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Active Vibration Control

of

Rotating Machines

by

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Abstract

Second order matrix equations arise in the description of real dynamical systems. Traditional modal control approaches utilise the eigenvectors of the undamped system to diagonalise the system matrices. Any remaining off-diagonal terms in the modal damping matrix are discarded. A regrettable automatic consequence of this action is the destruction of any notion of the skew-symmetry in the damping.

The methods presented in this thesis use the 'Lancaster Augmented Matrices' (LAMs) allowing state space representations of the second order systems. 'Structure preserving transformations' (SPTs) are used to manipulate the system matrices whilst preserving the structure within the LAMs. Utilisation of the SPTs permits the diagonalisation of the system mass, damping and stiffness matrices for non-classically damped systems. Thus a modal control method is presented in this thesis which exploits this diagonalisation. The method introduces independent modal control in which a separate modal controller is designed in modal space for each individual mode or pair of modes.

The modal displacements and velocities for the diagonalised systems are extracted from the physical quantities using first order SPT-based filters. Similarly the first order filters are used to translate the modal force into the physical domain. Derivation of the SPT-filters is presented together with a method by which one exploits the non-uniqueness of the diagonalising filters such that initially unstable filters are stabilised.

In the context of active control of rotating machines, standard optimal controller methods enable a trade-off to be made between (weighted) mean-square vibrations and (weighted) mean-square control forces, or in the case of a machines controlled using magnetic bearings the currents injected into the magnetic bearings. One shortcoming of such controllers for magnetic bearings is that no concern is devoted to the voltages required. In practice, the voltage available imposes a strict limitation on the maximum possible rate of change of control force (force slew rate). This thesis presents a method which removes the aforementioned existing shortcomings of traditional optimal control.

Case studies of realistic rotor systems are presented to illustrate the modal control and control force rate penalisation methods. The system damping matrices of the case studies contain skew-symmetric components due to gyroscopic forces typical of rotating machines. The SPT-based modal control method is used to decouple the non-classically damped equations of motion into n single degree of freedom systems. Optimal modal controllers are designed independently in the modal space such that the modal state, modal forces and modal force rates are weighted as required. The SPT-based modal control method is shown to yield superior results to the conventional notion of independent modal space control according to reasonable assessment.

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Contents

List of Publications

First Author

1. Houlston P.R., *Extracting Second Order System Matrices From State Space System*, Proceedings of the Institution of Mechanical Engineers, Part C, Journal of Mechanical Engineering Science, 2006, 220(8), pp. 1147 - 1149.

2. Houlston P.R., Garvey S.D. and Popov A.A., *Modal Control of Vibration in Rotating Machines and Other Generally Damped Systems*, ISMA 2006 International Conference on Noise and Vibration Engineering, Leuven, Belgium, 18-20 September 2006.

3. Houlston P.R., Garvey S.D. and Popov A.A., *Optimal Controller Designs for Rotating Machines - Penalising the Rate of Change of Control Forcing*, 7th IFToMM-Conference on Rotor Dynamics, Vienna, Austria, 25-28 September 2006.

4. Houlston P.R., Garvey S.D. and Popov A.A., *Modal Control of Vibration in Rotating Machines and Other Generally Damped Systems*, Journal of Sound and Vibration, 2007, 302(1-2), pp. 104-116.

5. Houlston P.R., Garvey S.D. and Popov A.A., *Electrical Equivalent Circuit Representation of a Vibrating Flexible Beam with Piezoelectric Transducers*, Submitted to Journal of Smart Materials and Structures, May 2007.

6. Houlston P.R., Garvey S.D. and Popov A.A., *The Implementation and Stabilisation of Structure Preserving Modal Filters for Control of Generally Damped Systems*, Submitted to Journal of Guidance, Control and Dynamics, June 2007.

Named Author

1. Khoo W.K.S., Garvey S.D., Kalita K. and Houlston P.R., *Vibration Control with Lateral Force Producing Electrical Machines*, Eigth International Conference on Vibrations in Rotating Machinery, Swansea, UK, 2004, Paper C623/074/2004, pp. 713-722.

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Abbreviations

AMB	Active Magnetic Bearing
DOF	Degree of Freedom
DSPT	Diagonalising Structure Preserving Transformation
FE	Finite Element
IMSC	Independent Modal Space Control
MIMSC	Modified Independent Modal Space Control
LAMs	Lancaster Augmented Matrices
MPC	Predictive Control
SDOF	Single Degree of Freedom
SPS	Structural Preserving Similarity
SPT	Structural Preserving Transformation
LQR	Linear Quadratic Regulator

Nomenclature

d	SDOF damping coefficient
$\mathbf{e}$	Vector of eigenvalues
$\mathbf{e}_{,\sigma}$	Vector of eigenvalue derivatives with respect to σ
$\mathbf{f}_A(t)$	Long vector of original generalised physical forces
$\mathbf{f}_A^*(t)$	Optimal force trajectory
$\mathbf{f}_{Ad}(k)$	Discrete time force vector at k^{th} time step
$\mathbf{f}_{Am}(t)$	Long vector of original master physical forces
$\mathbf{f}_{As}(t)$	Long vector of original slave physical forces
$\underline{\mathbf{f}_{Ak}}(t)$	State space vector of original generalised forcing
$\mathbf{f}_b(t)$	Long vector of balanced forces
$\mathbf{f}_B(t)$	Long vector of transformed generalised forces
$\mathbf{f}_C(t)$	Long vector of reformed second order system forces
g_d	SDOF controller gain proportional to velocity
$\mathbf{g}_d$	Controller gain matrices proportional to velocity
g_k	SDOF controller gain proportional to displacement
$\mathbf{g}_k$	Controller gain matrices proportional to displacement
$\mathbf{h}$	Vector defined by open-loop system and desired pole locations
i	Imaginary number, ($i = \sqrt{-1}$)
$\mathbf{i}$	Current applied to magnetic bearing
$\mathbf{i}_0$	Initial current
k	SDOF stiffness coefficient
m	SDOF mass coefficient
$\mathbf{q}_A(t)$	n-dimensional vector of original displacements

$\mathbf{q}_{Am}(t)$	Partition relating master co-ordinates
$\mathbf{q}_{As}(t)$	Partition relating slave co-ordinates
$\mathbf{q}_{A1}(t)$	Upper n dimensional partition of state vector $\underline{\mathbf{q}_A}$
$\mathbf{q}_{A2}(t)$	Lower n dimensional partition of state vector $\underline{\mathbf{q}_A}$
$\underline{\mathbf{q}_A}(t)$	$2n$-dimensional original state vector
$\underline{\mathbf{q}_A}^*(t)$	Optimal state trajectory
$\underline{\mathbf{q}_{Ad}}(k)$	$2n$-dimensional discrete time state vector
$\underline{\mathbf{q}_{aug}}(t)$	Augmented state vector
$\underline{\mathbf{q}_{br}}(t)$	Co-ordinates of balanced state to retain
$\underline{\mathbf{q}_{bd}}(t)$	Co-ordinates of balanced state to discard
$\underline{\mathbf{q}_{me}}(t)$	Mechanical-Electrical augmented state vector
$\mathbf{q}_B(t)$	Vector of transformed displacements, modal co-ordinates
$\mathbf{q}_{B1}(t)$	Upper n dimensional partition of transformed state vector $\underline{\mathbf{q}_B}$
$\mathbf{q}_{B2}(t)$	Lower n dimensional partition of transformed state vector $\underline{\mathbf{q}_B}$
$\underline{\mathbf{q}_B}(t)$	Transformed state vector
$\mathbf{q}_C(t)$	Vector of transformed displacements
$\mathbf{r}$	Magnetic bearing coil resistance
s	Laplace variable
$\underline{s}_L$	Single degree of freedom left reflexive SPT
$\underline{s}_R$	Single degree of freedom right reflexive SPT
t	Time
$\mathbf{u}_A(t)$	Short vector of original applied physical forces
$\underline{\mathbf{u}_A}(t)$	Augmented vector of applied physical forces and rate of change of forces
$\mathbf{u}_{Ad}(k)$	Discrete time vector of applied physical forces
$\mathbf{u}_B(t)$	Transformed vector of applied forces
$\mathbf{y}_A(t)$	Vector of observed original state
$\mathbf{u}_j$	Right eigenvector corresponding to j^{th} eigenvalue
$\mathbf{v}_j$	Left eigenvector corresponding to j^{th} eigenvalue
$\mathbf{x}_H$	vector corresponding to Hamiltonian system 0^{th} order state
$\mathbf{y}_C(t)$	Vector of observed reformed state

∂	Partial derivative symbol
$\boldsymbol{\gamma}(t)$	Lagrange multiplier
λ	Single eigenvalue
$\boldsymbol{\mu}(t)$	Lagrange multiplier
ρ	Density (kg/m^3)
σ	Reflexive SPT construction parameter
τ	Derivative with respect to time $(\frac{d}{dt})$
θ	angle (rad)
$\mathbf{A}$	n-dimensional matrix
$\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_l$	System matrices of l^{th} order system
$\mathbf{A}'$, $\mathbf{A}''$, $\mathbf{A}'''$	Transformed n-dimensional matrix
$\underline{\mathbf{A}_A}$	State space companion matrix of original system
$\underline{\mathbf{A}_{A,\,left}}$	Left state space companion matrix of original system
$\underline{\widetilde{\mathbf{A}}_A}$	Augmented state space companion matrix of original system
$\underline{\mathbf{A}_b}$	Balanced state space companion matrix
$\underline{\mathbf{A}_{bdd}}$	Discard-Discard partition of balanced state space companion matrix
$\underline{\mathbf{A}_{bdr}}$	Discard-Retain partition of balanced state space companion matrix
$\underline{\mathbf{A}_{brd}}$	Retain-Discard partition of balanced state space companion matrix
$\underline{\mathbf{A}_{brr}}$	Retain-Retain partition of balanced state space companion matrix
$\underline{\mathbf{A}_{con}}$	Companion matrix of closed loop system
$\underline{\mathbf{A}_d}$	Discrete time state space companion matrix
$\underline{\mathbf{A}_k}$	k^{th} orginial LAM $(k = 0, 1, 2)$
$\underline{\mathbf{B}_A}$	State space forcing matrix of original system
$\underline{\widetilde{\mathbf{B}}_A}$	Augmented state space forcing matrix of original system
$\underline{\mathbf{B}_b}$	Balanced state space forcing matrix
$\underline{\mathbf{B}_{bd}}$	Discard partition of balanced state space forcing matrix
$\underline{\mathbf{B}_{br}}$	Retained partition of balanced state space forcing matrix
$\underline{\mathbf{B}_d}$	Discrete time state space forcing matrix

$\underline{\mathbf{B}_k}$	k^{th} transformed LAM ($k = 0, 1, 2$)
$\underline{\mathbf{C}_A}$	State space observation matrix of original system
$\underline{\mathbf{C}_b}$	Balanced state space observation matrix
$\underline{\mathbf{C}_{bd}}$	Discarded partition of state space observation matrix
$\underline{\mathbf{C}_{br}}$	Retained partition of state space observation matrix
$\underline{\mathbf{C}_{con}}$	Controllability matrix
$\underline{\mathbf{C}_{obs}}$	Observability matrix
$\underline{\mathbf{C}_{RA}}$	Right original companion matrix
$\underline{\mathbf{C}_{RB}}$	Right transformed companion matrix
$\underline{\mathbf{C}_{LA}}$	Left original companion matrix
$\underline{\mathbf{C}_{LB}}$	Left transformed companion matrix
$\mathbf{D}_A$	Original second order damping matrix
$\widetilde{\mathbf{D}_A}$	$n-1$ dimensional partition of damping matrix
$\mathbf{D}_b$	Balanced second order damping matrix
$\mathbf{D}_B$	Transformed second order damping matrix, modal damping matrix
$\mathbf{D}_C$	Transformed second order damping matrix
$\mathbf{D}_g$	Damping matrix excluding contribution from gyroscopic components
E	Young's Modulus (Pa)
$\mathbf{F}_d$	Portion of modal displacement feedback gains
$\mathbf{F}_k$	Portion of modal velocity feedback gains
$\mathbf{F}_L$	Left SPT transformation construction parameters
$\mathbf{F}_R$	Right SPT transformation construction parameters
$\mathbf{G}$	Controller gain proportional to system state
$\mathbf{G}_d$	Controller gain proportional to velocity
$\mathbf{G}_g$	Matrix corresponding to modelled gyroscopic components
$\mathbf{G}_k$	Controller gain proportional to displacement
$\mathbf{G}_L$	Left SPT transformation construction parameters
$\mathbf{G}_R$	Right SPT transformation construction parameters
$\mathbf{H}$	Constrained optimisation function
$\mathbf{H}_1$	Unconstrained optimisation function

$\mathbf{H}_L(\tau)$	Left matrix polynomial
$\mathbf{H}_R(\tau)$	Right matrix polynomial
$\mathbf{I}$	Unit-identity matrix
$\mathbf{I}_b$	Magnetic bearing bias current
J	Cost defined by quadratic cost function
J_{aug}	Augmented optimal control cost
J_e	Cost due to distance between actual and desired eigenvalue locations
$J_{e,\sigma}$	Derivative of Cost due to distance between actual and desired eigenvalue locations with respect to σ
J_{ext}	Extended quadratic cost function
J_{ext_2nd}	Second order extended quadratic cost function
J_{lqr}	LQR optimal cost
J_{pi}	Quadratic cost resulting from pseudo-inverse optimal controller
$\overline{\mathbf{K}}$	Proportional matrix relating mechanical force and current for AMBs
$\mathbf{K}_A$	Original second order stiffness matrix
$\widetilde{\mathbf{K}_A}$	$n-1$ dimensional partition of stiffness matrix
$\mathbf{K}_{Amm}$	Master-Master partition of second order stiffness matrix
$\mathbf{K}_{Ams}$	Master-Slave partition of second order stiffness matrix
$\mathbf{K}_{Asm}$	Slave -Master partition of second order stiffness matrix
$\mathbf{K}_{Ass}$	Slave-Slave partition of second order stiffness matrix
$\mathbf{K}_b$	Balanced second order stiffness matrix
$\mathbf{K}_B$	Transformed second order stiffness matrix, modal stiffness matrix
$\mathbf{K}_C$	Transformed second order stiffness matrix
$\mathbf{L}$	Reflexive SPT construction parameter
L_0	Inductance of magnetic bearing coil
$\mathbf{L}_0$	Matrix of magnetic bearing inductances
$\mathbf{L}_A(\tau)$	Matrix polynomial of original equations of motion
$\mathbf{L}_B(\tau)$	Matrix polynomial of transformed equations of motion
$\mathbf{M}_A$	Original Mass Matrix
$\widetilde{\mathbf{M}_A}$	$n-1$ dimensional partition of mass matrix

$\mathbf{M}_b$	Balanced second order mass matrix
$\mathbf{M}_B$	Transformed Mass Matrix, modal mass matrix
$\mathbf{M}_C$	Transformed Mass Matrix
$\mathbf{N}$	Reflexive SPT construction parameter
$\underline{\mathbf{P}}$	Optimal co-state matrix
$\mathbf{P}_{A1}$, $\mathbf{P}_{A2}$	Second order observation matrices for displacement and velocities respectively
$\underline{\mathbf{P}_c}$	Controllability grammians
$\underline{\mathbf{P}_{cb}}$	Balanced controllability grammians
$\mathbf{P}_{C1}$, $\mathbf{P}_{C2}$	Reformed second order observation matrices for displacement and velocities respectively
$\underline{\mathbf{P}_o}$	Observability grammians
$\underline{\mathbf{P}_{ob}}$	Balanced observability grammians
$\underline{\mathbf{Q}}$	Optimal state weighting matrix
$\underline{\mathbf{Q}_{aug}}$	Optimal augmented state weighting matrix
$\underline{\mathbf{Q}_{me}}$	Optimal mechanical-electrical augmented state weighting matrix
$\mathbf{Q}_d$	Optimal displacement weighting matrix
$\mathbf{Q}_v$	Optimal velocity weighting matrix
$\mathbf{R}$	Optimal control force weighting matrix
$\mathbf{R}_r$	Matrix of magnetic bearing coil resistances
$\mathbf{R}_{me}$	Optimal control voltage weighting matrix
$\mathbf{R}_u$	Optimal control force weighting matrix
$\mathbf{R}_v$	Optimal rate of control force weighting matrix
$\mathbf{S}_A$	Selection matrix determining location of applied forces
$\mathbf{S}_b$	Balanced selection matrix determining location of applied forces
$\mathbf{S}_A$	Selection matrix determining location of applied forces
$\underline{\mathbf{S}_L}$	Left SPT reflexive transformation
$\underline{\mathbf{S}_R}$	Right SPT reflexive transformation
$\mathbf{T}_A$	n-dimensional transformation matrix
$\underline{\mathbf{T}_b}$	Balancing transformation

$\underline{\mathbf{T}_{b,\,2nd}}$	Second order balancing transformation
$\mathbf{T}_{b,\,disp}$	Second order displacement balancing transformation
$\mathbf{T}_{b,\,vel}$	Second order velocity balancing transformation
$\mathbf{T}_B$	n-dimensional transformation matrix
$\mathbf{T}_L$	Left n-dimensional transformation
$\underline{\mathbf{T}_L}$	Left SPT transformation
$\mathbf{T}_R$	Right n-dimensional transformation
$\underline{\mathbf{T}_R}$	Right SPT transformation
$\mathbf{U}_0$	0^{th} order right SPT-filter matrix
$\mathbf{U}_1$	1^{st} order right SPT-filter matrix
$\mathbf{V}_0$	0^{th} order left SPT-filter matrix
$\mathbf{V}_1$	1^{st} order left SPT-filter matrix
$\mathbf{V}_{dc}$	Finite voltage applied to active magnetic bearing
$\mathbf{W}(t)$	Generalised first order forcing vector
$\mathbf{X}$	Right hand eigenvectors
$\mathbf{X}_A$	Right SPT transformation matrix Kyrlov form construction matrix
$\mathbf{Y}_A$	Left SPT transformation matrix Kyrlov form construction matrix
$\mathbf{Z}$	Inverse-transpose of left eigenvectors
$\mathscr{H}_0$	0^{th} order matrix for second order Hamiltonian system
$\mathscr{H}_1$	1^{st} order matrix for second order Hamiltonian system
$\mathscr{H}_2$	2^{nd} order matrix for second order Hamiltonian system
$\mathscr{H}_{aug}$	Hamiltonian matrix for augmented system
Δ	Discrete time change of quantity
$\mathbf{\Gamma}$	Second order modal damping matrix
$\mathbf{\Gamma}_b$	Balanced controllability / observability grammians
$\mathbf{\Lambda}$	Diagonal matrix of eigenvalues
$\mathbf{\Lambda}_a$	Diagonal matrix of eigenvalues pairs, group 1
$\mathbf{\Lambda}_b$	Diagonal matrix of eigenvalues pairs, group 2

$\mathbf{\Lambda}_{,\sigma}$	Diagonal matrix of eigenvalues derivatives with respect to σ
Ω	Angular velocity (rad/s)
$\mathbf{\Phi}_L$	n-dimensional vector of left eigenvectors
$\underline{\mathbf{\Phi}_L}$	$2n$-dimensional vector of left eigenvectors
$\mathbf{\Phi}_R$	n-dimensional vector of right eigenvectors
$\underline{\mathbf{\Phi}_R}$	$2n$-dimensional vector of right eigenvectors
$\underline{\mathbf{\Psi}_L}$	$2n$-dimensional vector of left eigenvectors
$\underline{\mathbf{\Psi}_R}$	$2n$-dimensional vector of right eigenvectors
$\mathbf{\Theta}$	Diagonal matrix of angles (rad)

General Notation

- Underline notation used to differentiate $2n$-dimensional quantities from n-dimensional quantities.
- The dot above a vector denotes derivative with respect to time.
- Superscript$*$ denotes optimal quantities.
- Superscript $'$ denotes variation original vector/matrix (not conjugate transpose).
- Superscript T denotes transpose.
- Superscript H denotes the hermitian (conjugate) transpose.
- $\mathbb{R}$ denotes the set of all real numbers.
- $\mathbb{I}$ denotes imaginary numbers.
- $\Theta(x)$ is used to denote spectrum of x.
- Vectors are denoted by bold lower case characters.
- Matrices are denoted by bold uppercase characters.
- Scalars are represented by non-bold characters.

Chapter 1

Introduction

Many dynamic systems naturally present themselves in a second order form. Consider the equations of motion typical of a second order system

$$\mathbf{M}_A\,\ddot{\mathbf{q}}_A(t) + \mathbf{D}_A\,\dot{\mathbf{q}}_A(t) + \mathbf{K}_A\,\mathbf{q}_A(t) = \mathbf{S}_A\,\mathbf{u}_A(t) = \mathbf{f}_A(t) \tag{1.1}$$

where $\mathbf{M}_A, \mathbf{D}_A, \mathbf{K}_A \in \mathbb{R}^{n\times n}$ are the system mass, damping and stiffness matrices, respectively, and $\mathbf{q}_A(t) \in \mathbb{R}^n$ the vector of physical coordinates. The vector $\mathbf{u}_A(t) \in \mathbb{R}^r$ represents the short vector of applied forces with the matrix $\mathbf{S}_A \in \mathbb{R}^{n\times r}$ representing a selection matrix describing the locations of applied forces, and $\mathbf{f}_A(t) \in \mathbb{R}^{n\times r}$ represents the long vector of applied forces. The dot above $\mathbf{q}_A(t)$ denotes differentiation with respect to time. From here onwards the notation describing dependence on time has been removed.

The properties of the system determine the response of the system to external influence. There is an ever increasing call for lightweight dynamic structures with demanding properties. It is now increasingly common for dynamic systems to substitute real damping with artificially induced damping of the system using velocity dependent forces [S3]. This results in little physical damping being present in a system. Systems may also be artificially stiffened using displacement dependent forces.

Many methods exist to manipulate the physical properties of the system using control feedback such as proportional-integral-derivative action, pole allocation methods and modal control. It is the intent for this thesis to extend the knowledge in the area of control methodology available for application to rotating systems.

The basis for the areas explored in this project arises directly from the structure preserving transformations (SPTs) developed by Garvey *et al.* [G2, G3]. The study of the SPTs requires that the second order equations of motion be presented in a state space form

$$\begin{bmatrix} \mathbf{0} & \mathbf{K}_A \\ \mathbf{K}_A & \mathbf{D}_A \end{bmatrix} \begin{bmatrix} \mathbf{q}_A \\ \dot{\mathbf{q}}_A \end{bmatrix} - \begin{bmatrix} \mathbf{K}_A & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_A \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}}_A \\ \ddot{\mathbf{q}}_A \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_A \end{bmatrix} \tag{1.2}$$

$$\begin{bmatrix} \mathbf{K}_A & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_A \end{bmatrix} \begin{bmatrix} \mathbf{q}_A \\ \dot{\mathbf{q}}_A \end{bmatrix} - \begin{bmatrix} -\mathbf{D}_A & -\mathbf{M}_A \\ -\mathbf{M}_A & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}}_A \\ \ddot{\mathbf{q}}_A \end{bmatrix} = \begin{bmatrix} \mathbf{f}_A \\ \mathbf{0} \end{bmatrix} \tag{1.3}$$

$$\begin{bmatrix} \mathbf{0} & \mathbf{K}_A \\ \mathbf{K}_A & \mathbf{D}_A \end{bmatrix} \begin{bmatrix} \mathbf{q}_A \\ \dot{\mathbf{q}}_A \end{bmatrix} - \begin{bmatrix} -\mathbf{D}_A & -\mathbf{M}_A \\ -\mathbf{M}_A & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_A \\ \dddot{\mathbf{q}}_A \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{f}}_A \\ \mathbf{f}_A \end{bmatrix} \tag{1.4}$$

Two points arise from these particular representations of the equations of motion:

1. There exists a set of transformations for non-defective systems allowing the diagonalisation of all three system matrices for generally damped systems [G1]. The $2n$ dimensional state space representations allows a greater degree of manipulation than the n dimensional second order form thus one may fully transform all three system matrices as desired whilst retaining the block structure within the state space representation.

2. The specific form of equation (1.4) presents the equations of motion in a form which yields access to the rate of change of control force.

The first point raises the possibility of decoupling the equations of motion and applying modal control to a generally damped system. The second point highlights the potential to weight the relative importance on the rate of change of control action against the system response and applied forcing. The following sections elaborate on these points.

1.1 Modal Control

The SPTs allow the diagonalisation of the system matrices thus presenting the opportunity to decouple the equations of motion for non-classically damped systems. This creates the possibility to alter the properties of the system using modal control.

Modal control is a generic term used to describe the technique of controlling individual modes of vibration. This in effect represents n single degree of freedom second order systems to control where n is the number of modelled modes or simply the dimension of the system. Modal control preserves the mode shapes of the system thus tending to conserve control energy whilst moving the controlled poles of the open loop system to more desirable locations on the complex plane. The method presents itself intuitively since there is a direct relationship with the modal properties of the system.

Traditional modal control for second order systems utilises the mass-normalised left and right eigenvectors, $\mathbf{\Phi}_L$ and $\mathbf{\Phi}_R$ respectively, to diagonalise the system matrices. The coordinate transformation $\mathbf{q}_A = \mathbf{\Phi}_R\,\mathbf{q}_B$ is applied and the system matrices are pre-multiplied by the transpose of the left eigenvectors, ${\mathbf{\Phi}_L}^T$.

From

$$ {\mathbf{\Phi}_L}^T\,\mathbf{M}_A\,\mathbf{\Phi}_R\,\ddot{\mathbf{q}}_B + {\mathbf{\Phi}_L}^T\,\mathbf{D}_A\,\mathbf{\Phi}_R\,\dot{\mathbf{q}}_B + {\mathbf{\Phi}_L}^T\,\mathbf{K}_A\,\mathbf{\Phi}_R\,\mathbf{q}_B = {\mathbf{\Phi}_L}^T\,\mathbf{f}_A \tag{1.5} $$

one has

$$ \mathbf{I}\,\ddot{\mathbf{q}}_B + \mathbf{\Gamma}\,\dot{\mathbf{q}}_B + \mathbf{\Lambda}^2\,\mathbf{q}_B = {\mathbf{\Phi}_L}^T\,\mathbf{f}_A \tag{1.6} $$

with $\mathbf{q}_B$ representing the modal coordinates of the system.

The new damping matrix $\mathbf{\Gamma}$ is assumed to be of diagonal form. Convention dictates that one strips any remaining off-diagonal terms in $\mathbf{\Gamma}$ such that $\mathbf{\Gamma}$ is diagonal [G7]. However, the damping matrix typical of rotating machines contains skew-symmetry due to gyroscopic effects. The result of ignoring these off-diagonal terms for rotating systems has the consequence of ignoring the rotating nature of the system itself.

Meirovitch and Baruh introduced a first order modal control method using a state space representation of system containing skew-symmetry in the damping matrix [M2]. The modal contributions are extracted from the physical quantities using modal filters [M6] but the method does not derive an inverse modal filter to revert the modal quantities back to the physical domain. A backward transformation is defined which allows only one half of the modelled modes to be controlled. Meirovitch and Baruh proposed to control only the lower-order modelled modes with the justification for this being that the higher order modes are more difficult to excite hence do not contribute significantly to the system response. The method introduced in this thesis removes this constraint by defining an inverse filter making it possible to control all the modelled modes.

The first order modal control technique outlined by Meirovitch and Baruh expands the n-dimensional control problem into a $2n$-dimensional problem. Modern computers have enough computational capacity such that worries concerning the expansion of the control problem to $2n$ rather than an n-dimensional problem is not an issue for moderate values of n. However, redefining the second order equations of motion into a first order realisation has the disadvantage of destroying some properties such as symmetry and definiteness of the matrices describing the motion [R1]. The form of the first order state space model used by Meirovitch does not preserve any notion of the second order nature of the system hence is insufficient to truly describe a second order system. Here, direct second order techniques allow the retention of the natural form of dynamic systems arising from Newtonian mechanics.

In spite of the existence of a large body of literature referring to modal control little of it exploits the second order form of the equations of motion. Methods for dealing with modal control for classically damped systems exist but these are subject to constraints on the form of damping which are particularly unrealistic for rotating systems.

1.2 Optimal Control

Standard optimal control techniques allow a trade-off to be made between weighted system state error and weighted control force. From equation (1.4) one may notice that the state space representation contains rate of change of force in addition to the conventional force and state. A limitation of traditional optimal control is that no emphasis is placed on the rate of change of control force. Evidently control forces cannot be instantaneously changed and indeed several applications exist where the rate at which control forces can be modified is sufficiently important to warrant this work.

Consider magnetic bearings as a representative contemporary example of a control actuator for a dynamic system [C3]. It is usual to operate these bearings with a bias current such that the net force produced by the bearing in a direction is linearly proportional to the control currents injected into the bearing over a wide range of currents. The maximum force achievable by the bearing is dependent on the maximum control currents which can be injected, and the bearing force is identified as control input $\mathbf{f}_A$. The role of conventional optimal control in trying to keep $\mathbf{f}_A$ small is obvious here. Large mean-square currents would require thick conductors in the bearing and a higher current-rating in the power-amplifiers.

The rate of change of force in a magnetic bearing is dependent on the rate of change of current. All magnetic bearings have some inductance thus a finite rate of change of current requires a finite voltage in addition to the voltage required to drive a steady current. In many practical applications, the voltages associated with the rates of change of current are many times greater than the steady voltages. If the controller requires the magnetic bearing to produce very high rates of change of force then the power-amplifiers will require large internal voltages and the insulation between coils in the magnetic bearing will have to be large. Hence, for magnetic bearings, it is actually highly desirable to be able to develop controllers which minimise some cost function that is determined by both control input (mean square current) and rate of change of control input (mean square voltage).

1.3 Proposed Work

The proposed work for this PhD project may be introduced as two contributing areas:

1. Extension of conventional modal control techniques to include second order systems with substantial gyroscopic terms in the damping matrix for rotating machines. This will be achieved through the utilisation of the SPTs to simultaneously diagonalise all three system matrices for generally damped systems.

2. Extension of the optimal control method to encompass penalising the rate of change of force in addition to penalising the system state and force.

Figure 1.1: Rolls-Royce Trent 1000 Aero Engine

1.4 Applications

It is desirable at this point to highlight the applications of this project. Although the origins of this project stem from the investigation of the SPTs, this project is partly funded by Rolls-Royce plc. As a consequence of the source of funding, this project is concerned with tackling real issues encountered by Rolls-Royce in the design of new gas-turbine engines amongst which aero engines figure very importantly.

An important criterion of every new aircraft engine designed is to reduce lifetime cost to the customer. This can be achieved through numerous means such as increased efficiency and reduced weight. For example, every additional 1 kilogram in weight of an engine represents approximately £1,000 increased annual running cost per aircraft for a commercial airliner operating transatlantic journeys [R5]. For airline companies operating many aircraft this can amount to a significant annual cost.

A significant proportion of the weight of an aircraft engine is due to the main rotors which can be observed in the Rolls-Royce Trent 1000 engine in figure 1.1. As a consequence there is a certain desirability to run the engine with increasingly slender rotors which signify a reduction in the overall weight of the engine. Additional benefits of slender rotors are increased efficiency and reduced running noise. This can be achieved through stiffness gains made through progression in materials science but active control also presents the opportunity to artificially increase the stiffness and damping properties of the rotor [I2].

One possibility of incorporating the necessary actuators into a system is through the use of active electro-magnetic bearings which may perform the operation expected of a bearing as well as providing control forces to the system [U1]. Although at their current stage of evolution active electro-magnetic bearings cannot provide the force to weight ratio of conventional bearings the potential use of them is undeniably appealing. As outlined previously conventional optimal control provides inadequate controller design and hence the justification of extending the optimal control method is obvious here.

1.5 Summary of Thesis Work

A short summary is now presented of the work in this thesis. The author believes that an introduction to these findings will help the reader appreciate the intricacies of the work to follow. The summary is divided into separate modal and optimal control sections.

1.5.1 Summary of Modal Control Work

Theoretical application of structure preserving transformation based modal control is presented in this thesis. The SPT modal control method is not subjected to constraints on the structure of the damping matrix as conventional second order modal control methods are. This means that one may independently control modes of vibration for non-classically damped systems. Numerical examples are provided to justify this assertion.

As stated, the SPT modal control method has its origins in utilising the SPTs to simultaneously diagonalise the second order system mass, damping and stiffness matrices. Knowledge of the system eigenvalues are used to construct the diagonalised system matrices and subsequently use the eigenvectors of of the original and diagonalised state space systems to construct the diagonalising SPTs. Because the diagonalising SPTs are applied to state space representations of the second order system one must use first order modal filters to extract the modal co-ordinates from the physical co-ordinates. Conversely one must use an inverse modal filter to revert the modal force into a physical force such that true independent control of the system modes is realised.

The inverse modal filter used to transform the modal force into physical force results in both the force and time derivative of the force being available. For practical application of control methods one must implement the control system in discrete time. Thus one has the opportunity to smooth the discrete force though interpolation using knowledge of the rate of change of force. It is found that this may actually result in a more efficient controller with a lower associated cost of implementing the control. This is an advantage of the SPT based modal control over conventional modal control methods.

The modal filters are constructed directly from the definition of the diagonalising SPTs. A necessary requirement is that these filters must be stable. One may exploit the non-uniqueness of the SPTs using reflexive SPTs to attempt to stabilise the modal filters. The reflexive SPTs represent a non-trivial transformations such that one may map any diagonal system back onto itself. The reflexive SPTs are highly structured with $2n$ parameters available for manipulation. Application of the reflexive SPT results in a new diagonalising SPT which may or may not provide stable filters. One can exploit this possibility to try and move the filter eigenvalues into the stable half region by appropriately selecting the construction parameters of the reflexive SPT.

Two methods are presented by which one may *flow* the eigenvalues of the filter towards the stable half plane by determining the reflexive SPT construction parameters. The first method is the direct method where one selects the change to be made to the eigenvalues, and second method is the gradient method where one moves in the largest stable change of direction of the eigenvalues. Numerical examples are used to demonstrate the two methods and one finds that both have respective merits and disadvantages. The direct solution enables one to specify the the change of eigenvalues and find the corresponding reflexive parameters to give the change. The disadvantage is that one does not know to what extent the eigenvalues may be altered to stay within a linear range whilst maximising the change at each stage. The gradient method does not require one to specify the change in eigenvalues and correspondingly the determination of the parameters is numerically less intensive than the direct solution. However the gradient solution does not determine the optimum direction in which to change the eigenvalues only the direction in which the eigenvalues will become more stable.

One unfortunate consequence of any modal control method is that one can only control as many pairs of modes as actuators available. This is because each actuator is designated to control one pair of modes. This means that the system is subjected to control spillover in which the applied control forces may excite the uncontrolled and un-modelled modes. It will be shown that control spillover cannot destabilise the system for the SPT modal control method but may degrade the system performance adversely. Control spillover is an unfortunate side effect of the conventional modal control method as well as the SPT based method.

1.5.2 Summary of Optimal Control Work

As stated conventional optimal control may not provide adequate control in situations where strict limitations are placed on the rate at which control forces may be applied. This thesis first approaches this limitation of the optimal control problem by extending the conventional first order optimal control problem to augment the system state with the control force. This requires that one feeds back the rate of change of control force and subsequently integrate it. One obtains a conventional first order state space system for which the conventional optimal control method may be applied to the augmented system. One may consequently penalise the rate of change of control force by augmenting the state weighting matrix to include the weighting placed on the control force and use the conventional weighting matrix used to weight the control force to weight the importance of the rate of change of control force instead. Numerical example demonstrate that this method successfully penalises the rate at which control forces are applied to the system.

A sub-optimal control method is presented by which one can approach the extended optimal control problem using the second order matrices. The method requires that a non-unique pseudo-inverse is used to generate the feedback controller matrices meaning that the method presented does not provide the optimal solution. Despite the non-uniqueness of the solution, numerical example show that the controller provide a means to penalise the rate of change of control force in addition to the displacements, velocities and control forces of the system. One does indeed manage to provide substantial improvement to limiting the time derivative of the control force.

The author also considers the possibility of incorporating the actuator dynamics into the equations of motion. Through inclusion of the actuator dynamics one may apply the conventional optimal control problem and place appropriate weighting on the components which are directly responsible for imposing the physical limitations on the actuators. This would make the need to extend the optimal control problem obsolete but one would require exact knowledge of the actuator properties. It is shown that although this can be achieved through great effort, a more convenient (and almost equivalent) method would be to use one of the extended optimal control methods presented. This would negate the need to accurately model the actuator dynamics in the equations of motion.

1.6 Outline of Thesis

The remaining chapters of the thesis may be summarised as follows:

- Chapter 2 discusses currently available literature with regards to the various subtleties of this project.
- Chapter 3 introduces the structure preserving transformation as a basis for the modal control method.
- Chapter 4 discusses the stability of SPT-based filters necessary for the implementation of SPT-based modal control.
- Chapter 5 introduces the extended optimal control problem which incorporates the rate of change of force to the existing optimal control problem.
- Chapters 6 introduces a theoretical case study of a realistic rotor-turbine system.
- Chapter 7 presents the conclusions of the methods presented and proposed future work.

Chapter 2

Literature Review

The ambitions of this project are to extend the concept of modal control to rotating machinery. This chapter deals with the currently available literature with regards to the various subtleties of this project. The layout of this chapter may thus be summarised:

- Section 1 introduces the concept of rotating machine models and the system equations of motion.
- Section 2 introduces various types of co-ordinate transformations for matrix systems.
- Section 3 explores the ideas of model reduction applied to large scale systems with many degrees of freedom.
- Section 4 introduces the structure preserving transformations which form the backbone of this project. They allow the idea of decoupling non-classically damped system to be extended to the case of non-classically damped systems.
- Section 5 reviews the literature relevant to active control.
- Section 6 discusses the pole placement method for user defined assignment of system eigenvalues.
- Section 7 introduces the optimal control method for optimal determination of the controller design subject to user defined constraints.

- Section 8 introduces modal control which is the ability to individually control the modes of vibration making up the system physical response.
- Section 9 concludes the chapter.

2.1 Rotating Machine Modelling

The aim of this project is to control a dynamic rotor system using active control. Before the controller is applied to a real system it is first appropriate to apply a controller to a system in a theoretical setting. It therefore becomes necessary to build a valid model on which to base the control. The model needs to reproduce accurately the dynamic response of the real system over the frequency range of interest and also needs to be versatile enough to model variations of the rotor dynamic properties.

For the purposes of this project it is assumed that the rotor system flexibility can be modelled using Timoshenko beam elements which take into account transverse shear effects and rotary inertia. This is important for higher modes where contributions from shear and rotary inertia can be significant [R4]. Thus an appropriate method for modelling the rotor system is by 'finite element' (FE) modelling. FE modelling also allows the addition of discs to the rotor system. The discs inertias are added at the appropriate shaft nodes to take account of their contribution to the system dynamics.

An FE model of the rotor system is derived by discretising the system into a finite number of elements and describing the behaviour of each individual element. The cumulative effects of these elements are then taken into consideration by combining them into the system matrices. The books by Rao [R3] and Lalanne and Ferraris [L1] describe the discretisation of the system into components for the shaft, mass unbalance, bearings and discs. The model used for this project is based on the work by these authors. The results for simple and cantilever beams, clamped-clamped beams and free-free beams compare favourably with other analytical models as found in [B6].

The FE model results in second order equations of motion

$$\mathbf{M}_A\, \ddot{\mathbf{q}}_A(t) + (\mathbf{D}_g + \Omega\, \mathbf{G}_g)\ \dot{\mathbf{q}}_A(t) + \mathbf{K}_A\, \mathbf{q}_A(t) = \mathbf{S}_A\, \mathbf{u}_A(t) = \mathbf{f}_A(t) \tag{2.1}$$

with Ω denoting the shaft angular velocity in rad/s and $\mathbf{f}_A(t)$ is used to denote the full length vector of forces. The subscript A denotes the original system matrices.

The inclusion of the gyroscopic matrix $\mathbf{G}_g$ is due to the rotary inertia of the system. As apparent from equation (2.1) the significance of the effect increases with shaft speed. This has direct effect on the natural frequencies of the system which alter with shaft speed. For the purposes of this chapter it is assumed that shaft speed Ω is constant. Thus the damping matrix may be taken as fixed and the equations of motion simplified to

$$\mathbf{M}_A\,\ddot{\mathbf{q}}_A(t) + \mathbf{D}_A\,\dot{\mathbf{q}}_A(t) + \mathbf{K}_A\,\mathbf{q}_A(t) = \mathbf{f}_A(t) \tag{2.2}$$

where $\mathbf{D}_A := \mathbf{D}_g + \Omega\,\mathbf{G}_g$.

2.1.1 First Order System Representation

For many applications related to control and model reduction methods it is necessary to transform the second order equations of motion into a first order so called *linearised* form as defined by [G10]. Recognising the definition $\mathbf{q}_{A1}(t) = \mathbf{q}_A(t)$ and $\mathbf{q}_{A2}(t) = \dot{\mathbf{q}}_{A1}(t)$ one may form a first order state space representation of the equations of motion

$$\begin{bmatrix} \dot{\mathbf{q}}_{A1}(t) \\ \dot{\mathbf{q}}_{A2}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}_A^{-1}\,\mathbf{K}_A & -\mathbf{M}_A^{-1}\,\mathbf{D}_A \end{bmatrix} \begin{bmatrix} \mathbf{q}_{A1}(t) \\ \mathbf{q}_{A2}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{M}_A^{-1} \end{bmatrix} \mathbf{f}_A(t) \tag{2.3}$$

$$\mathbf{y}_A(t) = \begin{bmatrix} \mathbf{P}_{A1} & \mathbf{P}_{A2} \end{bmatrix} \begin{bmatrix} \mathbf{q}_{A1}(t) \\ \mathbf{q}_{A2}(t) \end{bmatrix} \tag{2.4}$$

where $\mathbf{y}_A(t)$ represents a vector of observed outputs comprising of displacement and velocity components $\mathbf{P}_{A1}, \mathbf{P}_{A2} \in \mathbb{R}^{p\times n}$. The underline notation is used to differentiate the $2n$-dimensional quantities from the n-dimensional quantities. The first order state space equation may be simplified to

$$\underline{\dot{\mathbf{q}}_A}(t) = \underline{\mathbf{A}_A}\,\underline{\mathbf{q}_A}(t) + \underline{\mathbf{B}_A}\,\mathbf{f}_A(t) \tag{2.5}$$

$$\mathbf{y}_A(t) = \underline{\mathbf{C}_A}\,\underline{\mathbf{q}_A}(t) \tag{2.6}$$

where the definitions are apparent. $\underline{\mathbf{A}_A}$ is referred to as the state space companion matrix, $\underline{\mathbf{B}_A}$ is referred to as the state space input matrix and $\underline{\mathbf{C}_A}$ is referred to as the state space output matrix.

The state space companion matrix, $\underline{\mathbf{A}_A}$, is considered to be the right form of the conventional state space form. One may also consider the left state space companion matrix which is equally valid and has the form

$$\underline{\mathbf{A}_{A,\,left}} = \begin{bmatrix} \mathbf{0} & -\mathbf{K}_A\,\mathbf{M}_A^{-1} \\ \mathbf{I} & -\mathbf{D}_A\,\mathbf{M}_A^{-1} \end{bmatrix} \tag{2.7}$$

2.2 Lancaster Augmented Matrices

The state space form illustrated by equation (2.3) is the most common form utilised when transforming systems of equations into first order form. However, there exists many equally valid state space forms such as those obtained from using the 'Lancaster Augmented Matrices' (LAMs).

For a second order system there exists three LAMs which can be produced by inspection to be,

$$\underline{\mathbf{A}_0} = \begin{bmatrix} -\mathbf{D}_A & -\mathbf{M}_A \\ -\mathbf{M}_A & \mathbf{0} \end{bmatrix} \quad , \; \underline{\mathbf{A}_1} = \begin{bmatrix} \mathbf{K}_A & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_A \end{bmatrix} \quad , \; \underline{\mathbf{A}_2} = \begin{bmatrix} \mathbf{0} & \mathbf{K}_A \\ \mathbf{K}_A & \mathbf{D}_A \end{bmatrix} \tag{2.8}$$

The LAMs allow the second order system to be represented in a reduced form as defined by Garvey *et al.* [G5]

$$\underline{\mathbf{A}_k}\,\underline{\mathbf{q}_A}(t) - \underline{\mathbf{A}_{k-1}}\,\underline{\dot{\mathbf{q}}_A}(t) = \underline{\mathbf{f}_{Ak}}(t) \qquad k = 1,2 \tag{2.9}$$

where

$$\underline{\mathbf{q}_A}(t) := \begin{bmatrix} \mathbf{q}_A(t) \\ \dot{\mathbf{q}}_A(t) \end{bmatrix}, \;\; \underline{\mathbf{f}_{A1}}(t) := \begin{bmatrix} \mathbf{f}_A(t) \\ \mathbf{0} \end{bmatrix}, \;\; \underline{\mathbf{f}_{A2}}(t) := \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_A(t) \end{bmatrix} \tag{2.10}$$

The form of the equations of motion as illustrated by equation (2.9) represents a strict linearisation of the second order system as defined by Gohberg *et al.* [G10]. The first form of the state space equations of motion in equation (2.9) is also known as Duncan's form.

2.3 Proportional and Classical Damping

The concept of proportional or classically damped systems is central to this project. The notion of whether or not the system is proportionally or classically damped is important if one wishes to diagonalise the system mass, damping and stiffness matrices. Simultaneous diagonalisation of the three mass, damping and stiffness matrices is usually attempted by pre- and post-multiplying the system matrices by the mass normalised left and right eigenvectors of the undamped system. One now explains the definition of these two terms.

One considers the situation where the damping can be defined as a proportion of the stiffness and/or mass matrices to be proportionally damped [R4]. This is simply where only linear proportions of the stiffness and mass contribute to the definition of the system damping as defined by

$$\mathbf{D}_A = \alpha \mathbf{K}_A + \beta \mathbf{M}_A \tag{2.11}$$

Proportional damping is often assumed in structural vibrations in simplify the analysis, see for example [G7]. However this type of simplification is inappropriate for rotating machines.

The definition of proportionally damped systems is very restrictive and there exists many situations where one may simultaneously diagonalise the system matrices where the definition outlined in equation (2.11) does not hold. This is when the system is *classically damped.*

Classically damped systems may be fully diagonalised only when the necessary requirement [C1]

$$\mathbf{K}_A \, \mathbf{M}_A^{-1} \, \mathbf{D}_A = \mathbf{D}_A \, \mathbf{M}_A^{-1} \, \mathbf{K}_A \tag{2.12}$$

is satisfied. The satisfaction of this constraint means that the system is classically damped although it is possible to quantify the extent to which the system is non-classically damped as shown for example by Prells and Friswell [P1].

The author considers proportional damping to be a specific, restrictive subset of classically damped system and therefore proportional damping is not mentioned further. The term classical damping is used to refer to the restriction on the structure of the damping matrix outlined in equation (2.12) and not with reference to the types of physical dampers used in the system itself.

2.4 Co-ordinate Transformations

Given a governing set of equations one may transform the system using mathematical techniques to obtain a more desirable form. Three possibilities exist of interest to this project: similarity transformations, congruence transformations and linear transformations [S5].

A similarity transformation is used to post- and pre-multiply a matrix by a transformation matrix $\mathbf{T}_A$ and its inverse $\mathbf{T}_A^{-1}$

$$\mathbf{A}' = \mathbf{T}_A^{-1}\,\mathbf{A}\,\mathbf{T}_A \tag{2.13}$$

Here the superscript $'$ denotes a new matrix obtained through the transformation applied and not an Hermitian transpose.

From equation (2.13), matrices $\mathbf{A}$ and $\mathbf{A}'$ are said to be similar. In essence the eigenvalues of the matrix are preserved such that

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det(\mathbf{A}' - \lambda\mathbf{I}) \qquad\qquad \forall\lambda \tag{2.14}$$

and all eigenvalues of the two matrices coincide. Similarity transformations can be applied to a system of equations preserving both the eigenvalues of the matrices themselves and the eigenvalues of the system as well. For example a similarity transformation can be applied to the un-forced second order equations of motion by making the substitution $\mathbf{q}_A(t) = \mathbf{T}_B\,\mathbf{q}_B(t)$ and pre-multiplying by $\mathbf{T}_B^{-1}$

$$\mathbf{T}_B^{-1}\,\mathbf{M}_A\,\mathbf{T}_B\,\ddot{\mathbf{q}}_B(t) + \mathbf{T}_B^{-1}\,\mathbf{D}_A\,\mathbf{T}_B\,\dot{\mathbf{q}}_B(t) + \mathbf{T}_B^{-1}\,\mathbf{K}_A\,\mathbf{T}_B\,\mathbf{q}_B(t) \tag{2.15}$$

to give

$$\mathbf{M}_B\,\ddot{\mathbf{q}}_B(t) + \mathbf{D}_B\,\dot{\mathbf{q}}_B(t) + \mathbf{K}_B\,\mathbf{q}_B(t) \tag{2.16}$$

Using Θ to denote the spectrum of the matrices, one finds that $\Theta(\mathbf{M}_A) = \Theta(\mathbf{M}_B)$, $\Theta(\mathbf{D}_A) = \Theta(\mathbf{D}_B)$, $\Theta(\mathbf{K}_A) = \Theta(\mathbf{K}_B)$ and $\Theta(\mathbf{K}_A, \mathbf{D}_A, \mathbf{M}_A) = \Theta(\mathbf{K}_B, \mathbf{D}_B, \mathbf{M}_B)$. Examples of similarity transformations may be found in the balanced reduction method.

Congruence transformations are similar in application to the similarity transformations except that the matrices are pre-multiplied by the transpose rather than the inverse of the transformation matrix

$$\mathbf{A}'' = \mathbf{T}_C^T\,\mathbf{A}\,\mathbf{T}_C \tag{2.17}$$

Thus applying the congruence transformations to the un-forced second order equations of motion as before one has

$$\mathbf{T}_C^T\,\mathbf{M}_A\,\mathbf{T}_C\,\ddot{\mathbf{q}}_C(t) + \mathbf{T}_C^T\,\mathbf{D}_A\,\mathbf{T}_C\,\dot{\mathbf{q}}_C(t) + \mathbf{T}_C^T\,\mathbf{K}_A\,\mathbf{T}_C\,\mathbf{q}_C(t) \tag{2.18}$$

to give

$$\mathbf{M}_C\,\ddot{\mathbf{q}}_C(t) + \mathbf{D}_C\,\dot{\mathbf{q}}_C(t) + \mathbf{K}_C\,\mathbf{q}_C(t) \tag{2.19}$$

One finds that the congruence transformations make no attempt to preserve the eigenvalues of the individual system matrices but preserve only the eigenvalues of the system itself. Thus retaining the definition of Θ to denote the spectrum one has $\Theta(\mathbf{M}_A) \neq \Theta(\mathbf{M}_C)$, $\Theta(\mathbf{D}_A) \neq \Theta(\mathbf{D}_C)$, $\Theta(\mathbf{K}_A) \neq \Theta(\mathbf{K}_C)$ <u>but</u> $\Theta(\mathbf{K}_A, \mathbf{D}_A, \mathbf{M}_A) = \Theta(\mathbf{K}_C, \mathbf{D}_C, \mathbf{M}_C)$. One also finds that the definiteness and symmetry, where appropriate, is retained through congruence transformations. This is known as Sylvester's law.

Linear transformations, by comparison, operate by pre- and post-multiplying the matrices by a left and right independent transformation such that

$$\mathbf{A}''' = \mathbf{T}_L^T\,\mathbf{A}\,\mathbf{T}_R \tag{2.20}$$

Again like the congruence transformations no attempt is made to preserve the eigenvalues of the individual matrices but instead only the eigenvalues of the system of equations. One may apply the linear transformations by making the substitution $\mathbf{q}_A(t) = \mathbf{T}_R\,\mathbf{q}_D$ using the right transformation and pre-multiplying by the transpose of the left transformation $\mathbf{T}_L^T$.

$$\mathbf{T}_L^T\,\mathbf{M}_A\,\mathbf{T}_R\,\ddot{\mathbf{q}}_D(t) + \mathbf{T}_L^T\,\mathbf{D}_A\,\mathbf{T}_R\,\dot{\mathbf{q}}_D(t) + \mathbf{T}_L^T\,\mathbf{K}_A\,\mathbf{T}_R\,\mathbf{q}_D(t) \tag{2.21}$$

to give

$$\mathbf{M}_D\,\ddot{\mathbf{q}}_D(t) + \mathbf{D}_D\,\dot{\mathbf{q}}_D(t) + \mathbf{K}_D\,\mathbf{q}_D(t) \tag{2.22}$$

2.5 Model Reduction

The dimension of the system generated by finite element modelling is generally fairly large. This can create problems when dealing with control systems due to the computational constraints imposed by computers. It is therefore evident that the dimension of the system must be reduced in size, and at the same time the reduced model must retain the dynamic frequency characteristics for the frequency range of interest. The user must therefore determine the range of interest for the reduced model and the characteristics beyond this range can be considered unimportant. An in-depth collection of papers concerning modern techniques such as the second order balanced truncation method described in section 2.5.4 and bench-mark tests to validate the methods may be found in reference [B3].

2.5.1 Static Reduction Methods

The most common model reduction method is that proposed by Guyan [G12]. Guyan reduction is the simplest method of model reduction reproducing the system flexibility exactly at zero frequency. Guyan reduction is a process of reducing the mass and stiffness matrices by eliminating the stiffness coordinates. Arranging the structural equations $\mathbf{f}_A(t) = \mathbf{K}_A\,\mathbf{q}_A(t)$ one may partitioned the stiffness matrix $\mathbf{K}_A$ into master (m) and slave (s) coordinates

$$\begin{bmatrix} \mathbf{f}_{Am}(t) \\ \mathbf{f}_{As}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{Amm} & \mathbf{K}_{Ams} \\ \mathbf{K}_{Asm} & \mathbf{K}_{Ass} \end{bmatrix} \begin{bmatrix} \mathbf{q}_{Am}(t) \\ \mathbf{q}_{As}(t) \end{bmatrix} \tag{2.23}$$

Setting $\mathbf{f}_{As}(t) = \mathbf{0}$ one may see that $\mathbf{q}_{As}(t) = \mathbf{K}_{Ass}^{-1}\,\mathbf{K}_{Asm}\,\mathbf{q}_{Am}(t)$ from the lower half of equation (2.23). Thus one may back substitute this result into equation (2.23) to find the transformation

$$\begin{bmatrix} \mathbf{q}_{Am}(t) \\ \mathbf{q}_{As}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{I} \\ \mathbf{K}_{Ass}^{-1}\,\mathbf{K}_{Asm} \end{bmatrix} \mathbf{q}_{Am}(t) \tag{2.24}$$

Thus the slave coordinates are removed. For the case where $\mathbf{f}_{As}(t) \neq \mathbf{0}$ one may make appropriate substitution such that the master co-ordinates $\mathbf{q}_{Am}(t)$ are in terms of $\mathbf{f}_{Am}(t)$ and $\mathbf{f}_{As}(t)$.

The effect of the Guyan reduction process is to decouple a single degree of freedom in the structural equations. The decoupled degree of freedom is subsequently removed. One may observe that the damping and inertial effects are ignored in this process. Redefining the transformation in equation (2.24) to be

$$\mathbf{T}_{guy} = \left[\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \hline \mathbf{K}_{Ass}^{-1}\,\mathbf{K}_{Asm} & 1 \end{array}\right] \tag{2.25}$$

such that the degree of freedom which was previously discarded is retained one finds that application of the transformation matrix to the stiffness matrix results in the structure

$$\mathbf{T}_{guy}^{T}\mathbf{K}_{A}\,\mathbf{T}_{guy} = \left[\begin{array}{c|c} \widehat{\mathbf{K}_{A}} & \mathbf{0} \\ \hline \mathbf{0} & \widehat{k} \end{array}\right] \tag{2.26}$$

where $\widehat{k}$ is the decoupled degree of freedom which may be subsequently removed. The same structure does not result in the mass or damping matrix in general.

One drawback of the conventional Guyan reduction method is that it is unclear which co-ordinates to keep if the user requires to keep more 'degrees of freedom' (DOF) in the system than the number of forces applied. This problem is identified by Henshell and Ong [H1] who proposed a method to identify the master and slave co-ordinates. They proposed to remove the co-ordinates with the lowest kinetic energy. Inertial energy for each co-ordinate was calculated by dividing the diagonal entries of the stiffness and mass matrices. They proposed to remove one degree of freedom at a time and repeat the process until the desired dimension of the system is attained. This enables the process of model reduction to become automated.

The raw Guyan reduction method has two significant flaws. The first of them is that currently no account is taken of the damping contributions in the reduction process. Indeed no knowledge of the damping is used in the construction of the reduction transformation meaning for example that rotating systems such as the ones of interest in this project lose vital information having undergone the reduction process.

The second significant flaw posed by the Guyan reduction method is that the reduced model is exact only at zero frequency. Indeed as presented in equation (2.1) the system is speed dependent via the gyroscopic matrix. As already stated this project considers rotating systems at a constant speed thus on is interested in the system operation at a specific frequency which is typically a long way from zero frequency.

