Classical and Quantum Causality in Quantum Field Theory

or

The Quantum Universe

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Thesis submitted to the University of Nottingham
for the degree of Doctor of Philosophy, April 2004

Abstract

Based on a number of experimentally verified physical observations, it is argued that the standard principles of quantum mechanics should be applied to the Universe as a whole. Thus, a paradigm is proposed in which the entire Universe is represented by a pure state wavefunction contained in a factorisable Hilbert space of enormous dimension, and where this statevector is developed by successive applications of operators that correspond to unitary rotations and Hermitian tests. Moreover, because by definition the Universe contains everything, it is argued that these operators must be chosen self-referentially; the overall dynamics of the system is envisaged to be analogous to a gigantic, self-governing, quantum computation. The issue of how the Universe could choose these operators without requiring or referring to a fictitious external observer is addressed, and this in turn rephrases and removes the traditional Measurement Problem inherent in the Copenhagen interpretation of quantum mechanics.

The processes by which conventional physics might be recovered from this fundamental, mathematical and global description of reality are particularly investigated. Specifically, it is demonstrated that by considering the changing properties, separabilities and factorisations of both the state and the operators as the Universe proceeds though a sequence of discrete computations, familiar notions such as classical distinguishability, particle physics, space, time, special relativity and endo-physical experiments can all begin to emerge from the proposed picture. A pregeometric vision of cosmology is therefore discussed, with all of physics ultimately arising from the relationships occurring between the elements of the underlying mathematical structure. The possible origins of observable physics, including physical objects positioned at definite locations in an arena of apparently continuous space and time, are consequently investigated for a Universe that incorporates quantum theory as a fundamental feature.

Overall, a framework for quantum cosmology is introduced and explored which attempts to account for the existence of time, space, matter and, eventually, everything else in the Universe, from a physically consistent perspective.

Keywords: *Quantum cosmology, Quantum computation, Pregeometry, Emergence, Factorisation and Entanglement, Qubit field theory, Quantum Causal sets, Discrete time, Information Exchange, Subregisters, Endo-physics, Self-Referential Quantum automata.*

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Acknowledgements

Due to his continual guidance, ideas, discussions and suggestions during the entire course of my research, I first thank my supervisor, George Jaroszkiewicz.

I would also especially like to thank my family for their neverending support and encouragement, not only over the past few years of my Ph.D., but also throughout the whole of my education and life.

On a further note, I am additionally grateful to Dr. Christopher Hide for help with LATEX code and document formatting, and to Drs. Nikolaos Demiris, Cornelius J. Griffin, Hilmar Hauer and Andrew Whitehead for illuminating discussions on many different aspects of mathematics. Appreciation also to Professor A. Sudbery and Professor V. Belavkin for constructive criticism and comments on the content of the thesis.

Lastly, special thanks to all those friends, colleagues, acquaintances and others who gave me the inspiration and often unexpected moments of insight that made this what it is; if you don't know who you are, at least I do.

I acknowledge the EPSRC (UK) for a research studentship.

To-morrow, and to-morrow, and to-morrow,
Creeps in this petty pace from day to day,
To the last syllable of recorded time;

- **W. Shakespeare**
  Macbeth

1 Introduction

Time permeates just about every sphere of life. Indeed, human civilisations are dominated by regularising and synchronising the events in the surrounding World, and it is easy to argue that without agreeing on a common standard of time, society would be unable to function in the way that it does.

In fact, time has played a crucial role in all of Mankind's development. From agricultural dependencies on the cyclicity of the Sun, to the machine timings that structured the Industrial Revolution; from the variable 't' present in most of the equations used in science and engineering, to the parameter vital in rationalising the study of history; from the timetables essential for efficient national and global travel, to the Time Machines of literature and imagination; from the rhythms governing the lives of animals and plants, to the cadences of music and speech; from the measurements necessary for road safety laws, to the quantity specified with pinpoint accuracy when coordinating extraterrestrial exploration; time plays a role in everything.

Moreover, human life often appears obsessed with the passage of time. Modern society frequently revolves around questions such as "When is...?", "What time did...?", "How long until...?", and so on, and the ever present threat of mortality seems to heighten the sense that time is a precious commodity to be 'saved' or 'made the most of' wherever possible. Time is something that employers buy, and the cautious bide. Time is something that 'waits for no man', but can appear to 'fly', drag' or 'stand still'. Time is even something whose effects medical research attempts to 'hold back'.

Even primitive Man recognised the importance of the nature of time for his continued existence. Basic subsistence and the quest for food relied heavily upon an understanding of the temporal durations of the seasons, and many archaeologists now believe that ancient monuments such as Stongehenge and the Egyptian Pyramids were used partly as astronomical calendars. Furthermore, the mystical significance often attached to the lunar cycle, and the magical rituals associated with mid-summer and mid-winter in, amongst others, druid and pagan cultures, indicate just how important the continual 're-birth' of time was taken to be in early society. Indeed, over 3000 years ago Greek mythology talked of Chronos, the personification of time and father to all Olympian gods. In fact, time still plays crucial roles in the sacred texts of the current major World religions: the Creation story of the Judaic texts is told to take place over a period of seven days, whilst the Eastern religions featured in the Hindu and Buddhist scriptures describe the eternal, cyclic nature of existence. Indeed, the act of Creation itself concerns the very origins of time; the Bible even starts with the words "In the beginning...".

