial infinity. We reduce the Chern-Simons formulation of the gravitational action, finding the reduced Hamiltonian for three special cases of the particle masses, in a phase space chart coordinatised by the geodesic distance between the two particles and its conjugate momentum. The dimension of the reduced phase space is two. At the threshold of black hole formation, the black hole mass depends linearly on the momentum, in agreement with previous analysis in a holonomy-based phase space chart. We use the reduced action to compute the semiclassical probability amplitude of two particles to tunnel out of the black hole, finding that the imaginary part of the action is equal to the Bekenstein-Hawking entropy of the hole.

We also study the form that conformal field theory (CFT) correlation functions take in coset spaces of $\mathrm{SL}(2,\mathbb{C})$. We realise the $\mathrm{SL}(2,\mathbb{C})$ twistor space $\mathbb{T}$ in two distinct but equivalent ways, deriving some important facts about this space, and we also give one representation of another coset space $\mathbb{B}$. We examine the form of CFT correlation functions in $\mathbb{T}$, $\mathbb{B}$ and two other related spaces using techniques from representation theory and make a number of comments on the twistor transform for $\mathbb{T}$.

Acknowledgements

I would like to thank my partner, Angharad, for her patience in delaying every plan she ever wanted to make until the thesis submission deadline had passed. Special thanks also go to my beautiful boy, Matthais, for smiling and "gooing" at me after the long days struggling with LaTeX.

I thank my parents for giving me the freedom to pursue a PhD and Ben, Ed and Gill for discussing all things unrelated to physics whenever we met.

I would also like to thank my immediate quantum gravity peers, Alejandro, for proving that a student in the QG group could finish within three and a half years, George, for discussing how to make ends meet, and Ileana, for convincing me that there is more to apples than the Golden Delicious.

Finally I thank my supervisors, Kirill Krasnov, for introducing me to twistors and conformal field theory, and Jorma Louko, whose patience, effort and attention to detail were invaluable throughout the latter parts of the PhD process.

Contents

Chapter 1

Introduction

Two of the most thought-provoking theories of modern physics to emerge from the twentieth century are quantum mechanics (QM) and general relativity (GR). QM, which is relevant on the atomic scale, accurately describes three of the four fundamental forces of nature, namely electromagnetism, the strong interaction and the weak interaction, but in its usual formulation does so on a fixed background spacetime. GR, which applies itself to large-scale structures such as stars and galaxies, describes gravitation, the fourth fundamental force, but crucially does so with a dynamical background spacetime.

Attempts to unify GR and QM into a so-called theory of quantum gravity (QG) have a long and complicated history of which it is not our intention to cover here; for an overview see [1]. A much more complete introduction to the subject is given in [2] and a progress report is given in [3]. It is the aim of this thesis to discuss aspects of two approaches to QG, namely canonical quantum gravity and twistor theory.

The thesis is split into two fairly unrelated parts — part I, consisting of Chapters 2 through 6, investigates point particles coupled to $(2+1)$-dimensional gravity and the Hamiltonian formulation thereof. Part II, consisting of Chapters 7 and 8, addresses $\mathrm{SL}(2,\mathbb{C})$ twistor space and some of its applications within conformal field theory. The results are summarised and discussed in chapter 9.

In the rest of this chapter we briefly introduce the specific topics to be investigated in Parts I and II and give a chapter-by-chapter outline of the thesis.

1.1 Canonical quantum gravity (CQG) and point particles

Canonical quantum gravity (CQG) is an attempt to quantise GR directly by writing GR in its canonical/Hamiltonian form and then quantising via a set of techniques invented by Dirac in 1950 [4]. The basic notions of CQG were established by DeWitt in 1967 [5]; the theory is written in terms of a set of configuration variables and canonically conjugate momentum variables describing the state of the system at some point in time. One can then obtain the time-evolution of both sets of variables from the Hamiltonian form of the action. In the "usual" way, the two sets of variables are then treated as operators obeying certain commutation relations in order to translate to the quantum theory.

Attempts to canonically quantize $(3+1)$-dimensional GR have historically run into many difficulties. The $(2+1)$-dimensional theory, however, provides us with a technically simplified setting while retaining many of the conceptual features of the $(3+1)$-dimensional theory. n-dimensional GR has $n\,(n-3)$ physical degrees of freedom per spacetime point [6], motivating the statement that $(2+1)$-dimensional GR is locally trivial, having zero (local) degrees of freedom. However, due to the technical simplicity of the theory in $2+1$ dimensions it can be consistently coupled to point particles and topologically nontrivial spacetimes can be constructed with a finite number of global degrees of freedom. Such spacetimes can be constructed in terms of holonomies around non-trivial loops which is nicely explained in the case of a zero and non-zero cosmological constant in [7] and [8] respectively. More recently [9] formulates and analyses the Hamiltonian dynamics of a pair of massive spinless point particles in $(2+1)$-dimensional Einstein gravity for the case where the cosmological constant is zero. The approach of [9] is to firstly work out the geometry of the spacetime at the spatial infinity and then anchor the particle trajectories to this geometry. They then use the description of two-particle spacetimes in terms of a piece of Minkowski geometry between the particle world lines [10], and translate this description into one that relates the worldlines of the particles to the spatial infinity. Finally they use the explicit form of the classical solutions anchored to the infinity to reduce the gravitational action and find the reduced Hamiltonian.

The quantisation of the Hamiltonian formulation obtained in [9] is considered in [11].

1.2 Point particles coupled to AdS_3 gravity

In Chapters 2 through 6 we would like to emulate the work done in [9] by generalising it to the case where we include a negative cosmological constant. We would like to obtain a Hamiltonian formulation for two massive point particles coupled to AdS_3 gravity. The main upshot of including a negative cosmological constant is that there are black hole solutions, meaning we can study the black hole formation and analyse the critical phenomena at the formation threshold. We will also be able to comment on the action for tunelling from the black hole.

Chapter 2 describes the geometrical details of the two-particle spacetimes. All the relevant coordinate conventions are established and the one-particle spacetimes are discussed. The two-particle spacetimes are then constructed where the spacetime has a non-zero spin parameter but assuming that the spacetime does not have a black hole. The last section in this chapter specialises to the case where the particles collide, being the setting for the rest of the work.

In Chapter 3 we discuss the first order action formalism of AdS_3 gravity. The gauge transformations of the theory are identified in order to make use of them in the Hamiltonian reduction in Chapter 6. We then specify a $(2+1)$-decomposition of the action and finally discuss the contributions to the action from the particles and from the boundary term at infinity.

Chapter 4 describes the embedding of the particle surface in relation to the two-particle spacetimes discussed in Chapter 2. We embed the surface containing the particles in a way that is consistent with the known classical solutions and the boundary conditions at the spatial infinity. We then use the details of the embedding to fix a gauge for the fields in a certain technical way in order to evaluate the reduced action in subsequent chapters.

Chapter 5 deals with the contribution to the action from the Liouville term using the details of the embedding and gauge choice from Chapter 4. We evaluate part of this contribution directly and convert the remaining part into a one-dimensional boundary integral using Stokes' theorem. The evaluation of the boundary integral is examined in some detail, and whilst we do not complete the analysis due to algebraic complications, we do establish the general form that the Liouville term takes.

In Chapter 6, the final chapter in this first part of the thesis, we first use the general form of the Liouville term found in Chapter 5 and our knowledge of the equations of motion to find the fully reduced action and write this action in Hamiltonian form. The phase space has dimension two. We then perform a canonical transformation to a phase space chart in which the "position" coordinate is the geodesic distance between the two particles. We use this action to analyse the black hole formation threshold and find the leading order critical exponent to be one, coinciding with the result obtained in [12]. Finally we study the black hole creation/annihilation as a quantum mechanical tunnelling process and find that the imaginary part of the action is equal to the Bekenstein-Hawking entropy of the black hole.

1.3 Twistor theory

Twistor theory in its original form was invented by Roger Penrose in 1967. His vision was that fundamental physics should be reformulated in terms of objects called twistors living in twistor space. Twistors could then be used to reconstruct spacetime in a prescribed mathematical manner. Penrose's popular monograph [13] describes the main ideas of twistor theory and an accessible technical introduction to twistor theory is given in [14]. Twistor theory was largely ignored by the wider theoretical physics community until 2003 when Edward Witten wrote a paper relating string theory and twistor geometry [15]. Twistor string theory was born and many papers followed, for example [16], bringing twistor theory once again into the limelight of mainstream research.

1.4 $\mathrm{SL}(2,\mathbb{C})$ twistor space and conformal field theory

One of the unique selling points of twistor theory is that solutions to the massless wave equation naturally arise using the methods of twistor geometry. The Penrose/twistor transform, whose details were first established in [17], is an integral transform from a certain subset of functions on twistor space to the space of solutions to the massless wave equation on compactified Minkowski space. The twistor transform is not, however, restricted to Penrose's twistor space; [18] shows how to construct the twistor transform for $\mathrm{SO}\,(1,n)$ rather than $\mathrm{SU}\,(2,2)$ twistors.

The AdS/CFT correspondence (or Maldacena duality) states there is an equivalence between a certain string theory living on $AdS \times K$, where K is a closed manifold, and a conformal field theory (CFT) living on the boundary of the AdS space. See [19], [20] and the review [21].

In Chapters 7 and 8 we draw upon the ideas of AdS/CFT and attempt to determine the relationship between CFT correlation functions on two spaces that have not been extensively studied in the literature. We elucidate properties of "$\mathrm{SL}(2,\mathbb{C})$ twistor space", $\mathbb{T}$, and attempt to construct the twistor transform corresponding to this space. We go on to explore the form of CFT correlation functions within $\mathbb{T}$ and various related spaces.

In Chapter 7 we describe a "$\mathrm{SL}(2,\mathbb{C})$ twistor space" that arises naturally from the Lie group $\mathrm{SL}\,(2,\mathbb{C})$. In contrast with Penrose's twistor space, which has complex dimension 4, $\mathrm{SL}(2,\mathbb{C})$ twistor space has complex dimension 2. We also construct a related space and discuss its global properties. Chapter 7 essentially sets the mathematical scene for physical applications within conformal field theory.

In Chapter 8 we examine the form of conformal field theory n-point functions in $\mathrm{SL}(2,\mathbb{C})$ twistor space and two related spaces. We also make a number of comments on the twistor transform for $\mathbb{T}$.

Chapter 2

One and two-particle AdS_3 geometry

In this chapter we establish the basic notation and conventions for the three-dimensional anti de Sitter space, AdS_3. We start by discussing one realisation of AdS_3 and how the various isometries act in this realisation. We then establish our coordinate conventions for AdS_3 and various related spaces in order to construct the one-particle spacetimes in the following section. The two-particle spacetimes are then constructed in all generality where the spacetime has a non-zero spin parameter, but assuming that the spacetime does not have a black hole. Finally the special case where this spin parameter is zero is presented as the setting for the work in the subsequent chapters.

2.1 AdS_3 hyperboloid and the isometry group

Here we initially follow the conventions used in [22] although our Killing vectors are defined with a slightly different orientation.

AdS_3 can be realised as an embedded hyperboloid in $\mathbb{R}^{2,2}$. The metric on $\mathbb{R}^{2,2}$ is

$$ds^2 = -dU^2 - dV^2 + dX^2 + dY^2. \tag{2.1.1}$$

For $x = (U, V, X, Y) \in \mathbb{R}^{2,2}$ and $l > 0$ the equation of the hyperboloid is

$$\langle x, x \rangle_{\mathbb{R}^{2,2}} = -U^2 - V^2 + X^2 + Y^2 = -l^2. \tag{2.1.2}$$

For simplicity we will only consider "unit" AdS_3 such that $l = 1$ to avoid littering the formulae with a scale parameter.

AdS_3 is a maximally symmetric space in that it admits six linearly independent Killing vector fields (KVF's). A (standard) set of such KVF's are

$$U\partial_V - V\partial_U, \tag{2.1.3a}$$
$$X\partial_Y - Y\partial_X, \tag{2.1.3b}$$
$$U\partial_X + X\partial_U, \tag{2.1.3c}$$
$$U\partial_Y + Y\partial_U, \tag{2.1.3d}$$
$$V\partial_X + X\partial_V, \tag{2.1.3e}$$
$$V\partial_Y + Y\partial_V, \tag{2.1.3f}$$

and the isometry group generated by them is $O_c(2, 2)$, where the subscript c stands for the connected component. The six isometries generated by (2.1.3) (with a parameter θ) read explicitly as follows:
(U, V) rotations:

$$\begin{pmatrix} U \\ V \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}, \tag{2.1.4}$$

(X, Y) rotations:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}, \tag{2.1.5}$$

(U, X) boosts:

$$\begin{pmatrix} U \\ X \end{pmatrix} \mapsto \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} U \\ X \end{pmatrix}, \tag{2.1.6}$$

(U, Y) boosts:

$$\begin{pmatrix} U \\ Y \end{pmatrix} \mapsto \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} U \\ Y \end{pmatrix}, \tag{2.1.7}$$

(V, X) boosts:

$$\begin{pmatrix} V \\ X \end{pmatrix} \mapsto \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} V \\ X \end{pmatrix}, \tag{2.1.8}$$

(V, Y) boosts:

$$\begin{pmatrix} V \\ Y \end{pmatrix} \mapsto \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} V \\ Y \end{pmatrix}, \tag{2.1.9}$$

For our purposes, it is convenient to write the $O_c(2,2)$ action on AdS_3 by using the group decomposition

$$O_c(2,2) \cong (\mathrm{SU}(1,1) \times \mathrm{SU}(1,1)) /\mathbb{Z}_2, \tag{2.1.10}$$

and expressing a general element of AdS_3 by a matrix, $W \in \mathrm{SU}(1,1)$, such that

$$W = \begin{pmatrix} U + iV & X + iY \\ X - iY & U - iV \end{pmatrix}, \tag{2.1.11}$$

where

$$U^2 + V^2 - X^2 - Y^2 = 1. \tag{2.1.12}$$

We denote elements of the isometry group by pairs; $(g_L, g_R) \in \mathrm{SU}(1,1) \times \mathrm{SU}(1,1)$ and find that the action $O_c(2,2) : AdS_3 \to AdS_3$ is equivalent to $\mathrm{SU}(1,1) \times \mathrm{SU}(1,1) : \mathrm{SU}(1,1) \to \mathrm{SU}(1,1)$. Concretely, for $(g_L, g_R) \in \mathrm{SU}(1,1) \times \mathrm{SU}(1,1)$, the action is

$$W \mapsto W' = g_L W g_R^{-1}. \tag{2.1.13}$$

Note that the hyperboloid condition (2.1.12) is implemented by $\det(W) = 1$ and this condition is invariant under the action (2.1.13). The invariant metric (2.1.1) is given in this matrix representation by $\frac{1}{2} Tr \left[\left(W^{-1} \, \mathrm{d} W \right)^2 \right]$.

We now list, for the reader's convenience, the elements of $\mathrm{SU}(1,1) \times \mathrm{SU}(1,1)$ that give the six isometries (2.1.4) to (2.1.9) in the form (2.1.13).

Most of the calculations of the two-particle spacetimes will use these matrices and various compositions thereof.

(U, V) rotations (2.1.4):

$$g_L = \begin{pmatrix} \exp\left(\frac{i\theta}{2}\right) & 0 \\ 0 & \exp\left(-\frac{i\theta}{2}\right) \end{pmatrix}, \tag{2.1.14a}$$

$$g_R = \begin{pmatrix} \exp\left(-\frac{i\theta}{2}\right) & 0 \\ 0 & \exp\left(\frac{i\theta}{2}\right) \end{pmatrix}. \tag{2.1.14b}$$

(X, Y) rotations (2.1.5):

$$g_L = \begin{pmatrix} \exp\left(\frac{i\theta}{2}\right) & 0 \\ 0 & \exp\left(-\frac{i\theta}{2}\right) \end{pmatrix}, \tag{2.1.15a}$$

$$g_R = \begin{pmatrix} \exp\left(\frac{i\theta}{2}\right) & 0 \\ 0 & \exp\left(-\frac{i\theta}{2}\right) \end{pmatrix}. \tag{2.1.15b}$$

(U, X) boosts (2.1.6):

$$g_L = \begin{pmatrix} \cosh\left(\frac{\theta}{2}\right) & \sinh\left(\frac{\theta}{2}\right) \\ \sinh\left(\frac{\theta}{2}\right) & \cosh\left(\frac{\theta}{2}\right) \end{pmatrix}, \tag{2.1.16a}$$

$$g_R = \begin{pmatrix} \cosh\left(\frac{\theta}{2}\right) & -\sinh\left(\frac{\theta}{2}\right) \\ -\sinh\left(\frac{\theta}{2}\right) & \cosh\left(\frac{\theta}{2}\right) \end{pmatrix}. \tag{2.1.16b}$$

(U, Y) boosts (2.1.7):

$$g_L = \begin{pmatrix} \cosh\left(\frac{\theta}{2}\right) & i\sinh\left(\frac{\theta}{2}\right) \\ -i\sinh\left(\frac{\theta}{2}\right) & \cosh\left(\frac{\theta}{2}\right) \end{pmatrix}, \tag{2.1.17a}$$

$$g_R = \begin{pmatrix} \cosh\left(\frac{\theta}{2}\right) & -i\sinh\left(\frac{\theta}{2}\right) \\ i\sinh\left(\frac{\theta}{2}\right) & \cosh\left(\frac{\theta}{2}\right) \end{pmatrix}. \tag{2.1.17b}$$

(V, X) boosts (2.1.8):

$$g_L = \begin{pmatrix} \cosh\left(\frac{\theta}{2}\right) & i\sinh\left(\frac{\theta}{2}\right) \\ -i\sinh\left(\frac{\theta}{2}\right) & \cosh\left(\frac{\theta}{2}\right) \end{pmatrix}, \tag{2.1.18a}$$

$$g_R = \begin{pmatrix} \cosh\left(\frac{\theta}{2}\right) & i\sinh\left(\frac{\theta}{2}\right) \\ -i\sinh\left(\frac{\theta}{2}\right) & \cosh\left(\frac{\theta}{2}\right) \end{pmatrix}. \tag{2.1.18b}$$

(V, Y) boosts (2.1.9):

$$g_L = \begin{pmatrix} \cosh\left(\frac{\theta}{2}\right) & -\sinh\left(\frac{\theta}{2}\right) \\ -\sinh\left(\frac{\theta}{2}\right) & \cosh\left(\frac{\theta}{2}\right) \end{pmatrix}, \tag{2.1.19a}$$

$$g_R = \begin{pmatrix} \cosh\left(\frac{\theta}{2}\right) & -\sinh\left(\frac{\theta}{2}\right) \\ -\sinh\left(\frac{\theta}{2}\right) & \cosh\left(\frac{\theta}{2}\right) \end{pmatrix}. \tag{2.1.19b}$$

Finally, note that the isometries can be written in terms of real-valued matrices if SU(1, 1) is replaced by the isomorphic group SL(2, ℝ) (see, for example, [23]). The use of SU(1, 1) has however certain computational advantages for our purposes.

2.2 Coordinate definitions

A set of coordinates (T, R, ϕ) that covers all of AdS_3 is defined by

$$U = (1 + R^2)^{\frac{1}{2}} \cos T, \tag{2.2.1a}$$

$$V = (1 + R^2)^{\frac{1}{2}} \sin T, \tag{2.2.1b}$$

$$X = R \cos \phi, \tag{2.2.1c}$$

$$Y = R \sin \phi. \tag{2.2.1d}$$

The metric reads

$$ds^2 = -\left(1 + R^2\right) dT^2 + \left(1 + R^2\right)^{-1} dR^2 + R^2 d\phi^2. \tag{2.2.2}$$

Note that there is a coordinate singularity at $R = 0$ but this does not concern us here. As T is periodic with period 2π, we see that the spacetime has closed timelike curves. If we unwrap T, we obtain the universal covering space $CAdS_3$. Let us now consider this done and with an abuse of notation refer to this covering space as AdS_3. The coordinate ranges for this space are $-\infty < T < \infty$, $R \geq 0$ and $0 \leq \phi < 2\pi$.

Coordinates which will be used extensively in the sequel are the so-called "sausage" coordinates (T, ρ, ϕ) defined by

$$R = \frac{2\rho}{1-\rho^2} \tag{2.2.3}$$

with $-\infty < T < \infty$, $0 \leq \rho < 1$ and $0 \leq \phi < 2\pi$, yielding

$$ds^2 = -\left(\frac{1+\rho^2}{1-\rho^2}\right)^2 dT^2 + \left(\frac{2}{1-\rho^2}\right)^2 \left(d\rho^2 + \rho^2 d\phi^2\right). \tag{2.2.4}$$

The coordinates (T, ρ, ϕ) yield a simply visualisable picture of AdS_3 as an infinite stack of Poincaré disks of constant T - see Figure 2.1.

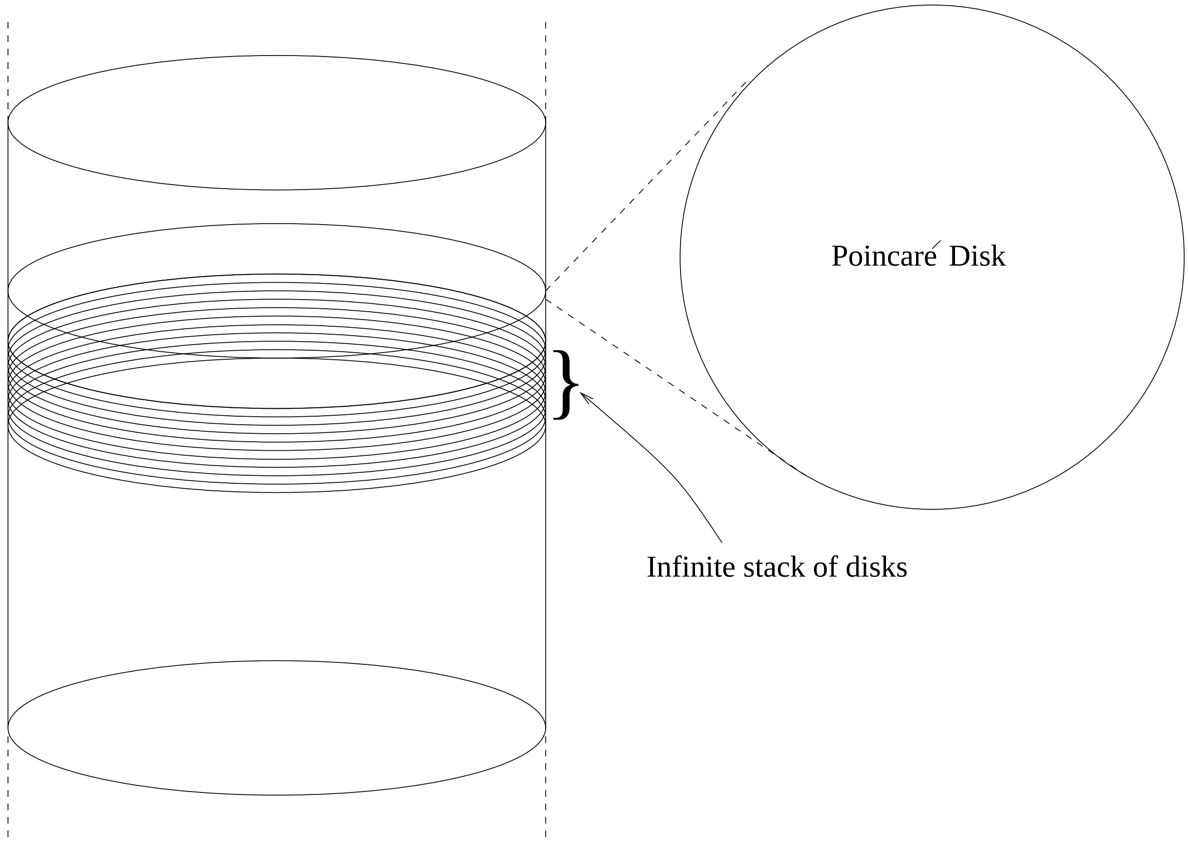

Figure 2.1: AdS_3 as an infinite stack of Poincaré disks. Each constant T slice of the cylinder has the metric (2.2.5).

Each constant T slice has the Poincaré disk metric

$$ds^2 = \left(\frac{2}{1-\rho^2}\right)^2 \left(d\rho^2 + \rho^2 d\phi^2\right), \tag{2.2.5}$$

which can be written in the more standard hyperbolic polar coordinates via the coordinate transformation $\rho = \tanh\left(\frac{w}{2}\right)$, yielding

$$ds^2 = dw^2 + \sinh^2 w \; d\phi^2. \tag{2.2.6}$$

As $\rho \to 1$ the metric (2.2.5) diverges - geodesic distances from points on the boundary of the disk to any other point on the disk are infinite.

Finally, we introduce the spinning BTZ coordinates (t, r, ψ), [22], via

$$T = \alpha t + S\psi, \tag{2.2.7a}$$

$$\phi = St + \alpha\psi, \tag{2.2.7b}$$

$$R^2 = \frac{r^2 + S^2}{\alpha^2 - S^2}, \tag{2.2.7c}$$

where α and S are parameters satisfying $\alpha > 0$ and $-\alpha < S < \alpha$. The metric reads

$$ds^2 = -\left(r^2 + S^2 + \alpha^2\right) dt^2 - 2S\alpha dt d\psi + \frac{r^2 dr^2}{\left(r^2 + S^2\right)\left(r^2 + \alpha^2\right)} + r^2 d\psi^2. \tag{2.2.8}$$

Setting $M = -\left(S^2 + \alpha^2\right)$ and $J = 2S\alpha$, the metric becomes

$$ds^2 = -\left(r^2 - M + \frac{J^2}{4r^2}\right) dt^2 + \frac{dr^2}{\left(r^2 - M + \frac{J^2}{4r^2}\right)} + r^2 \left(d\psi - \frac{J}{2r^2} dt\right)^2. \tag{2.2.9}$$

The metric (2.2.9) comes to us with the restriction $M < 0$. We could, however, start from (2.2.9) and ask what spacetime this metric describes for arbitrary values of M and J. The (partial) answer is that the continuation of (2.2.9) into the region where $M > 0$ but $|J| \leq M$, with the coordinates identified as $(t, r, \psi) \sim (t, r, \psi + 2\pi)$, describes the BTZ black hole analysed in [22]. We do not wish to say any more about this here but will return to the black hole in chapter 6.

2.3 Single spinning point particle - "AdS conical geometry"

We construct the single spinning particle spacetimes by adapting the discussion of [9] from Minkowski space to AdS_3. This will allow us to discuss the two-particle spacetimes and their structure at spacelike infinity.

We define AdS as the $(2+1)$-dimensional spacetime obtained by removing the timelike geodesic $R = 0$ from AdS_3 and $\widetilde{AdS}$ as the universal covering space of AdS. We introduce on $\widetilde{AdS}$ a set of global coordinates akin to the sausage coordinates (2.2.4), with the ϕ coordinate *unwound* around the particle worldline so that $T \in \mathbb{R}$, $0 < \rho < 1$ and $-\infty < \phi < \infty$. Due to the inhomogeneity introduced into the original space by removing a timelike geodesic, there are now only two independent isometries on $\widetilde{AdS}$, namely rotations in (U, V), generated by $\partial_T = -V\partial_U + U\partial_V$, and rotations in (X, Y), generated by $\partial_\phi = -Y\partial_X + X\partial_Y$. In the sausage coordinates these isometries are given by

$$J := \exp\left(2\pi S\partial_T + 2\pi\alpha\partial_\phi\right), \tag{2.3.1}$$

with the action

$$J : (T, \rho, \phi) \mapsto (T + 2\pi S, \rho, \phi + 2\pi\alpha) . \tag{2.3.2}$$

We now interpret $\widetilde{AdS}/\mathbb{Z}$, where the $\mathbb{Z}$ action is generated by (2.3.1), as the spacetime generated by a single spinning point particle at $\rho = 0$ [7–9, 24]. The mass of the particle equals $\pi(1-\alpha)$ and S gives the spin of the particle. $\widetilde{AdS}/\mathbb{Z}$ can be described in terms of a fundamental domain, D, and an identification across its boundaries, where the identification takes the form (2.3.2). If $\alpha < 1$, D can be embedded in AdS and the identification is a specific $O_c(2, 2)$ transformation of $\mathbb{R}^{2,2}$, namely a $2\pi\alpha$ rotation in (X, Y) (about the removed timelike geodesic) composed with a $2\pi S$ rotation in (U, V). We can then choose a wedge of the sausage for D - see Figure 2.2. The value $\alpha = 0$ is the threshold of black hole formation which we consider in chapter 6.

We introduce on $\widetilde{AdS}$ the coordinates (t, r, ψ) via (2.2.7) with the ψ coordinate unwound so that $-\infty < t < \infty$, $r > 0$ and $-\infty < \psi < \infty$. In these coordinates, the isometry J (2.3.1) reads

$$J := \exp\left(2\pi\partial_\psi\right), \tag{2.3.3}$$

$$J : (t, r, \psi) \mapsto (t, r, \psi + 2\pi) , \tag{2.3.4}$$

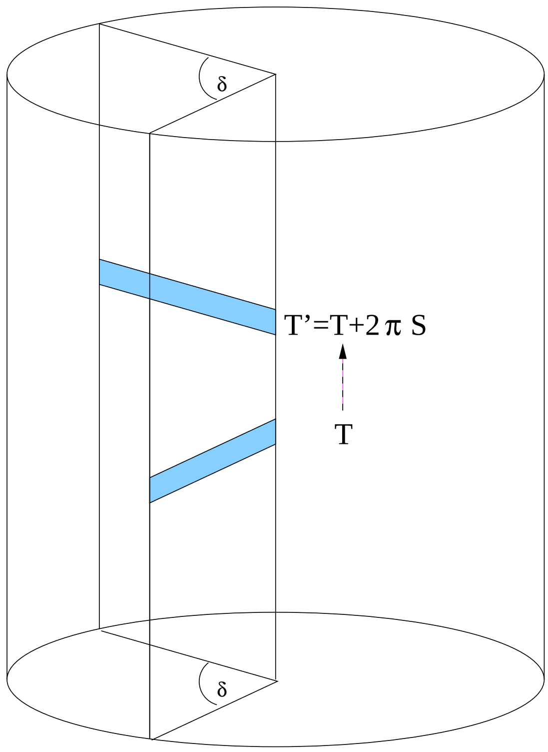

Figure 2.2: Cylindrical sausage with a particle wedge cut out. The wedge with angle $\delta := 2\pi\,(1-\alpha)$ is cut out of the spacetime leaving the fundamental domain for $\alpha < 1$ to the right of the cut out wedge. The identification of the timelike boundary is indicated by the shaded segments of the diagram. The particle mass is given by $\frac{\delta}{2}$ whereas the particle spin is given by S.

and we refer to the coordinates (t, r, ψ) with the identification $(t, r, \psi) \sim (t, r, \psi + 2\pi)$ as the AdS conical coordinates on $\widetilde{AdS}/\mathbb{Z}$.

2.4 Spacetime of two spinless particles

Now that we have established the main conventions for spacetimes with massive point particles we turn our attention to the main focus of this chapter — describing the geometry of two-particle AdS_3 spacetimes. (Note that this method of constructing spacetimes with particles can be extended to n particles for $n > 2$ but we will not do so here.)

We label the particles with an index $i \in \{1, 2\}$ and in a neighbourhood of each particle worldine the geometry is the spinless special case ($S = 0$) of section 2.3. We denote the defect angles of the particles by $\delta_i := 2\pi (1 - \alpha_i)$ and we define $c_i := \cos \frac{\delta_i}{2}$ and $s_i := \sin \frac{\delta_i}{2}$. The requirement that the particle masses are greater than zero give the inequalities $\delta_i > 0$. We also require that each particle is nothing more exotic than a point particle and so also set $\delta_i < 2\pi$. We further require that the geometry near the spacelike infinity is that of a single spinning point particle as described in section 2.3. This implies $\delta_1 + \delta_2 < 2\pi$ and $c_1 + c_2 > 0$, as in the case of a vanishing cosmological constant [9], and also a further condition, specific to a negative cosmological constant, which will emerge at the end of the section as (2.4.11).

What remains is to describe the geometry of the two-particle spacetime in terms of a fundamental domain $\widetilde{\Omega}_0$ - a piece of AdS_3 spacetime *between* the particles. We will first do so, but we will then translate this picture into an equivalent one in which the properties of the spacelike infinity are more apparent.

Without loss of generality we may assume the worldline of particle 1 to be at the centre of the sausage evolving *straight up* (as in Figure 2.2). We introduce the notation $B(w_1, w_2)$ for a boost parameter pair (w_1, w_2) as the composition of a (U, X) boost with parameter w_1 and a (V, Y) boost with parameter w_2;

$$B(w_1, w_2) : \begin{pmatrix} U \\ V \\ X \\ Y \end{pmatrix} \mapsto \begin{pmatrix} \cosh w_1 \, U + \sinh w_1 \, X \\ \cosh w_2 \, V + \sinh w_2 \, Y \\ \sinh w_1 \, U + \cosh w_1 \, X \\ \sinh w_2 \, V + \cosh w_2 \, Y \end{pmatrix}. \tag{2.4.1}$$

The worldline of particle 2 is obtained by taking a worldline at the centre of the sausage and transforming via the inverse of (2.4.1) with the pair (β, v), where $\beta \neq 0$ and $v \neq 0$;

$$B^{-1}(\beta, v) : \begin{pmatrix} U \\ V \\ X \\ Y \end{pmatrix} \mapsto \begin{pmatrix} \cosh\beta \, U - \sinh\beta \, X \\ \cosh v \, V - \sinh v \, Y \\ -\sinh\beta \, U + \cosh\beta \, X \\ -\sinh v \, V + \cosh v \, Y \end{pmatrix}. \tag{2.4.2}$$

The two defect angles combined with their relevant boost parameter pair (β, v) at $T = 0$ give us the *initial data* of the system. See Figure 2.3 for a cross-section of the sausage at $T = 0$ showing the beginning of the evolution. Note that we have chosen $\beta > 0$ and $v > 0$ for Figure 2.3 and all subsequent figures. The analysis in this section holds for $\beta \neq 0$ and $v \neq 0$ but the figures are drawn for $\beta > 0$ and $v > 0$.