O'Callaghan [O1] developed the 'improved reduction system' (IRS) in which an extra term is added to the static reduction model to make some allowance for the inertial terms. Thus the reduction transformation $\mathbf{T}_{irs}$ is created which comprises of the static reduction part ($\mathbf{T}_{static}$), obtained from Guyan reduction, and an inertial term ($\mathbf{T}_{inertial}$) to give

$$\mathbf{T}_{irs} = \mathbf{T}_{static} + \mathbf{T}_{inertial} \tag{2.27}$$

The original IRS method developed by O'Callaghan [O1] resulted in a stiffer stiffness matrix than that obtained from using Guyan reduction. Friswell *et al.* [F2, F3] presented an extension to the IRS method by proposing to use an iterative method in which a corrective term is generated iteratively using the current best model of the reduced model. If one continues the iterative process to a high degree of accuracy one starts to converge the full reduction model given by SEREP defined in reference [O2]. This allows the natural frequencies of the reduced model to converge to those of the full system.

The IRS methods still do not take into account damping but do allow the determination of a non-zero frequency window of operation where the accuracy is acceptable. This represents an improvement on the original Guyan reduction model.

An alternative method to the Guyan reduction is that proposed by Craig and Bampton [C5]. In the method presented they propose to decompose the system into boundary DOFs and internal DOFs. The boundary DOFs are retained and the internal DOFs are discarded or a smaller subset retained. Utilising the subscript m notation for the retained (master) co-ordinates and s for discarded (slave) co-ordinates the Craig-Bampton transformation is found to be

$$\begin{bmatrix} \mathbf{q}_{Am}(t) \\ \mathbf{q}_{As}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{\Phi}_{ext} & \mathbf{\Phi}_{int} \end{bmatrix} \mathbf{q}_{Am}(t) \tag{2.28}$$

where $\mathbf{\Phi}_{ext}$ is the Guyan reduction transformation defined previously ($\mathbf{K}_{Ass}^{-1}\mathbf{K}_{Asm}$) corresponding to the external DOFs and $\mathbf{\Phi}_{int}$ is the eigenvectors associated with the internal DOFs retained. As may be observed from the transformation matrix if no internal DOFs are retained then the Craig-Bampton method is identical to the Guyan reduction method.

The Craig-Bampton method mirrors the Guyan reduction closely but allows the system response to model exactly over the reduced system degrees of freedom. Indeed the method is accurate over a much larger range of frequencies than the Guyan reduction method and is not accurate solely at zero frequency. The Craig-Bampton method still does not resolve the issue associated with the damping.

2.5.2 System Deflation

Garvey *et al.* [G4] propose to use the 'structure preserving transformations' (SPTs) to deflate the system matrices. The second order matrices are contained within the 'Lancaster Augmented matrices' (LAMs) (introduced in section 2.2) allowing two distinct state space representations. The SPTs are real transformations used to transform the second order system matrices contained within the LAMs whilst retaining the block structure of the LAMs. The SPTs decouple a single mode from the system matrices at each stage such that the system matrices within the LAMs have the form

$$\mathbf{K}_B = \left[\begin{array}{ccc|c} & & & 0 \\ & \widetilde{\mathbf{K}}_A & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & k \end{array}\right], \quad \mathbf{D}_B = \left[\begin{array}{ccc|c} & & & 0 \\ & \widetilde{\mathbf{D}}_A & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & d \end{array}\right]$$

$$\mathbf{M}_B = \left[\begin{array}{ccc|c} & & & 0 \\ & \widetilde{\mathbf{M}}_A & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & m \end{array}\right] \tag{2.29}$$

The decoupled single degree of freedom systems represented by k, d, m may be removed from the transformed system matrices $\mathbf{K}_B, \mathbf{D}_B, \mathbf{M}_B$. Evidently the decoupled single degree of freedom systems must correspond to a pair of eigenvalues of the system thus their removal does not effect the eigenvalues of the retained subset. The second order matrices are deflated by a dimension of one at each turn. This process may be repeated as many times as required. The method can be applied to general second order system without ignoring the damping and retaining the second order structure of the system matrices. However it is currently unknown how to determine the transformation which yields the second order matrix structure given in equation (2.29) without one first solving the generalised eigenvalue problem.

2.5.3 First Order State Space Model Reduction

First order model reduction involves the transformation of the system into first order form as illustrated by equation (2.5). Perhaps, due to the intuitive appeal and relative ease of being applied to large order systems, the most widely utilised first order model reduction method is the balanced state space method. Gawronski discusses the balanced reduction method at length in the text [G7].

The balanced method works on the concept of determining how controllable or observable each individual mode of vibration is and then removing the modes which influence the system response the least. Grammians are derived to explore the relative energies required to control or observe a respective mode of vibration. The controllability ($\underline{\mathbf{P}_c}$) and observability ($\underline{\mathbf{P}_o}$) Grammians are defined to be [Z2]

$$\underline{\mathbf{P}_c} = \int_0^\infty \exp\left(\underline{\mathbf{A}_A}\, t\right)\ \underline{\mathbf{B}_A}\,\underline{\mathbf{B}_A}^T \exp\left(\underline{\mathbf{A}_A}^T\, t\right) dt \tag{2.30}$$

$$\underline{\mathbf{P}_o} = \int_0^\infty \exp\left(\underline{\mathbf{A}_A}^T\, t\right)\ \underline{\mathbf{C}_A}^T\,\underline{\mathbf{C}_A}\ \exp\left(\underline{\mathbf{A}_A}\, t\right) dt \tag{2.31}$$

Equation (2.30) and (2.31) yield the solutions to the Lyaponuv equations

$$\underline{\dot{\mathbf{P}}_c} = \underline{\mathbf{A}_A}\,\underline{\mathbf{P}_c} + \underline{\mathbf{P}_c}\,\underline{\mathbf{A}_A}^T + \underline{\mathbf{B}_A}\,\underline{\mathbf{B}_A}^T \tag{2.32}$$

$$\underline{\dot{\mathbf{P}}_o} = \underline{\mathbf{A}_A}^T\,\underline{\mathbf{P}_o} + \underline{\mathbf{P}_o}\,\underline{\mathbf{A}_A} + \underline{\mathbf{C}_A}^T\,\underline{\mathbf{C}_A} \tag{2.33}$$

The solutions are invariant under linear transformation thus the similarity transformation $\mathbf{T}_b$ may be applied to the system

$$\underline{\mathbf{T}_b}^{-1}\,\underline{\mathbf{A}_A}\,\underline{\mathbf{T}_b} \rightarrow \underline{\mathbf{A}_b} \quad , \quad \underline{\mathbf{T}_b}^{-1}\,\underline{\mathbf{B}_A} \rightarrow \underline{\mathbf{B}_b} \quad , \quad \underline{\mathbf{C}_A}\,\underline{\mathbf{T}_b} \rightarrow \underline{\mathbf{C}_b} \tag{2.34}$$

Through appropriate manipulation the transformation $\underline{\mathbf{T}_b}$ can be chosen such that the controllability and observability Grammians can be found equal and diagonal

$$\underline{\mathbf{P}_{cb}} = \underline{\mathbf{P}_{ob}} = \underline{\mathbf{\Gamma}_b} \tag{2.35}$$

This state is referred to as balanced and the new balanced Grammians $\underline{\mathbf{\Gamma}_b}$ represent the eigenvalues of the product of the original Grammians. For any one diagonal entry, the higher this Grammian value, the less energy is required to control or observe the mode in question hence the more controllable or observable a mode is. The less observable and controllable modes represented by smaller Grammian values are typically discarded.

Two methods are available to reduce the balanced forms, balanced realisation and balanced truncation methods [S4]. The balanced system may be compartmentalised into retained ($\underline{\mathbf{q}_{br}}(t)$) and discarded degrees of freedom ($\underline{\mathbf{q}_{bd}}(t)$).

$$\begin{bmatrix} \underline{\dot{\mathbf{q}}_{br}}(t) \\ \underline{\dot{\mathbf{q}}_{bd}}(t) \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{A}_{brr}} & \underline{\mathbf{A}_{brd}} \\ \underline{\mathbf{A}_{bdr}} & \underline{\mathbf{A}_{bdd}} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{q}_{br}}(t) \\ \underline{\mathbf{q}_{bd}}(t) \end{bmatrix} + \begin{bmatrix} \underline{\mathbf{B}_{br}} \\ \underline{\mathbf{B}_{bd}} \end{bmatrix} \mathbf{f}_b(t) \tag{2.36}$$

$$\mathbf{y}_b(t) = \begin{bmatrix} \underline{\mathbf{C}_{br}} & \underline{\mathbf{C}_{bd}} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{q}_{br}}(t) \\ \underline{\mathbf{q}_{bd}}(t) \end{bmatrix} \tag{2.37}$$

The balanced realisation method is accurate at zero frequencies and operates by creating a transformation between the discarded degrees of freedom and the retained degrees of freedom. Knowledge of the discarded part of the system is retained within the reduced model. The balanced truncation method is accurate at infinite frequency and operates by simply ignoring the discarded degrees of freedom completely.

Two problems arise with the balanced reduction methods: 1.) The destruction of the second order properties of the reduced system. 2.) It is not clear which Grammian corresponds to which pairs of eigenvalues?

The first problem arises due to the balanced form of the state space equations. Whilst the matrices corresponding to a second order system may be extracted from the balanced state using the inverse transformation $\mathbf{T}_b{}^{-1}$, once the model has been reduced in dimension this relationship has been lost. It is not entirely obvious what order the reduced system is [M8].

One possible method to overcome this problem is to re-establish the zeros and ones in the appropriate locations in the companion and forcing matrices yielding the notion of a second order system. The methods presented by Friswell *et al.* [F4], further simplified by Houlston [H2] for the multi-input case, propose a definition of a transformation to yield the appropriate form of the state space companion and forcing matrices. This allows the extraction of the mass normalised second order matrices

$$\mathbf{I}\,\ddot{\mathbf{q}}_C(t) + \mathbf{D}_C\,\dot{\mathbf{q}}_C(t) + \mathbf{K}_C\,\mathbf{q}_C(t) = \mathbf{f}_C(t) \tag{2.38}$$

$$\mathbf{y}_C(t) = \mathbf{P}_{C1}\,\mathbf{q}_C(t) + \mathbf{P}_{C2}\,\dot{\mathbf{q}}_C(t) \tag{2.39}$$

However one may note that the system output matrix $\mathbf{y}_C(t)$ is still in terms of $\mathbf{q}_C(t)$ and $\dot{\mathbf{q}}_C(t)$ meaning that full second order form has not been achieved. Indeed one may determine the structure of either the $\underline{\mathbf{A}_b}$ and $\underline{\mathbf{B}_b}$ matrices or the $\underline{\mathbf{A}_b}$ and $\underline{\mathbf{C}_b}$ matrices but not all three due to constraints on the number of variables available for manipulation. Thus one may not be able to re-establish the full second order form such that $\mathbf{P}_{C2} = \mathbf{0}$.

The second problem associated with the method is that it is not entirely apparent which Grammian applies to which pairs of eigenvalues. Work by Gawronski [G8] shows that the 2×2 block-diagonal entries of the balanced state space companion matrix $\underline{\mathbf{A}_b}$ are approximately equal to a specific modal form of the equations of motion defined by

$$\underline{\dot{\mathbf{q}}_m} = \underline{\mathbf{A}_m}\,\underline{\mathbf{q}_m} + \underline{\mathbf{B}_m}\,\mathbf{f}_m \tag{2.40}$$

where $\underline{\mathbf{A}_m}$ is block-diagonal comprising of 2×2 matrices

$$\underline{\mathbf{A}_{mj}} = \begin{bmatrix} -\gamma_j\,\omega_j & \omega_j \\ -\omega_j & -\gamma_j\,\omega_j \end{bmatrix} \quad , \quad j = 1, 2, \ldots, n \tag{2.41}$$

ω_j is the modal natural frequency and γ_j is the damping ratio of the specific modes of interest. This form is referred to as the modal type 2 form as defined by Gawronski [G8]. Thus it is apparent that the individual modes may be paired from the balanced state with their respective balanced controllability/observability Grammian.

2.5.4 Second Order Balanced Truncation

For the context of this project it is preferable to preserve the second order form of the equations of motion. The first order reduction has several distinct disadvantages [M8]: 1.) The new positional coordinates are a combination of both the old velocities and displacements thus obscuring physical intuition of the system; 2.) The procedure for establishing the second order form from the first order form is computationally intensive; 3.) The first order form does not exploit any elements of the second order structure.

Meyer and Srinivasan [M8] proposed a method of calculating the controllability and observability Grammians for second order systems where it is shown that the second order system results in four Grammians rather than the two Grammians in the first order form. These Grammians represent the controllability and observability for displacements and velocities respectively. The Grammians are obtained from the first order response of the system.

Chahlaoui *et al.* [C2] expanded on the work of Meyer and Srinivasan to balance the second order system such that the four Grammians for the velocities and displacements are equal and diagonal. This is obtained through balancing the individual pairs of Grammians to give balancing transformations for the displacements and velocities, $\mathbf{T}_{b,\,disp}$ and $\mathbf{T}_{b,\,vel}$ respectively. Thus one may form the balancing transformation $\underline{\mathbf{T}_{b,2nd}}$

$$\underline{\mathbf{T}_{b,\,2nd}} = \begin{bmatrix} \mathbf{T}_{b,\,disp} & 0 \\ 0 & \mathbf{T}_{b,\,vel} \end{bmatrix} \tag{2.42}$$

The second order equations of motion are represented in first order state space form and the balancing transformation $\underline{\mathbf{T}_{b,2nd}}$ is applied to yield the state space matrix containing the balanced second order equations of motion within

$$(\underline{\mathbf{T}_{b,\,2nd}})^{-1}\,\underline{\mathbf{A}_A}\,\underline{\mathbf{T}_{b,\,2nd}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K}_b & -\mathbf{D}_b \end{bmatrix} \tag{2.43}$$

$$(\underline{\mathbf{T}_{b,\,2nd}})^{-1}\,\underline{\mathbf{B}_A} = \begin{bmatrix} \mathbf{0} \\ \mathbf{S}_b \end{bmatrix} \tag{2.44}$$

$$\underline{\mathbf{C}_A}\,\underline{\mathbf{T}_{b,\,2nd}} = \begin{bmatrix} \mathbf{C}_{b,\,disp} & \mathbf{C}_{b,\,vel} \end{bmatrix} \tag{2.45}$$

The second order structure of the state space matrix $\underline{\mathbf{A}_b}$ is retained as shown in equations (2.43) such that the balanced second order equations of motion can be extracted accordingly.

The balanced truncation method can be applied directly to the balanced second order equations as outlined. Chahlaoui *et al.* state in the conclusions of their paper [C2] that it is not necessarily known whether or not the reduced model will always be stable when utilising the second order balanced truncation. This contrasts with the first order reduction method which always guarantees stability for the reduced order system. Indeed Gawronski and Williams not only proves the robustness of the first order reduction method but shows that the first order balanced reduction method is near optimal [G6].

2.6 Structure Preserving Transformations

The structure preserving transformation (SPTs) developed by Garvey *et al.* [G2, G3] are a set of transformations representing a bijective mapping between one linear system and another of the same Jordan form. For the context of this project all the eigenvalues are assumed to occur distinctly thus reference to the SPTs retaining the Jordan form of the system is somewhat excessive. Although this is a property of the SPTs from here onwards one refers to the SPTs retaining the same set of distinct eigenvalues.

A more mathematically precise name for the SPTs would perhaps be structure preserving equivalences but for the context of this project the SPT label is retained. The SPTs consist of left and right transformation matrices $\underline{\mathbf{T}_L}, \underline{\mathbf{T}_R} \in \mathbb{R}^{2n\times 2n}$ respectively, such that they may be used to pre- and post-multiply the original 'Lancaster Augmented Matrices' (LAMs) introduced in section 2.2 to yield a new set of LAMs corresponding to a new system whilst preserving the structure within the LAMs themselves. This allows the definition

$$\underline{\mathbf{T}_L}^T \, \underline{\mathbf{A}_k} \, \underline{\mathbf{T}_R} = \underline{\mathbf{B}_k} \quad , \quad k = 0, 1, 2 \tag{2.46}$$

The effect of a SPT is to transform the state space equations obtained from (2.9) for the original system into the form

$$\begin{bmatrix} \mathbf{K}_B & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_B \end{bmatrix} \begin{bmatrix} \mathbf{q}_{B1}(t) \\ \mathbf{q}_{B2}(t) \end{bmatrix} - \begin{bmatrix} -\mathbf{D}_B & -\mathbf{M}_B \\ -\mathbf{M}_B & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}}_{B1}(t) \\ \dot{\mathbf{q}}_{B2}(t) \end{bmatrix} = \underline{\mathbf{T}_L}^T \begin{bmatrix} \mathbf{f}_A(t) \\ \mathbf{0} \end{bmatrix} \tag{2.47}$$

$$\begin{bmatrix} \mathbf{0} & \mathbf{K}_B \\ \mathbf{K}_B & \mathbf{D}_B \end{bmatrix} \begin{bmatrix} \mathbf{q}_{B1}(t) \\ \mathbf{q}_{B2}(t) \end{bmatrix} - \begin{bmatrix} \mathbf{K}_B & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_B \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}}_{B1}(t) \\ \dot{\mathbf{q}}_{B2}(t) \end{bmatrix} = \underline{\mathbf{T}_L}^T \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_A(t) \end{bmatrix} \tag{2.48}$$

One may directly extract the new system matrices $\mathbf{M}_B, \mathbf{D}_B, \mathbf{K}_B$ from the new set of LAMs $\underline{\mathbf{B}_k}$ $(k = 0, 1, 2)$. However one may observe that the lower half of the state vector $\underline{\mathbf{q}_B}$ is no longer the derivative of the top half, $\mathbf{q}_{B2}(t) \neq \dot{\mathbf{q}}_{B1}(t)$ and that the structure of the forcing vector in general is not preserved.

The SPTs have the advantage of being able to transform the coordinates of a system as desired whilst being able to preserve the order of the structure being transformed. Indeed the SPTs exist for systems of any order and are not restricted to second order equations. It is equally possible to change a system of order l into another system of order m where $l \neq m$ which shares identical eigenvalues. These notions are further discussed in later subsections.

2.6.1 Higher Order Systems

As stated the SPTs are not limited to second order systems and one may use the SPTs to transform any order system. Consider the l^{th} order system

$$\left(\tau^{l}\,\mathbf{A}_{l}+\tau^{l-1}\,\mathbf{A}_{l-1}+\tau^{l-2}\,\mathbf{A}_{l-2}+\cdots+\tau^{0}\,\mathbf{A}_{0}\right)\,\mathbf{q}_{A}(\tau)=\mathbf{f}_{A}(\tau) \tag{2.49}$$

The operator $\tau \equiv \frac{d}{dt}$ is used to denote derivative with respect to time and the matrices $\mathbf{A}_k$ represent the system matrices.

There exist $(l+1)$ LAMs for an l^{th} order system which one again denotes $\underline{\mathbf{A}_k}$ where $k = 0, 1, \cdots, l$. For higher order cases it is not possible to identify the structure of the LAMs easily as for the second order case. Correspondingly one uses the definition from [P3] such that

$$\underline{\mathbf{A}_k}=\underline{\mathbf{A}_0}\,\left(\underline{\mathbf{C}_{RA}}\right)^{k}\quad,\qquad k=1,2,\cdots,l \tag{2.50}$$

Here $\underline{\mathbf{C}_{RA}}$ defines the right companion matrix defined by Gohberg *et al.* [G10] to be

$$\underline{\mathbf{C}_{RA}}=\begin{bmatrix}\mathbf{0} & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \cdots & \mathbf{I}\\ \vdots & & & & \vdots\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I}\\ -\mathbf{A}_l^{-1}\,\mathbf{A}_0 & -\mathbf{A}_l^{-1}\,\mathbf{A}_1 & -\mathbf{A}_l^{-1}\,\mathbf{A}_2 & \cdots & -\mathbf{A}_l^{-1}\,\mathbf{A}_{l-1}\end{bmatrix} \tag{2.51}$$

and $\underline{\mathbf{A}_0}$ defines the base LAM which has the upper triangular structure

$$\underline{\mathbf{A}_0}=\begin{bmatrix}-\mathbf{A}_1 & -\mathbf{A}_2 & -\mathbf{A}_3 & \cdots & -\mathbf{A}_{l-1} & -\mathbf{A}_l\\ -\mathbf{A}_2 & -\mathbf{A}_3 & -\mathbf{A}_4 & \cdots & -\mathbf{A}_l & \mathbf{0}\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ -\mathbf{A}_{l-1} & -\mathbf{A}_l & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0}\\ -\mathbf{A}_l & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0}\end{bmatrix} \tag{2.52}$$

Evidently the above definition of the LAMs is only valid when $\mathbf{A}_l$ is non-singular although other definitions do exist which do not rely on this constraint.

Equation (2.9) is still used to describe the system of equations although the definitions of $\underline{\mathbf{q}_A}$ need to be expanded to encompass the increased dimension of the system. The definition of $\underline{\mathbf{f}_{Ak}}$ can be found to be from

$$\underline{\mathbf{f}_{Ak}}(t) = \underline{\mathbf{A}_0}\,(\underline{\mathbf{C}_{RA}})^{k-1}\,\underline{\mathbf{f}_{A1}}(t) \quad , \qquad k = 2, 3, \cdots, l \tag{2.53}$$

2.6.2 Order Changing Structure Preserving Transformations

As briefly mentioned the SPTs can be used to transform an l^{th} order system into a m^{th} order system containing the same eigenvalues where $l \neq m$. Defining the l^{th} order system given in equation (2.49) one may change this to an m^{th} order system which is defined to be

$$\left(\tau^m\,\mathbf{B}_m + \tau^{m-1}\,\mathbf{B}_{m-1} + \tau^{m-2}\,\mathbf{B}_{m-2} + \cdots + \tau^0\,\mathbf{B}_0\right)\,\mathbf{q}_B(\tau) = \mathbf{f}_B(\tau) \tag{2.54}$$

Again $\tau \equiv \frac{d}{dt}$ is used here to act as an operator for derivative with respect to time.

The necessary question now arises as to how to calculate the order changing SPTs. Prells [P4] introduced the Krylov structure of the right SPT to be

$$\underline{\mathbf{T}_R} = \begin{bmatrix} \mathbf{X}_A \\ \mathbf{X}_A\,\underline{\mathbf{C}_{RA}} \\ \mathbf{X}_A\,\underline{\mathbf{C}_{RA}}^2 \\ \vdots \\ \mathbf{X}_A\,\underline{\mathbf{C}_{RA}}^{m-1} \end{bmatrix}^{-1} \tag{2.55}$$

$\underline{\mathbf{C}_{RA}}$ denotes the right companion matrix given in equation (2.51) of the original system defined by equation (2.49). The matrix $\mathbf{X}_A$ may be chosen to be any real arbitrary matrix for which the inverse shown in equation (2.55) is permitted.

By constructing the right transformation matrix $\underline{\mathbf{T}_R}$ one may form the right companion matrix of the m^{th} system such that

$$\underline{\mathbf{C}_{RB}} = \underline{\mathbf{T}_R}^{-1}\,\underline{\mathbf{C}_{RA}}\,\underline{\mathbf{T}_R} \tag{2.56}$$

Assuming that the new system is monic ($\mathbf{B}_m = \mathbf{I}$) one may directly extract the matrices $\mathbf{B}_k$ ($k = 0, 1, \cdots, m-1$) from $\underline{\mathbf{C}_{RB}}$ to form the base LAM $\underline{\mathbf{B}_0}$ for the transformed system.

One may define $\underline{\mathbf{T}_L}$ to be [P4]

$$\underline{\mathbf{T}_L} = \begin{bmatrix} \mathbf{Y}_A & \underline{\mathbf{C}_{RB}}\,\mathbf{Y}_A \end{bmatrix}^T \underline{\mathbf{B}_0}^T \tag{2.57}$$

where

$$\mathbf{Y}_A = \underline{\mathbf{T}_R}^{-1} \begin{bmatrix} \mathbf{0} \\ -\mathbf{M}_A^{-1} \end{bmatrix} \tag{2.58}$$

Thus the LAMs $\underline{\mathbf{A}_k}$ corresponding to the second order system can be converted to the m^{th} order system through the standard transformation given by equation (2.46).

Numerical Example 2.1

For the purpose of this numerical example one has an un-forced arbitrary 4 degree of freedom second order system with positive-definite matrices

$$\mathbf{M}_A = \begin{bmatrix} 164 & 82 & 158 & 150 \\ 82 & 95 & 106 & 126 \\ 158 & 106 & 227 & 220 \\ 150 & 126 & 220 & 242 \end{bmatrix}, \mathbf{D}_A = \begin{bmatrix} 15 & 8 & 18 & 6 \\ -8 & 6 & 11 & 2 \\ -18 & -11 & 23 & 6 \\ -6 & -2 & -6 & 4 \end{bmatrix} \tag{2.59}$$

$$\mathbf{K}_A = \begin{bmatrix} 171 & 111 & 134 & 153 \\ 111 & 94 & 116 & 109 \\ 134 & 116 & 155 & 139 \\ 153 & 109 & 139 & 145 \end{bmatrix}$$

The dimension and order of the given system permit two possible higher order changes to either order 4^{th} or 8^{th} order.

Consider the 4^{th} order conversion first with arbitrary $\mathbf{X}_A$ matrix to be

$$\mathbf{X}_A = \begin{bmatrix} 61 & 31 & 18 & 25 & 51 & 54 & 34 & 31 \\ 7 & 61 & 62 & 59 & 46 & 94 & 40 & 41 \end{bmatrix} \tag{2.60}$$

Correspondingly applying the SPS one extracts the 4^{th} matrices to be

$$\begin{aligned}
\mathbf{B}_0 &= \begin{bmatrix} 10.862 & -20.586 \\ 4.0833 & -8.7217 \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} 28.721 & -27.16 \\ 14.732 & -12.415 \end{bmatrix} \\
\mathbf{B}_2 &= \begin{bmatrix} -17.11 & -14.921 \\ -8.1046 & -7.1338 \end{bmatrix}, \quad \mathbf{B}_3 = \begin{bmatrix} -3.0679 & -1.4562 \\ -1.4705 & -0.69865 \end{bmatrix}
\end{aligned} \tag{2.61}$$

yielding the 4^{th} order system

$$\tau^4\,\mathbf{I} + \tau^3\,\mathbf{B}_3 + \tau^2\,\mathbf{B}_2 + \tau\,\mathbf{B}_1 + \mathbf{B}_0 \tag{2.62}$$

Consider now the 8^{th} order system. One defines matrix $\mathbf{X}_A$ to be

$$\mathbf{X}_A = \begin{bmatrix} 29 & 39 & 50 & 72 & 31 & 11 & 44 & 47 \end{bmatrix} \tag{2.63}$$

This definition of $\mathbf{X}_A$ permits a non-singular $\underline{\mathbf{T}_R}$ and correspondingly yields the characteristic polynomial

$$\tau^8+0.0021171\tau^7+0.096314\tau^6+0.66267\tau^5+2.072\tau^4+5.5908\tau^3+4.574\tau^2+5.6257\tau+2.1407 \tag{2.64}$$

This numerical example has demonstrated how to change the second order system into another system of higher order whilst retaining the same eigenvalues.

2.6.3 Structure Preserving Similarities

The family of structural transformations may be further expanded to encompass the 'Structure Preserving Similarities' (SPSs). The SPSs are a set of non-unique similarity transformations which transform the original state space system defined by equation (2.5) into a new state space system whilst retaining the appropriate structure in the right companion matrix $\underline{\mathbf{C}_{RA}}$. The origin of the SPSs become clearer when one redefines equation (2.9) to be

$$\underline{\dot{\mathbf{q}}_A}(t) = -(\underline{\mathbf{A}_{k-1}})^{-1}\,\underline{\mathbf{A}_k}\,\underline{\mathbf{q}_A}(t) + (\underline{\mathbf{A}_{k-1}})^{-1}\,\underline{\mathbf{f}_{Ak}}(t) \qquad k = 1, 2 \tag{2.65}$$

The result $-(\underline{\mathbf{A}_{k-1}})^{-1}\,\underline{\mathbf{A}_k}$ and $(\underline{\mathbf{A}_{k-1}})^{-1}\,\underline{\mathbf{f}_{Ak}}(t)$ gives identical composition to the $\underline{\mathbf{C}_{RA}}$ $(\equiv \underline{\mathbf{A}_A})$ and $\underline{\mathbf{B}_A}$ matrices defined in equation (2.5).

If one applies the SPTs to equation (2.65) then it becomes apparent that this is equivalent to pre- and post-multiplying the state space $\underline{\mathbf{C}_{RA}}$ $(\equiv \underline{\mathbf{A}_A})$ and $\underline{\mathbf{B}_A}$ matrices given in equation (2.5).

$$\left(\underline{\mathbf{T}_R}^{-1}\,(\underline{\mathbf{A}_{k-1}})^{-1}\,\underline{\mathbf{T}_L}^{-T}\right)\left(\underline{\mathbf{T}_L}^{T}\,\underline{\mathbf{A}_k}\,\underline{\mathbf{T}_R}\right) = \underline{\mathbf{T}_R}^{-1}\,(\underline{\mathbf{A}_{k-1}})^{-1}\,\underline{\mathbf{A}_k}\,\underline{\mathbf{T}_R} \tag{2.66}$$

$$\equiv \underline{\mathbf{T}_R}^{-1}\,\underline{\mathbf{C}_{RA}}\,\underline{\mathbf{T}_R} \tag{2.67}$$

Thus the origin of the SPSs becomes apparent and one may transform the state space companion matrix $\underline{\mathbf{C}_{RA}}$ to the transformed companion matrix $\underline{\mathbf{C}_{RB}}$.

The notion of the SPSs introduced above utilises only the right companion matrix form as defined by Gohberg [G10]. There also exists a left companion matrix of the form

$$\underline{\mathbf{C}_{LA}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -\mathbf{A}_0\,\mathbf{A}_l^{-1} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -\mathbf{A}_1\,\mathbf{A}_l^{-1} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} & -\mathbf{A}_2\,\mathbf{A}_l^{-1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & -\mathbf{A}_{l-1}\,\mathbf{A}_l^{-1} \end{bmatrix} \tag{2.68}$$

The left transformation matrix $\underline{\mathbf{T}_L}$ can be used as an SPS to the left companion matrix

$$\underline{\mathbf{C}_{LB}} = \underline{\mathbf{T}_L}^{T}\,\underline{\mathbf{C}_{LA}}\,\underline{\mathbf{T}_L}^{-T} \tag{2.69}$$

For a specific example of the applications of the SPSs see [H2].

2.6.4 Diagonalising Transformations

It was stated in section 2.3 that only classically or proportionally damped second order systems maybe diagonalised using the eigenvectors of the undamped system. For the situation where this constraint is not satisfied one may utilise the SPTs to diagonalise the system matrices by transforming the second order system matrices into state space form.

The SPTs enable all non-defective [F5] systems to be diagonalised [G1] without the necessity of fulfilling constraint (2.12). The left and right diagonalising SPT transformations also yield the diagonalising SPSs for the left and right companion matrices respectively. For classically damped systems the SPTs may become simply block-diagonal containing the eigenvectors although it is possible to have a diagonalising SPT for classically damped systems which is fully populated.

The conventional diagonalising transformation for systems satisfying (2.12) is accepted to represent the undamped physical modes of the system [G7]. Thus one may ask the question as to what the SPT diagonalised system matrices represent for non-proportionally damped systems? The diagonalised system matrices relate directly to the eigenvalues of the system and thus contain the necessary information to describe the system response fully. The diagonalised system may be thought of as a special modal representation of the damped modes of the system and although the modes have no physical accepted meaning they do comprise of physically-meaningful data. The SPTs do not require that the eigenvalues of the system occur only in conjugate pairs thus real eigenvalue pairs may be paired together as is appropriate. The mass-normalised diagonal matrices $\mathbf{K}_B, \mathbf{D}_B, \mathbf{M}_B$ have the relationship to the eigenvalues $\mathbf{\Lambda}_a$ with $\mathbf{\Lambda}_b$ representing the grouped pairs

$$\mathbf{M}_B = \mathbf{I}\,, \quad \mathbf{D}_B = -(\mathbf{\Lambda}_a + \mathbf{\Lambda}_b)\,, \quad \mathbf{K}_B = \mathbf{\Lambda}_a\,\mathbf{\Lambda}_b \tag{2.70}$$

The derivation of this result is found in appendix B and is analogous to that found by Datta *et al.* [D1] for a symmetric definite system.

Thus the SPT diagonalised system matrices are referred to as a modal form of equations.

Numerical Example 2.2

Consider the second order system matrices

$$\mathbf{M}_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{D}_A = \begin{bmatrix} 4.2 & 6.8 & 6.9 \\ -6.8 & 13.6 & 11 \\ -6.9 & -11 & 11.4 \end{bmatrix}, \quad \mathbf{K}_A = \begin{bmatrix} 116 & 92 & 76 \\ 92 & 137 & 139 \\ 76 & 139 & 197 \end{bmatrix} \tag{2.71}$$

The skew-symmetry in the damping matrix ensures that the system is non-classically damped thus it cannot be diagonalised using the mass-normalised eigenvectors of the undamped system. One finds the eigenvalue pairs of the system to be

$\boldsymbol{\Lambda}_a$	$\boldsymbol{\Lambda}_b$
-7.3989 + 23.091i	-7.3989 - 23.091i
-3.1042 + 7.2936i	-3.1042 - 7.2936i
-1.5190	-6.6747

Utilising equation (2.70) one finds the diagonal system matrices

$$\mathbf{M}_B = \operatorname{diag}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{D}_B = \operatorname{diag}\begin{bmatrix} 8.1937 \\ 6.2085 \\ 14.798 \end{bmatrix}, \quad \mathbf{K}_B = \operatorname{diag}\begin{bmatrix} 10.139 \\ 62.832 \\ 587.93 \end{bmatrix} \tag{2.72}$$

Thus one may form the LAMs for the original and diagonalised systems. Following the diagonalising process outlined in Chapter 3 one may form the diagonalising SPTs

$$\underline{\mathbf{T}_L} = \begin{bmatrix} -0.67282 & -0.8854 & 0.0021283 & -0.19728 & -0.033503 & 0.026871 \\ 1.1181 & 0.30609 & -0.72329 & 0.28719 & -0.0044738 & 0.008126 \\ -0.47817 & 0.13576 & -0.76367 & -0.10162 & 0.03399 & -0.018322 \\ 2.0002 & 2.1051 & -15.798 & 0.94359 & -0.6774 & -0.3955 \\ -2.9119 & 0.2811 & -4.7775 & -1.2351 & 0.33387 & -0.84354 \\ 1.0303 & -2.1357 & 10.772 & 0.35444 & -0.075272 & -0.49254 \end{bmatrix} \tag{2.73}$$

$$\underline{\mathbf{T}_R} = \begin{bmatrix} -0.294 & -0.67768 & -0.45789 & -0.05125 & 0.036303 & -0.011506 \\ 1.0742 & 0.064154 & -0.66684 & 0.2583 & 0.023218 & -0.010055 \\ -0.85762 & -0.20633 & -0.50495 & -0.24817 & -0.11798 & 0.020922 \\ 0.51962 & -2.281 & 6.7645 & 0.12592 & -0.90307 & -0.28763 \\ -2.6189 & -1.4588 & 5.9115 & -1.0422 & -0.079992 & -0.51806 \\ 2.5162 & 7.4128 & -12.3 & 1.1758 & 0.52613 & -0.81454 \end{bmatrix} \tag{2.74}$$

2.7 Control Methods

Control is a generic term for the process of using some means to alter the characteristics of a system. The characteristics may be altered in two conceivable ways. The first is through passive modification by addition of physical mass, damping or stiffness to the system and the second is through active control in which one may add artificial mass, damping and stiffness via actuators applying forces proportional to acceleration, velocity and/or displacement. The passive option requires permanent change to the physical system which will not allow for perturbations and changes in system parameters. The active control is traditionally more versatile allowing the control to be altered fairly simply. Mottershead and Ram [M10] provide a useful analysis of the two methods and the concept of structural vibrations in general.

Active control may be broken down into two broad approaches, open loop and closed loop control. Open loop control implies altering the system somehow with knowledge of only the current input and no knowledge of the current system state. This compares with closed loop control which has access to at least some of the current state of the system and thus has the potential to adapt to unforeseen circumstances. This project is concerned only with closed loop control and as such open loop control is not discussed further.

For the context of this project the notion of closed loop control is specified to mean altering the dynamic characteristics of a physical system utilising the knowledge of the current physical state of the system to allow the construction of a feedback force. The first order system introduced in equation (2.5)

$$\underline{\dot{\mathbf{q}}_A}(t) = \underline{\mathbf{A}_A}\,\underline{\mathbf{q}_A}(t) + \underline{\mathbf{B}_A}\,\mathbf{f}_A(t) \tag{2.75}$$

may be subjected to a feedback force proportional to the system state. This is represented pictorially in figure 2.1 where $\mathbf{G}$ represents the feedback gains matrix to determine the feedback force.

The methods considered in this project to determine the control of the feedback system fall into two general categories. The first category is concerned only with moving the unstable poles of a system thus paying no attention to the mode shapes of the system and the second category is concerned with moving the poles whilst preserving the mode shapes of the system. These methods represent pole placement and modal control techniques respectively. A subset of control problems exists within these methods such as optimal control which allow the quantification of the cost of moving the respective poles of the system.

The requirement for stabilising control is to move all the real components of the system poles into the left half of the complex plane [D4]. This is illustrated by considering the un-forced response of a linear system to an arbitrary initial condition $\underline{\mathbf{q}_A}(0)$

$$\underline{\mathbf{q}_A}(t) = \underline{\mathbf{\Phi}_R} \, \exp\left(\mathbf{\Lambda} \, t\right) \, \underline{\mathbf{\Phi}_R}^{-1} \, \underline{\mathbf{q}_A}(0) \tag{2.76}$$

where $\mathbf{\Lambda}$ represents a diagonal matrix containing the system complex poles, $\underline{\mathbf{\Phi}_R}$ contains the eigenvectors corresponding to $\mathbf{\Lambda}$ [D4].

Any diagonal entries in $\mathbf{\Lambda}$ having a positive real part will not permit the decay of the system state to zero as time advances. If there exists positive real parts on the diagonal matrix $\mathbf{\Lambda}$ then the response of the system $\underline{\mathbf{q}_A}(t)$ will increase even if the initial conditions are zero due to inevitable disturbance.

When one transforms the second order system into a first order form as illustrated by equation (2.3) one solves the general eigenvalue problem to find the system eigenvalues. However the second order equations of motion require the solution of the quadratic eigenvalue problem which has been shown to yield different solutions to the generalised problem [T1]. This creates a disparity between the notion of stability defined by the eigenvalues. Necessary and sufficient conditions for system stability of a first order system are readily available [M7]. Diwekar and Yedavalli [D3] extend the necessary and sufficient conditions for the stability of second order matrix systems under various loading types such as conservative and non-conservative forces.

The concept of the system decaying to zero state leads to the question about whether or not it is possible to control a given system thus giving rise to the notion of controllability. Controllability is defined as the ability to achieve any arbitrary required state in a finite time from any initial starting point. This may be more simply defined as whether or not it is possible to place a pole of a system at any arbitrary location on the complex plane [F6]. This definition may be expanded more formally such that if matrix $\underline{\mathbf{C}_{con}}$ is full rank then the system is controllable.

$$\underline{\mathbf{C}_{con}} = \left[\begin{array}{cccccc} \underline{\mathbf{B}_A} & \underline{\mathbf{A}_A}\,\underline{\mathbf{B}_A} & \underline{\mathbf{A}_A}^2\,\underline{\mathbf{B}_A} & \cdots & \underline{\mathbf{A}_A}^{2n-1}\,\underline{\mathbf{B}_A} \end{array} \right] \tag{2.77}$$

One may think of the system being controllable if one is able to excite all the modes of vibration using the available force inputs.

The concept of controllability presented here is depicted as merely as yes or no. This pays no attention as to the degree of uncontrollability of a system. For uncontrollable systems one may not be able to arbitrarily place a pole of a system but the regions of uncontrollability on the complex plane may not overlap with the desired locations where one wishes to place the pole. Thus one may only desire to move the poles of the system within the regions on the complex plane which are fully controllable. This essentially means that one may not be able to control all modes of vibration but often one does not require to control all modes of vibration. Often the higher order modes are left uncontrolled and are considered not to have a significant influence on the system performance. Thus the concept of the controllability Grammians introduced (see section 2.5.3) for the balanced representation of the state space system are more useful for quantifying how controllable or uncontrollable an individual pole is.

Parallel to the notion of controllability, observability of a system is defined based on whether every mode of vibration of a system may be observed. One formal criterion for a system to be a observable is for the matrix $\underline{\mathbf{C}_{obs}}$ to be full rank where

$$\underline{\mathbf{C}_{obs}} = \left[\begin{array}{ccccc} \underline{\mathbf{C}_A} & \underline{\mathbf{C}_A}\,\underline{\mathbf{A}_A} & \underline{\mathbf{C}_A}\,\underline{\mathbf{A}_A}^2 & \cdots & \underline{\mathbf{C}_A}\,\underline{\mathbf{A}_A}^{2n-1} \end{array} \right]^T \tag{2.78}$$

Once again this definition of observability is merely yes or no. The observability Grammians already introduced are a more useful means of quantifying the required observability as one does not always wish to observe the contributions of all the modes.

At this point it is necessary to elaborate why closed loop active control is so essential. The addition of active control to a system allows the introduction of artificial stiffness and damping. This has particular importance in space and aeronautical applications where the cost associated with operations is highly dependent on system weight. Applied physical damping in these applications is very costly due to the excessive weight associated with physical dampers [S3]. By contrast many lightweight smart materials are emerging which enable active control to change their physical properties allowing the addition of damping to the system for a fraction of the weight [I2].

Another example of the potential of active control is the development of self-levitating electrical motors [K3]. These essentially act as electro-magnetic bearings and allow access to apply forces to the rotor of a rotating machine so that vibrations in the machine can be cut substantially.

It remains to discuss the main control techniques in detail.

2.8 Pole Placement

Pole placement is the process by which the eigenvalues (poles) of a closed-loop system are moved to pre-determined locations [D4]. The closed loop poles are altered by utilising a feedback force proportional to the state such that artificial stiffness and damping are added to the system.

This concept can be defined more thoroughly through mathematical representation. Recalling the first order state space form from equation (2.5)

$$\underline{\dot{\mathbf{q}}_A}(t) = \underline{\mathbf{A}_A}\,\underline{\mathbf{q}_A}(t) + \underline{\mathbf{B}_A}\,\mathbf{f}_A(t) \tag{2.79}$$

For linear systems not subjected to external excitation the applied feedback force is traditionally proportional to the state through the relationship $\mathbf{f}_A(t) = -\mathbf{G}\,\underline{\mathbf{q}_A}(t)$. Substituting this relationship into the first order equations of motion yields the result

$$\underline{\dot{\mathbf{q}}_A}(t) = \left(\underline{\mathbf{A}_A} - \underline{\mathbf{B}_A}\,\mathbf{G}\right)\,\underline{\mathbf{q}_A}(t) = \underline{\mathbf{A}_{con}}\,\underline{\mathbf{q}_A}(t) \tag{2.80}$$

Thus the response of the system is now determined by the eigenvalues of closed loop matrix $\underline{\mathbf{A}_{con}}$ rather than $\underline{\mathbf{A}_A}$. Through appropriate calculation of the feedback gains matrix $\mathbf{G}$

one may arbitrarily place the eigenvalues of a fully controllable system where desired. Examples of pole placement methods may be found in references [K1], [M9] and [N1].

The pole placement method like most other control methods requires full state feedback. Full state feedback requires knowledge of all system states which is not realistic in many applications. Statistical means of reforming the unobserved states can be made using methods such as Kalman filters [B8] which provide estimates of the system state. Obviously these estimates are not exact and will introduce uncertainty into the system.

The pole placement method has two distinct disadvantages: 1.) There is no appreciation of the cost of moving the poles to their locations; 2.) The eigenvectors of the original system are destroyed.

Point one may be addressed by using a cost function to quantify the cost of moving a pole to its location. Typically the cost takes into account the response of the system and the control effort applied. One may use the same cost function to successive control designs to quantify which gives the most desirable result.

The second point associated with pole placement is rather more subjective. Pole placement methods make no attempt to preserve the eigenvectors of the controlled system thus intuition regarding the physical modes is lost. Preservation of the eigenvectors forms the basis of modal control which is discussed in section 2.9.

Pole placement does offer the possibility of moving a pole to coincide with a zero [M10]. This has the effect of eliminating a resonance of a system in at least one output from that system. This may prove to be more cost effective solution for poles which are reluctant to move towards more stable regions although this does depend on how controllable the particular pole to be moved is. However pole-zero cancellation is very sensitive to external excitation but perhaps more importantly an unstable system with the unstable poles eliminated using zeros is still unstable.

For single input systems pole placement may achieve arbitrary placement of the poles for a controllable system from a theoretical point of view. However there arise many practical problems [R2] such as sensitivity to perturbations and unrealistic magnitude of control forces. The multi-input case allows more choice over the way the poles are moved to a location, and hence the control may be more robust and control forces may be reduced in magnitude than for the single input case.

2.8.1 Placement of Poles for Second Order Systems

The conventional method as already outlined involves casting the second order equations of motion into first order state space form. This has the effect of increasing the dimension of the required system as well as destroying symmetry and definiteness of the system matrices. Kim *et al.* [K5] proposed a method to assign the poles of a system whilst retaining the second order nature of the system through the use of the Sylvester's equation. However the method requires the system to be classically damped such that the eigenvectors of the undamped system diagonalise all three system matrices. This is not appropriate for this project.

Ram and Elhay [R2] proposed a pole placement method for the second order system which placed a required subset of poles whilst retaining the remaining original poles.

For the single-input case they determined the control gains to be

$$\mathbf{g}_k = -\mathbf{K}_A\,\mathbf{\Phi}\,\mathbf{h} \qquad , \qquad \mathbf{g}_d = \mathbf{M}_A\,\mathbf{\Phi}\,\mathbf{\Lambda}\,\mathbf{h} \tag{2.81}$$

where $\mathbf{g}_k$ and $\mathbf{g}_d$ are the proportional and derivative controller components respectively, $\mathbf{h}$ is a vector defined by the eigenvalues of the open-loop system and the desired eigenvalues, $\mathbf{\Phi}$ is the matrix of eigenvectors and $\mathbf{\Lambda}$ is a diagonal matrix containing the eigenvalues on the diagonal. Thus the single-input controller is applied

$$\mathbf{M}_A\,\ddot{\mathbf{q}}_A(t) + \mathbf{D}_A\,\dot{\mathbf{q}}_A(t) + \mathbf{K}_A\,\mathbf{q}_A(t) = \mathbf{S}_A\ [\,\mathbf{g}_k\,\mathbf{q}_A(t) + \mathbf{g}_d\,\dot{\mathbf{q}}_A(t)\,] \tag{2.82}$$

where the number of actuator forces, r, is equal to 1 for this specific case.

For the multi-input case where $r \neq 1$ Ram and Elhay proposed to progressively move the pole location by solving r single-input pole placement problems. Thus one may move the pole to the desired location in r steps. This may be summarised by equation (2.83).

$$\mathbf{M}_A\,\ddot{\mathbf{q}}_A(t) + \mathbf{D}_A\,\dot{\mathbf{q}}_A(t) + \mathbf{K}_A\,\mathbf{q}_A(t) = \sum_{i=1}^{r} \mathbf{S}_{Ai}\ [\,\mathbf{g}_{ki}\,\mathbf{q}_A(t) + \mathbf{g}_{di}\,\dot{\mathbf{q}}_A(t)\,] \tag{2.83}$$

The effect of progressively changing the pole location in r steps rather than all at once is to reduce magnitude of the control forces and hence reduce the associated control burden.

2.9 Optimal Control

To discuss optimal control properly, one must first elaborate on the definition of optimal. In the context of this project optimal is defined as being the best possible cause of action subject to user supplied definition of some cost function and user defined constraints. In terms of optimal control this can be simplified further through the mathematical description [B8]

$$J(\underline{\mathbf{q}_A}^*, \mathbf{f}_A^*) \leq J(\underline{\mathbf{q}_A}^* + \delta\underline{\mathbf{q}_A}, \mathbf{f}_A^* + \delta\mathbf{f}_A) \qquad \forall\ \delta\underline{\mathbf{q}_A}, \delta\mathbf{f}_A \tag{2.84}$$

Here J is the result of a cost function and $\underline{\mathbf{q}_A}^*, \mathbf{f}_A^*$ are the optimal state and force. The δ represents a perturbation added to the optimal quantities. This equation simply states that any arbitrary perturbation to the system parameters will result in a cost function greater than or equal to the optimal cost.

As already stated optimal control finds the minimum cost of applying control when the user defines the relative importance of the state (or state error) and applied forces [Z2]. Many possible cost functions exist to define the optimal problem but for this project one defines the cost function to be

$$J = \int_0^\infty \underline{\mathbf{q}_A}(t)^T\, \underline{\mathbf{Q}}\, \underline{\mathbf{q}_A}(t) + \mathbf{f}_A(t)^T\, \mathbf{R}\, \mathbf{f}_A(t)\, dt \tag{2.85}$$

where $\underline{\mathbf{Q}}$ represents a positive semi-definite weighting matrix placed on the system state $\underline{\mathbf{q}_A}(t)$ and $\mathbf{R}$ represents a positive definite weighting matrix placed on the control force $\mathbf{f}_A(t)$. This form of the quadratic cost is used by the 'Linear Quadratic Regulator' (LQR) problem [B8].

The quadratic cost function illustrated in equation (2.85) has threefold implications:

- the positive and negative errors are weighted equally.

- the larger errors are penalised more harshly than smaller errors.
- The integral penalises the more persistent error more harshly than shorter term ones.

Optimal control has the direct benefit of quantifying the cost of moving the poles to a particular location. This can simplify the trial and error approach often resulting from pole placement. However it is often unclear how to select the optimal weighting matrices $\underline{\mathbf{Q}}$ and $\mathbf{R}$. Gawronski [G7] presents tools for calculating the optimal control gains, although the method is restricted to collocated sensors and actuators.

A distinct problem associated with optimal control is the requirement to find the solution to the non-linear Riccati equation [B7]. For systems of large dimension this may represent a serious numerical problem [M2]. However for the context of modal control in this project, single degree of freedom systems obtained from the modal equations are of concern. As shown by Meirovitch [M7], the decoupled equations of motion give rise to closed form solutions to the Riccati equation. Indeed for second order single degree of freedom systems the optimal feedback gain is found to be

$$\mathbf{G}_{opt} = \left[\; -k_j + \left(k_j^2 + \mathbf{Q}_{11}\right)^{\frac{1}{2}} \quad , \quad -d_j + \left(d_j^2 + \mathbf{Q}_{22} - 2\,k_j + 2\,\left(k_j^2 + \mathbf{Q}_{11}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} \; \right] \tag{2.86}$$

where $\mathbf{Q}$ is the 2×2 diagonal state weighting matrix, $\mathbf{R}$ was set equal to 1 and the second order single degree of freedom system is monic with stiffness k and damping d. The derivation for this result may be found in the appendix A.

One particular difference between pole placement and optimal control is the method by which each pole is relocated. Indeed, different pole placement methods yield different costs for the same pole locations. Optimal control offers the optimal allocation of the poles thus always beats or equals the pole placement methods in terms of cost for the same pole locations. This conclusion is supported by Saif [S1] who developed a pole placement method utilising the optimal control method. Saif proposed a method to calculate the optimal control weighting matrices $\underline{\mathbf{Q}}$ and $\mathbf{R}$ corresponding to user specified pole locations. The weighting matrices are then utilised in the optimal control problem to yield the optimal feedback gains matrix.

For this specific project it is desired to control directly the second order system. It would thus be beneficial to be able to calculate directly the optimal controller gains without resorting to the first order state space form as the conventional optimal control problem does. Ram and Inman [R1] achieve this for the single-input control problem. The method utilises the Euler-Lagrange equations to develop a linear fourth-order differential equation such that the minimisation of the cost function depends on second derivatives. Zhang [Z1] extends Ram and Inman's method to the multi-input system using a method in which the complex eigenvectors of the second order Hamiltonian equations are used to solve the optimal problem. Zhang finds that proportional and derivative controller gains, $\mathbf{G}_k$ and $\mathbf{G}_d$ respectively, to be

$$\begin{bmatrix} \mathbf{G}_k^T \\ \mathbf{G}_d^T \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \mathbf{\Phi}^T & \mathbf{\Lambda}\,\mathbf{\Phi}_1^T \end{bmatrix}^{-1} \mathbf{\Gamma}^T\,\mathbf{R}^{-1} \tag{2.87}$$

The eigenvalues of the Hamiltonian matrix are symmetric about the real axis [B8] such that half the roots are stable and half are unstable. Only the stable half are dealt with such that one extracts $\mathbf{\Phi}$ and $\mathbf{\Gamma}$ from the right eigenvectors of the second order Hamiltonian matrix and $\mathbf{\Lambda}$ is a diagonal matrix containing the stable half the Hamiltonian eigenvalues. The results from this method compare favourably with the first order results.

A notable feature of conventional optimal control problem is that no emphasis is placed on the rate at which control forces may be applied. Control forces cannot be instantaneously applied and indeed several applications exist where this can exert significant difficulties. One such example is electro-magnetic bearings. Houlston *et al.* [H3] have addressed this problem using an approach to augment the system state to include the force allowing the rate of change of force to be fed directly to the system and subsequently integrated. This allows the optimal problem to take account of the rate of change of control force but does have the drawback that integration of the force is required and the system has to be in first order form.

An alternative method of penalising the rate at which the control forces are applied is the 'model predicted control' (MPC) method as outlined by Maciejowski [M1]. The MPC method deals with the first order state space equations of motion in discrete time.

The MPC method determines the change in control force at each time step which is subsequently added to the current control action at such that

$$\mathbf{f}_{Ad}(k) = \mathbf{f}_{Ad}(k-1) + \Delta\mathbf{f}_{Ad}(k) \tag{2.88}$$

where $\mathbf{f}_{Ad}(k)$ is the discrete time forcing vector, k is the index for the current time step and $k-1$ is the previous time step. As is apparent from equation (2.88) the new control action at time sample k is the previous control action plus the calculated change in control action where Δ denotes change. Equation (2.88) may be combined with the discrete time state space equations to form the augmented first order system

$$\begin{bmatrix} \underline{\mathbf{q}_{Ad}}(k+1) \\ \mathbf{f}_{Ad}(k) \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{A}_d} & \underline{\mathbf{B}_d} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{q}_{Ad}}(k) \\ \mathbf{f}_{Ad}(k-1) \end{bmatrix} + \begin{bmatrix} \underline{\mathbf{B}_d} \\ \mathbf{I} \end{bmatrix} \Delta\mathbf{f}_{Ad}(k) \tag{2.89}$$

where $\underline{\mathbf{q}_{Ad}}(k)$ is the $2n$-dimensional discrete time state vector and $\underline{\mathbf{A}_d}$, $\underline{\mathbf{B}_d}$ are the discrete time state space companion and forcing matrices respectively.

The MPC method uses knowledge of the system dynamics and current state and forcing values to create a future prediction of the system state at time $k+N$. This future prediction can be used in a cost function in an attempt to determine the effect of the control parameters on the future behaviour of the system. This process is repeated at each time step and for the standard predictive control case one attempts to minimise the cost function

$$J_{MPC} = \sum_{k=0}^{N} \underline{\mathbf{q}_{Ad}}(k)^T \, \underline{\mathbf{Q}}(k) \, \underline{\mathbf{q}_{Ad}}(k) + \Delta\mathbf{f}_{Ad}(k)^T \, \mathbf{R}(k) \, \Delta\mathbf{f}_{Ad}(k) \tag{2.90}$$

The cost function in equation (2.90) is similar to the cost function given in equation (2.85) for the LQR problem. Here the discrete state and force weighting matrices are depicted as time dependent although one may fix the weighting matrices to be constant.

Cole *et al.* [C4] show that the MPC control method which minimises the cost function (2.90) at each time step for a suitable value of N yields a comparable result to the conventional infinite horizon LQR optimal control problem. Indeed the process of obtaining the minimum to the cost function in equation (2.90) is similar to that used in dynamic programming which is used to obtain optimal solutions to particular optimisation problems. A detailed account of dynamic programming may be found in references [B4, B5].

The predictive control method can be used to place physical limits on the rate of change of control force. These can take the form of soft or hard constraints where soft constraints allow the limit to be violated to a moderate degree and hard constraints represent an absolute limit. Both soft and hard MPC controllers result in a non-linear control action being applied to the system. Implementation of the limits placed on the rate of change of forcing require that the controller gains are time invariant and calculated at each time sample as a function of current state and force and predicted future state and force.

The MPC was originally developed for process control where the time samples are very slow. This raises the possibility of computational problems with systems requiring fast time samples such as rotating machines although this may be overcome using a method such as a lookup table. For relatively small dimensional systems requiring control or controllers such as the single degree of freedom controllers associated with modal control, significant computational problem will not arise.