The actual quantification of time is also something that has proved essential in the

development of society. Its accurate measurement has therefore keenly concerned inventors and engineers since ancient civilisation, from the historical use of sundials, waterclocks, hour glasses and candles, through to the mechanical world of clockwork gears and springs, and finally ending up in the modern age of digital chronographs and precision atomic clocks. Indeed, the second is considered one of the most important units of science, and its definition is given as one of the seven base quantities provided in the International System of physical standards.

In human development, too, are time and temporality important. Some psychologists believe that young children, for example, have very little temporal awareness, and it is not until about the age of two before a key stage in their cognitive growth occurs and they can appreciate the abstract concepts involved in 'today', 'tomorrow' and 'yesterday'. Indeed, even the development of language, a skill that has given humans a unique advantage over every other organism on Earth and is a central milestone in a child's progression towards maturity, is intrinsically linked with temporality, and whole sets of tenses are required in order for people to express events, ideas and plans that have happened, had happened, were happening, are happening, and will happen.

Overall, the concepts and measurements of time, rate and duration seem vital in humans' understanding, description and control of the reality in which they live.

Despite this, however, still nobody really knows what time actually *is.*

One ultimate aim of this thesis is to investigate the nature and origin of time in the physical Universe. In particular, a direction is taken that is based entirely upon empirical evidence, and is hence fully consistent with the experimentally verified principles of standard quantum mechanics.

To this end, it actually turns out that in attempting the above goal, a paradigm is proposed and developed that describes the entire Universe according to quantum principles. As will become apparent, the vision is of a fully quantum universe free from external observers, represented by a complex statevector in an enormous, but finite, factorisable Hilbert space, and developed by the successive applications of quantum operators.

Now, physicists do not of course perceive the Universe to be a single, complex vector in a Hilbert space. Instead, the Universe generally appears to contain an enormous number of classically distinct looking objects, consisting of molecules, atoms and, ultimately, elementary particles. Moreover, these objects appear to interact with one another in particular ways, and appear to exist at unique locations in a background of apparently continuous and classical space and time. The question immediately faced therefore asks how all of these features could arise from the proposed quantum state description, and it is by attempting to address this issue that a possible origin for a variety of physically

observed phenomena is provided.

So, from the initial aim of describing the possible mechanisms responsible for the existence of time in physics, a framework is proposed from which every feature of physical reality is hoped to emerge. Overall, the suggestion is that by adopting a fully quantum view of the Universe, the origins of time, space, matter and classicity in physics might be accounted for from a certain fundamental perspective.

Exactly how this could be achieved is the subject of this work.

The structure of this thesis is as follows. In Chapter 2, a number of the different traditional interpretations of time are introduced. Historical perspectives on the subject, including the role of time in various scientific disciplines, are briefly mentioned first, and this is then followed by a discussion of the nature of time in conventional physics.

In Chapter 3, the framework for the paradigm featured throughout this work is introduced, justified and discussed. The Universe is argued to be represented by a quantum state, and the constraints placed on this by physics are considered; the necessary features required for its dynamics are then defined. The paradigm was originally proposed in [1], and was also developed in [2].

In Chapter 4, the issue of obtaining classically distinguishable objects from the perspective of the fully quantum Universe is discussed. Specifically, the mathematical properties of factorisable Hilbert spaces and entangled/separable states are investigated; these ideas will form the basis for much of the work featured in the following chapters. Most of Chapter 4 was developed with G. Jaroszkiewicz, and is strongly similar to the work presented in [3].

In Chapter 5, the emergence of spatial degrees of freedom from the quantum universe paradigm is discussed. In particular, it is shown how quantum causal sets may be generated as the Universe proceeds through a sequence of stages, from considerations of both the changes in separability of the state, and from the changes in factorisability of the operators used to develop this state. The possibility of obtaining continuous, classical spacetime from such a treatment is suggested. As with Chapter 4, this work was developed with G. Jaroszkiewicz, and is congruent in content to [4].

Chapter 6 is split into two parts. First, a discussion of the development of low dimensional bit and qubit systems using CNOT operators is given; the parallels between computation, quantum computation and the quantum universe paradigm proposed in this thesis are then drawn. The second part of Chapter 6 addresses the issue of information flow in self-contained quantum systems, thereby defining the concepts of information change and information exchange. The definition of endo-physical measurements is then given, with the goal being to investigate quantum experiments in the circumstance where the observer is part of the subject under observation. Such a discussion is mandatory

for the proposed fully quantum paradigm, because physicists are by definition part of the Universe they attempt to measure. Much of Chapter 6 was developed in collaboration with G. Jaroszkiewicz, with the second part related to [5]. Chapter 6 is also supported by Appendix A, in which conventional (exo-physical) classical and quantum computation are discussed for completeness and comparison.

In Chapter 7, the possible emergence of quantum field theoretic concepts from the proposed paradigm is discussed; the generation of particle physics in a fully quantum Universe is hence suggested. Specifically, the Dirac field is investigated, and it is shown how the corresponding ladder, Hamiltonian, momentum and charge operators used in traditional collider physics may arise from the conjectured vision. The potential links between quantum computation, quantum field theory, and the quantum universe paradigm are thus explored. Much of this research was completed with G. Jaroszkiewicz, and will feature in a forthcoming article currently being prepared for publication. Chapter 7 is accompanied by Appendix B, which derives from first principles the conventional (phase space) Hamiltonian, momentum and charge operators for the relativistic Dirac equation.