From Figure 2.3 it is clear that we will want to define $\widetilde{\Omega}_0$ so that the particle worldlines are timelike geodesics on the boundary of $\widetilde{\Omega}_0$. We denote the worldline of particle i by P_i with corresponding proper time λ_i, with the zeroes chosen so that $\lambda_i = 0$ at $T = 0$. The P_i are given by

$$P_1 = \begin{pmatrix} U_1 \\ V_1 \\ X_1 \\ Y_1 \end{pmatrix} = \begin{pmatrix} \cos\frac{\lambda_1}{2} \\ \sin\frac{\lambda_1}{2} \\ 0 \\ 0 \end{pmatrix}, \tag{2.4.3}$$

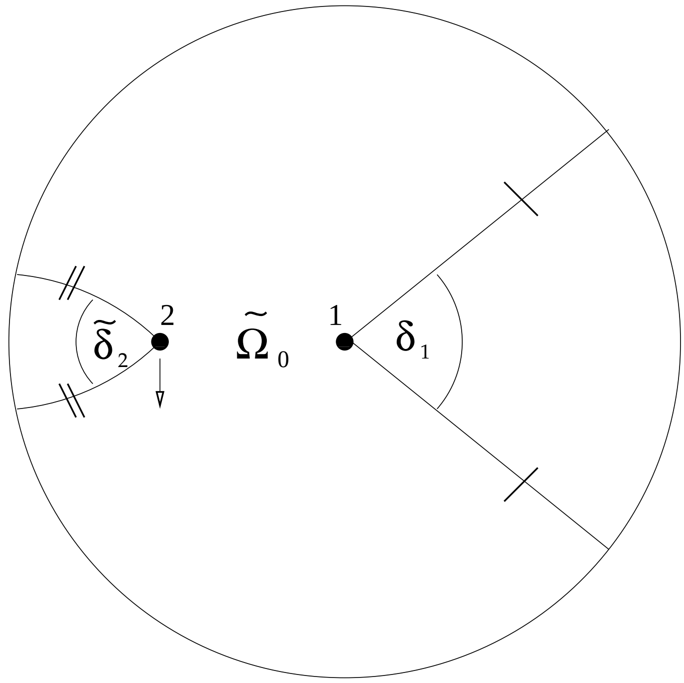

Figure 2.3: Initial data slice $T = 0$. The particles are located on this constant T slice as shown. The single and double stroked lines are the restriction of the relative single and double stroked boundaries of $\widetilde{\Omega}_0$ to this constant T slice. The single stroked boundaries are identified by an (X, Y) rotation on constant T slices. The double stroked boundaries are identified by an (X, Y) rotation conjugated by a boost (2.4.1) with boost-pair (β, v), in a way that, for $v \neq 0$, does not preserve the constant T slices and is discussed in more detail in section 2.5. $\widetilde{\delta_2}$ can be given in terms of δ_2 and the boost parameter pair (β, v) but is not particularly important for our subsequent analysis. The arrow attached to particle 2 indicates its velocity at $T = 0$. $\widetilde{\Omega}_0$ is shown on the figure between the removed wedges. Note that $\widetilde{\Omega}_0$ reaches the infinity in two disconnected parts and is therefore not well adapted to describing the spacelike infinity.

$$P_2 = \begin{pmatrix} U_2 \\ V_2 \\ X_2 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \cosh\beta\cos\frac{\lambda_2}{2} \\ \cosh\upsilon\sin\frac{\lambda_2}{2} \\ -\sinh\beta\cos\frac{\lambda_2}{2} \\ -\sinh\upsilon\sin\frac{\lambda_2}{2} \end{pmatrix}. \tag{2.4.4}$$

The particles evolve as shown in Figure 2.4. Elementary geometry shows that

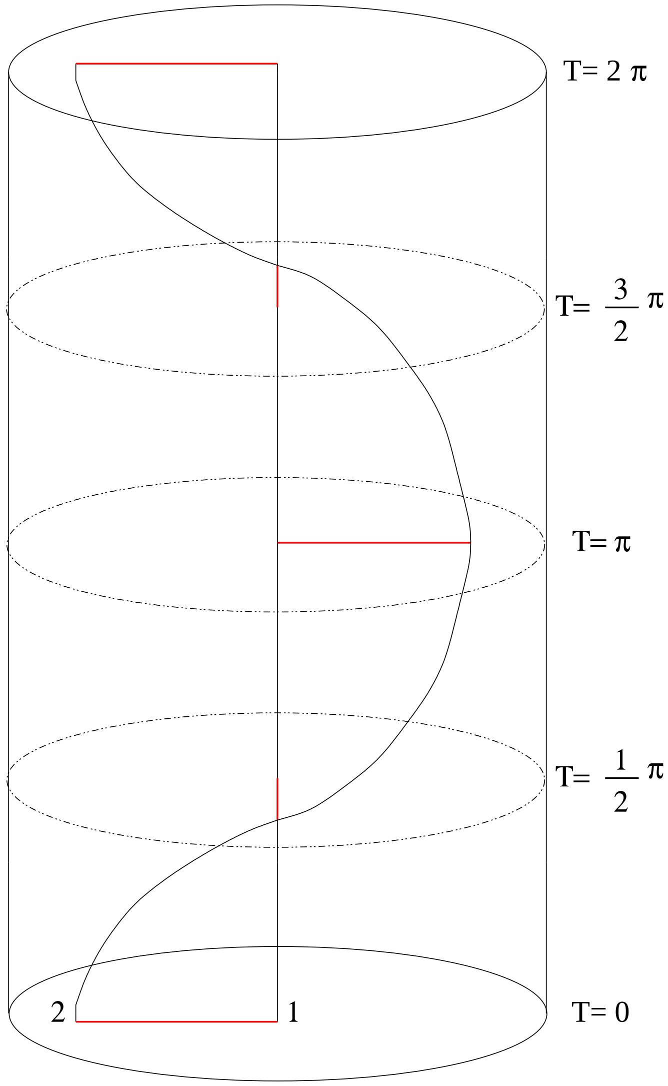

Figure 2.4: Evolution of the particles. Particle 2 orbits the worldline of particle 1 in a helix-like manner. The evolution is periodic with period $T = 2\pi$.

the geodesic distance s between points with given λ_1 and λ_2 is given by

$$\cosh s = \cos\frac{\lambda_1}{2}\cos\frac{\lambda_2}{2}\cosh\beta + \sin\frac{\lambda_1}{2}\sin\frac{\lambda_2}{2}\cosh v. \qquad (2.4.5)$$

We now wish to translate this description into one anchored to the AdS conical infinity. We omit the calculations but give a prescription by which the interested reader can reproduce them.

- Write the worldlines P_i in the SU$(1,1)$ matrix form W_i according to (2.1.11). Write the identification of the double-stroked boundaries in Figure 2.3 as a pair of SU$(1,1)$ matrices.

- Cut $\widetilde{\Omega}_0$ into two along a timelike surface connecting the particle worldlines in a way whose details will be specified in section 2.5. Rotate the two halves of $\widetilde{\Omega}_0$, via a pair of SU$(1,1)$ matrices, about the worldline of particle 1 so that the wedge originally at particle 1 closes and a new one opens. Keep track of the form of the W_i and the SU$(1,1)$ pair identifying the double-stroked boundaries. Figure 2.5 shows the new domain $\widetilde{\Omega}'_0$ after this "cut and rotate" process - Particle 2 is now at two positions in this new picture, labelled by 2 and $2'$.

- Perform a final isometry on the spacetime such that the double-stroked boundaries are now identified by a pair of matrices implementing the composition of an (X,Y) rotation and a (U,V) rotation. This identification is now in the form (2.3.1), anchoring the system to the infinity. This final isometry will be given by a transformation of the type (2.4.1) with a boost parameter pair (β_1, v_1). The W_i can now be expressed in terms of two new boost parameter pairs (β_1, v_1) and (β_2, v_2) where both pairs can be given in terms of the initial data. See Figure 2.6 for the initial data in the new fundamental domain $\widetilde{\Omega}$.

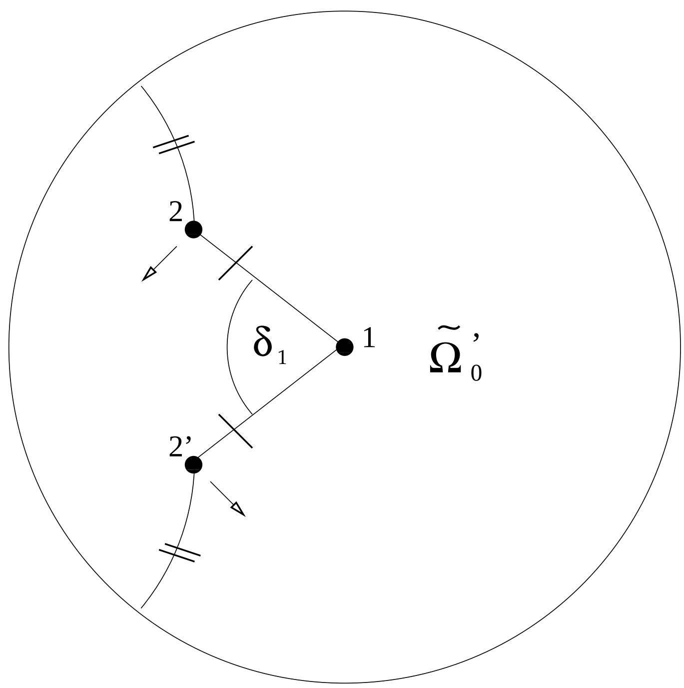

Figure 2.5: $T = 0$ slice after "cut and rotate" process. The single-stroked boundaries are still identified by an (X, Y) rotation whereas the double-stroked boundaries are identified as before but conjugated by an additional (X, Y) rotation. Again, the identification of the double stroked boundaries does not preserve the constant T slices when $v \neq 0$.

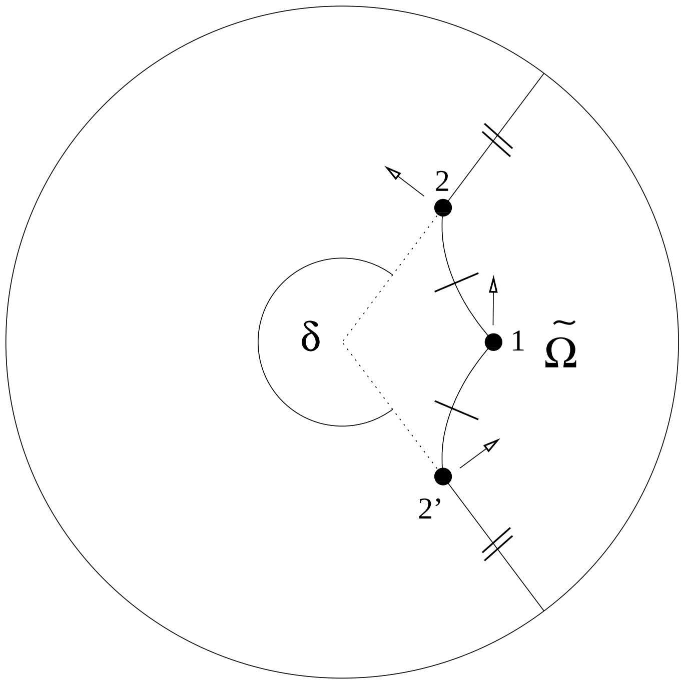

Figure 2.6: The "centre of mass" frame. The single-stroked boundaries of the fundamental domain $\widetilde{\Omega}$ are identified by an (X, Y) rotation conjugated by a boost (2.4.1) with boost-pair (β_1, v_1) given by (2.4.8). The double-stroked lines are identified by the composition of an (X, Y) rotation and a (U, V) rotation as shown by the grey segments in Figure 2.2. Note that particle 1 is in the $T = 0$ plane, but particles 2 and 2' are in this plane only for $v = 0$.

In $\widetilde{\Omega}$, the worldlines of the particles read

$$P_1 = \begin{pmatrix} U_1 \\ V_1 \\ X_1 \\ Y_1 \end{pmatrix} = \begin{pmatrix} \cos\frac{\lambda_1}{2}\cosh\beta_1 \\ \sin\frac{\lambda_1}{2}\cosh\upsilon_1 \\ \cos\frac{\lambda_1}{2}\sinh\beta_1 \\ \sin\frac{\lambda_1}{2}\sinh\upsilon_1 \end{pmatrix}, \tag{2.4.6}$$

$$P_{2,2'} = \begin{pmatrix} U_{2,2'} \\ V_{2,2'} \\ X_{2,2'} \\ Y_{2,2'} \end{pmatrix} = \begin{pmatrix} \cos\frac{\lambda_2}{2}\cosh\beta_2\cos\frac{\tau}{2} \mp \sin\frac{\lambda_2}{2}\cosh\upsilon_2\sin\frac{\tau}{2} \\ \pm\cos\frac{\lambda_2}{2}\cosh\beta_2\sin\frac{\tau}{2} + \sin\frac{\lambda_2}{2}\cosh\upsilon_2\cos\frac{\tau}{2} \\ -\cos\frac{\lambda_2}{2}\sinh\beta_2\cos\frac{\delta}{2} \mp \sin\frac{\lambda_2}{2}\sinh\upsilon_2\sin\frac{\delta}{2} \\ \pm\cos\frac{\lambda_2}{2}\sinh\beta_2\sin\frac{\delta}{2} - \sin\frac{\lambda_2}{2}\sinh\upsilon_2\cos\frac{\delta}{2} \end{pmatrix}, \tag{2.4.7}$$

where the upper (lower) signs pertain to particles 2 (2'), and the parameters β_i, υ_i, δ and τ are determined in terms of β, υ and δ_i by

$$\tanh(\upsilon_1 \pm \beta_1) = \frac{s_2\sinh(\upsilon \pm \beta)}{s_1c_2 + s_2c_1\cosh(\upsilon \pm \beta)}, \tag{2.4.8}$$

$$\tanh(\upsilon_2 \pm \beta_2) = \frac{s_1\sinh(\upsilon \pm \beta)}{s_2c_1 + s_1c_2\cosh(\upsilon \pm \beta)}, \tag{2.4.9}$$

$$\cos\left(\frac{\delta \pm \tau}{2}\right) = c_1c_2 - s_1s_2\cosh(\upsilon \pm \beta). \tag{2.4.10}$$

Equation (2.4.10) shows that the manipulations to specify the new fundamental domain $\widetilde{\Omega}$ are well defined provided

$$|c_1c_2 - s_1s_2\cosh(\upsilon \pm \beta)| < 1, \tag{2.4.11}$$

ensuring that Figure 2.6 exists. We assume (2.4.11) for now, but will relax this condition in Chapter 6 when we discuss the black hole parameter range.

2.5 Worldlines in other coordinatisations and the equations of motion

What is not portrayable in the spatial slices in Figures 2.3, 2.5 and 2.6 is the non-planar nature of the identified boundaries. For example, in Figure 2.6 the single-stroked boundary is identified on a slice of varying T where, if we take the line starting from particle 1 to be at $T = 0$, T increases (decreases) as we travel along the line from particle 1 to particle 2 (2'). The double-stroked boundary from 2 (2') to the edge of the disk is on a constant T slice where $T > 0$ ($T < 0$).

With this in mind we introduce a new parameter σ defined along the particle worldlines by

$$\tan\sigma = \tan\left(\frac{\lambda_1}{2}\right)\frac{\sinh v_1}{\sinh\beta_1} = \tan\left(\frac{\lambda_2}{2}\right)\frac{\sinh v_2}{\sinh\beta_2}, \tag{2.5.1}$$

such that

$$-\frac{\pi}{2} < \sigma < \frac{\pi}{2} \tag{2.5.2}$$

and $\sigma = 0$ at $T = 0$. Note that σ is well-defined only when $v \neq 0$. We consider the case where $v = 0$ in section 2.6.

We now rewrite the worldlines of the particles (2.4.6) and (2.4.7) in the sausage coordinates (2.2.4) in terms of σ, (abusing notation for the P_i somewhat),

$$P_1 = \begin{pmatrix} T_1 \\ \rho_1 \\ \phi_1 \end{pmatrix} = \begin{pmatrix} \arctan\left(\dfrac{\tan\sigma\tanh\beta_1}{\tanh v_1}\right) \\ \rho_1(\sigma) \\ \sigma \end{pmatrix}, \tag{2.5.3}$$

$$P_{2,2'} = \begin{pmatrix} T_{2,2'} \\ \rho_{2,2'} \\ \phi_{2,2'} \end{pmatrix} = \begin{pmatrix} \arctan\left(\dfrac{\tan\sigma\tanh\beta_2}{\tanh v_2}\right) \pm \frac{\tau}{2} \\ \rho_2(\sigma) \\ \sigma \pm \pi\alpha \end{pmatrix}, \tag{2.5.4}$$

where

$$\rho_i(\sigma) = \left(\frac{\left(\cosh^2\beta_i\sinh^2 v_i + \tan^2\sigma\cosh^2 v_i\sinh^2\beta_i\right)^{\frac{1}{2}} - \left(\sinh^2 v_i + \tan^2\sigma\sinh^2\beta_i\right)^{\frac{1}{2}}}{\left(\cosh^2\beta_i\sinh^2 v_i + \tan^2\sigma\cosh^2 v_i\sinh^2\beta_i\right)^{\frac{1}{2}} + \left(\sinh^2 v_i + \tan^2\sigma\sinh^2\beta_i\right)^{\frac{1}{2}}}\right)^{\frac{1}{2}}, \tag{2.5.5}$$

The equation for the geodesic distances between the particles (2.4.5), with the same value of σ on each worldline, becomes

$$\cosh s = \frac{\cos^2\sigma \cosh\beta \sinh v_1 \sinh v_2 + \sin^2\sigma \cosh v \sinh\beta_1 \sinh\beta_2}{\left(\cos^2\sigma \sinh^2 v_1 + \sin^2\sigma \sinh^2\beta_1\right)^{\frac{1}{2}} \left(\cos^2\sigma \sinh^2 v_2 + \sin^2\sigma \sinh^2\beta_2\right)^{\frac{1}{2}}}. \tag{2.5.6}$$

This specifies how $\widetilde{\Omega}_0$ was originally cut into two between the particles: the surface is formed from spacelike geodesics connecting the particles at the same value of σ at each end.

Finally we introduce the *AdS* conical coordinates (2.2.7a - 2.2.7c) in a neighbourhood of the infinity but replacing t by $t - t_0$ and ψ by $\psi - \psi_0$. The worldlines become

$$P_1 = \begin{pmatrix} t_1 \\ r_1 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} t_0 + \dfrac{\alpha \arctan\left(\dfrac{\tan\sigma \tanh\beta_1}{\tanh v_1}\right) - S\sigma}{\alpha^2 - S^2} \\ r_1(\sigma) \\ \psi_0 + \dfrac{-S \arctan\left(\dfrac{\tan\sigma \tanh\beta_1}{\tanh v_1}\right) + \alpha\sigma}{\alpha^2 - S^2} \end{pmatrix}, \tag{2.5.7}$$

$$P_{2,2'} = \begin{pmatrix} t_{2,2'} \\ r_{2,2'} \\ \psi_{2,2'} \end{pmatrix} = \begin{pmatrix} t_0 + \dfrac{\alpha \arctan\left(\dfrac{\tan\sigma \tanh\beta_2}{\tanh v_2}\right) - S\sigma}{\alpha^2 - S^2} \\ r_2(\sigma) \\ \psi_0 + \dfrac{-S \arctan\left(\dfrac{\tan\sigma \tanh\beta_2}{\tanh v_2}\right) + \alpha\sigma}{\alpha^2 - S^2} \pm \pi \end{pmatrix}, \tag{2.5.8}$$

where

$$r_i(\sigma) = \left(\frac{\alpha^2 \sinh^2 v_i \sinh^2\beta_i - S^2\left(\cos^2\sigma \sinh^2 v_i + \sin^2\sigma \sinh^2\beta_i + \sinh^2 v_i \sinh^2\beta_i\right)}{\cos^2\sigma \sinh^2 v_i + \sin^2\sigma \sinh^2\beta_i}\right)^{\frac{1}{2}}, \tag{2.5.9}$$

$$\alpha = 1 - \frac{\delta}{2\pi}, \tag{2.5.10}$$

$$S = 1 - \frac{\tau}{2\pi}. \tag{2.5.11}$$

Note also that

$$\tanh\beta_1 \tanh v_2 = \tanh\beta_2 \tanh v_1, \tag{2.5.12}$$

so that

$$t_1 = t_{2,2'} \tag{2.5.13}$$

and

$$\psi_1 = \psi_{2,2'} \mp \pi. \tag{2.5.14}$$

The constants $\{t_0, \psi_0\}$ give the AdS conical time and conical angle respectively when $\sigma = 0$. They encode the zero-point of time and the orientation of the two-particle system relative to the AdS conical coordinates.

We can extend the formulae (2.5.3) to (2.5.14) defined for (2.5.2) to the range $-\infty < \sigma < \infty$ (to describe the full evolution) by adding π $(-\pi)$ to T whenever σ increases (decreases) through the divergent points of $\tan\sigma$.

2.6 The colliding case

We now address the special case $v = 0$, in which the particles collide and which was not covered by the discussion in section 2.5.

The initial data of the system is given by the two defect angles and the boost parameter β. See Figure 2.7 for this initial configuration. The P_i are given by

$$P_1 = \begin{pmatrix} U_1 \\ V_1 \\ X_1 \\ Y_1 \end{pmatrix} = \begin{pmatrix} \cos\frac{\lambda_1}{2} \\ \sin\frac{\lambda_1}{2} \\ 0 \\ 0 \end{pmatrix}, \tag{2.6.1}$$

$$P_2 = \begin{pmatrix} U_2 \\ V_2 \\ X_2 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \cosh\beta\cos\frac{\lambda_2}{2} \\ \sin\frac{\lambda_2}{2} \\ -\sinh\beta\cos\frac{\lambda_2}{2} \\ 0 \end{pmatrix}. \tag{2.6.2}$$

The procedure to translate the fundamental domain into the centre-of-mass frame is essentially the same as before. $\widetilde{\Omega}_0$ is now initially cut along lines of constant T. The final fundamental domain, $\widetilde{\Omega}$, is shown in Figure 2.8. The

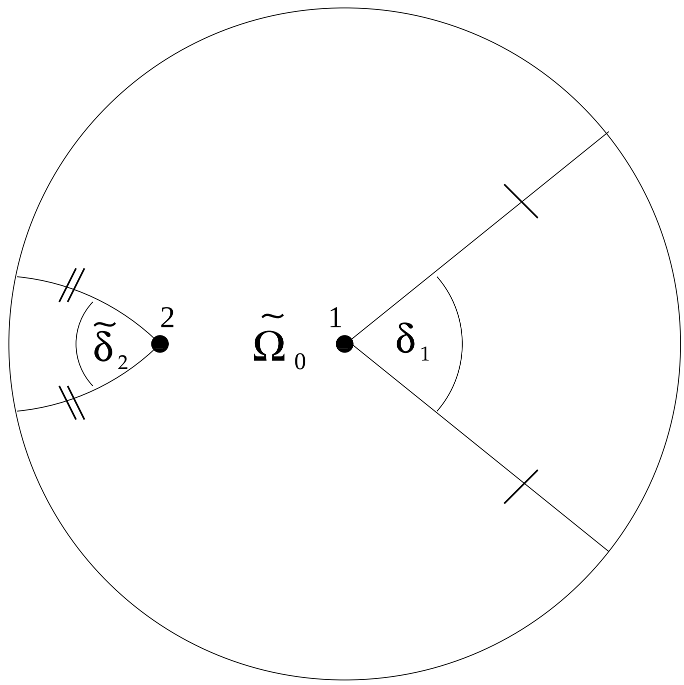

Figure 2.7: Initial data slice $T = 0$ for $v = 0$. The particles in the initial configuration are located as shown. The single and double stroked lines are geodesics identified *on* the Poincaré disk. The single stroked boundaries are identified by an (X, Y) rotation on constant T slices. The double stroked boundaries are identified by a (X, Y) rotation conjugated by the transformation (2.4.1) with $w_1 = \beta$, $w_2 = v = 0$ on constant T slices. The velocity of particle 2 at $T = 0$ is orthogonal to the $T = 0$ surface. $\widetilde{\Omega}_0$ is shown on the figure between the removed wedges.

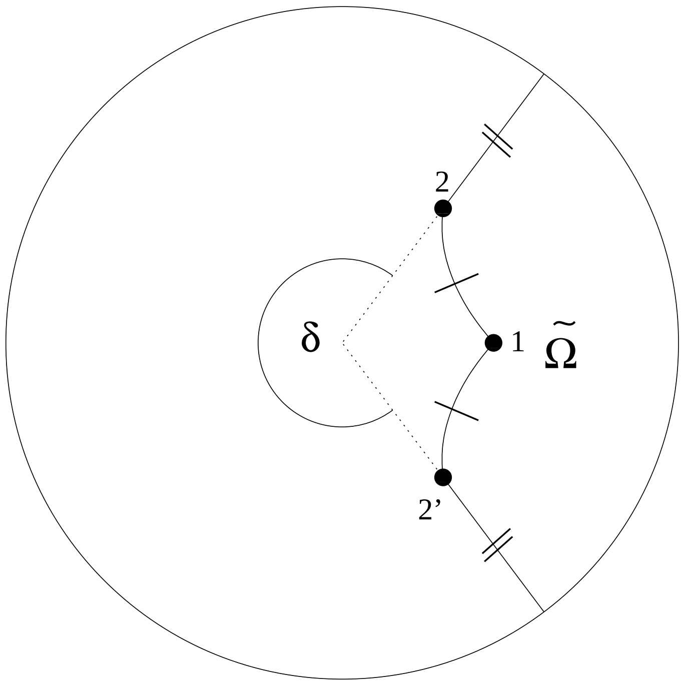

Figure 2.8: $T = 0$ slice in the "centre of mass" frame for $v = 0$. The single-stroked boundaries are identified by an (X, Y) rotation conjugated by a boost (2.4.1) with $w_1 = \beta_1$ and $w_2 = 0$ on constant T slices. The double stroked lines are identified by a simple (X, Y) rotation. At $T = 0$, the velocities of particle 1 and the two copies of particle 2 are orthogonal to the $T = 0$ surface.

worldlines of the particles on the boundary of $\widetilde{\Omega}$ become

$$P_1 = \begin{pmatrix} U_1 \\ V_1 \\ X_1 \\ Y_1 \end{pmatrix} = \begin{pmatrix} \cos\frac{\lambda_1}{2}\cosh\beta_1 \\ \sin\frac{\lambda_1}{2} \\ \cos\frac{\lambda_1}{2}\sinh\beta_1 \\ 0 \end{pmatrix}, \tag{2.6.3}$$

$$P_{2,2'} = \left(U_{2,2'}, V_{2,2'}, X_{2,2'}, Y_{2,2'}\right) = \begin{pmatrix} \cos\frac{\lambda_2}{2}\cosh\beta_2 \\ \sin\frac{\lambda_2}{2} \\ -\cos\frac{\lambda_2}{2}\sinh\beta_2\cos\frac{\delta}{2} \\ \pm\cos\frac{\lambda_2}{2}\sinh\beta_2\sin\frac{\delta}{2} \end{pmatrix}, \tag{2.6.4}$$

where the upper (lower) signs pertain to particle 2 (2'). The relevant boost parameters and the total deficit angle are given by

$$\tanh\beta_1 = \frac{s_2\sinh\beta}{s_1c_2 + s_2c_1\cosh\beta}, \tag{2.6.5}$$

$$\tanh\beta_2 = \frac{s_1\sinh\beta}{s_2c_1 + s_1c_2\cosh\beta}, \tag{2.6.6}$$

$$\cos\frac{\delta}{2} = c_1c_2 - s_1s_2\cosh\beta. \tag{2.6.7}$$

Note that (2.6.7) shows that the particle geometry near the infinity is the spinless $S = 0$ special case of the one-particle geometry. The dynamics are only defined in the range $-\frac{\pi}{2} \leq T \leq \frac{\pi}{2}$, where $\pm\frac{\pi}{2}$ are the values of T at which the particles collide respectively in the future and in the past.

In the sausage coordinates the particle worldlines take the form

$$P_1 = (T_1, \rho_1, \phi_1) = \begin{pmatrix} T \\ \rho_1(T) \\ 0 \end{pmatrix}, \tag{2.6.8}$$

$$P_{2,2'} = \left(T_{2,2'}, \rho_{2,2'}, \phi_{2,2'}\right) = \begin{pmatrix} T \\ \rho_2(T) \\ \pm\pi\alpha \end{pmatrix}, \tag{2.6.9}$$

where

$$\rho_i(T) = \left(\frac{\cosh\beta_i - \left(\cos^2 T + \sin^2 T \cosh^2 \beta_i\right)^{\frac{1}{2}}}{\cosh\beta_i + \left(\cos^2 T + \sin^2 T \cosh^2 \beta_i\right)^{\frac{1}{2}}} \right)^{\frac{1}{2}}, \tag{2.6.10}$$

which is well-defined for all $|T| < \frac{\pi}{2}$. Finally, in the AdS conical coordinates we replace t by $t - t_0$ and the worldlines take the form

$$P_1 = (t_1, r_1, \psi_1) = \begin{pmatrix} t_0 + \dfrac{T}{\alpha} \\ \dfrac{\alpha \cos T \sinh\beta_1}{\left(\cos^2 T + \sin^2 T \cosh^2 \beta_1\right)^{\frac{1}{2}}} \\ 0 \end{pmatrix}, \tag{2.6.11}$$

$$P_{2,2'} = \left(t_{2,2'}, r_{2,2'}, \psi_{2,2'}\right) = \begin{pmatrix} t_0 + \dfrac{T}{\alpha} \\ \dfrac{\alpha \cos T \sinh\beta_2}{\left(\cos^2 T + \sin^2 T \cosh^2 \beta_2\right)^{\frac{1}{2}}} \\ \pm\pi \end{pmatrix}. \tag{2.6.12}$$

Clearly, $t_1 = t_{2,2'}$ and $\psi_1 = \psi_{2,2'} \mp \pi$. The geodesic distance r_c between the particles in a constant T slice is

$$\cosh r_c = \frac{\cos^2 T \cosh\beta + \sin^2 T \cosh\beta_1 \cosh\beta_2}{\left(\cos^2 T + \sin^2 T \cosh^2 \beta_1\right)^{\frac{1}{2}} \left(\cos^2 T + \sin^2 T \cosh^2 \beta_2\right)^{\frac{1}{2}}}. \tag{2.6.13}$$

Note that t_0 is the AdS conical time when $T = 0$. The conical angle ψ_0 has been dropped as it will not be needed in the reduction of the action in Chapter 6.

Figures 2.9 and 2.10 show plots of the centre-of-mass fundamental domain $\widetilde{\Omega}$ with specific choices for the parameters. The evolution begins at $T = 0$ at the base of the plotted cylinder and evolves to collision at $T = \frac{\pi}{2}$ at the top of the cylinder.

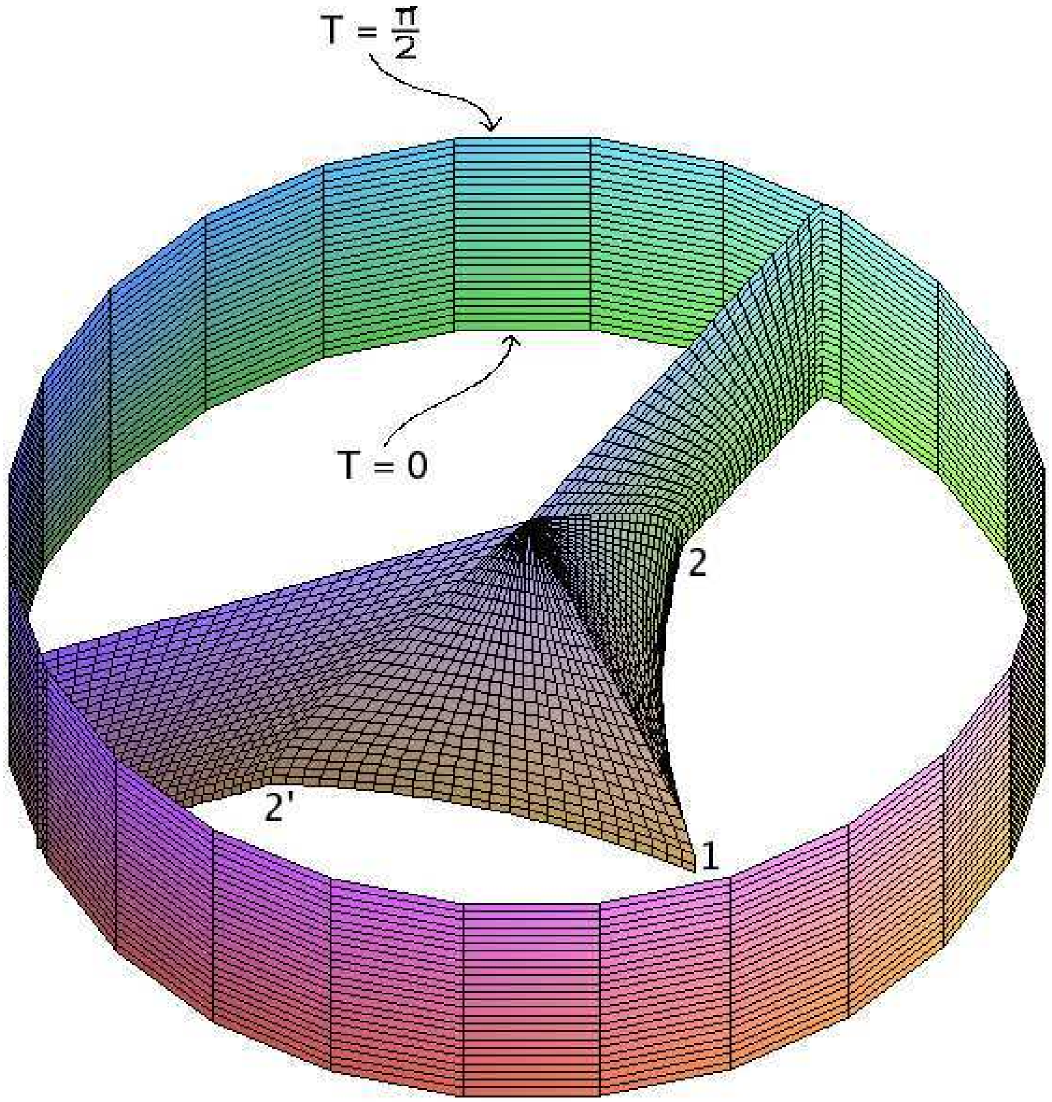

Figure 2.9: Two dimensional surface formed from drawing the boundary of the fundamental domain $\widetilde{\Omega}$ $\forall T \in [0, \frac{\pi}{2}]$, for $\delta_1 = \delta_2 = \frac{\pi}{2}$, $\beta = \ln(2 + \sqrt{3})$ and $S = 0$. $T = 0$ is at the base of the cylinder where the two particles are indicated. The point of collision is at $T = \frac{\pi}{2}$ at the top of the cylinder.