2.10 Modal Control

The physical response of any linear second order system can be shown to comprise the sum of a number of independent modal responses [G8]. One may relate the displacements $\mathbf{q}_A(t)$ to the modes of the system for free vibration through the relationship

$$\mathbf{q}_A(t) = \sum_{j=1}^{n} \mathbf{\Phi}_{Rj} \, \exp\left(i\lambda_j t\right) \quad , \qquad j = 1, 2, \cdots, 2n \tag{2.91}$$

where $\mathbf{\Phi}_{Rj}$ is the j^{th} right complex eigenvector corresponding to j^{th} complex eigenvalue λ_j and $i = \sqrt{-1}$.

For the specific case where the second order system is undamped one finds that the eigenvectors in equation (2.91) are real. Correspondingly one finds that there are only n distinct eigenvectors corresponding to n modes of vibration. However for the case when the system is damped one finds that there are $2n$ distinct complex eigenvectors corresponding to the $2n$ complex roots. One must accordingly group the $2n$ complex roots into mode-pairs. Due to this pairing one finds that it is impossible to excite only one half of the mode pairs and that the mode-pairs must be excited jointly. Correspondingly for control of damped systems one seeks to control sets of mode-pairs.

The relationship between the physical and modal properties of the system illustrated in equation (2.91) raises the question as to whether or not an independent mode-pair itself can be targeted. It transpires that it can and this is the essence of modal control.

Modal control is a particular type of control method in which the physical response of a system is divided into mode-pairs associated with their corresponding natural frequencies. A standard control approach is to move the natural frequencies and damping factors, or poles as they are collectively known, into a stable region. The essence of modal control is that since the eigenvectors of a system do not contribute to the asymptotic stability of a system then any effort expended on altering them represents wasted effort. This idea is supported by a study of conventional control versus modal control [M4].

The ambition of modal control is to design a controller that independently controls each mode-pair without influencing any other modes. For modal control each actuator is designated to control an individual mode-pair thus in general for second order modal control methods as many mode-pairs can be controlled independently as actuators available. For the first order modal control methods only half as many modes can be controlled as actuators available since the number of modes doubles due to representation in first order state space form. This presents the possibility of control spillover in which the uncontrolled modes are excited by the control forces [B1]. This area is explored later in more in section 2.10.3.

One finds that the modal transformations decouple the un-forced equations of motion. This form is referred to as internally decoupled since the internal degrees of freedom when not subjected to external excitation are entirely independent as may be observed from equation (2.92)

$$m_j\,\ddot{\mathbf{q}}_{Bj} + d_j\,\dot{\mathbf{q}}_{Bj} + k_j\,\mathbf{q}_{Bj} = 0 \tag{2.92}$$

where m_j, d_j and k_j correspond to the j^{th} single degree of freedom (SDOF) system matrices obtained from the diagonalised matrices $\mathbf{M}_B$, $\mathbf{D}_B$ and $\mathbf{K}_B$, and $\mathbf{q}_{Bj}$ is the j^{th} modal co-ordinate.

However when control forces are present one finds that in general the internally decoupled equations of motion are now coupled by the external control forces. As may be observed from equation (2.93) the external forces cannot independently affect only one SDOF system without necessarily affecting other co-ordinates.

$$m_j\,\ddot{\mathbf{q}}_{Bj} + d_j\,\dot{\mathbf{q}}_{Bj} + k_j\,\mathbf{q}_{Bj} = \sum_{k=1}^{n} \mathbf{f}_{Bk} \tag{2.93}$$

This is referred to as externally coupled since the right-hand side of equation (2.93) couples the internally decoupled system on the left hand side.

Modal control overcomes this problem by designing the control forces in the modal space ensuring that the system equations of motion are completely decoupled both internally and externally. A process to convert the modal forces into the physical domain is then utilised allowing the physical forces to be applied to the system. This is represented pictorially in figure 2.2

Very little literature is available on modal control of systems containing gyroscopic terms, see for example Meirovitch and Baruh [M2]. Lee and Chen [L3] proposed a method to create a feedback force to counteract the gyroscopic forces such that conventional second order modal control could be applied. This method seems counter-intuitive because the gyroscopic forces may prove stabilising in regions of interest. Indeed in their paper Lee and Chen state that if the stiffness matrix is positive definite and the damping matrix at least positive semi-definite then the gyroscopic terms cannot destabilise the system but can shift the eigenvalues in the complex plane. The method thus appears to yield little in direct benefit other than to permit the use of the conventional modal control methods.

Sawicki and Genta [S2] proposed to use the modal data to transform the gyroscopic system into decoupled 'single degree of freedom' (SDOF) systems. The method proposed placing the second order system into the state space form illustrated in equation (2.3). The modal data is manipulated to preserve the structure of the state space companion matrix to yield the complex decoupled form

$$\underline{\mathbf{A}_m} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\left(\overline{\mathbf{D}_m} - i\,\omega\overline{\mathbf{G}_m}\right) & -\left(\overline{\mathbf{K}_m} - i\,\omega\overline{\mathbf{C}_{mr}}\right) \end{bmatrix} \tag{2.94}$$

where $i = \sqrt{-1}$, ω is the spin-speed and the matrices $\overline{\mathbf{D}_m}$, $\overline{\mathbf{G}_m}$, $\overline{\mathbf{K}_m}$ and $\overline{\mathbf{C}_{mr}}$ represent the real and imaginary components of the Sawicki and Genta's modal damping and stiffness matrices definitions.

As may be seen from equation (2.94) the SDOF systems are indeed decoupled but result in a complex-valued form. The complex valued-form of the modal equations of motion require complex controllers thus do not readily yield themselves to conventional control methods. An additional constraint is placed on the value of the eigenvalues which are assumed to occur in conjugate pairs. For systems such as highly damped systems or systems with hydrodynamic bearings this assumption is invalid thus the method fails.

The vast majority of modal control literature is applicable to structural vibration problems which are reviewed here.

2.10.1 First Order Modal Control

Meirovitch and Baruh introduced a first order modal control method using a state space representation of system for non-classically damped systems [M2]. The first order representation of the equations of motion seen in equation (2.5) may be transformed into decoupled equations of motion by using the manipulated left and right eigenvectors, $\underline{\mathbf{\Psi}_L}$ and $\underline{\mathbf{\Psi}_R}$. The manipulated left and right eigenvectors , $\underline{\mathbf{\Psi}_L}$ and $\underline{\mathbf{\Psi}_R}$, are formed from the original left and right eigenvectors, $\underline{\mathbf{\Phi}_L}$ and $\underline{\mathbf{\Phi}_R}$. The eigenvalues are assumed to occur in n complex pairs such that there are n pairs of corresponding complex eigenvectors. Appropriate ordering ensures that the original complex left and right eigenvectors have the structure

$$\underline{\mathbf{\Phi}_L} = \left[\begin{array}{cccc} \mathbb{R}(\mathbf{\Phi}_{L1}) \pm \mathbb{I}(\mathbf{\Phi}_{L1}) & \mathbb{R}(\mathbf{\Phi}_{L2}) \pm \mathbb{I}(\mathbf{\Phi}_{L2}) & \cdots & \mathbb{R}(\mathbf{\Phi}_{Ln}) \pm \mathbb{I}(\mathbf{\Phi}_{Ln}) \end{array} \right] \tag{2.95}$$

$$\underline{\mathbf{\Phi}_R} = \left[\begin{array}{cccc} \mathbb{R}(\mathbf{\Phi}_{R1}) \pm \mathbb{I}(\mathbf{\Phi}_{R1}) & \mathbb{R}(\mathbf{\Phi}_{R2}) \pm \mathbb{I}(\mathbf{\Phi}_{R2}) & \cdots & \mathbb{R}(\mathbf{\Phi}_{Rn}) \pm \mathbb{I}(\mathbf{\Phi}_{Rn}) \end{array} \right] \tag{2.96}$$

Here $\mathbb{R}$ denotes the real components and $\mathbb{I}$ denotes the imaginary components.

One may realise only n columns of each of the eigenvectors in equations (2.95) and (2.96) contain distinct information. Consequently Meirovitch and Baruh propose to utilise only half of the complex eigenvectors to form the manipulated left and right eigenvectors $\underline{\mathbf{\Psi}_L}$ and $\underline{\mathbf{\Psi}_R}$. One finds the manipulated eigenvectors to have the structure

$$\underline{\mathbf{\Psi}_L} = \begin{bmatrix} \mathbb{R}(\mathbf{\Psi}_{L1}) & \mathbb{I}(\mathbf{\Psi}_{L1}) & \mathbb{R}(\mathbf{\Psi}_{L2}) & \mathbb{I}(\mathbf{\Psi}_{L2}) & \cdots & \mathbb{R}(\mathbf{\Psi}_{Ln}) & \mathbb{I}(\mathbf{\Psi}_{Ln}) \end{bmatrix} \tag{2.97}$$

$$\underline{\mathbf{\Psi}_R} = \begin{bmatrix} \mathbb{R}(\mathbf{\Psi}_{R1}) & \mathbb{I}(\mathbf{\Psi}_{R1}) & \mathbb{R}(\mathbf{\Psi}_{R2}) & \mathbb{I}(\mathbf{\Psi}_{R2}) & \cdots & \mathbb{R}(\mathbf{\Psi}_{Rn}) & \mathbb{I}(\mathbf{\Psi}_{Rn}) \end{bmatrix} \tag{2.98}$$

such that they fulfil the necessary scaling

$$\underline{\mathbf{\Psi}_L}^T \, \underline{\mathbf{\Psi}_R} = \mathbf{I} \tag{2.99}$$

Using the co-ordinate transformation $\underline{\mathbf{q}_A}(t) = \underline{\mathbf{\Psi}_R}\, \underline{\mathbf{q}_B}(t)$ and pre-multiplying the state space equations of motion given in equation (2.5) by $\underline{\mathbf{\Psi}_L}^T$, one has

$$\underline{\mathbf{\Psi}_L}^T \, \underline{\mathbf{\Psi}_R}\, \underline{\dot{\mathbf{q}}_B}(t) = \underline{\mathbf{\Psi}_L}^T \, \underline{\mathbf{A}_A}\, \underline{\mathbf{\Psi}_R}\, \underline{\mathbf{q}_B}(t) + \underline{\mathbf{\Psi}_L}^T \, \underline{\mathbf{B}_A}\, \mathbf{f}_A(t) \tag{2.100}$$

to give

$$\underline{\dot{\mathbf{q}}_B}(t) = \underline{\mathbf{\Lambda}_m}\, \underline{\mathbf{q}_B}(t) + \mathbf{W}(t) \tag{2.101}$$

where $\underline{\mathbf{\Lambda}_m}$ is assumed to be block diagonal with 2×2 block entries $\mathbf{\Lambda}_{mj}$ defined by

$$\mathbf{\Lambda}_{mj} = \begin{bmatrix} \alpha_j & -\beta_j \\ \beta_j & \alpha_j \end{bmatrix} \tag{2.102}$$

$j = 1, 2, \cdots, n$

Here $\lambda_{j(1,2)} = \alpha_j \pm i\beta_j$ represents the j^{th} complex conjugate eigenvalue pair and n the dimension of the second order system. An immediate observation may be seen that the eigenvalues are assumed to exist in conjugate pairs. As already stated for some gyroscopic systems and some systems with very high damping this is not the case. Thus for systems with only pairs of eigenvalues consisting of only real values which are different then the there are more than n distinct eigenvectors and the decoupling process outlined above will fail.

The modal contributions are extracted from the physical quantities using modal filters [M6] but the method does not derive an inverse modal filter to revert the modal quantities back to the physical domain. A backward transformation is defined which allows at most only half of the modelled modes to be controlled as illustrated by equation (2.103).

$$\mathbf{f}_A(t) = \mathbf{M}_A \left(\underline{\mathbf{\Psi}_{L,\,lower}}^T \right)^{\dagger} \mathbf{W}(t) \tag{2.103}$$

where $\underline{\mathbf{\Psi}_{L,\,lower}}$ represents the lower half of the manipulated left $2n$-dimensional eigenvectors $\underline{\mathbf{\Psi}_L}$ and the symbol $\dagger$ defines the pseudo-inverse.

One finds that an inverse rather than a pseudo-inverse can be used in the backward transformation if only half the modelled modes are controlled. Thus if only half the modelled modes are available for control it was proposed to control the lower frequency modelled modes and ignore the higher modes. This was justified through reasoning that the higher frequency modes are more difficult to excite hence do not contribute significantly to the system response [G8]. Indeed further support is given to this statement by acknowledging that the accuracy of finite element models wanes as one departs from the first few modes [L1]. This is again especially true for models reduced in dimension using Guyan reduction [G12] or related methods since these guarantee accuracy only at zero frequency.

An advantage of the first order modal control method is that a closed-form solution for the optimal control problem for high order systems can be obtained. This offers massive computational savings compared to traditional optimal control of the full $2n \times 2n$ state space models. This means that one has the opportunity to alter the controller gains real-time to soften or harden the control action if required. For some systems altering the gains real time can increase the effectiveness of the control action [P6], [K2].

For classically damped and undamped systems the use of one actuator to control one mode is shown to result in robust and effective control [M3]. The effect of control spillover in these problems is assumed to be non-destabilising due to not altering the closed loop eigenvalues of the controlled system [M5]. However for control of generally damped systems the first order method is shown by Lin and Chu [L4] to be potentially unstable and the assumption that r actuators can control r modes of interest is invalid. Thus the first order method is inappropriate for control of rotating systems.

Gawronski [G8] showed a method of utilising the first order modal control in his book which resulted in all modes accessible for control. Similar to Meirovitch he utilised first order state space control but used the eigenvectors of the un-forced second order equations of motion to ensure the decoupling of the modal states. The damping matrix was assumed classically damped in that the un-forced eigenvectors would diagonalise the damping matrix. However this method had the shortcoming that the damping matrix is rarely classically damped resulting in off-diagonal coupling terms in the modal damping matrix.

2.10.2 Second Order Modal Control

The second order IMSC technique stems from the paper [M6] in which the modal filters are introduced in depth. The left and right mass-normalised eigenvectors, $\mathbf{\Phi}_L$ and $\mathbf{\Phi}_R$ respectively, of the un-forced system are used to decouple the equations of motion. They are utilised by making the substitution to the second order equations of motion $\mathbf{q}_A(t) = \mathbf{\Phi}_R\,\mathbf{q}_B(t)$ and pre-multiplying by $\mathbf{\Phi}_L^T$

$$\mathbf{\Phi}_L{}^T\,\mathbf{M}_A\,\mathbf{\Phi}_R\,\ddot{\mathbf{q}}_B(t) + \mathbf{\Phi}_L{}^T\,\mathbf{D}_A\,\mathbf{\Phi}_R\,\dot{\mathbf{q}}_B(t) + \mathbf{\Phi}_L{}^T\,\mathbf{K}_A\,\mathbf{\Phi}_R\,\mathbf{q}_B(t) = \mathbf{\Phi}_L{}^T\,\mathbf{f}_A(t) \tag{2.104}$$

to yield monic system

$$\mathbf{I}\,\ddot{\mathbf{q}}_B(t) + \mathbf{\Gamma}\,\dot{\mathbf{q}}_B(t) + \mathbf{\Lambda}^2\,\mathbf{q}_B(t) = \mathbf{\Phi}_L{}^T\,\mathbf{f}_A(t) \tag{2.105}$$

here $\mathbf{q}_B(t)$ represents the modal coordinates of the system.

The new damping matrix $\mathbf{\Gamma}$ is assumed diagonal with any remaining off-diagonal terms in the modal damping matrix traditionally discarded [G7]. However, for rotating systems involving substantial gyroscopic terms ignoring these terms is in effect ignoring the gyroscopic terms themselves [S2].

Sawicki and Genta [S2] introduced a variation to the second order modal control method in which second order systems containing gyroscopic terms in the damping matrix are cast into first order state space form. The notion of the second order structure within the state space form is retained similar to that obtained using the structural preserving equivalences already introduced. They proposed to use the modal data to decouple the gyroscopic system as the second order IMSC technique does for classically damped systems. However the decoupled equations of motion are left in complex-valued form which destroys the real representation of the system thus requiring a complex controller. The form of Sawicki and Genta's decoupled system may be seen in equation (2.94).

2.10.3 Control Spillover

As already stated modal control techniques design the control forces independently in the modal space before transforming the forces into the physical domain. Typically modal control only allows as many modes to be controlled as there are actuators available. This presents the problem of control spillover in which the un-modelled or residual (uncontrolled) modes are excited by the control forces. The concept is best illustrated by considering the right-hand side of the second order IMSC control method given by equation (2.104).

$$\begin{bmatrix} \mathbf{f}_{Bc}(t) \\ \mathbf{f}_{Br}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_{L_{cc}}^T & \mathbf{\Phi}_{L_{rc}}^T \\ \mathbf{\Phi}_{L_{cr}}^T & \mathbf{\Phi}_{L_{rr}}^T \end{bmatrix} \begin{bmatrix} \mathbf{f}_A(t) \\ \mathbf{0} \end{bmatrix} \tag{2.106}$$

$\mathbf{f}_B(t)$ is the modal force, $\mathbf{f}_A(t)$ is the available physical force and $\mathbf{\Phi}_L$ represents the matrix of left eigenvectors used to transform the system into the modal state. The c and r subscripts denote controlled and residual portions respectively. One may see that for this particular problem making the substitution

$$\mathbf{\Phi}_L^{-T} = \mathbf{Z} = \begin{bmatrix} \mathbf{Z}_{cc} & \mathbf{Z}_{cr} \\ \mathbf{Z}_{rc} & \mathbf{Z}_{rr} \end{bmatrix} \tag{2.107}$$

one has

$$\begin{bmatrix} \mathbf{f}_A(t) \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_{cc} & \mathbf{Z}_{cr} \\ \mathbf{Z}_{rc} & \mathbf{Z}_{rr} \end{bmatrix} \begin{bmatrix} \mathbf{f}_{Bc}(t) \\ \mathbf{f}_{Br}(t) \end{bmatrix} \tag{2.108}$$

which yields the relationship between physical and modal forces to be

$$\mathbf{f}_A(t) = \mathbf{Z}_{cc}\, \mathbf{f}_{Bc}(t) \tag{2.109}$$

However one may also see from equation (2.108) that the uncontrolled modal forces are excited such that

$$\mathbf{f}_{Br}(t) = \mathbf{\Phi}_{Lcr}^{T}\,\mathbf{f}_A(t) \tag{2.110}$$

Thus for the second order IMSC method control spillover exists in the form illustrated by the above equation. Meirovitch and Baruh [M5] showed that the control spillover forces exciting the uncontrolled modes will not destabilise the system for classically damped systems. Consider that the second order modal equations of motion are obtained as illustrated in equation (2.105). The contributions of the controlled, residual (uncontrolled) and un-modelled modes, denoted by subscripts c, r and u respectively, are partitioned such that from equation (2.105) one has

$$\begin{bmatrix} (s^2\mathbf{I}_c + s\mathbf{\Gamma}_c + \mathbf{\Lambda}_c) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (s^2\mathbf{I}_r + s\mathbf{\Gamma}_r + \mathbf{\Lambda}_r) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (s^2\mathbf{I}_u + s\mathbf{\Gamma}_u + \mathbf{\Lambda}_u) \end{bmatrix} \begin{bmatrix} \mathbf{q}_c(s) \\ \mathbf{q}_r(s) \\ \mathbf{q}_u(s) \end{bmatrix} = \begin{bmatrix} \mathbf{f}_c \\ \mathbf{f}_r \\ \mathbf{f}_u \end{bmatrix} \tag{2.111}$$

where s is the Laplace variable.

Simplifying the right hand side of equation (2.105) one may define

$$\mathbf{f}_c = \mathbf{B}_c\mathbf{F}_m \tag{2.112}$$

$$\mathbf{f}_r = \mathbf{B}_r\mathbf{F}_m \tag{2.113}$$

$$\mathbf{f}_u = \mathbf{B}_u\mathbf{F}_m \tag{2.114}$$

where $\mathbf{F}_m$ is the modal controller defined to be

$$\mathbf{F}_m = \mathbf{G}_k\mathbf{q}_c + \mathbf{G}_d\dot{\mathbf{q}}_c \tag{2.115}$$

Substituting equation (2.115) into equation (2.111) one obtains

$$\begin{bmatrix} \left(s^2\mathbf{I}_c + s\left(\mathbf{\Gamma}_c - \mathbf{B}_c\mathbf{G}_d\right) + \left(\mathbf{\Lambda}_c - \mathbf{B}_c\mathbf{G}_k\right)\right) & \mathbf{0} & \mathbf{0} \\ -\left(s\mathbf{B}_r\mathbf{G}_d + \mathbf{B}_r\mathbf{G}_k\right) & \left(s^2\mathbf{I}_r + s\mathbf{\Gamma}_r + \mathbf{\Lambda}_r\right) & \mathbf{0} \\ -\left(s\mathbf{B}_u\mathbf{G}_d + \mathbf{B}_u\mathbf{G}_k\right) & \mathbf{0} & \left(s^2\mathbf{I}_u + s\mathbf{\Gamma}_u + \mathbf{\Lambda}_u\right) \end{bmatrix} \begin{bmatrix} \mathbf{q}_c(s) \\ \mathbf{q}_r(s) \\ \mathbf{q}_u(s) \end{bmatrix} = \mathbf{0} \tag{2.116}$$

One may observe from equation (2.116) that the determinant of the system remains unchanged thus the closed-loop poles of the uncontrolled and un-modelled modes remain unchanged. Therefore one may conclude that control spillover for the conventional second order IMSC cannot destabilise these modes.

Baz *et al.* [B1], exploit the possibility of sharing an actuator to control more than one mode. Their proposed method does not eliminate control spillover but does seek to control it. The method proposes to control the r modes with the highest energy resulting in switching between modes as uncontrolled modes become excited and controlled mode energy decreases. Indeed in reference [B2] it was demonstrated that the method could be practically applied.

Fang *et al.* [F1] proposed a method of designing the controller gains such that the modal control force is truly decoupled allowing the explicit control of r modes. The method presented creates a transformation such that the modal controller couples the modes in the modal domain but subsequent reversion to the physical domain using the backward filter decouples the modes. Recalling that one has

$$\mathbf{f}_B(t) = \mathbf{\Phi}_L{}^T\, \mathbf{f}_A(t) \tag{2.117}$$

where $\mathbf{\Phi}_L$ is the matrix of left eigenvectors one wishes to design a decoupled modal controller of the form

$$\mathbf{\Phi}_L{}^T\, \mathbf{G}_k = \left[\begin{array}{cccc} g_{k1} & 0 & \cdots & 0 \\ 0 & g_{k2} & \cdots & 0 \\ 0 & 0 & \cdots & g_{kr} \\ \hline 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{array}\right], \qquad \mathbf{\Phi}_L{}^T\, \mathbf{G}_d = \left[\begin{array}{cccc} g_{d1} & 0 & \cdots & 0 \\ 0 & g_{d2} & \cdots & 0 \\ 0 & 0 & \cdots & g_{dr} \\ \hline 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{array}\right] \tag{2.118}$$

Fang *et al.* make the definition

$$\mathbf{G}_k = \mathbf{M}_A\, \mathbf{\Phi}_R\, \mathbf{F}_k \quad \text{and} \quad \mathbf{G}_d = \mathbf{M}_A\, \mathbf{\Phi}_R\, \mathbf{F}_d \tag{2.119}$$

Pre-multiplying equation (2.119) by $\mathbf{\Phi}_L{}^T$ one recognises that the mass-normalised eigenvectors yield the identity matrix and correspondingly one may conclude that the controllers gains having the form

$$\mathbf{G}_k = \mathbf{M}_A\,\mathbf{\Phi}_R \left[\begin{array}{cccc} g_{k1} & 0 & \cdots & 0 \\ 0 & g_{k2} & \cdots & 0 \\ 0 & 0 & \cdots & g_{kr} \\ \hline 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{array}\right], \quad \mathbf{G}_d = \mathbf{M}_A\,\mathbf{\Phi}_R \left[\begin{array}{cccc} g_{d1} & 0 & \cdots & 0 \\ 0 & g_{d2} & \cdots & 0 \\ 0 & 0 & \cdots & g_{dr} \\ \hline 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{array}\right] \tag{2.120}$$

will yield diagonal controller matrices where g_{di}, g_{ki} are required modal controller gains. However one may also question whether the modal controller is actually providing the control action required because now part of the modal control action is used to eliminate the contributions to non-controlled modes. This method does not prevent control spillover in the un-modelled modes.

Perhaps a combination of the two methods proposed by Baz *et al.* and Fang *et al.* would allow the complete control across all the modelled modes.

2.11 Conclusions

This chapter has reviewed related areas of interest to this project and highlighted some of the pitfalls facing conventional techniques. This project attempts to address these pitfalls, notably modal control of non-classically damped systems and eliminate the reliance on first order linearisation for control techniques in which the second order notion of systems is destroyed.

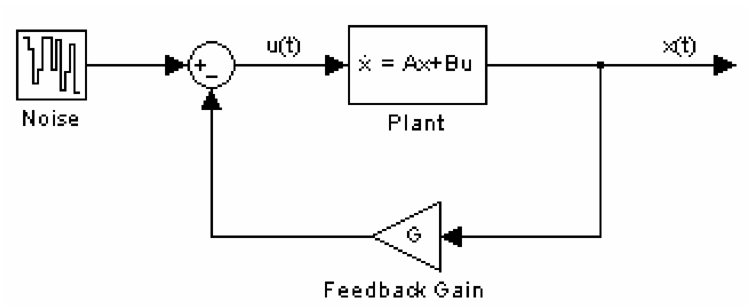


Figure 2.1: Negative Feedback System

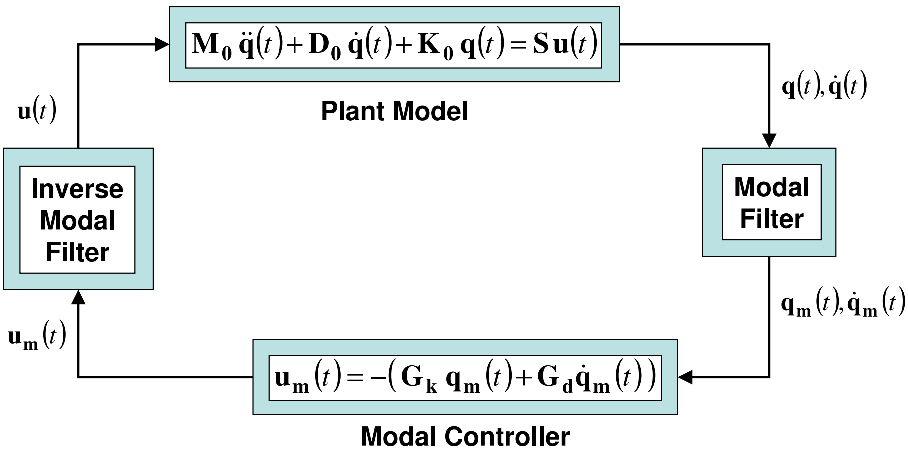


Figure 2.2: Modal Control Outline

Chapter 3

Modal Control of Vibration in Rotating Machines

Traditional control approaches, such as pole placement methods [K1], deal with the physical system in first order state space form. The ambitions of this project are to control the physical system in second order form. Very little literature is available in regards to direct second order control, see for example [D2]. Many obvious advantages over first order control are available [M8]: 1.) Physical insight of the system is preserved. 2.) Computational efficiency, since the dimension of the second order system is smaller than that of the state space form. 3.) Symmetry and structure of the systems can be preserved where desired.

Many structural and dynamic systems are described by the second order equations of motion

$$\mathbf{M}_A\,\ddot{\mathbf{q}}_A(t) + \mathbf{D}_A\,\dot{\mathbf{q}}_A(t) + \mathbf{K}_A\,\mathbf{q}_A(t) = \mathbf{S}_A\,\mathbf{u}_A(t) = \mathbf{f}_A(t) \tag{3.1}$$

where $\mathbf{M}_A, \mathbf{D}_A, \mathbf{K}_A \in \mathbb{R}^{n\times n}$ are the system mass, damping and stiffness matrices respectively, $\mathbf{q}_A(t) \in \mathbb{R}^n$ is the vector of physical coordinates, $\mathbf{S}_A \in \mathbb{R}^{n\times r}$ is a selection matrix describing the locations of applied forces and $\mathbf{u}_A(t) \in \mathbb{R}^r$ is the vector of applied forces. $\mathbf{f}_A(t) \in \mathbb{R}^{n\times r}$ is used to denote the full vector of applied forces. For the sake of brevity here it is assumed that forces are available at all coordinate locations ($r = n$). This constraint is later removed.

Modal control is a particular control method in which the physical response of a system is divided into mode-pairs associated with corresponding natural frequencies. A standard control approach is to move the system eigenvalues into a stable region. The essence of modal control is that since the eigenvectors of a system do not contribute to the asymptotic stability of a system then any effort expended on altering them represents wasted effort [M7]. More detail on the modal control method may be found in section 2.10.

Meirovitch and Baruh introduced a first order modal control method using a state space representation of system containing skew-symmetry in the damping matrix [M2]. The modal contributions are extracted from the physical quantities using modal filters [M6] but the method does not derive an inverse modal filter to revert the modal quantities back to the physical domain. A backward transformation is defined which allows only one half of the modelled modes to be controlled. Meirovitch and Baruh proposed to control only the lower order modelled modes with justification for this being that the higher order modes are more difficult to excite hence do not contribute significantly to the system response. The method proposed in this paper removes this constraint by defining an inverse filter making it possible to control all the modelled modes.

Modern computers have enough computational capacity such that worries concerning the expansion of the control problem to $2n$ rather than an n-dimensional problem is not an issue for moderate values of n. However, redefining the second order equations of motion into a first order realisation has the disadvantage of destroying some properties such as symmetry and definiteness of the matrices describing the motion [R1]. Here, direct second order techniques allow the retention of the natural form of dynamic systems arising from Newtonian mechanics.

Traditional modal control for second order systems such as the 'Independent Modal Space Control' (IMSC) method used by Baz *et al.* [B1] utilise the mass normalised left and right eigenvectors, $\mathbf{\Phi}_L$ and $\mathbf{\Phi}_R$, of the undamped system to diagonalise the system matrices. Although the method outlined is developed for self-adjoint systems the same method is applicable when this criterion is relaxed. The difference is that for the non-self-adjoint one uses distinct left and right eigenvectors of the undamped system to attempt the diagonalisation process. For the self-adjoint case one finds that $\mathbf{\Phi}_L = \mathbf{\Phi}_R$. The coordinate transformation $\mathbf{q}_A(t) = \mathbf{\Phi}_R\, \mathbf{q}_B(t)$ is applied and the system matrices pre-multiplied by the transpose of the left eigenvectors, ${\mathbf{\Phi}_L}^T$

From

$$ {\mathbf{\Phi}_L}^T\, \mathbf{M}_A\, \mathbf{\Phi}_R\, \ddot{\mathbf{q}}_B + {\mathbf{\Phi}_L}^T\, \mathbf{D}_A\, \mathbf{\Phi}_R\, \dot{\mathbf{q}}_B + {\mathbf{\Phi}_L}^T\, \mathbf{K}_A\, \mathbf{\Phi}_R\, \mathbf{q}_B = {\mathbf{\Phi}_L}^T \mathbf{f}_A \tag{3.2} $$

one assumes to have the diagonalised system matrices (for classical damping)

$$ \mathbf{I}\, \ddot{\mathbf{q}}_B + \mathbf{\Gamma}\, \dot{\mathbf{q}}_B + \mathbf{\Lambda}^2\, \mathbf{q}_B = {\mathbf{\Phi}_L}^T \mathbf{f}_A \tag{3.3} $$

where $\mathbf{q}_B(t)$ represents the modal coordinates of the system. For ease of reading the notation defining the dependence on time has been removed.

The new damping matrix $\mathbf{\Gamma}$ is assumed diagonal with any remaining off-diagonal terms in the modal damping matrix traditionally discarded [G7]. However, for rotating systems involving substantial gyroscopic terms ignoring these terms is in effect ignoring the gyroscopic terms themselves. Thus, it is proposed here to use the 'Structure Preserving Transformations' (SPTs) developed by Garvey *et al.* [G2, G3] to diagonalise the second order system matrices and decouple the system equations of motion without need to discard any terms involved in the description of the system.

The purpose of this chapter is present a detailed derivation of the SPT based modal control. One shows that general damped systems may be decoupled into equivalent single degree of freedom systems and each system independently controlled. For the case where one does not have sufficient actuators to control all the model modes it is shown that the effects of control spillover are non-destabilising. Numerical examples are used throughout to illustrate the methods.

3.1 Structure Preserving Transformations

The notion of the 'Lancaster Augmented Matrices' (LAMs) [G5] are now re-introduced here from section 2.2. The LAMs are used such that the second order system may be represented in state space form. For initially symmetric second order systems the LAMs permit two distinct symmetry-preserving linearisations. For a second order system there exist three LAMs which can be produced by inspection to be,

$$\underline{\mathbf{A}_0} = \begin{bmatrix} -\mathbf{D}_A & -\mathbf{M}_A \\ -\mathbf{M}_A & \mathbf{0} \end{bmatrix} \quad , \quad \underline{\mathbf{A}_1} = \begin{bmatrix} \mathbf{K}_A & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_A \end{bmatrix} \quad , \quad \underline{\mathbf{A}_2} = \begin{bmatrix} \mathbf{0} & \mathbf{K}_A \\ \mathbf{K}_A & \mathbf{D}_A \end{bmatrix} \tag{3.4}$$

The underline used in the notation of the LAMs is adopted to differentiate the $2n$-dimensional matrices from the n dimensional matrices.

The LAMs allow the second order system to be represented in a reduced form

$$\begin{bmatrix} \mathbf{K}_A & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_A \end{bmatrix} \begin{bmatrix} \mathbf{q}_{A1} \\ \mathbf{q}_{A2} \end{bmatrix} - \begin{bmatrix} -\mathbf{D}_A & -\mathbf{M}_A \\ -\mathbf{M}_A & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}}_{A1} \\ \dot{\mathbf{q}}_{A2} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_A \\ \mathbf{0} \end{bmatrix} \tag{3.5}$$

$$\begin{bmatrix} \mathbf{0} & \mathbf{K}_A \\ \mathbf{K}_A & \mathbf{D}_A \end{bmatrix} \begin{bmatrix} \mathbf{q}_{A1} \\ \mathbf{q}_{A2} \end{bmatrix} - \begin{bmatrix} \mathbf{K}_A & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_A \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}}_{A1} \\ \dot{\mathbf{q}}_{A2} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_A \end{bmatrix} \tag{3.6}$$

where $\mathbf{q}_{A1} = \mathbf{q}_A$ and $\mathbf{q}_{A2} = \dot{\mathbf{q}}_{A1}$. Equations (3.5) and (3.6) may generalised to

$$\underline{\mathbf{A}_k}\,\underline{\mathbf{q}_A} - \underline{\mathbf{A}_{k-1}}\,\underline{\dot{\mathbf{q}}_A} = \underline{\mathbf{f}_{Ak}} \tag{3.7}$$

$k = 1, 2$

A 'Structure Preserving Transformation' (SPT) is a coordinate transformation applied to the LAMs representing a bijective mapping between linear systems. The specific nature of the transformation allows the preservation of the appropriate structure within the LAMs. The SPTs are defined by left and right $2n \times 2n$ transformation matrices, $\underline{\mathbf{T}_L}$ and $\underline{\mathbf{T}_R}$ respectively, through

$$\underline{\mathbf{T}_L}^T\,\underline{\mathbf{A}_k}\,\underline{\mathbf{T}_R} = \underline{\mathbf{B}_k} \tag{3.8}$$

$k = 0, 1, 2$

The effect of applying an SPT is to transform equations (3.5) and (3.6) for the original system into the form

$$\begin{bmatrix} \mathbf{K}_B & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_B \end{bmatrix} \begin{bmatrix} \mathbf{q}_{B1} \\ \mathbf{q}_{B2} \end{bmatrix} - \begin{bmatrix} -\mathbf{D}_B & -\mathbf{M}_B \\ -\mathbf{M}_B & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}}_{B1} \\ \dot{\mathbf{q}}_{B2} \end{bmatrix} = \underline{\mathbf{T}_L}^T \begin{bmatrix} \mathbf{f}_A \\ \mathbf{0} \end{bmatrix} \tag{3.9}$$

$$\begin{bmatrix} \mathbf{0} & \mathbf{K}_B \\ \mathbf{K}_B & \mathbf{D}_B \end{bmatrix} \begin{bmatrix} \mathbf{q}_{B1} \\ \mathbf{q}_{B2} \end{bmatrix} - \begin{bmatrix} \mathbf{K}_B & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_B \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}}_{B1} \\ \dot{\mathbf{q}}_{B2} \end{bmatrix} = \underline{\mathbf{T}_L}^T \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_A \end{bmatrix} \tag{3.10}$$

One may directly extract the new real system matrices $\mathbf{M}_B, \mathbf{D}_B, \mathbf{K}_B$ from the new set of LAMs $\underline{\mathbf{B}_k}$ $(k = 0, 1, 2)$. However the lower half of the state vector $\underline{\mathbf{q}_B}$ is no longer the time derivative of the top half, $\mathbf{q}_{B2} \neq \dot{\mathbf{q}}_{B1}$. This fact necessitates the introduction of the SPT-modal filters which are presented in section 3.3.

The structure of the transformation matrices can be shown to have the following form

$$\underline{\mathbf{T}_L} = \begin{bmatrix} \mathbf{F}_L - \frac{1}{2}\,\mathbf{G}_L\,\mathbf{D}_A^T & -\mathbf{G}_L\,\mathbf{M}_A^T \\ \mathbf{G}_L\,\mathbf{K}_A^T & \mathbf{F}_L + \frac{1}{2}\,\mathbf{G}_L\,\mathbf{D}_A^T \end{bmatrix}^{-1} \tag{3.11}$$

$$\underline{\mathbf{T}_R} = \begin{bmatrix} \mathbf{F}_R - \frac{1}{2}\,\mathbf{G}_R\,\mathbf{D}_A & -\mathbf{G}_R\,\mathbf{M}_A \\ \mathbf{G}_R\,\mathbf{K}_A & \mathbf{F}_R + \frac{1}{2}\,\mathbf{G}_R\,\mathbf{D}_A \end{bmatrix}^{-1} \tag{3.12}$$

where $\mathbf{F}_L, \mathbf{F}_R, \mathbf{G}_L, \mathbf{G}_R \in \mathbb{R}^{n\times n}$ are arbitrary pre-defined matrices subject to the necessary constraint

$$\mathbf{F}_R\,\mathbf{G}_L^T + \mathbf{G}_R\,\mathbf{F}_L^T = 0 \tag{3.13}$$

3.2 Diagonalising Transformations

The SPT is said to be diagonalising if matrices of the transformed system, $\mathbf{M}_B, \mathbf{D}_B, \mathbf{K}_B$, are all diagonal. This diagonal system is not unique and equation (3.8) is not independent. For systems where $\mathbf{M}_A$ is invertible and where the transformation matrices are invertible, any two of the equations [H4] are sufficient to ensure that the third is also satisfied. The eigenvalues of the original system, $\mathbf{K}_A, \mathbf{D}_A, \mathbf{M}_A$, can be computed by solving the generalised eigenvalue problem involving either $\underline{\mathbf{A}_0}, \underline{\mathbf{A}_1}$ or $\underline{\mathbf{A}_1}, \underline{\mathbf{A}_2}$. Evidently, the eigenvalues of the new system $\mathbf{K}_B, \mathbf{D}_B, \mathbf{M}_B$ will be identical provided that $\underline{\mathbf{T}_L}, \underline{\mathbf{T}_R}$ are both invertible.

It is now shown how to obtain diagonalising transformation matrices $\underline{\mathbf{T}_L}, \underline{\mathbf{T}_R}$ from the complex modal data such that $\mathbf{K}_B, \mathbf{D}_B$ and $\mathbf{M}_B$ are real and diagonal. A 4-step process for calculating the diagonalising SPT is presented.

1. Calculate the left ($\underline{\mathbf{\Phi}_L}$) and right ($\underline{\mathbf{\Phi}_R}$) eigenvectors of reduced system

$$\underline{\mathbf{A}_1} - \lambda \underline{\mathbf{A}_0}$$

2. Calculate the n monic 'single degree of freedom' (SDOF) systems corresponding to conjugate eigenvalue pairs, $\lambda_{j(1,2)} = \alpha \pm i\beta$, found in part 1. For systems with real pairs of roots the same method applies through appropriate pairing

$$d_j = \lambda_{j1} + \lambda_{j2} \quad , \; k_j = \lambda_{j1}\,\lambda_{j2} \quad , \; m_j = 1 \tag{3.14}$$

   $j = 1, \cdots, n$.

   The derivation of this result may be found in appendix B and is analogous to that found by Datta *et al.* [D1] for a symmetric definite system.

3. Knowing the new diagonal system matrices form the new LAMs $\underline{\mathbf{B}_0}$ and $\underline{\mathbf{B}_1}$ representing the new diagonal system and calculate their corresponding left ($\underline{\mathbf{\Psi}_L}$) and right ($\underline{\mathbf{\Psi}_R}$) eigenvectors.

4. Since the two reduced systems have identical eigenvalues, appropriate scaling of the eigenvectors yields the following equality

$$\underline{\mathbf{\Phi}_L^T}\,\underline{\mathbf{A}_1}\,\underline{\mathbf{\Phi}_R} \quad = \underline{\mathbf{\Lambda}} = \quad \underline{\mathbf{\Psi}_L^T}\,\underline{\mathbf{B}_1}\,\underline{\mathbf{\Psi}_R} \tag{3.15}$$

$$\underline{\mathbf{\Phi}_L^T}\,\underline{\mathbf{A}_0}\,\underline{\mathbf{\Phi}_R} \quad = \underline{\mathbf{I}} = \quad \underline{\mathbf{\Psi}_L^T}\,\underline{\mathbf{B}_0}\,\underline{\mathbf{\Psi}_R} \tag{3.16}$$

   where $\underline{\mathbf{\Lambda}}$ is the diagonal matrix of corresponding eigenvalues and $\underline{\mathbf{I}}$ is the identity matrix. Thus one may recognise that to get from the original LAM to the new LAM the following condition must be satisfied

$$\left(\underline{\mathbf{\Psi}_L}^{-T}\,\underline{\mathbf{\Phi}_L}^T\right)\;\underline{\mathbf{A}_k}\;\left(\underline{\mathbf{\Phi}_R}\,\underline{\mathbf{\Psi}_R}^{-1}\right) = \underline{\mathbf{B}_k} \tag{3.17}$$

thus $\underline{\mathbf{T}_R} = \underline{\mathbf{\Phi}_R}\,\underline{\mathbf{\Psi}_R}^{-1}$ and $\underline{\mathbf{T}_L} = \underline{\mathbf{\Phi}_L}\,\underline{\mathbf{\Psi}_L}^{-1}$.

It may be noted that the above process for finding the diagonalising SPT requires only one eigenvalue solution problem. The eigenvectors of the diagonal LAMs, $\underline{\mathbf{\Psi}_L}$ and $\underline{\mathbf{\Psi}_R}$, have a sparse form such that their calculation is trivial.

3.3 Modal Filters

The premise of this chapter is to develop direct second order control of the equations of motion. In order to fully utilise modal control one needs to extract the modal quantities from the physical; this requires modal filters. The purpose of this section is to briefly summarise the modal filters developed in chapter 4.

It has been shown that from the original equations of motion

$$\mathbf{M}_A\,\ddot{\mathbf{q}}_A + \mathbf{D}_A\,\dot{\mathbf{q}}_A + \mathbf{K}_A\,\mathbf{q}_A = \mathbf{f}_A \tag{3.18}$$

one may use the notion of the LAMs and diagonalising SPTs to find the new second order equations of motion

$$\mathbf{M}_B\,\ddot{\mathbf{q}}_B + \mathbf{D}_B\,\dot{\mathbf{q}}_B + \mathbf{K}_B\,\mathbf{q}_B = \mathbf{f}_B \tag{3.19}$$

The vectors $\mathbf{q}_B$ and $\mathbf{f}_B$ are referred to as modal quantities since they comprise of physical meaningful modal data. Thus the necessary question is how to extract the second order modal contributions from the state space system. One may find the relationship between the old $(\mathbf{q}_A, \mathbf{f}_A)$ and the new coordinate sets $(\mathbf{q}_B, \mathbf{f}_B)$ as

$$\mathbf{q}_B = \mathbf{U}_0\,\mathbf{q}_A + \mathbf{U}_1\,\dot{\mathbf{q}}_A \tag{3.20}$$

$$\mathbf{f}_B = \mathbf{V}_0\,\mathbf{f}_A + \mathbf{V}_1\,\dot{\mathbf{f}}_A \tag{3.21}$$

with definitions

$$\begin{bmatrix} \mathbf{V}_0^T & \mathbf{V}_1^T \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \underline{\mathbf{T}_L} \tag{3.22}$$

$$\begin{bmatrix} \mathbf{U}_0 & \mathbf{U}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \underline{\mathbf{T}_R}^{-1} \tag{3.23}$$

The matrices $\mathbf{V}_0, \mathbf{V}_1$ and $\mathbf{U}_0, \mathbf{U}_1$ are the SPT filter matrices corresponding to the left and right filters respectively.

This definition of the filters is based on [L2] and the full derivation may be found in chapter 4. An alternative derivation of the filters may be found in appendix C. Equation (3.20) is referred to as the right filter and equation (3.21) is referred to as the left filter.

One finds that the filters allow the any SPT (whether it is a diagonalising SPT or not) to be interpreted in the Laplace frequency domain as follows [P2]

$$(\mathbf{V}_0 + s\mathbf{V}_1)\left(s^2\mathbf{M}_A + s\mathbf{D}_A + \mathbf{K}_A\right) = \left(s^2\mathbf{M}_B + s\mathbf{D}_B + \mathbf{K}_B\right)(\mathbf{U}_0 + s\mathbf{U}_1) \tag{3.24}$$

with s defining the Laplace variable and

$$\det\left(\mathbf{V}_0 + s\mathbf{V}_1\right) = 0 \Leftrightarrow \det\left(\mathbf{U}_0 + s\mathbf{U}_1\right) = 0 \tag{3.25}$$

With knowledge of the physical accelerations one may additionally obtain the modal velocities

$$\dot{\mathbf{q}}_B = \mathbf{U}_0\,\dot{\mathbf{q}}_A + \mathbf{U}_1\,\ddot{\mathbf{q}}_A \tag{3.26}$$

One may find an equally valid definition of the SPT filters using the lower half of the transformation matrices. Only the given definition of the filters is considered in this thesis although all methods not presented herein are equally valid.

The filters used to extract the modal quantities are first order thus a necessary constraint is that the real parts of the eigenvalues of the left and right filters must be in the stable region. Due to the definition of the filter the left and right filters contain the same eigenvalues, thus $\mathbf{V}_1^{-1}\,\mathbf{V}_0 = \mathbf{U}_1^{-1}\,\mathbf{U}_0 > 0$ must be true for a stable filter.

One may appropriately point out at this juncture that the diagonalising SPT is non-unique containing $2n$ independent parameters thus spanning a $2n$-dimensional space. It is possible to define a "reflexive" SPT $\underline{\mathbf{S}_L}, \underline{\mathbf{S}_R} \in \mathbb{R}^{2n\times 2n}$ for which maps the diagonal system directly back onto itself such that

$$\underline{\mathbf{S}_L}^T\,\underline{\mathbf{B}_k}\,\underline{\mathbf{S}_R} = \underline{\mathbf{B}_k} \tag{3.27}$$

$k = 0, 1, 2$

The non-uniqueness of the diagonalising SPT may be exploited to find a region in which the SPT-based filters are stable. The space spanned by the modal filters is not a linear vector (sub) space thus the search for a stable filter is not straight forward. When utilising the reflexive SPTs, numerical experiments suggest that it is always possible to find a stable filter. This remains to be proved formally but the author is content with the results from the numerical trials.

The concept of reflexive SPTs is dealt with in more detail in chapter 4 in which a technique is presented which progressively *flows* the filter eigenvalues into a more stable region.

3.4 True Independent Modal Control

To facilitate true independent modal control the modal equations of motion must be decoupled both internally for the un-forced system and externally when control forces are present [M7]. It has already been shown that the un-forced system may be decoupled using the diagonalising SPTs. However one finds that the decoupled internal degrees of freedom become coupled by the forcing parts.

$$\begin{bmatrix} s^2 m_1 + s\,d_1 + k_1 & 0 & \cdots & 0 \\ 0 & s^2 m_2 + s\,d_2 + k_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & s^2 m_n + s\,d_n + k_n \end{bmatrix} \mathbf{q}_B = \cdots$$

$$\cdots = \begin{bmatrix} \mathbf{V}_{0,11} + s\mathbf{V}_{1,11} & \mathbf{V}_{0,12} + s\mathbf{V}_{1,12} & \cdots & \mathbf{V}_{0,1r} + s\mathbf{V}_{1,1r} \\ \mathbf{V}_{0,21} + s\mathbf{V}_{1,21} & \mathbf{V}_{0,22} + s\mathbf{V}_{1,22} & \cdots & \mathbf{V}_{0,1r} + s\mathbf{V}_{1,2r} \\ \vdots & \vdots & & \vdots \\ \mathbf{V}_{0,n1} + s\mathbf{V}_{1,n1} & \mathbf{V}_{0,n2} + s\mathbf{V}_{1,n2} & \cdots & \mathbf{V}_{0,nr} + s\mathbf{V}_{1,nr} \end{bmatrix} \mathbf{f}_A \qquad (3.28)$$

where s is the Laplace variable.

This means that to design a practical decoupled controller one must design the controller in the modal space. The modal equations of motion can thus be defined

$$\mathbf{M}_B\,\ddot{\mathbf{q}}_B + \mathbf{D}_B\,\dot{\mathbf{q}}_B + \mathbf{K}_B\,\mathbf{q}_B = \mathbf{f}_B \qquad (3.29)$$

with $\mathbf{K}_B, \mathbf{D}_B, \mathbf{M}_B$ the diagonal modal system matrices and $\mathbf{q}_B \in \mathbb{R}^n$ the modal coordinates.

Equation (3.29) represents n SDOF systems with each one corresponding to a pair of modes of vibration. It is possible to use proportional-derivative control to affect directly the modal stiffness and damping properties of these modes. A controller of this form is introduced

$$\mathbf{f}_B = \mathbf{G}_k\,\mathbf{q}_B + \mathbf{G}_d\,\dot{\mathbf{q}}_B \qquad (3.30)$$

$\mathbf{G}_k, \mathbf{G}_d \in \mathbb{R}^{r\times n}$ represent the diagonal modal stiffness and damping gains matrices. Direct addition to the modal damping and stiffness matrices represents direct pole placement and has the advantage of being able to affect the poles of the system directly.

For modal control each actuator is designated to control an individual mode-pair thus in general for second order modal control methods as many mode-pairs can be controlled independently as actuators available. For conventional second order control the modal force can be typically converted back into the physical domain fairly easily as illustrated by Baz and Poh [B1]. For the SPT approach the left filter has already been defined and from equation (3.21) one can see that the physical and modal forces are related by the relationship

$$\mathbf{f}_B = \mathbf{V}_0\,\mathbf{f}_A + \mathbf{V}_1\,\dot{\mathbf{f}}_A \tag{3.31}$$

Rearrangement of equation (3.31) give the physical force in regards to the modal force

$$\dot{\mathbf{f}}_A = {\mathbf{V}_1}^{-1}\,\left(\mathbf{f}_B - \mathbf{V}_0\,\mathbf{f}_A\right) \tag{3.32}$$

Equation (3.32) constitutes a first order system. Consequently the real parts of the eigenvalues of $\left({\mathbf{V}_1}^{-1}\mathbf{V}_0\right)$ must be positive to ensure stability of the filter. Stability ensures that the response of the filter does not rapidly tend to infinity. The stability of the filter is further discussed in chapter 4.

3.5 Numerical Example 3.1

At this point it is appropriate to use a numerical example to highlight the benefits of the SPT-based method over the conventional second order modal control method which assume classical damping. Thus, consider the deliberate non-classically damped second order system with matrices

$$\mathbf{K}_A = \mathrm{diag}\begin{bmatrix} 50 \\ 70 \\ 90 \\ 10 \end{bmatrix}, \quad \mathbf{D}_A = \mathrm{diag}\begin{bmatrix} 11 & -2 & 0 & 3 \\ -2 & 16 & 5 & -1 \\ 0 & 5 & 11 & 2 \\ 3 & -1 & 2 & 14 \end{bmatrix}, \quad \mathbf{M}_A = \mathrm{diag}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{3.33}$$

subjected to initial displacements $\mathbf{q}_A(0) = \begin{bmatrix} 3 & 9 & 0 & 0 \end{bmatrix}^T$ and zero initial velocities. Forces may be applied at all locations and all the physical displacements are available for observation.

One finds the SPT-modal equations of motion to be

$$\begin{aligned}
\ddot{\mathbf{q}}_{B1_spt} &+ 9.4823\,\dot{\mathbf{q}}_{B1_spt} + 47.478\,\mathbf{q}_{B1_spt} = \mathbf{f}_{B1_spt} \\
\ddot{\mathbf{q}}_{B2_spt} &+ 7.4889\,\dot{\mathbf{q}}_{B2_spt} + 82.711\,\mathbf{q}_{B2_spt} = \mathbf{f}_{B2_spt} \\
\ddot{\mathbf{q}}_{B3_spt} &+ 14.434\,\dot{\mathbf{q}}_{B3_spt} + 10.168\,\mathbf{q}_{B3_spt} = \mathbf{f}_{B3_spt} \\
\ddot{\mathbf{q}}_{B4_spt} &+ 20.595\,\dot{\mathbf{q}}_{B4_spt} + 78.887\,\mathbf{q}_{B4_spt} = \mathbf{f}_{B4_spt}
\end{aligned} \tag{3.34}$$

and the IMSC modal equations of motion to be

$$\begin{aligned}
\ddot{\mathbf{q}}_{B1_imsc} &+ 14\,\dot{\mathbf{q}}_{B1_imsc} + 10\,\mathbf{q}_{B1_imsc} = \mathbf{f}_{B1_imsc} \\
\ddot{\mathbf{q}}_{B2_imsc} &+ 11\,\dot{\mathbf{q}}_{B2_imsc} + 50\,\mathbf{q}_{B2_imsc} = \mathbf{f}_{B2_imsc} \\
\ddot{\mathbf{q}}_{B3_imsc} &+ 16\,\dot{\mathbf{q}}_{B3_imsc} + 70\,\mathbf{q}_{B3_imsc} = \mathbf{f}_{B3_imsc} \\
\ddot{\mathbf{q}}_{B4_imsc} &+ 11\,\dot{\mathbf{q}}_{B4_imsc} + 90\,\mathbf{q}_{B4_imsc} = \mathbf{f}_{B4_imsc}
\end{aligned} \tag{3.35}$$

It is decided for illustrative purposes to remove the modal damping of the third mode-pair, $\mathbf{q}_{B3\_spt}$ and $\mathbf{q}_{B3\_imsc}$, and set the natural frequency of this mode-pair to 10 Hz. The SPT method is compared directly with the conventional IMSC method and conclusions drawn. For the respective methods one finds the modal controller matrices to be

$$\mathbf{f}_{B3\_spt} = -14.434\,\dot{\mathbf{q}}_{B3\_spt} + 3937.7\,\mathbf{q}_{B3\_spt} \tag{3.36}$$

$$\mathbf{f}_{B3\_imsc} = -16\,\dot{\mathbf{q}}_{B3\_imsc} + 3877.8\,\mathbf{q}_{B3\_imsc} \tag{3.37}$$

Figures 3.1 and 3.2 illustrate the physical and modal displacements of the system when the SPT-modal controller is off and on respectively. As may be observed from the comparison between the modal responses in figures 3.1 and 3.2 the third mode-pair is successfully made undamped with a natural frequency of 10 Hz. The modes are completely decoupled and the controller only effects mode-pair 3 and leaves the remaining modes unaltered. The physical effect of the controller may be observed from the physical displacements illustrated in these figures and the obvious effect is to make the system borderline stable, i.e. neither stable nor unstable.

Figures 3.3 and 3.4 illustrates the IMSC controller applied to the non-classically damped system. As may be observed the third mode-pair is again made undamped with a natural frequency of 10 Hz. However as may be observed the modal responses of the other modes is also effected. This is due to the coupling in the damping matrix which cannot be made diagonal using the undamped eigenvectors of the system. Thus the IMSC method does not allow the decoupling of the system matrices.

This numerical example may be simple but it does illustrate the advantage of the SPT method over the conventional IMSC method due to the fact that all three system matrices may be decoupled regardless of the structure of the damping. One may also observe that this results in different mode-shapes. The mode-shapes of the SPT method no longer match the undamped mode-shapes of the system as the IMSC modes do. However the SPT-modes do represent physically meaningful quantities and this is observed immediately in the physical response of the system in figure 3.2 when the SPT controller is on.

3.6 Numerical Example 3.2: Rotor System

A finite element model of a rotor-disc system is considered with four degrees of freedom at each node (2 translational, 2 torsional) and the limitation that $r = n$ is now lifted. The rotor-disc system is illustrated in Fig. 3.5.