Finally, in Chapter 8 the dynamics of developing quantum universes are discussed. On the basis that its state is developed by applying a quantum operator, and that there is no external physicist present to choose such an object, the various types of method that the Universe might employ to select this operator are first classified. Then, some of these types are investigated more thoroughly, with the aim being to discuss how the actual next operator used in a universe's development could depend on its current state. The physical results and limitations of the suggested 'self-referential' mechanisms are duly considered in turn, with the conclusion drawn that only certain types of method are able to provide valid dynamics for a universe's development. Lastly, a particular type of mechanism is proposed that appears able to automatically examine the current state, and then develop it according to what it is and what properties it might have; the application of this to the issue of endo-physical experimentation is discussed. Summarising, in Chapter 8 a number of algorithms are defined for the development of the quantum universe, and these are effectively seen as analogous to self-referential versions of the rules used in conventional physics to govern quantum computations.

A self-contained, developing, fully quantum Universe is thus proposed in the following, and the possible ways in which time, space, physics and matter could emerge from this paradigm are demonstrated.

2 History and Background

It is almost impossible to even attempt to provide a complete account of the background of the study of time. Indeed, time's 'ultimate nature' is a question that has concerned scientists and philosophers alike for many thousands of years. Moreover, because time pervades almost every aspect of human nature, it is perhaps not surprising that its study has drawn from a number of academic disciplines, and that many of these consider its definition in many different ways.

It is therefore beyond the scope of this thesis to detail every idea from such a wide range of sources. However, whilst this may be the case, in this chapter a number of the conventional perspectives on time are briefly introduced.

2.1 Time in Mathematics, Philosophy and Biology

In mathematics, time is generally just the 'parameter t' that is used simply as a suitable variable when describing the changes of a developing system. So, from absolute Newtonian time to proper time in relativity, temporality is normally only employed merely as a convenient label to distinguish events. In Newton's *Principia Mathematica,* for example, time is barely mentioned apart from as the t in the equations, and is regarded as an almost ethereal notion without further discussion or justification; to quote: "*Absolute time flows equably, without regard to anything else*".

Time in mathematics is therefore most often used simply as a coordinate reference, that is, as a linear, real axis stretching from $-\infty$ (the past) to $+\infty$ (the future) via the origin (now). Although this idea may be extended to cover imaginary time as well (for example, in the path integral approach in quantum field theory, where time t is mapped to $i\tau$ in order to prevent divergences [6]), the linearity of the coordinate axis is normally still preserved. In fact, by reversing this line of thinking, since time is often interpreted as a linear sequence of moments, which it generally is despite the possibility of some Zeno-type paradoxes, it is easy to see why it is so frequently linked to the number-line. Indeed, Kant even thought that time and number were inseparable.

A mathematical construction of continuous time may be built up from some fairly basic, logical arguments. One particular method is given in [7]; if T is a 'temporal continuum', and if p, q, and r are 'instants' or 'moments' defined to be members of this set, then the following statements are assumed to hold true:

1. Mutual exclusivity: either p and q are simultaneous, or p precedes q, or q precedes p.

2. If q precedes r and p precedes q, then p precedes r.

3. T is a *dense set;* if p precedes r, there exists at least one event q between p and r.

4. If T contains non-empty subsets T_1 and T_2, where $T \equiv T_1 \cup T_2$ and all elements of T_1 precede all elements of T_2, there exists at least one 'instant' t for which any instant preceding t is in T_1, but any instant after t is in T_2.

5. Between any two members of T there is at least one instant which is a member of a denumerable subset of T; the relation between time and the number line is again drawn.

From a set of axioms such as these, the suggestion is that a mathematical framework for past, present and future may be derived; a linear temporal parameter may consequently be generated. The logical framework, however, does not actually begin to describe what this time might actually *be*.

The logical approach to time may be argued to have its roots in philosophy. Indeed, the definition of time has naturally been a subject pertinent to many philosophers, too numerous to mention, since at least the World of Ancient Greece. Plato, for example, thought that time could not actually be described by mathematics, because he believed that only things that exist eternally were real, and time, unlike numbers, is transient. His student Aristotle, on the other hand, thought that time was just "*a measure of motion*".

Historically downstream, William of Alnwick suggested in the Fourteenth century the idea of a discrete time, referring to an "*indivisible of motion*". One hundred and fifty years later, Leonardo Da Vinci contended that: "*an instant has no time. Time is made of movements of the instant, and instants are the boundaries of time*". Later still, Immanuel Kant believed that time can neither have a beginning nor be eternal, and that Mankind's understanding of what time might be will always, ultimately, be inadequate.

Some of the great Eighteenth and Nineteenth century mathematicians and physicists also contributed to the philosophical interpretation of time. Hamilton, for example, linked time with algebra in an analogous way to how space is linked with geometry. Conversely, Leibniz and Laplace lived in a deterministic world, so effectively 'removed' the need for time because they believed that everything could be determined from initial conditions.

Leibniz and Laplace's opinions are not reconciled from every modern perspective, particularly from the current belief that the universe incorporates stochastic quantum laws and so is not strictly and classically determined. Indeed, Penrose, for example, even argues that classical determinism is broken by quantum mechanical effects in the brain, and this has profound implications for discussions of time. This 'consciousness debate' is moreover congruent to the belief of Hobbes, who contended that time is a decay of the Before and

After in the mind. In fact in many ways, this type of interpretation may be summarised by the words of Einstein: "*Objective reality* is *and does not happen. Only by consciousness does the world come to be as an image in space continuously changing in time*".

It is by invoking such notions of consciousness that provides a bridge between philosophical discussions of time, and those present in the biological sciences.

Time in biology splits broadly into two categories: on the one hand, there exists the temporal rhythms of nature; on the other, there is the subjective, conscious experience.