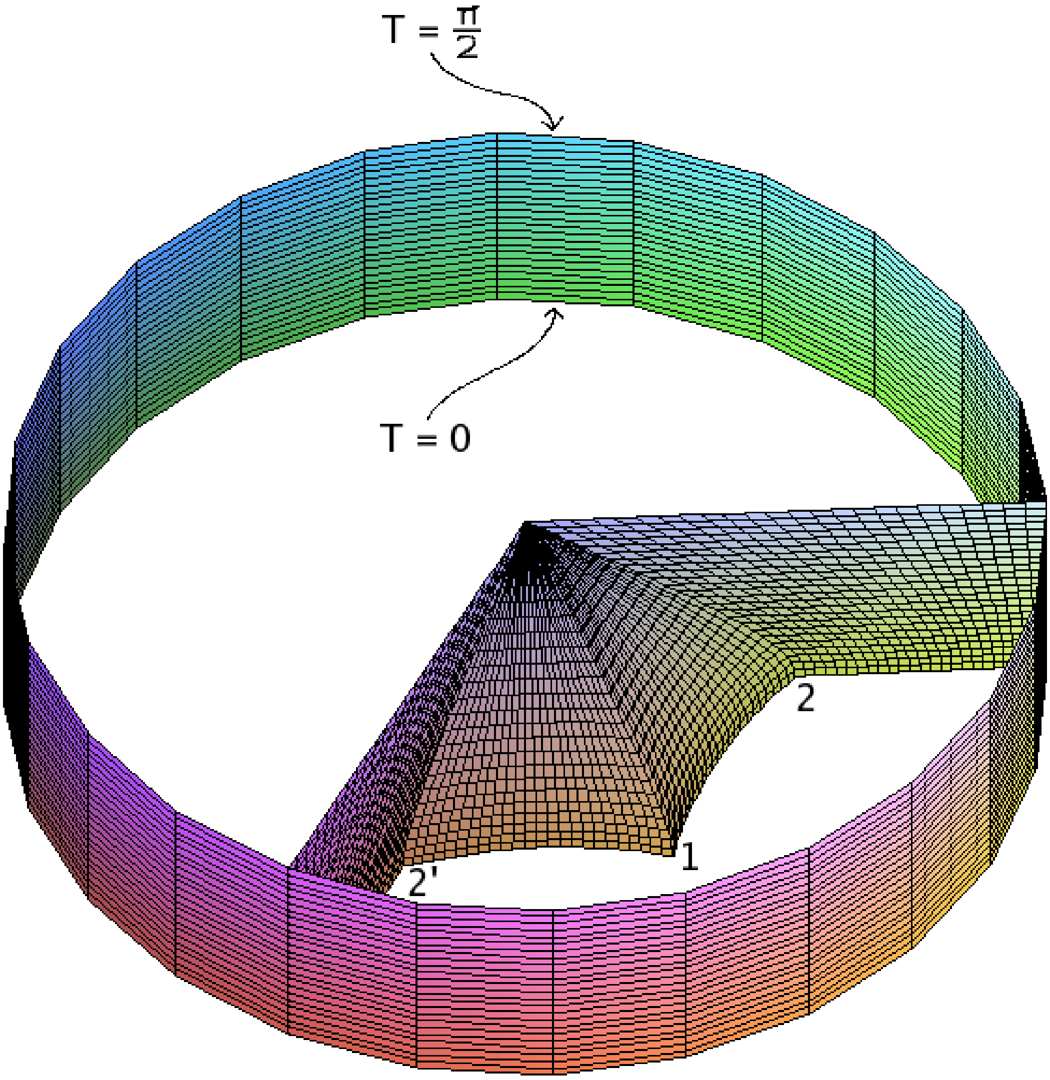

Figure 2.10: Two dimensional surface formed from drawing the boundary of the fundamental domain $\widetilde{\Omega}$ $\forall T \in [0, \frac{\pi}{2}]$, for $\delta_1 = \delta_2 = \frac{\pi}{4}$, $\beta = 2$ and $S = 0$. $T = 0$ is at the base of the cylinder where the two particles are indicated. The point of collision is at $T = \frac{\pi}{2}$ at the top of the cylinder.

Chapter 3

Action for $(2+1)$-dimensional gravity with a negative cosmological constant

In this chapter we discuss the first order formulation of $(2+1)$-dimensional Einstein gravity with a negative cosmological constant in terms of two Chern-Simons connections. We identify the gauge transformations for the Chern-Simons connections, in order to make use of them in the Hamiltonian reduction in Chapters 4 and 5. Finally we discuss the contributions to the action from the point particles and from the boundary term at infinity.

3.1 First order formalism

We consider a three dimensional manifold M. The basic dynamical variables in the first order formalism are the co-triad e_a^I and the $O(2,1)$ connection A_a^I on M. The upper case latin letters $I, J, K, \ldots$ denote internal indices taking values in $\{0,1,2\}$, pertaining to a 3-dimensional fixed internal vector space V. The internal indices are raised and lowered with a fixed Minkowski metric η_{IJ} with signature $(-,+,+)$. The lower case latin letters $a, b, c, \ldots$ are spacetime indices.

We assume from now on that the co-triad is non-degenerate, $\det\left(e_a^I\right) \neq 0$. At any point p in M the co-triad provides then a linear isomorphism between the tangent space of M and the internal space V, and we can construct from the co-triad a spacetime metric of signature $(-,+,+)$ by $g_{ab} = \eta_{IJ} e_a^I e_b^J$. We use g_{ab} to raise and lower the spacetime indices. For generalisations to a degenerate co-triad, see [25].

We write the gravitational bulk action as a function of the dynamical variables (following the notation in [9]) as

$$S_{\text{bulk}} = \frac{1}{2\pi} \int_M d^3x \; \tilde{\eta}^{abc} e_{aI} \left(F_{bc}^I + \frac{1}{3} \epsilon^I{}_{JK} e_b^J e_c^K \right), \tag{3.1.1}$$

where our units are such that the cosmological constant $\Lambda = -\frac{1}{l^2} = -1$ and $8G = 1$ (following [22] with $l = 1$). $\tilde{\eta}^{abc}$ is the Levi-Civita density according to $d^3x \; \tilde{\eta}^{abc} = dx^a \wedge dx^b \wedge dx^c$ and F_{bc}^I is the curvature of the connection,

$$F_{bc}^I = 2\partial_{[b} A_{c]}^I + \epsilon^I{}_{JK} A_b^J A_c^K, \tag{3.1.2}$$

where

$$A_{[a} B_{b]} := \frac{1}{2} \left(A_a B_b - A_b B_a \right). \tag{3.1.3}$$

Here ϵ_{IJK}, with all lower indices, is the totally antisymmetric symbol with $\epsilon_{012} = 1$ and the indices are raised and lowered with the Minkowski metric η_{IJ}.

The equations of motion obtained by varying the action (3.1.1) with respect to A_a^I and e_a^I are the condition that the connection is torsion free,

$$\partial_{[b} e_{c]}^I + \epsilon^I{}_{JK} A_{[b}^J e_{c]}^K = 0, \tag{3.1.4}$$

and the constant negative curvature condition,

$$F_{bc}^I = -\epsilon^I{}_{JK} e_b^J e_c^K. \tag{3.1.5}$$

Taken together, [6], these equations are equivalent to Einstein's equation for the metric g_{ab} with $\Lambda = -1$,

$$R_{ab} - \frac{1}{2} R g_{ab} - g_{ab} = 0. \tag{3.1.6}$$

As an example consider the region in a neighbourhood of the spatial infinity of the colliding geometry, section 2.6. The co-triad one-forms and connection one-forms as given by $e^I = e^I_a dx^a$ and $A^I = A^I_a dx^a$ are

$$e^0 = \left(r^2 + \alpha^2\right)^{\frac{1}{2}} dt, \qquad A^0 = \left(r^2 + \alpha^2\right)^{\frac{1}{2}} d\psi, \tag{3.1.7a}$$

$$e^1 = \left(r^2 + \alpha^2\right)^{-\frac{1}{2}} dr, \qquad A^1 = 0, \tag{3.1.7b}$$

$$e^2 = r d\psi, \qquad A^2 = r dt, \tag{3.1.7c}$$

and as can easily be checked, these fields satisfy the equations of motion.

See the section "2+1 Palatini theory coupled to a cosmological constant" in [26] for a detailed overview of the bulk action (3.1.1).

3.2 Chern-Simons formulation of the action

Following [27] we split the bulk action (3.1.1) into two Chern-Simons (C-S) type actions via the use of two $O(2,1)$ C-S connections,

$${}^{\pm}\mathcal{A}^I_a = A^I_a \pm e^I_a, \tag{3.2.1}$$

yielding

$$S_{\text{bulk}} = {}^{+}S_{\text{bulk}}\left({}^{+}\mathcal{A}^I_a\right) - {}^{-}S_{\text{bulk}}\left({}^{-}\mathcal{A}^I_a\right), \tag{3.2.2}$$

where

$${}^{\pm}S_{\text{bulk}}\left({}^{\pm}\mathcal{A}^I_a\right) = \frac{1}{8\pi}\int_M d^3x\ \tilde{\eta}^{abc}\,{}^{\pm}\mathcal{A}_{aI}\left({}^{\pm}\mathcal{F}^I_{bc} - \frac{1}{3}\epsilon^I{}_{JK}\,{}^{\pm}\mathcal{A}^J_b\,{}^{\pm}\mathcal{A}^K_c\right). \tag{3.2.3}$$

The ${}^{\pm}\mathcal{F}^I_{bc}$ are the curvatures of the two C-S connections and the equations of motion are simply the condition that both connections are flat,

$${}^{\pm}\mathcal{F}^I_{bc} = 2\partial_{[b}\,{}^{\pm}\mathcal{A}^I_{c]} + \epsilon^I{}_{JK}\,{}^{\pm}\mathcal{A}^J_b\,{}^{\pm}\mathcal{A}^K_c = 0. \tag{3.2.4}$$

As an example, we can transform the co-triad and connection one-forms

(3.1.7) into their C-S counterparts, with the result

$$ {}^{+}\!\mathcal{A}^0 = \left(r^2+\alpha^2\right)^{\frac{1}{2}}(dt+d\psi)\,, \qquad {}^{-}\!\mathcal{A}^0 = \left(r^2+\alpha^2\right)^{\frac{1}{2}}(-dt+d\psi)\,, \tag{3.2.5a} $$

$$ {}^{+}\!\mathcal{A}^1 = \left(r^2+\alpha^2\right)^{-\frac{1}{2}}dr, \qquad {}^{-}\!\mathcal{A}^1 = -\left(r^2+\alpha^2\right)^{-\frac{1}{2}}dr, \tag{3.2.5b} $$

$$ {}^{+}\!\mathcal{A}^2 = r\,(dt+d\psi)\,, \qquad {}^{-}\!\mathcal{A}^2 = r\,(dt-d\psi)\,. \tag{3.2.5c} $$

3.3 Gauge transformations in $\mathrm{SU}(1,1)$ representation

3.3.1 Finite gauge transformations

Three dimensional gravity with a negative cosmological constant is related to a gauge theory with gauge group $G = O_c(2,2) \cong (\mathrm{SU}(1,1)\times\mathrm{SU}(1,1))/\mathbb{Z}_2$, [27]. The $(2+1)$-dimensional gravitational field in the first order formalism is a connection form A_a in a G-bundle over M, taking values in $\mathfrak{o}(2,2) \cong \mathfrak{su}(1,1)\oplus\mathfrak{su}(1,1)$. We make use of the group decomposition (2.1.10) to split A_a into a linear sum of the C-S connections,

$$ \mathsf{A}_a = {}^{+}\!\mathcal{A}_a^I\,{}^{+}a_I + {}^{-}\!\mathcal{A}_a^I\,{}^{-}a_I, \tag{3.3.1} $$

where the ${}^{\pm}a_I$ are bases for the two distinct copies of $\mathfrak{su}(1,1)$ where we have chosen

$$ a_0 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{3.3.2} $$

$$ a_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.3.3} $$

$$ a_2 = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \tag{3.3.4} $$

with the commutation relations

$$ \begin{aligned} [{}^{\pm}a_I, {}^{\pm}a_J] =&\ \epsilon^K{}_{IJ}\,{}^{\pm}a_K, \\ [{}^{+}a_I, {}^{-}a_J] =&\ 0. \end{aligned} \tag{3.3.5} $$

We take

$$ g = \left({}^{+}g\ ,\ {}^{-}g\right) \in (\mathrm{SU}(1,1)\times\mathrm{SU}(1,1)) \tag{3.3.6} $$

with the group composition law

$$\left({}^{+}g_2\ ,\ {}^{-}g_2\right) \circ \left({}^{+}g_1\ ,\ {}^{-}g_1\right) = \left({}^{+}g_2{}^{+}g_1\ ,\ {}^{-}g_2{}^{-}g_1\right) \tag{3.3.7}$$

and the inverse

$$\left({}^{+}g\ ,\ {}^{-}g\right)^{-1} = \left({}^{+}g^{-1}\ ,\ {}^{-}g^{-1}\right). \tag{3.3.8}$$

Writing $\mathsf{A} = \mathsf{A}_a dx^a$ we find [28] that the gauge transformations leaving (3.2.3) invariant are

$$\mathsf{A} \mapsto g^{-1}\mathsf{A}g + g^{-1}dg \tag{3.3.9}$$

and after some relatively simple algebra using (3.3.7) and (3.3.8) we find that the gauge transformations for the C-S connections are

$${}^{\pm}\mathcal{A}^I_a a_I \mapsto {}^{\pm}g^{-1}{}^{\pm}\mathcal{A}^I_a a_I{}^{\pm}g + {}^{\pm}g^{-1}\partial_a{}^{\pm}g. \tag{3.3.10}$$

It is now a simple calculation to check that (3.2.3) is invariant under transformations of the type given by (3.3.10).

3.3.2 Infinitesimal gauge transformations

We define the infinitesimal gauge parameter u as

$$u := {}^{+}\tau^I{}^{+}a_I + {}^{-}\tau^I{}^{-}a_I. \tag{3.3.11}$$

Writing the element of the gauge group to first order,

$${}^{\pm}g = \exp\left({}^{\pm}\tau^I{}^{\pm}a_I\right) = 1 + {}^{\pm}\tau^I{}^{\pm}a_I, \tag{3.3.12}$$

we find the infinitesimal form of the gauge transformation is

$$\delta{}^{\pm}\mathcal{A}^I_a = \partial_a{}^{\pm}\tau^I + \epsilon^{IJ}_K{}^{\pm}\mathcal{A}_{Ja}{}^{\pm}\tau^K. \tag{3.3.13}$$

We will use this result in Chapter 4.

3.4 Splitting the action

We start from the bulk action in the form (3.2.2) and assume that the spacetime manifold has the form $M = \Sigma \times \mathbb{R}$ where Σ is a two-dimensional manifold. The $2+1$ decomposition of each C-S bulk action (3.2.3) is

$$ {}^{\pm}S_{\text{bulk}}\left({}^{\pm}\mathcal{A}^I_a\right) = \frac{1}{4\pi}\int dt \int_\Sigma d^2x \left({}^{\pm}\widetilde{\mathcal{A}}^j_I \partial_t {}^{\pm}\mathcal{A}^I_j + \tilde{\epsilon}^{ij}\,{}^{\pm}\mathcal{A}_{tI}\,{}^{\pm}\mathcal{F}^I_{ij}\right), \tag{3.4.1} $$

where $i, j, ...$ are spatial indices corresponding to the two-dimensional surface Σ and t is the coordinate on $\mathbb{R}$. $\tilde{\epsilon}^{ij}$ is the Levi-Civita density on Σ given by $\tilde{\epsilon}^{ij} = \tilde{\eta}^{tij}$, ${}^{\pm}\mathcal{A}^I_j$ is the pull-back of ${}^{\pm}\mathcal{A}^I_a$ to Σ and ${}^{\pm}\widetilde{\mathcal{A}}^j_I$ is a connection density given by ${}^{\pm}\widetilde{\mathcal{A}}^j_I = \tilde{\epsilon}^{ji}\,{}^{\pm}\mathcal{A}_{iI}$. The curvature of the pulled-back connection ${}^{\pm}\mathcal{A}^I_j$ is given by

$$ {}^{\pm}\mathcal{F}^I_{ij} = 2\partial_{[i}\,{}^{\pm}\mathcal{A}^I_{j]} + \epsilon^I{}_{JK}\,{}^{\pm}\mathcal{A}^J_i\,{}^{\pm}\mathcal{A}^K_j \tag{3.4.2} $$

and the ${}^{\pm}\mathcal{A}_{tI}$ act as Lagrange multipliers enforcing the constraints that the pulled-back connections are flat,

$$ {}^{\pm}\mathcal{F}^I_{ij} = 0. \tag{3.4.3} $$

The total bulk action can therefore be rewritten in terms of the two C-S connections as

$$ S_{\text{bulk}} = \frac{1}{4\pi}\int dt \int_\Sigma d^2x\ \mathcal{L}_{\text{int}}, \tag{3.4.4} $$

where

$$ \mathcal{L}_{\text{int}} = {}^{+}\widetilde{\mathcal{A}}^j_I \partial_t {}^{+}\mathcal{A}^I_j + \tilde{\epsilon}^{ij}\,{}^{+}\mathcal{A}_{tI}\,{}^{+}\mathcal{F}^I_{ij} - {}^{-}\widetilde{\mathcal{A}}^j_I \partial_t {}^{-}\mathcal{A}^I_j - \tilde{\epsilon}^{ij}\,{}^{-}\mathcal{A}_{tI}\,{}^{-}\mathcal{F}^I_{ij}. \tag{3.4.5} $$

3.5 Particle actions

For a summary of the particle actions see [9]. In the $\Lambda = -1$ setting the analysis is exactly the same and the upshot is that after the Hamiltonian reduction in Chapter 6 the particle action terms will not contribute to the reduced action.

3.6 Boundary term from the spatial infinity

When we vary the action integrand (3.4.5) in a neighbourhood of the spatial infinity such that (3.2.5) holds we get the variation term

$$\delta\mathcal{L}_{\text{int}} = -2\ \tilde{\epsilon}^{r\psi}\ \partial_r \left(\delta\left(\alpha^2\right)\right). \tag{3.6.1}$$

Therefore, the variation of (3.4.4) acquires from the spatial infinity the boundary term

$$-\int dt\ \delta\left(\alpha^2\right). \tag{3.6.2}$$

We can cancel this boundary term by adding to S_{bulk} the boundary term S_∞ given by

$$S_\infty = \int dt\ \ \left(\alpha^2 + C\right), \tag{3.6.3}$$

where the constant C can be chosen at will. We shall from now on take $C = 0$: this choice has become standard in the literature, and it has the property that S_∞ vanishes at the threshold of black hole formation, $\alpha \to 0$ [22, 23].

Chapter 4

Reduction of the action I: Gauge choice

In this chapter we reduce the action (3.4.4) by imposing the constraints on the connection density pair ${}^{\pm}\widetilde{\mathcal{A}}^{j}_{I}$ and also fixing the gauge. We let F denote a (for the moment fictitious) spacetime of the form discussed in section 2.6 of Chapter 2 and Σ denote a surface within this spacetime containing the particles. The ${}^{\pm}\widetilde{\mathcal{A}}^{j}_{I}$ live on Σ and we impose the constraints by embedding Σ into F in such a way that the embedding is smooth and remains consistent with the identified boundaries of F. We then fix the gauge of ${}^{\pm}\widetilde{\mathcal{A}}^{j}_{I}$ using the knowledge of the embedding. The δ_i are considered fixed and so F is specified completely by α or β through (2.6.7).

4.1 Embedding of the particle surface Σ

The embedding of Σ in F is most easily understood by introducing a simply-connected fundamental half-strip Ω on Σ coordinatised by (λ, ω) such that $\Omega := \{(\lambda, \omega) \mid \lambda > 0, -\pi < \omega < \pi\}$ - see Figure 4.1. The boundaries of Ω at $\omega = \pm\pi$ are identified via $(\lambda, \omega) \sim (\lambda, \omega + 2\pi)$. Particle 1 is on the boundary of Ω at $\lambda = 0$ whereas particle 2 (2') is on the boundary of Ω at $(1, \pm\pi)$.

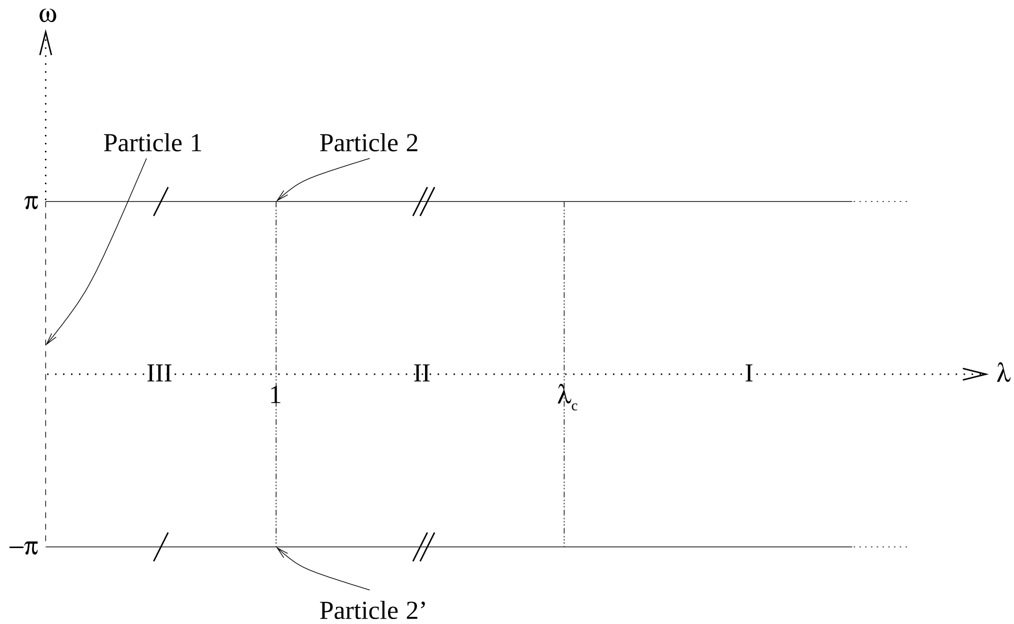

Figure 4.1: The fundamental half strip Ω. Regions I, II and III and the line between particles 2 and $2'$ as indicated on the diagram are explained in the text.

We now specify the embedding so that near the infinity (λ, ω) are the spatial *AdS* conical coordinates of F, (2.2.7b) and (2.2.7c) with $S = 0$, while near the particles (λ, ω) are suitably adapted to the particle motion.

We introduce $\lambda_c > 1$ and where $\lambda > \lambda_c$ (region I) we take the embedding to be in the surface of constant *AdS* conical time t.

Region II, where $1 < \lambda < \lambda_c$ is the region from particle two up to the spatial infinity neighbourhood, whereas region III, where $0 < \lambda < 1$ is the region *between* particles one and two.

In order to specify the embedding of Σ in regions II and III we need to consider the embedding of F into the fundamental domain $\widetilde{\Omega}$ discussed in section 2.6. In terms of this embedding the single and double-stroked boundaries of Ω lie at the corresponding single and double-stroked boundaries on the constant T spacelike sections of $\widetilde{\Omega}$ shown in Figure 2.8.

4.1.1 Region I, $\lambda > \lambda_c$

In region I we embed Σ in a surface of constant AdS conical time in F by taking

$$\lambda = r, \tag{4.1.1a}$$

$$\omega = \psi, \tag{4.1.1b}$$

where r and ψ are the spatial conical coordinates (2.2.7b) and (2.2.7c).

4.1.2 Region II, $1 < \lambda \leq \lambda_c$

Everywhere near and on the double-stroked boundary and near and on the line at $\lambda = \lambda_c$ in region II we set

$$\partial_t = \dot{T}\partial_T, \tag{4.1.2a}$$

$$\partial_\lambda = f(\lambda)\,\partial_\rho, \tag{4.1.2b}$$

$$\partial_\omega = \alpha\partial_\phi, \tag{4.1.2c}$$

such that $f(\lambda)$ is a positive function with domain $1 < \lambda \leq \lambda_c$ obeying the condition

$$\int_1^{\lambda_c} f(\lambda)\,d\lambda = \rho_c - \rho_2, \tag{4.1.3}$$

ensuring that $\rho = \rho_2$ at $\lambda = 1$ and $\rho = \rho_c$ at $\lambda = \lambda_c$. ρ_c is the value of ρ where regions I and II meet.

4.1.3 Region III, $0 < \lambda \leq 1$

The single-stroked boundary segments on $\widetilde{\Omega}$ are geodesics on the Poincaré disk dependent on the initial data of the system (see [29] for the details concerning the geodesics). Using the disk metric in the form (2.2.5) we obtain the geodesic Lagrangian

$$\mathcal{L} = \left(\frac{2}{1-\rho^2}\right)^2 \left(\left(\frac{d\rho}{du}\right)^2 + \rho^2\left(\frac{d\phi}{du}\right)^2\right). \tag{4.1.4}$$

Noting that ϕ is cyclic we choose to parameterise the coordinates via the proper distance s. The Euler-Lagrange equation for ϕ then yields

$$\frac{d\phi}{ds} = \frac{K\left(1-\rho^2\right)^2}{2\rho^2}, \tag{4.1.5}$$

where we have chosen the constant of integration to be $4K$. Using this we can now solve the Euler-Lagrange equation for $\rho(s)$ to find that

$$\rho(s,w_0) = \left(\frac{e^s(1+w_0) - 2(1-w_0) + e^{-s}(1+w_0)}{e^s(1+w_0) + 2(1-w_0) + e^{-s}(1+w_0)}\right)^{\frac{1}{2}}, \tag{4.1.6}$$

where

$$K = \frac{\sqrt{w_0}}{1-w_0}, \tag{4.1.7}$$

and geometrically $\sqrt{w_0}$ is the point of closest approach of the geodesic to the centre of the disk. $\sqrt{w_0}$ can be obtained from the intital data via

$$\sqrt{w_0} = \frac{A^+ - A^-}{2\rho_1\rho_2\sin\phi}, \tag{4.1.8}$$

where

$$A^{\pm} = \left(\left(1\pm\rho_1^2\right)^2\rho_2^2\sin^2\left(\frac{\delta}{2}\right) + \left(\rho_1\left(1+\rho_2^2\right) + \rho_2\cos\left(\frac{\delta}{2}\right)\left(1+\rho_1^2\right)\right)^2\right)^{\frac{1}{2}}, \tag{4.1.9}$$

δ can be obtained from (2.6.7) and the ρ_i are given by (2.6.10).

Solving (4.1.5) for ϕ we find

$$\phi(s,w_0) = \pm\arctan\left(\frac{\left(f(w_0)+e^{2s}\right)(1+w_0)^2}{4\sqrt{w_0}(1-w_0)}\right) + C, \tag{4.1.10}$$

where

$$f(w_0) = \frac{-\left(1-6w_0+w_0^2\right)}{(1+w_0)^2}, \tag{4.1.11}$$

C is the constant of integration (chosen appropriately to produce the Figures 2.9 and 2.10) and the upper (lower) sign in (4.1.10) pertain to the boundary segments from 1 to 2 (2').

Having found the form of ρ and ϕ in terms of the proper distance and the initial data we can now specify the embedding of the single-stroked boundary in region III. Everywhere near and on the single-stroked boundary segments of reqion III we set

$$\partial_\lambda = r_c d_s = r_c \left(\partial_s \rho \partial_\rho \pm \partial_s \phi \partial_\phi\right). \tag{4.1.12}$$

The factor r_c is the geodesic distance between the particles, (2.6.13), and is introduced so that $\| \partial_\lambda \| = r_c$. The coefficients in (4.1.12) can be computed from (4.1.5), (4.1.6) and (4.1.7).

We require ∂_ω to be orthogonal to ∂_λ on the single-stroked boundary and continuous on Σ across the identified boundaries of Ω. It can be verified that this is achieved by setting

$$\partial_\omega = \pm k_1 \partial_T \pm k_2 \partial_\rho + k_3 \partial_\phi, \tag{4.1.13}$$

where

$$k_1 = \frac{SVY}{\left((-SX+CU)^2 + V^2 - 1\right)^{\frac{1}{2}} \left(U^2+V^2\right)}, \tag{4.1.14a}$$

$$k_2 = \frac{-SUY}{\left((-SX+CU)^2 + V^2 - 1\right)^{\frac{1}{2}} \left(U^2+V^2\right)^{\frac{1}{2}} \left(U^2+V^2-1\right)^{\frac{1}{2}} \left(\left(U^2+V^2\right)^{\frac{1}{2}}+1\right)}, \tag{4.1.14b}$$

$$k_3 = \frac{C\left(X^2+Y^2\right) - SUX}{\left((-SX+CU)^2 + V^2 - 1\right)^{\frac{1}{2}} \left(X^2+Y^2\right)}, \tag{4.1.14c}$$

$$S = \sinh \beta_1, \tag{4.1.15a}$$

$$C = \cosh \beta_1, \tag{4.1.15b}$$

and X, Y, U and V are given by the sausage coordinates (2.2.1) and (2.2.3) with ρ and ϕ given by (4.1.6) and (4.1.10).

To summarise, everywhere near and on the single-stroked boundary in region III the embedding is given by

$$\partial_t = \dot{T} \partial_T, \tag{4.1.16a}$$

$$\partial_\lambda = r_c \left(\partial_s \rho \partial_\rho \pm \partial_s \phi \partial_\phi\right), \tag{4.1.16b}$$

$$\partial_\omega = \pm k_1 \partial_T \pm k_2 \partial_\rho + k_3 \partial_\phi, \tag{4.1.16c}$$

where the upper (lower) sign in (4.1.16) indicates the boundary segments from 1 to 2 (2').

The embedding of Σ in F is now specified at and near the boundaries of regions I and II and everywhere in region III. As we will see in the following chapter that is all that will be required for the computation of the Liouville term in the reduced action. As the embedding at and near the boundaries is based on the form of $\widetilde{\Omega}$ it is continuous across the identified boundaries of Ω. We can choose a smooth embedding of Σ everywhere except at the particles and we now consider that done.

4.2 Gauge choice

We now choose a gauge for the C-S fields ${}^{\pm}\!\mathcal{A}^I$ to coincide with the embedding given in the previous section.

In a neighbourhood of the spatial infinity the fields take the form (3.2.5). To choose a gauge in region I we transform the spatial projection of these fields to (λ, ω) coordinates via (4.1.1). The resulting gauge is

$$ {}^{+}\!\mathcal{A}^0 = \left(\lambda^2 + \alpha^2\right)^{\frac{1}{2}} d\omega, \qquad {}^{-}\!\mathcal{A}^0 = \left(\lambda^2 + \alpha^2\right)^{\frac{1}{2}} d\omega, \tag{4.2.1a} $$

$$ {}^{+}\!\mathcal{A}^1 = \left(\lambda^2 + \alpha^2\right)^{-\frac{1}{2}} d\lambda, \qquad {}^{-}\!\mathcal{A}^1 = -\left(\lambda^2 + \alpha^2\right)^{-\frac{1}{2}} d\lambda, \tag{4.2.1b} $$

$$ {}^{+}\!\mathcal{A}^2 = \lambda d\omega, \qquad {}^{-}\!\mathcal{A}^2 = -\lambda d\omega. \tag{4.2.1c} $$

The choice of gauge in regions II and III is substantially more complicated. The task is to choose a gauge in which the fields are smooth across the identification of the boundaries of Ω. We shall do this by writing the fields as a β_1-dependent gauge transformation of a reference configuration that is independent of β_1.

As a preliminary, we first introduce on the fundamental domain $\widetilde{\Omega}$ of F

the fields

$$e^0 = \left(\frac{1+\rho^2}{1-\rho^2}\right) dT, \qquad A^0 = \left(\frac{1+\rho^2}{1-\rho^2}\right) d\phi, \tag{4.2.2a}$$

$$e^1 = \left(\frac{2}{1-\rho^2}\right) d\rho, \qquad A^1 = 0, \tag{4.2.2b}$$

$$e^2 = \left(\frac{2\rho}{1-\rho^2}\right) d\phi, \qquad A^2 = \left(\frac{2\rho}{1-\rho^2}\right) dT, \tag{4.2.2c}$$

where the e^I reproduce the sausage metric (2.2.4) via $ds^2 = \eta_{IJ} e^I e^J$ and the A^I are the connection components compatible with the e^I. Rewriting (4.2.2) in terms of $\mathbb{R}^{2,2}$ coordinates, we find that the C-S counterparts are

$$ {}^{\pm}_{r}\mathcal{A}^0 = \frac{K}{K^2-1}\left(XdY - YdX\right) \pm \frac{1}{K}\left(UdV - VdU\right), \tag{4.2.3a}$$

$$ {}^{\pm}_{r}\mathcal{A}^1 = \pm \frac{1}{K\left(K^2-1\right)^{\frac{1}{2}}}\left(UdU + VdV\right), \tag{4.2.3b}$$

$$ {}^{\pm}_{r}\mathcal{A}^2 = \frac{\left(K^2-1\right)^{\frac{1}{2}}}{K^2}\left(UdV - VdU\right) \pm \frac{1}{\left(K^2-1\right)^{\frac{1}{2}}}\left(XdY - YdX\right), \tag{4.2.3c}$$

where

$$K = \left(U^2 + V^2\right)^{\frac{1}{2}}. \tag{4.2.4}$$

The pre-subscript r refers to the invariance of ${}^{\pm}_{r}\mathcal{A}^I$ under the (X, Y) rotations. Note that ${}^{\pm}_{r}\mathcal{A}^I$ are independent of β_1. We also introduce a set of zero fields,

$$ {}^{\pm}_{0}\mathcal{A}^I = 0 \quad \forall I. \tag{4.2.5}$$

Finally, we set

$$\begin{aligned} {}^{\pm}_{b}\mathcal{A}^0 =& \frac{K_n}{K_n^2-1}\left[\left(CX - SU\right)dY - Y\left(CdX - SdU\right)\right] \\ & \pm \frac{1}{K_n}\left[\left(CU - SX\right)dV - V\left(CdU - SdX\right)\right], \end{aligned} \tag{4.2.6a}$$

$$ {}^{\pm}_{b}\mathcal{A}^1 = \pm \frac{1}{K_n\left(K_n^2-1\right)^{\frac{1}{2}}}\left[\left(CU - SX\right)\left(CdU - SdX\right) + VdV\right], \tag{4.2.6b}$$

$$\begin{aligned} {}^{\pm}_{b}\mathcal{A}^2 =& \frac{\left(K_n^2-1\right)^{\frac{1}{2}}}{K_n^2}\left[\left(CU - SX\right)dV - V\left(CdU - SdX\right)\right] \\ & \pm \frac{1}{\left(K_n^2-1\right)^{\frac{1}{2}}}\left[\left(CX - SU\right)dY - Y\left(CdX - SdU\right)\right], \end{aligned} \tag{4.2.6c}$$

where

$$K_n = \left((CU - SX)^2 + V^2 \right)^{\frac{1}{2}}, \tag{4.2.7}$$

and C and S are given by (4.1.15). Note that ${}^{\pm}_{b}\mathcal{A}^I$ is the pull-back of ${}^{\pm}_{r}\mathcal{A}^I$ by the boost (2.4.1) with $(w_1, w_2) = (\beta_1, 0)$. ${}^{\pm}_{b}\mathcal{A}^I$ is hence invariant under rotations about the worldline of particle 1 on the boundary of $\widetilde{\Omega}$.

Next, we wish to write ${}^{\pm}_{b}\mathcal{A}^I$ as a gauge transformation of ${}^{\pm}_{r}\mathcal{A}^I$. For this, we take a short interlude to review the relationship of diffeomorphisms and gauge transformations in the Chern-Simons formulation.