The system is constructed from steel with Young's modulus, $E = 200$ GPa and density $\rho = 7800$ kg/m^3. The model is split into 13 equal-length elements of 0.1m and the discs have dimensions given in table 3.1.

The bearings at each end of the rotor system are deliberately anisotropic with stiffness and damping properties given in table 3.2.

Control forces can be applied at node 8 in the x and y-directions and similarly the displacements in the x-direction at this node are observed. For computational ease Guyan reduction [G12] is used to reduce the model to 6 degrees of freedom. The system is operated at 2,500 rpm and the uncontrolled response is illustrated in figure 3.6.

Modal control dictates that each actuator controls individual mode-pairs of vibration resulting in the number of mode-pairs to be controlled the same as the number of actuators available. The model allows for 2 mode-pairs to be controlled. It is decided to control the first two mode-pairs of vibration since these dominate the system response.

The single degree of freedom systems corresponding to the first two mode-pairs in modal space are

$$\ddot{\mathbf{q}}_{B1} + 0.37850\,\dot{\mathbf{q}}_{B1} + 1.4467 \times 10^5\,\mathbf{q}_{B1} = \mathbf{f}_{B1} \tag{3.38}$$

$$\ddot{\mathbf{q}}_{B2} + 0.32708\,\dot{\mathbf{q}}_{B2} + 1.5772 \times 10^5\,\mathbf{q}_{B2} = \mathbf{f}_{B2} \tag{3.39}$$

Optimal control is used to minimise the modal kinetic and potential energies such that controller gains are

$$\mathbf{G}_k = \begin{bmatrix} 4.999913 & 0 & 0 & \cdots & 0 \\ 0 & 4.999921 & 0 & \cdots & 0 \end{bmatrix} \tag{3.40}$$

$$\mathbf{G}_d = \begin{bmatrix} 4.1096 & 0 & 0 & \cdots & 0 \\ 0 & 4.1570 & 0 & \cdots & 0 \end{bmatrix} \tag{3.41}$$

The response of the system with the controller on is illustrated in Fig. 3.7.

As expected the response of the system decays much faster than that for the uncontrolled system with the displacement converging to zero much more rapidly. This is due to targeting the first two mode-pairs of vibration of the system which dominate the system response. The modal control technique is indeed successfully applied to bring the system under control.

3.7 Control Spillover

The SPT-based modal control method has been introduced with the restriction that the forcing vector $\mathbf{f}_A$ has been assumed to be a n-dimensional vector. This describes a system in which actuators are available at all modelled locations and consequently all modelled modes of vibration may be controlled. For practical reasons this is unrealistic for real world situations due to restrictions on where actuators may be located.

An infinity of modes exists for any real structure although only a finite number are ever modelled. As stated, usually only a subset of the modelled modes are controlled due to restrictions on the number of actuators available for control. This raises the possibility of control spillover in which the un-modelled and uncontrolled modes are excited by the control forces. This phenomenon is introduced here for the SPT-based modal control method.

Recalling the inverse filter from equation (3.32) one may separate the filter into controlled, residual (uncontrolled) and un-modelled segments denoted by the subscripts c, r, u respectively.

$$\begin{bmatrix} \mathbf{f}_{Bc} \\ \mathbf{f}_{Br} \\ \mathbf{f}_{Bu} \end{bmatrix} = \begin{bmatrix} \mathbf{V}_{0cc} + s\mathbf{V}_{1cc} & \mathbf{V}_{0cr} + s\mathbf{V}_{1cr} & \mathbf{V}_{0cu} + s\mathbf{V}_{1cu} \\ \mathbf{V}_{0rc} + s\mathbf{V}_{1rc} & \mathbf{V}_{0rr} + s\mathbf{V}_{1rr} & \mathbf{V}_{0ru} + s\mathbf{V}_{1ru} \\ \mathbf{V}_{0uc} + s\mathbf{V}_{1uc} & \mathbf{V}_{0ur} + s\mathbf{V}_{1ur} & \mathbf{V}_{0uu} + s\mathbf{V}_{1uu} \end{bmatrix} \begin{bmatrix} \mathbf{f}_c \\ \mathbf{f}_r \\ \mathbf{f}_u \end{bmatrix} \tag{3.42}$$

where s is the Laplace variable.

One must now realise the physical constraint $\mathbf{f}_r = \mathbf{f}_u = 0$ must necessarily be true such that equation (3.42) becomes

$$\mathbf{f}_{Bc} = \mathbf{V}_{0cc}\,\mathbf{f}_A + \mathbf{V}_{1cc}\,\dot{\mathbf{f}}_A \tag{3.43}$$

$$\mathbf{f}_{Br} = \mathbf{V}_{0rc}\,\mathbf{f}_A + \mathbf{V}_{1rc}\,\dot{\mathbf{f}}_A \tag{3.44}$$

$$\mathbf{f}_{Bu} = \mathbf{V}_{0uc}\,\mathbf{f}_A + \mathbf{V}_{1uc}\,\dot{\mathbf{f}}_A \tag{3.45}$$

From equation (3.43) one now finds the physical force to be

$$\dot{\mathbf{f}}_A = \mathbf{V}_{1cc}^{-1}\,(\,\mathbf{f}_{Bc} - \mathbf{V}_{0cc}\,\mathbf{f}_A\,) \tag{3.46}$$

From analysis of equations (3.44) and (3.45) one may observe that a relationship exists between the physical force and the residual and un-modelled modal forces. Thus it may be seen that the applied physical force generates modal forces applied to excite the uncontrolled and un-modelled modes. This defines what the control spillover problem is.

Meirovitch and Baruh [M5] showed that control spillover for the conventional second order modal control method does not cause instability in the un-modelled and uncontrolled modes (see section 2.10.3). One may follow the same process for the SPT modal control method to show that the effects of control spillover are non-destabilising for the closed loop uncontrolled and un-modelled poles.

Recalling that for the second order equations of motion one may decouple the equations of motion using the diagonalising SPT-filters such that from

$$\left(\mathbf{V}_0 + s\,\mathbf{V}_1\right)\left(s^2\mathbf{M}_A + s\,\mathbf{D}_A + \mathbf{K}_A\right)\left(\mathbf{U}_0 + s\,\mathbf{U}_1\right)^{-1}\mathbf{q}_B(s) = \left(\mathbf{V}_0 + s\,\mathbf{V}_1\right)\,\mathbf{f}_A(s) \qquad (3.47)$$

one obtains the diagonalised system which may be compartmentalised into controlled, residual and un-modelled components

$$\begin{bmatrix} (s^2\mathbf{M}_{Bc} + s\,\mathbf{D}_{Bc} + \mathbf{K}_{Bc}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (s^2\mathbf{M}_{Br} + s\,\mathbf{D}_{Br} + \mathbf{K}_{Br}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (s^2\mathbf{M}_{Bu} + s\,\mathbf{D}_{Bu} + \mathbf{K}_{Bu}) \end{bmatrix} \begin{bmatrix} \mathbf{q}_{Bc} \\ \mathbf{q}_{Br} \\ \mathbf{q}_{Bu} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{Bc} \\ \mathbf{f}_{Br} \\ \mathbf{f}_{Bu} \end{bmatrix} \qquad (3.48)$$

A modal controller is used that is dependent only on the controlled degrees of freedom such that the modal controller is of the form

$$\mathbf{f}_{Bc} = \left(\mathbf{G}_k + s\,\mathbf{G}_d\right)\,\mathbf{q}_{Bc} \qquad (3.49)$$

Recalling equations (3.43)-(3.45) one may substitute the definition of the modal controller into equation (3.48) to give

$$\begin{bmatrix} s^2\left(\mathbf{M}_{Bc} - \mathbf{V}_{1cc}\mathbf{G}_d\right) + s\left(\mathbf{D}_{Bc} - \mathbf{V}_{1cc}\mathbf{G}_k - \mathbf{V}_{0cc}\mathbf{G}_d\right) + \left(\mathbf{K}_{Bc} - \mathbf{V}_{0cc}\mathbf{G}_k\right) & \mathbf{0} & \mathbf{0} \\ -\left(s^2\mathbf{V}_{1rc}\mathbf{G}_d + s\left(\mathbf{V}_{1rc}\mathbf{G}_k + \mathbf{V}_{0rc}\mathbf{G}_d\right) + \mathbf{V}_{0rc}\mathbf{G}_k\right) & s^2\mathbf{M}_{Br} + s\mathbf{D}_{Br} + \mathbf{K}_{Br} & \mathbf{0} \\ -\left(s^2\mathbf{V}_{1uc}\mathbf{G}_d + s\left(\mathbf{V}_{urc}\mathbf{G}_k + \mathbf{V}_{0uc}\mathbf{G}_d\right) + \mathbf{V}_{0uc}\mathbf{G}_k\right) & \mathbf{0} & s^2\mathbf{M}_{Bu} + s\mathbf{D}_{Bu} + \mathbf{K}_{Bu} \end{bmatrix} \begin{bmatrix} \mathbf{q}_{Bc} \\ \mathbf{q}_{Br} \\ \mathbf{q}_{Bu} \end{bmatrix} = \mathbf{0} \qquad (3.50)$$

The determinant of a triangular matrix is equal to product of the diagonal entries. Thus one may observe from equation (3.50) that the effects of the control spillover on the uncontrolled and un-modelled modes does not alter the diagonal entries on the system matrices hence their closed loop poles remain unaltered. Therefore one can conclude that the effect of control spillover cannot destabilise these modes. This is a analogous result to that obtained by Meirovitch and Baruh for the conventional modal control problem [M5].

The control spillover problem for the conventional second order IMSC technique has been tackled by several authors such as Baz *et al.* [B1] and Fang *et al.* [F1]. The approach described by Fang *et al.* is inappropriate for the first order form of the filters presented for the SPT method. The approach introduced by Baz *et al.* illustrates the ability to control several mode-pairs through a sharing strategy in which the actuators can switch between the mode-pairs they control. This offers a potential solution for control spillover in the SPT modal control approach although initial investigations report that the strategy destabilises the SPT filters.

The effect of control spillover may degrade the performance of the controlled system and it is difficult to immediately quantify the effect control spillover has on a system. A numerical example now demonstrates the effects of control spillover.

3.8 Numerical Example 3.3: Spillover

Consider a simple, non-classically damped 2×2 system with equations of motion

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ddot{\mathbf{q}}_A + \begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix} \dot{\mathbf{q}}_A + \begin{bmatrix} 20 & -10 \\ -10 & 20 \end{bmatrix} \mathbf{q}_A = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbf{f}_A \tag{3.51}$$

The modal controller gains applied to the first mode of vibration are arbitrarily decided to be

$$\mathbf{G}_k = \begin{bmatrix} 20 & 0 \end{bmatrix} \quad , \quad \mathbf{G}_d = \begin{bmatrix} 7.5 & 0 \end{bmatrix} \tag{3.52}$$

Figure 3.8 illustrates the response of the system to initial conditions and Figure 3.9 illustrates the response of the system when the controller is applied. The intent of the controller was to bring the response of mode 1 under control and leave mode 2 uncontrolled. As may be seen from Figure 3.9 the response of mode 1 is successfully brought under control as it settles to zero much faster than in Figure 3.8. However, comparing the response of mode 2 in Figures 3.8 and 3.9 respectively shows that the response of mode

2 is actually greater in the controlled case than the uncontrolled case. Thus it may be concluded that the response of mode 2 is actually excited by spillover.

3.9 Discrete Force Smoothing

The modal control method described in this chapter is implemented using continuous time. For practical systems the control methods must necessarily be implemented discretely thus resulting in instantaneous incremental changes in the applied control forces. The concept of independent modal control is that the control forces are designed in the modal space before being converted into the physical domain. The SPT-based modal filter used to convert the modal force to the physical force results in both the control force and time derivative of control force being extracted from the modal force as may be observed from the definition of the left filter

$$\dot{\mathbf{f}}_A = {\mathbf{V}_1}^{-1} \left(\mathbf{f}_B - \mathbf{V}_0\, \mathbf{f}_A\right) \tag{3.53}$$

Since one obtains the rate of control force as well as the control force from the left filter it yields the possibility to smooth out the applied physical force through interpolation. Indeed through using knowledge of the rate of change of force one may construct a projected path over the finite increment at which no controller action is calculated. This is demonstrated numerically.

3.10 Numerical Example 3.4 - Smoothing Discrete Force

The previous numerical example is used such that the system is described by equation (3.51) and the modal controller gains defined by equation (3.52). A coarse time step of 0.05s is used by the controller where the minimum required time constant as defined by the Nyquist criterion [G8] is 0.63410s. Whilst this choice of time step is too coarse for practical application it serves well to illustrate the method. A cost function of the form

$$J = \int_0^{t_f} \mathbf{q}_A^T \, \mathbf{q}_A + \dot{\mathbf{q}}_A^T \, \dot{\mathbf{q}}_A + \mathbf{f}_A^T \, \mathbf{f}_A \;\; dt \tag{3.54}$$

is adopted to measure the expense of the discrete and smoothed controllers from zero time to final time t_f. The cost function measures the system displacement and velocity responses as well as the applied control. The applied forces of the two controllers are shown in Figure 3.10. The discrete controller yields a cost of 1903 versus 1349 for the smoothed controller. Thus the smoothed controller cost is approximately 71% of the discrete controller representing a massive saving in cost.

3.11 Summary

Theoretical application of structure preserving transformation based modal control has been presented in this chapter. The SPT modal control method is not subjected to constraints on the structure of the damping matrix as conventional second order modal control methods are. This means that one may independently control modes of vibration for non-classically damped systems. This chapter has provided numerical examples to justify this assertion.

The SPT modal control method has its origins in utilising the SPTs to simultaneously diagonalise the second order system mass, damping and stiffness matrices. As demonstrated one may use knowledge of the system eigenvalues to construct the diagonalised system matrices. One may then use the eigenvectors of the original and diagonalised state space systems to construct the diagonalising SPTs.

Because the diagonalising SPTs are applied to state space representations of the second order system one must use first order modal filters to extract the modal co-ordinates from the physical co-ordinates. Conversely one must use an inverse modal filter to revert the modal force into a physical force and consequently true independent control of the system modes is realised.

The inverse modal filter used to transform the modal force into physical force results in both the force and time derivative of the force being available. For practical application of control methods one must implement the control system in discrete time. Thus one has the opportunity to smooth the discrete force though interpolation using knowledge of the rate of change of force. A numerical example was used to illustrate that this may actually result in a more efficient controller with a lower associated cost of implementing the control. This is an advantage of the SPT modal control method over the conventional modal control methods.

One unfortunate consequence of any modal control method is that one can only control as many pairs of modes as actuators available. This is because each actuator is designated to control one pair of modes. This means that the system is subjected to control spillover in which the applied control forces may excite the uncontrolled and un-modelled modes. It has been shown in this chapter that control spillover cannot destabilise the system for the SPT modal control method but may degrade the system performance adversely. Control spillover is an unfortunate side effect of the conventional modal control method as well as the SPT based method.

3.12 Conclusions

This chapter has presented a method to apply modal control to a general non-defective second order system of no specific structure. More specifically the method presented does not insist that the system be 'classically damped' such that the undamped eigenvectors of the system be used to simultaneously diagonalise all three system matrices. It has been demonstrated how the SPTs may be used to diagonalise the system matrices of a generally damped system. Direct application of the diagonalising SPTs decouples the unforced equations of motion. However, the diagonalised system is still coupled externally via the forcing. Utilising the notion of independent modal space control permits the determination of the control force independently in the modal space before reverting the modal force back into the physical domain. This permits one to target individual mode-pairs of vibration for controlled obtained from non-classically damped systems.

SPT-based modal filters have been presented to show the relationship between the physical and modal systems. This allows the extraction of the modal displacements from the physical displacements and velocities. The modal velocities may be correspondingly obtained through knowledge of the physical velocities and accelerations. Frequently one does not possess knowledge of all the system states or physical accelerations. Correspondingly one must use techniques such as Kalman filters [B8] to reconstruct the system state (displacements and velocities). However one may not directly reform the system accelerations required for SPT-modal velocities which may potentially pose a problem for practical application.

The potential implications of control spillover in which the uncontrolled modes of vibration are excited by the applied control forces has been highlighted. It has been demonstrated through numerical example that the phenomenon does indeed exist and has the potential to detract from the performance of the control system. This is indeed something that requires further study.

Utilising the SPT-based filters to revert from the modal force back into the physical domain results in the both the control force and rate of control force being obtained. This presents the opportunity to smooth the physical force through interpolation if the modal control method is implemented in discrete time. Practical applications require discrete time implementation using digital equipment thus the opportunity arises to exploit the smoothing potential of the SPT-based filters. Numerical example has shown that the smoothing of the force can actually decrease a cost function which measures the effect of applied control and system response. Thus SPT-modal control may offer additional benefits over conventional modal control methods.

Disc	Disc 1	Disc 2	Disc 3
Node	3	6	11
Thickness (m)	0.05	0.05	0.06
Inner diameter (m)	0.10	0.10	0.10
Outer diameter (m)	0.24	0.40	0.40

Table 3.1: Numerical Example 3.2 Disc Properties

.

Bearing	Bearing 1	Bearing 2
Stiffness K_{xx} (MN/m)	50	50
Stiffness K_{yy} (MN/m)	70	70
Damping D_{xx} (N/m/s)	500	500
Damping D_{yy} (N/m/s)	700	700

Table 3.2: Numerical Example 3.2 Bearing Properties

.

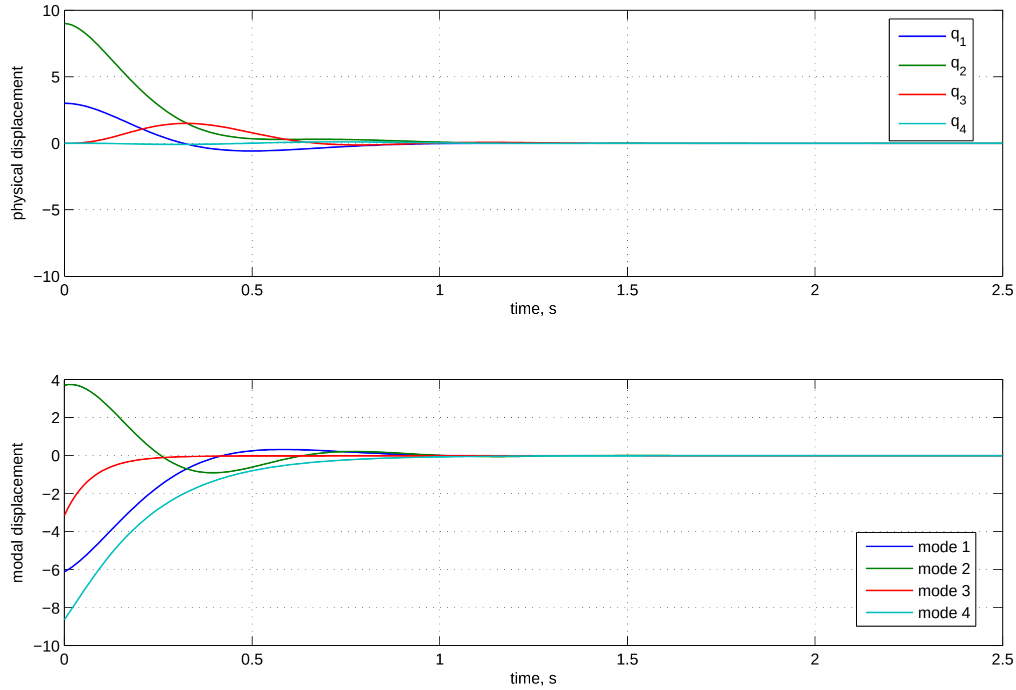


Figure 3.1: Numerical example 3.1 - Uncontrolled physical and modal responses, SPT method

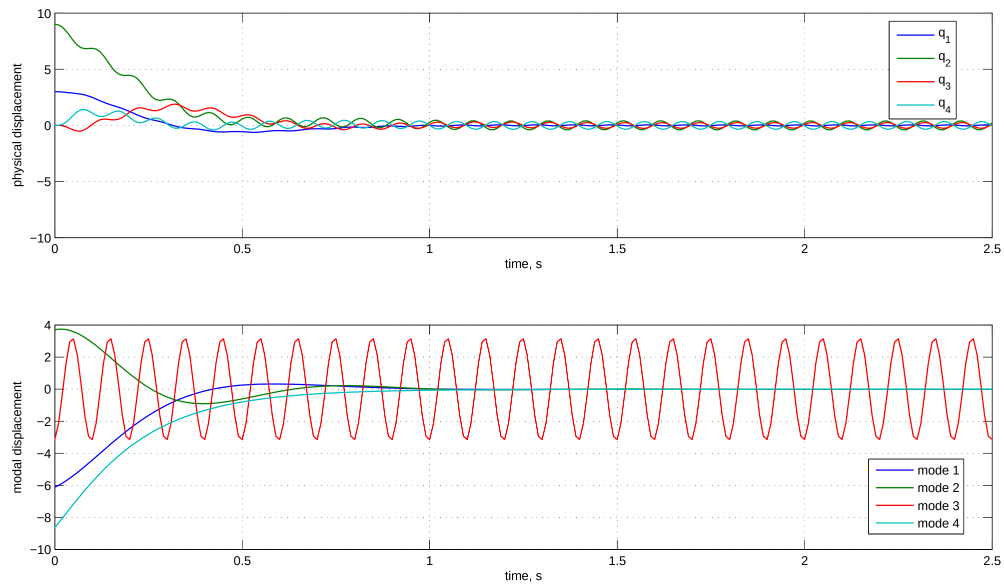


Figure 3.2: Numerical example 3.1 - Controlled physical and modal responses, SPT method

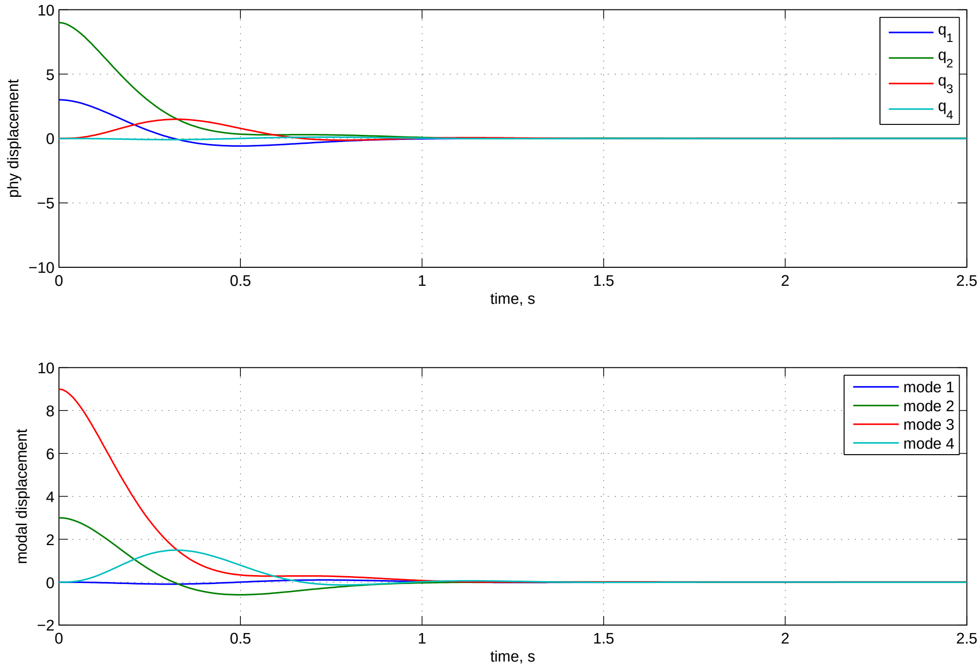


Figure 3.3: Numerical example 3.1 - Uncontrolled physical and modal responses, IMSC method

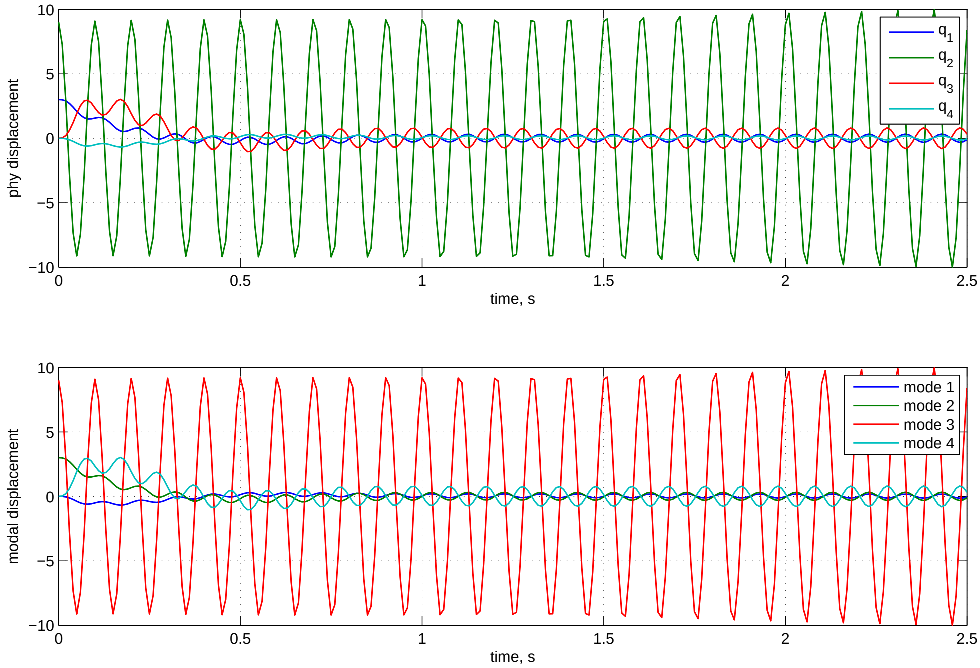


Figure 3.4: Numerical example 3.1 - Controlled physical and modal responses, IMSC method

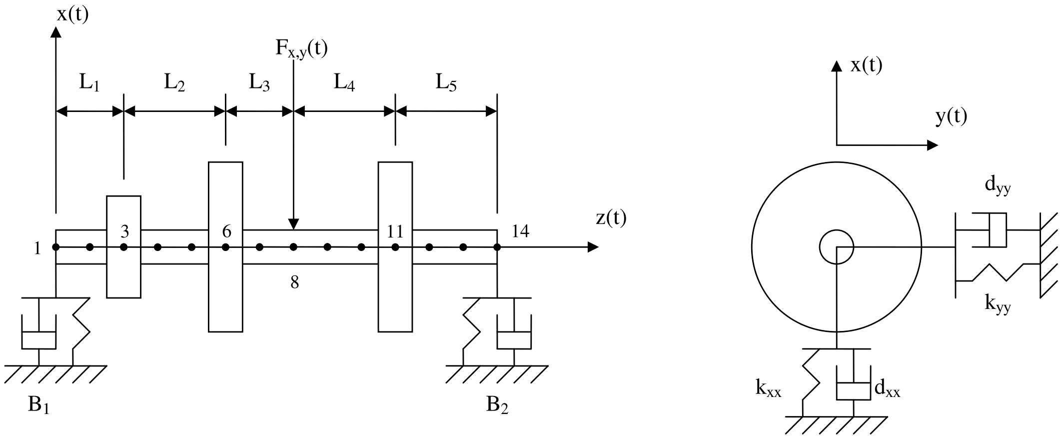


Figure 3.5: Numerical example 3.2 - Rotor-Disc system

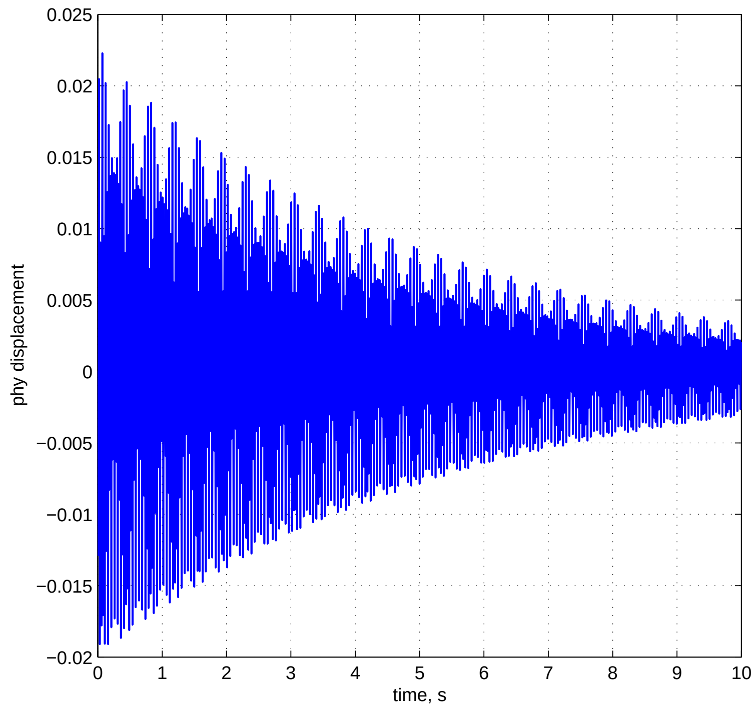


Figure 3.6: Numerical example 3.2 - SPT response to initial conditions: control off

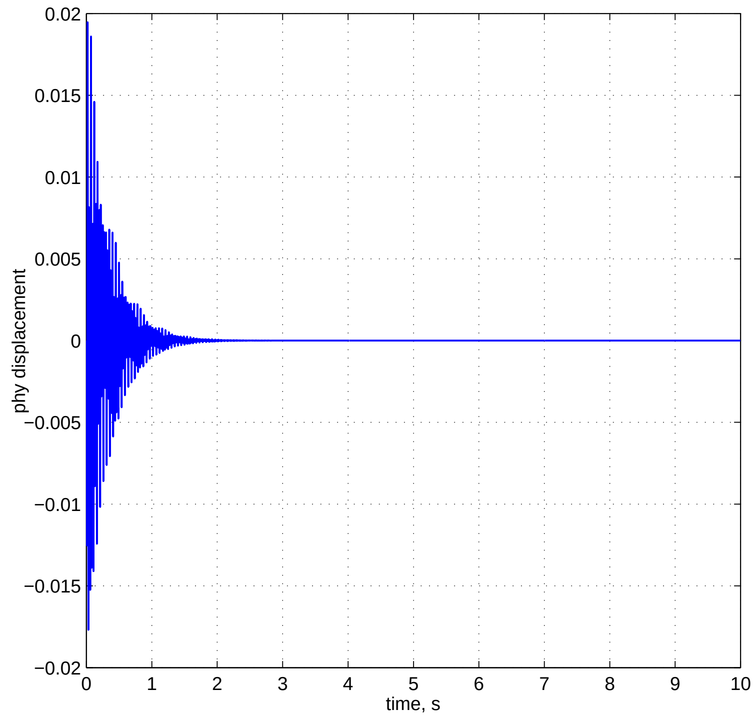


Figure 3.7: Numerical example 3.2 - SPT response to initial conditions: control on

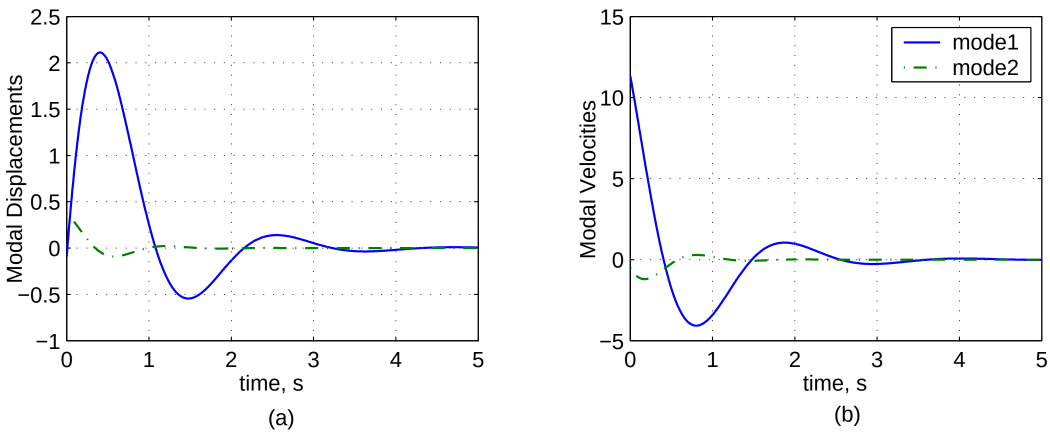


Figure 3.8: Numerical example 3.3 - Modal response of 2×2 system - controller off

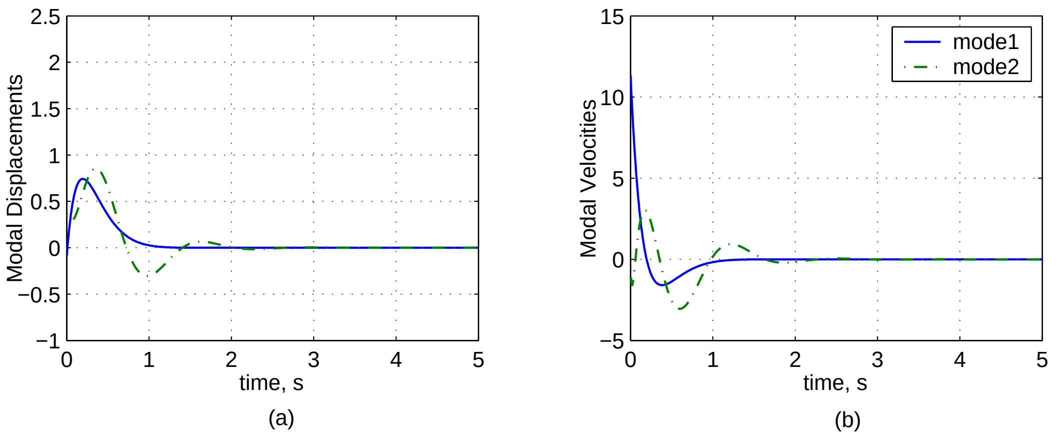


Figure 3.9: Numerical example 3.3 - Modal response of 2 × 2 system - controller on

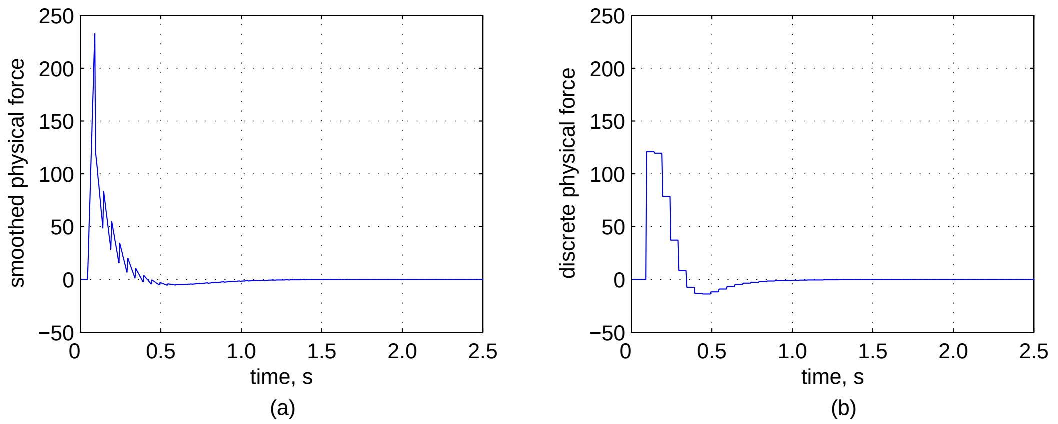


Figure 3.10: Numerical example 3.4 - (a) Smoothed and (b) discrete force

Chapter 4

Reflexive SPTs and Stable Filters

Consider the general second order equations of motion of a dynamic system

$$\mathbf{M}_A\,\ddot{\mathbf{q}}_A(t) + \mathbf{D}_A\,\dot{\mathbf{q}}_A(t) + \mathbf{K}_A\,\mathbf{q}_A(t) = \mathbf{S}_A\,\mathbf{u}_A(t) = \mathbf{f}_A(t) \tag{4.1}$$

Here the vector $\mathbf{q}_A(t) \in \mathbb{R}^n$ represents the generalised displacements of the system and the dot above represents derivative with respect to time. The matrices $\mathbf{M}_A, \mathbf{D}_A, \mathbf{K}_A \in \mathbb{R}^{n\times n}$ represent the system mass, damping and stiffness matrices respectively with the subscript A representing the original untransformed matrices. The vector $\mathbf{u}_A(t) \in \mathbb{R}^r$ represents the short vector of applied physical forces and $\mathbf{S}_A \in \mathbb{R}^{n\times r}$ represents a matrix defining the locations of the applied forces. The generalised forcing is denoted by the long vector $\mathbf{f}_A(t) \in \mathbb{R}^{n\times r}$ for simplicity.

For many practical problems such as model reduction or control it is often preferable to transform the original equations of motion into a more desirable form. Conventional transformations available to second order systems consist of linear transformations in which the system matrices are pre and post-multiplied by left ($\mathbf{T}_L$) and right ($\mathbf{T}_R$) transformations such that new transformed equations of motion are obtained. Thus for the second order system already introduced one may make the substitution $\mathbf{q}_A(t) = \mathbf{T}_R\,\mathbf{q}_B(t)$ and pre-multiply by $\mathbf{T}_L^T$ such that from

$$\mathbf{T}_L^T\,\mathbf{M}_A\,\mathbf{T}_R\,\ddot{\mathbf{q}}_B + \mathbf{T}_L^T\,\mathbf{D}_A\,\mathbf{T}_R\,\dot{\mathbf{q}}_B + \mathbf{T}_L^T\,\mathbf{K}_A\,\mathbf{T}_R\,\mathbf{q}_B = \mathbf{T}_L^T\,\mathbf{f}_A \tag{4.2}$$

ones finds the transformed system

$$\mathbf{M}_B\,\ddot{\mathbf{q}}_B + \mathbf{D}_B\,\dot{\mathbf{q}}_B + \mathbf{K}_B\,\mathbf{q}_B = \mathbf{f}_B \tag{4.3}$$

Thus the original co-ordinate basis $\mathbf{q}_A, \mathbf{f}_A$ is transformed into $\mathbf{q}_B, \mathbf{f}_B$. For ease of reading the notation showing the variables as a function of time has been removed.

Conventional transformations as illustrated above have the limitation that the $3n^2$ space spanned by the independent system matrices exceeds the $2n^2$ space available for manipulation by the transformations. Thus one may not fully manipulate the system matrices as desired. A notable example of this is the inability to diagonalise all three system matrices when the system is non-classically damped [P1]. This pitfall may be overcome by making special use of certain state space representations of the second order system such that one may fully manipulate all three system matrices. This is the notion of 'structure preserving transformations' (SPTs) [G2, G3].

For the conventional transformations illustrated in equation (4.2) one may readily discover the relationship between the the original and transformed co-ordinates vectors. However for the state space representations used by the SPTs this becomes less obvious and indeed one finds that first order filters are involved where constant matrices had previously served. This raises fundamental questions regarding stability of these filters. Thus one may summarise the ambitions of the chapter to be derivation of SPT-based filters and introduction of a method by which one may stabilise initially unstable filters.

4.1 Structure Preserving Transformations

As stated before it is possible to introduce two unique state-space representations of the second order equations of motion introduced in equation (4.1) such that

$$\begin{bmatrix} \mathbf{K}_A & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_A \end{bmatrix} \begin{bmatrix} \mathbf{q}_{A1} \\ \mathbf{q}_{A2} \end{bmatrix} - \begin{bmatrix} -\mathbf{D}_A & -\mathbf{M}_A \\ -\mathbf{M}_A & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}}_{A1} \\ \dot{\mathbf{q}}_{A2} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_A \\ \mathbf{0} \end{bmatrix} \tag{4.4}$$

$$\begin{bmatrix} \mathbf{0} & \mathbf{K}_A \\ \mathbf{K}_A & \mathbf{D}_A \end{bmatrix} \begin{bmatrix} \mathbf{q}_{A1} \\ \mathbf{q}_{A2} \end{bmatrix} - \begin{bmatrix} \mathbf{K}_A & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_A \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}}_{A1} \\ \dot{\mathbf{q}}_{A2} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_A \end{bmatrix} \tag{4.5}$$

where $\mathbf{q}_{A1} = \mathbf{q}_A$ and $\mathbf{q}_{A2} = \dot{\mathbf{q}}_{A1}$.

One may simplify equations (4.4) and (4.5) such that one has

$$\underline{\mathbf{A}_1}\,\underline{\mathbf{q}_A} - \underline{\mathbf{A}_0}\,\underline{\dot{\mathbf{q}}_A} \;=\; \underline{\mathbf{f}_{A1}} \tag{4.6}$$

$$\underline{\mathbf{A}_2}\,\underline{\mathbf{q}_A} - \underline{\mathbf{A}_1}\,\underline{\dot{\mathbf{q}}_A} \;=\; \underline{\mathbf{f}_{A2}} \tag{4.7}$$

The matrices $\underline{\mathbf{A}_0}$, $\underline{\mathbf{A}_1}$ and $\underline{\mathbf{A}_2}$ are referred to as the Lancaster Augmented Matrices (LAMs) as introduced in section 2.2 and the underlined notation is used again here to differentiate the $2n$-dimensional quantities from the n-dimensional quantities. Equations (4.6) and (4.7) may be generalised to

$$\underline{\mathbf{A}_k}\,\underline{\mathbf{q}_A} - \underline{\mathbf{A}_{k-1}}\,\underline{\dot{\mathbf{q}}_A} = \underline{\mathbf{f}_{Ak}} \qquad k = 1, 2 \tag{4.8}$$

The ‘structure preserving transformations’ (SPTs) [G2, G3] developed by Garvey *et al.* are a real set of co-ordinate transformations applied to the LAMs such that the eigenvalues contained of the LAMs and the structure within the LAMs is preserved. The SPTs consist of a left ($\underline{\mathbf{T}_L}$) and right ($\underline{\mathbf{T}_R}$) transformation applied such that

$$\underline{\mathbf{B}_k} = \underline{\mathbf{T}_L}^T\,\underline{\mathbf{A}_k}\,\underline{\mathbf{T}_R} \qquad k = 0, 1, 2 \tag{4.9}$$

The SPTs can be shown to have the structure

$$\underline{\mathbf{T}_L} \;=\; \begin{bmatrix} \mathbf{F}_L - \frac{1}{2}\,\mathbf{G}_L\,\mathbf{D}_A^T & -\mathbf{G}_L\,\mathbf{M}_A^T \\ \mathbf{G}_L\,\mathbf{K}_A^T & \mathbf{F}_L + \frac{1}{2}\,\mathbf{G}_L\,\mathbf{D}_A^T \end{bmatrix}^{-1} . \tag{4.10}$$

$$\underline{\mathbf{T}_R} \;=\; \begin{bmatrix} \mathbf{F}_R - \frac{1}{2}\,\mathbf{G}_R\,\mathbf{D}_A & -\mathbf{G}_R\,\mathbf{M}_A \\ \mathbf{G}_R\,\mathbf{K}_A & \mathbf{F}_R + \frac{1}{2}\,\mathbf{G}_R\,\mathbf{D}_A \end{bmatrix}^{-1} \tag{4.11}$$

where $\mathbf{F}_L, \mathbf{F}_R, \mathbf{G}_L, \mathbf{G}_R \in \mathbb{R}^{n\times n}$ are arbitrary pre-defined matrices subject to the necessary constraint

$$\mathbf{F}_R\,\mathbf{G}_L^T + \mathbf{G}_R\,\mathbf{F}_L^T = 0 \tag{4.12}$$

The effect of an SPT is to transform equations (4.4) and (4.5) for the original system into the form

$$\begin{bmatrix} \mathbf{K}_B & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_B \end{bmatrix} \begin{bmatrix} \mathbf{q}_{B1} \\ \mathbf{q}_{B2} \end{bmatrix} - \begin{bmatrix} -\mathbf{D}_B & -\mathbf{M}_B \\ -\mathbf{M}_B & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}}_{B1} \\ \dot{\mathbf{q}}_{B2} \end{bmatrix} = \underline{\mathbf{T}_L}^T \begin{bmatrix} \mathbf{f}_A \\ \mathbf{0} \end{bmatrix} \tag{4.13}$$

$$\begin{bmatrix} \mathbf{0} & \mathbf{K}_B \\ \mathbf{K}_B & \mathbf{D}_B \end{bmatrix} \begin{bmatrix} \mathbf{q}_{B1} \\ \mathbf{q}_{B2} \end{bmatrix} - \begin{bmatrix} \mathbf{K}_B & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_B \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}}_{B1} \\ \dot{\mathbf{q}}_{B2} \end{bmatrix} = \underline{\mathbf{T}_L}^T \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_A \end{bmatrix} \tag{4.14}$$

One may directly extract the new real system matrices $\mathbf{M}_B, \mathbf{D}_B, \mathbf{K}_B$ from the new set of LAMs $\underline{\mathbf{B}_k}$ $(k = 0, 1, 2)$ to form the new second order equations of motion

$$\mathbf{M}_B\, \ddot{\mathbf{q}}_B + \mathbf{D}_B\, \dot{\mathbf{q}}_B + \mathbf{K}_B\, \mathbf{q}_B = \mathbf{f}_B \tag{4.15}$$

However one may note that the lower half of the state vector $\underline{\mathbf{q}_B}$ is no longer the derivative of the top half, $\mathbf{q}_{B2} \neq \dot{\mathbf{q}}_{B1}$.

The SPT is said to be diagonalising if matrices of the transformed system, $\mathbf{M}_B$, $\mathbf{D}_B$, $\mathbf{K}_B$, are all diagonal. This diagonal system is not unique. For systems where $\mathbf{M}_A$ is invertible and where the transformation matrices are invertible, any two of the equations [H4] are sufficient to ensure that the third is also satisfied. The eigenvalues of the original system, $\mathbf{K}_A, \mathbf{D}_A, \mathbf{M}_A$, can be computed by solving the generalised eigenvalue problem involving either $\underline{\mathbf{A}_0}, \underline{\mathbf{A}_1}$ or $\underline{\mathbf{A}_1}, \underline{\mathbf{A}_2}$. Evidently, the eigenvalues of the new system $\mathbf{K}_B, \mathbf{D}_B, \mathbf{M}_B$ will be identical provided that $\underline{\mathbf{T}_L}, \underline{\mathbf{T}_R}$ are both invertible. Houlston *et al.* [H5] have shown how to obtain diagonalising transformation matrices $\underline{\mathbf{T}_L}, \underline{\mathbf{T}_R}$ from the complex modal data.

Due to the now diagonal nature of equation (4.15) one refers to the new vectors $\mathbf{q}_B$ and $\mathbf{f}_B$ as the *modal* displacements and forces respectively since they comprise of the system modal data. It now becomes necessary to define how one obtains the modal quantities $\mathbf{q}_B, \mathbf{f}_B$ from $\mathbf{q}_A, \mathbf{f}_A$? For this modal filters are required and their derivation is given in the next section.

4.2 Modal Filters

This section shows how the modal contributions may be extracted from the state space system given in equation (4.8). The derivation of the SPT-based modal filters is presented now [L2]. For an alternative mechanical derivation of the SPT-based filters please see appendix C. Correspondingly the un-forced second order equations of motion from equation (4.1) may be written as a matrix polynomial in operator form

$$\mathbf{L}_A(\tau) = \tau^2\mathbf{M}_A + \tau\,\mathbf{D}_A + \mathbf{K}_A \tag{4.16}$$

where the definition $\tau \equiv \dfrac{d}{dt}$ is made.

For the case $k = 1$ equation (4.8) yields the two forms

$$\left(\underline{\mathbf{A}_1} - \tau\underline{\mathbf{A}_0}\right)\begin{bmatrix}\mathbf{I}\\ \tau\mathbf{I}\end{bmatrix} = \begin{bmatrix}\mathbf{L}_A(\tau)\\ \mathbf{0}\end{bmatrix} \tag{4.17}$$

$$\begin{bmatrix}\mathbf{I} & \tau\mathbf{I}\end{bmatrix}\left(\underline{\mathbf{A}_1} - \tau\underline{\mathbf{A}_0}\right) = \begin{bmatrix}\mathbf{L}_A(\tau) & \mathbf{0}\end{bmatrix} \tag{4.18}$$

where $\mathbf{I} \in \mathbb{R}^{n\times n}$ is the identity operator.

The SPTs are used to transform the original LAMs $\underline{\mathbf{A}_k}$ to the new block diagonalised LAMs $\underline{\mathbf{B}_k}$. Thus the matrix polynomial for the diagonalised system may be given as

$$\mathbf{L}_B(\tau) = \tau^2\mathbf{M}_B + \tau\mathbf{D}_B + \mathbf{K}_B \tag{4.19}$$

It may be observed that equation (4.9) produces the result

$$\underline{\mathbf{T}_L}^T\left(\underline{\mathbf{A}_1} - \tau\underline{\mathbf{A}_0}\right) = \left(\underline{\mathbf{B}_1} - \tau\underline{\mathbf{B}_0}\right)\underline{\mathbf{T}_R}^{-1} \tag{4.20}$$

Thus one may form

$$\begin{bmatrix}\mathbf{I} & \tau\mathbf{I}\end{bmatrix}\underline{\mathbf{T}_L}^T\left(\underline{\mathbf{A}_1} - \tau\underline{\mathbf{A}_0}\right)\begin{bmatrix}\mathbf{I}\\ \tau\mathbf{I}\end{bmatrix} = \begin{bmatrix}\mathbf{I} & \tau\mathbf{I}\end{bmatrix}\left(\underline{\mathbf{B}_1} - \tau\underline{\mathbf{B}_0}\right)\underline{\mathbf{T}_R}^{-1}\begin{bmatrix}\mathbf{I}\\ \tau\mathbf{I}\end{bmatrix} \tag{4.21}$$

and substituting using equations (4.17) and (4.18) gives

$$\begin{bmatrix}\mathbf{I} & \tau\mathbf{I}\end{bmatrix}\underline{\mathbf{T}_L}^T\begin{bmatrix}\mathbf{L}_A(\tau)\\ \mathbf{0}\end{bmatrix} = \begin{bmatrix}\mathbf{L}_B(\tau) & \mathbf{0}\end{bmatrix}\underline{\mathbf{T}_R}^{-1}\begin{bmatrix}\mathbf{I}\\ \tau\mathbf{I}\end{bmatrix} \tag{4.22}$$

The left and right polynomials, $\mathbf{H}_L(\tau)$ and $\mathbf{H}_R(\tau)$ respectively, may be accordingly defined

$$\mathbf{H}_L(\tau) = \begin{bmatrix}\mathbf{I} & \tau\mathbf{I}\end{bmatrix}\underline{\mathbf{T}_L}^T\begin{bmatrix}\mathbf{I}\\ \mathbf{0}\end{bmatrix} \tag{4.23}$$

$$\mathbf{H}_R(\tau) = \begin{bmatrix}\mathbf{I} & \mathbf{0}\end{bmatrix}\underline{\mathbf{T}_R}^{-1}\begin{bmatrix}\mathbf{I}\\ \tau\mathbf{I}\end{bmatrix} \tag{4.24}$$

such that

$$\mathbf{H}_L(\tau)\,\mathbf{L}_0(\tau) = \mathbf{L}_1(\tau)\,\mathbf{H}_R(\tau) \tag{4.25}$$

One may simplify equations (4.23) and (4.24) to

$$\mathbf{H}_L(\tau) = \mathbf{V}_0 + \tau\mathbf{V}_1 \tag{4.26}$$

$$\mathbf{H}_R(\tau) = \mathbf{U}_0 + \tau\mathbf{U}_1 \tag{4.27}$$

with obvious definitions

$$\begin{bmatrix} \mathbf{V}_0^T & \mathbf{V}_1^T \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \underline{\mathbf{T}_L} \tag{4.28}$$

$$\begin{bmatrix} \mathbf{U}_0 & \mathbf{U}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \underline{\mathbf{T}_R}^{-1} \tag{4.29}$$

Returning to equation (4.1) one may observe that the left and right polynomials may be used to transform the equations of motion. Letting $\mathbf{q}_A = (\mathbf{U}_0 + \tau\mathbf{U}_1)^{-1}\,\mathbf{q}_B$ and pre-multiplying by $(\mathbf{V}_0 + \tau\mathbf{V}_1)$ yields

$$(\mathbf{V}_0 + \tau\mathbf{V}_1)\left(\tau^2\mathbf{M}_A + \tau\mathbf{D}_A + \mathbf{K}_A\right)(\mathbf{U}_0 + \tau\mathbf{U}_1)^{-1}\,\mathbf{q}_B = (\mathbf{V}_0 + \tau\mathbf{V}_1)\,\mathbf{f}_A \tag{4.30}$$

to give

$$\left(\tau^2\mathbf{M}_B + \tau\mathbf{D}_B + \mathbf{K}_B\right)\,\mathbf{q}_B = \mathbf{f}_B \tag{4.31}$$

The relationship between the old $(\mathbf{q}_A, \mathbf{f}_A)$ and the new coordinate sets $(\mathbf{q}_B, \mathbf{f}_B)$ may thus be defined

$$\mathbf{q}_B = (\,\mathbf{U}_0 + \tau\mathbf{U}_1\,)\;\mathbf{q}_A \tag{4.32}$$

$$\mathbf{f}_B = (\,\mathbf{V}_0 + \tau\mathbf{V}_1\,)\;\mathbf{f}_A \tag{4.33}$$

One also finds that the filter matrices have the property

$$\det(\,\mathbf{U}_0 + \tau\,\mathbf{U}_1\,) = 0 \;\Leftrightarrow\; \det(\,\mathbf{V}_0 + \tau\,\mathbf{V}_1\,) = 0 \tag{4.34}$$

With knowledge of the physical accelerations one may additionally obtain the modal velocities

$$\tau\mathbf{q}_B = \left(\,\tau\mathbf{U}_0 + \tau^2\mathbf{U}_1\,\right)\;\mathbf{q}_A \tag{4.35}$$

From the definition of the filters presented one may observe that only half the information contained within the transformation matrices need be used.

$$\underbrace{\left[\begin{array}{c|c} \mathbf{V}_0^T & \mathbf{V}_1^T \\ \hline \vdots & \vdots \end{array}\right]}_{\mathbf{T}_L} \quad , \quad \underbrace{\left[\begin{array}{c|c} \mathbf{U}_0 & \mathbf{U}_1 \\ \hline \vdots & \vdots \end{array}\right]}_{\mathbf{T}_R^{-1}} \tag{4.36}$$

One may find an equally valid definition of the SPT filters using the lower half of the transformation matrices. Only the given definition of the filters is considered in this thesis although all possible definitions are equally valid.

4.2.1 Stability of Filters

The previous section derived the relationship between original and transformed co-ordinate basis through the use of filters. From the definitions of the filters given in equations (4.32) and (4.33) one must realise a necessary requirement for usage of the filters is for them to be stable. Accordingly one may state that the real components of the eigenvalues of the filters must be stable. Since the SPT-based filters preserve the eigenvalues of the second order system then it is apparent that the left and right SPT-filters must have the same spectrum. This may be observed by recalling equation (4.25)

$$(\mathbf{V}_0 + \tau\mathbf{V}_1)\left(\tau^2\mathbf{M}_A + \tau\mathbf{D}_A + \mathbf{K}_A\right) = \left(\tau^2\mathbf{M}_A + \tau\mathbf{D}_A + \mathbf{K}_A\right)(\mathbf{U}_0 + \tau\mathbf{U}_1) \tag{4.37}$$

where $\tau \equiv \frac{d}{dt}$.

The eigenvalues of the left and right sides of equation (4.37) are equal hence the contributions of the filters must cancel. Thus $\mathbf{U}_1^{-1}\mathbf{U}_0 = \mathbf{V}_1^{-1}\mathbf{V}_0 > 0$ must be necessarily true in order that the filters will be stable. Since the diagonalising SPTs introduced are non-unique an infinity of different SPTs may be used to diagonalise the system matrices. A subset of this infinity of diagonalising SPTs may yield pairs of stable filters; one may use the concept of *reflexive* SPTs to exploit this possibility.

4.3 Reflexive Transformations

One now introduces the concept of *reflexive* SPTs to exploit the non-uniqueness of the diagonalising SPTs. A reflexive SPT is a non-trivial SPT which maps any LAM back onto itself such that one has

$$\underline{\mathbf{A}_k} = \underline{\mathbf{S}_L}^T \underline{\mathbf{A}_k}\, \underline{\mathbf{S}_R} \tag{4.38}$$

The matrices $\underline{\mathbf{S}_L}, \underline{\mathbf{S}_R} \in \mathbb{R}^{2n\times 2n}$ correspond to the left and right reflexive SPTs respectively. The section shows how to construct the reflexive SPTs for diagonal systems such that one may effectively change the original diagonalising SPT non-trivially.