Temporal rhythms are generally governed by the responses to external time stimuli, and imply that living organisms often appear to possess inbuilt 'biological clocks'. Such bio-clocks are often synchronised to well regulated outside 'cues': these may be variations in light intensity, length of daylight, average temperature, lunar cycles, tidal effects, etc. So, examples include diurnal rhythms (e.g. differences in day/night mental activity), monthly rhythms (e.g. menstrual cycle), annual rhythms (e.g. in perennial plants), and so on. As a consequence of these periodic patterns, human beings are often able to get the false impression that time, ultimately, is cyclic in nature.

'Conscious time' is more complicated, partly perhaps because it is uncertain exactly what consciousness is, and hence partly also because scientists cannot easily ascertain its existence in other organisms. Moreover, it is also experienced in different ways that are subject to context; a human's perception of duration, simultaneity and time elapse are all highly dependent on the particular individual, her state of mind, her age, her memory, and the physical stimulus itself, etc.

There is also, of course, clearly a fine line between scientific evaluation of psychological and neurophysiological time, and the philosophical question of mind itself.

2.2 Time in Physics

Despite the appearance of time in mathematics, philosophy and biology, it is perhaps the physical sciences that should be most concerned with providing a definition for what time actually is. After all, time evidently seems to be a physical phenomenon.

In addition to this, it is also noted that time is present in most of the equations of physics. Indeed, some might argue that it is the purpose of physics to either predict the future outcome of an experiment from a given set of initial conditions, or to reconstruct the past from results that exist in the present. The question, then, of what past, present and future actually are should consequently be taken to be of prime importance. In fact, given that time is surely one of the most fundamental physical phenomena there is, its

ultimate origin and definition would have to be explained by any law that pertains to be a Theory of Everything in order for such a suggestion be accepted as truly satisfactory.

So, having mentioned these points, it is perhaps surprising to consider just how little physicists seem to understand about the true nature of time. Going further, it is perhaps equally surprising to observe just how seldom this issue is even addressed. In only a few areas of physics, for example, is time actually viewed as a fundamental quantity, instead of just as a convenient 'yardstick' to measure against or label events. Rarely does physics really consider what this yardstick might actually be, or what this label might mean.

One possible reason for this lack of definition might be because it does not normally appear to matter what time actually is, as long as its effects may be accounted for. Newton's laws, for example, are entirely symmetric with respect to a reversal of time: a ball rolling without friction from A to B and back to A appears exactly equivalent in 'reverse'; so, in this situation time is reduced to a mere coordinate that provides a convenient parameter useful in defining dynamics. In short, as long as the continuous variable 't' may be employed in the equations of motion with accurate results, the issues concerning what it might actually be are often ignored.

However, not every phenomenon of science is time symmetric. Thus, time might most interestingly be discussed in situations where its direction does seem to play a distinguishing part. As examples, cosmology, particle theory, thermodynamics and quantum mechanics each contain such asymmetries, and each of these appears to introduce important comments regarding the role of time. So, it is these issues that are discussed in turn now.

The cosmological development of the Universe is intrisically linked to a number of questions regarding time. Indeed for a start, its evolution as a whole is manifestly time asymmetric.

Specifically, current thinking is that the Universe began as a Big Bang about 12 billion years ago, and has been expanding ever since; indeed, most relativistic cosmologists believe that the Big Bang actually marked the very beginning of time. Now, this scenario has two implications for the present discussion. The first point concernes the Universe's fate and future: either it will stay expanded forever, or else it will collapse back to a Big Crunch, depending on its density. If the former is true, there is an immediate asymmetry associated with a finite past but an infinite future, and the question is provoked as to how and why the Universe actually began. Moreover, given that this question might naturally be rephrased as "what was it that changed and caused the Universe?", it is remarked both that the notion of 'change' itself implies a reference to an external time, and that the concepts of cause and effect rely on a sense of before and after, whereas none of these are

defined at the Big Bang.

Conversely, if the latter situation is true and the Universe will eventually collapse back on itself, the conclusion may be drawn that either the Big Crunch is different to the Big Bang, leading to another asymmetry, or that time is somehow reversed during the contraction phase of the Universe. This second point is immediately undesirable, because it could imply that entropy might decrease, stars would 'suck in' light, particles should disappear from event horizons, etc.

The remaining implication of an expanding Universe scenario is that the frame of reference in which it is expanding may be considered to be 'preferred'. In this case, such a frame's time component could then naturally be linked with an absolute or universal time, and this appears at odds with the generally accepted principles of relativity. Furthermore, although such a hypothetical frame is often taken in the literature to be the frame in which the Cosmic Microwave Background Radiation (CMBR) is isotropic, it is still debateable as to whether this really provides a genuinely preferred frame; it is hence unclear as to what the consequences of this might mean.

As discussed above, problems concerning time exist on the largest scales of physics. However, difficulties also arise at some of the smallest scales.

As an example, it is noted that the equations describing elementary particles are expected to be invariant under the combined operations of Charge conjugation, Parity reversal and Time reversal (CPT). It appears to be an empirical fact, however, that the decay of the electrically neutral kaons K^0 and $\overline{K}^0$ via the weak interaction appears to violate Charge-Parity (CP) conservation, and so this decay is also expected to violate time reversal if the overall CPT operation is to remain invariant. There is currently no satisfactory explanation for this effect, and it is therefore believed by some physicists that its investigation might shed light on the true nature of time.

Discussions of time, however, are not just limited to the scales of cosmology or fundamental particles. In fact, one important area of 'everyday' physics that exhibits time asymmetry occurs in thermodynamics. Indeed, even from the outset, the equations of thermodynamics do not obviously appear reversible: heat always flows from a hot body to a cooler one.