4.2.1 Relating the gauge transformations to diffeomorphisms

In [27] it is shown, by analysing the generators of the gauge transformations in relation to the generators of diffeomorphisms, that the Chern-Simons gauge transformations do coincide with the usual transformations of $(2+1)$-dimensional gravity for $\Lambda = 0$. The key point is that the generator of the gauge transformations, τ, is dependent on the vector field generating the diffeomorphisms, v, *and* the field configuration being transformed. Adapting the discussion in [27] to the case where $\Lambda < 0$ we find that a vector field v generates the infinitesimal transformation

$$\widetilde{\delta}\,{}^{\pm}\mathcal{A}_a^I = v^b \partial_{[b}\,{}^{\pm}\mathcal{A}_{a]}^I + \partial_a \left(v^b\,{}^{\pm}\mathcal{A}_b^I \right), \tag{4.2.8}$$

and this agrees with the gauge transformation (3.3.13) iff

$${}^{\pm}\tau^I = v^a\,{}^{\pm}\mathcal{A}_a^I. \tag{4.2.9}$$

Consider in particular the one-parameter family ${}^{\pm}\mathcal{A}_a^I(t)$ of field configurations obtained by acting on the configuration ${}^{\pm}\mathcal{A}_a^I(0)$ by the one-parameter family of diffeomorphisms generated by the vector field v. We wish to write ${}^{\pm}\mathcal{A}_a^I(t)$ as the t-dependent gauge transformation of ${}^{\pm}\mathcal{A}_a^I(0)$. To do this, we observe from (4.2.9) that the generator of the gauge transformation is given by

$${}^{\pm}\tau^I(t) = v^a\,{}^{\pm}\mathcal{A}_a^I(t). \tag{4.2.10}$$

The required gauge group element ${}^{\pm}g\left(t\right)$ is thus obtained by integrating

$$ {}^{\pm}g^{-1}\left(t\right){}^{\pm}\dot{g}\left(t\right)={}^{\pm}\tau^{I}\left(t\right)a_{I}, \tag{4.2.11} $$

where the dot indicates differentiation with respect to t, with the initial condition

$$ {}^{\pm}g\left(0\right)=1. \tag{4.2.12} $$

A simple example

As an example let us consider $v=(0,0,Y,-X)$, which generates an (X,Y) rotation, and choose the field configuration to be given by (4.2.3), which is invariant under the (X,Y) rotation. Denoting the parameter of the gauge transformation by θ, from (4.2.10) we find

$$ {}^{\pm}\tau^{0}\left(\theta\right)=-K, \tag{4.2.13a} $$

$$ {}^{\pm}\tau^{1}\left(\theta\right)=0, \tag{4.2.13b} $$

$$ {}^{\pm}\tau^{2}\left(\theta\right)=\mp\left(K^{2}-1\right)^{\frac{1}{2}}, \tag{4.2.13c} $$

which are all independent of θ. The solution of (4.2.11) such that (4.2.12) holds is therefore simply

$$ {}^{\pm}g\left(\theta\right)=\exp\left(\theta\,{}^{\pm}\tau^{I}a_{I}\right), \tag{4.2.14} $$

and a small calculation yields

$$ {}^{\pm}g\left(\theta\right)=\begin{pmatrix}\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}K & \mp i\sin\frac{\theta}{2}\left(K^{2}-1\right)^{\frac{1}{2}}\\ \pm i\sin\frac{\theta}{2}\left(K^{2}-1\right)^{\frac{1}{2}} & \cos\frac{\theta}{2}+i\sin\frac{\theta}{2}K\end{pmatrix}. \tag{4.2.15} $$

As a check, it is a straightforward (but long-winded) task to verify that the gauge transformation implemented by (3.3.10) with ${}^{\pm}g$ given by (4.2.15) does leave the field configuration (4.2.3) invariant.

4.2.2 Gauge transformation from ${}^{\pm}_{r}\mathcal{A}^{I}$ to ${}^{\pm}_{b}\mathcal{A}^{I}$

To find the gauge transformation from ${}^{\pm}_{r}\mathcal{A}^{I}$ (4.2.3) to ${}^{\pm}_{b}\mathcal{A}^{I}$ (4.2.6), we recall that ${}^{\pm}_{b}\mathcal{A}^{I}$ is the pull-back of ${}^{\pm}_{r}\mathcal{A}^{I}$ by the boost (2.4.1) with $(w_1,w_2)=(\beta_1,0)$, and this boost is generated by the vector field $v = (X, 0, U, 0)$. We may therefore use (4.2.10) with ${}^{\pm}\mathcal{A}^I = {}^{\pm}_{b}\mathcal{A}^I(\beta_1)$, $v = (X, 0, U, 0)$ and $t = \beta_1$. We find

$$ {}^{\pm}\tau^0 = \frac{K_n}{K_n^2 - 1} Y(CU - SX) \pm \frac{1}{K_n} V(CX - SU), \tag{4.2.16a} $$

$$ {}^{\pm}\tau^1 = \mp \frac{1}{K_n (K_n^2 - 1)^{\frac{1}{2}}} (CX - SU)(CU - SX), \tag{4.2.16b} $$

$$ {}^{\pm}\tau^2 = \pm \frac{1}{(K_n^2 - 1)^{\frac{1}{2}}} Y(CU - SX) + \frac{(K_n^2 - 1)^{\frac{1}{2}}}{K_n^2} V(CX - SU). \tag{4.2.16c} $$

The τ^I are now dependent on β_1. To solve (4.2.10), we first observe from (4.2.16) that

$$ {}^{\pm}\tau^2 = \pm \frac{(K_n^2 - 1)^{\frac{1}{2}}}{K_n} {}^{\pm}\tau^0. \tag{4.2.17} $$

As $|{}^{\pm}\tau^0| > |{}^{\pm}\tau^2|$, we may find a pair of matrices ${}^{\pm}h \in \mathrm{SU}(1,1)$ such that the internal vector ${}^{\pm}\sigma^I$ defined by

$$ {}^{\pm}\sigma^I a_I = {}^{\pm}h \, {}^{\pm}\tau^I a_I \, {}^{\pm}h^{-1}, \tag{4.2.18} $$

satisfies ${}^{\pm}\sigma^2 = 0$. Choosing ${}^{\pm}h$ to be a pure boost in the internal (02) plane, we find

$$ {}^{\pm}h = \frac{1}{\sqrt{2}} \begin{pmatrix} (K_n + 1)^{\frac{1}{2}} & \pm (K_n - 1)^{\frac{1}{2}} \\ \pm (K_n - 1)^{\frac{1}{2}} & (K_n + 1)^{\frac{1}{2}} \end{pmatrix}, \tag{4.2.19} $$

and

$$ {}^{\pm}\sigma^0 = \frac{1}{K_n^2 - 1} Y(CU - SX) \pm \frac{1}{K_n^2} V(CX - SU), \tag{4.2.20a} $$

$$ {}^{\pm}\sigma^1 = \mp \frac{1}{K_n (K_n^2 - 1)^{\frac{1}{2}}} (CX - SU)(CU - SX), \tag{4.2.20b} $$

$$ {}^{\pm}\sigma^2 = 0. \tag{4.2.20c} $$

Now, writing

$$ {}^{\pm}g = {}^{\pm}k \, {}^{\pm}h \tag{4.2.21} $$

and substituting (4.2.21) into (4.2.11) we find that the equation for ${}^{\pm}k$ is

$$ {}^{\pm}k^{-1} \, {}^{\pm}\dot{k} = {}^{\pm}\sigma - {}^{\pm}\dot{h} \, {}^{\pm}h^{-1} = {}^{\pm}\gamma^I a_I, \tag{4.2.22} $$

where

$$^{\pm}\gamma^0 = \frac{1}{K_n^2 - 1} Y\,(CU - SX) \pm \frac{1}{K_n^2} V\,(CX - SU)\,, \tag{4.2.23a}$$

$$^{\pm}\gamma^1 = 0, \tag{4.2.23b}$$

$$^{\pm}\gamma^2 = 0. \tag{4.2.23c}$$

The general solution to (4.2.22) is

$$^{\pm}k = {}^{\pm}\widetilde{k} \begin{pmatrix} ^{\pm}f & 0 \\ 0 & ^{\pm}f^{-1} \end{pmatrix}, \tag{4.2.24}$$

where $^{\pm}\widetilde{k} \in \mathrm{SU}\,(1,1)$ is independent of β_1 and $^{\pm}f$ satisfies

$$^{\pm}\dot{f} - \frac{i}{2}{}^{\pm}f^{\pm}\gamma^0 = 0. \tag{4.2.25}$$

The solution to (4.2.25) is

$$^{\pm}f = {}^{\pm}\mathcal{C} \exp\left(\frac{i}{2}\int {}^{\pm}\gamma^0 d\beta_1\right), \tag{4.2.26}$$

where $^{\pm}\mathcal{C}$ is a constant of integration satisfying $|^{\pm}\mathcal{C}| = 1$. Evaluating this integral, we find

$$^{\pm}f = \left\{\frac{[Y - i\,(CX - SU)]\,[V \mp i\,(CU - SX)]}{[Y + i\,(CX - SU)]\,[V \pm i\,(CU - SX)]}\right\}^{\frac{1}{4}}, \tag{4.2.27}$$

where the principal branch of the fractional power is understood and we have set $^{\pm}\mathcal{C} = 1$ without loss of generality. $^{\pm}f$ has a branch point singularity at particle one, where $Y = 0$ and $CX = SU$, but as we shall see in Chapter 5, this will not affect us when computing the Liouville term as we can choose the branch cut such that the contour of integration will never cross it.

Finally the constant matrix pair $^{\pm}\widetilde{k}$ is fixed by the initial condition

$$^{\pm}g\,(\beta_1 = 0) = 1. \tag{4.2.28}$$

This gives the final gauge element in the form

$$^{\pm}g = {}^{\pm}m^{\pm}n, \tag{4.2.29}$$

where

$$ {}^{\pm}m = {}^{\pm}B_0^{-1}\,{}^{\pm}R_0^{-1}, \tag{4.2.30a} $$

$$ {}^{\pm}n = {}^{\pm}R\,{}^{\pm}B, \tag{4.2.30b} $$

and

$$ {}^{\pm}B_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} (K+1)^{\frac{1}{2}} & \pm(K-1)^{\frac{1}{2}} \\ \pm(K-1)^{\frac{1}{2}} & (K+1)^{\frac{1}{2}} \end{pmatrix}, \tag{4.2.31a} $$

$$ {}^{\pm}R_0 = \begin{pmatrix} {}^{\pm}f|_{\beta_1=0} & 0 \\ 0 & {}^{\pm}f^{-1}|_{\beta_1=0} \end{pmatrix}, \tag{4.2.31b} $$

$$ {}^{\pm}R = \begin{pmatrix} {}^{\pm}f & 0 \\ 0 & {}^{\pm}f^{-1} \end{pmatrix}, \tag{4.2.31c} $$

$$ {}^{\pm}B = \frac{1}{\sqrt{2}} \begin{pmatrix} (K_n+1)^{\frac{1}{2}} & \pm(K_n-1)^{\frac{1}{2}} \\ \pm(K_n-1)^{\frac{1}{2}} & (K_n+1)^{\frac{1}{2}} \end{pmatrix}. \tag{4.2.31d} $$

Again it is a straightforward task to verify that the gauge transformation implemented by (3.3.10) with ${}^{\pm}g$ given by (4.2.29) transforms ${}^{\pm}_{r}\mathcal{A}^I$ (4.2.3) into ${}^{\pm}_{b}\mathcal{A}^I$ (4.2.6).

We further find by virtue of the forms of ${}^{\pm}m$ and ${}^{\pm}n$ that

$$ {}^{\pm}m^{-1} : {}^{\pm}_{0}\mathcal{A} \to {}^{\pm}_{r}\mathcal{A}, \tag{4.2.32} $$

and

$$ {}^{\pm}n : {}^{\pm}_{0}\mathcal{A} \to {}^{\pm}_{b}\mathcal{A}. \tag{4.2.33} $$

4.2.3 Gauge choice in regions II and III

After all the above preperation, we can now state the gauge choice in regions II and III. We choose a gauge near and at the double-stroked boundaries and the line $\lambda = \lambda_c$ of region II according to ${}^{\pm}m^{-1}\left({}^{\pm}_{0}\mathcal{A}\right)$. We also choose a gauge near and at the single-stroked boundaries and the line $\lambda = 0$ of region III according to ${}^{\pm}n\left({}^{\pm}_{0}\mathcal{A}\right)$. As will be seen in the following chapter, this partial specification of the gauge (see Figure 4.2) will be enough to evaluate the Liouville term.

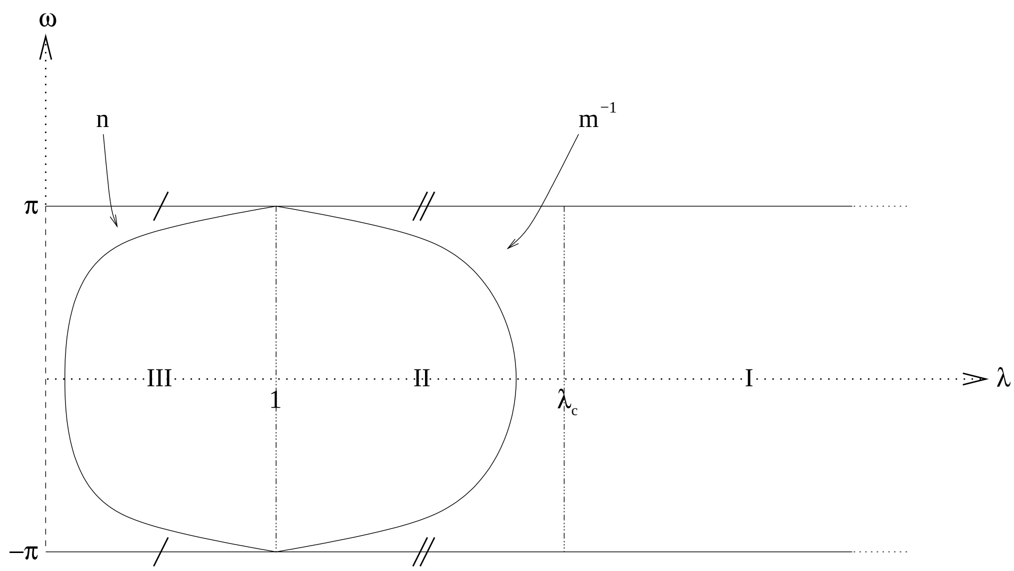

Figure 4.2: The gauge is chosen via the gauge elements n and m^{-1} in the two regions indicated as explained in the text.

4.3 Continuity of the gauge choice

The gauge choice for all three regions defined in the previous section can be extended from Ω to Σ. In order to verify this we need to show that the fields are continuous across the double-stroked (region II) and single-stroked (region III) boundaries. In region I the fields are extendable to Σ by definition. In region II on the double-stroked boundaries the non-vanishing components of ${}^{\pm}\mathcal{A}^I_j$ are

$$ {}^{\pm}\mathcal{A}^0_\omega = \alpha\left(\frac{1+\rho^2}{1-\rho^2}\right), \tag{4.3.1a}$$

$$ {}^{\pm}\mathcal{A}^1_\lambda = \pm f(\lambda)\left(\frac{2}{1-\rho^2}\right), \tag{4.3.1b}$$

$$ {}^{\pm}\mathcal{A}^2_\omega = \pm\alpha\left(\frac{2\rho}{1-\rho^2}\right), \tag{4.3.1c}$$

which are all continuous across the identification. A similar analysis reveals that the non-vanishing components of ${}^{\pm}\mathcal{A}^I_j$ in region III are continuous across the identification of the single-stroked boundaries.

The gauge has therefore been specified on and near the boundaries of the fundamental domain Ω. What remains is to evaluate the reduced action which we shall address in Chapters 5 and 6.

Chapter 5

Reduction of the action II: The Liouville term

In Chapter 4 we explained the gauge fixing procedure and with this choice of gauge we now evaluate the reduced action. The constraint terms in (3.4.5) vanish by the gauge choice, and so do the particle contributions as discussed in section 3.5. The boundary term at infinity was found in section 3.6 and reads

$$S_\infty = \int dt\ \alpha^2. \tag{5.0.1}$$

What remains are the Liouville terms in (3.4.5), given by

$$L = \frac{1}{4\pi}\int_\Sigma d^2x \left({}^{+}\widetilde{\mathcal{A}}^j_I \partial_t {}^{+}\mathcal{A}^I_j - {}^{-}\widetilde{\mathcal{A}}^j_I \partial_t {}^{-}\mathcal{A}^I_j \right). \tag{5.0.2}$$

For convenience we will rewrite (5.0.2) as the trace over Lie-algebra valued fields,

$$L = {}^{+}L - {}^{-}L, \tag{5.0.3}$$

where

$${}^{\pm}L = \frac{1}{2\pi}\int_\Sigma d^2x\ \tilde{\epsilon}^{ji}\ \mathrm{Tr}\left({}^{\pm}\mathcal{A}_i \partial_t {}^{\pm}\mathcal{A}_j \right), \tag{5.0.4}$$

and

$${}^{\pm}\mathcal{A}_i = {}^{\pm}\mathcal{A}^I_i a_I, \tag{5.0.5}$$

and we have used the identity $\eta_{IJ} = 2\ \mathrm{Tr}\,(a_I a_J)$. The purpose of this chapter is to analyse these Liouville terms (5.0.4).

5.1 Direct evaluation in region I

In region I on Ω the fields are given by (4.2.1) and the integrand in (5.0.4) is identically zero. There is no contribution to the action from the Liouville term in this region.

5.2 Integral conversion

In the other two regions of Ω we will show that due to the form of the gauge choice described in Chapter 4 we can write the Liouville term as an integral of a total derivative. By Stokes' theorem, we can then convert the two-dimensional integral in (5.0.4) into a one-dimensional integral over the boundary of each region.

We introduce ${}^{\pm}_{r}\mathcal{A}_i$ and ${}^{\pm}_{b}\mathcal{A}_i$ to denote the Lie-algebra valued spatial projection of (4.2.3) and (4.2.6) respectively. Using (4.2.29) we find that

$$
{}^{\pm}_{x}\mathcal{A}_i = {}^{\pm}l^{-1}\partial_i{}^{\pm}l, \tag{5.2.1}
$$

where $x = r$ and ${}^{\pm}l = {}^{\pm}m^{-1}$ for region II whereas $x = b$ and ${}^{\pm}l = {}^{\pm}n$ for region III. (5.0.4) now becomes

$$
\begin{aligned}
{}^{\pm}L =& \frac{1}{2\pi}\int_{\Sigma} d^2x\ \tilde{\epsilon}^{ji}\ \mathrm{Tr}\left[\left({}^{\pm}l^{-1}\partial_i{}^{\pm}l\right)\partial_t\left({}^{\pm}l^{-1}\partial_j{}^{\pm}l\right)\right] && (5.2.2)\\
=& \frac{1}{2\pi}\int_{\Sigma} d^2x\ \tilde{\epsilon}^{ji}\ \mathrm{Tr}\left[\partial_j\left({}^{\pm}l^{-1}\partial_i\partial_t{}^{\pm}l\right) - {}^{\pm}l^{-1}\partial_i{}^{\pm}l{}^{\pm}l^{-1}\partial_t{}^{\pm}l{}^{\pm}l^{-1}\partial_j{}^{\pm}l\right]. && (5.2.3)
\end{aligned}
$$

In both regions ${}^{\pm}l$ can be written as the product of two $SU(1,1)$ matrices. For example, in region III, (the analysis in region II is analogous)

$$
{}^{\pm}l = \exp\left(a_0{}^{\pm}\phi\right)\exp\left(a_1{}^{\pm}\chi\right), \tag{5.2.4}
$$

for some ${}^{\pm}\phi$ and ${}^{\pm}\chi$. Therefore

$$
{}^{\pm}l^{-1}\partial_a{}^{\pm}l = G^I_a a_I, \tag{5.2.5}
$$

where

$$G_a^0 = \cosh {}^{\pm}\chi \partial_a {}^{\pm}\phi, \tag{5.2.6a}$$

$$G_a^1 = \partial_a {}^{\pm}\chi, \tag{5.2.6b}$$

$$G_a^2 = \sinh {}^{\pm}\chi \partial_a {}^{\pm}\phi. \tag{5.2.6c}$$

The second term in (5.2.3) can therefore be written as

$$-\frac{1}{8\pi}\int_\Sigma d^2x \; \tilde{\epsilon}^{ji}\epsilon_{IJK}G_i^I G_t^J G_j^K, \tag{5.2.7}$$

where we have used the identity

$$\mathrm{Tr}\,(a_I a_J a_K) = \frac{1}{4}\epsilon_{IJK}. \tag{5.2.8}$$

Evaluating the integrand in (5.2.7) using (5.2.6) reveals that it is zero, independently of the functions ${}^{\pm}\phi$ and ${}^{\pm}\chi$. We hence obtain

$${}^{\pm}L = \frac{1}{2\pi}\int_\Sigma d^2x \; \tilde{\epsilon}^{ji}\partial_j \; \mathrm{Tr}\left({}^{\pm}l^{-1}\partial_i\partial_t {}^{\pm}l\right), \tag{5.2.9}$$

where the integrand is now a total derivative as promised. The integral given by (5.0.4) can therefore be converted into a integral over the boundary of each region. The orientation of the boundary is acquired from the orientation of the three-dimensional spacetime; the boundaries of regions II and III are oriented in an anti-clockwise direction with respect to Figure 4.1. We now consider the relevant parts of the boundary in each region.

In region II we have the line at $\lambda = \lambda_c$, the double-stroked lines at $\omega = \pm\pi$, two small quarter-circles about the singular points 2 and 2' and the line at $\lambda = 1$. In region III we have the line at $\lambda = 0$, the single-stroked lines at $\omega = \pm\pi$, two small quarter-circles about the singular points 2 and 2' and the line at $\lambda = 1$. The contribution from the line at $\lambda = 1$ in region II will cancel with the contribution from the line at $\lambda = 1$ in region III due to the orientation of the boundaries and the continuity of the gauge choice. We are left with five distinct parts of the boundary we need to consider;

1. The line at $\lambda = \lambda_c$ on the boundary of region II ,

2. The double-stroked lines at $\omega = \pm\pi$ on the boundary of region II ,

3. The single-stroked lines at $\omega = \pm\pi$ on the boundary of region III ,

4. The line at $\lambda = 0$ on the boundary of region III ,

5. Two small half-circles about the singular points 2 and 2' .

We will write L_i where $i \in \{1, 2, 3, 4, 5\}$ to denote the contribution from the Liouville term for each of the five parts and address all of these contributions in turn in the following section.

5.3 Evaluation of the contributions

For each contribution to the Lagrangian from evaluating (5.2.9) on the relevant boundaries we will write

$$^{\pm}W_i = {}^{\pm}l^{-1}\partial_i\partial_t{}^{\pm}l. \tag{5.3.1}$$

We also note that according to our previously defined conventions we have

$$\tilde{\eta}^{t\lambda\omega} = \tilde{\epsilon}^{\lambda\omega} = +1 \tag{5.3.2}$$

In order to evaluate the resulting one-dimensional integrals we need to use the details of the embedding and gauge elements elucidated in Chapter 4.

5.3.1 $\lambda = \lambda_c$ and the double-stroked boundaries

The embedding at the boundaries is given by (4.1.2) and the gauge element is ${}^{\pm}l = {}^{\pm}m^{-1}$.

On the line at $\lambda = \lambda_c$ the contribution is

$$^{\pm}L_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega \ \mathrm{Tr}\ {}^{\pm}W_\omega. \tag{5.3.3}$$

Upon evaluating ${}^{\pm}W_\omega$ we find

$$^{\pm}W_\omega = \mp\frac{1}{4}\mathbb{I}_2\alpha\dot{T}, \tag{5.3.4}$$

where $\mathbb{I}_2$ is the identity matrix. Taking the trace and evaluating the integral we obtain the finite contribution to the Lagrangian from this boundary being

$$L_1 = -\alpha\dot{T}. \tag{5.3.5}$$

On the double-stroked boundaries the contribution is

$$-\frac{1}{2\pi}\int_1^{\lambda_c} d\lambda \ \mathrm{Tr}\ {}^{\pm}W_\lambda, \tag{5.3.6}$$

along the boundary at $\omega = -\pi$ and

$$-\frac{1}{2\pi}\int_{\lambda_c}^{1} d\lambda \ \mathrm{Tr}\ {}^{\pm}W_\lambda, \tag{5.3.7}$$

along the boundary at $\omega = \pi$. Upon evaluating ${}^{\pm}W_\lambda$ we find that it is proportional to a Lie-algebra valued matrix. Taking the trace yields zero for the integrand. The contribution to the Lagrangian from the double-stroked boundaries is zero,

$$L_2 = 0. \tag{5.3.8}$$

5.3.2 The single-stroked boundaries

The embedding at the boundaries is given by (4.1.16) and the gauge element is ${}^{\pm}l = {}^{\pm}n$.

On the single-stroked boundaries the contribution is

$$-\frac{1}{2\pi}\int_0^{1} d\lambda \ \mathrm{Tr}\ {}^{\pm}W_\lambda, \tag{5.3.9}$$

along the boundary at $\omega = -\pi$ and

$$-\frac{1}{2\pi}\int_1^{0} d\lambda \ \mathrm{Tr}\ {}^{\pm}W_\lambda, \tag{5.3.10}$$

along the boundary at $\omega = \pi$. It is easiest to evaluate these integrals by parameterising in terms of the proper distance, s, as opposed to λ. We now have to evaluate ${}^{\pm}W_s$ by considering the gauge element ${}^{\pm}n$ in the form (4.2.30b) and by differentiating (4.2.31d) and (4.2.31c) with respect to t and s we note the intermediate results,

$$\partial_t {}^{\pm}B = {}^{\pm}d \; a_1 {}^{\pm}B, \tag{5.3.11a}$$

$$\partial_t {}^{\pm}R = {}^{\pm}e \; a_0 {}^{\pm}R, \tag{5.3.11b}$$

$$\partial_s {}^{\pm}B = {}^{\pm}b \; a_1 {}^{\pm}B, \tag{5.3.11c}$$

$$\partial_s {}^{\pm}R = {}^{\pm}r \; a_0 {}^{\pm}R, \tag{5.3.11d}$$

where

$${}^{\pm}d = \pm \frac{(CX - SU)\left(SV\dot{T} - (CU - SX)\,\dot{\beta}_1\right)}{K_n \left(K_n^2 - 1\right)^{\frac{1}{2}}}, \tag{5.3.12a}$$

$${}^{\pm}e = \pm \frac{V\,(CX - SU)\,\dot{\beta}_1 + \left(C\left(U^2 + V^2\right) - SUX\right)\dot{T}}{K_n^2} + \frac{Y\left((CU - SX)\,\dot{\beta}_1 - SV\dot{T}\right)}{K_n^2 - 1}, \tag{5.3.12b}$$

$${}^{\pm}b = \pm \frac{(CU - SX)\,(C\partial_s U - S\partial_s X) + V\partial_s V}{K_n \left(K_n^2 - 1\right)^{\frac{1}{2}}}, \tag{5.3.12c}$$

$${}^{\pm}r = \frac{(CX - SU)\,\partial_s Y - Y\,(C\partial_s X - S\partial_s U)}{K_n^2 - 1} \pm \frac{(CU - SX)\,\partial_s V - V\,(C\partial_s U - S\partial_s X)}{K_n^2}. \tag{5.3.12d}$$

A further calculation reveals that

$$\mathrm{Tr}\left({}^{+}W_s - {}^{-}W_s\right) = \frac{1}{2}\left({}^{-}e\,{}^{-}r - {}^{+}e\,{}^{+}r\right) = f\,(s)\,, \tag{5.3.13}$$

where

$$\begin{aligned} f\,(s) = -\frac{1}{K_n^2\left(K_n^2 - 1\right)} \Big\{ & \left(V\,(CX - SU)\,\dot{\beta}_1 + \left(C\left(U^2 + V^2\right) - SUX\right)\dot{T}\right) \times \\ & ((CX - SU)\,\partial_s Y - Y\,(C\partial_s X - S\partial_s U)) \\ & + Y\left((CU - SX)\,\dot{\beta}_1 - SV\dot{T}\right) \times \\ & ((CU - SX)\,\partial_s V - V\,(C\partial_s U - S\partial_s X)) \Big\}. \end{aligned} \tag{5.3.14}$$

Using the property that $f\,(s)$ switches sign across the identification of the boundary ($Y \rightarrow -Y$), we can combine (5.3.9) and (5.3.10) into one simple expression evaluated on the boundary at $\omega = \pi$,

$$L_3 = \frac{1}{\pi} \int_{s_i}^{s_f} ds \; f\,(s)\,. \tag{5.3.15}$$

To evaluate this integral we must first parameterise the integrand explicitly in terms of s and w_0. The parameterisation for U, V, X and Y is

$$U = \left(\frac{1+w_0}{1-w_0}\right)\cosh s \cos T, \tag{5.3.16a}$$

$$V = \left(\frac{1+w_0}{1-w_0}\right)\cosh s \sin T, \tag{5.3.16b}$$

$$X = \frac{(1+w_0)}{2\,(1-w_0)\left(e^{2s_i}+2f+e^{-2s_i}\right)^{\frac{1}{2}}}\left(\left(e^{s+s_i}+e^{-s-s_i}\right)+f\left(e^{s-s_i}+e^{-s+s_i}\right)\right), \tag{5.3.16c}$$

$$Y = \frac{2w_0^{\frac{1}{2}}}{(1+w_0)\left(e^{2s_i}+2f+e^{-2s_i}\right)^{\frac{1}{2}}}\left(e^{s-s_i}-e^{-s+s_i}\right), \tag{5.3.16d}$$

where we have used (4.1.10) and also inverted (4.1.10) to obtain s_i and s_f, the initial and final proper distances,

$$s_i = -\log\left(\frac{(1-w_0w_1)^{\frac{1}{2}}+(w_1-w_0)^{\frac{1}{2}}}{(1-w_0w_1)^{\frac{1}{2}}-(w_1-w_0)^{\frac{1}{2}}}\right), \tag{5.3.17a}$$

$$s_f = +\log\left(\frac{(1-w_0w_2)^{\frac{1}{2}}+(w_2-w_0)^{\frac{1}{2}}}{(1-w_0w_2)^{\frac{1}{2}}-(w_2-w_0)^{\frac{1}{2}}}\right), \tag{5.3.17b}$$

where w_i is the square of the distance of particle i from the centre of the disk. w_0, w_1 and w_2 can be rewritten in a similar form in terms of the initial data and T,

$$w_0 = \frac{\left(\sin^2\frac{\delta}{2}+s_1^2s_2^2\sinh^2\beta_1\right)^{\frac{1}{2}}-\left(\sin^2\frac{\delta}{2}+s_1^2s_2^2\sinh^2\beta_1\sin^2 T\right)^{\frac{1}{2}}}{\left(\sin^2\frac{\delta}{2}+s_1^2s_2^2\sinh^2\beta_1\right)^{\frac{1}{2}}+\left(\sin^2\frac{\delta}{2}+s_1^2s_2^2\sinh^2\beta_1\sin^2 T\right)^{\frac{1}{2}}}, \tag{5.3.18a}$$

$$w_i = \frac{\left(s_i^2\sin^2\frac{\delta}{2}+s_1^2s_2^2\sinh^2\beta_1\right)^{\frac{1}{2}}-\left(s_i^2\sin^2\frac{\delta}{2}+s_1^2s_2^2\sinh^2\beta_1\sin^2 T\right)^{\frac{1}{2}}}{\left(s_i^2\sin^2\frac{\delta}{2}+s_1^2s_2^2\sinh^2\beta_1\right)^{\frac{1}{2}}+\left(s_i^2\sin^2\frac{\delta}{2}+s_1^2s_2^2\sinh^2\beta_1\sin^2 T\right)^{\frac{1}{2}}}. \tag{5.3.18b}$$

Using (5.3.17) and (5.3.18) we can now rewrite (5.3.16),

$$U = A\cosh s, \tag{5.3.19a}$$

$$V = B\cosh s, \tag{5.3.19b}$$

$$X = D\cosh s + E\sinh s, \tag{5.3.19c}$$

$$Y = F\cosh s + G\sinh s, \tag{5.3.19d}$$

where.

$$A = \left(\frac{\sin^2 \frac{\delta}{2} + s_1^2 s_2^2 \sinh^2 \beta_1}{\sin^2 \frac{\delta}{2} + s_1^2 s_2^2 \sinh^2 \beta_1 \sin^2 T} \right)^{\frac{1}{2}} \cos T, \tag{5.3.20a}$$

$$B = \left(\frac{\sin^2 \frac{\delta}{2} + s_1^2 s_2^2 \sinh^2 \beta_1}{\sin^2 \frac{\delta}{2} + s_1^2 s_2^2 \sinh^2 \beta_1 \sin^2 T} \right)^{\frac{1}{2}} \sin T, \tag{5.3.20b}$$

$$D = \frac{s_1 s_2 \sinh \beta_1 \cos T \left(s_1^2 \sin^2 \frac{\delta}{2} + s_1^2 s_2^2 \sinh^2 \beta_1 \right)^{\frac{1}{2}}}{\left(\sin^2 \frac{\delta}{2} + s_1^2 s_2^2 \sinh^2 \beta_1 \right)^{\frac{1}{2}} \left(\sin^2 \frac{\delta}{2} + s_1^2 s_2^2 \sinh^2 \beta_1 \sin^2 T \right)^{\frac{1}{2}}}, \tag{5.3.20c}$$

$$E = \frac{-c_1 \sin \frac{\delta}{2}}{\left(\sin^2 \frac{\delta}{2} + s_1^2 s_2^2 \sinh^2 \beta_1 \right)^{\frac{1}{2}}}, \tag{5.3.20d}$$

$$F = \frac{c_1 s_1 s_2 \sin \frac{\delta}{2} \sinh \beta_1 \cos T}{\left(\sin^2 \frac{\delta}{2} + s_1^2 s_2^2 \sinh^2 \beta_1 \right)^{\frac{1}{2}} \left(\sin^2 \frac{\delta}{2} + s_1^2 s_2^2 \sinh^2 \beta_1 \sin^2 T \right)^{\frac{1}{2}}}, \tag{5.3.20e}$$

$$G = \left(\frac{s_1^2 \sin^2 \frac{\delta}{2} + s_1^2 s_2^2 \sinh^2 \beta_1}{\sin^2 \frac{\delta}{2} + s_1^2 s_2^2 \sinh^2 \beta_1} \right)^{\frac{1}{2}}. \tag{5.3.20f}$$

We now reparameterise the integral via

$$p = \tanh \frac{s}{2}, \tag{5.3.21}$$

so that

$$\cosh s = \frac{1 + p^2}{1 - p^2}, \tag{5.3.22a}$$

$$\sinh s = \frac{2p}{1 - p^2}, \tag{5.3.22b}$$

We now have

$$L_3 = -\frac{2}{\pi \phi} \int_{p_i}^{p_f} dp \; \frac{P(p)}{\left(p^2 - p_a^2\right) \left(p^2 - p_b^2\right) \left(p^2 - p_c^2\right)^2}. \tag{5.3.23}$$

where

$$\phi = \left((CA-SD)^2+B^2\right)\left((CD-SA)^2+F^2\right), \tag{5.3.24a}$$

$$p_f = \left(\frac{w_2-w_0}{1-w_0w_2}\right)^{\frac{1}{2}}, \tag{5.3.24b}$$

$$p_i = -\left(\frac{w_1-w_0}{1-w_0w_1}\right)^{\frac{1}{2}}, \tag{5.3.24c}$$

$$p_a^2 = \frac{2SE}{CA-SD+iB}-1, \tag{5.3.24d}$$

$$p_b^2 = \frac{2SE}{CA-SD-iB}-1 = \bar{p}_a^2, \tag{5.3.24e}$$

$$p_c^2 = -\frac{2\left[CE\left(CD-SA\right)+FG\right]}{\left(CD-SA\right)^2+F^2}-1, \tag{5.3.24f}$$

$$P\left(p\right) = K_0+K_1p\left(1-p^4\right)+K_2p^2\left(1-p^2\right), \tag{5.3.24g}$$

$$K_0 = SEB\left[F\left(CA-SD\right)\dot{\beta}_1-SBF\dot{T}\right], \tag{5.3.24h}$$

$$K_1 = 2SEB\left[\left(G\left(CA-SD\right)-SEF\right)\dot{\beta}_1-SBG\dot{T}\right], \tag{5.3.24i}$$

$$K_2 = SEB\left[\left(F\left(CA-SD\right)-4SEG\right)\dot{\beta}_1-SBF\dot{T}\right]. \tag{5.3.24j}$$

The integral (5.3.23) can be evaluated in terms of elementary functions. The result is

$$\begin{aligned} L_3 = -\frac{2}{\pi\phi}\Big[& A^+\mathrm{Log}\left(p+p_a\right)+A^-\mathrm{Log}\left(p-p_a\right) \\ & + B^+\mathrm{Log}\left(p+p_b\right)+B^-\mathrm{Log}\left(p-p_b\right) \\ & + C^+\mathrm{Log}\left(p+p_c\right)+C^-\mathrm{Log}\left(p-p_c\right) \\ & - \frac{K^+}{p+p_c}-\frac{K^-}{p-p_c}\Big]\Bigg|_{p_i}^{p_f}. \end{aligned} \tag{5.3.25}$$

where

$$A^{\pm} = \frac{\mp P\left(\mp p_a\right)}{2p_a\left(p_a^2-p_b^2\right)\left(p_a^2-p_c^2\right)^2}, \tag{5.3.26a}$$

$$B^{\pm} = \frac{\mp P\left(\mp p_b\right)}{2p_b\left(p_b^2-p_a^2\right)\left(p_b^2-p_c^2\right)^2}, \tag{5.3.26b}$$

$$C^{\pm} = \frac{P^{\pm}\left(p_a^2,p_b^2,p_c^2\right)}{4p_c^3\left(p_c^2-p_a^2\right)^2\left(p_c^2-p_b^2\right)^2}, \tag{5.3.26c}$$

$$K^{\pm} = \frac{P\left(\mp p_c\right)}{4p_c^2\left(p_c^2-p_a^2\right)\left(p_c^2-p_b^2\right)}, \tag{5.3.26d}$$

and

$$
\begin{aligned}
P^{\pm}\left(p_a^2, p_b^2, p_c^2\right) = &\pm K_0\left(-3p_c^2\left(p_a^2+p_b^2\right)+p_a^2p_b^2+5p_c^4\right) \\
&- 2p_c^3K_1\left(2p_a^2p_b^2p_c^2-p_a^2-p_b^2-p_b^2p_c^4-p_a^2p_c^4+2p_c^2\right) \\
&\pm p_c^2K_2\left(-p_a^2p_b^2+3p_c^4-p_a^2p_c^2-p_b^2p_c^4-p_a^2p_c^4-p_c^6-p_b^2p_c^2+3p_a^2p_b^2p_c^2\right).
\end{aligned} \tag{5.3.27}
$$

We will leave the solution in the form (5.3.25) for now.