It is found that the 'single degree of freedom' (SDOF) systems generated from the diagonalising SPTs may be mapped back precisely onto themselves by a reflexive SPT which is not the identity. Establishing the LAMs for a SDOF system

$$\underline{\mathbf{a}_0} = \begin{bmatrix} 0 & k_j \\ k_j & d_j \end{bmatrix}, \quad \underline{\mathbf{a}_1} = \begin{bmatrix} k_j & 0 \\ 0 & -m_j \end{bmatrix}, \quad \underline{\mathbf{a}_2} = \begin{bmatrix} -d_j & -m_j \\ -m_j & 0 \end{bmatrix} \tag{4.39}$$

one may define the 2×2 reflexive SPT for the SDOF system from equations (4.10) and (4.11) to be

$$\underline{\mathbf{s}_L} \;=\; \begin{bmatrix} f+\frac{1}{2}g\,d_j & g\,m_j \\ -g\,k_j & f-\frac{1}{2}g\,d_j \end{bmatrix} \tag{4.40}$$

$$\underline{\mathbf{s}_R} \;=\; \begin{bmatrix} f-\frac{1}{2}g\,d_j & -g\,m_j \\ g\,k_j & f+\frac{1}{2}g\,d_j \end{bmatrix} \tag{4.41}$$

$j = 1, 2, \cdots, n.$

k_j, d_j and m_j are the j^{th} SDOF stiffness, damping and mass values obtained from the diagonal system matrices $\mathbf{K}_B, \mathbf{D}_B$ and $\mathbf{M}_B$ and f, g are arbitrary scalars subject to the constraint

$$\det(\underline{\mathbf{s}_L}) = \det(\underline{\mathbf{s}_R}) = 1 \tag{4.42}$$

or equivalently

$$f^2 + g^2\left(k_j\,m_j - \frac{1}{4}d_j^2\right) = 1 \tag{4.43}$$

Constraint (4.43) ensures that a one to one mapping is retained for the scalar system values. If the constraint is not upheld then although equations (4.40) and (4.41) still define a valid SPT the scaling of the values contained within the SDOF LAMs are not retained. Ensuring that the constraint is upheld ensures that the values within the SDOF LAMs are identical before and after the SDOF SPTs are applied.

It is worth noting that the structure of the SDOF reflexive SPTs defined in equations (4.40) and (4.41) do not involve an inverse as was required for the conventional SPT structure. One may realise that the inverse is unnecessary because one may define the SPTs equally valid as

$$\underline{\mathbf{s}_L}^{-T}\underline{\mathbf{b}_j}\,\underline{\mathbf{s}_R}^{-1} = \underline{\mathbf{a}_j} \tag{4.44}$$

since $\underline{\mathbf{a}_j} = \underline{\mathbf{b}_j}$.

If any one diagonalising SPT has been found for a system having n degrees of freedom it is clear that this actually represents a space of diagonalising SPTs of dimension $2n$ since a further reflexive SPT may be applied independently to each SDOF system individually after diagonalisation. Thus one may form the $2n$-dimensional reflexive SPT, $\underline{\mathbf{S}_L}, \underline{\mathbf{S}_R} \in \mathbb{R}^{2n\times 2n}$, such that

$$\underline{\mathbf{S}_L}^T\,\underline{\mathbf{B}_k}\,\underline{\mathbf{S}_R} = \underline{\mathbf{B}_k} \tag{4.45}$$

The reflexive SPTs are applied to change the diagonalising SPTs such that

$$\underline{\mathbf{T}_L}' = \underline{\mathbf{T}_L}\,\underline{\mathbf{S}_L} \quad , \quad \underline{\mathbf{T}_R}' = \underline{\mathbf{T}_R}\,\underline{\mathbf{S}_R} \tag{4.46}$$

(The superscript $'$ represents a change to the original matrix and not a (conjugate) transpose.)

Only the left filter is dealt with from this point onwards since ensuring stability of the left filter necessarily ensures stability of the right filter. Using equation (4.46) and recalling the definition of the left filter from equation (4.28) one may find the new left filter matrices to be

$$\mathbf{V}_0{}' = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \underline{\mathbf{S}_L}^T \begin{bmatrix} \mathbf{V}_0 \\ \mathbf{V}_1 \end{bmatrix} \tag{4.47}$$

$$\mathbf{V}_1{}' = \begin{bmatrix} \mathbf{0} & \mathbf{I} \end{bmatrix} \underline{\mathbf{S}_L}^T \begin{bmatrix} \mathbf{V}_0 \\ \mathbf{V}_1 \end{bmatrix} \tag{4.48}$$

4.3.1 Structure of Reflexive SPTs

The purpose of the reflexive SPTs is to alter the eigenvalues of the pair of filter matrices $\{\mathbf{V}_0, \mathbf{V}_1\}$. However the utilisation of the reflexive SPTs causes the eigenvalues to change non-linearly. By operating over a small range one may linearise the change in filter eigenvalues to find the change of the eigenvalues with respect to the reflexive SPT parameters.

Referring to the structure of the SDOF reflexive SPTs defined in equations (4.40) and (4.41) one may define the left and right $2n$-dimensional reflexive SPTs to be

$$\underline{\mathbf{S}_L} = \begin{bmatrix} (\mathbf{I} + \sigma\mathbf{N}) + \frac{1}{2}\sigma\mathbf{L}\,\mathbf{D}_B & \sigma\mathbf{L}\,\mathbf{M}_B \\ -\sigma\mathbf{L}\,\mathbf{K}_B & (\mathbf{I} + \sigma\mathbf{N}) - \frac{1}{2}\sigma\mathbf{L}\,\mathbf{D}_B \end{bmatrix} \tag{4.49}$$

$$\underline{\mathbf{S}_R} = \begin{bmatrix} (\mathbf{I} + \sigma\mathbf{N}) - \frac{1}{2}\sigma\mathbf{L}\,\mathbf{D}_B & -\sigma\mathbf{L}\,\mathbf{M}_B \\ \sigma\mathbf{L}\,\mathbf{K}_B & (\mathbf{I} + \sigma\mathbf{N}) + \frac{1}{2}\sigma\mathbf{L}\,\mathbf{D}_B \end{bmatrix} \tag{4.50}$$

$\mathbf{N}, \mathbf{L} \in \mathbb{R}^{n\times n}$ are diagonal matrices specified by the user to transform the SPTs and the parameter σ determines the magnitude of change made.

4.4 Eigenvalue Derivative

The eigenvalues of the pair of SPT-filter matrices vary non-linear with respect to the construction parameters. Thus the relationship between the construction parameters of the reflexive SPTs and the eigenvalues of the transformed filter matrices become very complicated. One possible solution to this predicament is to assume that the eigenvalues of the filter matrices alter linearly over a finite range such that one may move the eigenvalues into a more stable region. This process may be continued until all the eigenvalues of the diagonalising filters become stable.

One may find the change of the j^{th} eigenvalue λ_j of the pair of filter matrices $\{\mathbf{V}_0, \mathbf{V}_1\}$ with respect to parameter σ. The subscript notation $,\sigma$ is used to denote differentiation with respect to parameter σ, for example, $\frac{\partial \mathbf{V}_1}{\partial \sigma} \equiv \mathbf{V}_{1,\sigma}$. The standard eigenvalue derivative equation is then

$$\lambda_{j,\sigma} = -\mathbf{v}_j^T \left(\mathbf{V}_{0,\sigma} + \lambda_j \mathbf{V}_{1,\sigma}\right) \mathbf{u}_j \qquad j = 1, \cdots, n \tag{4.51}$$

Here, $\mathbf{v}_j, \mathbf{u}_j$ represent the left and right eigenvectors corresponding to eigenvalue λ_j of the filter matrices scaled such that $\mathbf{v}_j^T\, \mathbf{V}_1\, \mathbf{u}_j = 1$. The derivation of equation (4.51) may be found in appendix D and is based on the paper by Adhikari and Friswell [A1].

Equation (4.51) may be generalised for all n eigenvalues to give

$$\mathbf{e}_{,\,\sigma} = \mathrm{diag}\,\left(\mathbf{\Phi}_L^T\, \mathbf{V}_{0,\sigma}\, \mathbf{\Phi}_R + \mathbf{\Phi}_L^T\, \mathbf{V}_{1,\sigma}\, \mathbf{\Phi}_R\, \mathbf{e}\right) \tag{4.52}$$

where

$$\begin{aligned} \mathbf{\Phi}_L &= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}, & \mathbf{e} &= \mathrm{diag}\begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{bmatrix} \\ \mathbf{\Phi}_R &= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix}, & \mathbf{e}_{,\,\sigma} &= \mathrm{diag}\begin{bmatrix} \dfrac{\partial \lambda_1}{\partial \sigma} & \dfrac{\partial \lambda_2}{\partial \sigma} & \cdots & \dfrac{\partial \lambda_n}{\partial \sigma} \end{bmatrix} \end{aligned} \tag{4.53}$$

One may substitute knowledge from equations (4.47) and (4.48) into equation (4.52) to yield

$$\mathbf{e}_{,\,\sigma} = \mathrm{diag}\left(\begin{bmatrix} \mathbf{\Phi}_L^T & \mathbf{0} \end{bmatrix} \underline{\mathbf{S}_{L,\,\sigma}}^T \begin{bmatrix} \mathbf{V}_0 \\ \mathbf{V}_1 \end{bmatrix} \mathbf{\Phi}_R + \begin{bmatrix} \mathbf{0} & \mathbf{\Phi}_L^T \end{bmatrix} \underline{\mathbf{S}_{L,\,\sigma}}^T \begin{bmatrix} \mathbf{V}_0 \\ \mathbf{V}_1 \end{bmatrix} \mathbf{\Phi}_R\, \mathbf{e}\right) \tag{4.54}$$

Taking the transpose of equation (4.54) and grouping together terms yields

$$\mathbf{e}_{,\,\sigma}^T = \mathbf{e}_{,\,\sigma} = \mathrm{diag}\left(\mathbf{\Phi}_R^T \begin{bmatrix} \mathbf{V}_0^T & \mathbf{V}_1^T \end{bmatrix} \underline{\mathbf{S}_{L,\,\sigma}} \begin{bmatrix} \mathbf{\Phi}_L \\ \mathbf{\Phi}_L\, \mathbf{e} \end{bmatrix}\right) \tag{4.55}$$

Thus one may form

$$\mathbf{e}_{,\,\sigma} = \mathrm{diag}\left(\mathbf{W}\, \underline{\mathbf{S}_{L,\,\sigma}}\, \mathbf{X}\right) \tag{4.56}$$

where the definitions for $\mathbf{W}$ and $\mathbf{X}$ are obvious from (4.55). One may readily show that the derivative of the reflexive SPT with respect to σ must be

$$\underline{\mathbf{S}_{L,\,\sigma}} = \begin{bmatrix} \mathbf{N} + \frac{1}{2}\mathbf{L}\,\mathbf{D}_B & \mathbf{L}\,\mathbf{M}_B \\ -\mathbf{L}\,\mathbf{K}_B & \mathbf{N} - \frac{1}{2}\mathbf{L}\,\mathbf{D}_B \end{bmatrix} \tag{4.57}$$

Extracting the definition of $\underline{\mathbf{S}_{L,\,\sigma}}$ one may redefine equation (4.56) as

$$\mathbf{e}_{,\,\sigma} = \begin{bmatrix} \mathbf{Y} & \mathbf{Z} \end{bmatrix} \begin{bmatrix} \operatorname{diag}(\mathbf{N}) \\ \operatorname{diag}(\mathbf{L}) \end{bmatrix} \tag{4.58}$$

with $\mathbf{Y}$ and $\mathbf{Z}$ constructed element-wise as

$$\mathbf{Y}(j,k) \quad = \quad \mathbf{W}(j,k)\mathbf{X}(k,j) + \mathbf{W}(j,k+n)\mathbf{X}(k+n,j) \tag{4.59}$$

$$\begin{aligned} \mathbf{Z}(j,k) \quad &= \quad \left(\frac{1}{2}\mathbf{D}_B(k,k)\mathbf{W}(j,k) - \mathbf{K}_B(k,k)\mathbf{W}(j,k+n)\right)\mathbf{X}(k,j) \\ &- \quad \left(\frac{1}{2}\mathbf{D}_B(k,k)\mathbf{W}(j,k+n) - \mathbf{M}_B(k,k)\mathbf{W}(j,k)\right)\mathbf{X}(k+n,j) \end{aligned} \tag{4.60}$$

4.4.1 Linear Flows

So far the eigenvalue derivative technique has been presented in a discrete, iterative context. This means that one specifies the magnitude of travel using σ and direction of travel using $\mathbf{N}$ and $\mathbf{L}$. One may consider the situation where the magnitude of travel tends to zero ($\sigma \to 0$) such that a continuous process emerges. This represents the notion of *flows* where the filter eigenvalues are ‘flowed’ into the desired location rather than discontinuously pushed. An example of using *flows* may be found in the notion of *isospectral flows* by Garvey *et al.* [G5] where one *flows* the parameters of the original SPT to alter the structure of the second order matrices towards a desired form.

Thus one may flow the reflexive SPTs, and correspondingly the SPT-filters, through integrating the derivative of the reflexive SPTs

$$\underline{\mathbf{S}_L}(\sigma) = \int \underline{\mathbf{S}_{L,\,\sigma}}(\sigma)\ \ d\sigma \tag{4.61}$$

Equation (4.61) allows one to integrate the filter matrices over a specified time period representing a smooth, continuous flow of the eigenvalues of the filter.

4.4.2 Limitations of the Eigenvalue Derivative

Before one approaches the task of determining the reflexive parameters $\mathbf{N}$ and $\mathbf{L}$ it is appropriate to highlight some of the limitations of the eigenvalue derivative method.

The eigenvalue derivative method assumes that the eigenvalues change approximately linearly over a finite range. This approximation introduces inaccuracies when moving the eigenvalues potentially leading to significant errors should one choose an inappropriate range over which to move the eigenvalues. Fortunately the method utilised in this project uses the integration functions provided in Matlab to move the eigenvalues. The functions used automatically adjust the integration step in accordance with the linearised range.

A further assumption made is that the eigenvectors remain unaltered over this finite range. Indeed this is only an approximation and the eigenvectors do change over the finite linearised range adding further. The paper by Adhikari and Friswell [A1] present eigenvector derivatives as well the eigenvalue derivatives already introduced. However it was found during the implementation of the method that calculating the eigenvector derivatives required more computational effort than actually solving the eigenvector problem using the inbuilt Matlab functions. This is a constraint with the Matlab programming environment rather than the code developed by the author.

4.5 Choosing the Reflexive Parameters

Numerical trials suggest that one is much more likely to obtain an unstable filter than a stable one from a diagonalising transformation. This means that one will probably start with an initially unstable filter and attempt to move the eigenvalues into a stable region. The previous section has shown how the eigenvalues of the filter varied with respect to the diagonalised reflexive parameters $\mathbf{N}$ and $\mathbf{L}$. One now considers how to select the parameters $\mathbf{N}, \mathbf{L}$ such that the reflexive SPT has the effect of making the diagonalising SPT more stable.

Simplifying equation (4.58) gives

$$\mathbf{p} = \mathbf{C}\,\mathbf{z} \tag{4.62}$$

where the definitions of $\mathbf{C}$ and $\mathbf{z}$ are apparent and $\mathbf{p} = \mathbf{e}_{,\sigma}$.

There exist two obvious methods by which one may move the eigenvalues into more stable regions. The first method is to solve equation (4.62) to give the $\mathbf{N}$ and $\mathbf{L}$ parameters assuming one knows how they wish to alter the eigenvalues such that $\mathbf{p}$ is known. The second method is to determine the direction of travel in which the eigenvalues move in the stable direction

$$\mathbf{C}\,\mathbf{z} > \mathbf{0} \tag{4.63}$$

where $> \mathbf{0}$ means that the $2n$ entries in the vector result on the left hand side of equation (4.63) are greater than zero in this instance. The stability criterion of the SPT-based filters utilises reverse convention when using the eigenvalues to define stability. In the case of the SPT-filters one defines the real parts of the eigenvalues to be positive rather than negative for stability to be ensured. This reverse convention arises from the definition of the left inverse SPT-filter. Rearranging equation (4.33) one finds the left inverse filter to be

$$\dot{\mathbf{f}}_A = -\mathbf{V}_1^{-1}\left(\mathbf{V}_0\,\mathbf{f}_A - \mathbf{f}_B\right) \tag{4.64}$$

One may observe that the eigenvalues of the inverse filter in equation (4.64) are defined by $-\mathbf{V}_1^{-1}\mathbf{V}_0$. However the negative sign has been neglected to define that stability is determined when talking about $\mathbf{V}_1^{-1}\mathbf{V}_0$ thus the sign of the eigenvalues is reversed and positive real parts in the system eigenvalues are required such that the response will decay to zero.

The two methods outlined in this thesis to determine parameters $\mathbf{N}$ and $\mathbf{L}$ are referred to as the direct and gradient methods respectively. One can consider both methods to have relative strengths and weaknesses and the choice of the method depends on which the user feels will provide the best solution.

The direct solution has the advantage that over an appropriately defined linear range one can specify the the change of eigenvalues, $\mathbf{e}_{,\sigma}$, and find the reflexive parameters which will give this change. The disadvantage is how to optimise the choice of $\mathbf{e}_{,\sigma}$ such that one stays within the linear range whilst maximising the change at each stage.

The gradient method does not require the user to specify the change in eigenvalues required thus one less parameter is required. The determination of the parameters is numerically less intensive than the direct solution which for larger values n may offer a computational advantage. However the gradient solution by itself does not determine the optimum direction in which to change the eigenvalues only the direction in which the eigenvalues will definitely result in more stable filter. Thus one requires to determine the steepest gradient which may require relatively many more iterations to achieve the same eigenvalue locations as the direct solution. Therefore the computation cost saving of the gradient method may not be realised.

Both methods are now presented.

4.5.1 Direct Solution

Recalling the definition of equation (4.62) one may make the substitution

$$\mathbf{z} = \mathbf{C}^T \mathbf{g} \tag{4.65}$$

such that one has

$$\mathbf{p} = \mathbf{C}\,\mathbf{C}^T \mathbf{g} \tag{4.66}$$

Correspondingly one may solve for $\mathbf{g}$.

$$\mathbf{g} = \left(\mathbf{C}\,\mathbf{C}^T\right)^{-1} \mathbf{p} \tag{4.67}$$

Substituting the value of $\mathbf{g}$ back into equation (4.65) one finds

$$\mathbf{z} = \mathbf{C}^T \left(\mathbf{C}\,\mathbf{C}^T\right)^{-1} \mathbf{p} \tag{4.68}$$

Recalling the definitions of $\mathbf{C}$, $\mathbf{z}$ and $\mathbf{p}$ one finds

$$\begin{bmatrix} \operatorname{diag}(\mathbf{N}) \\ \operatorname{diag}(\mathbf{L}) \end{bmatrix} = \begin{bmatrix} \mathbf{Y}^T \\ \mathbf{Z}^T \end{bmatrix} \left(\mathbf{Y}\,\mathbf{Y}^T + \mathbf{Z}\,\mathbf{Z}^T\right)^{-1} \mathbf{e}_{,\sigma} \tag{4.69}$$

A necessary requirement is for the matrix $\begin{bmatrix} \mathbf{Y} & \mathbf{Z} \end{bmatrix}$ to be full rank.

4.5.2 Steepest Gradient Solution

Recalling equation (4.63) one may make the substitution

$$\mathbf{z} = \mathbf{C}^T \left(\mathbf{C}\,\mathbf{C}^T\right)^{-1} \mathbf{g} \tag{4.70}$$

such that one has the new inequality

$$\begin{aligned} \mathbf{C}\,\mathbf{C}^T \left(\mathbf{C}\,\mathbf{C}^T\right)^{-1} \mathbf{g} &> 0 & (4.71) \\ \mathbf{g} &> 0 & (4.72) \end{aligned}$$

Thus for any positive $\mathbf{g}$ will permit the equality in equation (4.72) to be satisfied. Correspondingly, solving for equation (4.70) gives the required solution.

It would advantageous if one could determine the direction of the steepest gradient such as to maximise the change of eigenvalues at each stage. Thus one would be able to move in the direction in which the rate of change of eigenvalues with respect to σ was greatest. This would have the obvious effect of maximising the change at each step.

Defining a quadratic cost function which is determined by the distance of the filter eigenvalues from a desired location

$$J_e = (\mathbf{e} - \underline{\boldsymbol{\delta}}\,)^H\,(\mathbf{e} - \underline{\boldsymbol{\delta}}\,) \tag{4.73}$$

Here the vector $\underline{\boldsymbol{\delta}} \in \mathbb{R}^{n\times 1}$ is occupied by a constant real user specified value such that all entries are identical and the superscript H defines the hermitian (conjugate) transpose

One may differentiate the cost function (4.73) with respect to the reflexive movement parameter σ to obtain

$$\frac{\partial J_e}{\partial \sigma} = 2\,(\mathbf{e} - \underline{\boldsymbol{\delta}}\,)^H\,\frac{\partial\,(\mathbf{e} - \underline{\boldsymbol{\delta}}\,)}{\partial \sigma} \tag{4.74}$$

Realising that $\underline{\boldsymbol{\delta}}$ is constant and reverting to existing notation one has

$$J_{e,\,\sigma} = 2\,(\mathbf{e} - \underline{\boldsymbol{\delta}}\,)^H\;\mathbf{e}_{,\,\sigma} \tag{4.75}$$

Thus substituting the definition of $\mathbf{e}_{,\,\sigma}$ from equation (4.58) into equation (4.75) yields

$$J_{e,\,\sigma} = 2\,(\mathbf{e} - \underline{\boldsymbol{\delta}}\,)^H\;\mathbf{C}\,\mathbf{z} = \mathbf{E}^T\,\mathbf{z} \tag{4.76}$$

where the definition of $\mathbf{E}$ is apparent.

One wishes to maximise the extent of each change to the filter eigenvalues using the reflexive parameters. It is therefore required to find and move in the direction of steepest gradient with regards to the eigenvalue derivative. This requires that one maximises

$$\mathbf{X} = \frac{\mathbf{E}^T\mathbf{z}}{\sqrt{\mathbf{z}^T\mathbf{z}}} \tag{4.77}$$

Maximising $\mathbf{X}$ is obtained by differentiating with respect to $\mathbf{z}$ and setting $\mathbf{X}_{,\mathbf{z}} = 0$. From this one finds that the steepest gradient is obtained when $\mathbf{z} = \mathbf{E}$. This gives the $\mathbf{N}$ and $\mathbf{L}$ parameters as

$$\mathbf{N} = \mathrm{diag}\,(\,\mathbf{E}(1)\,) \tag{4.78}$$

$$\mathbf{L} = \mathrm{diag}\,(\,\mathbf{E}(2)\,) \tag{4.79}$$

where the brackets refer to $n \times 1$ segment of vector $\mathbf{E}$.

4.6 Numerical Examples

The preceding sections have demonstrated how to determine the parameters $\mathbf{N}$ and $\mathbf{L}$ of the reflexive SPT such that one may *flow* the eigenvalues of the filter in a stable direction. It has been shown that one may always make the eigenvalues more stable over a finite range. However one should consider the possibility that the change in eigenvalues may go asymptotic such that the change in eigenvalues tends towards zero. This means that the eigenvalues will never cross into a stable region even though they are being made increasingly stable. Numerical experimentation suggests that this is not the case but this remains to be proved formally.

One now introduces numerical examples to demonstrate the methods presented in this chapter.

4.6.1 Numerical Example 4.1 - Direct Method

A non-classically damped second order system is generated with mass, damping and stiffness matrices

$$\mathbf{M}_A = \text{diag}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{D}_A = \begin{bmatrix} 5.0 & 3.0 & 6.8 \\ -3.0 & 1.9 & 3.0 \\ -6.8 & -3.0 & 5.4 \end{bmatrix}, \quad \mathbf{K}_A = \text{diag}\begin{bmatrix} 50 \\ 70 \\ 90 \end{bmatrix} \tag{4.80}$$

The left and right diagonalising SPT transformations are found which yield unstable filters. The starting eigenvalues of the SPT-filters are given in table 4.1 and the SPT transformations matrices may be reported to be

$$\underline{\mathbf{T}_L} = \begin{bmatrix} -0.31222 & 0.18625 & 0.7055 & -0.11608 & -0.014316 & 0.023817 \\ -0.21191 & -0.34496 & 0.36451 & -0.027467 & 0.094153 & 0.024569 \\ -0.31124 & 0.27861 & -0.29911 & 0.020399 & -0.038622 & 0.057803 \\ 2.8801 & 1.001 & -4.3248 & 0.0075105 & 0.22459 & 0.54194 \\ 0.68146 & -6.5833 & -4.4613 & -0.13626 & -0.59716 & 0.1958 \\ -0.5061 & 2.7005 & -10.496 & -0.36743 & 0.38206 & -0.69605 \end{bmatrix} \tag{4.81}$$

and

$$\underline{\mathbf{T}_R} = \begin{bmatrix} 0.038836 & 0.35321 & 0.5673 & 0.13975 & 0.037269 & -0.024664 \\ -0.2861 & -0.57954 & -0.12697 & -0.0088132 & -0.092151 & -0.032701 \\ -0.38832 & 0.20464 & -0.68062 & -0.036465 & 0.013133 & -0.052799 \\ -3.4672 & -2.6059 & 4.4786 & -0.34608 & 0.25338 & 0.73666 \\ 0.21866 & 6.4433 & 5.9379 & -0.26183 & -0.33271 & 0.09759 \\ 0.90471 & -0.91829 & 9.5874 & -0.28788 & 0.16946 & -0.31805 \end{bmatrix} \tag{4.82}$$

These SPTs yield the diagonal system matrices

$$\mathbf{M}_B = \text{diag}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{D}_B = \text{diag}\begin{bmatrix} 2.7543 \\ 2.6786 \\ 6.8671 \end{bmatrix}, \quad \mathbf{K}_B = \text{diag}\begin{bmatrix} 24.8101 \\ 69.9209 \\ 181.5830 \end{bmatrix} \tag{4.83}$$

The starting eigenvalues of the first order SPT-filter are unstable thus it is necessary to integrate the flow of the reflexive SPTs to stabilise the filters. Using the direct method one may set the change of eigenvalues to be equal one if the roots are unstable and zero if roots are already stable. This means that one attempts to move only the unstable eigenvalues. Thus,

$$\mathbf{e}_{,\sigma} = \mathrm{sign}\big(\ \mathrm{real}(\mathbf{e})\ \big) < \mathbf{0} \tag{4.84}$$

The integration process is implemented using the MATLAB integrator function 'ode45.' The choice of integrator is dependent on the condition number of the filters. The 'ode45' function is an integrator for non-stiff systems.

The integrator determines the necessary step size σ at each time step. Thus one only need determine the sign of the vector $\mathbf{e}_{,\sigma}$ rather than the magnitude because the integrator will determine the step size σ accordingly. The results of the integration process for this numerical example are illustrated in figure 4.1 and the finishing eigenvalues may be found in table 4.1. The finishing left reflexive SPT may be reported to be

$$\underline{\mathbf{S}_L} = \begin{bmatrix} 0.98171 & 0 & 0 & -0.012069 & 0 & 0 \\ 0 & 0.82175 & 0 & 0 & -0.054202 & 0 \\ 0 & 0 & 0.4458 & 0 & 0 & -0.058531 \\ 0.29943 & 0 & 0 & 1.015 & 0 & 0 \\ 0 & 3.7899 & 0 & 0 & 0.96694 & 0 \\ 0 & 0 & 10.628 & 0 & 0 & 0.84774 \end{bmatrix} \tag{4.85}$$

with the structure of the right reflexive SPT having the form

$$\underline{\mathbf{S}_R} = \begin{bmatrix} \underline{\mathbf{T}_{Lr,\,22}} & -\underline{\mathbf{T}_{Lr,\,12}} \\ -\underline{\mathbf{T}_{Lr,\,21}} & \underline{\mathbf{T}_{Lr,\,11}} \end{bmatrix} \tag{4.86}$$

Figure 4.1 illustrates the integration process used to flow the eigenvalues of the filter into stable region for this numerical example. Once all the eigenvalues are flowed into the stable region it may be observed that the process suspends because the required change in eigenvalues, $\mathbf{e}_{,\sigma}$, becomes zero. The interesting point to observe from figure 4.1 is that although the required change for the stable third eigenvalue is zero the process still moves the eigenvalue into the stable region. This illustrates that it is highly difficult to isolate one eigenvalue of the filter matrices for change without necessarily affecting the other eigenvalues. This concept is precisely why the stabilisation of the SPT-filters becomes quite complicated and may prove to be highly non-linear is various regions. However this numerical example has demonstrated that one may flow the eigenvalues into the stable region using the methods outlined in this chapter.

4.6.2 Numerical Example 4.2 - Steepest Gradient

One now uses the steepest gradient method outlined in section 4.5.2 to stabilise the initially unstable filters obtained from diagonalising SPTs in (4.81) and (4.82). The desired location for the eigenvalues is decided to be 1 such that

$$\underline{\boldsymbol{\delta}} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T \tag{4.87}$$

The integration process is again implemented using the MATLAB integrator function 'ode45' and the integration process is illustrated in figure 4.2. The starting and final filter eigenvalues are given in table 4.2.

From the flow process one obtains the stabilising reflexive SPTs

$$\underline{\mathbf{S}_L} = \begin{bmatrix} 0.56192 & 0 & 0 & -0.13778 & 0 & 0 \\ 0 & 0.81035 & 0 & 0 & -0.05625 & 0 \\ 0 & 0 & 0.34118 & 0 & 0 & -0.063604 \\ 3.4184 & 0 & 0 & 0.94142 & 0 & 0 \\ 0 & 3.9331 & 0 & 0 & 0.96102 & 0 \\ 0 & 0 & 11.549 & 0 & 0 & 0.77795 \end{bmatrix} \tag{4.88}$$

with the $\underline{\mathbf{S}_R}$ structured as shown in equation (4.86)

As may be observed from figure 4.2 the integration time required for the steepest gradient integration process to stabilise is much greater than that required for the direct method. However the computational burden required at each integration point for the steepest gradient method is less hence negating some of the burden of requiring a longer integration horizon for stabilisation to occur.

4.7 Summary

The structure preserving transformations enable one to simultaneously diagonalise the matrices of a generally damped second order system. This is performed by transforming the second order equations of motion into a specific state space representation. By applying a set of left and right transformations to the state space system one obtains a new state space containing diagonal matrices corresponding to a new second order system. One can extract these real, diagonal second order matrices from the state space system which contain identical eigenvalues to the original system. However one must use first order filters to obtain the modal co-ordinates from the physical displacements and velocities. A necessary requirement is that these filters must be stable.

Because the filters are determined by the structure of the SPTs one may exploit the non-uniqueness of the SPTs using reflexive SPTs. The reflexive SPTs represent a non-trivial transformations such that one may map any diagonal system back onto itself. The reflexive SPTs are highly structured with $2n$ parameters available for manipulation. Application of the reflexive SPT results in a new diagonalising SPT which may or may not provide a stable filter. One can exploit this possibility to try and move the filter eigenvalues into the stable half region by appropriately selecting the construction parameters of the reflexive SPT.

This chapter has demonstrated a method by which one can *flow* the eigenvalues of the filter towards the stable half plane. Two methods have been presented by which one can determine the parameters with which to move the eigenvalues; the direct method where one selects the change to be made to the eigenvalues, and the gradient method where one moves in the largest stable change of direction of the eigenvalues. Numerical examples have demonstrated the two methods and one finds that both have respective merits and disadvantages. The direct solution enables one to specify the the change of eigenvalues and find the corresponding reflexive parameters to give the change. The disadvantage is that one does not know to what extent the eigenvalues may be altered to stay within a linear range whilst maximising the change at each stage. The gradient method does not require one to specify the change in eigenvalues and correspondingly the determination of the parameters is numerically less intensive than the direct solution. However the gradient solution does not determine the optimum direction in which to change the eigenvalues only the direction in which the eigenvalues will become more stable.

4.8 Conclusion

The SPTs may be used to transform any second order system into another second order system whilst retaining the same eigenvalues. This chapter has demonstrated how to extract the new displacements and velocities from the original physical co-ordinates by the use of SPT-based filters. The SPT-based filters are first order in nature and consequently a necessary requirement is that their eigenvalues reside on the stable half plane.

For the specific case when the transformed system matrices are real and diagonal, a method has been developed in which one may *flow* the eigenvalues towards the stable region through the use of reflexive SPTs. Consequently one may always make a filter increasingly stable. A numerical example has successfully demonstrated the method for a non-classically damped 3 degree of freedom model with initially unstable filters. However for the general case it is currently unknown whether the change in eigenvalues may go asymptotic, in which case the eigenvalues may never cross into the stable half. Numerical experimentation suggests that this is not the case and that it is always possible to find a stable filter. This remains to be proved formally.

Eigenvalue	Initial	Final
λ_1	6.8345	16.558
λ_2	-3.8693	0.01077 + 0.042692i
λ_3	-7.1993	0.01077 - 0.042692i

Table 4.1: Numerical example 4.1 - Eigenvalues of filter, direct method

Eigenvalue	Initial	Final
λ_1	6.8345	55.124
λ_2	-3.8693	2.7591
λ_3	-7.1993	0.20493

Table 4.2: Numerical example 4.2 - Eigenvalues of filter, steepest gradient method

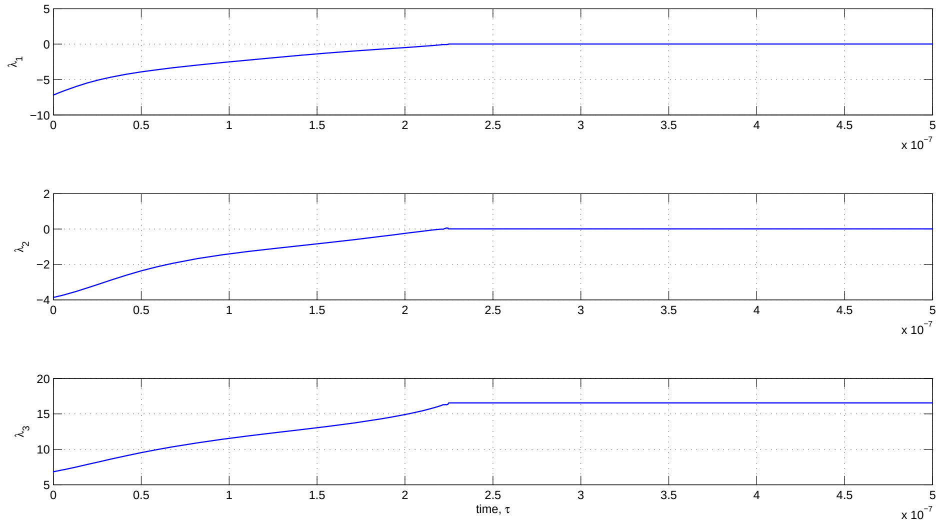


Figure 4.1: Num. example 4.1, Eig.val. variation during integration, direct method

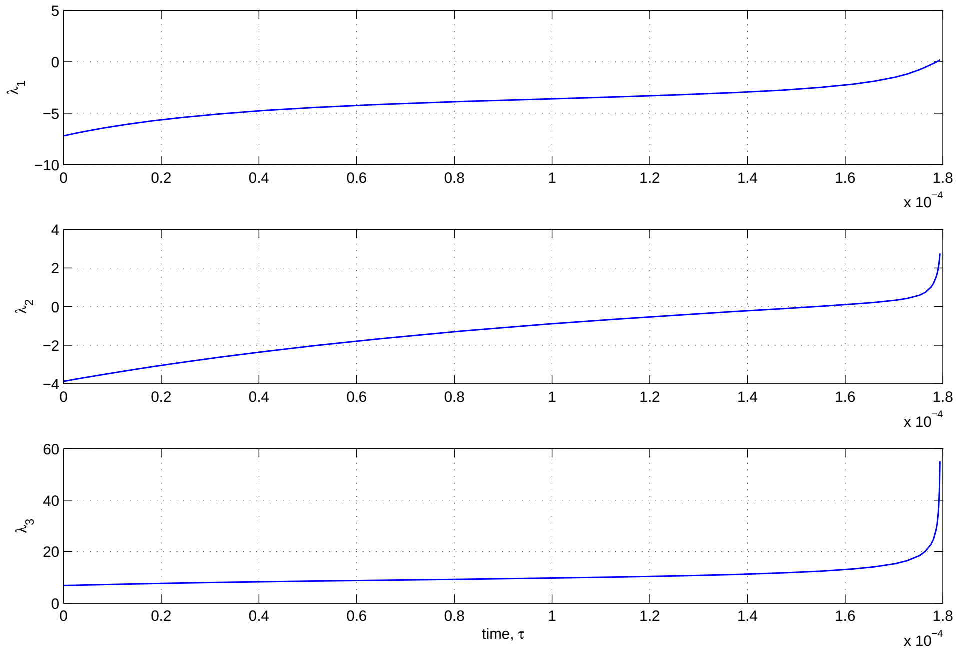


Figure 4.2: Num. example 4.2, Eig.val. variation during integration, steepest gradient

Chapter 5

Optimal Controller Designs for Rotating Machines

Consider a second order system with the equations of motion

$$\mathbf{M}_A\,\ddot{\mathbf{q}}_A(t) + \mathbf{D}_A\,\dot{\mathbf{q}}_A(t) + \mathbf{K}_A\,\mathbf{q}_A(t) = \mathbf{S}_A\,\mathbf{u}_A(t) \tag{5.1}$$

where $\mathbf{M}_A, \mathbf{D}_A, \mathbf{K}_A \in \mathbb{R}^{n\times n}$ are the system mass, damping and stiffness matrices, $\mathbf{q}_A(t) \in \mathbb{R}^n$ is the vector of displacements, $\mathbf{u}_A(t) \in \mathbb{R}^r$ the vector of applied forces and $\mathbf{S}_A \in \mathbb{R}^{n\times r}$ is a selection matrix describing the locations of applied forces. The dot above $\mathbf{q}_A(t)$ denotes its derivative with respect to time.

In the context of active control of rotating machines, standard optimal controller methods enable a trade-off to be made between weighted mean-square vibrations and weighted mean-square control force. A major drawback of the traditional approach to optimal control is that no emphasis is placed on the rate at which the control effort can be applied when designing the controller. Control forces cannot be applied instantaneously and indeed several applications exist where the rate at which control forces can be applied is sufficiently important to warrant this work. One such area is in the field of magnetic bearings.

Consider magnetic bearings as a representative contemporary example of a control actuator for a dynamic system [C3]. It is usual to operate these bearings with a bias current such that the net force produced by the bearing in a direction is linearly proportional to the control current injected into the bearing. The maximum force achievable by the bearing is dependent on the maximum control current which can be injected and could be identified as control input, $\mathbf{u}_A(t)$. The role of conventional optimal control in trying to keep $\mathbf{u}_A(t)$ small is obvious here. Large currents require thick conductors in the bearing and a higher current-rating in the power-amplifiers.

However, the maximum rate of change of force in a magnetic bearing is dependent on the rate of change of current. All magnetic bearings have some inductance and thus a finite rate of change of current requires a finite voltage additional to the voltage required to drive a steady current. In many practical applications, the voltages associated with the rates of change of current are many times greater than the steady "IR" voltages. If the controller requires the magnetic bearing to produce very high rates of change of force then the power-amplifiers will require large internal voltages and the insulation around the windings in the magnetic bearing will have to be thick. Hence, for magnetic bearings, it is actually highly desirable to be able to develop controllers which minimise some cost function that is determined by both control input and rate of control input. Thus conventional optimal control does not provide adequate controller design.

The purpose of this chapter is to explore and present new methods by which one can extend the conventional optimal control method to include penalising the rate of change of force in addition to penalising the state response and applied force. One presents a method which utilises the conventional optimal control applied to an augmented system and a sub-optimal control solution in which one deals directly with the second order system. One also considers the inclusion of the actuator dynamics in the equations of motion such that the standard optimal control problem may be sufficient. Numerical examples are used throughout the chapter where appropriate.

5.1 Conventional Optimal Control

The second order equations of motion may be represented in first order state space form such that

$$\begin{bmatrix} \dot{\mathbf{q}}_{A1}(t) \\ \dot{\mathbf{q}}_{A2}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}_A^{-1}\mathbf{K}_A & -\mathbf{M}_A^{-1}\mathbf{D}_A \end{bmatrix} \begin{bmatrix} \mathbf{q}_{A1}(t) \\ \mathbf{q}_{A2}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{M}_A^{-1}\mathbf{S}_A \end{bmatrix} \mathbf{u}_A(t) \tag{5.2}$$

with $\mathbf{q}_{A1}(t) = \mathbf{q}_A(t)$ and $\mathbf{q}_{A2}(t) = \dot{\mathbf{q}}_{A1}(t)$. Equation (5.2) may be simplified to

$$\underline{\dot{\mathbf{q}}_A}(t) = \underline{\mathbf{A}_A}\,\underline{\mathbf{q}_A}(t) + \underline{\mathbf{B}_A}\,\mathbf{u}_A(t) \tag{5.3}$$

where the definitions are apparent. For ease of reading the notation denoting dependence on time is removed from here onwards.

A feedback gain, $\mathbf{G}$, may be calculated such that more desirable system properties can be found. This is represented pictorially in figure 5.1.

One approach to determining the feedback gains matrix, $\mathbf{G}$, is to utilise so-called optimal control methods. Optimal control is best summarised by calculating an optimal feedback force which minimises a quadratic form defined by

$$J = \frac{1}{2}\int_0^\infty \left(\underline{\mathbf{q}_A}^T\, \underline{\mathbf{Q}}\, \underline{\mathbf{q}_A} + \mathbf{u}_A^T\, \mathbf{R}_\mathrm{u}\, \mathbf{u}_A \right)\, dt \tag{5.4}$$

where $\underline{\mathbf{Q}} \in \mathbb{R}^{2n\times 2n}$ represents a symmetric semi-definite weighting matrix governing the relative importance of the system state at time, t, and similarly $\mathbf{R}_\mathrm{u} \in \mathbb{R}^{r\times r}$ represents a symmetric positive definite matrix to weight the control effort.

The quadratic expression of equation (5.4) has threefold implication:

- The positive and negative errors are weighted equally.
- The larger errors are penalised more harshly than smaller errors.
- The integral penalises the more persistent error more harshly.

The standard approach to solving the optimal control problem is well understood and a great deal of literature is available for the problem. The usual procedure is to solve the

Riccati equation [Z2] but methods exist to calculate the optimal control for second order systems [Z1] without the need to deal with the state space approach. The optimal control method described here is frequently referred to as the Linear Quadratic Regulator (LQR) control as it applies to linear problems and involves a quadratic cost function.

One may use a second order state-space representation of the second order equations of motion such that one has

$$\begin{bmatrix} \mathbf{0} & \mathbf{K}_A \\ \mathbf{K}_A & \mathbf{D}_A \end{bmatrix} \begin{bmatrix} \mathbf{q}_{A1} \\ \mathbf{q}_{A2} \end{bmatrix} - \begin{bmatrix} -\mathbf{D}_A & -\mathbf{M}_A \\ -\mathbf{M}_A & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_{A1} \\ \ddot{\mathbf{q}}_{A2} \end{bmatrix} = \begin{bmatrix} \mathbf{S}_A & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_A \end{bmatrix} \begin{bmatrix} \dot{\mathbf{u}}_A \\ \mathbf{u}_A \end{bmatrix} \tag{5.5}$$

with $\mathbf{q}_{A1}(t) = \mathbf{q}_A(t)$ and $\mathbf{q}_{A2}(t) = \dot{\mathbf{q}}_{A1}(t)$.

The state space representation of equation (5.5) utilises the 'Lancaster Augmented Matrices' (LAMs) first introduced in section 2.2. Using the notation given in section 2.6 one simplifies equation (5.5) to be

$$\underline{\mathbf{A}_2}\,\underline{\mathbf{q}_A} - \underline{\mathbf{A}_0}\,\underline{\ddot{\mathbf{q}}_A} = \underline{\mathbf{S}_A}\,\underline{\mathbf{u}_A} \tag{5.6}$$

where the definitions of $\underline{\mathbf{A}_0}$, $\underline{\mathbf{A}_2}$ and $\underline{\mathbf{S}_A}$ are apparent and the augmented forcing vector is defined as $\underline{\mathbf{u}_A} \equiv \begin{bmatrix} \dot{\mathbf{u}}_A & \mathbf{u}_A \end{bmatrix}^T$.

As apparent from equation (5.5), the equations of motion contain both the control vector $\mathbf{u}_A$ and time derivative of control vector $\dot{\mathbf{u}}_A$. The description of the system by the LAMs raises the possibility of controlling the force rate in addition to the forces applied to a vibrating system.

It is proposed in this chapter to design a controller to minimise the extended cost function of the form

$$J_{ext} = \frac{1}{2} \int_0^\infty \left(\underline{\mathbf{q}_A}^T\,\underline{\mathbf{Q}}\,\underline{\mathbf{q}_A} + \mathbf{u}_A^T\,\mathbf{R}_\mathrm{u}\,\mathbf{u}_A + \dot{\mathbf{u}}_A^T\,\mathbf{R}_\mathrm{v}\,\dot{\mathbf{u}}_A \right)\, dt \tag{5.7}$$

where $\mathbf{R}_\mathrm{v} \in \mathbb{R}^{r\times r}$ is a symmetric positive-definite weighting matrix to penalise rate of change of control. It is assumed that full state feedback is available.

A possible method to penalise the rate of change of control is to find numerically the optimal force which minimises the extended cost function when the system is subjected to a known initial condition $\underline{\mathbf{q}_A}(0)$. The forcing function, $\mathbf{u}_A$, used to minimise the cost function (5.7) is constructed using a summation of forcing wavelets. Appendix E presents the numerical penalisation method and a numerical example is presented. The example compares the numerical penalisation method against the standard LQR method and illustrates that it is indeed possible to penalise the rate of change of control whilst bringing the response of the system under control.

The numerical penalisation method shows that it is possible to construct an optimal feedback force subjected to constraints on the state of the system and the force and rate of change of force applied to the system. The numerical idea presented utilises the fact that the cost functional is quadratic and contains only a single minimum. Whilst this method is usable to identify an appropriate feedback force for a particular initial condition it is neither numerically efficient nor general enough for use in an active control system. The method identifies that a solution to the minimisation problem exists and the obvious next stage is to present an analytical solution.

5.2 Optimal Eigenvalue Locations

The original idea concerning the penalisation of the control rate arose through the description of the second order system using the LAMs. Two immediate necessary constraints become obvious from equation (5.5): the top half of the augmented control vector $\underline{\mathbf{u}_A}$ is equal to the time derivative of the bottom half, and the bottom half of the state vector $\underline{\mathbf{q}_A}$ is equal to the time derivative of the top half. This may be represented

$$\begin{bmatrix} \mathbf{0} & \mathbf{I} \end{bmatrix} \underline{\mathbf{q}_A} - \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \underline{\dot{\mathbf{q}}_A} = 0 \tag{5.8}$$

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \underline{\mathbf{u}_A} - \begin{bmatrix} \mathbf{0} & \mathbf{I} \end{bmatrix} \underline{\dot{\mathbf{u}}_A} = 0 \tag{5.9}$$

In addition knowledge of the physical system must be introduced

$$\underline{\mathbf{A}_2}\, \underline{\mathbf{q}_A} - \underline{\mathbf{A}_0}\, \underline{\ddot{\mathbf{q}}_A} - \underline{\mathbf{S}_A}\, \underline{\mathbf{u}_A} = 0 \tag{5.10}$$

Thus the constrained optimisation problem of equation (5.6) may be transformed into an unconstrained optimisation problem through incorporating the three system constraints.

A desirable result for this project is to retain the second order structure of the system where possible. Rather than dealing with the optimal control problem by the first order state space approach it is possible to utilise the second order problem. Recalling the definition of the second order system

$$\mathbf{M}_A\,\ddot{\mathbf{q}}_A + \mathbf{D}_A\,\dot{\mathbf{q}}_A + \mathbf{K}_A\,\mathbf{q}_A = \mathbf{S}_A\,\mathbf{u}_A \tag{5.11}$$

The extended quadratic cost function defined in equation (5.7) may be found equivalent to

$$J_{ext\_2nd} = \frac{1}{2}\int_0^\infty \left\{\mathbf{q}_A^T\,\mathbf{Q}_\mathrm{d}\,\mathbf{q}_A + \dot{\mathbf{q}}_A^T\,\mathbf{Q}_\mathrm{v}\,\dot{\mathbf{q}}_A + \mathbf{u}_A^T\,\mathbf{R}_\mathrm{u}\,\mathbf{u}_A + \mathbf{v}_A^T\,\mathbf{R}_\mathrm{v}\,\mathbf{v}_A\right\}\,dt \tag{5.12}$$

where $\mathbf{v}_A = \dot{\mathbf{u}}_A$ and $\mathbf{Q}_\mathrm{d}, \mathbf{Q}_\mathrm{v} \in \mathbb{R}^{n\times n}$ are the positive semi-definite weighting matrices placed on the physical displacements and velocities respectively. Equation (5.12) presupposes that there exist no cross terms involving $\mathbf{q}$ and $\dot{\mathbf{q}}$.

Defining $\mathbf{H} = \dfrac{dJ}{dt}$ and introducing the time dependent Lagrange multiplier vectors $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}$ into $\mathbf{H}$ in order to create an unconstrained optimisation problem

$$\begin{aligned}\mathbf{H}_1 &= \frac{1}{2}\left[\mathbf{q}_A^T\,\mathbf{Q}_\mathrm{d}\,\mathbf{q}_A + \dot{\mathbf{q}}_A^T\,\mathbf{Q}_\mathrm{v}\,\dot{\mathbf{q}}_A + \mathbf{u}_A^T\,\mathbf{R}_\mathrm{u}\,\mathbf{u}_A + \mathbf{v}_A^T\,\mathbf{R}_\mathrm{v}\,\mathbf{v}_A\right] \\ &+ \boldsymbol{\mu}^T\left[\mathbf{M}_A\,\ddot{\mathbf{q}}_A + \mathbf{D}_A\,\dot{\mathbf{q}}_A + \mathbf{K}_A\,\mathbf{q}_A - \mathbf{S}_A\,\mathbf{u}_A\right] \\ &+ \boldsymbol{\gamma}^T\left[\mathbf{v}_A - \dot{\mathbf{u}}_A\right]\end{aligned} \tag{5.13}$$

$\mathbf{H}_1$ contains the information necessary to find the minimum of function (5.12). The Euler-Lagrange equations [G11] may be used to derive the necessary conditions for the global minimum

$$\frac{\partial \mathbf{H}_1}{\partial \mathbf{q}} - \frac{d}{dt}\left(\frac{\partial \mathbf{H}_1}{\partial \dot{\mathbf{q}}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial \mathbf{H}_1}{\partial \ddot{\mathbf{q}}}\right) = 0 \tag{5.14}$$

$$\frac{\partial \mathbf{H}_1}{\partial \mathbf{u}} - \frac{d}{dt}\left(\frac{\partial \mathbf{H}_1}{\partial \dot{\mathbf{u}}}\right) = 0 \tag{5.15}$$

$$\frac{\partial \mathbf{H}_1}{\partial \mathbf{v}} = 0 \tag{5.16}$$

Equations (5.14) to (5.16) yield the results

$$\begin{aligned} \mathbf{0} &= \left(\mathbf{M}_A^T\,\ddot{\boldsymbol{\mu}} - \mathbf{D}_A^T\,\dot{\boldsymbol{\mu}} + \mathbf{K}_A^T\,\boldsymbol{\mu}\right) - \left(\mathbf{Q}_{\mathrm{v}}\,\ddot{\mathbf{q}}_A - \mathbf{Q}_{\mathrm{d}}\,\mathbf{q}_A\right) &\quad (5.17)\\ \mathbf{u}_A &= \mathbf{R}_{\mathrm{u}}^{-1}\,\mathbf{S}_A^T\,\boldsymbol{\mu} - \mathbf{R}_{\mathrm{u}}^{-1}\,\dot{\boldsymbol{\gamma}} &\quad (5.18)\\ \mathbf{v}_A &= -\mathbf{R}_{\mathrm{v}}^{-1}\,\boldsymbol{\gamma} &\quad (5.19) \end{aligned}$$

The results obtained from the Euler-Lagrange equations may also be obtained from other approaches to optimal control, notably calculus of variations [Y1] or dynamic programming [B4, B5]. Combining the results above with the constraint equations it is possible to form the system

$$\begin{bmatrix} -\mathbf{Q}_{\mathrm{v}} & \mathbf{M}_A^T & \mathbf{0} \\ \mathbf{M}_A & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{R}_{\mathrm{u}}^{-1} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_A \\ \ddot{\boldsymbol{\mu}} \\ \ddot{\boldsymbol{\gamma}} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & -\mathbf{D}_A^T & \mathbf{0} \\ \mathbf{D}_A & \mathbf{0} & \mathbf{S}_A\,\mathbf{R}_{\mathrm{u}}^{-1} \\ \mathbf{0} & -\mathbf{R}_{\mathrm{u}}^{-1}\,\mathbf{S}_A^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}}_A \\ \dot{\boldsymbol{\mu}} \\ \dot{\boldsymbol{\gamma}} \end{bmatrix} + \cdots$$

$$\cdots + \begin{bmatrix} \mathbf{Q}_d & \mathbf{K}_A^T & \mathbf{0} \\ \mathbf{K}_A & -\mathbf{S}_A\,\mathbf{R}_{\mathrm{u}}^{-1}\,\mathbf{S}_A^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{R}_{\mathrm{v}}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{q}_A \\ \boldsymbol{\mu} \\ \boldsymbol{\gamma} \end{bmatrix} = 0 \quad (5.20)$$

This set of equations may be simplified to

$$\mathscr{H}_2\,\ddot{\mathbf{x}}_H + \mathscr{H}_1\,\dot{\mathbf{x}}_H + \mathscr{H}_0\,\mathbf{x}_H = 0 \quad (5.21)$$

where the definitions of $\mathscr{H}_j$ $(j = 0, 1, 2)$ and $\mathbf{x}_H$ are apparent.

The system illustrated by equation (5.20) is referred to as a Hamiltonian system because it contains a set of differential equations which may be written in the form of Hamilton's equations [G11]. Hamilton's equation are derived from investigating how the Lagrangian equations change with respect to time.

One may notice that the $\mathscr{H}_2$ and $\mathscr{H}_0$ matrices in equation (5.21) are symmetric and the $\mathscr{H}_1$ matrix is skew-symmetric. The result of this structure is that the eigenvalues of this Hamiltonian system are symmetric about the imaginary axis [B8] and Hamiltonian system contains the optimal eigenvalues for the control problem. The stable half of the eigenvalues represent the pole locations for the optimal control problem defined by the cost function. However, there are $2n + r$ stable eigenvalues associated with the Hamiltonian matrix. This is dissimilar to the $2n$ eigenvalues associated with the system equations of motion. Thus there is a disparity of dimension r. The dissimilarity in dimension can be explained from the inclusion of the additional constraint to account for the rate of change of control. Thus it seems apparent that the system must be extended by dimension r to accommodate the additional constraints imposed on the time derivative of the control force. The question arises how to do this practically? Indeed a very novel and practical way presents itself as discussed in the next section.

5.3 A Tangent to Conventional Optimal Control

The approach pursued in this chapter is to re-think the pictorial representation of the system illustrated by figure 5.1. Suppose that an augmented plant can be constructed such that the input to this augmented plant is the time derivative of control rather than the normal control vector. A feedback gains matrix could be calculated to find the optimal rate of change of control force applied to the system. This concept is represented pictorially in figure 5.2.

As illustrated by figure 5.2 an augmented state $\underline{\mathbf{q}_{aug}}$ can be formed including the control vector which is defined as,

$$\underline{\mathbf{q}_{aug}} = \begin{bmatrix} \mathbf{u}_A \\ \underline{\mathbf{q}_A} \end{bmatrix} \tag{5.22}$$

Thus the augmented plant can be shown to have the following equations of motion

$$\begin{bmatrix} \dot{\mathbf{u}}_A \\ \underline{\dot{\mathbf{q}}_A} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \underline{\mathbf{B}_A} & \underline{\mathbf{A}_A} \end{bmatrix} \begin{bmatrix} \mathbf{u}_A \\ \underline{\mathbf{q}_A} \end{bmatrix} + \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix} \mathbf{v}_A \tag{5.23}$$

This may be simplified to

$$\underline{\dot{\mathbf{q}}_{aug}} = \underline{\widetilde{\mathbf{A}}_A}\,\underline{\mathbf{q}_{aug}} + \underline{\widetilde{\mathbf{B}}_A}\,\mathbf{v}_A \tag{5.24}$$

where the definitions of $\underline{\widetilde{\mathbf{A}}_A}, \underline{\widetilde{\mathbf{B}}_A}$ are obvious. One may observe that in addition to the initial state the initial force is required for the integration of the system equations of motion.