Of course, the above observation may be phrased more precisely by stating that the entropy of a system always increases with time. In other words, a system that is initially macroscopically heterogeneous becomes microscopically heterogeneous (or, equivalently, macroscopically homogeneous) over time. Moreover, the converse of this is not in general true, and this has lead some authors to conject that it is such an effect that defines the arrow of time. Thus, the 'direction' of increasing entropy is consequently taken to define

the 'direction' of the increase in time.

Others authors [8], however, contend that classical entropy is, in fact, really reversible (at least in principle), because its microscopic scale is still governed by deterministic kinetic theory, and hence time symmetric laws. The argument is then that real irreversibility only comes from quantum effects, by introducing a random 'ingredient' into an observer's knowledge of the kinematics. In short, because the particles' positions and velocities are ultimately indefinite in the quantum case, the argument is that they can no longer develop deterministically. An irreversibility is therefore introduced into the dynamics, and it is this that is eventually taken to provide a direction for the 'flow' of time.

So, the suggestion here is that it is quantum mechanics that ultimately provides an explanation for the origin of asymmetric time.

The above comment introduces perhaps the most important conflict in the history of the study of time.

Without exaggeration, much of fundamental physics in the Twentieth century was founded on two great pillars, namely, relativity and quantum mechanics. Each of these tremendous theories says something profound about the nature of time, and, moreover, each is ultimately incompatible with the other.

Special and General relativity are based on the notion of coordinate time. In other words, time is assumed to be a dimension, as real and linear as length, breadth and width. In fact, the temporal parameter is placed on a completely equal footing to the spatial coordinates, and this has led to a vision of physics existing in a four dimensional spacetime. So, in the relativistic approach, spacetime is imagined to be a four dimensional 'fabric' which is then curved and distorted by the presence of matter. Moreover, the resulting 'shape' of this fabric may be described by a metric, and this is a continuous function of the temporal and spatial coordinates. Physical objects consequently describe trajectories, or worldlines, through this background arena of spacetime.

Thus, relativity adopts a '*Block universe*' perspective: time and space are effectively equivalent, extended dimensions. It appears, moreover, to just be a 'biological accident' that humans appear to perceive a three dimensional space evolving temporally; according to relativity, each of the temporal and spatial dimensions is just as special as the others, with metric signatures providing the only difference.

Of course, this interpretation leaves a number of questions unanswered. Why does time appear to be special for humans? Why can objects move back and forth in space with complete freedom, but appear unable to travel backwards in time at all? Why can humans only 'go forwards' in time at an apparently fixed rate? Indeed, why can an object not remain at one position in time, just as it appears able to rest at a single spatial location?

As an outcome of the association of time with dimension, in the Block universe approach past, present and future all exist in an equal way. The only distinction between them, in fact, arises from the point of view of a particular observer: two observers in different frames of reference witnessing the same set of events cannot necessarily agree on their order. Specifically, an event that lies in the future of one observer (or more technically, in the future lightcone of one observer) could be the present for a second observer, but could lie in the past (lightcone) of a third observer. Thus, two events that may appear simultaneous in one frame of reference may be temporally separated in a second

Moreover, this analysis then implies that the relativistic universe is effectively deterministic: any object in such a universe has its future 'mapped out', because future events on its worldline might be in the past of an observer in a different frame of reference. For any given moment in the Block universe approach, the past exists just as much as the present does, and a pre-determined future is already 'out there' waiting to be discovered.

In quantum mechanics, however, the situation is a little different.

In classical mechanics, on which relativity is based, it is acceptable to observe an object and expect it to remain unchanged. Thus, two observers can not only measure the same event, but they can also measure it simultaneously, confident in the knowledge that neither is affecting or altering it.

In quantum mechanics, however, the same is not true: the act of measurement generally destroys the state under observation, and replaces it with a new state that is an eigenvector of whichever operator is used to represent the test. Thus, quantum mechanics provides another example of a situation in which physics behaves time asymmetrically. Specifically, although in the absence of observation the evolution of a quantum state is deterministic and time reversible according to the Schrödinger equation, the state vector reduction (or 'collapse') occurring at the act of measurement is time asymmetric: the wavefunction discontinuously, randomly and irreversibly 'jumps' into one of the eigenstates of the Hermitian operator used to test the state.

As a consequence of state collapse, quantum theory introduces a problem into the earlier discussion on relativity. Namely, since by measuring a quantum object its state is irrevocably changed, this act automatically specifies a definite 'time of observation'. Certainly, it could not be observed again in its original state by a second observer after it has been altered. The temporal order of other events can then be compared to this known moment, thereby implying a strict causal order.

Herein lies the problem. Consider two spacelike separated observers, Alice, A, and Bob, B, and consider an 'object' initially prepared in a state O (which could, for the sake of illustration, be imagined to be an entangled state extended across a large region of

space). Consider also the point of view of Alice, and assume that if she were to measure O, she would randomly change its state to O_a, which is one of the eigenvectors of some test.

Now consider Bob's point of view. To start, assume that the event representing the measurement of the object by Alice is later than the event representing the measurement of the object by Bob, from the perspective of a particular extended frame of reference (and as given in the standard literature on relativity). Then, Bob would be able to measure the state O, a course of action that could cause it to collapse to O_b. However, this would then mean that Alice would be unable to observe the object in its original form O, and would instead only be able to measure the changed state O_b.