5.3.3 The line at $\lambda = 0$

On the line at $\lambda = 0$ we need to perform the integration for the contour shown in Figure 5.1.

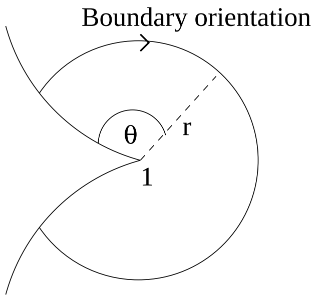

Figure 5.1: Contour of integration about particle 1.

The parameterisation for the $\mathbb{R}^{2,2}$ coordinates is

$$U = \left(\frac{1+(\rho_1+r\cos\theta)^2+r^2\sin^2\theta}{1-(\rho_1+r\cos\theta)^2-r^2\sin^2\theta}\right)\cos T, \tag{5.3.28a}$$

$$V = \left(\frac{1+(\rho_1+r\cos\theta)^2+r^2\sin^2\theta}{1-(\rho_1+r\cos\theta)^2-r^2\sin^2\theta}\right)\sin T, \tag{5.3.28b}$$

$$X = \frac{2\left(\rho_1+r\cos\theta\right)}{1-(\rho_1+r\cos\theta)^2-r^2\sin^2\theta}, \tag{5.3.28c}$$

$$Y = \frac{2r\sin\theta}{1-(\rho_1+r\cos\theta)^2-r^2\sin^2\theta}, \tag{5.3.28d}$$

where r and θ are shown in Figure 5.1. After the integration we will take the limit $r \to 0$. The function we integrate is similar to (5.3.14) with s replaced by θ. After performing a similar analysis to the previous section we find that the contribution to the Lagrangian from this contour is

$$L_4 = \frac{1}{\pi}\left[\frac{\cosh\beta_1}{\sqrt{\chi}}\left(\sqrt{\chi}\delta_1 - \theta_i\right)\dot{T} - \frac{\cot T}{\sqrt{\chi}\sinh\beta_1}\left(\sqrt{\chi}\frac{\delta_1}{2} - \theta_i\right)\dot{\beta}_1\right], \quad (5.3.29)$$

where

$$\theta_i = \arctan\left(\sqrt{\chi}\tan\frac{\delta_1}{2}\right), \quad (5.3.30)$$

and

$$\chi = \cos^2 T + \sin^2 T\cosh^2\beta_1. \quad (5.3.31)$$

5.3.4 The two small half-circles about the singular points 2 and 2'

The final contribution to the Lagrangian is from the two small half-circles about the singular points 2 and 2'. The gauge choice in this region is determined by first specifying a group-valued function in the interval $\rho \in (\rho_2 - \tau, \rho_2 + \epsilon)$ where τ and ϵ are small, such that it takes the value ${}^{\pm}n$ at $\rho = \rho_2 - \tau$ and ${}^{\pm}m^{-1}$ at $\rho = \rho_2 + \epsilon$. A prospective choice for this function is provided by the formula

$${}^{\pm}p = \begin{pmatrix} {}^{\pm}\widetilde{f} & 0 \\ 0 & {}^{\pm}\widetilde{f}^{-1} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \left(\widetilde{K}+1\right)^{\frac{1}{2}} & \pm\left(\widetilde{K}-1\right)^{\frac{1}{2}} \\ \pm\left(\widetilde{K}-1\right)^{\frac{1}{2}} & \left(\widetilde{K}+1\right)^{\frac{1}{2}} \end{pmatrix}, \quad (5.3.32)$$

where

$$\widetilde{f} = \frac{\left(\widetilde{Y} - i\left(\widetilde{C}\widetilde{X} - \widetilde{S}\widetilde{U}\right)\right)^{\frac{1}{4}}\left(\widetilde{V} \mp i\left(\widetilde{C}\widetilde{U} - \widetilde{S}\widetilde{X}\right)\right)^{\frac{1}{4}}}{\left(\widetilde{Y} + i\left(\widetilde{C}\widetilde{X} - \widetilde{S}\widetilde{U}\right)\right)^{\frac{1}{4}}\left(\widetilde{V} \pm i\left(\widetilde{C}\widetilde{U} - \widetilde{S}\widetilde{X}\right)\right)^{\frac{1}{4}}}, \tag{5.3.33a}$$

$$\widetilde{C} = \cosh\left(\beta_1\left(1-p\right)\right), \tag{5.3.33b}$$

$$\widetilde{S} = \sinh\left(\beta_1\left(1-p\right)\right), \tag{5.3.33c}$$

$$\widetilde{K} = \left(\widetilde{C}\widetilde{U} - \widetilde{S}\widetilde{X}\right)^2 + \widetilde{V}^2, \tag{5.3.33d}$$

$$\widetilde{U} = \left(\frac{1+\widetilde{\rho}^2}{1-\widetilde{\rho}^2}\right)\cos T, \tag{5.3.33e}$$

$$\widetilde{V} = \left(\frac{1+\widetilde{\rho}^2}{1-\widetilde{\rho}^2}\right)\sin T, \tag{5.3.33f}$$

$$\widetilde{X} = \left(\frac{2\widetilde{\rho}}{1-\widetilde{\rho}^2}\right)\cos\phi, \tag{5.3.33g}$$

$$\widetilde{Y} = \left(\frac{2\widetilde{\rho}}{1-\widetilde{\rho}^2}\right)\sin\phi, \tag{5.3.33h}$$

$$\widetilde{\rho} = \rho_2 + \epsilon p - \tau\left(1-p\right). \tag{5.3.33i}$$

We have introduced the parameter $p \in [0,1]$ to interpolate between the two values in the interim region. $p = 0$ at $\rho = \rho_2 - \tau$ and $p = 1$ at $\rho = \rho_2 + \epsilon$. However, formula (5.3.32) could conceivably fail if it becomes singular somewhere in the domain bounded by the two small semicircles and the lines $p = 0$ and $p = 1$. As we will only be needing the gauge choice on the two small semicircles it is possible to resolve this issue by an indirect argument as follows.

Consider the gauge choice ${}^{\pm}p$ (5.3.32) on the two small semicircles and on the lines $p = 0$ and $p = 1$. When tracing over the closed curve formed by these four lines the gauge function ${}^{\pm}p$ traces, by construction, a closed curve in $\mathrm{SU}(1,1) \cong \mathbb{R}^2 \times S^1$. If this closed curve in $\mathrm{SU}(1,1)$ is homotopically trivial, that is, does not wind around the S^1 factor, then there exists an extension of the gauge function into the domain bounded by the four lines and we may use ${}^{\pm}p$ on the small semicircles. If this closed curve in $\mathrm{SU}(1,1)$ is not homotopically trivial, we modify ${}^{\pm}p$ by including on one or both of the small semicircles a factor that does the requisite unwinding to make the new curve homotopically trivial. We then use this modified choice for ${}^{\pm}p$ on the small semicircles.

The author has not been able to complete this analysis and the contribution L_5 remains undetermined. However, as the idea behind all of this is to obtain a "nice" form for the reduced action we are not overly worried about L_5 mainly due to the intractable nature of L_3. A "nice" form for the reduced action will be obtained in a different way in Chapter 6.

Chapter 6

The reduced action

In this chapter we first obtain the reduced Hamiltonian action in a phase space chart in which the "position" coordinate is the geodesic distance between the two particles. We then use this action to analyse the black hole formation threshold in the language of critical phenomena, and in particular we compute the critical exponent of the black hole mass at this threshold. We also use the Hamiltonian action to analyse the black hole creation/annihilation as a quantum mechanical tunnelling process, finding that the imaginary part of the action is equal to the Bekenstein-Hawking entropy.

6.1 The equations of motion

Although algebraic complications prevented us from computing the Liouville terms in the reduced Hamiltonian action in Chapter 5, the analysis of Chapter 5 did show that the phase space is two-dimensional and can be coordinatised by the (a priori non-canonical) chart (α, T) as can be seen from the form of L_1 (5.3.5), L_3 (5.3.25) and the equations relating α to β_1 (2.6.7) and (2.6.5). α refers to a two-particle spacetime, T refers to a particular spacelike slice within the spacetime, and both parameters are here understood as functions of the BTZ coordinate time t. The analysis also showed that the value of the Hamiltonian is $-\alpha^2$ by virtue of the form of S_∞ (5.0.1). Further, it follows from the definition of T, α and the BTZ time t that the equations of motion in this chart take the simple form

$$\dot{\alpha} = 0, \tag{6.1.1a}$$

$$\dot{T} = \alpha, \tag{6.1.1b}$$

where the dot indicates differentiation with respect to t. We shall now use these facts to indirectly determine the Liouville terms in the action.

6.1.1 Reproducing the E.O.M

We wish to find an action, S, that reproduces (6.1.1), with the Hamiltonian $H = -\alpha^2$ and with Liouville terms of the form obtained in Chapter 5. The action must take the form

$$S = \int dt \left[\dot{T}F(\alpha, T) + \dot{\alpha}\,G(\alpha, T) + \alpha^2\right] \tag{6.1.2}$$

where the functions $F(\alpha, T)$ and $G(\alpha, T)$ are to be determined.

We can simplify (6.1.2) somewhat by noting that there exists a function $h(\alpha, T)$ such that

$$G(\alpha, T) = \frac{\partial h(\alpha, T)}{\partial \alpha}. \tag{6.1.3}$$

The second term in the integrand in (6.1.2) can thus be rewritten as

$$\dot{\alpha}G(\alpha, T) = \frac{d}{dt}h(\alpha, T) - \dot{T}\frac{\partial h}{\partial T}. \tag{6.1.4}$$

As a total derivative in the integrand in (6.1.2) will not affect the E.O.M, it suffices to consider the action

$$S = \int dt \left[\dot{T}f(\alpha, T) + \alpha^2\right], \tag{6.1.5}$$

where the function $f(\alpha, T)$ is to be determined.

The Euler-Lagrange equation for α combined with (6.1.1b) yields a differential equation for f,

$$\frac{\partial f}{\partial \alpha} = -2, \tag{6.1.6}$$

which is easily solved,

$$f = -2\alpha + \phi(T). \tag{6.1.7}$$

The Euler-Lagrange equation for T combined with (6.1.1a) shows that $\phi(T)$ is a constant, and we may assume without loss of generality that $\phi(T) = 0$. A Lagrangian reproducing (6.1.1) is therefore

$$L = -2\alpha\dot{T} + \alpha^2. \tag{6.1.8}$$

Rewriting α in terms of the new variable

$$p_T := -2\alpha, \tag{6.1.9}$$

the action takes the Hamiltonian form

$$S = \int dt \left[p_T\dot{T} - H\right], \tag{6.1.10}$$

where

$$H = -\frac{p_T^2}{4}. \tag{6.1.11}$$

The phase space has thus dimension two, and the pair (T, p_T) provides a canonical chart, with $-\frac{\pi}{2} < T < \frac{\pi}{2}$ and $-2 < p_T < 0$.

6.2 Massless particles

As seen above, the theory has a two-dimensional phase space and (T, p_T) is a canonical chart. The task now is to undertake a canonical transformation $(T, p_T) \to (r_c, p_{r_c})$ where r_c is the geodesic distance between the particles and p_{r_c} is its conjugate momentum. In this section we shall do this in the limit where the particles are massless, $\delta_1 \to 0$ and $\delta_2 \to 0$. As we have until now assumed both δ_1 and δ_2 to be strictly positive, we first need to take this limit in the formulas of Chapter 2.

To analyse the limit we write $\delta_1 = \epsilon m_1$ and $\delta_2 = \epsilon m_2$ where m_i are regarded as positive constants and ϵ is a positive parameter that will eventually be taken to zero. We anticipate that in the limit $\epsilon \to 0$, β should go to infinity. An ansatz which turns out to give the correct scaling is to write

$$\cosh\beta = \frac{4\mu}{\epsilon^2 m_1 m_2}, \tag{6.2.1}$$

where μ is independent of ϵ. We then find, after taking $\epsilon \to 0$, that

$$c_i = 1, \tag{6.2.2a}$$

$$s_i = 0, \tag{6.2.2b}$$

$$\cos\frac{\delta}{2} = 1 - \mu. \tag{6.2.2c}$$

We can rearrange (6.2.2c) to express α as

$$\alpha = \frac{1}{\pi}\arccos\left(\mu - 1\right). \tag{6.2.3}$$

Note that pure AdS_3 occurs when $\mu = 0$ and the threshold of black hole formation occurs at $\mu = 2$.

For the geodesic distance between the particles, taking the $\epsilon \to 0$ limit of (2.6.13) gives

$$\cosh r_c = 1 + \frac{(2-\mu)}{\tan^2 T}. \tag{6.2.4}$$

6.2.1 The canonical transformation

For brevity of notation, we now drop the subscript c and let r stand for the geodesic distance between the particles and p_r for its conjugate momentum. As a first step towards the canonical transformation we find, from (6.1.9), (6.2.3) and (6.2.4)

$$\sinh\frac{r}{2} = -\frac{\sin\left(\frac{\pi p_T}{4}\right)}{\tan T}, \tag{6.2.5}$$

We have chosen a minus sign in (6.2.5) as $-2 < p_T < 0$ and we're assuming that we are in the colliding particle regime where $0 < T < \frac{\pi}{2}$. Rearranging (6.2.5) gives

$$p_T = -\frac{4}{\pi}\arcsin\left(\tan T \sinh\frac{r}{2}\right). \tag{6.2.6}$$

The criterion for the transformation from $(T, p_T) \to (r, p_r)$ to be canonical can be written as [30]

$$p_r dr - p_T dT = df, \tag{6.2.7}$$

where p_r and p_T are regarded as functions of T and r, and f is an arbitrary function of T and r. To find a $p_r(T, r)$ that satisfies (6.2.7), we note that (6.2.7) is equivalent to the condition

$$\frac{\partial p_r}{\partial T} = -\frac{\partial p_T}{\partial r}. \tag{6.2.8}$$

After a fairly lengthy straightforward calculation using (6.2.6) we find that

$$\frac{\partial p_r}{\partial T} = -\frac{2}{\pi}\frac{\partial}{\partial T}\operatorname{arccosh}\left(\frac{\cos T}{\tanh\frac{r}{2}}\right). \tag{6.2.9}$$

The general solution is

$$p_r = -\frac{2}{\pi}\operatorname{arccosh}\left(\frac{\cos T}{\tanh\frac{r}{2}}\right) + g(r), \tag{6.2.10}$$

where the function $g(r)$ is arbitrary. We choose $g(r) = 0$ for reasons that will become apparent shortly.

The new phase space coordinates (r, p_r) are then defined implicitly by

$$-\sin\left(\frac{\pi p_T}{4}\right) = \sinh\left(\frac{r}{2}\right)\tan T, \tag{6.2.11a}$$

$$\cos T = \cosh\left(\frac{\pi p_r}{2}\right)\tanh\left(\frac{r}{2}\right). \tag{6.2.11b}$$

Eliminating T from (6.2.11) we find

$$p_T = \frac{4}{\pi}\arccos\left(-\cosh\left(\frac{r}{2}\right)\tanh\left(\frac{\pi p_r}{2}\right)\right). \tag{6.2.12}$$

In the new chart the Hamiltonian takes therefore the form

$$H = -\frac{4}{\pi^2}\left[\arccos\left(-\cosh\left(\frac{r}{2}\right)\tanh\left(\frac{\pi p_r}{2}\right)\right)\right]^2. \tag{6.2.13}$$

Note that in the limit $p_r \to 0$, we have $H \to -1$, and the spacetime becomes pure AdS_3. This was the reason to choose $g(r) = 0$ in (6.2.10).

In deriving (6.2.13) we have used the information that we are in the colliding particle regime where $p_r < 0$ and $0 < T < \frac{\pi}{2}$. We can generalise (6.2.13) to cover both the colliding ($p_r < 0$, $0 < T < \frac{\pi}{2}$) and the expanding ($p_r > 0$, $-\frac{\pi}{2} < T < 0$) particle regime by writing

$$H = -\frac{4}{\pi^2}\left[\arccos\left(\cosh\left(\frac{r}{2}\right)\tanh\left(\frac{\pi|p_r|}{2}\right)\right)\right]^2. \tag{6.2.14}$$

When $\cosh\left(\frac{r}{2}\right)\tanh\left(\frac{\pi|p_r|}{2}\right) > 1$, the Hamiltonian (6.2.14) continues analytically to

$$H = \frac{4}{\pi^2}\left[\operatorname{arccosh}\left(\cosh\left(\frac{r}{2}\right)\tanh\left(\frac{\pi|p_r|}{2}\right)\right)\right]^2, \tag{6.2.15}$$

which takes positive values. The geometry near the infinity is then the BTZ geometry (2.2.9) with $M = H > 0$, $S = 0$ and ψ periodic with period 2π. This is the spinless BTZ black hole with mass $M = H$ [22, 23].

We note that our Hamiltonian (6.2.15) differs from that obtained in [31], even though both use the geodesic distance as the configuration variable. The reason is that the time coordinate used in [31] (see equation (3.21) therein) is related to the BTZ time coordinate by a rescaling that depends on M.

6.2.2 Threshold of black hole formation

The threshold of black hole formation is where $H = 0$. We wish to examine how the mass of the black hole depends on p_r near this threshold.

From (6.2.14) we can see that $H = 0$ occurs when

$$\cosh\left(\frac{r}{2}\right)\tanh\left(\frac{\pi|p_r|}{2}\right) = 1. \tag{6.2.16}$$

We set

$$\cosh\left(\frac{r}{2}\right)\tanh\left(\frac{\pi|p_r|}{2}\right) = 1 + \epsilon, \tag{6.2.17}$$

for small $\epsilon > 0$. The behaviour of H in terms of ϵ is

$$H = \frac{8}{\pi^2}\epsilon + O\left(\epsilon^2\right). \tag{6.2.18}$$

For fixed r, ϵ behaves as

$$\epsilon = K\left(p_r - p_0\right), \tag{6.2.19}$$

where p_0 is the value of p_r at which $H = 0$ and K is constant. We hence find

$$H = \frac{8K}{\pi^2} (p_r - p_0) + O (p_r - p_0)^2 , \tag{6.2.20}$$

Near the black hole formation threshold, the mass therefore depends linearly on the momentum of the particles. In the language of critical exponents, we can say that the mass behaves linearly as a function of the initial data, or has the critical exponent 1. This agrees with what was found in [12] in a formulation that parameterises the phase space in terms of the time-independent $O_c (2, 2)$ holonomies of the two-particle spacetime.

6.2.3 Action for tunnelling from the black hole

In [32] the authors evaluate the classical action S of the positive definite section of the Schwarzschild geometry. They postulate that $Z \approx \exp (-S)$ is the correct partition function to use, and show that this partition function indeed reproduces the Bekenstein-Hawking entropy by the usual formulae of the canonical ensemble. See also [33] and the review in [34].

Motivated by this and subsequent work, in [35] the authors perform a tunnelling calculation for ripping a pair of magnetically charged black holes out of the vacuum. Some clarifying comments are in [36].

As we have formulated a classical action for our point particle system, we are motivated to consider a quantum mechanical process analogous to the one considered in [35] in which the particles semiclassically tunnel out of the black hole.

To begin, suppose we are in the black hole regime, $H > 0$, and consider the action in the new chart,

$$S = \int dt \ \ (p_r \dot{r} - H) , \tag{6.2.21}$$

where H is given by (6.2.15). Assume further r to be so large that the particles are outside the horizon, and consider the expanding case $p_r > 0$. One E.O.M is simply that H is constant. The other E.O.M is that

$$\dot{r} = \frac{\partial H}{\partial p_r}, \tag{6.2.22}$$

which after some algebra yields

$$\dot{r} = \frac{2\sqrt{H}}{\sinh\left(\frac{\pi\sqrt{H}}{2}\right)\cosh\left(\frac{r}{2}\right)}\left[\sinh^2\left(\frac{r}{2}\right) - \sinh^2\left(\frac{\pi\sqrt{H}}{2}\right)\right]. \tag{6.2.23}$$

The expression in square brackets is zero when

$$r = r_{\text{horizon}} = \pi\sqrt{H}, \tag{6.2.24}$$

which is the value of r at which both particles are at the horizon of the black hole. It would take an infinite amount of coordinate time t to reach r_{horizon} as one would expect by the form of the BTZ metric (2.2.9) with positive mass.

The solution to (6.2.23) is found by first making the substitutions

$$A = \sinh\left(\frac{\pi\sqrt{H}}{2}\right), \tag{6.2.25a}$$

$$S = \sinh\left(\frac{r}{2}\right), \tag{6.2.25b}$$

so that after some manipulation

$$2\sqrt{H}dt = d\ln\left(\frac{S-A}{S+A}\right). \tag{6.2.26}$$

Choosing a suitable zero for t, we end up with

$$\sinh\left(\frac{r}{2}\right) = -\coth\left(\sqrt{H}t\right)\sinh\left(\frac{\pi\sqrt{H}}{2}\right), \tag{6.2.27}$$

where $t < 0$. Note that $r \to \infty$ as $t \to 0_-$ and $r \to r_{\text{horizon}}$ as $t \to -\infty$.

We now wish to examine the tunnelling process from $r = 0$ to $r > r_{\text{horizon}}$. The trajectory will have to be complex but it should have a well-defined action. We wish to evaluate this action and see whether its imaginary part is related to the Bekenstein-Hawking entropy.

We can express (6.2.21) in terms of an integral over r;

$$S = \int dr\ \left(p_r - \frac{H}{\dot{r}}\right). \tag{6.2.28}$$

The first term in the integrand is

$$p_r = \frac{2}{\pi} \operatorname{arctanh} \left(\frac{\cosh\left(\frac{\pi\sqrt{H}}{2}\right)}{\cosh\left(\frac{r}{2}\right)} \right). \tag{6.2.29}$$

The integral we are interested in is

$$\int_0^\infty p_r dr = -\frac{1}{\pi} \int_0^\infty dr \ln \left[\frac{\cosh\left(\frac{r}{2}\right) - \cosh\left(\frac{\pi\sqrt{H}}{2}\right)}{\cosh\left(\frac{r}{2}\right) + \cosh\left(\frac{\pi\sqrt{H}}{2}\right)} \right], \tag{6.2.30}$$

and evaluating this gives the imaginary contribution

$$-i\eta\pi\sqrt{H}, \tag{6.2.31}$$

where $\eta = +1$, $(\eta = -1)$ if r has a small positive (negative) part around the singularity at $r = r_{\text{horizon}}$.

The second term in the integrand is

$$\frac{H}{\dot{r}} = \frac{A\sqrt{H}}{2} \frac{\cosh\left(\frac{r}{2}\right)}{\sinh^2\left(\frac{r}{2}\right) - A^2}, \tag{6.2.32}$$

so that the integral becomes

$$-\int_0^\infty \frac{H}{\dot{r}} dr = -\frac{\sqrt{H}}{2} \left[\ln\left(\frac{S-A}{S+A} \right) \right]_{r=0}^{r=\infty}. \tag{6.2.33}$$

Evaluating this gives an imaginary contribution of

$$i\eta\pi\frac{\sqrt{H}}{2}. \tag{6.2.34}$$

Collecting both imaginary terms together we get the imaginary contribution to the total action of

$$-i\eta\frac{2\pi\sqrt{H}}{4} = -i\eta\frac{A}{4}, \tag{6.2.35}$$

where A is the horizon circumference,

$$A = 2\pi\sqrt{H}, \tag{6.2.36}$$

The imaginary part of the action is hence equal to the Bekenstein-Hawking entropy of the black hole [22].

6.3 Massive particles

In this section we keep the masses of the two particles strictly positive. We start with generic values of the masses but specialise early on to equal masses.

6.3.1 The canonical transformation

For two massive particles the algebra in finding a canonical transformation is substantially more difficult. The geodesic distance between the particles is given by (2.6.13),

$$\cosh r = \frac{\cos^2 T \cosh\beta + \sin^2 T \cosh\beta_1 \cosh\beta_2}{\left(\cos^2 T + \sin^2 T \cosh^2\beta_1\right)^{\frac{1}{2}} \left(\cos^2 T + \sin^2 T \cosh^2\beta_2\right)^{\frac{1}{2}}}, \tag{6.3.1}$$

where β_i are given by (2.6.5) and (2.6.6) and β is related to α through (2.6.7).

The first task is to invert (6.3.1) to find an expression for T in terms of r and p_T. To do this we write

$$v = \cos\left(\frac{\pi p_T}{2}\right) = -\cos\left(\frac{\delta}{2}\right), \tag{6.3.2}$$

where $-1 < v < 1$, and

$$\sigma = \sin^2 T, \tag{6.3.3}$$

and find (after a lengthy calculation) that

$$\tan^2 T = \frac{2g\left(R^2 - v^2\right)}{R^2(m+n) - 2vh - R\left(1-v^2\right)\left[R^2\left(c_1^2 - c_2^2\right)^2 + 4c_1c_2\sqrt{m}\sqrt{n}\right]^{\frac{1}{2}}} - 1, \tag{6.3.4}$$

where

$$R = \cosh r, \tag{6.3.5a}$$

$$g = c_1^2 + c_2^2 - 1 + 2vc_1c_2 + v^2, \tag{6.3.5b}$$

$$h = v\left(c_1^2 + c_2^2\right) + c_1c_2\left(1+v^2\right), \tag{6.3.5c}$$

$$m = \left(c_1 + c_2 v\right)^2, \tag{6.3.5d}$$

$$n = \left(c_2 + c_1 v\right)^2. \tag{6.3.5e}$$

We shall only consider the special case of equal masses, $\delta_1 = \delta_2$.

6.3.2 Equal masses

Specialising to equal masses, $c_1 = c_2 := c$, where $0 < c < 1$, (6.3.4) simplifies to

$$\tan^2 T = \left(\frac{u^2}{S^2} + 1\right)\left(\frac{c^2 - u^2}{c^2\,(1 - u^2)}\right) - 1, \tag{6.3.6}$$

where we have written

$$u = -\sqrt{\frac{1-v}{2}}, \tag{6.3.7a}$$

$$S = \sinh\frac{r}{2} = \sqrt{\frac{R-1}{2}}. \tag{6.3.7b}$$

The range of u is such that $-c < u < 0$. Note that when we take the massless limit $c \to 1$ we recover (6.2.11a).

We now consider p_r and T as functions of p_T and r. The condition for the transformation from (T, p_T) to (r, p_r) to be canonical is then

$$p_r dr + T dp_T = d\widetilde{f}, \tag{6.3.8}$$

where $\widetilde{f}$ is a function of r and p_T. An equivalent form is

$$\frac{\partial p_r}{\partial p_T} = \frac{\partial T}{\partial r}. \tag{6.3.9}$$

The R.H.S is easily computed and we find

$$\frac{\partial p_r}{\partial p_T} = -\frac{u^2 C}{2S^3\left(1 + \frac{u^2}{S^2}\right)\left[\left(1 + \frac{u^2}{S^2}\right)\left(\frac{c^2-u^2}{c^2(1-u^2)}\right) - 1\right]^{\frac{1}{2}}}, \tag{6.3.10}$$

where

$$C = \cosh\left(\frac{r}{2}\right), \tag{6.3.11}$$

and we have assumed we are in the range $0 < T < \frac{\pi}{2}$. Changing the differentiation variable to u and simplifying the expression in the square brackets we find

$$\frac{\partial p_r}{\partial u} = \frac{2Ccu}{\pi\,(u^2 + S^2)\,[1 - s^2 C^2 - u^2]^{\frac{1}{2}}}, \tag{6.3.12}$$

where

$$s = \sqrt{1 - c^2}. \tag{6.3.13}$$

We can integrate this directly by changing variables. The answer is

$$p_r = \frac{1}{\pi} \ln \left[\frac{Cc - \left(1 - s^2C^2 - u^2\right)^{\frac{1}{2}}}{Cc + \left(1 - s^2C^2 - u^2\right)^{\frac{1}{2}}} \right], \tag{6.3.14}$$

where we have set the arbitrary function of r obtained when integrating to zero in order that $p_r = 0$ at $T = 0$. We now rearrange this expression to find

$$\left(1 - u^2\right)^{\frac{1}{2}} = C \left(s^2 + c^2 \tanh^2\left(\frac{\pi p_r}{2}\right)\right)^{\frac{1}{2}}. \tag{6.3.15}$$

Collecting everything together we can now express the Hamiltonian in the new chart as

$$H = -\frac{4}{\pi^2} \left\{ \arccos \left[C \left(s^2 + c^2 \tanh^2\left(\frac{\pi p_r}{2}\right)\right)^{\frac{1}{2}} \right] \right\}^2. \tag{6.3.16}$$

The analytic continuation of (6.3.16) to the black hole regime, $H > 0$, is

$$H = \frac{4}{\pi^2} \left\{ \operatorname{arccosh} \left[C \left(s^2 + c^2 \tanh^2\left(\frac{\pi p_r}{2}\right)\right)^{\frac{1}{2}} \right] \right\}^2, \tag{6.3.17}$$

where $C\left(s^2 + c^2 \tanh^2\left(\frac{\pi p_r}{2}\right)\right)^{\frac{1}{2}} > 1$. To compute (6.3.16) we have assumed that $0 < T < \frac{\pi}{2}$ and $p_r < 0$. A similar analysis shows that (6.3.16) holds also when $-\frac{\pi}{2} < T < 0$ and $p_r > 0$. The result thus holds for $-\frac{\pi}{2} < T < \frac{\pi}{2}$.

6.3.3 Threshold of black hole formation

The threshold of black hole formation is where $H = 0$. As in the massless case the analysis yields the same formula for the Hamiltonian (6.2.20) showing that the critical exponent is 1, again in agreement with [12].

6.3.4 Action for tunnelling from the black hole

The analysis in this section is qualitatively similar to the action analysis for the massless case. Again, one E.O.M is simply that H is constant. After a lengthy calculation the other E.O.M yields

$$\frac{\sinh^2\left(\frac{r}{2}\right)}{\sinh^2\left(\frac{\pi\sqrt{H}}{2}\right)} = \frac{\frac{s^2}{c^2}\tanh^2\left(\frac{\pi\sqrt{H}}{2}\right) + 1}{\frac{s^2}{c^2}\tanh^2\left(\frac{\pi\sqrt{H}}{2}\right) + \tanh^2\left(\sqrt{H}t\right)}. \tag{6.3.18}$$

The particles again reach the black hole horizon when $r_{\text{horizon}} = \pi\sqrt{H}$. It would take an infinite amount of coordinate time t to reach r_{horizon}.

After evaluating the imaginary contribution of the action we find it to be again given by formulas (6.2.35) and (6.2.36), and hence equal to the Bekenstein-Hawking entropy of the black hole.

6.4 One massive and one massless particle

To end this chapter, we consider briefly the special case when we have one massive and one massless particle. We will just state the result, the analysis being entirely analogous with the previous sections in this chapter.

Taking particle 2 to be massless, we have $s_2 = 0$ and $c_2 = 1$. The Hamiltonian in the black hole regime is

$$\begin{aligned} H =& \frac{4}{\pi^2}\Big\{ \operatorname{arccosh}\Big[- c_1 \left(1 - \tanh^2\left(\frac{\pi p_r}{2}\right)\right) \\ & + \cosh(r)\tanh\left(\frac{\pi|p_r|}{2}\right)\left(s_1^2 + c_1^2\tanh^2\left(\frac{\pi p_r}{2}\right)\right)\Big]\Big\}^2. \end{aligned} \tag{6.4.1}$$

We did not complete the analysis for the critical exponent and the tunnelling action but have no reason to expect the results would be any different to those obtained in the massless and equal massive cases.

According to the AdS/CFT correspondence, [20], processes happening inside AdS space should be describable by a conformal field theory on the boundary of AdS. In particular, the processes involving point particles we have considered in the first part of the thesis should be able to be described in terms of a CFT, see [37] and [38]. Although we do not follow up on this here, we now draw motivation from the AdS/CFT correspondence to discuss CFT in a different setting in the second part of the thesis.

Chapter 7

$\mathrm{SL}(2,\mathbb{C})$ twistor space

This chapter is concerned with homogeneous spaces constructed from the Lie group $G = \mathrm{SL}(2,\mathbb{C})$ via coset space techniques. We realise "$\mathrm{SL}(2,\mathbb{C})$ twistor space", showing that it has complex dimension two, a metric with signature $(+,+,-,-)$ and an interesting complex structure. We also consider a second coset space of G and discuss its global properties. Essentially we would like to set the mathematical scene for physical applications within conformal field theory.