The extended quadratic cost illustrated by equation (5.7) is equivalent to

$$J_{aug} = \frac{1}{2}\int_0^{\infty} \underline{\mathbf{q}_{aug}}^T\, \underline{\mathbf{Q}_{aug}}\, \underline{\mathbf{q}_{aug}} + \mathbf{v}_A^T\, \mathbf{R}_{\mathrm{v}}\, \mathbf{v}_A \;\; dt \tag{5.25}$$

Here the new definition of $\underline{\mathbf{Q}_{aug}}$ can be seen clearly to be,

$$\underline{\mathbf{Q}_{aug}} = \begin{bmatrix} \mathbf{R}_{\mathrm{u}} & \mathbf{0} \\ \mathbf{0} & \underline{\mathbf{Q}} \end{bmatrix} \tag{5.26}$$

5.3.1 Calculating the Optimal Control Gain

Equation (5.24) represents the dynamics of a first order system subjected to the constraint that the cost function given by equation (5.25) must be minimal over an infinite time horizon. This situation represents a constrained variational problem. In order to solve the optimal control problem the system must first be converted into an unconstrained control problem using the introduction of the co-state vector, $\boldsymbol{\mu} \in \mathbb{R}^n$. The function $\underline{\mathbf{H}_{aug}}$ may be defined,

$$\begin{aligned} \underline{\mathbf{H}_{aug}}\left(\underline{\mathbf{q}_{aug}}, \mathbf{u}_A, \boldsymbol{\mu}\right) &= \frac{1}{2}\,\underline{\mathbf{q}_{aug}}^T\, \underline{\mathbf{Q}_{aug}}\, \underline{\mathbf{q}_{aug}} + \frac{1}{2}\,\mathbf{v}_A^T\, \mathbf{R}_{\mathrm{v}}\, \mathbf{v}_A \\ &+ \boldsymbol{\mu}^T \left\{ \underline{\widetilde{\mathbf{A}}_A}\, \underline{\mathbf{q}_{aug}} + \underline{\widetilde{\mathbf{B}}_A}\, \mathbf{v}_A - \underline{\dot{\mathbf{q}}_{aug}} \right\} \end{aligned} \tag{5.27}$$

As already introduced, the necessary conditions to produce a minimum for the optimal problem are given by the Euler-Lagrange equations [G11]

$$\frac{\partial \underline{\mathbf{H}_{aug}}}{\partial \underline{\mathbf{q}_{aug}}} - \frac{d}{dt}\left(\frac{\partial \underline{\mathbf{H}_{aug}}}{\partial \underline{\dot{\mathbf{q}}_{aug}}}\right) = 0 \tag{5.28}$$

$$\frac{\partial \underline{\mathbf{H}_{aug}}}{\partial \mathbf{v}} = 0 \tag{5.29}$$

Substituting the definition of $\underline{\mathbf{H}_{aug}}$ into equations (5.28) and (5.29) yields the results

$$\dot{\boldsymbol{\mu}} = -\underline{\mathbf{Q}_{aug}}\, \underline{\mathbf{q}_{aug}} - \underline{\widetilde{\mathbf{A}}_A}\, \boldsymbol{\mu} \tag{5.30}$$

$$\mathbf{v}_A = -\mathbf{R}_{\mathrm{v}}\, \underline{\widetilde{\mathbf{B}}_A}^T\, \boldsymbol{\mu} \tag{5.31}$$

It is worth reminding the reader here that the matrices $\underline{\mathbf{Q}_{aug}}$ and $\mathbf{R}_{\mathrm{v}}$ are symmetric hence equations (5.30) and (5.31) are simplified further by recognising that $\underline{\mathbf{Q}_{aug}}$ and $\mathbf{R}_{\mathrm{v}}$ equal their own transposes.

Equation (5.31) is substituted into the equations of motion represented by equation (5.24) and then combined with equation (5.30) to form the Hamiltonian system

$$\begin{bmatrix} \underline{\dot{\mathbf{q}}_{aug}} \\ \dot{\boldsymbol{\mu}} \end{bmatrix} = \begin{bmatrix} \underline{\widetilde{\mathbf{A}}_A} & -\underline{\widetilde{\mathbf{B}}_A}\,\mathbf{R}_{\mathrm{v}}^{-1}\,\underline{\widetilde{\mathbf{B}}_A}^T \\ -\underline{\mathbf{Q}_{aug}} & -\underline{\widetilde{\mathbf{A}}_A}^T \end{bmatrix} \begin{bmatrix} \underline{\mathbf{q}_{aug}} \\ \boldsymbol{\mu} \end{bmatrix} = \underline{\mathscr{H}_{aug}} \begin{bmatrix} \underline{\mathbf{q}_{aug}} \\ \boldsymbol{\mu} \end{bmatrix} \tag{5.32}$$

Conventional optimal control asserts that the co-state vector $\boldsymbol{\mu}$ is related to the augmented state $\underline{\mathbf{q}_{aug}}$ of the system through the linear relationship [B8]

$$\boldsymbol{\mu} = \underline{\mathbf{P}}\,\underline{\mathbf{q}_{aug}} \tag{5.33}$$

where $\underline{\mathbf{P}}$ is referred to as the co-state matrix.

This chapter deals with the specific infinite horizon problem where the desired end condition is assumed to be equal to zero so that the final conditions may be $\boldsymbol{\mu}(\infty)$. It is relatively simple to subject the system to a final settling state but this is not addressed here. Therefore knowing the initial and final states of the system it is possible to solve the Hamiltonian by utilising equation (5.33).

5.3.2 The Riccati Equation

A more robust method of finding the optimal control is to form the Riccati equation which may be solved backwards through time to give the matrix $\underline{\mathbf{P}}$. The relationship between the co-state vector and control vector can be combined to yield the feedback control

$$\mathbf{v}_A = -\mathbf{R}_{\mathrm{v}}^{-1}\,\underline{\widetilde{\mathbf{B}}_A}^T\,\boldsymbol{\mu} = -\mathbf{R}_{\mathrm{v}}\,\underline{\widetilde{\mathbf{B}}_A}^T\,\underline{\mathbf{P}}\,\underline{\mathbf{q}_{aug}} = -\mathbf{G}_{aug}\,\underline{\mathbf{q}_{aug}} \tag{5.34}$$

The Riccati equation may be formed by differentiating equation (5.33)

$$\dot{\boldsymbol{\mu}} = \underline{\dot{\mathbf{P}}}\,\underline{\mathbf{q}_{aug}} + \underline{\mathbf{P}}\,\underline{\dot{\mathbf{q}}_{aug}} \tag{5.35}$$

Substituting in equation (5.30) for $\dot{\boldsymbol{\mu}}$ and the equations of motion for $\underline{\dot{\mathbf{q}}_{aug}}$ yields

$$\left(\underline{\dot{\mathbf{P}}} + \underline{\mathbf{P}}\,\underline{\widetilde{\mathbf{A}}_A} + \underline{\widetilde{\mathbf{A}}_A}^T\,\underline{\mathbf{P}} + \underline{\mathbf{Q}_{aug}} - \underline{\mathbf{P}}\,\underline{\widetilde{\mathbf{B}}_A}\,\mathbf{R}_{\mathrm{v}}^{-1}\,\underline{\widetilde{\mathbf{B}}_A}^T\,\underline{\mathbf{P}}\right)\,\underline{\mathbf{q}_{aug}} = 0 \tag{5.36}$$

Equation (5.36) holds for any arbitrary state $\underline{\mathbf{q}_{aug}}$ starting with known initial conditions. Therefore the dependence of the state $\underline{\mathbf{q}_{aug}}$ can be removed which implies that $\underline{\mathbf{P}}$ must satisfy

$$\dot{\underline{\mathbf{P}}} = -\underline{\mathbf{P}}\,\underline{\widetilde{\mathbf{A}}_A} - \underline{\widetilde{\mathbf{A}}_A}^T\,\underline{\mathbf{P}} - \underline{\mathbf{Q}_{aug}} + \underline{\mathbf{P}}\,\underline{\widetilde{\mathbf{B}}_A}\,\mathbf{R}_{\mathrm{v}}^{-1}\,\underline{\widetilde{\mathbf{B}}_A}^T\,\underline{\mathbf{P}} \tag{5.37}$$

This result is known as the Riccati equation and may be solved backwards through time [Z2] knowing the desired final conditions for $\underline{\mathbf{P}}$.

5.3.3 Numerical Example 5.1

A spring-mass system with no damping is constructed such that equations of motion are governed by

$$\mathbf{M}_A\,\ddot{\mathbf{q}} + \mathbf{K}_A\,\mathbf{q} = \mathbf{S}_A\,\mathbf{u}_A \tag{5.38}$$

with matrices

$$\mathbf{M}_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \;,\; \mathbf{K}_A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \;,\; \mathbf{S}_A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{5.39}$$

subjected to initial conditions $\mathbf{q}_A(0) = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}^T$ and $\dot{\mathbf{q}}_A(0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$. The un-forced response to the initial conditions of mass 3 is illustrated in figure 5.3.

The second order system can be converted into first order state space form such that

$$\underline{\mathbf{A}_A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}_A^{-1}\,\mathbf{K}_A & \mathbf{0} \end{bmatrix} \;,\; \underline{\mathbf{B}_A} = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}_A^{-1}\,\mathbf{S}_A \end{bmatrix} \tag{5.40}$$

The weighting matrices are chosen to minimise the kinetic and potential energies of the system. The control and rate of control vectors are chosen arbitrarily

$$\underline{\mathbf{Q}} = \begin{bmatrix} \mathbf{K}_A & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_A \end{bmatrix} \;,\; \mathbf{R}_{\mathrm{u}} = 10^{-1}\,\mathbf{I} \quad,\; \mathbf{R}_{\mathrm{v}} = 10^{2}\,\mathbf{I} \tag{5.41}$$

where $\mathbf{I}$ is the identity matrix of appropriate dimension.

The augmented plant is constructed utilising the form shown by equation (5.24) and the new weighting matrices satisfy equation (5.25). Solving the Riccati equation and substituting the result into equation (5.34) one finds the optimal controller gain for the augmented system to be,

$$\mathbf{G}_{aug} = \begin{bmatrix} 0.23965 & 0.11317 & -0.2171 & 0.051166 & 0.022016 & 0.034619 & 0.027036 & 0.014187 \\ 0.11317 & 0.33345 & 0.054257 & -0.19381 & 0.069238 & 0.037823 & 0.0615 & 0.029084 \end{bmatrix} \tag{5.42}$$

The response to the initial conditions of mass 3 is illustrated in figure 5.4. This gives the quadratic cost, $J_{aug} = 161$.

The LQR optimal control gain for the first order system illustrated by equation (5.40) is

$$\mathbf{G}_{lqr} = \begin{bmatrix} 2.8722 & -0.26984 & -0.74947 & 3.9676 & -0.046734 & -0.14134 \\ -0.12322 & 4.8708 & -3.0562 & -0.046734 & 4.4429 & 1.9914 \end{bmatrix} \tag{5.43}$$

Subjecting the traditional optimal controller to the same quadratic cost function illustrated by equation (5.25) yields the cost $J_{lqr} = 24{,}835$. The response is illustrated in figure 5.5.

As is apparent from figures 5.4 and 5.5, the LQR approach provides no weighting to the rate at which the force is applied so the force is applied more quickly bringing the state of the system under control much quicker than that of the augmented plant method. But examination of the costs defined from equation (5.25) of the two systems alone illustrates the expense of doing so. The LQR cost is approximately 154 times larger than that of the augmented system.

The assertion was made that the Hamiltonian system of equation (5.20) is optimal therefore the eigenvalues of the controlled augmented system must yield the same values if the augmented system represents the optimal problem. This is indeed the case and the values are illustrated in table 5.1.

5.3.4 Numerical Example 5.2

A rotor dynamic system illustrated by figure 5.6 is modelled using finite elements with 4 degrees of freedom at each nodal point representing rotational and displacement coordinates. The system model is reduced in size using Guyan reduction [G12] to 6 degrees of freedom and the optimal control system outlined in this document is applied. All dimensions marked on the figure are in millimetres (mm) and each element is 10 mm in length. The steel shaft is of diameter 30 mm and the diameters of the aluminium discs illustrated at points Out 1 and Out 2 are of diameter 150 mm. Bearings 1 and 2 constrain the system such that the rotor displacements at these points are zero.

The initial conditions are such that the rotor system has an initial velocity for the entire system being equal to 10 m/s. The control forces are applied at the location of the arrow as indicated. The output displacements are calculated at the centre of the two discs.

The weighting matrices are chosen to penalise the displacements of the system at the locations of the discs more harshly than other locations. Thus the **Q**-matrix is set equal to the identity matrix except for nodes 1, 2, 3, 31, 32, 33, 34, 35 and 36, corresponding to the disc locations, and are weighted such that they are equal to 100. The weighting on the force and force rate are given values $10^2\mathbf{I}$ and $10^{-2}\mathbf{I}$, respectively, where $\mathbf{I}$ is the identity matrix of appropriate dimension. The eigenvalues of the optimal augmented system versus the second order Hamiltonian system are given in table 5.2. It may be observed that a few rounding errors exist although these are deemed minor.

The quadratic costs for the augmented and LQR control approaches are 4.4580×10^{10} (69.2%) and 6.4463×10^{10} (100%) respectively. This means that the augmented approach represents a sizeable reduction, approximately 30%, in cost compared with the LQR problem. Much of this reduction in cost is due to the peak time derivative control rate vector for the augmented system being approximately 40% of the peak time derivative control rate for the LQR system as illustrated in figure 5.7. This has immediate relevance to the magnitude of the force applied to the system resulting in a sizeable reduction in the magnitude of peak force again contributing to the reduction in quadratic cost.

5.4 Second Order Analytical Solution

The ambitions sought in this chapter was to identify a method which yields the optimal controller which incorporates the rate of change of forcing into the standard optimal cost function. Through numerical examples and analysis of the system eigenvalues, this ambition has been shown to have been achieved. However, the intent of this thesis is to extend the literature in regards to direct second order control techniques and it is desirable to obtain a controller of the form

$$\mathbf{u}_A = \mathbf{G}_k\,\mathbf{q}_A + \mathbf{G}_d\,\dot{\mathbf{q}}_A \tag{5.44}$$

Following the approach given by Zhang [Z1] where the solution to the optimal control problem for the second order system is discussed one may establish a solution to the optimal control problem which incorporates the rate of change of force. Recalling the Hamiltonian system from equation (5.20) the associated eigenvalue problem may be expressed

$$\begin{bmatrix} -(\,\lambda^2\mathbf{Q}_{\mathrm{v}} - \mathbf{Q}_d\,) & (\,\lambda^2\mathbf{M}_A^T - \lambda\mathbf{D}_A^T + \mathbf{K}_A^T\,) & \mathbf{0} \\ (\,\lambda^2\mathbf{M}_A + \lambda\mathbf{D}_A + \mathbf{K}_A\,) & -\mathbf{S}_A\,\mathbf{R}_{\mathrm{u}}^{-1}\,\mathbf{S}_A^T & \lambda\mathbf{S}_A\,\mathbf{R}_{\mathrm{u}}^{-1} \\ \mathbf{0} & -\lambda\mathbf{R}_{\mathrm{u}}^{-1}\,\mathbf{S}_A^T & (\,\lambda^2\mathbf{R}_{\mathrm{u}}^{-1} - \mathbf{R}_{\mathrm{v}}^{-1}\,) \end{bmatrix} \begin{bmatrix} \mathbf{X}_{\mathbf{q}_A} \\ \mathbf{X}_{\boldsymbol{\mu}} \\ \mathbf{X}_{\boldsymbol{\gamma}} \end{bmatrix} = \mathbf{0} \tag{5.45}$$

where $\mathbf{X}_{\mathbf{q}_A}$, $\mathbf{X}_{\boldsymbol{\mu}}$ and $\mathbf{X}_{\boldsymbol{\gamma}}$ are the associated sub-vectors of the eigenvector related to eigenvalue λ.

It can be shown from [I1] that equation (5.45) has general solution in terms of the eigenvalues and eigenvectors by the modal expansion theorem

$$\mathbf{q}_A = \sum_{i=1}^{4n+2r} a_i\,e^{\lambda_i t}\mathbf{X}_{\mathbf{q}_{Ai}} \tag{5.46}$$

$$\boldsymbol{\mu} = \sum_{i=1}^{4n+2r} a_i\,e^{\lambda_i t}\mathbf{X}_{\boldsymbol{\mu}_i} \tag{5.47}$$

$$\boldsymbol{\gamma} = \sum_{i=1}^{4n+2r} a_i\,e^{\lambda_i t}\mathbf{X}_{\boldsymbol{\gamma}_i} \tag{5.48}$$

with coefficients a_i determined by the initial conditions.

As stated previously the roots of the Hamiltonian system are symmetric about the imaginary axis. One may thus consider only the stable $2n + r$ eigenvalues. Equations

(5.46) - (5.48) may be accordingly rewritten

$$\mathbf{q}_A = \sum_{j=1}^{2n+r} a_j\, e^{\lambda_i t}\mathbf{X}_{\mathbf{q}_{Aj}} = \mathbf{d}\,\mathbf{E}\,\boldsymbol{\Phi}_s \tag{5.49}$$

$$\boldsymbol{\mu} = \sum_{j=1}^{2n+r} a_j\, e^{\lambda_i t}\mathbf{X}_{\boldsymbol{\mu}_j} = \mathbf{d}\,\mathbf{E}\,\boldsymbol{\Psi}_s \tag{5.50}$$

$$\boldsymbol{\gamma} = \sum_{j=1}^{2n+r} a_j\, e^{\lambda_i t}\mathbf{X}_{\boldsymbol{\gamma}_j} = \mathbf{d}\,\mathbf{E}\,\boldsymbol{\Theta}_s \tag{5.51}$$

with the definitions

$$\begin{aligned}
\mathbf{d} &= \begin{bmatrix} a_1 & a_2 & \cdots & a_{2n+r} \end{bmatrix}^T, & \mathbf{E} &= \mathrm{diag}\begin{bmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} & \cdots & e^{\lambda_{2n+r} t} \end{bmatrix} \\
\boldsymbol{\Phi}_s &= \begin{bmatrix} \mathbf{X}_{\mathbf{q}_{A1}} & \mathbf{X}_{\mathbf{q}_{A2}} & \cdots & \mathbf{X}_{\mathbf{q}_{A,2n+r}} \end{bmatrix}, & \boldsymbol{\Psi}_s &= \begin{bmatrix} \mathbf{X}_{\boldsymbol{\mu}_1} & \mathbf{X}_{\boldsymbol{\mu}_2} & \cdots & \mathbf{X}_{\boldsymbol{\mu}_{2n+r}} \end{bmatrix} \\
\boldsymbol{\Theta}_s &= \begin{bmatrix} \mathbf{X}_{\boldsymbol{\gamma}_1} & \mathbf{X}_{\boldsymbol{\gamma}_2} & \cdots & \mathbf{X}_{\boldsymbol{\gamma}_{2n+r}} \end{bmatrix}
\end{aligned}$$

λ_j, $(j = 1, 2, \ldots, 2n + r)$ are the $2n + r$ stable eigenvalues and $\boldsymbol{\Phi}_s$, $\boldsymbol{\Psi}_s$ and $\boldsymbol{\Theta}_s$ are the corresponding sub-vectors of the eigenvectors.

Recalling the definition of $\mathbf{u}_A$ from equation (5.18) and the desired controller feedback form from equation (5.44) one may equate these two equations such that

$$\mathbf{u}_A = \mathbf{R}_{\mathrm{u}}^{-1}\,\mathbf{S}_A^T\,\boldsymbol{\mu} - \mathbf{R}_{\mathrm{u}}^{-1}\,\dot{\boldsymbol{\gamma}} = \mathbf{G}_k\,\mathbf{q}_A + \mathbf{G}_d\,\dot{\mathbf{q}}_A \tag{5.52}$$

Recognising that

$$\dot{\mathbf{q}}_A = \mathbf{d}\,\mathbf{E}\,\boldsymbol{\Phi}_{s1}\,, \quad \boldsymbol{\Phi}_1 = \begin{bmatrix} \lambda_1\,\mathbf{X}_{\mathbf{q}_{A1}} & \lambda_2\,\mathbf{X}_{\mathbf{q}_{A2}} & \cdots & \lambda_{2n+r}\,\mathbf{X}_{\mathbf{q}_{A,2n+r}} \end{bmatrix} \tag{5.53}$$

$$\dot{\boldsymbol{\gamma}} = \mathbf{d}\,\mathbf{E}\,\boldsymbol{\Theta}_{s1}\,, \quad \boldsymbol{\Theta}_1 = \begin{bmatrix} \lambda_1\,\mathbf{X}_{\boldsymbol{\gamma}_1} & \lambda_2\,\mathbf{X}_{\boldsymbol{\gamma}_2} & \cdots & \lambda_{2n+r}\,\mathbf{X}_{\boldsymbol{\gamma}_{2n+r}} \end{bmatrix} \tag{5.54}$$

one may introduce these results and the general solutions of $\mathbf{q}_A$, $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}$ into equation (5.52) such that

$$\mathbf{u}_A = \mathbf{R}_{\mathrm{u}}^{-1}\,\mathbf{S}_A^T\,\mathbf{d}\,\mathbf{E}\,\boldsymbol{\Psi}_s - \mathbf{R}_{\mathrm{u}}^{-1}\,\mathbf{d}\,\mathbf{E}\,\boldsymbol{\Theta}_{s1} = \mathbf{G}_k\,\mathbf{d}\,\mathbf{E}\,\boldsymbol{\Phi}_s + \mathbf{G}_d\,\mathbf{d}\,\mathbf{E}\,\boldsymbol{\Phi}_{s1} \tag{5.55}$$

Rearranging yields the result

$$\begin{bmatrix} \mathbf{G}_k & \mathbf{G}_d \end{bmatrix} = \mathbf{R}_{\mathrm{u}}^{-1}\left[\mathbf{S}_A^T\,\boldsymbol{\Psi}_s - \boldsymbol{\Theta}_{s1}\right]\begin{bmatrix} \boldsymbol{\Phi}_s \\ \boldsymbol{\Phi}_{s1} \end{bmatrix}^{\dagger} \tag{5.56}$$

where † denotes a pseudo-inverse.

It may be observed that equation (5.56) does not yield a unique solution due to the pseudo-inverse thus one may conclude that the method cannot yield the optimal solution. The method does however retain the second order structure and the author has observed from numerical trials that for the single degree of freedom system the method does obtain satisfactory results compared to the augmented and LQR approaches. This observation is particularly pleasing because the modal control part of this thesis deals with single degree of freedom systems. A numerical example now highlights this assertion.

5.5 Numerical Example 5.3

A single degree of freedom system is considered with the equations of motion

$$\ddot{\mathbf{q}}_A + 0.81\dot{\mathbf{q}}_A + 9\mathbf{q}_A = \mathbf{u}_A \tag{5.57}$$

The single degree of freedom systems are the form of equation typically dealt with during modal control analysis.

It is desired to minimise the potential and kinetic energies of the system and to place a large weighting on the rate of change of control compared with the control. The corresponding weighting matrices are given by

$$\underline{\mathbf{Q}} = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \quad , \ \mathbf{R}_{\mathrm{u}} = 10^{-3} \quad , \ \mathbf{R}_{\mathrm{v}} = 10^{-1} \tag{5.58}$$

The system is subjected to initial conditions $\mathbf{q}_A(0) = 1$ and $\dot{\mathbf{q}}_A(0) = 0$. The response of the displacement $\mathbf{q}_A$ is observed.

The response of the system to the initial conditions is plotted for three optimal controllers: (1) Pseudo-Inverse (second order) method. (2) Augmented method. (3) Standard LQR placing no emphasis on rate of change of control. The responses, force and force rates are illustrated in figures 5.8 to 5.10.

The respective costs for the three methods are: Augmented Approach, 7.6335; Pseudo-Inverse Approach, 9.0784; and LQR Approach, 5.0968 $\times 10^4$.

As may be observed from these costs the augmented and pseudo-inverse methods yield similar costs which are substantially lower than the LQR cost. One may see from figures 5.8 to 5.10 the response of the augmented and pseudo-inverse methods are similar in magnitude and behaviour but not identical hence yielding slightly different costs. The LQR method yields a substantially different controller and attempts no penalisation of the rate of change of control hence the sizeable difference in cost. Due to the non-uniqueness of the pseudo-inverse, the pseudo-inverse method can only ever match the augmented system in terms of performance and can never exceed it due to the augmented approach representing the optimal controller.

5.6 Numerical Example 5.4

The pseudo-inverse second order analytical method is now applied to the system presented in numerical example 5.1. One may observe the un-forced response of the third mass in figure 5.3. Recalling that the weighting matrices were chosen to be

$$\mathbf{Q}_{\mathrm{d}} = \mathbf{K}_A, \quad \mathbf{Q}_{\mathrm{v}} = \mathbf{M}_A, \quad \mathbf{R}_{\mathrm{u}} = 10^{-2}\mathbf{I}, \quad \mathbf{R}_{\mathrm{v}} = 10^{2}\mathbf{I} \tag{5.59}$$

One may solve for equation (5.56) such that the controller gains are found to be

$$\mathbf{G}_k = \begin{bmatrix} 1.5173 & -0.87023 & 0.078398 \\ -0.80025 & 0.95397 & -0.25301 \end{bmatrix} \tag{5.60}$$

$$\mathbf{G}_d = \begin{bmatrix} -0.15566 & -0.028823 & -0.029775 \\ -0.12628 & -0.26609 & -0.070203 \end{bmatrix} \tag{5.61}$$

Applying the pseudo-inverse controller to the system described by equation (5.39) one obtains the response given in figure 5.11. This may be directly compared to the augmented optimal control approach and conventional LQR approach given in figures 5.4 and 5.5 respectively. Utilising the extended cost function described by equation (5.12) one finds that quadratic cost of the pseudo-inverse controller is $J_{pi} = 1031.2$. This compares to $J_{aug} = 161$ for the augmented approach and $J_{lqr} = 24,835$ for the LQR approach. The cost from the pseudo-inverse method is 6.4 times larger than the augmented optimal approach but 24 times smaller than the associated LQR cost. Correspondingly one may observe that the pseudo-inverse method has successfully penalised the time derivative of the control force in addition to the conventional parameters. Although one has not strictly obtained the optimal control as may be found for the augmented method, the pseudo-inverse method has retained the second order structure of the control problem whilst providing more suitable control than the LQR method in this instance.

5.7 Modelling of the Actuator Dynamics

So far this chapter has presented new methods in which one tries to weight relative importance of the rate of change of control force against system response and control action. One may ask if the same can be achieved through conventional optimal control if one includes the actuator dynamics in the system equations of motion.

Consider magnetic bearings as a representative contemporary example of a control actuator for a dynamic system. Figure 5.12 shows the schematic of an 8 pole 'active magnetic bearing' (AMB) [C3]. It is usual to operate these bearings with a bias current, $\mathbf{I}_b$ such that the net force produced by the bearing in a direction is linearly proportional to the control currents ($\mathbf{i}$) injected into the bearing. Thus, for the 8 pole AMB given in figure 5.12 one finds the linearised force equations in horizontal (x) and vertical (y) directions to be

$$\mathbf{u}_x = \frac{2\,L_{0,13}}{g}\cos(\alpha)\,\mathbf{I}_{b13}\,\mathbf{i}_{13} \tag{5.62}$$

$$\mathbf{u}_y = \frac{2\,L_{0,24}}{g}\sin(\alpha)\,\mathbf{I}_{b24}\,\mathbf{i}_{24} \tag{5.63}$$

where g is the air gap, α is the angle angular position of the poles and the subscripts 13 and 24 refer to the magnet pairs 1 and 3, and 2 and 4 illustrated in figure 5.12. L_0 is the induction of the coil defined as

$$L_0 = \frac{N^2\mu_0 wl}{2(g-x)} \tag{5.64}$$

N is the number of turns of the coil, μ_0 is permeability of free space ($4\pi \times 10^{-7}$ H/m), w and l is the width and length of the of the air gap and x is the deviation of the air gap from g.

Assuming constant mechanical values one may simplify equations (5.62) and (5.63) such that

$$\mathbf{u}_{amb} = \begin{bmatrix} \mathbf{u}_x \\ \mathbf{u}_y \end{bmatrix} = \frac{2}{g} \begin{bmatrix} L_{0,13} \cos(\alpha)\, \mathbf{I}_{b13} & 0 \\ 0 & L_{0,24} \cos(\alpha)\, \mathbf{I}_{b24} \end{bmatrix} \begin{bmatrix} \mathbf{i}_{13} \\ \mathbf{i}_{24} \end{bmatrix} = \overline{\mathbf{K}}\, \mathbf{i} \tag{5.65}$$

As stated AMBs have some inductance thus the change of current requires additional voltage on top of the voltage required to drive a steady current. Thus one finds that the current is related to the applied voltage in the windings as

$$\mathbf{V}_{dc} = \frac{d}{dt} (\mathbf{i} - \mathbf{i}_0)\, L_0 + \mathbf{r}\, \mathbf{i} \tag{5.66}$$

where $\mathbf{r}$ is the resistance of the coil and $\mathbf{i}_0$ is the initial current. Rearranging equation (5.66) and realising $\frac{d}{dt}\, \mathbf{i}_0 = 0$ yields

$$\frac{d}{dt}\mathbf{i} = \frac{1}{L_0}\mathbf{V}_{dc} - \frac{\mathbf{r}}{L_0}\, \mathbf{i} \tag{5.67}$$

Recalling figure 5.12 one may combine the equations for magnet pairs 13 and 24 to give

$$\begin{bmatrix} \frac{d}{dt}\mathbf{i}_{13} \\ \frac{d}{dt}\mathbf{i}_{24} \end{bmatrix} = \begin{bmatrix} \frac{1}{L_{0,13}} & 0 \\ 0 & \frac{1}{L_{0,24}} \end{bmatrix} \begin{bmatrix} \mathbf{V}_{dc13} \\ \mathbf{V}_{dc24} \end{bmatrix} - \begin{bmatrix} \frac{1}{L_{0,13}} & 0 \\ 0 & \frac{1}{L_{0,24}} \end{bmatrix} \begin{bmatrix} \mathbf{r}_{13} & 0 \\ 0 & \mathbf{r}_{24} \end{bmatrix} \begin{bmatrix} \mathbf{i}_{13} \\ \mathbf{i}_{24} \end{bmatrix} \tag{5.68}$$

which may be simplified to

$$\frac{d}{dt}\mathbf{i} = \mathbf{L}_0^{-1}\mathbf{V}_{dc} - \mathbf{L}_0^{-1}\mathbf{R}_r\, \mathbf{i} \tag{5.69}$$

Equations (5.65) and (5.69) contain the necessary information which describes the actuator equations of motion. One may incorporate these equations with the state space equations of motion given in equation (5.2) such that one has

$$\begin{bmatrix} \frac{d}{dt}\mathbf{i} \\ \underline{\dot{\mathbf{q}}_{A1}} \\ \underline{\dot{\mathbf{q}}_{A2}} \end{bmatrix} = \begin{bmatrix} -\mathbf{R}_r\, \mathbf{L}_0^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{M}_A^{-1}\mathbf{S}_A\, \overline{\mathbf{K}} & -\mathbf{M}_A^{-1}\mathbf{K}_A & -\mathbf{M}_A^{-1}\mathbf{D}_A \end{bmatrix} \begin{bmatrix} \mathbf{i} \\ \underline{\mathbf{q}_{A1}} \\ \underline{\mathbf{q}_{A2}} \end{bmatrix} + \begin{bmatrix} \mathbf{L}_0^{-1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{V}_{dc} \tag{5.70}$$

One may further simplify the mechanical-electrical equations of motion to give

$$\underline{\dot{\mathbf{q}}_{me}} = \overline{\mathbf{A}}\, \underline{\mathbf{q}_{me}} + \overline{\mathbf{B}}\, \mathbf{V}_{dc} \tag{5.71}$$

One may notice that the combined system and actuator equations of motion are similar to those obtained for the augmented first order system obtained in equation (5.23).

The model illustrated by equation (5.70) contains all the necessary information of the mechanical and electrical system. Since the rate of change of force is directly dependent on the rate of change of current which in turn is directly dependent of the finite voltage $\mathbf{V}_{dc}$ supplied to the windings, one may penalise the rate of change of control force by penalising the voltage applied. Likewise one may also penalise the control force by placing appropriate weighting on the current which is augmented in the new system state vector, $\underline{\mathbf{q}_{me}}$. Thus one finds that the cost function outlined in equation (5.7) is equivalent to

$$J = \int_0^\infty \frac{1}{2}\,\underline{\mathbf{q}_{me}}^T \underline{\mathbf{Q}_{me}}\,\underline{\mathbf{q}_{me}} + \frac{1}{2}\mathbf{V}_{dc}{}^T\mathbf{R}_{me}\,\mathbf{V}_{dc}\;\, dt \tag{5.72}$$

for the combined mechanical-electrical system.

The obvious question now arises concerning the relevance of previous sections when one may potentially achieve the same result through including the actuator dynamics in the equations of motion? This question may be addressed by considering some of the necessary requirements imposed on AMBs.

The coil resistance of the magnetic bearings is often very small resulting in the combined equations of motion of equation (5.70) becoming stiff. For example in numerical example 5.2 of the book by Chiba *et al.* [C3] an AMB with a copper coil is presented in which the coil resistance is calculated to be $\mathbf{r} = 0.38\Omega$. Thus magnetic bearing controllers are often used to overcome this problem to provide better frequency bandwidth, disturbance rejection and increased stability for the bearings. This further complicates the combined equations of motion and requires one to know in advance the controller chosen for the AMBs.

The main argument for the preceding sections is that one does not require knowledge of the actuator dynamics to provide adequate control. As has been shown in this section the combined mechanical-electrical equations of motion is almost equivalent to that obtained from the augmented approach outlined in section 5.3. Thus there may be no advantage to modelling the actuator dynamics, indeed by not including the actuator dynamics in the model negates the requirement of knowing the exact properties of the AMB and the bearing controller action applied to it.

5.8 Summary

Conventional optimal control enables a trade off to be made between the applied control force and the response of the system subjected to arbitrary initial conditions. This enables one to judge the relative importance of each variable. However there may exist limitations on how quickly the control forces may be altered meaning that conventional optimal control may not provide the optimal response if one tries to exceed these limitations. For certain applications such as electro-magnetic bearings this represents a real constraint and thus it becomes essential to account for this limitation in the formulation of the optimal control problem. This chapter has achieved this.

This chapter has approached this limitation of the optimal control problem by first extending the conventional first order optimal control problem to augment the system state with the control force. This requires that one feeds back the rate of change of control force and subsequently integrate it. One obtains a conventional first order state space system for which the conventional optimal control method may be applied to the augmented system. One may consequently penalise the rate of change of control force by augmenting the state weighting matrix to include the weighting placed on the control force and use the conventional weighting matrix used to weight the control force to weight the importance of the rate of change of control force instead. Numerical example has demonstrated that this method successfully penalises the rate at which control forces are applied to the system.

The purpose of this thesis has been to extend direct second order control techniques. A sub-optimal control method has been presented by which one can approach the extended optimal control problem using the second order matrices. One uses a non-unique pseudo-inverse to generate the feedback controller matrices meaning that the method presented does not provide the optimal solution. Despite the non-uniqueness of the solution, numerical example has shown that the controller provide a means to penalise the rate of change of control force in addition to the displacements, velocities and control forces of the system. One does indeed manage to provide substantial improvement to limiting the time derivative of the control force.

This chapter has considered the possibility of incorporating the actuator dynamics into the equations of motion. Through inclusion of the actuator dynamics one may apply the conventional optimal control problem and place appropriate weighting on the components which are directly responsible for imposing the physical limitations on the actuators. This would make the need to extend the optimal control problem obsolete but one would require exact knowledge of the actuator properties. This chapter has shown that although this can be achieved through great effort, a more convenient (and almost equivalent) method would be to use one of the extended optimal control method presented in this chapter. This would negate the need to accurately model the actuator dynamics in the equations of motion.

5.9 Conclusions

In this chapter an extension to the optimal control problem has been presented in which the time derivative of the control vector has been incorporated. Numerical examples have been presented and compared with the traditional LQR approach.

For the first numerical example presented it could be argued that the weighting placed on the rate of change of force is substantially larger than the weighting placed on the force so the LQR approach is immediately disadvantaged due to no inclusion of the rate to the control problem. This is precisely the key message that the author is trying to present because there exist applications where the weighting on the rate at which the force can be applied will be much higher than the weighting placed on the force itself. The second numerical example illustrates that the method yields itself to practical situations and again appropriately penalises the time derivative of force as required.

As already stated the eigenvalues of the Hamiltonian system are symmetric about the imaginary axis and contain the eigenvalues of the final optimal system. Thus solving the optimal control problem to yield a feedback controller, the eigenvalues of the closed loop system must match that of the Hamiltonian system. In addition one may conclude that the Hamiltonian system illustrated by equation (5.20) and the Hamiltonian $\underline{\mathscr{H}_{aug}}$ of the augmented system must have the same eigenvalues since they represent the same optimal control problem. Indeed for the numerical examples presented, this assertion has been shown to be correct.

Augmented System	Second Order System
-0.069231	-0.069231
-0.14455	-0.14455
-0.097087 ± 0.46466i	-0.097087 ± 0.46466i
-0.045546 ± 1.2486i	-0.045546 ± 1.2486i
-0.037027 ± 1.8027i	-0.037027 ± 1.8027i

Table 5.1: Numerical example 5.2 - Eigenvalues of augmented versus second order Hamiltonian system

Augmented System	Second Order System
-0.0038066 ± 348.12i	-0.003807 ± 348.12i
-0.0079546 ± 269.68i	-0.0079547 ± 269.68i
-0.0084182 ± 191.43i	-0.0084181 ± 191.43i
-0.019247 ± 63.613i	-0.019247 ± 63.613i
-0.018714 ± 54.194i	-0.018714 ± 54.194i
-100 ± 0i	-100 ± 0i

Table 5.2: Numerical example 5.3 - Eigenvalues of augmented versus second order Hamiltonian system

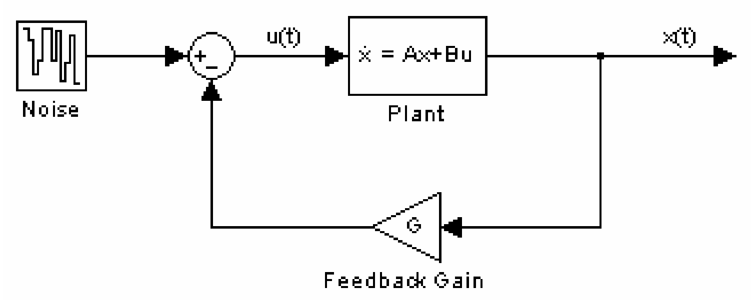


Figure 5.1: Pictorial representation of control system

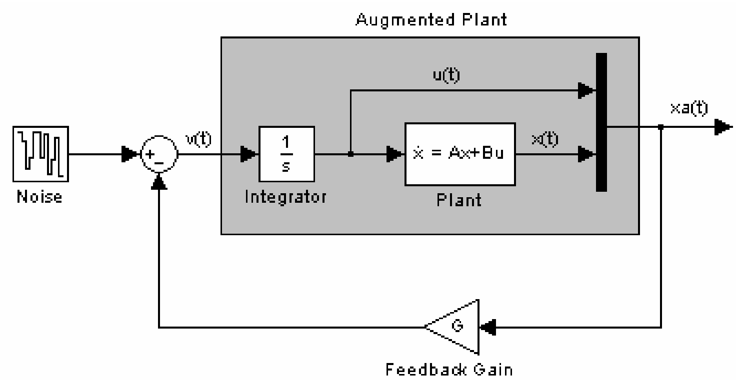


Figure 5.2: Augmented plant

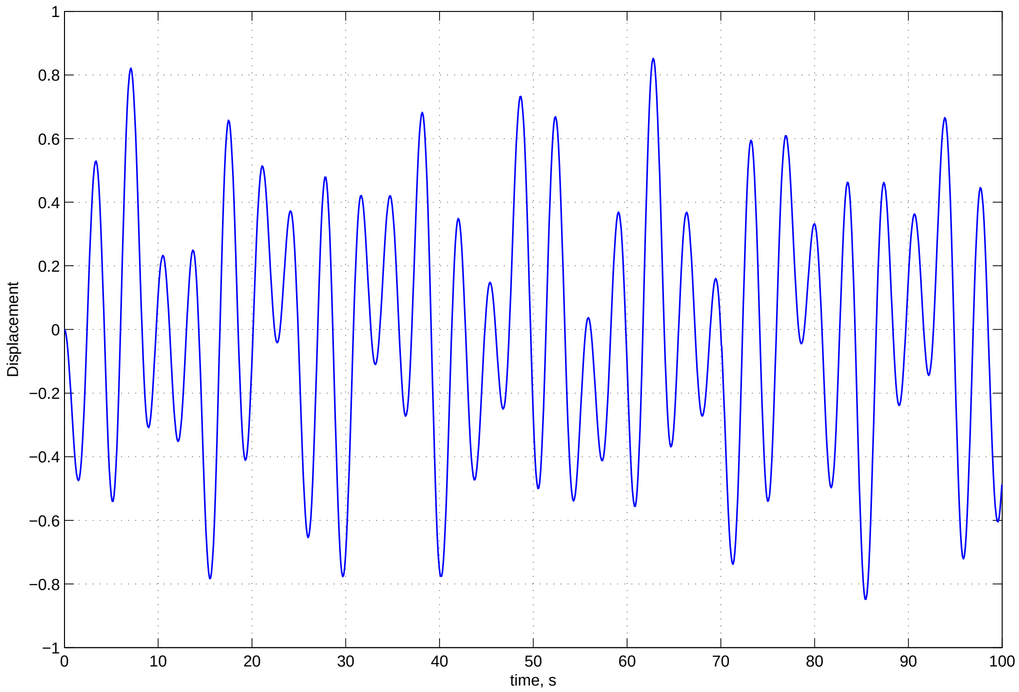


Figure 5.3: Numerical Example 5.1 un-forced response to initial conditions

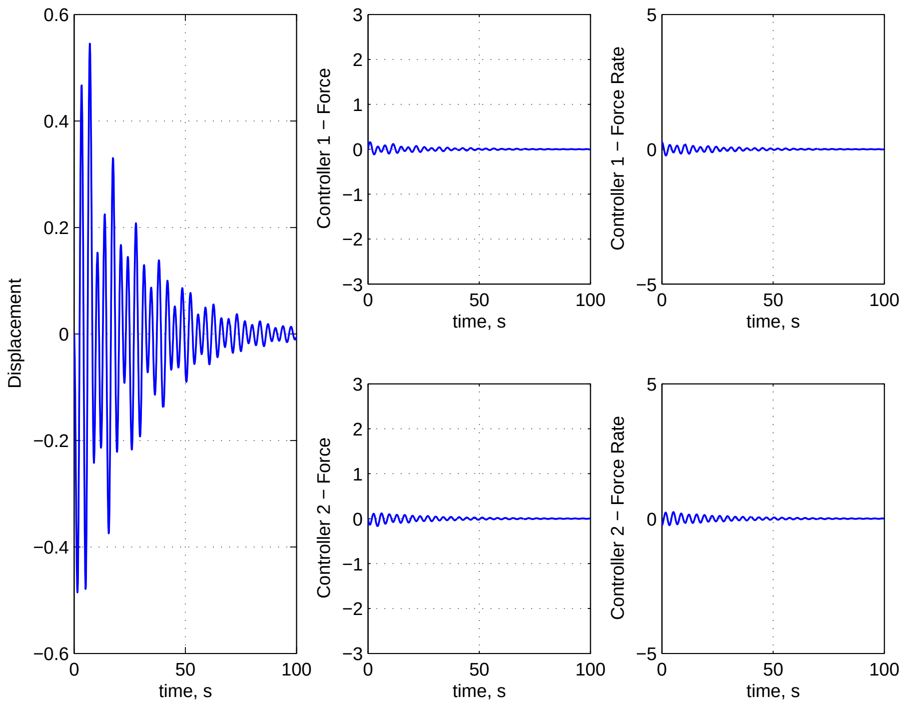


Figure 5.4: Optimal augmented system response to initial conditions

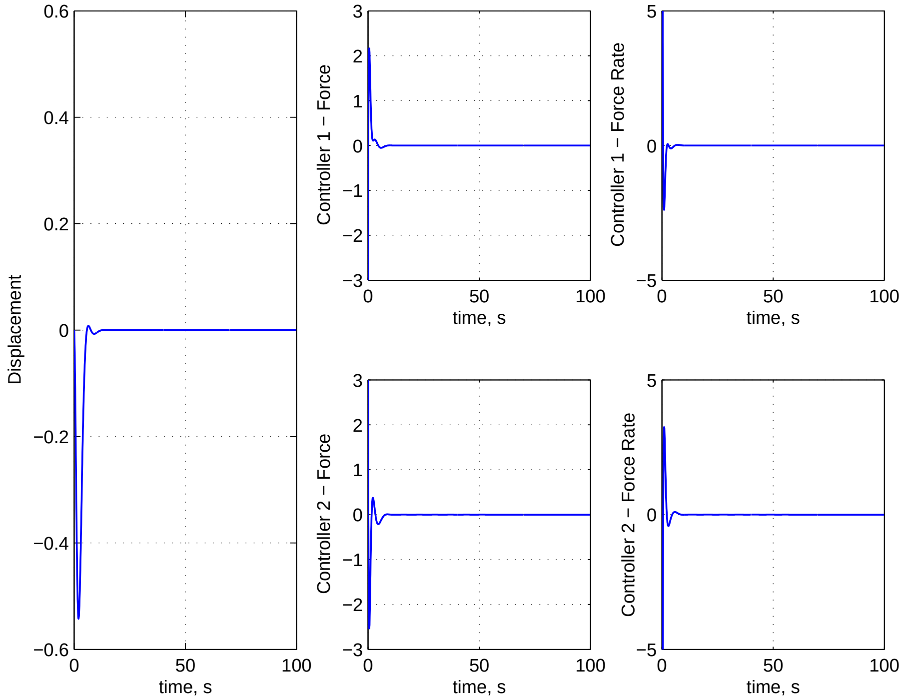


Figure 5.5: LQR system response to initial conditions

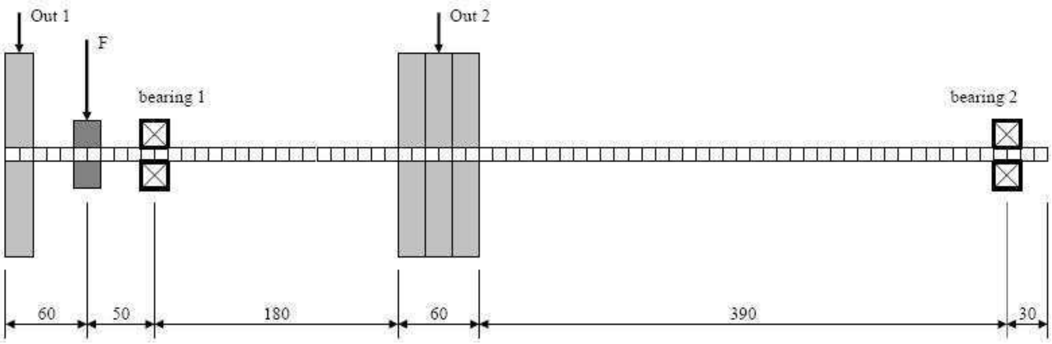


Figure 5.6: Rotor-dynamic model

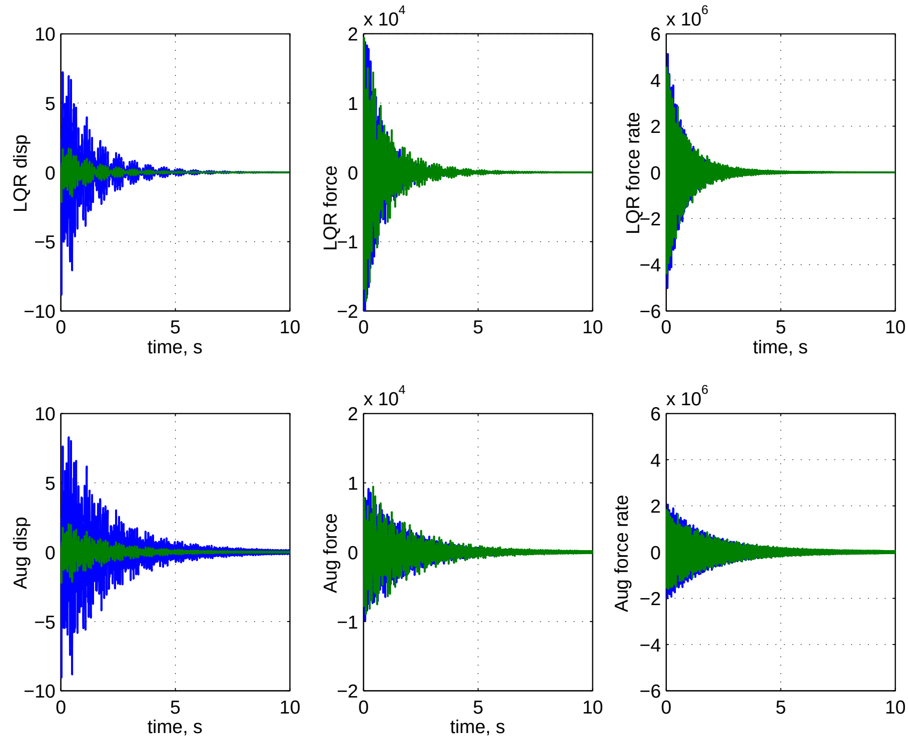


Figure 5.7: Rotor Dynamic Model Response to Control

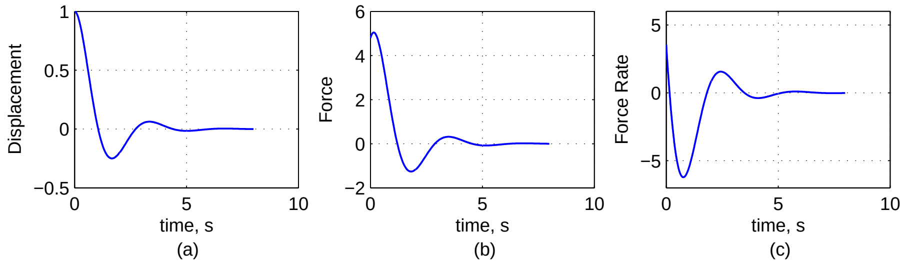


Figure 5.8: Pseudo-Inverse Controller: (a) response, (b) force, (c) force rate

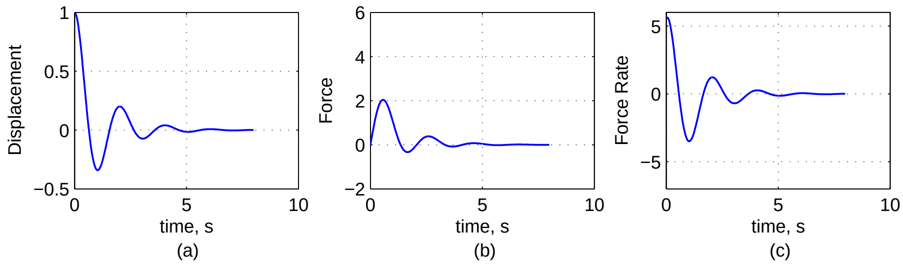


Figure 5.9: Augmented Controller: (a) response, (b) force, (c) force rate

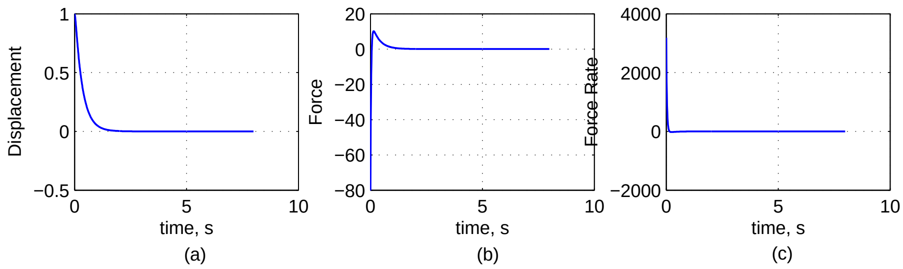


Figure 5.10: LQR Controller: (a) response, (b) force, (c) force rate

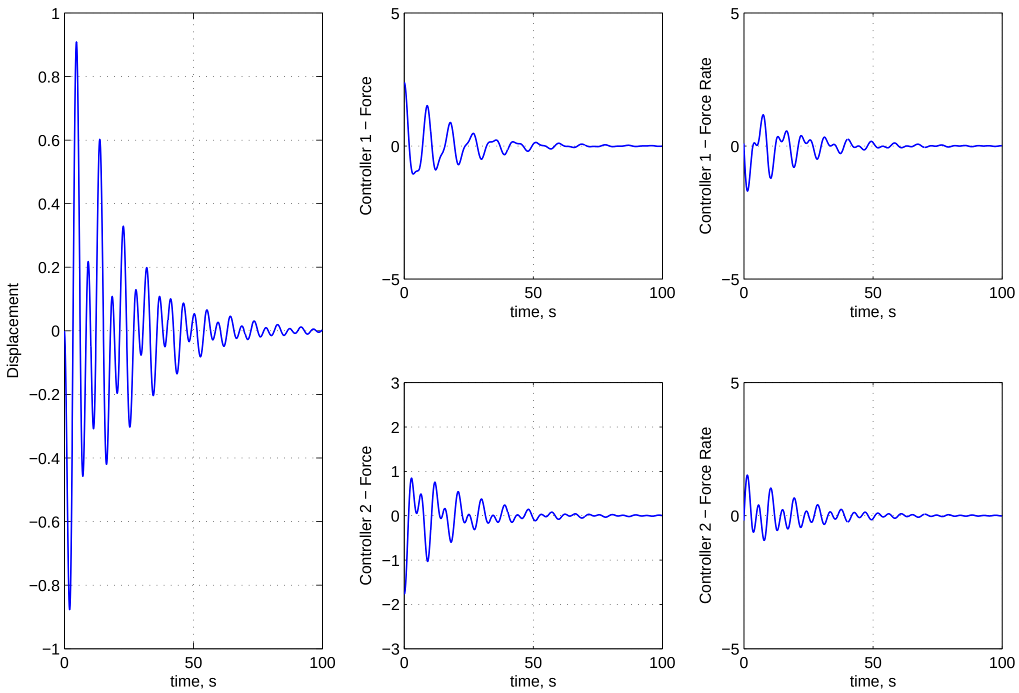


Figure 5.11: Numerical Example 5.4 Pseudo Inverse Control On

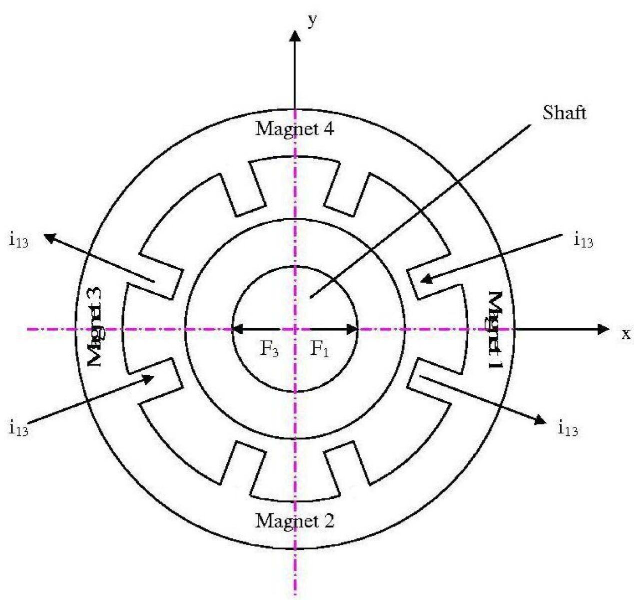


Figure 5.12: Schematic of 8 pole active magnetic bearing

Chapter 6

Theoretical Case Study

The purpose of this chapter is to present a detailed case study of a rotor-dynamic control problem. A rotor-system is presented to which the 'structure preserving transformation' (SPT) based modal control technique is used to bring the vibrations of the system under control. The controlled and uncontrolled responses are compared to show the effect of the control applied to the system.

The SPT-based method is compared with the conventional 'independent modal space control' (IMSC) method. Although the two methods are not directly comparable due to differences in the definition of the modes the comparison shows the advantages of the SPT method over the IMSC method.

6.1 Rotor Model

Consider the over-hung rotor system shown in figure 6.1. The rotor comprises 6 equal length elements of length 0.2 m with 4 modelled 'degrees of freedom' (DOF) at each node (2 translational, 2 rotational). The shaft of the rotor has diameter 0.05 m and is constructed from steel with Young's modulus, $E = 200$ GPa and density, $\rho = 7800$ kg/m^3.

The shaft of the rotor is supported by bearings at nodes 1 and 5. The bearings are chosen deliberately to be orthotropic with properties shown in table 6.5.

At node 7 a disc of diameter 0.3 m and axial thickness 0.04 m is situated. The disc is made from the same material as the shaft. The nature of the disc compared to the bearings ensures that the modes are well coupled due to the Coriollis effects. This essentially means that the modes of the system are dependent on the spin speed of the rotor. This may be represented using the Campbell diagram [G9] shown in figure 6.2. The Campbell diagram shows how the imaginary parts of the eigenvalues vary with respect to shaft speed. As may be seen in figure 6.2 the natural frequencies split representing the forward and backward whirl modes. The splitting of the forward and backward modes is due to the spin-dependent gyroscopic forces which are proportional to velocity. Correspondingly these cross-coupling forces are represented by skew-symmetric components residing in the system damping matrix. This structure of the damping matrix ensures that the over-hung rotor system in this case study is very non-classically damped.