Moreover, and by applying again the usual relativistic arguments, it could be the case that in a different frame of reference, Alice and Bob are such that the event representing the measurement of the object by B is later than the event representing the measurement of the object by A. So, the conclusion would consequently be that Alice is able to measure the object first, thereby changing its state from O to O_a, such that Bob cannot therefore measure the object in its original form O.

The point is that each of Alice and Bob could believe that the event representing the measurement of the object by the other observer is in their own personal future, from the point of view of different frames of reference. Each would therefore believe that they could measure O, consequently changing it, such that the other observer could only measure the changed state. Clearly, this reasoning leads to a paradox[1].

Now, the above problem would obviously not arise in classical physics. In classical mechanics, the measurement of the object O by either A or B leaves it in the same state O, such that the other observer may then also measure it in its original form. In quantum physics, however, this is no longer the case, because only one of Alice or Bob may measure the original state; the act of observation destroys the state, thereby enforcing a strict, absolute and global causal order for the observation events. So, although classical, relativistic arguments might suggest that neither of the spacelike separated observers can fundamentally be said to measure a classical object first (because the order of such events may be different in different frames of reference), such a symmetry is broken when quantum effects are included. By incorporating quantum theory, a frame independent causal order must be placed upon the events that represent the measurements of the object by the

[1]Relativity theory is in fact riddled with potential contradictions, the 'Grandfather Paradox' of closed timelike curves in general relativity being a famous example. Perhaps these difficulties are themselves sufficient to suggest that relativity does not provide a completely consistent paradigm for physics, and highlights the general principle that just because something is mathematically possible, it does not make it physical reality. Introducing *ad hoc* caveats such as Hawking's Chronology Protection Theory do little to avail these conclusions.

spacelike separated observers, and this appears contrary to the standard principles of special relativity.

In a Universe containing quantum mechanics, there cannot be a situation in which in some frames A measures O and B measures O_a whilst in others B measures O and A measures O_b; in reality, only one of A or B actually measures O, and this reality is independent of the choice of frame. A conclusion, then, is that a relativistic, Block universe approach to physics does not seem immediately compatible with the accepted principles of quantum measurement.

A second difficulty faced by attempts to reconcile the principles of quantum theory with those of relativity is that the existence of a 'concrete' past, present and future is disputed in quantum mechanics.

The results of the Kochen-Specker theorem and the violation of the Bell inequality (discussed in the next chapter), for example, conclusively demonstrate that prior to any measurement, a quantum object cannot be described as possessing any fixed set of pre-existing physical characteristics, such as those featuring in theories of classical Hidden Variables. In physics, a quantum state does not have any pre-existing attributes just waiting to be discovered by observers.

The point is that if a quantum object does not exist in any definite form prior to an observation, its present cannot be given any real, concrete, definite existence. In particular, if the Universe contains quantum objects, which it certainly appears to, it is not possible to know everything about the moment of the present.

It is consequently difficult to imagine how the future could be granted such a status either, because it is difficult to accept that the future is somehow 'more real' than the present.

Specifically, then, if Charlie is a quantum object in the Universe, and if the Block universe approach to physics were to be accepted, he would have to conclude that even though according to the empirically verified laws of quantum mechanics his present is undefined, unknown and unknowable, his future is somehow fixed, definite and pre-determined. Clearly, this would be a strange position. Moreover, to accept such a conclusion would also imply that the result of a measurement on a quantum object is deterministic, and not the random, probabilistic outcome it is experimentally known to be.

Of course, it is perhaps not surprising that standard relativity runs into difficulties when attempting to describe a Universe incorporating quantum principles. After all, historically, Einstein's theory of special relativity pre-dates Schrödinger's quantum mechanics by about twenty years.

The theory of relativity is based upon, and generally framed in terms of, the relation-

ships arising between sets of classical observers as they witness classical events. But, the overriding lesson learnt from quantum mechanics is that physicists' notions of observation must be radically redefined: measurements cannot be performed non-invasively, and sets of observers cannot observe the same object in the same way. It consequently seems difficult to believe how any theory based upon classical observation could really be taken to provide a truly fundamental picture of a quantum reality, and this therefore obviously raises the question as to whether the results and conclusions of classical relativity should ever be accepted as completely reliable, at least as far as constraints on the ultimate physics of the Universe are concerned.

Quantum theory taught physicists that, fundamentally, reality does not quite behave as they thought it did. So, any theory originating from the pre-quantum era of science can really only be an incomplete vision of a better, quantum perspective of physics, and this includes any classical view or comment regarding the nature of time.

A Block universe interpretation of time is unable to account for quantum principles, because such an approach assumes the presence of an 'eternal', pre-existing and fixed past, present and future. The conclusion, then, is that the standard, Block universe vision of physics that arises from relativity is fundamentally incompatible with the standard principles of quantum theory. Consequently, if quantum mechanics cannot support such notions, it is suggested that a Block universe model is not the correct way to analyse a Universe that undoubtedly contains quantum objects. This perhaps explains why attempts at deriving theories of quantum gravity by directly quantising classical general relativity have predominantly been so unsuccessful.

In quantum theory only the moment of the present can be granted any real existence, and even this is limited. Thus, instead of adopting a Block universe approach to physics, quantum theory suggests that a '*Process*' interpretation of time is required; only 'Now' may be given any physical significance.

Moreover, in fact, any attempt to describe time and physics from a point of view that *is* compatible with the empirically verified priciples of quantum mechanics must consequently also assume this interpretation. It is therefore such a perspective that is adopted in this thesis.