7.1 The twistor coset, $\mathbb{T}$ - matrix representation

We consider the Lie group $G = \mathrm{SL}(2,\mathbb{C})$. We define the twistor coset $\mathbb{T}$ as a homogeneous space via a quotient space construction,

$$\mathbb{T} := G/A = \{gA \mid g \in G\}\,, \tag{7.1.1}$$

where A is the diagonal subgroup,

$$A = \left\{ \begin{pmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{pmatrix} \mid \delta \in \mathbb{C} \setminus \{0\} \right\}. \tag{7.1.2}$$

Elements of G/A are equivalence classes on G where the equivalence relation is $g \sim g'$ if $\exists\ a \in A \mid g' = ga$, and so $[g] = [ga]\ \forall a \in A$. The relationship between elements $g \in G$ and elements $[g] \in \mathbb{T}$ is, for

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{7.1.3}$$

$$[g] = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] = \left\{ \begin{pmatrix} a\delta^{-1} & b\delta \\ c\delta^{-1} & d\delta \end{pmatrix} \mid \delta \in \mathbb{C} \setminus \{0\} \right\}. \tag{7.1.4}$$

When $d \neq 0$, we may choose complex coordinates (x_1, x_2) on $\mathbb{T}$ by setting $\delta = d^{-1}$, $x_1 = \frac{b}{d}$ and $x_2 = \frac{c}{a}$ in (7.1.4), so that a unique representative of a class in $\mathbb{T}$ is given by

$$t(x_1, x_2) = \begin{pmatrix} \dfrac{1}{1 - x_1 x_2} & x_1 \\ \dfrac{x_2}{1 - x_1 x_2} & 1 \end{pmatrix}, \tag{7.1.5}$$

where $x_i \in \mathbb{C}$, and

$$x_1 x_2 \neq 1. \tag{7.1.6}$$

An alternative parameterisation is

$$\bar{t}(\xi_1, \xi_2) = \begin{pmatrix} \dfrac{\xi_2}{\xi_2 - \xi_1} & \xi_1 \\ \dfrac{1}{\xi_2 - \xi_1} & 1 \end{pmatrix}, \tag{7.1.7}$$

where $\xi_i \in \mathbb{C}$, and

$$\xi_1 \neq \xi_2. \tag{7.1.8}$$

The equivalence classes where $d = 0$ form a subset of $\mathbb{T}$ of complex dimension 1 and can be understood as singular limits in the parameterisations (7.1.5) or (7.1.7).

Among the infinitely many parameterisations of $\mathbb{T}$ available these particular two give "nice" transformations of the coordinates when the action of the covering group on the space is considered as will be done in the following subsection.

7.2 G-action on $\mathbb{T}$

We show here that the natural left action of G on $\mathbb{T}$ is by distinct fractional linear transformations on the two complex coordinates. In deriving this result we use the fact that the action of G on elements of G/A is the same as the class of the action of G on any representative of G/A, namely (for $g \in G$)

$$g[t] = [gt]. \tag{7.2.1}$$

We denote an element of A by

$$\widetilde{a}\,(\delta) = \begin{pmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{pmatrix}, \tag{7.2.2}$$

and use a generic representative h of G/A by leaving δ freely specifiable for the moment,

$$\begin{aligned} h(x_1, x_2, \delta) &= t(x_1, x_2) \circ \widetilde{a}\,(\delta) \\ &= \begin{pmatrix} \dfrac{\delta^{-1}}{1 - x_1 x_2} & \delta x_1 \\ \dfrac{\delta^{-1} x_2}{1 - x_1 x_2} & \delta \end{pmatrix}. \end{aligned} \tag{7.2.3}$$

The action is

$$\begin{aligned} g[t(x_1, x_2)] = g[h(x_1, x_2, \delta)] &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \left[\begin{pmatrix} \dfrac{\delta^{-1}}{1 - x_1 x_2} & \delta x_1 \\ \dfrac{\delta^{-1} x_2}{1 - x_1 x_2} & \delta \end{pmatrix} \right] \\ &= \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \dfrac{\delta^{-1}}{1 - x_1 x_2} & \delta x_1 \\ \dfrac{\delta^{-1} x_2}{1 - x_1 x_2} & \delta \end{pmatrix} \right] \\ &= \left[\begin{pmatrix} \dfrac{(a + b x_2)(c x_1 + d)}{1 - x_1 x_2} & \dfrac{a x_1 + b}{c x_1 + d} \\ \dfrac{(c + d x_2)(c x_1 + d)}{1 - x_1 x_2} & 1 \end{pmatrix} \right] \\ &= [t(x_1', x_2')], \end{aligned} \tag{7.2.4}$$

where in the last step we have chosen $\delta = (cx_1 + d)^{-1}$. The explicit coordinate transformation is given by

$$(x_1, x_2) \mapsto (x_1', x_2') = \left(\frac{ax_1 + b}{cx_1 + d}, \frac{dx_2 + c}{bx_2 + a} \right). \tag{7.2.5}$$

In the alternative coordinatisation the transformation is

$$(\xi_1, \xi_2) \mapsto (\xi_1', \xi_2') = \left(\frac{a\xi_1 + b}{c\xi_1 + d}, \frac{a\xi_2 + b}{c\xi_2 + d} \right). \tag{7.2.6}$$

We now study the one-parameter subgroups of G and calculate the corresponding left invariant vector fields on G.

We use the generic representative (7.2.3), denoted by $h = t \circ \widetilde{a}$. The natural left action of G is via $h \mapsto h' = g \circ h$ where $g \in G$. If we parameterise the element g in the same way as (7.2.3) with coordinates (z_1, z_2, β) the explicit transformation of the coordinates is

$$x_1' = \frac{\beta^2 z_1 (1 - z_1 z_2) + x_1}{\beta^2 (1 - z_1 z_2) + z_2 x_1}, \tag{7.2.7a}$$

$$x_2' = \frac{\beta^2 x_2 (1 - z_1 z_2) + z_2}{\beta^2 x_2 z_1 (1 - z_1 z_2) + 1}, \tag{7.2.7b}$$

$$\delta' = \frac{\delta \left(\beta^2 (1 - z_1 z_2) + z_2 x_1\right)}{\beta (1 - z_1 z_2)}, \tag{7.2.7c}$$

with corresponding inverse transformations

$$x_1 = \frac{\beta^2 (1 - z_1 z_2)(x_1' - z_1)}{(1 - z_2 x_1')}, \tag{7.2.8a}$$

$$x_2 = \frac{x_2' - z_2}{\beta^2 (1 - z_1 z_2)(1 - z_1 x_2')}, \tag{7.2.8b}$$

$$\delta = \frac{\delta' (1 - z_2 x_1')}{\beta (1 - z_1 z_2)}. \tag{7.2.8c}$$

The transformation of basis vectors is

$$\partial_{x_1'} = \frac{\left(\beta^2 (1 - z_1 z_2) + z_2 x_1\right)^2}{\beta^2 (1 - z_1 z_2)^2} \partial_{x_1} - \frac{z_2 \delta \left(\beta^2 (1 - z_1 z_2) + z_2 x_1\right)}{\beta^2 (1 - z_1 z_2)^2} \partial_\delta, \tag{7.2.9a}$$

$$\partial_{x_2'} = \frac{\left(\beta^2 x_2 z_1 (1 - z_1 z_2) + 1\right)^2}{\beta^2 (1 - z_1 z_2)^2} \partial_{x_2}, \tag{7.2.9b}$$

$$\partial_{\delta'} = \frac{\beta (1 - z_1 z_2)}{\beta^2 (1 - z_1 z_2) + z_2 x_1} \partial_\delta, \tag{7.2.9c}$$

with similar inverse transformations.

We now take the canonical basis for the complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{7.2.10}$$

with the commutation relations

$$[\sigma_+,\sigma_-] = \sigma_3, \qquad [\sigma_3,\sigma_+] = 2\sigma_+, \qquad [\sigma_3,\sigma_-] = -2\sigma_-. \tag{7.2.11}$$

The real one-parameter subgroups of $\mathrm{SL}(2,\mathbb{C})$ corresponding to these basis elements are

$$g_+(t) = e^{t\sigma_+} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad \tilde{g}_+(t) = e^{it\sigma_+} = \begin{pmatrix} 1 & it \\ 0 & 1 \end{pmatrix}, \tag{7.2.12a}$$

$$g_-(t) = e^{t\sigma_-} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \qquad \tilde{g}_-(t) = e^{it\sigma_-} = \begin{pmatrix} 1 & 0 \\ it & 1 \end{pmatrix}, \tag{7.2.12b}$$

$$g_3(t) = e^{t\sigma_3} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \qquad \tilde{g}_3(t) = e^{it\sigma_3} = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}, \tag{7.2.12c}$$

where $t \in \mathbb{R}$. The corresponding complex vector fields invariant under the left $\mathrm{SL}(2,\mathbb{C})$ action are

$$X_1 = \frac{x_2}{\delta(1-x_1x_2)}\partial_\delta + \frac{1}{\delta^2}\partial_{x_1}, \tag{7.2.13a}$$

$$X_2 = \delta^2(1-x_1x_2)^2\partial_{x_2}, \tag{7.2.13b}$$

$$X_3 = -\delta\partial_\delta, \tag{7.2.13c}$$

$$X_4 = \frac{ix_2}{\delta(1-x_1x_2)}\partial_\delta + \frac{i}{\delta^2}\partial_{x_1}, \tag{7.2.13d}$$

$$X_5 = i\delta^2(1-x_1x_2)^2\partial_{x_2}, \tag{7.2.13e}$$

$$X_6 = -i\delta\partial_\delta, \tag{7.2.13f}$$

with the correspondence

$$X_1 \longleftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \sigma_+, \tag{7.2.14a}$$

$$X_2 \longleftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \sigma_-, \tag{7.2.14b}$$

$$X_3 \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3, \tag{7.2.14c}$$

$$X_4 \longleftrightarrow \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} = i\sigma_+, \tag{7.2.14d}$$

$$X_5 \longleftrightarrow \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} = i\sigma_-, \tag{7.2.14e}$$

$$X_6 \longleftrightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_3. \tag{7.2.14f}$$

We form real left-invariant vector fields from the X_i via

$$Y_i = X_i + \bar{X}_i, \tag{7.2.15}$$

and define a set of dual, left-invariant one-forms $\{\omega_i\}$ by $\langle \omega_i \mid Y_j \rangle = \delta_{ij}$. They read

$$\omega_1 = \frac{1}{2}\left(\delta^2 dx_1 + \bar{\delta}^2 d\bar{x}_1\right), \tag{7.2.16a}$$

$$\omega_2 = \frac{1}{2}\left(\frac{dx_2}{\delta^2(1-x_1x_2)^2} + \frac{d\bar{x}_2}{\bar{\delta}^2(1-\bar{x}_1\bar{x}_2)^2}\right), \tag{7.2.16b}$$

$$\omega_3 = \frac{1}{2}\left(\frac{x_2 dx_1}{1-x_1x_2} + \frac{\bar{x}_2 d\bar{x}_1}{1-\bar{x}_1\bar{x}_2} - \frac{d\delta}{\delta} - \frac{d\bar{\delta}}{\bar{\delta}}\right), \tag{7.2.16c}$$

$$\omega_4 = \frac{i}{2}\left(-\delta^2 dx_1 + \bar{\delta}^2 d\bar{x}_1\right), \tag{7.2.16d}$$

$$\omega_5 = \frac{i}{2}\left(-\frac{dx_2}{\delta^2(1-x_1x_2)^2} + \frac{d\bar{x}_2}{\bar{\delta}^2(1-\bar{x}_1\bar{x}_2)^2}\right), \tag{7.2.16e}$$

$$\omega_6 = \frac{i}{2}\left(-\frac{x_2 dx_1}{1-x_1x_2} + \frac{\bar{x}_2 d\bar{x}_1}{1-\bar{x}_1\bar{x}_2} + \frac{d\delta}{\delta} - \frac{d\bar{\delta}}{\bar{\delta}}\right). \tag{7.2.16f}$$

7.3 Map to $O_c(1,3)$

There is a well known two-to-one group homomorphism $H : \mathrm{SL}(2,\mathbb{C}) \to O_c(1,3)$, where the subscript c stands for the connected component. The explicit form of the homomorphism is

$$\{H(g)\}_{jk} = \frac{1}{2} Tr(\sigma_j g \sigma_k g^\dagger), \tag{7.3.1}$$

where the choice of Pauli matrices is

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{7.3.2a}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.3.2b}$$

The kernel of H is the subgroup $\{\pm\mathbb{I}_2\} \subset A$, and the image of A under H is the direct product $O_c(1,1) \times O_c(2)$. It follows that $\mathbb{T}$ has a realisation as the homogeneous space

$$\mathbb{T} = O_c(1,3)/\left[O_c(1,1) \times O_c(2)\right]. \tag{7.3.3}$$

The action of H on our representative element (7.1.5) of $\mathbb{T}$ is given by

$$H : t(x_1,x_2) \mapsto T(x_1,x_2) = (V_0, V_1, V_2, V_3), \tag{7.3.4}$$

where $[T(x_1,x_2)] \in O_c(1,3)/\left[O_c(1,1) \times O_c(2)\right]$ and the V_0, V_1, V_2, V_3 are columns given by

$$V_0 = \frac{1}{2|1 - x_1x_2|^2} \begin{pmatrix} 1 + |x_2|^2 + |1 - x_1x_2|^2(1 + |x_1|^2) \\ x_2 + \bar{x}_2 + |1 - x_1x_2|^2(x_1 + \bar{x}_1) \\ i\left[\bar{x}_2 - x_2 + |1 - x_1x_2|^2(x_1 - \bar{x}_1)\right] \\ 1 - |x_2|^2 + |1 - x_1x_2|^2(-1 + |x_1|^2) \end{pmatrix}, \tag{7.3.5a}$$

$$V_1 = \frac{1}{2|1 - x_1x_2|^2} \begin{pmatrix} (1 - \bar{x}_1\bar{x}_2)(\bar{x}_1 + x_2) + (1 - x_1x_2)(x_1 + \bar{x}_2) \\ (1 - \bar{x}_1\bar{x}_2)(1 + \bar{x}_1x_2) + (1 - x_1x_2)(1 + x_1\bar{x}_2) \\ i\left[(1 - \bar{x}_1\bar{x}_2)(1 - \bar{x}_1x_2) + (1 - x_1x_2)(-1 + x_1\bar{x}_2)\right] \\ (1 - \bar{x}_1\bar{x}_2)(\bar{x}_1 - x_2) + (1 - x_1x_2)(x_1 - \bar{x}_2) \end{pmatrix}, \tag{7.3.5b}$$

$$V_2 = \frac{1}{2|1 - x_1x_2|^2} \begin{pmatrix} i\left[(1 - \bar{x}_1\bar{x}_2)(-\bar{x}_1 - x_2) + (1 - x_1x_2)(x_1 + \bar{x}_2)\right] \\ i\left[(1 - \bar{x}_1\bar{x}_2)(-1 - \bar{x}_1x_2) + (1 - x_1x_2)(1 + x_1\bar{x}_2)\right] \\ (1 - \bar{x}_1\bar{x}_2)(1 - \bar{x}_1x_2) + (1 - x_1x_2)(1 - x_1\bar{x}_2) \\ i\left[(1 - \bar{x}_1\bar{x}_2)(x_2 - \bar{x}_1) + (1 - x_1x_2)(x_1 - \bar{x}_2)\right] \end{pmatrix}, \tag{7.3.5c}$$

$$V_3 = \frac{1}{2|1 - x_1x_2|^2} \begin{pmatrix} 1 + |x_2|^2 + |1 - x_1x_2|^2(-1 - |x_1|^2) \\ x_2 + \bar{x}_2 + |1 - x_1x_2|^2(-x_1 - \bar{x}_1) \\ i\left[\bar{x}_2 - x_2 + |1 - x_1x_2|^2(\bar{x}_1 - x_1)\right] \\ 1 - |x_2|^2 + |1 - x_1x_2|^2(1 - |x_1|^2) \end{pmatrix}. \tag{7.3.5d}$$

7.3.1 Lorentz-orthonormal basis in $\mathbb{R}^{1,3}$

If we denote the Minkowski inner product in $\mathbb{R}^{1,3}$ by brackets;

$$(X, Y)_{\mathbb{R}^{1,3}} = -X_0Y_0 + X_1Y_1 + X_2Y_2 + X_3Y_3, \tag{7.3.6}$$

then

$$(V_0, V_0)_{\mathbb{R}^{1,3}} = -1, \qquad (V_0, V_i)_{\mathbb{R}^{1,3}} = 0, \qquad (V_i, V_j)_{\mathbb{R}^{1,3}} = \delta_{ij}, \tag{7.3.7}$$

so that the vectors obtained from the columns of T form a Lorentz-orthonormal basis in the Minkowski space $\mathbb{R}^{1,3}$. In particular, at $x_1 = x_2 = 0$ these vectors form the standard frame

$$V_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad V_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad V_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad V_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{7.3.8}$$

7.3.2 A special vector

We now form the complexification of $\mathbb{R}^{1,3}$, namely $\mathbb{R}^{1,3^{\mathbb{C}}} \cong \mathbb{C}^{1,3}$, and recall that elements of this space are formed from complex linear combinations of the real subspace, namely for $X \in \mathbb{R}^{1,3}$ and $Y \in \mathbb{R}^{1,3}$ we have

$$Z = (X + iY) \in \mathbb{C}^{1,3}. \tag{7.3.9}$$

We can also extend the domain of the Minkowski inner product $(\quad , \quad)_{\mathbb{R}^{1,3}}$ to the complexified space (for $Z = X + iY,\ W = U + iV$) via

$$(Z, W)_{\mathbb{C}^{1,3}} = (X, U)_{\mathbb{R}^{1,3}} - (Y, V)_{\mathbb{R}^{1,3}} + i\left[(Y, U)_{\mathbb{R}^{1,3}} + (X, V)_{\mathbb{R}^{1,3}}\right], \tag{7.3.10}$$

where the complexified Minkowski inner product is the complex bilinear form defined, for $Z, W \in \mathbb{C}^{1,3}$, by

$$(Z, W)_{\mathbb{C}^{1,3}} = -Z_0W_0 + Z_1W_1 + Z_2W_2 + Z_3W_3. \tag{7.3.11}$$

We now introduce a certain complex linear combination of the vectors V_1 and V_2 by

$$Z(x_1, x_2) := \frac{1}{\sqrt{2}}(V_1 - iV_2) = \frac{1}{\sqrt{2}(1 - \bar{x}_1\bar{x}_2)} \begin{pmatrix} x_1 + \bar{x}_2 \\ 1 + x_1\bar{x}_2 \\ i(1 - x_1\bar{x}_2) \\ x_1 - \bar{x}_2 \end{pmatrix}. \tag{7.3.12}$$

Z has the properties

$$(Z, Z)_{\mathbb{C}^{1,3}} = 0, \qquad (Z, \bar{Z})_{\mathbb{C}^{1,3}} = 1. \tag{7.3.13}$$

The reason why precisely this linear combination is considered is that the group $H(A) = O_c(1,1) \times O_c(2)$ when acting on $\widetilde{Z} = Z(0,0)$ only changes $\widetilde{Z}$ by a phase, and so $H(A) \circ \widetilde{Z}$ still satisfies the conditions (7.3.13). We will use the result (7.3.12) when discussing a different realisation of $\mathbb{T}$ in section 7.7.

7.4 Tangent space in $O_c(1,3)$ representation

Under the group homomorphism (7.3.1) the six one-parameter subgroups (7.2.12a) to (7.2.12c) of $\mathrm{SL}(2,\mathbb{C})$ are mapped to their corresponding subgroups in $O_c(1,3)$. The resulting Lie algebra elements in $O_c(1,3)$ representation are given by the matrices

$$a_+ = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad a_- = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \tag{7.4.1}$$

$$\tilde{a}_+ = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad \tilde{a}_- = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{7.4.2}$$

$$a_3 = 2\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \tilde{a}_3 = 2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{7.4.3}$$

where a_i is the algebra element corresponding to the 1-parameter subgroup $H \circ g_i(t)$.

We form linear combinations of the algebra elements such that

$$e_1 = \frac{1}{2}(a_+ + a_-) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{7.4.4}$$

$$e_2 = \frac{1}{2}(\tilde{a}_- - \tilde{a}_+) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{7.4.5}$$

$$f_1 = \frac{1}{2}(a_- - a_+) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \tag{7.4.6}$$

$$f_2 = \frac{1}{2}(\tilde{a}_+ + \tilde{a}_-) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{7.4.7}$$

These algebra elements in $O_c(1,3)$ representation provide us with a basis for vectors in the tangent space at the origin e of $\mathbb{T}$ where $x_1 = x_2 = 0$. Note that a_3 and $\tilde{a}_3$ are not required since δ is not a bona-fide coordinate on $\mathbb{T}$. If we represent an element V of the tangent space $T_e\mathbb{T}$ via the combination $V = m+n$ where $n = n^i e_i$, $m = m^i f_i$ and $i \in \{1, 2\}$ we have

$$V = \begin{pmatrix} 0 & n^1 & n^2 & 0 \\ n^1 & 0 & 0 & m^1 \\ n^2 & 0 & 0 & m^2 \\ 0 & -m^1 & -m^2 & 0 \end{pmatrix}. \tag{7.4.8}$$

We now parameterise the isotropy group in the $\mathrm{SL}(2,\mathbb{C})$ representation as $\delta = e^{\frac{t}{2}} e^{\frac{i\theta}{2}}$ where $t \in \mathbb{R}$ and $\theta \in (-2\pi, 2\pi]$. The corresponding isotropy group in $O_c(1,3)$ representation is given by

$$H(A) = \begin{pmatrix} \cosh t & 0 & 0 & \sinh t \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ \sinh t & 0 & 0 & \cosh t \end{pmatrix}. \tag{7.4.9}$$

The adjoint action of $H(A)$ (7.4.9) on V is $V \mapsto V' = H(A) \circ V \circ H^{-1}(A)$, so that

$$V' = \begin{pmatrix} 0 & n'^1 & n'^2 & 0 \\ n'^1 & 0 & 0 & m'^1 \\ n'^2 & 0 & 0 & m'^2 \\ 0 & -m'^1 & -m'^2 & 0 \end{pmatrix}, \tag{7.4.10}$$

where

$$n'^1 = \cosh t(n^1\cos\theta + n^2\sin\theta) - \sinh t(m^1\cos\theta + m^2\sin\theta), \tag{7.4.11}$$

$$n'^2 = \cosh t(n^2\cos\theta - n^1\sin\theta) + \sinh t(m^1\sin\theta - m^2\cos\theta), \tag{7.4.12}$$

$$m'^1 = -\sinh t(n^1\cos\theta + n^2\sin\theta) + \cosh t(m^1\cos\theta + m^2\sin\theta), \tag{7.4.13}$$

$$m'^2 = -\sinh t(n^2\cos\theta - n^1\sin\theta) - \cosh t(m^1\sin\theta - m^2\cos\theta). \tag{7.4.14}$$

We now choose to represent the components of our vector field $V \in T_e\mathbb{T}$ as a 2×2 matrix

$$V_c = \begin{pmatrix} n^1 & n^2 \\ m^1 & m^2 \end{pmatrix}. \tag{7.4.15}$$

Thus, the action of the isotropy group A on V_c in this representation splits according to

$$V'_c = B(t) \circ V_c \circ R(\theta), \tag{7.4.16}$$

where

$$V'_c = \begin{pmatrix} n'^1 & n'^2 \\ m'^1 & m'^2 \end{pmatrix}, \tag{7.4.17}$$

$$B(t) = \begin{pmatrix} \cosh t & -\sinh t \\ -\sinh t & \cosh t \end{pmatrix}, \tag{7.4.18}$$

and

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{7.4.19}$$

If we introduce a 2-dimensional Minkowski metric η such that

$$\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{7.4.20}$$

we find, by virtue of the pseudo-orthogonality condition $B^T \eta B = \eta$, that a bilinear form invariant under the action of the isotropy group is given by

$$Tr\left(V'^T_c \eta \widetilde{V}'_c\right) = Tr\left(R^T V_c^T B^T \eta B \widetilde{V}_c R\right) = Tr\left(V_c^T \eta \widetilde{V}_c\right). \tag{7.4.21}$$

7.5 Metric on $\mathbb{T}$

From the previous construction, we find that a bilinear form in the tangent space at the origin, invariant under the action of A, is given by

$$g(V, \widetilde{V}) = (n, \widetilde{n}) - (m, \widetilde{m}), \tag{7.5.1}$$

where $(m, \widetilde{m}) = m^1 \widetilde{m}^1 + m^2 \widetilde{m}^2$ is the usual inner product on $\mathbb{R}^2$.

If we recall the relation between basis vectors in the tangent space at the origin and left invariant vector fields, we have the correspondence

$$e_1 = \frac{1}{2}(a_+ + a_-) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longleftrightarrow \frac{1}{2}(X_1 + X_2) =: U_1, \tag{7.5.2}$$

$$e_2 = \frac{1}{2}(\tilde{a}_- - \tilde{a}_+) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longleftrightarrow \frac{1}{2}(X_5 - X_4) =: U_2, \tag{7.5.3}$$

$$f_1 = \frac{1}{2}(a_- - a_+) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \longleftrightarrow \frac{1}{2}(X_2 - X_1) =: U_3, \tag{7.5.4}$$

$$f_2 = \frac{1}{2}(\tilde{a}_+ + \tilde{a}_-) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \longleftrightarrow \frac{1}{2}(X_4 + X_5) =: U_4. \tag{7.5.5}$$

The duality relations between vectors and co-vectors in this new basis is $\langle \rho_i \mid U_j \rangle = \delta_{ij}$, where the dual co-vectors are

$$\rho_1 = \omega_1 + \omega_2, \tag{7.5.6}$$

$$\rho_2 = \omega_5 - \omega_4, \tag{7.5.7}$$

$$\rho_3 = \omega_2 - \omega_1, \tag{7.5.8}$$

$$\rho_4 = \omega_4 + \omega_5. \tag{7.5.9}$$

The invariant bilinear form (7.5.1) thus allows us to introduce on $\mathbb{T}$, with the possible exception of the points in $\mathbb{T}$ not covered by the parameterisation

(7.1.5), a metric by

$$\begin{aligned} g_{\mathbb{T}} &= \rho_1 \otimes \rho_1 + \rho_2 \otimes \rho_2 - \rho_3 \otimes \rho_3 - \rho_4 \otimes \rho_4 \\ &= \frac{dx_1 \otimes dx_2 + dx_2 \otimes dx_1}{(1 - x_1 x_2)^2} + \frac{d\bar{x}_1 \otimes d\bar{x}_2 + d\bar{x}_2 \otimes d\bar{x}_1}{(1 - \bar{x}_1 \bar{x}_2)^2}. \end{aligned} \tag{7.5.10}$$

The metric has signature $(+,+,-,-)$ and is, by construction, invariant under the $\mathrm{SL}(2,\mathbb{C})$ action on $\mathbb{T}$.

7.6 Complex structure of $\mathbb{T}$

The almost complex structure on a manifold is completely determined by the action of a linear map $J_p : T_pM \to T_pM$ satisfying $J_p^2 = -id$. For reasons which will become apparent we consider the simple case of a 4-dimensional Euclidean space $\mathbb{R}^4 = \{(x^1, y^1, x^2, y^2) \mid x^\mu \in \mathbb{R}, y^\mu \in \mathbb{R}\} \cong \mathbb{C}^2$ and specify an almost complex structure via $J_p(\partial_{x^\mu}) = \partial_{y^\mu}$, $J_p(\partial_{y^\mu}) = -\partial_{x^\mu}$, corresponding to a $\frac{\pi}{2}$ anti-clockwise rotation in the planes $\{(x^1, y^1)\}$ and $\{(x^2, y^2)\}$. Note that the J_p operator satisfies $J_p^2 = -id$ and that J_p is compatible with the Euclidean metric, g, on $\mathbb{R}^4$; $g(J_pV, J_p\tilde{V}) = g(V, \tilde{V})$. Writing $\partial_{z^\mu} = \frac{1}{2}(\partial_{x^\mu} - i\partial_{y^\mu})$ and $\partial_{\bar{z}^\mu} = \frac{1}{2}(\partial_{x^\mu} + i\partial_{y^\mu})$ we find the action of J_p on complex basis vector fields is $J_p(\partial_{z^\mu}) = i\partial_{z^\mu}$, $J_p(\partial_{\bar{z}^\mu}) = -i\partial_{\bar{z}^\mu}$, that is, multiplication by $\pm i$. Note finally that this choice of an almost complex structure corresponds to the usual conjugation operation on the complex coordinates $z^\mu = x^\mu + iy^\mu$, $\bar{z}^\mu = x^\mu - iy^\mu$. If we had defined the almost complex structure to correspond to a $\frac{\pi}{2}$ *clockwise* rotation, the eigenvalues of J_p would be interchanged.

Bearing all of this in mind we seek an almost complex structure on $\mathbb{T}$ which is invariant under the action of the isotropy group and also compatible with the metric (7.5.10).

7.6.1 Invariance under the action of A

A acts on components of vector fields via

$$A : V_c \mapsto V_c'; \qquad V_c' = BV_cR. \tag{7.6.1}$$

where B and R are defined by (7.4.18) and (7.4.19). We specify the vector field on $\mathbb{T}$ via

$$V = Tr(V_cU_b), \tag{7.6.2}$$

where U_b is the matrix of basis vector fields, (7.5.2) to (7.5.5), given by

$$U_b = \begin{pmatrix} U_1 & U_3 \\ U_2 & U_4 \end{pmatrix}. \tag{7.6.3}$$

The action of A on the vector field is then

$$A : V \mapsto V' = Tr(BV_cRU_b), \tag{7.6.4}$$

so the action of A is equivalently specified on U_b via

$$A : U_b \mapsto U_b' = RU_bB. \tag{7.6.5}$$

The almost complex structure operator acts on basis vector fields. The most general action of J_p is specified via

$$J_p : U_b \mapsto U_b' = J_LU_bJ_R, \tag{7.6.6}$$

where

$$J_L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{7.6.7}$$

and

$$J_R = \begin{pmatrix} e & f \\ g & h \end{pmatrix}, \tag{7.6.8}$$

are 2×2 matrices with complex-valued entries.

Invariance of the almost complex structure under the action of A means that $A \circ J_p \circ U_b = J_p \circ A \circ U_b$, corresponding to

$$RJ_LU_bJ_RB = J_LRU_bBJ_R, \tag{7.6.9}$$

which is necessary and sufficient for

$$[R, J_L] = 0, \qquad [B, J_R] = 0, \tag{7.6.10}$$

or equivalently

$$J_L = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \qquad J_R = \begin{pmatrix} e & f \\ f & e \end{pmatrix}. \tag{7.6.11}$$

Further restrictions on the entries of J_L and J_R follow from the idempotency condition of J_p. In terms of our matrices we consider two cases:

$$\begin{array}{lll} \text{(I)} & J_L^2 = 1, & J_R^2 = -1; \\ \text{(II)} & J_L^2 = -1, & J_R^2 = 1. \end{array} \tag{7.6.12}$$

Case (I) gives the possibilities

$$\begin{array}{lll} E_1: & a = \pm 1, & b = 0; \\ F_1: & a = 0, & b = \pm i; \\ G_1: & e = \pm i, & f = 0; \\ H_1: & e = 0, & f = \pm i; \end{array} \tag{7.6.13}$$

whilst case (II) gives the possibilities

$$\begin{array}{lll} E_2: & a = \pm i, & b = 0; \\ F_2: & a = 0, & b = \pm 1; \\ G_2: & e = \pm 1, & f = 0; \\ H_2: & e = 0, & f = \pm 1. \end{array} \tag{7.6.14}$$

and the eight possible permutations are

$$E_i \cap G_i, \quad E_i \cap H_i, \quad F_i \cap G_i, \quad F_i \cap H_i. \quad i \in \{1, 2\}. \tag{7.6.15}$$

Although the entries of J_L and J_R can be complex valued we require the total transformation to be real which rules out the four cases

$$E_i \cap G_i, \quad E_i \cap H_i. \quad i \in \{1, 2\}. \tag{7.6.16}$$

We also note that the choice of $i \in \{1,2\}$ merely changes the action by an overall minus sign so it suffices to consider the two cases $F_1 \cap G_1$ and $F_1 \cap H_1$. $F_1 \cap G_1$ yields the equation

$$\begin{pmatrix} J_p(U_1) & J_p(U_3) \\ J_p(U_2) & J_p(U_4) \end{pmatrix} = \pm \begin{pmatrix} -U_2 & -U_4 \\ U_1 & U_3 \end{pmatrix}, \tag{7.6.17}$$

whereas $F_1 \cap H_1$ yields

$$\begin{pmatrix} J_p(U_1) & J_p(U_3) \\ J_p(U_2) & J_p(U_4) \end{pmatrix} = \pm \begin{pmatrix} -U_4 & -U_2 \\ U_3 & U_1 \end{pmatrix}. \tag{7.6.18}$$

Writing

$$\alpha = \frac{1}{\delta^2}\partial_{x_1}, \tag{7.6.19a}$$

$$\beta = \delta^2 \left(1 - x_1 x_2\right)^2 \partial_{x_2}, \tag{7.6.19b}$$

$$\bar{\alpha} = \frac{1}{\bar{\delta}^2}\partial_{\bar{x}_1}, \tag{7.6.19c}$$

$$\bar{\beta} = \bar{\delta}^2 \left(1 - \bar{x}_1 \bar{x}_2\right)^2 \partial_{\bar{x}_2}, \tag{7.6.19d}$$

we find

$$U_1 = \alpha + \bar{\alpha} + \beta + \bar{\beta}, \tag{7.6.20a}$$

$$U_2 = i\left(-\alpha + \bar{\alpha} + \beta - \bar{\beta}\right), \tag{7.6.20b}$$

$$U_3 = -\alpha - \bar{\alpha} + \beta + \bar{\beta}, \tag{7.6.20c}$$

$$U_4 = i\left(\alpha - \bar{\alpha} + \beta - \bar{\beta}\right). \tag{7.6.20d}$$

For the case $F_1 \cap G_1$ we find

$$J_p\left(\alpha + \bar{\alpha} + \beta + \bar{\beta}\right) = \mp i\left(-\alpha + \bar{\alpha} + \beta - \bar{\beta}\right), \tag{7.6.21a}$$

$$J_p\left(-\alpha - \bar{\alpha} + \beta + \bar{\beta}\right) = \mp i\left(\alpha - \bar{\alpha} + \beta - \bar{\beta}\right), \tag{7.6.21b}$$

$$J_p\left(-\alpha + \bar{\alpha} + \beta - \bar{\beta}\right) = \mp i\left(\alpha + \bar{\alpha} + \beta + \bar{\beta}\right), \tag{7.6.21c}$$

$$J_p\left(\alpha - \bar{\alpha} + \beta - \bar{\beta}\right) = \mp i\left(-\alpha - \bar{\alpha} + \beta + \bar{\beta}\right). \tag{7.6.21d}$$

Forming appropriate linear combinations of these and using the linearity of J_p, we can write this complex structure as

$$J_p\left(\partial_{x_1}\right) = \pm i\partial_{x_1}, \tag{7.6.22a}$$

$$J_p\left(\partial_{x_2}\right) = \mp i\partial_{x_2}, \tag{7.6.22b}$$

$$J_p\left(\partial_{\bar{x}_1}\right) = \mp i\partial_{\bar{x}_1}, \tag{7.6.22c}$$

$$J_p\left(\partial_{\bar{x}_2}\right) = \pm i\partial_{\bar{x}_2}. \tag{7.6.22d}$$

For the case $F_1 \cap H_1$ similar reasoning shows that we can write the complex structure as

$$J_p\left(\partial_{x_1}\right) = \pm i\partial_{x_1}, \tag{7.6.23a}$$

$$J_p\left(\partial_{x_2}\right) = \pm i\partial_{x_2}, \tag{7.6.23b}$$

$$J_p\left(\partial_{\bar{x}_1}\right) = \mp i\partial_{\bar{x}_1}, \tag{7.6.23c}$$

$$J_p\left(\partial_{\bar{x}_2}\right) = \mp i\partial_{\bar{x}_2}. \tag{7.6.23d}$$

The almost complex structure specified by (7.6.22) is compatible with the metric (7.5.10) as $g\left(J_p(V), J_p(\widetilde{V})\right) = g\left(V, \widetilde{V}\right)$ whereas the almost complex structure specified by (7.6.23) satisfies $g\left(J_p(V), J_p(\widetilde{V})\right) = -g\left(V, \widetilde{V}\right)$ and hence is not compatible with the metric. The unique almost complex structure on $\mathbb{T}$ (up to an overall minus sign) is given by (7.6.22). Furthermore, the Nijenhuis tensor N_{J_p} defined by its action on vector fields X and Y, [39],

$$N_{J_p}\left(X, Y\right) = \left[X, Y\right] + J_p\left[J_p X, Y\right] + J_p\left[X, J_p Y\right] - \left[J_p X, J_p Y\right], \tag{7.6.24}$$

vanishes for J_p defined by (7.6.22). Therefore the almost complex structure is in fact a complex structure.