The rotor is assumed to be driven by a motor situated at node 3. The motor has weight of 10.4 kg and acts as point mass on the rotor. It is assumed that parallel windings are added to the motor such that lateral forces can be applied using the motor as described by Khoo *et al*, [K4]. Thus the motor can be used to apply lateral forces in the $x-$ and y-directions respectively and the motor is assumed to act as a self-sensing magnetic bearing such that displacements at this location may also be observed.

Finite element (FE) modelling is used to generate a suitable description of the rotor system. A 28-dimensional FE model is obtained requiring the necessary step of dimensional reduction to a more manageable dimension. The rotor is thus reduced in dimension to 6 DOFs using the Guyan reduction [G12]. The degrees of freedom retained are determined by the automated process described by Henshell and Ong [H1] and are found to correspond to the 4 translation degrees of freedom at the actuator location and disc location and the rotational degrees of freedom at the disc.

Analysis of the Bode magnitude plots of the full and reduced systems, found in figures 6.3 and 6.4, show that the response is dominated by the lower order mode-pairs and less influenced by the higher order mode-pairs. The reduced order model retains the lower order modes of significance. Further justification for the order of the reduced model can be found from comparing the impulse responses of the full and reduced systems found in figures 6.5 and 6.6. As may be observed from figures 6.5 and 6.6 the reduced system exhibits the characteristics of the full system quite accurately.

The initial displacements of the rotor are assumed to be identical to the first mass-normalised undamped mode shape in the x-direction for a rotor speed of $5,000$ rpm. The initial velocities are zero. The physical response to these initial conditions may be observed in figure 6.7. As may be observed from figure 6.7 the rotor response is not heavily damped and the response is not noticeably decaying, thus it could be brought under control quickly using control techniques.

6.2 SPT-Control

An SPT based controller is designed to control the rotor system. One finds that for the SPT diagonalising transformation one finds the modal equations of motion to be

$$\begin{array}{lllllll} \ddot{\mathbf{q}}_{m1\_spt} & + & 0.0081577\,\dot{\mathbf{q}}_{m1\_spt} & + & 28834\,\mathbf{q}_{m1\_spt} & = & \mathbf{u}_{m1\_spt} \\ \ddot{\mathbf{q}}_{m2\_spt} & + & 0.016809\,\dot{\mathbf{q}}_{m2\_spt} & + & 44516\,\mathbf{q}_{m2\_spt} & = & \mathbf{u}_{m2\_spt} \\ \ddot{\mathbf{q}}_{m3\_spt} & + & 0.60207\,\dot{\mathbf{q}}_{m3\_spt} & + & 4.8788\times 10^{5}\mathbf{q}_{m3\_spt} & = & \mathbf{u}_{m3\_spt} \\ \ddot{\mathbf{q}}_{m4\_spt} & + & 0.55289\,\dot{\mathbf{q}}_{m4\_spt} & + & 5.1457\times 10^{5}\mathbf{q}_{m4\_spt} & = & \mathbf{u}_{m4\_spt} \\ \ddot{\mathbf{q}}_{m5\_spt} & + & 2.6745\,\dot{\mathbf{q}}_{m5\_spt} & + & 2.0422\times 10^{6}\mathbf{q}_{m5\_spt} & = & \mathbf{u}_{m5\_spt} \\ \ddot{\mathbf{q}}_{m6\_spt} & + & 3.7297\,\dot{\mathbf{q}}_{m6\_spt} & + & 5.138\times 10^{6}\mathbf{q}_{m6\_spt} & = & \mathbf{u}_{m6\_spt} \end{array} \tag{6.1}$$

If one considers single degree of freedom (SDOF) modal equations to have the classical form

$$\ddot{\mathbf{q}}_{mj\_spt} + 2\gamma_j\omega_j\dot{\mathbf{q}}_{mj\_spt} + \omega_j^2\mathbf{q}_{mj\_spt} \tag{6.2}$$

where $\mathbf{q}_{mj\_spt}$ is the displacement of the j^{th} mode ($j = 1, 2, \ldots, n$) and γ_j and ω_j are damping ratio and natural frequency of the j^{th} mode, then one may find the damping ratios for the modal equations as shown in table 6.5.

From table 6.5 one may see that the damping ratio of the first two pairs of modes is significantly less than the other modelled modes meaning that the response from these mode-pairs will take much longer to decay. The Bode magnitude plot in figure 6.4 shows that the physical response of the system is dominated by these first two modes of vibration and it is decided to control these mode-pairs.

The controller for the mode-pairs is designed using optimal control so as to minimise the kinetic and potential modal energies of the system. One chooses the weighting matrix on the modal forces is to be 10^{-3} for both mode-pairs. It was found that this level of weighting on the modal force ensured that the modal responses were brought under control sufficiently quickly without requiring a very large control burden due to the force. The resulting modal controllers for the first and second mode-pairs are found to be (see section 3.4 for further details)

$$\mathbf{u}_{m1\_spt} = -\left(495.74\,\mathbf{q}_{m1\_spt} + 44.618\,\dot{\mathbf{q}}_{m1\_spt}\right) \tag{6.3}$$

$$\mathbf{u}_{m2\_spt} = -\left(497.22\,\mathbf{q}_{m2\_spt} + 44.642\,\dot{\mathbf{q}}_{m2\_spt}\right) \tag{6.4}$$

The physical response of the system when the SPT-modal controller is on may be observed in figure 6.8. As expected, the physical response of the system is affected by the SPT controller. The initial displacements are rapidly reduced to a value of approximately 0.018 where the physical response proceeds to decay much more slowly. The reasons for this response may be found in the modal responses of the uncontrolled and controlled modal responses found in figures 6.10 and 6.11 respectively. One observes that the first two mode-pairs are rapidly brought under control and the remaining modes are not controlled. Consequently, the physical effects of the first two mode-pairs are removed from the physical response and the remaining modal responses dictate the longer term physical response shown in figure 6.8.

The modal responses of the system for the uncontrolled and controlled cases are perhaps more useful in the analysis of the controller. The ambition of the SPT-controller was to bring the first two mode-pairs under control and leave the remaining modes uncontrolled. By comparing the modal responses for mode-pairs 1 and 2 in figures 6.10 and 6.11 one can readily see that these mode-pairs are brought under control. As may be observed the remaining uncontrolled modes are excited by control spillover effects although the effects are not significant. The important observation is to note that modes do not excite or couple with each other meaning that full, independent modal control has been obtained.

6.3 IMSC Control

To fully appreciate the advantages of the SPT-modal control method one must contrast the SPT method with the conventional IMSC method [B1]. The IMSC method uses the eigenvectors of the undamped system to diagonalise the system matrices. Any remaining off-diagonal terms in the damping matrix are stripped away such that they are discarded. This has the unfortunate consequence of ignoring any notion of gyroscopic coupling in the system model. The modal equations of motion for the IMSC method are found to be

$$\begin{aligned}
\ddot{\mathbf{q}}_{m1_imsc} &+ 0.013806\,\dot{\mathbf{q}}_{m1_imsc} + 36042\,\mathbf{q}_{m1_imsc} = \mathbf{u}_{m1_imsc} \\
\ddot{\mathbf{q}}_{m2_imsc} &+ 0.010035\,\dot{\mathbf{q}}_{m2_imsc} + 36448\,\mathbf{q}_{m2_imsc} = \mathbf{u}_{m2_imsc} \\
\ddot{\mathbf{q}}_{m3_imsc} &+ 0.66172\,\dot{\mathbf{q}}_{m3_imsc} + 4.909\times 10^{5}\,\mathbf{q}_{m3_imsc} = \mathbf{u}_{m3_imsc} \\
\ddot{\mathbf{q}}_{m4_imsc} &+ 0.51071\,\dot{\mathbf{q}}_{m4_imsc} + 5.1367\times 10^{5}\,\mathbf{q}_{m4_imsc} = \mathbf{u}_{m4_imsc} \\
\ddot{\mathbf{q}}_{m5_imsc} &+ 3.4203\,\dot{\mathbf{q}}_{m5_imsc} + 3.1096\times 10^{6}\,\mathbf{q}_{m5_imsc} = \mathbf{u}_{m5_imsc} \\
\ddot{\mathbf{q}}_{m6_imsc} &+ 2.9676\,\dot{\mathbf{q}}_{m6_imsc} + 3.2826\times 10^{6}\,\mathbf{q}_{m6_imsc} = \mathbf{u}_{m6_imsc}
\end{aligned} \tag{6.5}$$

From the modal equations of motion given in equation (6.5) one may observe the effect of stripping the modal damping matrix in table 6.3 by comparing the original system eigenvalues with those of the diagonalised IMSC eigenvalues.

As may be observed from table 6.3 the eigenvalues are substantially altered by discarding the off-diagonal terms in the modal damping matrix. Additionally any concept of coupling the modes by the damping matrix is ignored. Ignoring the coupling in the damping modal matrix cannot be justified in many cases and the coupling of the modes generated by the IMSC method can have significant effect as will be shown for this example.

One now designs the equivalent optimal IMSC modal controller so as to minimise the kinetic and potential modal energies. Because the IMSC method generates different definitions of the mode-pairs due to discarding the off-diagonal and skew-symmetry in the damping matrix, the IMSC diagonalised system matrices are different from the SPT diagonalised matrices. Correspondingly different optimal modal controllers are obtained and for the first two pairs of modes these are found to be

$$\mathbf{u}_{m1\_imsc} = -\left(496.58\,\mathbf{q}_{m1\_imsc} + 44.631\,\dot{\mathbf{q}}_{m1\_imsc}\right) \tag{6.6}$$

$$\mathbf{u}_{m2\_imsc} = -\left(496.62\,\mathbf{q}_{m2\_imsc} + 44.636\,\dot{\mathbf{q}}_{m2\_imsc}\right) \tag{6.7}$$

The physical effect of the IMSC controller may be observed in figure 6.9. The physical response of the system is not dissimilar to that of the SPT-method. The physical response of the system rapidly descends to an approximate value of 0.018 before decaying more slowly as a consequence of the remaining modal responses belonging to the uncontrolled modes. The differences between the SPT and IMSC methods become more apparent when one looks at the IMSC modal responses of the uncontrolled and controlled cases.

One now examines the modal responses to the uncontrolled and controlled systems represented in figures 6.12 and 6.13 respectively. The response of the first two mode-pairs are brought under control as was the case for the SPT-method. However when the uncontrolled modes are examined it becomes obvious that they are affected by the modal control force. This is particularly apparent when one examines the responses of mode-pairs 3 and 4. Thus the proficiency of the controller can be questioned because it is apparent from the modal response to the controller that the modes cannot be independently controlled for the IMSC method.

6.4 Summary

This chapter has presented a case study of a rotor system subjected to initial conditions. This rotor was modelled using the finite element method and the mathematical model generated was reduced in dimension using Guyan reduction. The initial conditions applied to the reduced model caused a vibrational response which decayed over a period of time. It was shown that the physical response was dominated by the first two modes of vibration and the SPT and IMSC methods were applied to reduce the first two modal responses to zero and consequently bringing the physical response of the system to minimum energy state more quickly.

Application of the SPT method enabled one to fully decouple the rotor system into decoupled systems whilst retaining the same eigenvalues of the system. This contrasts to the IMSC method where the undamped eigenvectors were used to create a pseudo-diagonalised system in which the remaining off-diagonal terms in the damping matrix (corresponding to the rotational nature of the system) were discarded as convention dictates. The effects of ignoring these off-diagonal damping terms was first demonstrated by comparing the eigenvalues of the IMSC and original system; substantial alteration of the system eigenvalues was observed. Perhaps more importantly the application of the IMSC modal controller showed that one could not control the modes independently resulting in the controlled modes exciting the uncontrolled modes. Both methods suffered from the effects of control spillover.

6.5 Conclusions

The purpose of this chapter was to present a detailed case study of a rotor-dynamic control problem. A rotor system has been presented in which the vibrations caused by initial conditions have been controlled using an SPT-based modal controller. The example has shown that the response of the system has been successfully teased apart into independent decoupled systems. The SPT-method has been directly contrasted to the conventional IMSC method in which true independent control was not achieved because of the non-classical damping associated with the skew-symmetry in the damping matrix. Although the two methods are not directly comparable due to differences in the definition of the modes the comparison shows that the SPT-method holds distinct advantage over the IMSC method.

Bearing	Bearing 1	Bearing 2
Stiffness $\mathbf{K}_{xx}$ (MN/m)	50	50
Stiffness $\mathbf{K}_{yy}$ (MN/m)	70	70
Damping $\mathbf{D}_{xx}$ (Ns/m)	500	500
Damping $\mathbf{D}_{yy}$ (Ns/m)	700	700

Table 6.1: Rotor system bearing properties

mode	damping ratio (γ)
1	0.000024021
2	0.000039834
3	0.000430983
4	0.000385378
5	0.002959126
6	0.002601638

Table 6.2: Damping ratios of SPT modes

Original	IMSC
-0.0040788 ± 169.8i	-0.006903 ± 189.85i
-0.0084045 ± 210.99i	-0.0050176 ± 190.91i
-0.30104 ± 698.48i	-0.33086 ± 700.64i
-0.27644 ± 717.34i	-0.25535 ± 716.71i
-1.3373 ± 1429.1i	-1.7102 ± 1763.4i
-1.8648 ± 2266.7i	-1.4838 ± 1811.8i

Table 6.3: Case study, comparison of original and IMSC eigenvalues

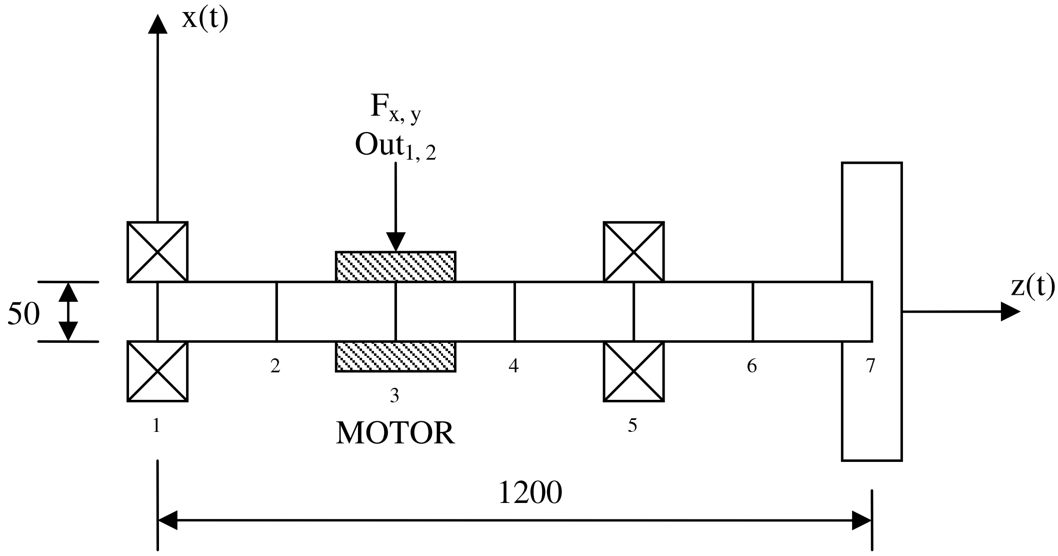


Figure 6.1: Schematic of over-hung rotor system

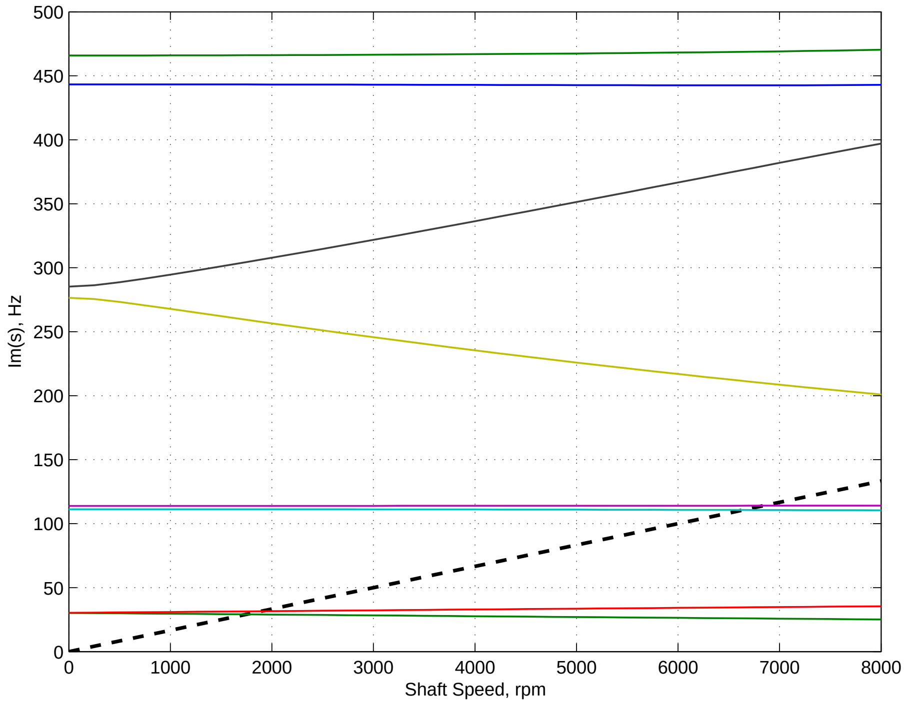


Figure 6.2: Campbell diagram of rotor system

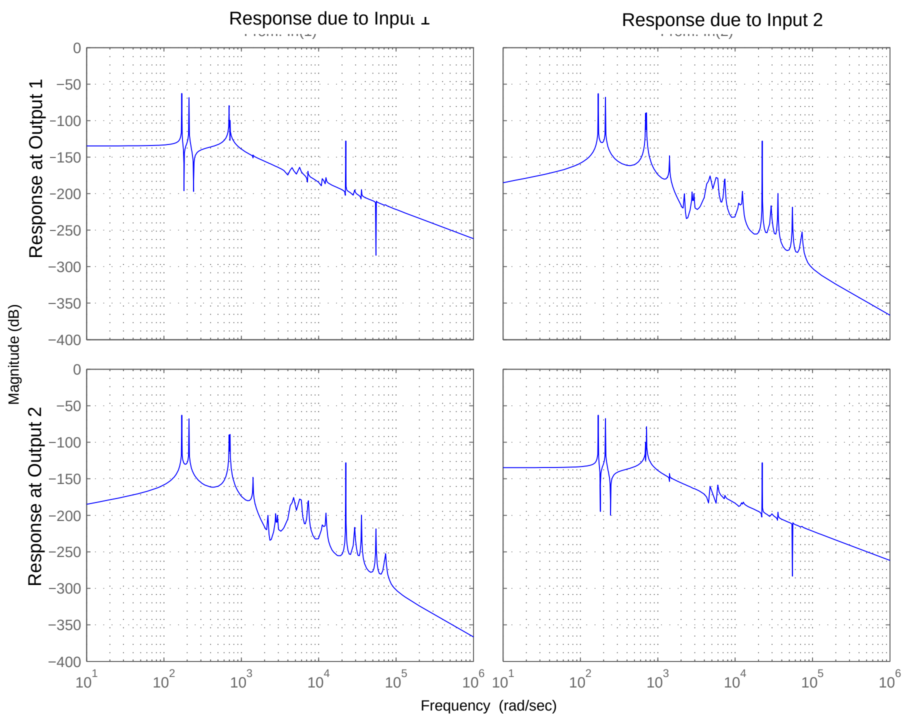


Figure 6.3: Bode magnitude plot of the full order system

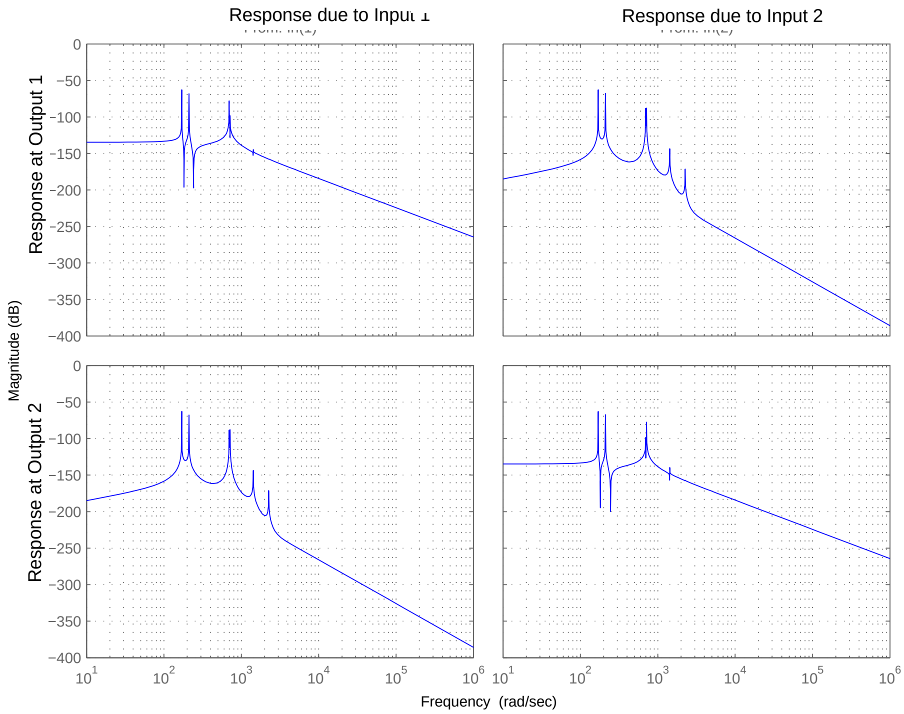


Figure 6.4: Bode magnitude plot of the reduced order system

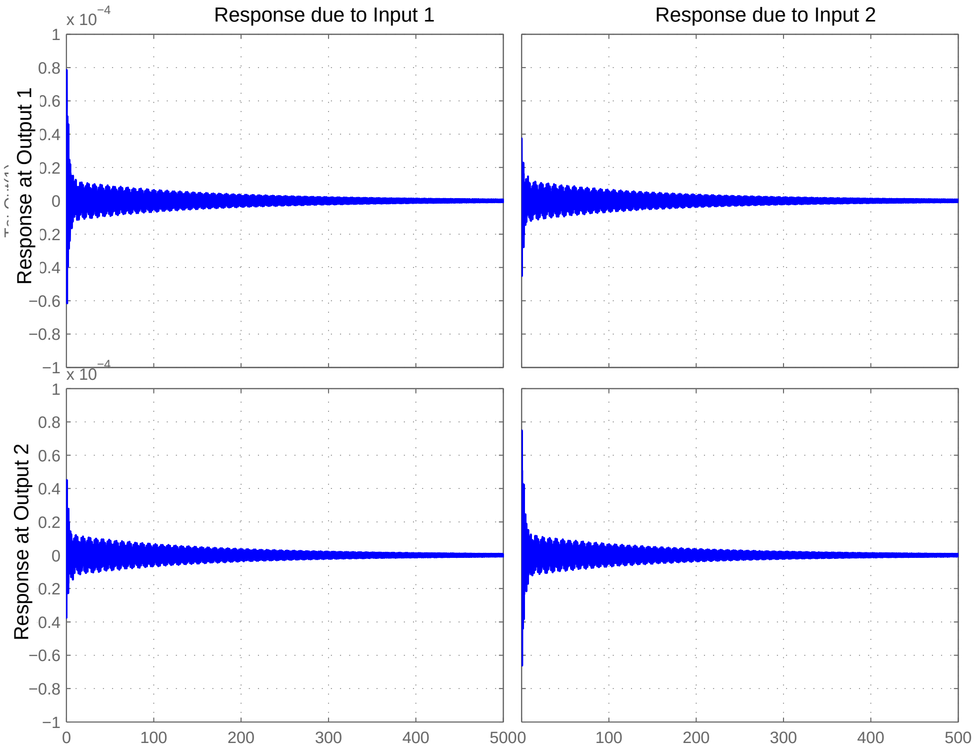


Figure 6.5: Impulse applied to full system

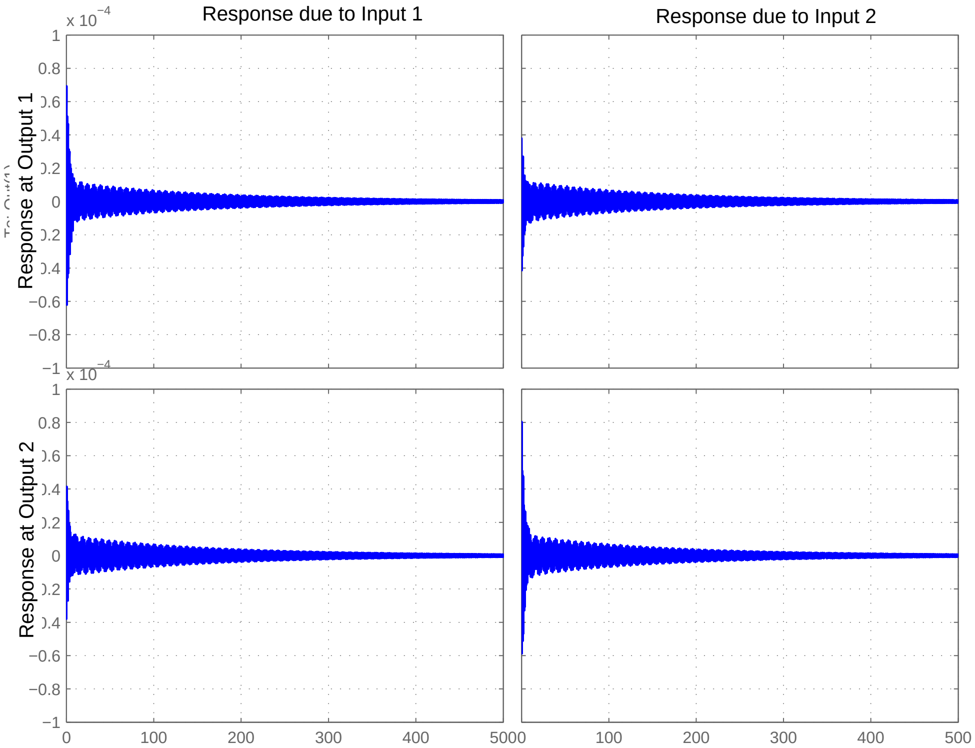


Figure 6.6: Impulse applied to Guyan reduced system

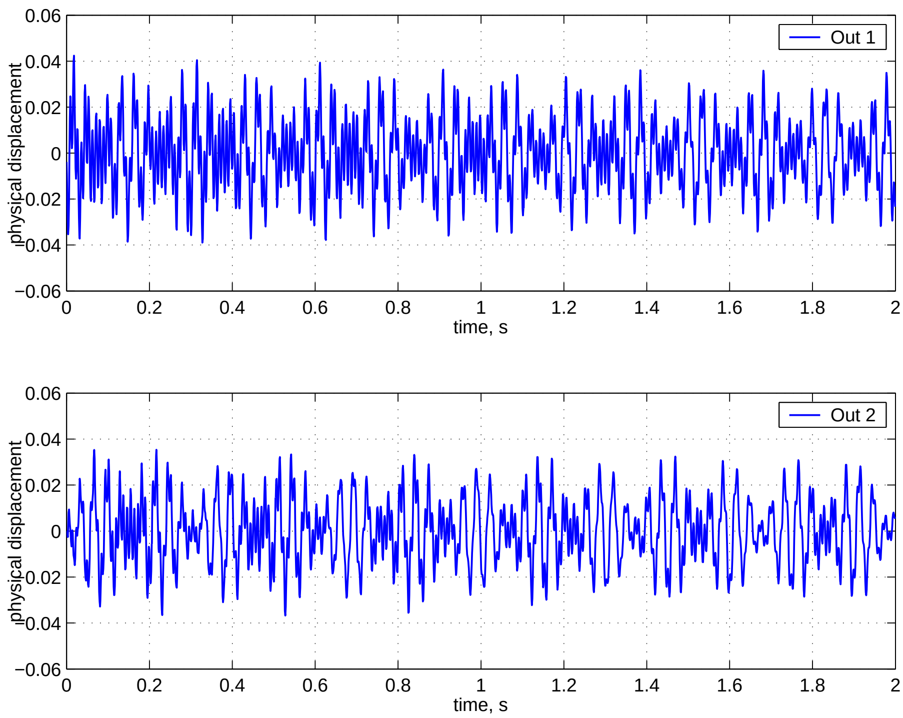


Figure 6.7: Free response of rotor to initial conditions

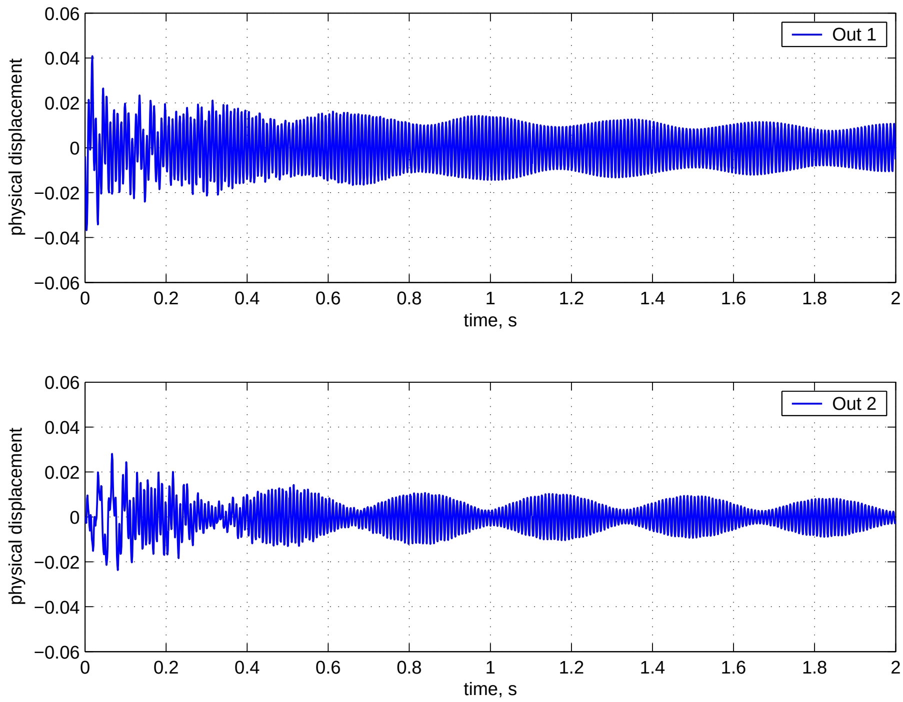


Figure 6.8: SPT-modal controlled response of rotor to initial conditions

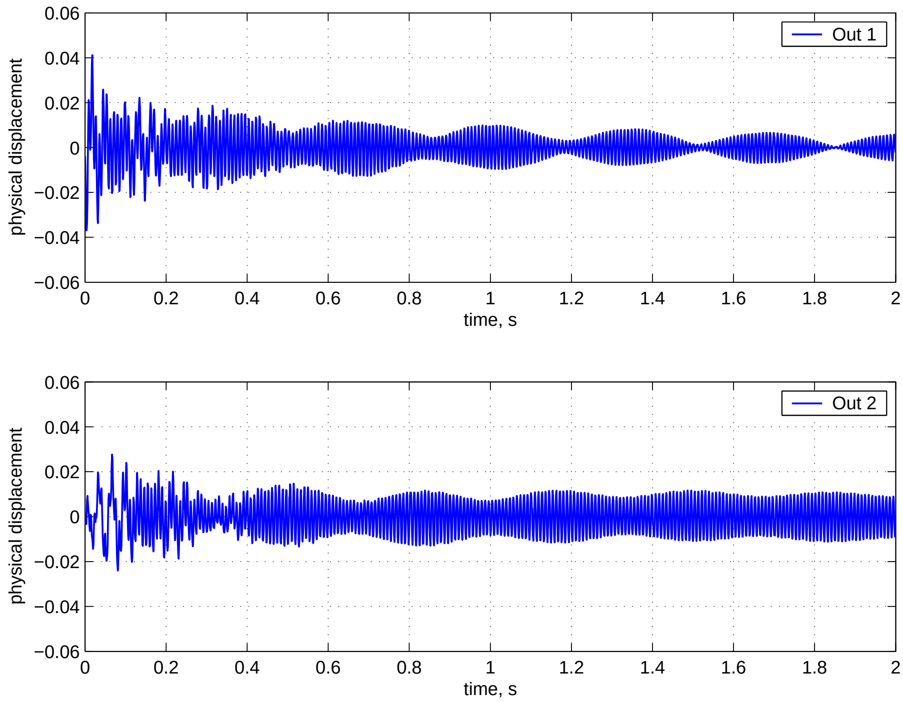


Figure 6.9: IMSC-modal controlled response of rotor to initial conditions

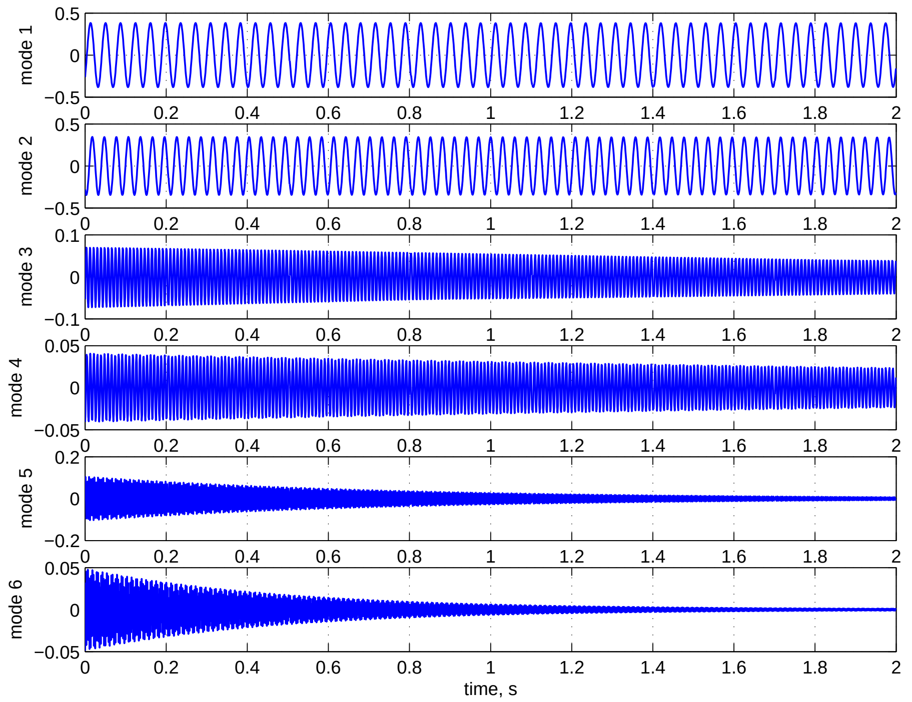


Figure 6.10: SPT modal response, control off

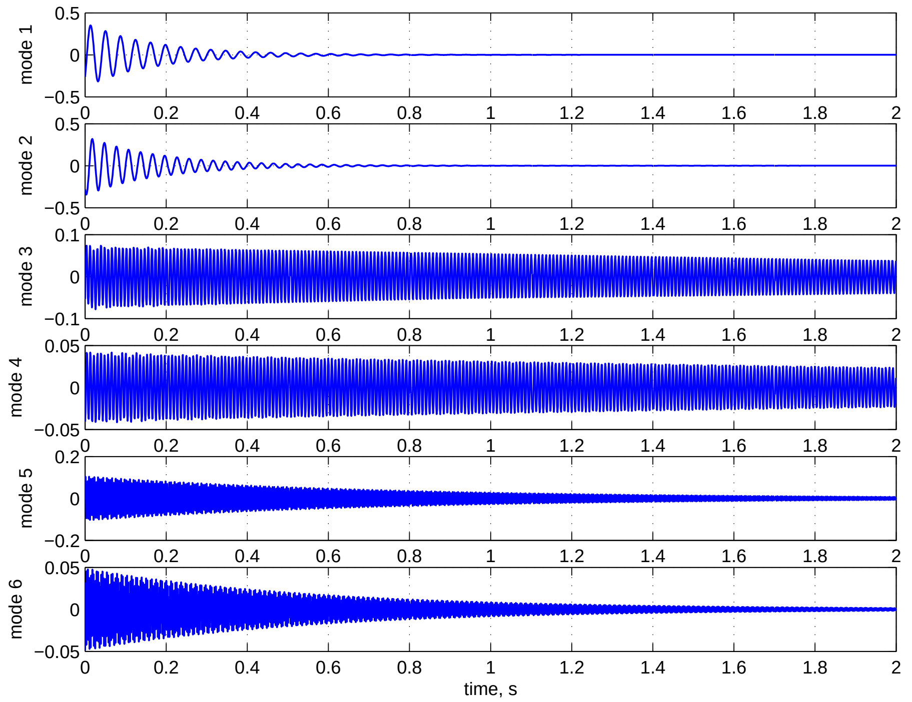


Figure 6.11: SPT modal response, control on

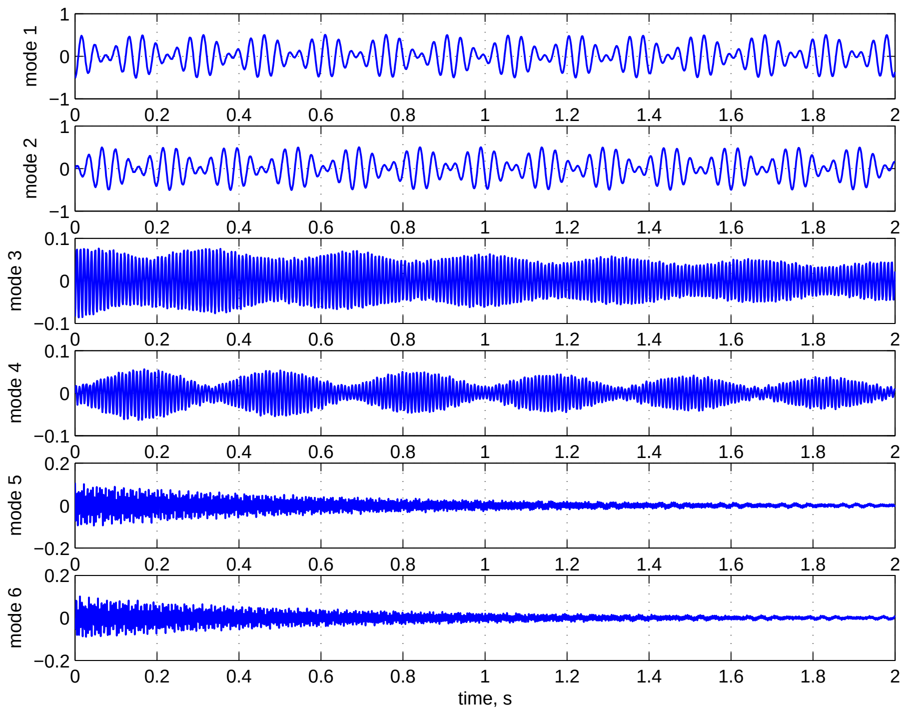


Figure 6.12: IMSC modal response, control off

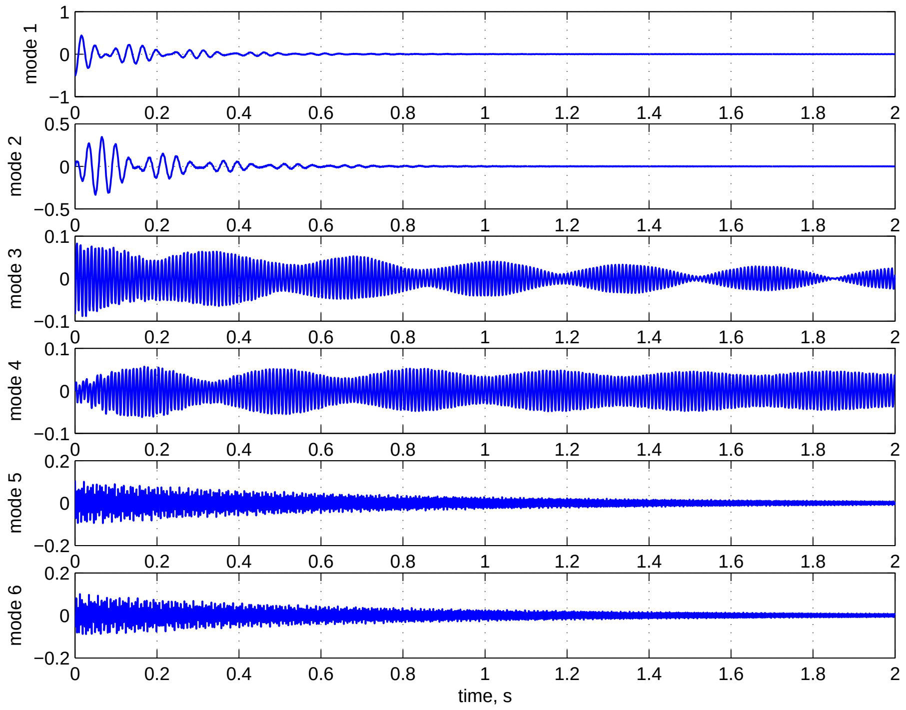


Figure 6.13: IMSC modal response, control on

Chapter 7

Conclusions and Further Work

The ambition of this project was to extend knowledge in the field of control of rotating machinery. Two new methods have been presented - modal control of generally damped systems and optimal control incorporating the rate of change of force. The origins of both methods reside in the study of the structure preserving transformations. Numerical examples have illustrated the two methods successfully and they have shown that the methods result in superior control than conventional techniques permit. The two methods are now concluded in detail.

7.1 Structure Preserving Based Modal Control

The modal control method presented in this thesis involves the use of the 'structure preserving transformations' (SPTs) to diagonalise the system matrices. Specifically the method presented does not insist that the system be 'classically damped' such that the eigenvectors of the undamped system be used to diagonalise simultaneously all three system matrices. It has been shown how to form a diagonalising SPT such that one may diagonalise the system matrices of a generally damped system. One may use the diagonalising SPTs to decouple the un-forced equations of motion but the diagonalised system is still coupled externally via the forcing matrix.

The applied control is designed independently in the modal space before being transformed back into the physical domain. This allows decoupling of the second order equations of motion of the system both internally for the un-forced system and externally when control forces are present. The SPT-method does not constrain the system damping to be 'classically damped' as conventional second order modal control does. Theory and numerical examples have shown that the SPT-based modal control method provides superior control for rotating systems.

The SPTs are used to transform any second order system into another second order system whilst retaining the same Jordan form. This thesis has demonstrated how to extract the new displacements and velocities from the original physical co-ordinates through the use of SPT-based filters. Convention dictates that the modes obtained for control through modal filtering should match the real physical modes of vibration. For the SPT-modal control approach this wisdom is dismissed and the modes obtained are no longer the undamped modes of vibration formed by the mass and stiffness matrices. However, the modal form utilised in this project does indeed contain physically meaningful data enabling the application of the SPT modal method to systems for which traditional modal control methods would yield inadequate control.

The SPT-based filters are dynamic (first order systems) and consequently a necessary requirement is that their eigenvalues reside on the stable half-plane. For the specific case when the transformed system matrices are real and diagonal a method has been developed in which one may flow the eigenvalues towards the stable region through the use of reflexive SPTs. The transformed system matrices remain unaltered and numerical experiments suggest that one may always make a filter increasingly stable. Numerical example has successfully demonstrated the method for non-classically damped systems with initially unstable filters. However for the general case it is currently unknown whether the change in eigenvalues may go asymptotic, in which case the eigenvalues may never cross into the stable half-plane. Numerical experimentation suggests that this is not the case and that it is always possible to find a stable filter. This remains to be proved formally.

Utilising the SPT-based filters to transform the modal force into the physical domain results in the both the force and rate of change of force being obtained. This presents the opportunity to smooth the physical force through interpolation if the modal control method is implemented in discrete time. For theoretical problems in which the method can be implemented continuously, this is not an issue but practical application of any control system requires discrete implementation through the use of digital equipment. This thesis has shown the potential improvement in performance through smoothing the control force using the rate of change of control force. Thus SPT-modal control may offer additional benefits over conventional modal control methods. This is something in need of further investigation.

7.1.1 Further Work

Analysis of the SPT-modal control method presented in the thesis raises several subtleties which remain to be addressed. The method may be directly compared to results from the conventional techniques as found in the literature and potential areas of development can be highlighted.

Like all modal control methods the SPT-method theoretically requires knowledge of all system states to be utilised. It has not been the intent of this project to extend the concept of state reconstruction and the author has relied upon existing techniques such as Kalman filters [B8] or Luenberger observers [M7] to satisfy this requirement. In the literature Oz and Meirovitch [O3] developed modal state reconstruction filters for the first order IMSC technique such that the modal state could be directly reconstructed given discrete observations of the system state.

The first order state space approach favoured by Oz and Meirovitch uses a 0^{th} order modal "filter" to obtain the modal state from the physical state. The SPT-approach however requires that in addition to the state the derivative of the state is used as well in the definition of the SPT-modal parameters. This results in a much more complex definition of the modal state parameters and consequently the modal Kalman filters developed by Oz and Meirovitch are much simpler than the first order filters required for the SPT-modal approach. It is suggested that something similar to the modal Kalman filters could be developed for the SPT modal control and this is an area where further work should be considered.

The SPT modal control technique suffers from the effects of control spillover in which the physical force causes excitation of the uncontrolled and un-modelled modes. This problem is not unique to the SPT-modal control method. The control spillover problem has the potential to cause significant degradation to the control applied to the system and hence the control objectives may not be met. It was shown in chapter 3 that the control spillover problem for the SPT modal control method is non-destabilising since the eigenvalues of the open loop uncontrolled mode-pairs remain unaltered. The significance of the control spillover to degrade the system performance has not been investigated.

The control spillover problem for the conventional second order IMSC technique has been tackled by several authors such as Baz *et al.*, [B1] and Fang *et al.*, [F1]. The approach described by Fang *et al.*, is inappropriate for the first order form of the filters presented for the SPT method. The approach introduced by Baz *et al.*, illustrates the ability to control several mode-pairs through a sharing strategy in which the actuators can switch between the mode-pairs they control. This offers a potential solution for control spillover in the SPT modal control approach although initial investigations report that the strategy destabilises the SPT filters. For the long term establishment of the SPT modal control method it is advised that the control spillover problem is something that is in need of refinement.

The predominant difference between the modal control method presented in this project and other control methods is due to the first order nature of the modal filters. The modal filters introduce an additional dimension of work into the modal control problem due to the necessity that they must be stable. Unstable filters will not permit the conversion of the modal force into the physical domain and the physical control force indicated will rapidly tend to infinity. This project has illustrated a method by which the filters can be 'flowed' into a more stable region through the use of reflexive SPTs exploiting the non-uniqueness of the diagonalising SPTs. This allows the filters to be made progressively more stable. It has not yet been formally proven that a stable filter is always available for a diagonalising SPT. Indeed it seems statistically more probable to obtain an unstable filter rather than a stable filter although numerical experimentation suggests that a stable filter always exist. This problem seems an obvious point to address and indeed the author has spent much time trying to formally prove the existence of stable filters for all diagonalising transformations.

7.1.2 Summary of Further Work

Following the various issues highlighted above the future areas for development for the SPT modal control method may be hereby summarised:

- Development of modal Kalman filters to reconstruct complete modal state from discrete sensor measurements.
- The development of control spillover strategies to minimise the effects of exciting uncontrolled modes.
- Development of a formal proof verifying that there exists stable SPT filters for all diagonalising SPTs.
- Investigation of the benefits of smoothing the applied force using the obtained rate of change of control force for digital controllers.

7.2 Extended Optimal Control

In this thesis an extension to the optimal control problem has been presented in which the rate of change of control has been incorporated into the cost function. Numerical examples have been successfully demonstrated to show the implementation of the method and comparison drawn with the conventional LQR approach. The rationale behind the extended optimal control method was to establish an analytical method by which weighting could be placed on the system response, applied control force and corresponding rate of change of control force. Justification of this ambition has been made from the use of electro-magnetic bearings as a contemporary example. The extended optimal control method outlined in this thesis makes use of optimal control theory to impose a limit on the rate at which forces can be applied to a system. The method presented in this project yields a usable analytical solution to the extended optimal control problem and the usability of the method has been demonstrated through the use of example.

7.2.1 Further Work

The analytical method obtained in this thesis augments the system state to include the control force. This requires that the rate of change of force be fed back and subsequently integrated. This raises various questions concerning the stability and robustness of the system which have not been addressed in this project. It would thus be beneficial to transform the problem such that the input to the system is in a more conventional form.

The only other method of penalising the rate at which the control forces of which the author is aware is predictive control [M1]. The predictive control method uses a first order discrete time representation of the dynamic system and calculates the required change of force at each time sample. In order to deal with the required change in the discrete force vector one must augment the system state through the addition of the force vector. This is analogous to the continuous time augmented state used in the augmented optimal control method presented in chapter 5.

The predictive control method uses knowledge of the system and current state to create a prediction of how the system is going to react in the future. The predictive control method then attempts to minimise a quadratic cost function, equivalent to the quadratic cost function used in the 'linear quadratic regulator' (LQR) problem, spanning a finite number of time samples in the future. Cole *et al.* [C4] showed that the predictive control method yields a comparable result to the conventional optimal control problem.

The possibility arises to extend the predictive control problem to solve the problem of optimally penalising system state, forcing and rate of change of forcing. However implementation of the predictive control method requires that the controller gains are time variant and calculated at each time sample as a function of current state and force and predicted future state and force. Predictive control was originally developed for process control where the time samples are very slow. This raises the possibility of computational problems for systems represented by larger dimensional models requiring fast time samples such as rotating machines. This problem may be overcome using a solution such as a lookup table such that the controller action is selected from pre-computed controller gains.

The predictive control method can also be used to place physical limits on the rate of change of control force. These can take the form of soft or hard constraints where soft constraints allow the limit to be violated to a moderate degree and hard constraints represent an absolute limit. Both soft and hard predictive control controllers result in a non-linear controller when the control action nears the physical limits designed into the controller.

A useful comparison would be to compare the results from the predictive control method to the continuous time augmented method presented in this thesis.

The theme of this project has been to extend the notion of direct second order control. For the conventional optimal control Zhang [Z1] has shown that it is possible to solve the optimal control problem for second order systems through utilising the eigenvalue and eigenvectors of the Hamiltonian system. This raises the valid assumption that the same may be possible for the extended optimal control problem if the system is presented in an appropriate form. The second order process illustrated in chapter 5 results requires the use of a pseudo-inverse. This is undesirable due to the non-uniqueness of the solution. Thus one suggests that it may be possible to solve the method to yield a unique solution that is applicable to the second order system directly. The results for the augmented system suggest that an optimal solution does indeed exist for the first order system, thus it may be possible to find an analytical solution for the second order system.

7.2.2 Summary of Further Work

From the areas highlighted above the areas of future work for the extended optimal control method may be accordingly summarised:

- Removal of necessity to feed-back rate of change of control force.
- Extension of the extended optimal control problem to second order form.
- Perform an assessment of the predictive control method usefulness in penalising the rate of change of force.

7.3 Final Remark

It is the author's opinion that the areas highlighted for future areas of development represent the significant remaining areas of interest and potential weaknesses of this project. The areas for development are drawn from immediate comparison to existing techniques that this project seeks to extend. Despite these weaknesses the author believes that a usable and exciting contribution has been made to the field of control of rotating machines.

Bibliography

[A1] Adhikari S. and Friswell M.I., *Eigenderivative Analysis of Asymmetric Non-Conservative Systems*, International Journal for Numerical Methods in Engineering, 2001, 51(6), pp. 709-733.

[B1] Baz. A, Poh S. and Studer P., *Modified Independent Modal Space Control Method for Active Control of Flexible Systems*, Proceedings of the Institute of Mechanical Engineers Part C, 1989, 203(2), pp. 103-112.

[B2] Baz A. and Poh S., *Experimental Implementation of the Modified Independent Modal Space Control Method*, Journal of Sound and Vibration, 1990, 139(1), pp. 133-149.

[B3] Benner P., Mehrmann V. and Sorensen D., *Dimension reduction of large scale systems*, Springer-Verlag, Berlin, 2005, ISBN: 3540245456.

[B4] Bertsekas D.P., *Dynamic programming and optimal control Vol. 1*, Athena Scientific Belmont, Mass, 1995, ISBN: 1886529124.

[B5] Bertsekas D.P., *Dynamic programming and optimal control Vol. 2*, Athena Scientific, Belmont, Mass, 1995, ISBN: 1886529132.

[B6] Blevins R.D., *Formulas for natural frequency and mode shapes*, Krieger Publishing, Malabar, 1995, ISBN: 0894648942.

[B7] Brogan W.L., *Modern control theory*, 3rd Edition, Prentice Hall, 1990, ISBN: 0135904153.

[B8] Burl J., *Linear optimal control : H2 and H[infinity] methods*, Addison Wesley Longman, Menlo Park, Calif., 1999, ISBN: 0201808684.

[C1] Caughey T.K. and O'Kelly M.E.J., *Classical Normal Modes in Damped Linear Dynamic Systems*, Journal of Applied Mechanics, 1965, 32(3) pp. 583.

[C2] Chahlaoui Y., Lemonnier D., Vandendorpe A. and Van Dooren P., *Second-Order Balanced Truncation*, Linear Algebra and its Applications, 2006, 415(2-3), pp. 373-384.

[C3] Chiba A., Fukao T., Ichikawa O., Oshima M., Takemoto M. and Dorrell D.G., *Magnetic bearings and bearingless drives*, Newnes, Oxford, 2005, ISBN: 0750657278.

[C4] Cole D.J., Pick A.J. and Odhams A.M.C., *Predictive and Linear Quadratic Methods for Potential Application to Modelling Driver Steering Control*, Vehicle System Dynamics, 2006, 44(3), pp. 259-284.

[C5] Craig R.R. and Bampton M.C.C., *Coupling of Substructures for Dynamic Analyses*, AIAA Journal, 1968, 6(7), pp. 1313-1319.

[D1] Datta B.N., Elhay S. and Ram Y.M., *Orthogonality and Partial Pole Assignment for the Symmetric Definite Quadratic Pencil*, Linear Algebra and Its Applications, 1997, 257(1), pp. 29-48.

[D2] Diwekar A.M. and Yedavalli R.K., *Robust Controller Design for Matrix Second-Order Systems with Structured Uncertainty*, IEEE Transactions on Automatic Control, 1999, 44(2), pp. 401-4045.

[D3] Diwekar A.M. and Yedavalli R.K., *Stability of Matrix Second Order Systems: New Conditions and Perspectives*, IEEE Transactions on Automatic Control, 1999, 44(9), pp. 1773-1777.

[D4] Dorf R.C. and Bishop R.H., *Modern control systems*, 10th Edition, Pearson-Prentice Hall, Upper Saddle River, N.J, 2005, ISBN: 0131277650.

[F1] Fang J.Q., Li Q.S. and Jeary A.P., *Modified Independent Modal Space Control of Multi Degree of Freedom Systems*, Journal of Sound and Vibation, 2003, 261(3), pp. 421-441.

[F2] Friswell M.I., Garvey S.D. and Penny J.E.T, *Model Reduction Using Dynamic and Iterated IRS Techniques*, Journal of Sound and Vibration, 1995, 186(2), pp. 311-323.

[F3] Friswell M.I., Garvey S.D. and Penny J.E.T, *The Convergence of the Iterated IRS Method*, Journal of Sound and Vibration, 1998, 211(1), pp. 123-132.

[F4] Friswell M.I., Garvey S.D. and Penny J.E.T., *Extracting Second Order Systems From State-Space Representations*, AIAA Journal, 1999, 37(1), pp. 132-135.

[F5] Friswell M.I. and Prells U., *Relationship Between Defective Systems and Unit-Rank Modification of Classical Damping*, Journal of Vibration and Acoustics, Transactions of the ASME, 2000, 122(2), pp. 180-183.

[F6] Fuller C.R., Elliot S.J. and Nelson P.A., *Active control of vibration*, Academic Press, London, 1996, ISBN: 0122694406.

[G1] Garvey S.D., Prells U. and Friswell M.I., *Diagonalising Coordinate Transformations for Systems with General Viscous Damping*, 2001, 19th International Modal Analysis Conference, Orlando, Florida, USA 1(1), pp. 622-627.

[G2] Garvey S.D., Friswell M.I. and Prells U., *Coordinate Transformations for Second Order Systems, Part 1: General Transformations*, Journal of Sound and Vibration, 2002, 258(5), pp. 885-909.

[G3] Garvey S.D., Friswell M.I. and Prells U., *Coordinate Transformations for Second Order Systems, Part II: Elementary Structure-Preserving Transformations*, Journal of Sound and Vibration, 2002, 258(5), pp. 911-930.

[G4] Garvey S.D., Friswell M.I. and Prells U., *Deflating Second Order Systems*, ISMA International Conference on Noise and Vibration Engineering, 27, Leuven, Belgium, 2002, pp. 579-588.

[G5] Garvey S.D., Prells U., Friswell M.I. and Chen Z., *General Isospectral Flows for Linear Dynamic Systems*, Linear Algebra and its Applications, 2004, 385, pp. 335-368.

[G6] Gawronski W. and Williams T., *Model Reduction for Flexible Space Structures*, Journal of Guidance, Control and Dynamics, 1991, 14(1), pp.68-76.