3 The Quantum Universe

One aim of this thesis is to investigate how the observed physics of the Universe, especially including the concept of a continuous time parameter, might arise from a certain fundamental perspective of reality.

The obvious starting point is therefore to specify what this fundamental perspective might be. In fact, the viewpoint discussed in this thesis will itself be shown to follow quite naturally from considering a set of observations regarding the actual nature of the Universe.

To this end, the intention of this chapter is to define, describe and, where necessary, justify the observations, assumptions, consequences and conjectures present in the following work.

3.1 Quantum Mechanics

The first observation is summed up by the following statement:

Quantum Mechanics is a *valid* theory.

In other words, the argument is that the 'standard' quantum theory of Bohr, Heisenberg, Schrödinger, Dirac *et al* is the correct theory to use when describing certain physical, microscopic systems. Specifically, the implication is that a physical system may indeed be represented by a complex, linearly superposed statefunction, that this state may undergo unitary evolution in some sense, and that by measuring the state it is 'collapsed' into an eigenvector of an observable represented by an Hermitian operator.

The evidence cited to justify such a statement is the overwhelmingly universal success of quantum theory in science. In chemistry and biology, for example, quantum equations have allowed scientists to confidently model the properties and reactions of many types of molecule and atom. In high energy physics, the development of quantum field theory has allowed physicists to accurately predict the characteristics of particles that may not have existed since the era of the Big Bang. Even in the everyday world, the essential 21^{st} Century technologies behind optical telecommunication and computer science would not work if it were not for an understanding of the quantum laws governing the laser and silicon chips.

A mathematical demonstration of the validity of quantum mechanics was provided by J. S. Bell [9], based on an analysis of the correlations produced in a system similar to that described in the thought experiment proposed by Einstein, Podolsky and Rosen [11]. A

number of derivations of Bell's result are available in the literature; the approach outlined below roughly follows the treatment given in [12] and [13].

Consider the decay of a spinless, neutral pion into an electron-positron pair $\pi^0 \longrightarrow e^- + e^+$. Electrons and positrons possess spin components of $\pm\frac{1}{2}$ (denoted by 'up' and 'down', or equivalently '+' and '-', in some frame), so by conservation of angular momentum, a spin-up electron is partnered by a spin-down positron, and vice versa. Consider also a frame of reference $\mathcal{F}$ parameterised by Cartesian coordinates (x, y, z). With the aid of a suitably orientated Stern-Gerlach apparatus, it is possible to measure the component of angular momentum of either electron or positron in any direction in $\mathcal{F}$.

Bell's argument is the following. *If* the system ultimately obeyed classical instead of quantum mechanics, the assertion would be that after the decay of the pion the electron and positron would each have definite and independently measurable angular momenta, pointing in the general directions $\underline{n}$ and $-\underline{n}$ respectively. It would also be possible to non-invasively measure the same particle many times to obtain its component of angular momentum in any direction.

Consider measuring the spin of such a 'classical electron' in three different directions $\underline{a}$, $\underline{b}$ and $\underline{c}$, noting that $\underline{a}$, $\underline{b}$ and $\underline{c}$ need not be perpendicular. If, without loss of generality, it may be assumed that $\underline{n}$ is not orthogonal to any of $\underline{a}$, $\underline{b}$ or $\underline{c}$, the component of $\underline{n}$ in each direction will either be $+ve$ or $-ve$. So, by measuring the electron's spin first in the a-direction, then the b-direction, then the c-direction, the overall result will be one of eight possibilities: $\{a$ result, b result, c result$\} = \{+,+,+\}$ or $\{+,+,-\}$ or $\{+,-,+\}$ or $\{+,-,-\}$ or $\{-,+,+\}$ or $\{-,+,-\}$ or $\{-,-,+\}$ or $\{-,-,-\}$. If the orientation of $\underline{n}$ is random and may point in any direction, then depending on the choice of $\underline{a}$, $\underline{b}$ and $\underline{c}$, each of these eight results has a certain probability $P\{\pm,\pm,\pm\}$ of occurring, with total probability summing to unity.

Consider now a measurement of the electron by a Stern-Gerlach apparatus orientated along one of the directions $\underline{a}$, $\underline{b}$ or $\underline{c}$, followed by a measurement of the positron by a second Stern-Gerlach orientated along a different one of the directions $\underline{a}$, $\underline{b}$ or $\underline{c}$. If $P_c(+\underline{a}, +\underline{b})$ is defined as the classical probability that the component of the electron's spin along $\underline{a}$ is found to be $+ve$ *and* that the component of the positron's spin along $\underline{b}$ is also found to be $+ve$, then by conservation of angular momentum $P_c(+\underline{a}, +\underline{b})$ is equally defined as the probability that the component of the electron's spin along $\underline{a}$ is $+ve$ but its component if measured along $\underline{b}$ would be $-ve$. Note that this is clearly a classical result, as expected, because the implication is that two spin components of the electron have been measured even though it is only disturbed once.

By considering the eight possibilities given above, $P_c(+\underline{a}, +\underline{b})$ is given by the sum

$$P_c(+\underline{a}, +\underline{b}) = P\{+,-,+\} + P\{+,-,-\}, \tag{3.1}$$

which reflects the experimenter's ignorance of the spin component of the electron (or positron) in the direction of $\underline{c}$. Similarly the relations $P_c(+\underline{a}, +\underline{c}) = P\{+,-,-\}+P\{+,+,-\}$ and $P_c(+\underline{b}, +\underline{c}) = P\{+,+,-\} + P\{-,+,-\}$ are readily obtained.