We make a choice of sign for this complex-structure and compare the structure with the complex manifold $\mathbb{C}^2$ considered at the start of this section. For the $\mathbb{C}^2$ case the canonical complex structure was equivalent to the usual complex conjugation on coordinates. In the $\mathbb{T}$ case, however, matching the eigenvalues of J_p leads us to introduce a $*$ operation on coordinates, distinct from complex conjugation, given by

$$* : \left(x_1, x_2, \bar{x}_1, \bar{x}_2\right) \mapsto \left(x_2, x_1, \bar{x}_2, \bar{x}_1\right). \tag{7.6.25}$$

7.7 $\mathbb{T}$ as a complex quadric

We wish to show how $\mathbb{T}$ can be realised as a quadric embedded in $\mathbb{C}P^3$. To do this we make a short digression into the realms of projective geometry. The 3-dimensional complex projective space $\mathbb{C}P^3$ is defined as $\mathbb{C}P^3 = (\mathbb{C}^4 \setminus \{0\})/\sim$ where the equivalence relation on $\mathbb{C}^4$ is $Z \sim Z'$ if $\exists p \in \mathbb{C} \mid Z' = pZ$ and $p \neq 0$. Geometrically this corresponds to the space of complex lines through the origin in $\mathbb{C}^4$. We denote the projection $\mathbb{C}^4 \setminus \{0\} \to \mathbb{CP}^3$ by π. The homogeneous coordinates (z_0, z_1, z_2, z_3) on $\mathbb{C}^4$ no longer provide us with an independent set of coordinates on the quotient space as the quotient kills one complex dimension. However, we can define a set of independent coordinates by firstly specifying a set of charts U_μ, where $\mu \in \{0, 1, 2, 3\}$, such that U_μ is the set of lines in $\mathbb{C}^4$ where $z_\mu \neq 0$. Note that $\mathbb{C}P^3 = \bigcup_{\mu=0}^{3} U_\mu$. In a chart U_μ we define the inhomogeneous coordinates by

$$\xi_\nu^{(\mu)} = \begin{cases} \frac{z_\nu}{z_\mu} & \text{if } \nu \leq \mu - 1 \\ \frac{z_{\nu+1}}{z_\mu} & \text{if } \nu \geq \mu \end{cases} \tag{7.7.1}$$

where μ labels the specific chart and $\nu \in \{0, 1, 2\}$ labels the inhomogeneous coordinate. In $U_\mu \cap U_\nu$ the transition functions $\Psi_{\mu\nu} : \mathbb{C}P^3 \to \mathbb{C}P^3$ are given by $\xi_\lambda^{(\nu)} \mapsto \xi_\lambda^{(\mu)} = \frac{z_\mu}{z_\nu} \xi_\lambda^{(\nu)}$ and are necessarily holomorphic.

We now define the subset $Q_2(\mathbb{C}) \subset \mathbb{C}P^3$ as the projection of the quadric $(Z, Z)_{\mathbb{C}^{1,3}} = 0$:

$$Q_2(\mathbb{C}) = \left\{\pi(Z) \mid Z \in \mathbb{C}^4 \setminus \{0\}, (Z, Z)_{\mathbb{C}^{1,3}} = 0\right\}. \tag{7.7.2}$$

As the complex vector $Z\,(x_1, x_2)$ (7.3.12) is on this quadric, we obtain an embedding of $\mathbb{T}$ in $Q_2(\mathbb{C})$ by

$$q(x_1, x_2) := \pi(Z(x_1, x_2)) \in Q_2(\mathbb{C}), \tag{7.7.3}$$

with the possible exception of the points in $\mathbb{T}$ not covered by the parameterisation (7.1.5).

$Q_2(\mathbb{C})$ inherits a complex structure from its embedding in $\mathbb{C}P^3$. Parameterising the coordinates on $\mathbb{C}P^3$ in terms of the coordinates (x_1, x_2) we can calculate the pushforward, to $Q_2(\mathbb{C})$, of the basis vectors spanning $\mathbb{C}P^3$. The canonical complex structure on $\mathbb{C}P^3$ then reveals that the complex structure on $Q_2(\mathbb{C})$ is compatible with the complex structure given by (7.6.22).

7.8 The coset space $\mathbb{B}$

We now define another 2-dimensional subspace of G, which we denote by $\mathbb{B}$, discussing its global structure and the action of G on $\mathbb{B}$.

We define $\mathbb{B}$ in a similar way to $\mathbb{T}$ via a quotient space construction

$$\mathbb{B} := N_+ \backslash G, \tag{7.8.1}$$

where N_+ is the 1-parameter subgroup of upper triangular matrices,

$$N_+ = \left\{ \begin{pmatrix} 1 & n_+ \\ 0 & 1 \end{pmatrix} \mid n_+ \in \mathbb{C} \right\}. \tag{7.8.2}$$

and the $\backslash$ in (7.8.1) is understood as denoting a set of right cosets of N_+ w.r.t G.

7.8.1 Global structure of $\mathbb{B}$

A general element of $\mathbb{B}$ is a class

$$[N_+ g] = \left[\begin{pmatrix} a + n_+ c & b + n_+ d \\ c & d \end{pmatrix} \right], \tag{7.8.3}$$

where $n_+ \in \mathbb{C}$. In the equivalence classes (7.8.3), c and d cannot both be zero. When $c \neq 0$, we can choose a unique representative by setting $n_+ = -\frac{a}{c}$: writing $c = z_1 \in \mathbb{C} \setminus \{0\}$ and $d = z_2 \in \mathbb{C}$, this representative reads

$$b_1(z_1, z_2) = \begin{pmatrix} 0 & -\dfrac{1}{z_1} \\ z_1 & z_2 \end{pmatrix}. \tag{7.8.4}$$

Similarly, when $d \neq 0$, we can choose a unique representative by setting $n_+ = -\frac{b}{d}$: writing $d = z_2 \in \mathbb{C} \setminus \{0\}$ and $c = z_1 \in \mathbb{C}$, this representative reads

$$b_2(z_1, z_2) = \begin{pmatrix} \frac{1}{z_2} & 0 \\ z_1 & z_2 \end{pmatrix} . \tag{7.8.5}$$

These two parameterisations show that $\mathbb{B}$ can be covered by the two charts U_1 and U_2, where

$$U_1 = \left\{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 \neq 0\right\}, \tag{7.8.6a}$$

$$U_2 = \left\{(z_1, z_2) \in \mathbb{C}^2 \mid z_2 \neq 0\right\}, \tag{7.8.6b}$$

such that the transition function on $U_1 \cap U_2$ is the identity. We hence have $\mathbb{B} \cong U_1 \cup U_2 = \mathbb{C}^2 \setminus \{(0, 0)\}$.

7.8.2 G-action on $\mathbb{B}$

The canonical action of G on itself from the right induces an action of G on $\mathbb{B}$. In the parameterisation $\mathbb{B} \cong \mathbb{C}^2 \setminus \{(0, 0)\}$, this action is the linear transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (z_1, z_2) \mapsto (z_1', z_2') = (az_1 + cz_2, bz_1 + dz_2). \tag{7.8.7}$$

We have realised two distinct coset spaces of $\mathrm{SL}(2, \mathbb{C})$, $\mathbb{T}$ and $\mathbb{B}$, and established various properties of these spaces. We will now go on to discuss conformal field theory n-point functions with reference to $\mathbb{T}$ and $\mathbb{B}$.

Chapter 8

Conformal field theory (CFT)

This chapter describes relevant aspects of conformal field theory (CFT) in various function spaces and the relationships between said spaces using techniques from representation theory.

8.1 CFT n-point functions

A CFT on the sphere (which we consider as the boundary of hyperbolic three-space) is defined by the set of all its n-point correlation functions. We summarise here, for convenience, the 2, 3 and 4 point functions of fields as given in [40].

On the sphere we are interested in the 2-point function of fields which are correlated only if they have the same scaling dimension Δ, [40], namely functions of the form

$$K_{\Delta}^{\mathbb{C}^*}(z, y) = \frac{C_{12}}{|z-y|^{2\Delta}}, \tag{8.1.1}$$

where the superscript $\mathbb{C}^*$ denotes the one-point compactification of $\mathbb{C}$ into the Riemann sphere, $z, y \in \mathbb{C}^*$ are the two points on the sphere and C_{12} is a multiplicative constant corresponding to the normalisation of the field. This function is also known as the boundary-to-boundary propagator [41].

The 3-point function of fields (with differing scaling dimensions Δ_1, Δ_2

and Δ_3), can be given in the form

$$C^{\mathbb{C}^*}_{\Delta_1\Delta_2\Delta_2}(z,u,v) = \frac{C_{123}}{|z-u|^{\Delta_1+\Delta_2-\Delta_3}|z-v|^{\Delta_1+\Delta_3-\Delta_2}|u-v|^{\Delta_2+\Delta_3-\Delta_1}}. \tag{8.1.2}$$

The 4-point function in two dimensions is unique only up to a multiplicative function dependent on one anharmonic/cross ratio. The function therefore has the form

$$\begin{aligned} S^{\mathbb{C}^*}_{\Delta_1\Delta_2\Delta_3\Delta_4} = \phi\left(\frac{|z-y||u-v|}{|z-u||y-v|}\right) \times \\ |z-y|^{\frac{\alpha}{3}}|z-u|^{\frac{\beta}{3}}|z-v|^{\frac{\gamma}{3}}|y-u|^{\frac{\delta}{3}}|y-v|^{\frac{\epsilon}{3}}|u-v|^{\frac{\zeta}{3}}, \end{aligned} \tag{8.1.3}$$

where ϕ is a multiplicative function of the cross ratio, and

$$\alpha = \Delta_3 + \Delta_4 - 2\Delta_1 - 2\Delta_2, \tag{8.1.4a}$$

$$\beta = \Delta_2 + \Delta_4 - 2\Delta_1 - 2\Delta_3, \tag{8.1.4b}$$

$$\gamma = \Delta_2 + \Delta_3 - 2\Delta_1 - 2\Delta_4, \tag{8.1.4c}$$

$$\delta = \Delta_1 + \Delta_4 - 2\Delta_2 - 2\Delta_3, \tag{8.1.4d}$$

$$\epsilon = \Delta_1 + \Delta_3 - 2\Delta_2 - 2\Delta_4, \tag{8.1.4e}$$

$$\zeta = \Delta_1 + \Delta_2 - 2\Delta_3 - 2\Delta_4. \tag{8.1.4f}$$

According to the "conformal bootstrap" idea, [40], the n-point function of fields for $n > 4$ can be constructed out of these three simpler functions.

8.2 Spaces of functions

We are interested in mapping specific functions between specific mathematical spaces. We denote the collection of *all* functions on a manifold M by $\mathcal{F}(M)$ and we wish to explore the following diagram

$$\begin{array}{ccc} \mathcal{F}(\mathbb{C}^*) & \longleftrightarrow & \mathcal{F}(H_3) \\ \updownarrow & & \updownarrow \\ \mathcal{F}(\mathbb{B}) & \longleftrightarrow & \mathcal{F}(\mathbb{T}) \end{array} \; . \tag{8.2.1}$$

More specifically, there will be various constraints placed on the possible functions in a space and we will consider functions on n copies of a manifold (or n-point functions). Also, we shall understand "functions" to include also densities of specified (complex) weights.

To begin to explore this diagram we need to turn to representation theory of $\mathrm{SL}(2,\mathbb{C})$ which has been extensively studied in the literature and of which we only use the details here which are of interest to us.

8.3 Representation theory of $\mathrm{SL}(2,\mathbb{C})$

We follow [42] for the majority of this section taking advantage of the isomorphism $\mathbb{B} \cong \mathbb{C}^2 \setminus \{(0,0)\}$ obtained in the previous chapter. We derive some facts about representation theory of $\mathrm{SL}(2,\mathbb{C})$ where the representations act in spaces of importance to us. We take the defining equation of a representation to be

$$T(g_1)T(g_2) = T(g_1g_2), \tag{8.3.1}$$

where the $T(g)$ are understood to act on a specified linear space, g is an element of the group, and the representation of the identity element corresponds to the identity transformation on the linear space. Various realisations of the $T(g)$ are available, the most trivial being where g is represented by itself and acts on $\mathbb{C}^2$. We wish to explore more interesting examples. In what follows, for simplicity we shall denote functions of d complex variables by

$$f(z_1,\ldots,z_d) := f(z_1,\ldots,z_d;\bar{z}_1,\ldots,\bar{z}_d). \tag{8.3.2}$$

8.3.1 The infinite-dimensional linear space $H\mathcal{F}(\mathbb{B})$

We first take a subspace of $\mathcal{F}(\mathbb{B})$, namely the space of homogeneous functions of bidegree (λ,μ) which we denote $H\mathcal{F}(\mathbb{B})$ where we have supressed the (λ,μ) dependence for notational convenience. Elements of this infinite dimensional linear space are characterised by the condition

$$f(pz_1, pz_2) = p^{\lambda}\bar{p}^{\mu} f(z_1, z_2), \tag{8.3.3}$$

where $p \in \mathbb{C} \setminus \{0\}$ and $(\lambda - \mu) \in \mathbb{Z}$ in order that the above equation is single-valued.

8.3.2 Representation of $\mathrm{SL}(2, \mathbb{C})$ acting on $H\mathcal{F}(\mathbb{B})$

$\mathrm{SL}(2, \mathbb{C})$ acts on $\mathbb{B}$ from the right via $(z_1, z_2) \mapsto (az_1 + cz_2, bz_1 + dz_2)$ (7.8.7) and can be considered to induce a transformation on $H\mathcal{F}(\mathbb{B})$ which we specify in terms of a representation action via

$$\left(T(g)f\right)(z_1, z_2) = f^g(z_1, z_2) = f(az_1 + cz_2, bz_1 + dz_2). \tag{8.3.4}$$

It is immediate that $f \in H\mathcal{F}(\mathbb{B}) \implies f^g \in H\mathcal{F}(\mathbb{B})$ and also

$$\begin{aligned}\left(T(g_1 g_2)f\right)(z_1, z_2) &= f\big[\left(a_1 a_2 + b_1 c_2\right) z_1 + \left(c_1 a_2 + d_1 c_2\right) z_2, \\ &\qquad \left(a_1 b_2 + b_1 d_2\right) z_1 + \left(c_1 b_2 + d_1 d_2\right) z_2\big] \\ &= \left(T(g_1)T(g_2)f\right)(z_1, z_2),\end{aligned} \tag{8.3.5}$$

showing that the $T(g)$ satisfy the functional equation (8.3.1). Also the representation of the identity element is the identity transformation on $H\mathcal{F}(\mathbb{B})$, and the equation defining the representation (8.3.4) depends continuously on $g \in G$. In all generality we have the pairing $(T(g), H\mathcal{F}(\mathbb{B}))$ which (abusing terminology) we call the representation and we understand this representation in terms of (8.3.3) and (8.3.4).

8.3.3 $\mathcal{F}(\mathbb{C}^*)$ and its relation to $H\mathcal{F}(\mathbb{B})$

We now wish to relate functions defined on the Riemann sphere $\mathcal{F}(\mathbb{C}^*)$ to $H\mathcal{F}(\mathbb{B})$.

Firstly we note [42] that elements of $H\mathcal{F}(\mathbb{B})$ are uniquely determined by their values on a contour in $\mathbb{B}$ that crosses once each complex line of the form

$$a_1 z_1 + a_2 z_2 = 0, \tag{8.3.6}$$

where

$$(a_1, a_2) \neq (0, 0)\,. \tag{8.3.7}$$

The complex contour $z_2 = 1$ crosses each complex line of the form (8.3.6) once, with $a_1 \neq 0$, and so elements of $H\mathcal{F}(\mathbb{B})$ are determined uniquely by their values on this contour, except for the values on the lines with $a_1 = 0$. To this effect we define

$$\psi(z) := f(z, 1). \tag{8.3.8}$$

Note that $\psi(z)$ is well-defined on the complex plane $\mathbb{C}$. Similarly we can consider the contour $z_1 = 1$ and define

$$\hat{\psi}(m) = f(1, m), \tag{8.3.9}$$

which is also well-defined on $\mathbb{C}$. Using the homogeneity of f it is then simple to specify elements of $H\mathcal{F}(\mathbb{B})$ in terms of $\psi(z)$ and $\hat{\psi}(m)$, in the charts U_1 and U_2 we defined in Chapter 7, by

$$U_1: \;\; f(z_1, z_2) = z_1^{\lambda} \bar{z}_1^{\mu} \hat{\psi}\left(\frac{z_2}{z_1}\right), \tag{8.3.10a}$$

$$U_2: \;\; f(z_1, z_2) = z_2^{\lambda} \bar{z}_2^{\mu} \psi\left(\frac{z_1}{z_2}\right). \tag{8.3.10b}$$

On the intersection of the charts, the two functions (8.3.8) and (8.3.9) are related by

$$\psi(z) = z^{\lambda} \bar{z}^{\mu} \hat{\psi}\left(\frac{1}{z}\right), \tag{8.3.11a}$$

$$\hat{\psi}(m) = m^{\lambda} \bar{m}^{\mu} \psi\left(\frac{1}{m}\right). \tag{8.3.11b}$$

Equation (8.3.11) shows that we are really dealing with functions on the Riemann sphere $\mathbb{C}^*$, not just on $\mathbb{C}$; further, these functions are not scalar-valued but must be understood as densities whose holomorphic and antiholomorphic weights are specified by λ and μ. Through (8.3.10) we can transform the density-valued functions on $\mathbb{C}^*$ into the corresponding homogeneous functions on $\mathbb{B}$. Conversely, any homogeneous function on $\mathbb{B}$ can be transformed into a density-valued function on $\mathbb{C}^*$. For brevity, we shall refer to the elements of $\mathcal{F}(\mathbb{C}^*)$ simply as functions.

In order to make these details a little more clear we consider the simple case of a specific homogeneous polynomial f of degree $(2,2)$ on $\mathbb{B}$ where

$$f(z_1,z_2) = z_1^2(\bar{z}_1^2 + \bar{z}_2^2). \qquad (8.3.12)$$

The corresponding function on the sphere (again depending on the chart in $\mathbb{B}$) is

$$U_1: \ \hat{\psi}(m) = 1 + \bar{m}^2, \qquad (8.3.13a)$$

$$U_2: \ \psi(z) = z^2(\bar{z}^2 + 1), \qquad (8.3.13b)$$

$$U_1 \cap U_2: \ \psi(z) = z^2\bar{z}^2\hat{\psi}\left(\frac{1}{z}\right). \qquad (8.3.13c)$$

We have thus established an isomorphic mapping of the two function spaces

$$H\mathcal{F}(\mathbb{B}) \cong \mathcal{F}(\mathbb{C}^*), \qquad (8.3.14)$$

where the isomorphism is given explicitly by (8.3.10).

The representation (8.3.4) is realised in this isomorphic vector space via the equations

$$\big(T(g)\hat{\psi}\big)(m) = (a+cm)^\lambda(\bar{a}+\bar{c}\bar{m})^\mu\hat{\psi}\left(\frac{b+dm}{a+cm}\right), \qquad (8.3.15a)$$

$$\big(T(g)\psi\big)(z) = (bz+d)^\lambda(\bar{b}\bar{z}+\bar{d})^\mu\psi\left(\frac{az+c}{bz+d}\right), \qquad (8.3.15b)$$

where use has been made of the linear property of the $T(g)$. Again, it is a simple task to verify that the $T(g)$ satisfy the functional equation (8.3.1) and the representation of the identity element is the identity transformation on $\mathcal{F}(\mathbb{C}^*)$. Once again, the pairing $(T(g), \mathcal{F}(\mathbb{C}^*))$ is called the representation and is understood via (8.3.15).

8.4 CFT correlation functions as functions in $\mathbb{B}$

All of the previous representation theory has been concerned with only one copy of the underlying manifold. To map n-point functions we need the generalised representation theory in the context of n copies of the relevant manifold. We can, however, immediately generalise the previous theory to n-point functions on (n copies of) the two spaces of interest to us, $\mathbb{B}$ and $\mathbb{C}^*$. We denote n copies of $\mathbb{B}$ by

$$\mathbb{B}^n = \underbrace{\mathbb{B}\times, \ldots, \times\mathbb{B}}_{\text{n copies}}, \tag{8.4.1}$$

and similarly for n copies of $\mathbb{C}^*$. We further denote the function spaces by $H\mathcal{F}_{\otimes_{i=1}^n(\lambda_i,\mu_i)}(\mathbb{B}^n)$ (for each copy of $\mathbb{B}$ we specify the appropriate degree of homogeneity) and $\mathcal{F}(\mathbb{C}^{*n})$ and note they are both linear spaces.

8.4.1 The linear space $H\mathcal{F}_{\otimes_{i=1}^n(\lambda_i,\mu_i)}(\mathbb{B}^n)$

In an analogous way to the 1-point functions defined previously we denote elements of $H\mathcal{F}_{\otimes_{i=1}^n(\lambda_i,\mu_i)}(\mathbb{B}^n)$ by

$$f\left(\underbrace{(w_1, w_2), (x_1, x_2), \ldots, (z_1, z_2)}_{\text{n pairs}}\right), \tag{8.4.2}$$

with

$$f\big((p_1w_1, p_1w_2), \ldots, (p_nz_1, p_nz_2)\big) = p_1^{\lambda_1}\bar{p}_1^{\mu_1}\cdots p_n^{\lambda_n}\bar{p}_n^{\mu_n}\times$$
$$f\big((w_1, w_2), \ldots, (z_1, z_2)\big). \tag{8.4.3}$$

The natural representation of G on this space is

$$(T(g)f)\big((w_1, w_2), \ldots, (z_1, z_2)\big) = f\big((aw_1 + cw_2, bw_1 + dw_2), \ldots$$
$$\ldots, (az_1 + cz_2, bz_1 + dz_2)\big). \tag{8.4.4}$$

8.4.2 $\mathcal{F}(\mathbb{C}^{*n})$ and its relation to $H\mathcal{F}_{\otimes_{i=1}^{n}(\lambda_i,\mu_i)}(\mathbb{B}^n)$

The relation between the n-point function spaces is

$$\begin{aligned} U_1^n: \quad \hat{\psi}(p,\ldots,m) &= w_1^{-\lambda_1}\bar{w_1}^{-\mu_1}\cdots z_1^{-\lambda_n}\bar{z_1}^{-\mu_n}\times \\ &\qquad f\big((w_1,w_2),\ldots,(z_1,z_2)\big), && (8.4.5a) \\ U_2^n: \quad \psi(w,\ldots,z) &= w_2^{-\lambda_1}\bar{w_2}^{-\mu_1}\cdots z_2^{-\lambda_n}\bar{z_2}^{-\mu_n}\times \\ &\qquad f\left((w_1,w_2),\ldots,(z_1,z_2)\right), && (8.4.5b) \end{aligned}$$

where $w = \frac{w_1}{w_2},\ldots,z=\frac{z_1}{z_2}$ and $p=\frac{w_2}{w_1},\ldots,n=\frac{z_2}{z_1}$. Note that the n-point functions are restricted to n copies of a specific chart. On the intersection of the charts the n-point transition functions are given by the analagous version of (8.3.11). The representation acting on this space is given by

$$\begin{aligned} \big(T(g)\hat{\psi}\big)(p,\ldots,m) &= (a+cp)^{\lambda_1}(\bar{a}+\bar{c}\bar{p})^{\mu_1}\cdots(a+cm)^{\lambda_n}(\bar{a}+\bar{c}\bar{m})^{\mu_n}\times \\ &\qquad \psi\left(\frac{b+dp}{a+cp},\ldots,\frac{b+dm}{a+cm}\right), && (8.4.6a) \\ \big(T(g)\psi\big)(w,\ldots,z) &= (bw+d)^{\lambda_1}\left(\bar{b}\bar{w}+\bar{d}\right)^{\mu_1}\cdots(bz+d)^{\lambda_n}\left(\bar{b}\bar{z}+\bar{d}\right)^{\mu_n}\times \\ &\qquad \psi\left(\frac{aw+c}{bw+d},\ldots,\frac{az+c}{bz+d}\right). && (8.4.6b) \end{aligned}$$

In the following context of CFT correlation functions we will only use the subset of $H\mathcal{F}_{\otimes_{i=1}^{n}(\lambda_i,\mu_i)}(\mathbb{B}^n)$ where $\lambda_i=\mu_i$ so that without loss of generality we can denote $H\mathcal{F}_{\otimes_{i=1}^{n}(\lambda_i,\lambda_i)}(\mathbb{B}^n)$ by $H\mathcal{F}(\mathbb{B}^n)$. We will also use the fact that the conformal scaling dimension is equal to "minus the degree of homogeneity" [41], $\Delta_i=-\lambda_i$.

8.4.3 2,3 and 4-point functions

We map the 2-point function on $\mathbb{C}^*$ (8.1.1), using (8.4.5b) with $n=2$, into the corresponding 2-point function on U_2 whereby

$$K_\lambda^{U_2}(z_1,z_2;y_1,y_2) = C_{12}|\left(z_1y_2-z_2y_1\right)|^{2\lambda}. \qquad (8.4.7)$$

We map the 3-point function (8.1.2), using (8.4.5b) with $n = 3$, into the corresponding function on U_2 whereby

$$C^{U_2}_{\lambda_1\lambda_2\lambda_3} = C_{123}|\,(z_1u_2 - z_2u_1)\,|^{\lambda_3-\lambda_1-\lambda_2}\times \\ |\,(z_1v_2 - z_2v_1)\,|^{\lambda_2-\lambda_1-\lambda_3}|\,(u_1v_2 - u_2v_1)\,|^{\lambda_1-\lambda_2-\lambda_3}. \tag{8.4.8}$$

We map the 4-point function (8.1.3), using (8.4.5b) with $n = 4$, and find the function takes the following form

$$S^{U_2}_{\lambda_1\lambda_2\lambda_3\lambda_4} = \phi\left(\frac{|z_1y_2 - y_1z_2||u_1v_2 - v_1u_2|}{|z_1u_2 - u_1z_2||y_1v_2 - v_1y_2|}\right)\times \\ |z_1y_2 - y_1z_2|^{\frac{\alpha}{3}}|z_1u_2 - u_1z_2|^{\frac{\beta}{3}}|z_1v_2 - v_1z_2|^{\frac{\gamma}{3}}\times \\ |y_1u_2 - u_1y_2|^{\frac{\delta}{3}}|y_1v_2 - v_1y_2|^{\frac{\epsilon}{3}}|u_1v_2 - v_1u_2|^{\frac{\zeta}{3}}, \tag{8.4.9}$$

where

$$\alpha = -\lambda_3 - \lambda_4 + 2\lambda_1 + 2\lambda_2, \tag{8.4.10a}$$

$$\beta = -\lambda_2 - \lambda_4 + 2\lambda_1 + 2\lambda_3, \tag{8.4.10b}$$

$$\gamma = -\lambda_2 - \lambda_3 + 2\lambda_1 + 2\lambda_4, \tag{8.4.10c}$$

$$\delta = -\lambda_1 - \lambda_4 + 2\lambda_2 + 2\lambda_3, \tag{8.4.10d}$$

$$\epsilon = -\lambda_1 - \lambda_3 + 2\lambda_2 + 2\lambda_4, \tag{8.4.10e}$$

$$\zeta = -\lambda_1 - \lambda_2 + 2\lambda_3 + 2\lambda_4. \tag{8.4.10f}$$

8.5 Integral transform from $\mathbb{B}$ to $\mathbb{T}$

We have determined the left hand arrow of the diagram (8.2.1) (with suitable constraints placed on the functions on $\mathbb{B}$) and mapped the functions of interest to us into $\mathbb{B}$. We wish to do the same thing for the bottom arrow of the diagram, namely transform n-point functions into $\mathbb{T}^n$. As an attempt at doing this, an integral transform finding $f^{\mathbb{T}}_p(z^0_1, z^0_2)$ (i.e. a function at a specific point in $\mathbb{T}$)

when $f^{\mathbb{B}}(z_1, z_2)$ is known, is given in [42] as

$$f_p^{\mathbb{T}}(z_1^0, z_2^0) = \frac{i}{2}\int_{\mathbb{C}} |a_1 a_2|\chi^{-\frac{1}{2}}\left(\frac{a_1}{a_2}, \frac{a_2}{a_1}\right) \times \partial_{\mu\bar{\mu}}\left\{|\mu|^2 f^{\mathbb{B}}(\mu(a_1 a_2)^{\frac{1}{2}} z, \mu(a_1 a_2)^{\frac{1}{2}})\right\}_{\mu=1} dz d\bar{z}, \tag{8.5.1}$$

where

$$\mu \in \mathbb{C}, \tag{8.5.2a}$$

$$a_j = \frac{z_j^0}{z_j^0 - z}, \tag{8.5.2b}$$

$$\chi(z_1, z_2) = z_1^{n_1} \bar{z}_1^{n_2} z_2^{m_1} \bar{z}_2^{m_2}. \tag{8.5.2c}$$

We generalise this to generate n-point functions on $\mathbb{T}$,

$$f_p^{\mathbb{T}^n}(w_1^0, w_2^0; \ldots; z_1^0, z_2^0) = \left(\frac{i}{2}\right)^n \underbrace{\int \cdots \int}_{n} |\underbrace{a_1 a_2 \ldots d_1 d_2}_{\text{n pairs}} \times |\chi^{-\frac{1}{2}}\left(\frac{a_1}{a_2}, \frac{a_2}{a_1}\right) \ldots \chi^{-\frac{1}{2}}\left(\frac{d_1}{d_2}, \frac{d_2}{d_1}\right) \times \partial_{\mu_1\bar{\mu}_1 \cdots \mu_n \bar{\mu}_n}\{|\mu_1|^2 \ldots |\mu_n|^2 \times f^{\mathbb{B}^n}(\mu_1(a_1 a_2)^{\frac{1}{2}} z, \ldots, \mu_n(d_1 d_2)^{\frac{1}{2}})\}_{\mu_i=1} \times dz d\bar{z} \ldots dw d\bar{w}, \tag{8.5.3}$$

where

$$\mu_n \in \mathbb{C} \quad \forall n \in \mathbb{Z}, \tag{8.5.4a}$$

$$a_j = \frac{z_j^0}{z_j^0 - z}, \ldots, d_j = \frac{w_j^0}{w_j^0 - w}, \tag{8.5.4b}$$

$$\chi(z_1, z_2) = z_1^{n_1} \bar{z}_1^{n_2} z_2^{m_1} \bar{z}_2^{m_2}. \tag{8.5.4c}$$

To make the formula (8.5.3) a little more clear we consider the transformation of the 2-point function and find that the integral takes the form

$$f_p^{\mathbb{T}^2}(y_1^0, y_2^0; z_1^0, z_2^0) = -\frac{1}{4}(1-\Delta)^4 C_{12} \int\int \left(\frac{z_1^0 y_1^0}{(z_1^0 - z)(y_1^0 - y)}\right)^{\frac{1}{2}(1-\Delta+m_1-n_1)} \times$$
$$\left(\frac{\bar{z}_1^0 \bar{y}_1^0}{(\bar{z}_1^0 - \bar{z})(\bar{y}_1^0 - \bar{y})}\right)^{\frac{1}{2}(1-\Delta+m_2-n_2)} \times$$
$$\left(\frac{z_2^0 y_2^0}{(z_2^0 - z)(y_2^0 - y)}\right)^{\frac{1}{2}(1-\Delta-m_1+n_1)} \times$$
$$\left(\frac{\bar{z}_2^0 \bar{y}_2^0}{(\bar{z}_2^0 - \bar{z})(\bar{y}_2^0 - \bar{y})}\right)^{\frac{1}{2}(1-\Delta-m_2+n_2)} \times$$
$$(z-y)^{-\Delta}(\bar{z}-\bar{y})^{-\Delta}\, dy d\bar{y} dz d\bar{z}. \tag{8.5.5}$$

We wish to find the values of Δ for which the integral (8.5.5) converges. Possible points of divergence are where $z = z_1^0, z = z_2^0, z = y$ and $z \to \infty$, and similarly for y.

For an integral such as

$$I = \frac{i}{2}\int_{\mathbb{C}} \frac{dz \wedge d\bar{z}}{(z-x)^a (\bar{z}-\bar{x})^{\bar{a}}}, \tag{8.5.6}$$

where x is a fixed complex number, we reparameterise via coordinates adapted to the possible point of divergence $z = x + re^{i\phi}$ and find that

$$I = \int_0^{2\pi}\int_0^{\infty} r^{1-a-\bar{a}} e^{i\phi(\bar{a}-a)} dr \wedge d\phi. \tag{8.5.7}$$

The ϕ integral will converge for all a. The r integral will converge at $r = 0$ if and only if $1 - a - \bar{a} > -1$, i.e. $a + \bar{a} < 2$.

Returning to the form of the 2-point function (8.5.5) on $\mathbb{T}^2$ we find that for the cases $z = z_1^0$ and $z = z_2^0$, Δ has to obey $\Delta > -1$ for the integral to converge. However, the $z \to \infty$ limiting case yields an even bigger problem as Δ drops out of the dominating term and we find the integral diverges logarithmically. In conclusion, the 2-point function on $\mathbb{T}^2$ derived in this way diverges $\forall \Delta$ and we conclude that the formula given in [42] yields divergent functions on $\mathbb{T}$.

8.6 The bulk-to-bulk propagator in H_3

We now turn our attention to the top-most arrow of the diagram (8.2.1). In [43] the authors calculate the bulk-to-bulk propagator in H_{d+1}. For our purposes $d = 2$, so that $H_{d+1} = H_3$. In the upper half-space model of H_3 individual points (with subscript i) can be denoted by $\xi_i = \left(\xi_i^0, \xi_i\right) = \left(\xi_i^0, \xi_i^1, \xi_i^2\right)$ where $\xi_i^0 > 0$. Viewing H_3 in the unit ball model it is simple to see the topological boundary is the Riemann sphere which we coordinatise by $\left(x^1, x^2\right)$. We now introduce a bulk-to-boundary propagator [43] which is a function (of a given conformal dimension Δ) dependent on a point in the bulk and a point on the boundary

$$K_\Delta \left(\xi, x\right) = \frac{(\xi^0)^\Delta}{((\xi^0)^2 + |\xi - x|^2)^\Delta}. \tag{8.6.1}$$

Note that $|\xi - x|^2 = (\xi^1 - x^1)^2 + (\xi^2 - x^2)^2$ denoting the Euclidean distance between the projection of bulk points onto the boundary and the boundary point itself.