[G7] Gawronski W.K., *Dynamics and control of structures*, Springer-Verlag, New York, 1998, ISBN: 0387985271.

[G8] Gawronski W.K., *Advanced structural dynamics and active control of structures*, Springer-Verlag, New York, 2004, ISBN: 0387406492.

[G9] Genta G., *Dynamics of rotating systems*, Springer-Verlag, New York, 2004, ISBN: 0387209360.

[G10] Gohberg I., Lancaster P. and Rodman L., *Matrix polynomials*, Academic Press, London, 1982, ISBN: 012287160.

[G11] Goldstein H., Poole, C. and Safko J., *Classical mechanics*, Addison-Wesley, San Fransico, Calif., 2002, ISBN: 0321188977.

[G12] Guyan R.J., *Reduction of Stiffness and Mass Matrices*, AIAA Journal, 1964, 3(2), pp. 380.

[H1] Henshell R.D. and Ong J.H., *Automatic Masters for Eigenvalue Economization*, Earthquake Engineering and Structural Dynamics, 1975, 3, pp. 375-383.

[H2] Houlston P.R., *Extracting Second Order System Matrices From State Space System*, Proceedings of the Institution of Mechanical Engineers, Part C, Journal of Mechanical Engineering Science, 2006, 220(8), pp. 1147 - 1149.

[H3] Houlston P.R., Garvey S.D. and Popov A.A., *Optimal Controller Designs for Rotating Machines - Penalising the Rate of Change of Control Forcing*, 7th IFToMM-Conference on Rotor Dynamics, Vienna, Austria, 25-28 September 2006.

[H4] Houlston P.R., Garvey S.D. and Popov A.A., *Modal Control of Vibration in Rotating Machines and Other Generally Damped Systems*, ISMA 2006 International Conference on Noise and Vibration Engineering, Leuven, Belgium, 18-20 September 2006.

[H5] Houlston P.R., Garvey S.D. and Popov A.A., *Modal Control of Vibration in Rotating Machines and Other Generally Damped Systems*, Journal of Sound and Vibration, in press.

[I1] D. J. Inman, *Vibration with control, measurement and stability*, Englewood Cliffs, NJ: Prentice-Hall, 1989, ISBN: 0139427988.

[I2] Inman D.J., *Active Modal Control for Smart Structures*, Philosophical Transactions of the Royal Society of London Series A - Mathematical Physical and Engineering Sciences, 2001, 359(1778), pp. 205-219.

[K1] Kautsky J., Nichols N.K. and Van Dooren V., *Robust Pole Assignment in Linear State Feedback*, International Journal of Control, 1985, 41(5), pp. 1129-1155.

[K2] Keogh P.S., Sahinkaya M.N., Burrows C.R. and Prabhakar S., *Wavelet Based Adaptation of H-infinity Control in Flexible Rotor/Magnetic Bearing Systems*, 7th IFToMM-Conference on Rotor Dynamics, Vienna, Austria, 25-28 September 2006.

[K3] Khoo W.K.S., Garvey S.D., Kalita K. and Houlston P.R., *Vibration Control with Lateral Force Producing Electrical Machines*, Eigth International Conference on Vibrations in Rotating Machinery, Swansea, UK, 2004, Paper C623/074/2004, pp. 713-722.

[K4] Khoo W.K.S., Garvey S.D. and Kalita K., *Controlled UMP in Electrical Machines Can Stabilise Otherwise Unstable Machines*, Third International Symposium on Stability Control of Rotating Machinery, Cleveland, Ohio, USA, 19-23 September 2005.

[K5] Kim Y., Kim, H.S. and Junkins J.L., *Eigenstructure Assignment Algorithm for Mechanical Second Order Systems*, Journal of Guidance and Control, 1999, 22(5), pp. 729-731.

[L1] Lalanne M. and Ferraris G., *Rotordynamics prediction in engineering*, John Wiley, Chichester, 1998, ISBN: 0471972886.

[L2] Lancaster P., *Private Communication*, April 2006.

[L3] Lee A.C. and Chen S.T., *Optimal Vibration Control for Flexible Rotor With Gyroscopic Effects*, JSME International Journal Series III, 1992, 35(3), pp. 446-455.

[L4] Lin Y.H. and Chu C.L., *A New Design for Independent Modal Space Control of General Dynamic Systems*, Journal of Sound and Vibration, 1995, 180(2), pp. 351-361.

[M1] Maciejowski J.M., *Predictive Control with Constraints*, Prentice Hall, 2001, ISBN: 0201398230.

[M2] Meirovitch L. and Baruh H., *Optimal Control of Damped Flexible Gyroscopic Systems*, Journal of Guidance and Control, 1981, 4(2), pp. 157-163.

[M3] Meirovitch L. and Baruh H., *Control of Self-Adjoint Distributed Parameter Systems*, Journal of Guidance, Control and Dynamics, 1982, 5(1), pp. 60-66.

[M4] Meirovitch L., Baruh H. and Oz H., *A Comparison of Control Techniques for Large Flexible Systems*, Journal of Guidance, Control and Dynamics, 1983, 6(4), pp. 302-310.

[M5] Meirovitch L. and Baruh H., *On the Problem of Observation Spillover in Self-Adjoint Distributed Parameter Systems*, Journal of Optimization Theory and Applications, 1983, 39(2), pp. 269-291.

[M6] Meirovitch L. and Baruh H., *The Implementation of Modal Filters for Control of Structures*, Journal of Guidance, Control and Dynamics, 1985, 8(6), pp. 707-716.

[M7] Meirovitch L., *Dynamics and control of structures*, Wiley, New York, 1990, ISBN: 0471628581.

[M8] Meyer D.G. and Srinivasan S., *Balancing and Model Reduction for Second Order Form Linear Systems*, IEEE Transactions of Automatic Control, 1996, 41(11), pp. 1632-1644.

[M9] Miminis G.S. and Paige C.C., *An Algorithm for Pole Assignment of Time Invariant Linear Systems*, International Journal of Control, 1982, 35(2), pp. 341-354.

[M10] Mottershead J.E. and Ram Y.M., *Inverse Eigenvalue Problems in Vibration Absorbtion: Passive Modifications and Active Control*, Mechanical Systems and Signal Processing, 2006, 20(1), pp. 5-44.

[N1] Nichols N.K. and Kautsky J., *Robust Eigenstructure Assignment in Quadratic Matrix Polynomials: Non-Singular Case*, Siam J. Matrix Anal. Appl., 2001, 23(1), pp. 77-102.

[O1] O'Callahan J.C., *A procedure for an improved reduced system (IRS) model*, Proceedings of the 6th International Modal Analysis Conference, Las Vegas, January 1989, pp. 17-21.

[O2] O'Callahan, Avitabile P. and Riemer R., *System equivalent reduction expansion process (SEREP)*, Proceedings of the 6th International Modal Analysis Conference, Las Vegas, January 1989, pp. 29-37.

[O3] Oz H. and Meirovitch L., *Stochastic Independent Modal Space Control of Distributed Parameter Systems*, Journal of Optimization Theory and Applications, 1983, 40(1), pp. 121-154.

[P1] Prells U. and Friswell M.I., *A Measure of Non-Proportional Damping*, Mechanical Systems and Signal Processing, 2000, 14(2), pp. 125-137.

[P2] Prells U., *A view of Structure Preserving Transformations in terms of Linear Filters. Unpublished presentation at a colloquium on Structured Eigenvalue Problems*, University of Manchester, November 8th, 2002.

[P3] Prells U. and Lancaster P., *Isospectral Vibrating Systems, Part 2: Structure Preserving Transformations*, Operator Theory: Advances and Applications, 163(1), 2005, pp. 275-298.

[P4] Prells U., *Private Communication*, 2005.

[P6] Pumhossel T. and Springer H., *Active Vibration Control of Nonlinear Cantilever Shaped Blades in Turbomachinery*, 7th IFToMM-Conference on Rotor Dynamics, Vienna, Austria, 25-28 September 2006.

[R1] Ram Y.M and Inman D.J., *Optimal Control for Vibrating Systems*, Mechanical Systems and Signal Processing, 1999, 13(6), pp. 879-892.

[R2] Ram. Y.M. and Elhay S., *Pole Assignment in Vibratory Systems by Multi-Input Control*, Journal of Sound and Vibation, 2000, 230(2), pp. 309-321.

[R3] Rao J.S., *Rotor dynamics*, 2nd Edition, Wiley, New York, 1991, ISBN: 0470217871.

[R4] Rao S.S., *Mechanical vibrations*, 3rd Edition, Addison-Wesley, Reading, Mass., 1995, ISBN: 0201984962.

[R5] UTC Annual meeting, the University of Nottingham, April 2006.

[S1] Saif M., *Optimal Modal Controller Design by Entire Eigenstructure Assigment*, IEEE Proceedings Part D: Control Theory and Application, 1989, 136(6), pp. 341-344.

[S2] Sawicki J.T. and Genta G., *Modal Uncoupling of Damped Gyroscopic Systems*, Journal of Sound and Vibration, 2001, 244(3), pp. 431-451.

[S3] Singh S.P., Pruthi H.S. and Agarwal V.P., *Efficient Modal Control Strategies for Active Control of Vibrations*, Journal of Sound and Vibration, 2003, 262(3), pp. 563-575.

[S4] Skogestad S. and Postlethwaite I., *Multivariable feedback control: analysis and design, 2nd Edition*, Wiley, Chichester, 2005, ISBN: 0470011688.

[S5] Strang G., *Linear algebra and its applications*, Harcourt College Publishers, Fort Worth, 1988, ISBN: 0155510053.

[T1] Tisseur F. and Meerbergen K., *The Quadratic Eigenvalue Problem*, Society for Industrial and Applied Mathematics (SIAM) Review, 2001, 43(2), pp. 235-286.

[U1] Ulbrich, H. and Ginzinger L., *Stabilisation of a Rubbing Rotor Using a Robust Feedback Control*, Paper 306.00, 7th IFToMM-Conference on Rotor Dynamics, Vienna, Austria, 25-28 September 2006.

[Y1] Young L.C., *Lectures on the calculus of variations and optimal control theory*, London, Saunders, 1969, ISBN: 0721696406.

[Z1] Zhang J.F., *Optimal Control for Mechanical Vibration Systems Based on Second Order Matrix Equations*, Mechanical Systems and Signal Processing, 2002, 16(1), pp. 61-67.

[Z2] Zhou K., Doyle J.C. and Glover K., *Robust and optimal control*, Prentice Hall, Upper Saddle River, N.J., 1996, ISBN: 0134565673.

Appendix A

Closed Form Optimal Modal Controller Gains

For the context of this project the aim to design a controller to give adequate dynamic response. The question of how to balance design needs versus practicalities obviously arises. Pole placement methods yield direct consequence to the system response but the effect of the pole placement is not known prior to application. It is possible to measure the cost of applying the controller and then a decision made as to whether or not the controller is appropriate. However working backwards one may decide how to measure the cost and correspondingly how to generate a controller to minimise it. This approach generally defines what the optimal control method is.

Optimal control may be defined as the design of a controller which yields the minimum cost subjected to user-defined constraints. Typically the cost function is a function of system state ($\underline{\mathbf{q}_A}(t)$) and applied force ($\mathbf{u}_A(t)$) and is quadratic in nature. The cost function utilised in this section may be defined as

$$J = \frac{1}{2} \int_0^\infty \underline{\mathbf{q}_A}^T(t)\, \underline{\mathbf{Q}}\, \underline{\mathbf{q}_A}(t) + \mathbf{u}_A^T(t)\, \mathbf{R}\, \mathbf{u}_A(t)\; dt \tag{A.1}$$

where $\underline{\mathbf{Q}} \in \mathbb{R}^{2n \times 2n}$ represents a positive semi-definitive weighting matrix to determine the relative importance of the system state at time t and $\mathbf{R} \in \mathbb{R}^{r \times r}$ is a positive definite weighting matrix to determine the relative importance of the applied force at time t.

The quadratic nature ensures equal weighting between positive and negative values and the integral ensures that the more persistent error is penalised more harshly [B8]. The underlined notation denoting the $2n \times 2n$ dimensional matrices is relaxed here.

The optimal control law is sought such that a linear combination of the system state may be used to minimise the cost function illustrated by equation (A.1). A controller of this form is defined

$$\mathbf{u}_A = \mathbf{G}_{opt}\, \underline{\mathbf{q}_A}(t) \tag{A.2}$$

It is well established that the solution to the optimal control problem may be found by solving the Riccati equation [Z2] backward through time to yield time dependent matrix $\mathbf{P}(t)$

$$\underline{\dot{\mathbf{P}}}(t) = -\underline{\mathbf{Q}} - \underline{\mathbf{P}}(t)\, \underline{\mathbf{A}_A} - \underline{\mathbf{A}_A}^T\, \underline{\mathbf{P}}(t) + \underline{\mathbf{P}}(t)\, \underline{\mathbf{B}_A}\, \mathbf{R}^{-1} \underline{\mathbf{B}_A}^T\, \underline{\mathbf{P}}(t) \tag{A.3}$$

Here the matrices $\underline{\mathbf{A}_A}$ and $\underline{\mathbf{B}_A}$ are the state space system companion and forcing matrices respectively.

The system in question is assumed to be controllable. As a consequence of this the system state at infinite time is assumed to have reached steady state resulting with the matrix $\underline{\dot{\mathbf{P}}}(\infty) = 0$. This is referred to as the infinite horizon problem and allows a solution to yield a constant matrix $\underline{\mathbf{P}}$. Thus a relationship between the infinite horizon solution to the Riccati equation and the controller matrix $\mathbf{G}_{opt}$ may be established.

$$\mathbf{G}_{opt} = \mathbf{R}^{-1}\, \underline{\mathbf{B}_A}^T\, \underline{\mathbf{P}} \tag{A.4}$$

As shown the modal control method resorts to controlling r monic SDOF systems. Meirovitch showed that it was possible to yield r-closed form optimal controller gains for the first order IMSC technique [M2]. The method presented by Meirovitch is extended to find a closed form solution to the optimal control problem for the monic single degree of freedom systems. Defining the state space system for the single degree of freedom system to be

$$\begin{bmatrix} \underline{\dot{\mathbf{q}}_{B1j}}(t) \\ \underline{\dot{\mathbf{q}}_{B2j}}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_j & -d_j \end{bmatrix} \begin{bmatrix} \underline{\mathbf{q}_{B1j}}(t) \\ \underline{\mathbf{q}_{B2j}}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}(t) \tag{A.5}$$

where $\underline{\mathbf{q}_{B1j}}(t) = \mathbf{q}_{Bj}(t)$ and $\underline{\mathbf{q}_{B2j}}(t) = \underline{\dot{\mathbf{q}}_{B1j}}(t)$. Equation (A.5) may be simplified to

$$\underline{\dot{\mathbf{q}}_{Bj}}(t) = \underline{\mathbf{A}_{Bj}}\,\underline{\mathbf{q}_{Bj}}(t) + \underline{\mathbf{B}_{Bj}}\,\mathbf{u}_{Bj}(t) \tag{A.6}$$

The interest is in finding the 2×2 Riccati matrix $\underline{\mathbf{P}_j}$ which may be partitioned

$$\underline{\mathbf{P}_j} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \tag{A.7}$$

The problem may be further simplified by first noting that the solution to the Riccati equation $\mathbf{P}_j$ is symmetric hence $p_{12} = p_{21}$ and setting the scalar value $\mathbf{R}_j = 1$ without loss of generality since this represents only a relative relationship with the matrix $\underline{\mathbf{Q}_j}$. So that there is no coupling between the modal state variables $\underline{\mathbf{q}_{B1j}}(t)$ and $\underline{\mathbf{q}_{B2j}}(t)$ the 2×2 matrix $\underline{\mathbf{Q}_j}$ is assumed to have diagonal form

$$\mathbf{Q}_j = \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix} \tag{A.8}$$

One may thus acknowledge the infinite horizon problem to solve the Riccati equation

$$-\underline{\mathbf{Q}_j} - \underline{\mathbf{P}_j}\,\underline{\mathbf{A}_{Bj}} - \underline{\mathbf{A}_{Bj}}^T\,\underline{\mathbf{P}_j} + \underline{\mathbf{P}_j}\,\underline{\mathbf{B}_{Bj}}\,\mathbf{R}_j^{-1}\underline{\mathbf{B}_{Bj}}^T\,\underline{\mathbf{P}_j} = 0 \tag{A.9}$$

Algebraic manipulation of equation (A.9) yields three equations which may be used to give the 2×2 Riccati matrix $\underline{\mathbf{P}_j}$.

$$\begin{aligned} p_{21} \;=\; p_{12} &= -k_j + \sqrt{k_j^2 + q_{11}} \\ p_{22} &= -d_j + \sqrt{d_j^2 + q22 + 2p_{21}} \\ p_{11} &= k_j\,p_{22} + d_j\,p_{21} + p_{21}p_{22} \end{aligned} \tag{A.10}$$

Recognising the form of the control problem and one may separate the optimal controller gains matrix $\mathbf{G}_{opt\_j}$ into its two constitute parts to give

$$\mathbf{G}_{opt\_j} = \begin{bmatrix} \mathbf{G}_{kj} & \mathbf{G}_{dj} \end{bmatrix} \tag{A.11}$$

where $\mathbf{G}_{kj}$ and $\mathbf{G}_{dj}$ were introduced in the previous section such that the may add contributions to the modal stiffness and damping matrices respectively. Using the definitions from equations (A.4) and (A.5) and recalling that we set $\mathbf{R} = 1$, the closed form solutions are found to be

$$\mathbf{G}_{kj} = -k_j + \sqrt{k_j^2 + q_{11}} \tag{A.12}$$

$$\mathbf{G}_{dj} = -d_j + \sqrt{d_j^2 + q_{22} - 2\,k_j + 2\,\sqrt{k_j^2 + q_{11}}} \tag{A.13}$$

A.1 Numerical Example: Closed Form Optimal Gains

Utilising the spring-mass system from the numerical example presented in section 3.5 the closed form optimal control gains are used to calculated such to minimise the modal potential and kinetic energies of the system. The modal weighting matrices may thus be shown to be

$$\underline{\mathbf{Q}_j} = \begin{bmatrix} k_j & 0 \\ 0 & m_j \end{bmatrix} \quad , \ \mathbf{R}_j = 1 \tag{A.14}$$

$j = 1, 2, \cdots, n$

Accordingly, from the closed loop method the optimal modal controller gains are found to be

$$\mathbf{G}_k = \begin{bmatrix} 0.49988 & 0 & 0 \\ 0 & 0.49995 & 0 \\ 0 & 0 & 0.49998 \end{bmatrix} \ , \ \mathbf{G}_d = \begin{bmatrix} 0.30459 & 0 & 0 \\ 0 & 0.09965 & 0 \\ 0 & 0 & 0.080481 \end{bmatrix} \tag{A.15}$$

These results may be compared to the conventional results obtained from solving the Riccati equation. The norm of the difference between the gains obtained for two methods is 3.7228×10^{-13}. This illustrates that the solutions obtained through the closed loop equations give adequate accuracy and the advantage of avoiding the need to solve the Riccati equation for the modal system.

A.2 Conclusions

A necessary requirement of any control method is the determination of the feedback controller. It has been chosen to use optimal control for this project because it yields direct comparison to the original IMSC method [M2] and enables the determination of closed loop solutions to the feedback controller gains. Traditional pole placement methods do not necessary yield an intuitive sense of what direct addition to the modal stiffness and damping will yield. The closed loop solution optimal control method removes this question through a relative inexpensive computation.

Appendix B

Constructing a Diagonal System from System Eigenvalues

The purpose of this chapter is to show how to obtain the diagonal second order system matrices from the original second order system matrices. A 3 step process is followed.

1. Calculate the $2n$ eigenvalues λ_i $(i = 1, 2, \cdots, 2n)$ of the original second order systems $\mathbf{K}_A, \mathbf{D}_A, \mathbf{M}_A$.

2. Group the $2n$ eigenvalues into n appropriate pairings. Thus one now wishes to find the parameters k_j, d_j, m_j which correspond to the roots of the quadratic equation.

3. One may use the quadratic formula $z_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to find the roots of a quadratic equation. Thus assuming a monic equation such that $m_j = 1$

$$\begin{aligned}
\lambda_j &= \frac{1}{2}\left(-d_j + \sqrt{d_j^2 - 4k_j}\right) && \text{(B.1)} \\
\lambda_{j+1} &= \frac{1}{2}\left(-d_j - \sqrt{d_j^2 - 4k_j}\right) && \text{(B.2)}
\end{aligned}$$

   Solving for the unknowns in equations (B.1) and (B.2) one finds

$$m_j = 1, \quad d_j = (\lambda_j + \lambda_{j+1}), \quad k_j = \lambda_j \, \lambda_{j+1}, \quad j = 1, 3, 5, \cdots, 2n-1 \tag{B.3}$$

Appendix C

Alternative Derivation of SPT Modal Filters

Many structural and dynamic systems are described by the second order equations of motion

$$\mathbf{M}_A\,\ddot{\mathbf{q}}_A(t) + \mathbf{D}_A\,\dot{\mathbf{q}}_A(t) + \mathbf{K}_A\,\mathbf{q}_A(t) = \mathbf{f}_A(t) \tag{C.1}$$

where $\mathbf{M}_A, \mathbf{D}_A, \mathbf{K}_A \in \mathbb{R}^{n\times n}$ are the system mass, damping and stiffness matrices respectively, $\mathbf{q}_A(t) \in \mathbb{R}^n$ the vector of physical coordinates and $\mathbf{f}_A(t) \in \mathbb{R}^r$ is the generalised vector of applied forces. For the sake of brevity this paper assumes that forces are available at all locations and as a consequence $r = n$ and the notation depicting dependence on time has been removed.

The notion of the 'Lancaster Augmented Matrices' (LAMs) are introduced here such that the system may be represented in state space form. For a second order system there exists three LAMs which can be produced by inspection to be,

$$\underline{\mathbf{A}_0} = \begin{bmatrix} -\mathbf{D}_A & -\mathbf{M}_A \\ -\mathbf{M}_A & \mathbf{0} \end{bmatrix} \quad , \; \underline{\mathbf{A}_1} = \begin{bmatrix} \mathbf{K}_A & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_A \end{bmatrix} \quad , \; \underline{\mathbf{A}_2} = \begin{bmatrix} \mathbf{0} & \mathbf{K}_A \\ \mathbf{K}_A & \mathbf{D}_A \end{bmatrix} \tag{C.2}$$

The LAMs allow the second order system to be represented in a reduced form

$$\underline{\mathbf{A}_k}\,\underline{\mathbf{q}_A} - \underline{\mathbf{A}_{k-1}}\,\underline{\dot{\mathbf{q}}_A} = \underline{\mathbf{f}_{Ak}} \qquad k = 1, 2 \tag{C.3}$$

The vectors $\underline{\mathbf{q}_A}$ and $\underline{\mathbf{f}_{Ak}}$ may be defined

$$\underline{\mathbf{q}_A} := \begin{bmatrix} \mathbf{q}_A \\ \dot{\mathbf{q}}_A \end{bmatrix} \quad \underline{\mathbf{f}_{A1}} := \begin{bmatrix} \mathbf{f}_A \\ \mathbf{0} \end{bmatrix} \quad \underline{\mathbf{f}_{A2}} := \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_A \end{bmatrix} \tag{C.4}$$

A ‘Structure Preserving Transformation’ (SPT) is a coordinate transformation applied to the LAMs representing a bijective mapping between linear systems. The specific nature of the transformation allows the preservation of the appropriate structure within the LAMs. The SPTs are defined simply by left and right $2n \times 2n$ transformation matrices, $\underline{\mathbf{T}_L}$ and $\underline{\mathbf{T}_R}$ respectively, allowing the definition

$$\underline{\mathbf{T}_L}^T \, \underline{\mathbf{A}_k} \, \underline{\mathbf{T}_R} = \underline{\mathbf{B}_k} \quad \forall \quad k = 0, 1, 2 \;\; \cdot \tag{C.5}$$

Thus the new LAMs are represented by $\underline{\mathbf{B}_k}$ containing the new second order system matrices $\mathbf{K}_B, \mathbf{D}_B, \mathbf{M}_B$. The structure of the SPTs can be shown to have the following form

$$\underline{\mathbf{T}_L} = \begin{bmatrix} \mathbf{F}_L - \frac{1}{2}\,\mathbf{G}_L\,\mathbf{D}_A^T & -\mathbf{G}_L\,\mathbf{M}_A^T \\ \mathbf{G}_L\,\mathbf{K}_A^T & \mathbf{F}_L + \frac{1}{2}\,\mathbf{G}_L\,\mathbf{D}_A^T \end{bmatrix}^{-1} \quad \underline{\mathbf{T}_R} = \begin{bmatrix} \mathbf{F}_R - \frac{1}{2}\,\mathbf{G}_R\,\mathbf{D}_A & -\mathbf{G}_R\,\mathbf{M}_A \\ \mathbf{G}_R\,\mathbf{K}_A & \mathbf{F}_R + \frac{1}{2}\,\mathbf{G}_R\,\mathbf{D}_A \end{bmatrix}^{-1} \tag{C.6}$$

where $\mathbf{F}_L, \mathbf{F}_R, \mathbf{G}_L, \mathbf{G}_R \in \mathbb{R}^{n \times n}$ are arbitrary pre-defined matrices subject to the necessary constraint

$$\mathbf{F}_R\,\mathbf{G}_L^T + \mathbf{G}_R\,\mathbf{F}_L^T = 0 \;\; \cdot \tag{C.7}$$

C.1 Modal Filters

The premise of this paper is to develop direct second order control of the equations of motion. Thus the necessary question is how to extract the second order modal contributions from the state space system. The derivation of the modal filters for SPT-based control is presented here. For the purpose of this section one creates the definition $\tau \equiv \frac{\partial}{\partial t}$.

Introducing the partitioning

$$\underline{\mathbf{q}_A} =: \begin{bmatrix} \underline{\mathbf{q}_A}(1) \\ \underline{\mathbf{q}_A}(2) \end{bmatrix} \quad , \underline{\mathbf{f}_{A1}} =: \begin{bmatrix} \underline{\mathbf{f}_{A1}}(1) \\ \underline{\mathbf{f}_{A1}}(2) \end{bmatrix} \quad , \underline{\mathbf{f}_{A2}} =: \begin{bmatrix} \underline{\mathbf{f}_{A2}}(1) \\ \underline{\mathbf{f}_{A2}}(2) \end{bmatrix} \tag{C.8}$$

it is possible to extract a definition of the original second order system from the state space representation.

$$\mathbf{q}_A = \underline{\mathbf{q}_A}(1) \;, \qquad \mathbf{f}_A = \underline{\mathbf{f}_{A1}}(1) + \tau\, \underline{\mathbf{f}_{A1}}(2) \tag{C.9}$$

Equation (C.9) has been generalised such that it is assumed that the forcing part of the state space representation, $\underline{\mathbf{f}_{A1}}$, is fully populated. Whilst for the original system this is clearly not the case, the definition allows the extension to the transformed problem which $\underline{\mathbf{f}_{B1}}$ in general is fully populated. Equation (C.9) can be proved mechanistically for the second system. Expanding equation (C.3) for $k = 1$ and $k = 2$ yields

$$\mathbf{K}_A\Big(\underline{\mathbf{q}_A}(2) - \tau\, \underline{\mathbf{q}_A}(1)\Big) \;=\; \underline{\mathbf{f}_{A2}}(1) \tag{C.10}$$

$$\mathbf{D}_A\Big(\underline{\mathbf{q}_A}(2) - \tau\, \underline{\mathbf{q}_A}(1)\Big) \;=\; \underline{\mathbf{f}_{A2}}(2) - \underline{\mathbf{f}_{A1}}(1) \tag{C.11}$$

$$\mathbf{M}_A\Big(\underline{\mathbf{q}_A}(2) - \tau\, \underline{\mathbf{q}_A}(1)\Big) \;=\; -\underline{\mathbf{f}_{A1}}(2) \tag{C.12}$$

By substituting $\underline{\mathbf{f}_{A2}}(2)$ from equation (C.3) into equation (C.11) and subtracting τ multiplied by equation (C.12) yields

$$(\mathbf{K}_A + \tau\, \mathbf{D}_A + \tau^2\, \mathbf{M}_A)\, \underline{\mathbf{q}_A}(1) = \underline{\mathbf{f}_{A1}}(1) + \tau\, \underline{\mathbf{f}_{A1}}(2) \tag{C.13}$$

Hence equations (C.9) are proved. It is prudent at this juncture to point out that following similar methodology the equations of motion may also be represented in terms of $\underline{\mathbf{q}_A}(2)$.

By applying the SPTs one has the new transformed equations of motion

$$\Big(\underline{\mathbf{B}_k} - \tau \underline{\mathbf{B}_{k-1}}\Big)\; \underline{\mathbf{q}_B} = \underline{\mathbf{T}_L}^T\, \underline{\mathbf{f}_{Ak}} = \underline{\mathbf{f}_{Bk}} \tag{C.14}$$

Thus one may extract the new second order system of equations in terms of partitions of the new state variable $\underline{\mathbf{q}_B}$.

$$\big(\mathbf{K}_B + \tau\, \mathbf{D}_B + \tau^2\, \mathbf{M}_B\big)\, \mathbf{q}_B = \mathbf{f}_B \tag{C.15}$$

Knowing the definition from equation (C.9) the following are defined

$$\mathbf{q}_B = \underline{\mathbf{q}_B}(1) \quad , \quad \mathbf{f}_B = \underline{\mathbf{f}_{B1}}(1) + \tau\, \underline{\mathbf{f}_{B1}}(2) \tag{C.16}$$

The obvious question now arises, what is the relationship between the old and new coordinate sets? Acknowledging that the system coordinates are transformed using the definition $\underline{\mathbf{q}_B} = \underline{\mathbf{Z}_R}\,\underline{\mathbf{q}_A}$ where $\underline{\mathbf{Z}_R} = \underline{\mathbf{T}_R}^{-1}$ one has

$$\begin{bmatrix} \underline{\mathbf{q}_B}(1) \\ \underline{\mathbf{q}_B}(2) \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{Z}_R}_{11} & \underline{\mathbf{Z}_R}_{12} \\ \underline{\mathbf{Z}_R}_{21} & \underline{\mathbf{Z}_R}_{22} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{q}_A}(1) \\ \underline{\mathbf{q}_A}(2) \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{Z}_R}_{11}\,\underline{\mathbf{q}_A}(1) + \underline{\mathbf{Z}_R}_{12}\,\underline{\mathbf{q}_A}(2) \\ \underline{\mathbf{Z}_R}_{21}\,\underline{\mathbf{q}_A}(1) + \underline{\mathbf{Z}_R}_{22}\,\underline{\mathbf{q}_A}(2) \end{bmatrix} \tag{C.17}$$

The definition of $\mathbf{q}_B$ from equation (C.16) is used to see that the new coordinate set has the relationship to the old through the definition

$$\mathbf{q}_B = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \underline{\mathbf{Z}_R}^{-1}\,\underline{\mathbf{q}_A} \tag{C.18}$$

This result allows the introduction of a right filter of the form

$$\mathbf{q}_B = \left(\underline{\mathbf{Z}_R}_{11} + \tau\,\underline{\mathbf{Z}_R}_{12}\right)\,\mathbf{q}_A \tag{C.19}$$

The vector $\mathbf{q}_B$ represents the modal displacement obtained through the right filter using knowledge of physical displacements and velocities. For SPT modal control one also requires the modal velocities $\tau\mathbf{q}_B$. However the lower half of the state vector $\underline{\mathbf{q}_B}$ is in general not the derivative of the top half as was the case before the SPTs were applied. Thus the modal velocity cannot be extracted directly from the lower half of the state vector the way the modal displacement was extracted from the top half. Thus realising the relationship for the modal velocity $\tau\mathbf{q}_B = \tau\,\underline{\mathbf{q}_B}(1)$ one may rearrange equation (C.12) to find that the modal velocities are related to the modal displacements through

$$\tau\,\mathbf{q}_B = \tau\,\underline{\mathbf{q}_B}(1) = \underline{\mathbf{q}_B}(2) + \mathbf{M}_B^{-1}\,\underline{\mathbf{f}_{B1}}(2) \tag{C.20}$$

Introducing the known structure of the original untransformed equations of motion one has

$$\tau\,\mathbf{q}_B = \left(\underline{\mathbf{Z}_R}_{21} + \tau\,\underline{\mathbf{Z}_R}_{22}\right)\,\mathbf{q}_A + \mathbf{M}_B^{-1}\,\underline{\mathbf{T}_L}_{12}^{T}\,\mathbf{f}_A \tag{C.21}$$

Accordingly the modal velocities are obtained so long as $\mathbf{M}_B$ is non-singular.

As may be observed from equation (C.21) one requires knowledge of the forces applied to the system. For a real system one can only know the control forces applied to the system and one cannot know the forces due to external influences. If a linear proportional-derivative modal controller is used of the form

$$\mathbf{f}_B = (\mathbf{G}_k + \tau\, \mathbf{G}_d)\ \mathbf{q}_B \tag{C.22}$$

then one may determine the modal control forces to be

$$\mathbf{f}_B = \Big[\big(\mathbf{G}_k\, \underline{\mathbf{Z}_R}_{11} + \mathbf{G}_d\, \underline{\mathbf{Z}_R}_{21}\big) + \tau\big(\mathbf{G}_k\, \underline{\mathbf{Z}_R}_{12} + \mathbf{G}_d\, \underline{\mathbf{Z}_R}_{22}\big)\Big]\, \mathbf{q}_A + \mathbf{M}_B^{-1}\, \underline{\mathbf{T}_L}_{12}^{T}\, \mathbf{f}_A \tag{C.23}$$

Thus one may determine that the influence from $\mathbf{f}_A$ may reduce the effectiveness of the modal controller but will not alter closed loop stability of the controlled system. A possible course of action is to treat the external forces and noise applied to the system as an uncertainty.

It is now necessary to introduce the left filter to allow the relationship between new and old forcing vectors to be established.

From the result of equation (C.14) it is clear that the vector $\underline{\mathbf{f}_{B1}} = \underline{\mathbf{T}_L}^T \underline{\mathbf{f}_{A1}}$. Knowing from definition given in equation (C.4) that $\underline{\mathbf{f}_{A1}}(1) = \mathbf{f}_A$ and $\underline{\mathbf{f}_{A1}}(2) = 0$ it can be deduced that

$$\begin{bmatrix} \underline{\mathbf{f}_{B1}}(1) \\ \underline{\mathbf{f}_{B1}}(2) \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{T}_L}_{11}^T & \underline{\mathbf{T}_L}_{21}^T \\ \underline{\mathbf{T}_L}_{12}^T & \underline{\mathbf{T}_L}_{22}^T \end{bmatrix} \begin{bmatrix} \mathbf{f}_A \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{T}_L}_{11}^T\, \underline{\mathbf{f}_{A1}}(1) \\ \underline{\mathbf{T}_L}_{12}^T\, \underline{\mathbf{f}_{A1}}(1) \end{bmatrix} \tag{C.24}$$

Thus we may see define the left filter

$$\mathbf{f}_B = \big(\underline{\mathbf{T}_L}_{11}^T + \tau\, \underline{\mathbf{T}_L}_{12}^T\big)\ \mathbf{f}_A \tag{C.25}$$

C.2 Conclusions

It has been presented how to relate the modal displacements, velocities and forces with their physical counterparts. The definitions presented equate to a specific definition but it should be highlighted that other definitions of the filters can be found such as extracting the modal velocities from the state vector and subsequently find a relationship to form the modal displacements. The definitions presented here within are retained for the SPT-modal control method and other definitions of the modal filters are not considered further.

Appendix D

Eigenvalue Derivative

From reference [L2] one may find the left SPT-based filter to be

$$\mathbf{V}(\lambda, \sigma) = \mathbf{V}_0 + \lambda \mathbf{V}_1 \tag{D.1}$$

with $\mathbf{V}_0, \mathbf{V}_1$ defined as

$$\begin{bmatrix} \mathbf{V}_0^T & \mathbf{V}_1^T \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \underline{\mathbf{T}_L} \tag{D.2}$$

The right and left eigenvalue problems are defined

$$\begin{aligned} \mathbf{V}(\lambda, \sigma)\, \mathbf{u}_j &= 0 & \text{(D.3)} \\ \mathbf{v}_j^T \mathbf{V}(\lambda, \sigma) &= 0 & \text{(D.4)} \end{aligned}$$

where $\mathbf{v}_j, \mathbf{u}_j$ are the left and right eigenvectors corresponding to eigenvalue λ_j. $\mathbf{v}_j$ and $\mathbf{u}_j$ are normalised such to give

$$\mathbf{v}_j^T \mathbf{V}_1 \mathbf{u}_j = 1 \tag{D.5}$$

One may differentiate equation (D.3) to yield

$$\frac{\partial \mathbf{V}(\lambda, \sigma)}{\partial \sigma} \mathbf{u}_j + \mathbf{V}(\lambda, \sigma) \frac{\partial \mathbf{u}_j}{\partial \sigma} = 0 \tag{D.6}$$

Differentiating equation (D.1) with respect to σ

$$\frac{\partial \mathbf{V}(\lambda, \sigma)}{\partial \sigma} = \lambda_j \frac{\partial V_1}{\partial \sigma} + \frac{\partial V_0}{\partial \sigma} + \frac{\partial \lambda}{\partial \sigma} \mathbf{V}_1 \tag{D.7}$$

Pre-multiplying equation (D.6) by $\mathbf{v}_j^T$ one yields

$$\mathbf{v}_j^T \frac{\partial \mathbf{V}(\lambda, \sigma)}{\partial \sigma} \mathbf{u}_j = 0 \tag{D.8}$$

since from equation (D.4) one has $\mathbf{v}_j^T \mathbf{V}(\lambda, \sigma) = 0$. Substituting the result of equation (D.7) into this and rearranging yields

$$\frac{\partial \lambda_j}{\partial \sigma} = -\mathbf{v}_j^T \left(\lambda_j \frac{\partial V_1}{\partial \sigma} + \frac{\partial V_0}{\partial \sigma} \right) \mathbf{u}_j \tag{D.9}$$

Thus one finds the derivative of eigenvalue λ_j with respect to parameter σ.

Appendix E

Numerical Penalisation of Control Rate

Consider a second order system with the equations of motion

$$\mathbf{M}_A\,\ddot{\mathbf{q}}_A(t) + \mathbf{D}_A\,\dot{\mathbf{q}}_A(t) + \mathbf{K}_A\,\mathbf{q}_A(t) = \mathbf{S}_A\,\mathbf{u}_A(t) \tag{E.1}$$

where $\mathbf{M}_A, \mathbf{D}_A, \mathbf{K}_A \in \mathbb{R}^{n\times n}$ are the system mass, damping and stiffness matrices, $\mathbf{q}_A(t) \in \mathbb{R}^n$ is the vector of displacements, $\mathbf{u}_A(t) \in \mathbb{R}^r$ the vector of applied forces and $\mathbf{S}_A \in \mathbb{R}^{n\times r}$ is a selection matrix describing the locations of applied forces. The dot above $\mathbf{q}_A(t)$ denotes derivative with respect to time. From this point onwards the notation showing the dependence on time is removed.

Equation (E.1) may be converted into first-order state space form such that

$$\begin{bmatrix} \underline{\dot{\mathbf{q}}_{A1}} \\ \underline{\dot{\mathbf{q}}_{A2}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}_A^{-1}\mathbf{K}_A & -\mathbf{M}_A^{-1}\mathbf{D}_A \end{bmatrix} \begin{bmatrix} \underline{\mathbf{q}_{A1}} \\ \underline{\mathbf{q}_{A2}} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{M}_A^{-1}\mathbf{S}_A \end{bmatrix} \mathbf{u}_A \tag{E.2}$$

with $\underline{\mathbf{q}_{A1}} = \mathbf{q}_A$ and $\underline{\mathbf{q}_{A2}} = \underline{\dot{\mathbf{q}}_{A1}}$. Equation (E.2) may be simplified to

$$\underline{\dot{\mathbf{q}}_A} = \underline{\mathbf{A}_A}\,\underline{\mathbf{q}_A} + \underline{\mathbf{B}_A}\,\mathbf{u}_A \tag{E.3}$$

where the definitions are apparent. The underline notation depicts $2n$-dimensional quantities rather than n-dimensional quantities.

In the context of active control of rotating machines, standard optimal controller methods enable a trade-off to be made between weighted mean-square vibrations and weighted mean-square control force. A major drawback of the traditional approach to optimal control is that no emphasis is placed on the rate at which the control effort can be applied when designing the controller. Control forces cannot be instantaneously applied and indeed several applications exist where the rate at which control forces can be applied is sufficiently important to warrant this work.

It is thus desired to find the minimum of a quadratic cost function of the form

$$J_{ext} = \frac{1}{2} \int_0^\infty \left\{ \underline{\mathbf{q}_A}^T \, \underline{\mathbf{Q}} \, \underline{\mathbf{q}_A} + \mathbf{u}_A^T \, \mathbf{R}_\mathrm{u} \, \mathbf{u}_A + \dot{\mathbf{u}}_A^T \, \mathbf{R}_\mathrm{v} \, \dot{\mathbf{u}}_A \right\} \, dt \tag{E.4}$$

where $\underline{\mathbf{Q}} \in \mathbb{R}^{2n \times 2n}$ represents a symmetric semi-definite weighting matrix governing the relative importance of the system state at time, t, and similarly $\mathbf{R}_\mathrm{u} \in \mathbb{R}^{r \times r}$ and $\mathbf{R}_\mathrm{v} \in \mathbb{R}^{r \times r}$ represents a symmetric positive definite matrix to weight the control force and control force rate respectively. It is assumed that full state feedback is available.

A possible method to find the minimum of cost function (E.4) is to find numerically the optimal force which minimises the extended cost function when the system is subjected to a known initial condition $\underline{\mathbf{q}_A}(0)$. The forcing function, $\mathbf{u}_A$, used to minimise the cost function (E.4) is assumed to be constructed using a summation of forcing wavelets. The collection of wavelets allows a smooth function to be constructed which has a user-specified frequency content. This allows a sufficient freedom of choice to construct a suitable forcing function. The forcing function can be defined as

$$\mathbf{u}_A = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij} \frac{\sin[\beta_i (t - s_j)]}{(t - s_j)} = \mathbf{F}_p \, \mathbf{\Theta}_p \tag{E.5}$$

where

$$\mathbf{F}_p = \begin{bmatrix} \frac{\sin[\beta_1(t-s_1)]}{(t-s_1)} \\ \vdots \\ \frac{\sin[\beta_1(t-s_m)]}{(t-s_m)} \\ \frac{\sin[\beta_2(t-s_1)]}{(t-s_1)} \\ \vdots \\ \frac{\sin[\beta_2(t-s_m)]}{(t-s_m)} \\ \vdots \\ \frac{\sin[\beta_n(t-s_m)]}{(t-s_m)} \end{bmatrix} \qquad \mathbf{\Theta}_p = \begin{bmatrix} \alpha_{11} \\ \vdots \\ \alpha_{1m} \\ \alpha_{21} \\ \vdots \\ \alpha_{2m} \\ \vdots \\ \alpha_{nm} \end{bmatrix} \qquad \begin{array}{l} i = 1, 2, ..., m \\ j = 1, 2, ..., n \end{array} \tag{E.6}$$

Here β_i represents the frequency content of the wavelet and s_j represents an offset in time. An illustrative example of the forcing function where $m = n = 1$ may be found in figure E.1.

The motivation for choosing a forcing function of this type is that the time offset and frequency content of the expression can be readily controlled. This allows an expression to be tailored to the specific system in question and it creates a suitably large basis for minimisation. Additionally, the expression for the force in equation (E.5) readily yields the expression for the rate of force

$$\dot{\mathbf{u}}_A = \mathbf{G}_p \, \mathbf{\Theta}_p \tag{E.7}$$

where

$$\mathbf{G}_p = \begin{bmatrix} \frac{\beta_1 \cos[\beta_1(t-s_1)]}{(t-s_1)} - \frac{\sin[\beta_1(t-s_1)]}{(t-s_1)^2} \\ \vdots \\ \frac{\beta_1 \cos[\beta_1(t-s_m)]}{(t-s_m)} - \frac{\sin[\beta_1(t-s_m)]}{(t-s_m)^2} \\ \frac{\beta_2 \cos[\beta_2(t-s_1)]}{(t-s_1)} - \frac{\sin[\beta_2(t-s_1)]}{(t-s_1)^2} \\ \vdots \\ \frac{\beta_2 \cos[\beta_2(t-s_m)]}{(t-s_m)} - \frac{\sin[\beta_2(t-s_m)]}{(t-s_m)^2} \\ \vdots \\ \frac{\beta_n \cos[\beta_n(t-s_m)]}{(t-s_m)} - \frac{\sin[\beta_n(t-s_m)]}{(t-s_m)^2} \end{bmatrix} \qquad \begin{array}{l} i = 1, 2, ..., m \\ j = 1, 2, ..., n \end{array} \tag{E.8}$$

β_i and s_j are fixed prior to optimisation and do not represent variables in the minimisation process. Only $\mathbf{\Theta}_p$ is available for alteration which represents the set of coefficients of the individual wavelets.

The cost functional of equation (E.4) is a quadratic form and variation of $\mathbf{\Theta}_p$ is permitted. Thus, a sufficient criteria to find the global minimum of a quadratic function is

$$\frac{dJ_{ext}}{d\mathbf{\Theta}_p} = 0 \tag{E.9}$$

The extended quadratic cost function of equation (E.4) may be compartmentalised into individual components relating to state, force and force rate

$$J_{ext} := J_{\underline{\mathbf{q}_A}} + J_{\mathbf{u}_A} + J_{\dot{\mathbf{u}}_A} \tag{E.10}$$

The individual parts illustrated in the compartmentalised form of the cost function (E.10) from initial time t = 0 can be differentiated with respect to $\mathbf{\Theta}_p$ and combined.

$$\begin{aligned} J_{\mathbf{u}_A} &= \frac{1}{2}\int_0^\infty \mathbf{u}_A^T\,\mathbf{R}_\mathrm{u}\,\mathbf{u}_A\,dt && \text{(E.11)} \\ &= \frac{1}{2}\int_0^\infty \mathbf{\Theta}_p^T\,\mathbf{F}_p^T\,\mathbf{R}_\mathrm{u}\,\mathbf{F}_p\,\mathbf{\Theta}_p\,dt && \text{(E.12)} \\ \frac{dJ_{\mathbf{u}_A}}{d\mathbf{\Theta}_p} &= \mathbf{\Theta}_p^T\int_0^\infty \mathbf{F}_p^T\,\mathbf{R}_\mathrm{u}\,\mathbf{F}_p\,dt && \text{(E.13)} \end{aligned}$$

The component containing the derivative of force rate similarly yields

$$\frac{dJ_{\dot{\mathbf{u}}_A}}{d\mathbf{\Theta}_p} = \mathbf{\Theta}_p^T\int_0^\infty \mathbf{G}_p^T\,\mathbf{R}_\mathrm{v}\,\mathbf{G}_p\,dt \tag{E.14}$$

The component of the cost function constructed using the system state is a little more complicated. The state can be obtained using a form of the convolution integral [M7] where $\underline{\mathbf{q}_A}(0)$ represents the initial state of the system, and $\underline{\mathbf{A}_A}$ and $\underline{\mathbf{B}_A}$ represent the first order system state space companion and forcing matrices respectively

$$\begin{aligned} \underline{\mathbf{q}_A} &= \exp^{\underline{\mathbf{A}_A}t}\,\underline{\mathbf{q}_A}(0) + \int_0^\infty \exp^{\underline{\mathbf{A}_A}(t-\tau)}\;\underline{\mathbf{B}_A}\,\mathbf{u}_A(\tau)\,d\tau \\ &= \exp^{\underline{\mathbf{A}_A}t}\,\underline{\mathbf{q}_A}(0) + \int_0^\infty \exp^{\underline{\mathbf{A}_A}(t-\tau)}\;\underline{\mathbf{B}_A}\,\mathbf{F}_p(\tau)\,\mathbf{\Theta}_p\,d\tau && \text{(E.15)} \end{aligned}$$

Thus it is found

$$\frac{d}{d\mathbf{\Theta}_p}\int_0^\infty \underline{\mathbf{q}_A}^T\,\underline{\mathbf{Q}}\,\underline{\mathbf{q}_A}\,dt = \int_0^\infty \left(\underline{\mathbf{q}_A}(0)^T\ \exp^{\underline{\mathbf{A}_A}^T t}\ \underline{\mathbf{Q}}\int_0^t \left[\exp^{\underline{\mathbf{A}_A}\,(t-\tau)}\ \underline{\mathbf{B}_A}\,\mathbf{F}_p(\tau)\right]d\tau\right)\,dt\,+$$
$$\cdots + \mathbf{\Theta}_p^T\int_0^\infty\left(\int_0^t\left[\mathbf{F}_p(\tau)^T\,\underline{\mathbf{B}_A}^T\ \exp^{\underline{\mathbf{A}_A}^T\,(t-\tau)}\ \underline{\mathbf{Q}}\ \exp^{\underline{\mathbf{A}_A}\,(t-\tau)}\ \underline{\mathbf{B}_A}\,\mathbf{F}_p(\tau)\right]d\tau\right)\,dt \tag{E.16}$$

The differentiated components of the cost function can be grouped to satisfy the constraint for a global minimum and give an equation of the form

$$\mathbf{X} + \mathbf{Y}\,\mathbf{\Theta}_p = 0 \tag{E.17}$$

Thus equation (E.17) is fully determined and one may solve $\mathbf{\Theta}_p$ accordingly.

E.1 Numerical Example 1

Consider an undamped spring-mass system as illustrated in figure E.2. Masses are interconnected by springs and can only move horizontally. The displacement response of mass 1 is observed.

This example has the equation of motion

$$\mathbf{M}_A\,\ddot{\mathbf{q}}_A + \mathbf{K}_A\,\mathbf{q}_A = \mathbf{S}_A\,\mathbf{u}_A \tag{E.18}$$

and the system matrices are assumed to be

$$\mathbf{M}_A = \mathrm{diag}\begin{bmatrix}1\\1\\2\end{bmatrix}\ ,\ \mathbf{K}_A = 10^3\begin{bmatrix}4 & -2 & 0\\ -2 & 3 & -1\\ 0 & -1 & 4\end{bmatrix}\ ,\ \mathbf{S}_A = \begin{bmatrix}1\\0\\0\end{bmatrix} \tag{E.19}$$

The system is initially at rest and subjected to initial displacements $\mathbf{q}_A(0) = \begin{bmatrix}0 & 0 & 2.5\end{bmatrix}^T$. The response $\mathbf{q}_1$ from the initial conditions for the uncontrolled system yields the response shown in figure E.3.

30 ($n = 30$) values of β_i and 30 ($m = 30$) values of s_j are chosen to provide a suitably large basis for the numerical penalisation process. The β_i and s_j values are given through

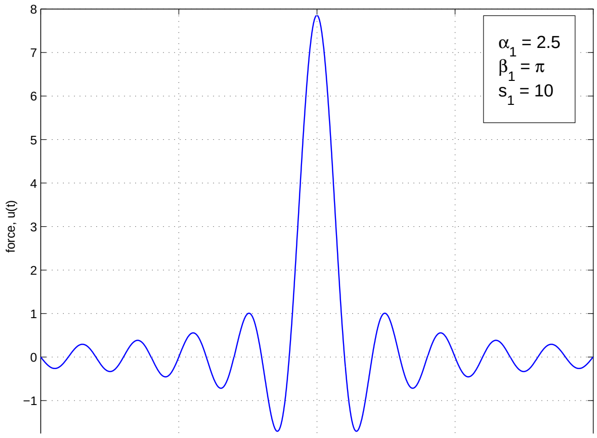
the relationship

$$\beta_i = \frac{3\pi}{\max t}\begin{bmatrix} 2 & 4 & \cdots & 2n \end{bmatrix}^T \tag{E.20}$$

$$s_j = \frac{\max t}{3m}\begin{bmatrix} 0 & 1 & \cdots & m-1 \end{bmatrix}^T \tag{E.21}$$

Using the given values of β_i and s_j one may solve $\mathbf{\Theta}_p$ in equation (E.17) to give the appropriate values of α_{ij}. Using the results obtained for the numerical solution one may plot the response of the system to the initial conditions when the numerically obtained force is applied. The response is shown in figure E.4.

As may be observed from figure E.4 the displacement is decaying and a measurable decrease in the amplitude may be observed from the unforced response to the initial conditions as illustrated in figure E.3. The important comparison is made to the conventional LQR solution which is plotted in figure E.5. As is apparent the LQR response decays much quicker than the response from the numerically obtained force but the control force and control force rate are significantly greater. Thus the penalisation of the control force rate has indeed been obtained.



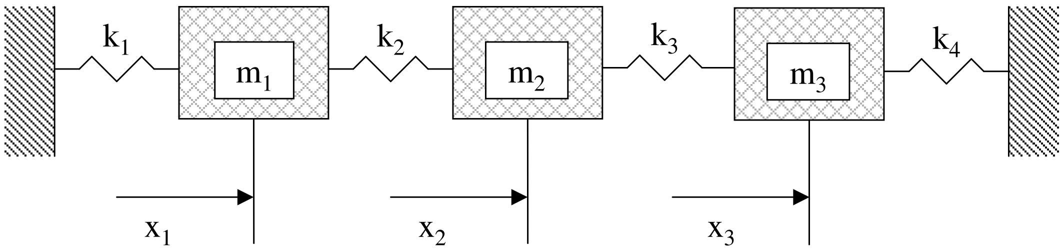


Figure E.2: Spring Mass System

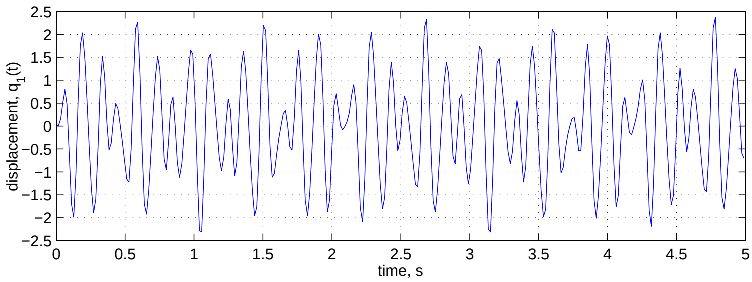


Figure E.3: Unforced response of system to initial conditions

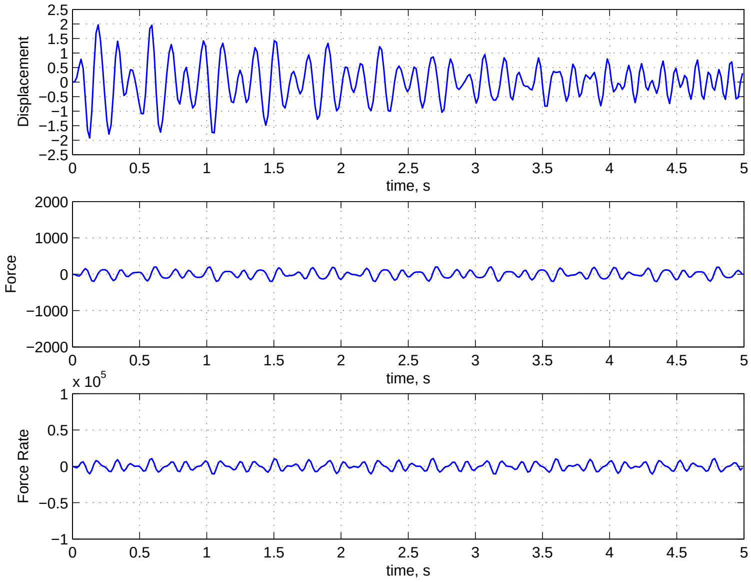


Figure E.4: Response of system to initial conditions - Numerical penalisation method

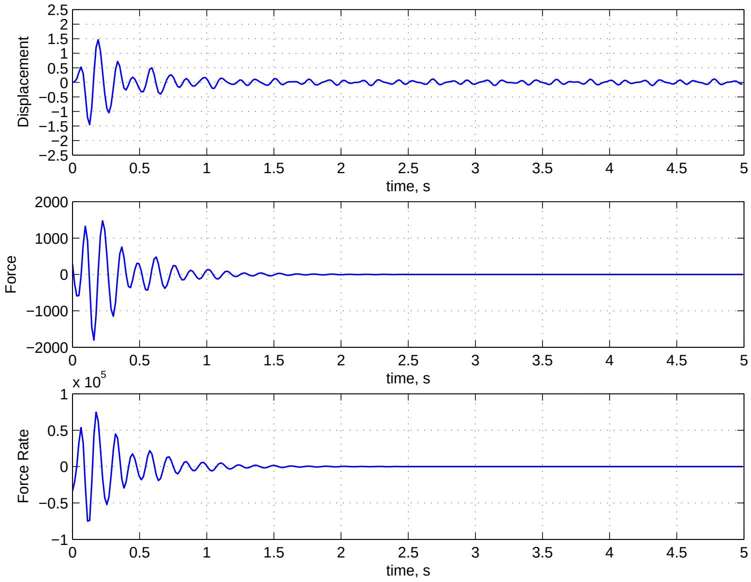


Figure E.5: Response of system to initial conditions - LQR controlled