The 2-point function, or bulk-to-bulk propagator is then found by integrating over the boundary two bulk-to-boundary propagators (of representation weight Δ and $\bar{\Delta}$) connected on the boundary at the same point, namely

$$K_\Delta^{H_3} \left(\xi_1, \xi_2\right) = \int_{S^2} d^2x K_\Delta \left(\xi_1, x\right) K_{\bar{\Delta}} \left(\xi_2, x\right), \tag{8.6.2}$$

which converges $\forall \Delta \in \mathbb{C}$. Following [43] we will restrict ourselves to Type I representations where

$$\Delta = 1 + i\rho. \tag{8.6.3}$$

After a lengthy calculation (see [43] for the details) we obtain the final result for the 2-point function on H_3,

$$K_\rho^{H_3} \left(\xi_1, \xi_2\right) = \frac{\pi}{\rho} \frac{\sin \rho l}{\sinh l}, \tag{8.6.4}$$

where

$$l = \log \left(\frac{\xi_1^0}{\xi_2^0}\right). \tag{8.6.5}$$

We do not compute the 3 and 4-point functions on H_3 using these techniques explicitly as they are quite complex objects.

8.7 Integral transform from H_3 to $\mathbb{T}$ - the "twistor transform"

Having elucidated the details of two of the arrows on the diagram (8.2.1) (and attempted one of the others unsuccesfully) we now turn our attention to the right-most arrow. This transformation (like the $\mathbb{B}$ to $\mathbb{T}$ transform) is also an integral transform. The fascinating thing about the transform is that when we integrate a function over a certain space of hyperplane sections in $\mathbb{T}$ we are supposed to obtain a function on $H_3^{\mathbb{C}}$ that automatically satisfies the massless wave equation.

Reference [18] develops twistor constructions for $\mathrm{SO}(1;n)$ and specifically gives the details of the Penrose transform. Here we present some original work on this transformation using the formulae from [18] but unfortunately conclude that the transformation in this parameterisation is still ill-understood.

We wish first to explain the types of dual spaces we are interested in.

8.7.1 First-kind and second-kind coupled spaces

We define a *first-kind coupled* space to $\mathbb{T}$ very simply as the complex hyperboloid $H_3^{\mathbb{C}}$,

$$H_3^{\mathbb{C}} = \left\{\xi \in \mathbb{C}^{1,3} \mid \langle \xi, \xi \rangle = 1\right\}. \tag{8.7.1}$$

Second-kind coupled spaces are a little more tricky to visualise, so we make a brief aside here in order to alert the reader to how the constructions work. We consider, for visualisable simplicity, the field of real numbers although the construction works equally well for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. The application we have in mind is for 3-dimensional complex projective space.

The simplest case to consider is the Euclidean plane $\mathbb{R}^2$ and elements x thereof. We firstly note that the 1-dimensional real projective space $\mathbb{R}P^1$ is defined as $\mathbb{R}P^1 = (\mathbb{R}^2 \setminus \{0\})/\sim$ where the equivalence relation on $\mathbb{R}^2$ is $x \sim x'$ if $\exists p \in \mathbb{R} \mid x' = px$ and $p \neq 0$. Geometrically this corresponds to the space of real lines through the origin in $\mathbb{R}^2$. We denote elements of $\mathbb{R}P^1$ by $[x]$.

We take the usual inner product on $\mathbb{R}^2$ and specify elements of a subset (in this case a linear subspace) of $\mathbb{R}^2$ by

$$\mathcal{L}_{[x]} = \left\{ y \in \mathbb{R}^2 \mid \langle x, y \rangle = 0 \right\}. \tag{8.7.2}$$

Geometrically, $\mathcal{L}_{[x]}$ is the real line orthogonal to x, but note that $\mathcal{L}_{[x]}$ is only defined up to projective rescalings of x so that $\mathcal{L}_{[x]}$ is orthogonal to $[x]$ explaining the notation. Due to this fact we have the following duality

$$[x] \leftrightarrow \mathcal{L}_{[x]}. \tag{8.7.3}$$

In words, points in $\mathbb{R}P^1$ are dual to origin intersecting lines in $\mathbb{R}^2$, i.e points in $\mathbb{R}P^1$. In this 2-dimensional case the duality is geometrically very trivial. However, when we consider $\mathbb{R}^3$ and elements x and y thereof we immediately see through the condition

$$\langle x, y \rangle = 0, \tag{8.7.4}$$

that points in $\mathbb{R}P^2$ are dual to origin-intersecting 2-dimensional planes in $\mathbb{R}^3$ which are completely specified by the original point in $\mathbb{R}P^2$. Although the planes are no longer elements of $\mathbb{R}^3$ or $\mathbb{R}P^2$ we can consider the manifold formed from these hyperplane sections to be isomorphic to $\mathbb{R}P^2$. The duality in 3 dimensions takes the same form as (8.7.3) where now $[x] \in \mathbb{R}P^2$ and $\mathcal{L}_{[x]}$ is an element of the space of hyperplane sections.

8.7.2 Generalisations

We can immediately generalise to an $(n+1)$-dimensional covering space with the corresponding n-dimensional projective geometry. We obtain the result that the manifold formed from hyperplane sections according to (8.7.4) is isomorphic to the n-dimensional projective space with the duality of the elements given by (8.7.3). With the particular quadric condition (8.7.4) the hyperplane sections are flat and all intersect the origin. It is important to note, however, that the underlying geometrical dimension of an element of the "hyperplane section" manifold is n in comparison with the usual "point" manifolds where the geometrical dimension of a single element is zero. We can also generalise to an $(n+1)$-dimensional covering Minkowski space with the same quadric condition (8.7.4) but now with respect to the Minkowski inner product with $(-,+,...,+)$ signature. All the previous analysis holds (including the origin intersecting condition) but, of course, the notion of perpendicularity changes.

We now generalise to the complex case by merely replacing $\mathbb{R}$ by $\mathbb{C}$ in the previous paragraphs! A specialisation which will be of particular importance to us will be the complex case where $n = 3$.

8.7.3 The transform

We first define a complex form of bidegree (r, s), or an (r, s)-form, on M as [39]

$$\omega = \frac{1}{r!s!}\omega_{\mu_1 \ldots \mu_r \bar{\nu}_1 \ldots \bar{\nu}_s} dz^{\mu_1} \wedge \cdots \wedge dz^{\mu_r} \wedge d\bar{z}^{\nu_1} \wedge \cdots \wedge d\bar{z}^{\nu_s}, \tag{8.7.5}$$

so that $\omega \in \Omega^{r,s}(M)$. The action of the exterior derivative d on elements of $\Omega^{r,s}(M)$ splits according to its destinations by

$$d = \partial + \bar{\partial}, \tag{8.7.6}$$

where

$$\partial : \Omega^{r,s}(M) \to \Omega^{r+1,s}(M), \tag{8.7.7a}$$

$$\bar{\partial} : \Omega^{r,s}(M) \to \Omega^{r,s+1}(M). \tag{8.7.7b}$$

By ∂-closed we mean forms ω such that $\partial\omega = 0$ and similarly, $\bar{\partial}$-closed means forms ω such that $\bar{\partial}\omega = 0$. We further define a homogeneous form of bidegree (λ, μ) to be a form ω whose component functions obey (8.3.3).

We work in the complex quadric representation of $\mathbb{T}$, (7.7.2). The integral transform as given in [18] is

$$\hat{\psi}(\xi) = \int_S \psi \wedge \omega_\xi, \tag{8.7.8}$$

where ψ is a $\bar{\partial}$-closed $(0,1)$ form on $\mathbb{T}$ homogeneous of bidegree $(0,-1)$. This is the function on $\mathbb{T}$ we wish to integrate with some additional form dependence. ω_ξ is a ∂-closed $(1,0)$ form on the hyperplane section S of $\mathbb{T}$ that we integrate over. S is found by considering the intersection of $\mathbb{T}$ with a second kind dual object corresponding to a point in the first kind dual object. For this complex variable case, the point in the first kind dual object is an element ξ of $H_3^{\mathbb{C}}$, which via the duality condition

$$\langle z, \xi \rangle = 0, \tag{8.7.9}$$

has a 3-dimensional complex hyperplane associated with it $\mathcal{L}_{[\xi]} \in \mathbb{C}P_\xi^3$. The intersection of this hyperplane with $\mathbb{T}$ gives us a 1-dimensional complex hyperplane section $S = \mathbb{T} \cap \mathcal{L}_{[\xi]}$ which we integrate over.

Reference [18] gives the formula for the invariant one-form as

$$\omega_\xi = \frac{[u, v, z, dz]}{\langle u, z \rangle \langle \xi, v \rangle}. \tag{8.7.10}$$

The notation $[a, b, c, d]$ denotes the determinant of a 4×4 matrix with the 4-component objects a, b, c, d as the columns. u, v, z, ξ are all $\mathbb{C}^{1,3}$ vectors obeying the conditions

$$\langle z, z \rangle = 0, \tag{8.7.11}$$

specifying the quadric and

$$\langle z, \xi \rangle = 0, \tag{8.7.12}$$

specifying the duality condition. $\langle u, z \rangle \neq 0$ and $\langle \xi, v \rangle \neq 0$ ensure that the form is well-defined.

Note that this form is indeed invariant under the action of the group $O_c(1,3)$ as was checked via Maple. Crucially this form is dependent on a point in the first kind dual space $\xi \in H_3^{\mathbb{C}}$ so that the result $\hat{\psi}$ is a function on $H_3^{\mathbb{C}}$.

However, contrary to the claim in [18], the measure is dependent on u and v. We do not want to have a transformation defined only up to some dependence on two $\mathbb{C}^{1,3}$ vectors so we redefine the one-form via

$$\omega_\xi = \frac{[u, v, z, dz]}{\langle u, z\rangle \langle \xi, v\rangle - \langle u, \xi\rangle \langle z, v\rangle}, \tag{8.7.13}$$

which is again $O_c(1,3)$ invariant but now is independent of u and v. It is this second form (8.7.13) which we use in the computations.

8.7.4 Real twistor transform

As a first step on the path to the full complex twistor transform we first consider the simpler case of real variables. (In this context we change the nomenclature of the variables; $z \to x$, $\xi \to y$). The specific advantage of considering this simpler case is that all the geometry is visualisable. The complex quadric prescription of $\mathbb{T}$ takes on a well known form when we restrict our attention to real variables. Consider the equation defining the quadric

$$\langle x, x\rangle = 0, \tag{8.7.14}$$

where $x \in \mathbb{R}^{1,3}$. This equation defines the 3-dimensional light cone (where we consider only the part with $x_0 > 0$) so that

$$L_+ = \left\{x \in \mathbb{R}^{1,3} \mid \langle x, x\rangle = 0,\ x_0 > 0\right\}, \tag{8.7.15}$$

and when we remember that we are working in projective space we obtain the projective light cone,

$$\mathcal{P}L_+ = L_+/\sim, \tag{8.7.16}$$

where $\sim$ is the equivalence relation $x \sim x' \Leftrightarrow x = \lambda x'$ for some $\lambda \in \mathbb{R}^+$. We have obtained the result that the "real" twistor space is simply the projective light cone, which is isomorphic to the 2-sphere,

$$\mathbb{T}^{\mathbb{R}} = \mathcal{P}L_+ \cong S^2. \tag{8.7.17}$$

The first-kind dual space in the context of real twistor space is simply de Sitter 3-space dS_3, specified by elements $y \in \mathbb{R}^{1,3}$ s.t $\langle y, y \rangle = 1$. The duality condition (8.7.3) gives us (for $y \in dS_3$) timelike 3-planes orthogonal to y,

$$\mathcal{L}_{[y]} = \{x \mid \langle x, y \rangle = 0\} . \tag{8.7.18}$$

The hyperplane section of $\mathbb{T}^{\mathbb{R}}$ we integrate over is the intersection of this timelike 3-plane with the 2-sphere which gives us a circle on the sphere. For example if we choose $y = (0,0,0,1) \in dS_3$ then the condition (8.7.4) places no restriction on x_0, x_1, x_2 but sets $x_3 = 0$ to give us a timelike 3-plane $\mathcal{L}_{[y]}$. Taking advantage of the equivalence relation in (8.7.16) we can parameterise $\mathcal{P}L_+$ by $x_0 = 1$ and find that the intersection of $\mathcal{L}_{[x]}$ and $\mathcal{P}L_+$ is

$$\mathcal{L}_x \cap \mathcal{P}L_+ = \left\{x \in L_+ \mid x_0 = 1, x_1^2 + x_2^2 = 1, x_3 = 0\right\} , \tag{8.7.19}$$

being a specific circle $S^1 \subset S^2$. Other points in dS_3 give rise to other circles on the sphere. Finally we note that y and $-y$ in dS_3 specify the same circle on the 2-sphere. Defining an equivalence relation $\sim$ by $y \sim y' \Leftrightarrow y = -y'$ we obtain the result that $dS_3/\sim$ is isomorphic through (8.7.4) to the space of circles on the sphere. The real version of the one-form (8.7.13) now gives us an invariant measure on this space of circles. We wish to parameterise this one-form on a general circle.

To parameterise the one-form we consider inverse stereographic projection from the 2-plane to the 2-sphere $\sigma^{-1} : \mathbb{R}^2 \to S^2$ so that

$$\begin{aligned}\sigma^{-1}(X_1, X_2) &= \left(1, \frac{2X_1}{1+X^2}, \frac{2X_2}{1+X^2}, \frac{X^2-1}{1+X^2}\right) \\ &\sim \left(\frac{1+X^2}{2}, X_1, X_2, \frac{X^2-1}{2}\right) = x \in \mathbb{R}^{1,3}.\end{aligned} \tag{8.7.20}$$

as x is defined only up to projective rescalings. Here $X = (X_1, X_2) \in \mathbb{R}^2$ and $X^2 = X_1^2 + X_2^2 = \langle X, X \rangle_{\mathbb{R}^2}$. We also choose a coordinate system for dS_3 with $y \in dS_3$ such that

$$y = (\sinh t, \cosh t\ Y) , \tag{8.7.21}$$

where $Y = (y_1, y_2, y_3) \in \mathbb{R}^3$ and $\Sigma_i y_i^2 = 1$ so that $\langle y, y \rangle = 1$ is automatically satisfied. For a fixed y the duality condition (8.7.4) gives us the equation of a circle on $\mathbb{R}^2$,

$$\left(X_1 - \frac{y_1}{p - y_3}\right)^2 + \left(X_2 - \frac{y_2}{p - y_3}\right)^2 = \frac{1 - p^2}{(p - y_3)^2}, \tag{8.7.22}$$

with centre

$$\frac{(y_1, y_2)}{p - y_3}, \tag{8.7.23}$$

and radius

$$\frac{\left(1 - p^2\right)^{1/2}}{p - y_3}, \tag{8.7.24}$$

where

$$p = \tanh t. \tag{8.7.25}$$

We introduce a polar angle χ along this circle by

$$(X_1, X_2) = \frac{1}{p - y_3}\left(y_1 + \left(1 - p^2\right)^{1/2}\cos\chi, y_2 + \left(1 - p^2\right)^{1/2}\sin\chi\right), \tag{8.7.26}$$

which parameterises the circle for $y \in dS_3$.

We now compute the real version of the one-form (8.7.13) and find

$$\omega_y = \frac{d\chi}{\sinh t - \cosh t\left(1 - y_1^2 - y_2^2\right)^{1/2}}, \tag{8.7.27}$$

which is independent of u and v in (8.7.13) and gives us the correct measure to use when integrating functions over this circle on S^2. One point to note is that using this form of the sterographic projection, ω_y is ill-defined for circles intersecting the north pole as the stereographic projection will send this point to infinity.

Using ω_y we now wish to integrate functions of an appropriate degree of homogeneity on the sphere, over this circle, with the desire that the resulting function automatically satisfies the wave equation in dS_3.

8.8 Solutions to the wave equation on dS_3

We now consider solutions to the wave equation in general and give a result concerning the solution in dS_3.

The Laplacian Δ is defined using the exterior derivative and its adjoint as a linear map $\Delta : \Omega^r(M) \to \Omega^r(M)$, where $\Delta = \left(d + d^\dagger\right)^2$ [39]. For the specific case where $r = 0$ the Laplacian is a linear map between functions given by the expression,

$$\Delta = \frac{1}{\sqrt{|g|}}\partial_\nu\left(\sqrt{|g|}g^{\nu\mu}\partial_\mu\right), \tag{8.8.1}$$

where g is the metric on the manifold.

In the context of the real twistor transform we wish to obtain functions automatically satisfying the wave equation on dS_3 which has the metric

$$ds^2 = -dt^2 + \cosh^2 t\; d\Omega^2, \tag{8.8.2}$$

where $d\Omega^2$ is the metric on the 2-sphere. This metric is induced from the metric on $\mathbb{R}^{1,3}$ and has induced signature $(-,+,+)$. On dS_3 the Laplacian takes the form

$$\Delta_{dS^3} = \frac{1}{\cosh^2 t}\left(-\partial_t\left(\cosh^2 t\; \partial_t\right) + \Delta_{S^2}\right), \tag{8.8.3}$$

where Δ_{S^2} is the Laplacian on the 2-sphere.

We search for separable solutions to the massless wave equation of the form $f = T(t)Y(\theta,\phi)$ and see that the Y part of the solution takes the usual form of a spherical harmonic on the 2-sphere

$$\Delta_{S^2}Y_l^m(\theta,\phi) = -l(l+1)Y_l^m(\theta,\phi), \tag{8.8.4}$$

$$Y_l^m(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi}, \tag{8.8.5}$$

where the $P_l^m(\cos\theta)$ are the associated Legendre polynomials. We are thus left with the T part of the solution satisfying the differential equation

$$\left(\partial_t(\cosh^2 t\partial_t) + l(l+1)\right)T(t) = 0. \tag{8.8.6}$$

Via the change of variables $\sigma = -e^{2t}$ we transform to

$$\sigma \left(1-\sigma\right) T'' - 2\sigma T' - \frac{l(l+1)}{1-\sigma} T = 0, \tag{8.8.7}$$

where differentiation is indicated w.r.t σ.

We now make the substitution $T = \cosh^l t \; e^{(l+2)t} F$ so that F satisfies the hypergeometric equation

$$\sigma \left(1-\sigma\right) F'' + \left(c - \left(1+a+b\right)\sigma\right) F' - abF = 0, \tag{8.8.8}$$

with $a = l+1, b = l+2, c = 2$. One solution to this equation is given by the hypergeometric series, [44], $F_{l+1,l+2} = F\left(l+1, l+2, 2, -e^{2t}\right)$, which can be simplified to give

$$F_{l+1,l+2} = \begin{cases} \left(1+e^{2t}\right)^{-1} & l = 0, \\ \left(l+1\right)\left(1+e^{2t}\right)^{-2l-1} \sum_{c=0}^{l-1} \binom{l+1}{c+1}\binom{l-1}{c}\left(-e^{2t}\right)^{c} & l \geq 1. \end{cases} \tag{8.8.9}$$

In [45] the two linearly independent solutions to the massive, $m \neq 0$, wave equation on dS_3 are given. The result (8.8.9) is the massless special case of the result obtained in [45], with the hypergeometric part written explicitly as a polynomial. We obtain the final result that one solution to the massless wave equation on dS_3 is given (for an integer value of l and corresponding integer values of m) by the product of the hypergeometric function (8.8.9), $\cosh^l t \; e^{(l+2)t}$ and spherical harmonics Y_l^m. We expect the result of the real twistor transform to give (at least) a subset of these functions and possibly intersect with the whole set.

Returning to the twistor transform we now use (8.8.3) in the form

$$\Delta_{dS_3} = -\partial_t^2 - 2\tanh t \partial_t + \frac{1}{\cosh^2 t}\left(\partial_\theta^2 + \cot\theta\partial_\theta + \frac{1}{\sin^2\theta}\partial_\phi^2\right), \tag{8.8.10}$$

where we have chosen $Y = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$. Unfortunately, we could not make functions $\hat{\psi}$ transformed using the real version of (8.7.8) with the form ω_y (8.7.27) satisfy $\Delta_{dS_3}\hat{\psi} = 0$.

8.9 Complex twistor transform

Of course, it is the complex version which we are really interested in to understand the right-most arrow of the diagram (8.2.1). It is a relatively simple extension where we now use inverse complex sterographic projection $\sigma_{\mathbb{C}}^{-1} : \mathbb{C}^2 \to \mathcal{P}L_+^{\mathbb{C}}$, ϕ, t and y are now complex parameters and we specify the first kind dual manifold as $\langle \xi, \xi \rangle = 1$. The invariant form on the space of hyperplane sections is

$$\omega_\xi = \frac{d\phi}{\cosh t - \sinh t \left(1 - y_1^2 - y_2^2\right)^{1/2}}, \tag{8.9.1}$$

which is a ∂-closed $(1,0)$-form.

The problems afflicting the real twistor transform are present in the complex case as well - the transformation does not give us a set of functions satisfying the massless wave equation on $H_3^{\mathbb{C}}$. One possible problem is that the form (8.7.13) is an ansatz which turned out to be explicitly u, v independent. If we were to return to this problem we would compute the measure as given in [18] from first principles as the residue of a closed form with a simple pole.

Chapter 9

Conclusions

The thesis was split into two parts; chapters 2 through 6 discussed the Hamiltonian formulation of two massive point particles coupled to AdS_3 gravity whereas chapters 7 and 8 discussed $\mathrm{SL}(2,\mathbb{C})$ twistor space and conformal field theory n-point functions.

9.1 Point particles coupled to AdS_3 gravity

In one sense GR in $2+1$ dimensions is trivial when compared with its $(3+1)$-dimensional counterpart; the triviality abounding from the fact that a $(2+1)$-dimensional spacetime has no local degrees of freedom, physically meaning there are no gravitational waves. However, in $2+1$ dimensions the theory can be consistently coupled to point particles, providing us with one method of constructing topologically nontrivial spacetimes which have a finite number of global degrees of freedom. There are a number of (classically equivalent) ways to analyse $(2+1)$-dimensional gravitational theories being that of geometric structures, the Chern-Simons formulation and the ADM formalism [46]. There are also many ways in which to formulate the dynamics of such a system with point particles present, particularly with respect to the boundary conditions of the theory, see references in [9].

In the first part of the thesis we formulated and analysed the Hamiltonian for a pair of massive point-particles coupled to AdS_3 gravity.

We began by discussing the geometry of two particles coupled to AdS_3. We chose, following [9], to anchor the dynamics to the asymptotically AdS conical infinity, and in doing so described the geometry of the system in the relativistic analogue to the Newtonian centre-of-mass frame. We did this by firstly describing the geometry of the two-particle spacetimes in terms of a piece of AdS_3 spacetime *between* the particles and then translating this description into one in which we could relate the spacetime dynamics to the infinity. This anchoring procedure also led us to use the BTZ time as the time coordinate in our discussion of the Hamiltonian formulation. We made a substantial technical simplification at this point by specialising to the case of zero angular momentum yielding a spacetime containing colliding particles.

We chose further to use the Chern-Simons formulation of gravity in order to discuss the bulk action and how it naturally splits into two Chern-Simons (C-S) type actions. We then went on to discuss the gauge transformations of the theory which we used when discussing how to fix the gauge. We then split the spacetime manifold according to $M = \Sigma \times \mathbb{R}$ and calculated the $2 + 1$ decomposition of the bulk action in order to discover what form the Liouville term took. We also briefly discussed the particle actions and chose the action on the boundary at the spatial infinity.

In order to reduce the action we imposed the constraints and fixed the gauge of the theory by embedding Σ into a fictitious spacetime of the form already discussed. We then used the details of the embedding and gauge fixing to evaluate the Liouville term. We evaluated the term through a combination of direct evaluation and conversion of the term into a one-dimensional boundary integral by Stokes' theorem. The evaluation of this boundary integral presented the largest problem within the whole body of work and although we did not complete the reduction using these techniques, we did carry it out to a stage where it was possible to sidestep the remaining technicalities and use consistency with the known equations of motion to complete the analysis.

We thus evaluated the reduced action of the theory, for three special cases of the masses of the two particles, and obtained a two dimensional reduced phase space. The dimensionality of the phase space was due to the fact that we were in the colliding particle regime and thus needed only one position and one momentum coordinate to describe the system fully. We would expect to obtain a four dimensional reduced phase space if the analysis were completed for the spinning particle regime as is the case in [9]. We performed a canonical transformation to a phase space chart coordinatised by the geodesic distance between the particles, being analagous to the reduced position vector of a Newtonian two-body system in the centre-of-mass frame, and its relative conjugate momentum, and wrote the Hamiltonian in terms of these variables.

In contrast to [9] our theory included a negative cosmological constant meaning we had a certain regime in which the spacetime contained a black hole. We continued the Hamiltonian analytically to the black hole regime and also analysed the threshold of black hole formation $H = 0$. We found that near this threshold the mass of the black hole depended linearly on the momentum of the particles. In the language of critical phenomena, this equates to the mass scaling with critical exponent 1 in agreement with what was found in [12] by a method that uses the constants of motion as coordinates on the phase space.

We also used the action to compute the semiclassical tunnelling probability amplitude of two particles out of the black hole. We found that the imaginary part of the action was equal to the Bekenstein-Hawking entropy of the black hole. In a similar analysis for a spherical shell in four dimensions [47], the imaginary part of the action was found to be half of the Bekenstein-Hawking entropy S_{BH}, leading to the factor $\exp(-S_{\mathrm{BH}})$ on taking the modulus squared of the semiclassical probablility amplitude. The reason for the factor of 2 difference between our result and that of [47] appears to be in the different choices of the time coordinate [48].

In summary, we have formulated and analysed a Hamiltonian for three specific cases for two-particle AdS_3 spacetimes. We have described the geometry of such spacetimes and used the Chern-Simons formulation of gravity in discussing the action. We fixed the gauge and evaluated the reduced action of the theory. Finally we performed a canonical transformation to find a Hamiltonian for each of the three cases.

For two of the cases (two massless particles and two particles with equal positive masses) the critical exponent for the threshold of black hole formation was shown to coincide with the results in [12]. Also for these two cases the equations of motion have been analysed and used to calculate the imaginary contribution to the action which has been shown to coincide with the Bekenstein-Hawking entropy. The author has no reason to expect that the results for arbitrary vlues of the masses would be different.

We could extend the research in a variety of ways. One thing to consider would be when the spacetime contained $n > 2$ particles. In theory we could proceed with the analysis of the spacetime geometry in the same way although in practice the calculations would get increasingly more difficult. We could also consider evaluating the reduced action for the spinning particles case although it may be worthwhile to refine the techniques used to carry out the reduction first. One other area worthy of further study would be to consider the quantisation of the various Hamiltonians obtained. The quantisation of the Hamiltonian for the zero cosmological constant case has been considered in [11] and it would be worthwhile to consider the quantisation of the negative cosmological constant case along similar lines. We leave this question open for further studies.

9.2 $\mathrm{SL}(2,\mathbb{C})$ twistor space and conformal field theory

In the second part of the thesis we analysed the properties of $\mathrm{SL}(2,\mathbb{C})$ twistor space, $\mathbb{T}$, and explored the form that conformal field theory takes within $\mathbb{T}$ and various related spaces. We gave some details on the twistor transform for $\mathbb{T}$.

We were first concerned with coset spaces constructed from $\mathrm{SL}(2,\mathbb{C})$. We realised "$\mathrm{SL}(2,\mathbb{C})$ twistor space", showing that it has complex dimension two, a metric with signature $(+,+,-,-)$ and an interesting complex structure. We also realised one representation of the coset space $\mathbb{B}$. We went on to examine the form of conformal field theory n-point functions in S_2, H_3, $\mathbb{T}$ and $\mathbb{B}$. In order to translate the n-point functions to the various spaces we needed to make use of two separate integral transforms, one being the twistor transform for $\mathbb{T}$. We gave the details of the 2, 3 and 4-point correlation functions defined on the Riemann sphere and used some techniques from representation theory to transform the functions of interest into $\mathbb{B}$. We then analysed the integral transform from $\mathbb{B}$ to $\mathbb{T}$ as given in [42] but unfortunately concluded that the formula given in [42] yields divergent functions on $\mathbb{T}$.

In the attempt to understand the twistor transform from $\mathbb{T}$ to H_3 our work enabled us to write the hypergeometric part of the solution to the massless wave equation on dS_3 explicitly as a polynomial. Unfortunately the computation of the twistor transform did not yield functions automatically satisfying the wave equation (being the main motivation for this work). If we were to return to this problem we would compute the measure as given in [18] from first principles as the residue of a closed form with a simple pole. However, we unfortunately had to conclude that the twistor transform in this context is still ill understood.

References

[1] L. Smolin. *Three roads to quantum gravity.* Weidenfeld and Nicolson, 2000.

[2] C. Rovelli. *Quantum gravity.* Cambridge University Press, 2004.

[3] S. Carlip. Quantum gravity: A progress report. *Rept. Prog. Phys.*, 64:885, 2001.

[4] P. Dirac. Generalized Hamiltonian dynamics. *Can. J. Math.*, 2:129–48, 1950.

[5] B. S. DeWitt. Quantum theory of gravity. I: The canonical theory. *Phys. Rev.*, 160:1113–1148, 1967.

[6] S. Carlip. *Quantum gravity in 2+1 dimensions.* University Press, Cambridge, 1998.

[7] S. Deser, R. Jackiw, and G. 't Hooft. Three-dimensional Einstein gravity: Dynamics of flat space. *Ann. Phys.*, 152:220–235, 1984.

[8] S. Deser and R. Jackiw. Three-dimensional cosmological gravity: Dynamics of constant curvature. *Ann. Phys.*, 153:405–416, 1984.

[9] J. Louko and H-J. Matschull. (2+1)-dimensional Einstein-Kepler problem in the centre-of-mass frame. *Class. Quant. Grav.*, 17:1847–1873, 2000.

[10] S. Deser, R. Jackiw, and G. 't Hooft. Physical cosmic strings do not generate closed timelike curves. *Phys. Rev. Lett.*, 68:267, 1992.

[11] J. Louko and H-J. Matschull. The (2+1) Kepler problem and its quantization. *Class. Quant. Grav.*, 18:2731–2784, 2001.

[12] D. Birmingham. Choptuik scaling and quasinormal modes in the AdS/CFT correspondence. *Phys. Rev. D*, 64:309–317, 2001.

[13] R. Penrose. *The road to reality: A complete guide to the laws of the universe*. Knopf, 2005.

[14] S. A. Huggett and K. P. Tod. *An introduction to twistor theory*. London Mathematical Student Texts, 1994.

[15] E. Witten. Perturbative gauge theory as a string theory in twistor space. *Commun. Math. Phys.*, 252:189–258, 2004.

[16] E. Witten. Parity invariance for strings in twistor space. *Adv. Theor. Math. Phys.*, 8:779–796, 2004.

[17] R. Penrose and M. A. H. MacCallum. Twistor theory: An approach to the quantisation of fields and space-time. *Phys. Rept.*, 6:241–315, 1973.

[18] S. Gindikin. SO(1;n)-twistors. *Journal of Geometry and Physics*, 26:26–36, 1998.

[19] J. Maldacena. The large N limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys*, 2:231–252, 1998.

[20] E. Witten. Anti de Sitter space and holography. *Adv. Theor. Math. Phys*, 2:253–291, 1998.

[21] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz. Large N field theories, string theory and gravity. *Phys. Rept.*, 323:183–386, 2000.

[22] M. Banados, M. Hennaux, C. Teitelboim, and J. Zanelli. Geometry of the 2+1 black hole. *Phys. Rev. D*, 48:1506–1525, 1993.

[23] S. Carlip. The (2+1)-dimensional black hole. *Class. Quant. Grav.*, 12: 2853–2880, 1995.

[24] H-J. Matschull and M. Welling. Quantum mechanics of a point particle in 2+1 dimensional gravity. *Class. Quant. Grav.*, 15:2981–3030, 1998.

[25] H-J. Matschull. On the relation between 2+1 Einstein gravity and Chern Simons theory. *Class. Quant. Grav.*, 16:2599–2609, 1999.

[26] J. D. Romano. Geometrodynamics vs. connection dynamics. *Gen. Rel. Grav.*, 25:759–854, 1993.

[27] E. Witten. 2+1 dimensional gravity as an exactly soluble system. *Nucl. Phys. B*, 311:46–78, 1988.

[28] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry Vol. 2*. John Wiley and Sons, Inc., New York, 1969.

[29] J. W. Anderson. *Hyperbolic Geometry*. Springer-Verlag, London, 2005.

[30] H. Goldstein. *Classical Mechanics*. Addison-Wesley, Massachusetts, 1980.

[31] A. Jevicki and J. Thaler. Dynamics of black hole formation in an exactly solvable model. *Phys. Rev. D*, 66:024041, 2002.

[32] G. W. Gibbons and S. W. Hawking. Action integrals and partition functions in quantum gravity. *Phys. Rev. D*, 15:2752, 1977.

[33] S. W. Hawking. *General Relativity: An Einstein Centenary survey*. Cambridge University Press, Cambridge, England, 1979.

[34] D. N. Page. *Black Hole Physics*. Kluwer Academic Publishers, Dordrecht, 1992.

[35] D. Garfinkle, S. B. Giddings, and A. Strominger. Entropy in black hole pair production. *Phys. Rev. D*, 49:958, 1994.

[36] S. W. Hawking, G. T. Horowitz, and S. F. Ross. Entropy, area, and black hole pairs. *Phys. Rev. D*, 51:4302, 1995.

[37] K. Krasnov. Lambda<0 quantum gravity in 2+1 dimensions i: Quantum states and stringy s-matrix. *Class. Quant. Grav.*, 19:3977–3998, 2002.

[38] K. Krasnov. Lambda<0 quantum gravity in 2+1 dimensions ii: Black hole creation by point particles. *Class. Quant. Grav.*, 19:3999–4028, 2002.

[39] M. Nakahara. *Geometry, topology and physics.* Institute of physics publishing, Bristol and Philadelphia, 2003.

[40] P. Di Francesco, P. Mathieu, and D. Senechal. *Conformal field theory.* Springer-Verlag, Inc., New York, 1997.

[41] K. Krasnov. Holography for the Lorentz group Racah coefficients. *Class. Quant. Grav.*, 22:1933–1944, 2005.

[42] I. M. Gelfand, M.I Graev, and N. Ya. Vilenkin. *Integral geometry and representation theory, generalized functions vol. 5.* Academic Press Inc, London, 1966.

[43] K. Krasnov and J. Louko. SO(1,d+1) Racah coefficients: Type I representations. *J. Math. Phys.*, 47:334–359, 2006.

[44] N. N. Lebedev. *Special functions and their applications.* Prentice-Hall Inc, New Jersey, 1965.

[45] R. Bousso, A. Maloney, and A. Strominger. Conformal vacua and entropy in de Sitter space. *Phys. Rev. D*, 65:104039, 2002.

[46] S. Carlip. Lectures on (2+1)-dimensional gravity. *J. Korean Phys. Soc.*, 28:S447–S467, 1995.

[47] M. K. Parikh and F. Wilczek. Hawking radiation as tunneling. *Phys. Rev. Lett.*, 85:5042, 2000.

[48] A. J. M. Medved. A brief editorial on de Sitter radiation via tunneling. <arXiv:hep-th/0802.3796>, 2008.