The results may be summed,

$$\begin{aligned} P_c(+\underline{a}, +\underline{b}) + P_c(+\underline{b}, +\underline{c}) &= \{+,-,+\} + \{+,-,-\} + \{+,+,-\} + \{-,+,-\} \\ &= \{+,-,+\} + P_c(+\underline{a}, +\underline{c}) + \{-,+,-\}, \end{aligned} \tag{3.2}$$

such that, since all probabilities are positive, it is possible to produce the classical inequality

$$P_c(+\underline{a}, +\underline{b}) + P_c(+\underline{b}, +\underline{c}) \geq P_c(+\underline{a}, +\underline{c}). \tag{3.3}$$

So, if the system is governed fully by classical mechanics, i.e. if prior to any measurement the electron definitely possesses angular momentum in the direction of $\underline{n}$, then any set of measurements must necessarily satisfy this relation. That is, if the spin components of the correlated electrons and positrons in a large number of identically prepared systems are measured along any set of directions $\underline{a}$, $\underline{b}$ and $\underline{c}$, the classical probabilities evaluated from the statistics of the results would obey the inequality (3.3).

However, it may be shown that if the electron-positron pair instead obey the laws of quantum mechanics, the probabilities of obtaining certain results may violate this inequality.

In quantum theory, a system does not have any pre-existing or definite properties prior to an observation. Before a measurement, a particle's component of angular momentum only has the potential to be either $+ve$ or $-ve$ in some direction, and it is the measurement itself that forces the system to 'choose' one of these states to collapse into. In this sense, therefore, it may be said that prior to an observation each particle is in both potential spin states simultaneously, and the system is represented by an entangled state $|\psi\rangle$ described in obvious notation by the antisymmetric linear superposition

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_e \otimes |\downarrow\rangle_p - |\downarrow\rangle_e \otimes |\uparrow\rangle_p \right) \tag{3.4}$$

where, for example, $|\downarrow\rangle_p$ represents the state of a positron that is spin-down in some direction.

Now, a measurement of the spin of either the electron or positron destroys the entanglement. If, for example, the electron is measured and found to be in the state $|\uparrow\rangle_e$, it can obviously no longer be described as potentially being in the state $|\downarrow\rangle_e$, and the

wavefunction of the electron-positron system collapses to $|\phi_\uparrow\rangle = |\uparrow\rangle_e \otimes |\downarrow\rangle_p$. Subsequent measurements of the spin of the positron in this direction, with its state now prepared as part of this new state $|\phi_\uparrow\rangle$, must then produce the result $|\downarrow\rangle_p$.

Alternatively, if the first measurement had instead found the electron to be in the state $|\downarrow\rangle_e$, it would imply a collapse of the initial entangled state into the product state $|\phi_\downarrow\rangle = |\downarrow\rangle_e \otimes |\uparrow\rangle_p$, and later observations of the positron would find it to be spin-up in this direction, $|\uparrow\rangle_p$.

Consider now the quantum probability $P(+\underline{a}, +\underline{b})$, defined analogously to the classical probability $P_c(+\underline{a}, +\underline{b})$. In quantum mechanics, the evaluation of this requires two measurements to be performed on each of a statistical number of identically prepared systems: firstly the electron's spin is measured in the direction of $\underline{a}$, and secondly the positron's spin is then measured in the direction of $\underline{b}$. Consequently, this process necessarily involves a collapse of the initial entangled state $|\psi\rangle$ into a product state $|\phi\rangle$ when the electron is measured, followed by a projection of the 'positron part' of this new state $|\phi\rangle$ in the direction of $\underline{b}$ when the positron is measured. The overall result $P(+\underline{a}, +\underline{b})$ is then given by the products of the probabilities obtained from these two measurements.

To illustrate how this may be achieved, consider a particular choice of the vectors $\underline{a}$ and $\underline{b}$. For simplicity, and without loss of generality, $\underline{a}$ may be chosen to lie in the direction of the z-axis, and $\underline{b}$ may be chosen to be a vector in the $x-z$ plane that subtends an angle θ_{ab} to $\underline{a}$ (or z). When the electron is measured, and its spin component in the direction of $\underline{a}$ is found to be either $+ve$ or $-ve$, the entangled state[2] ψ collapses into either the state $|a_+\rangle = |+\rangle_e \otimes |-\rangle_p$ or the state $|a_-\rangle = |-\rangle_e \otimes |+\rangle_p$. Clearly, the probability that the spin of the electron is found to be $+ve$ in the $\underline{a}$ direction is $\frac{1}{2}$, because both the states $|a_+\rangle$ and $|a_-\rangle$ are equally likely, as is evident from the initial entangled state.

For later convenience, note that $|+\rangle$ may alternatively be written in the matrix form $\binom{1}{0}$, whilst $|-\rangle$ may be written as $\binom{0}{1}$.

The operator $\hat{S}_{\theta_{ab}}$ representing the subsequent measurement of the positron by the Stern-Gerlach apparatus orientated along the direction $\underline{b}$, i.e. at an angle θ_{ab} to the z-axis, is given by

$$\hat{S}_{\theta_{ab}} = \hat{S}_z \cos\theta_{ab} + \hat{S}_x \sin\theta_{ab} \tag{3.5}$$

where $\hat{S}_z = \frac{1}{2}\hbar\hat{\sigma}_z$, $\hat{S}_x = \frac{1}{2}\hbar\hat{\sigma}_x$ and $\hat{\sigma}_z$ and $\hat{\sigma}_x$ are the Pauli spin matrices in the z and x directions with matrix representations

$$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.6}$$

[2]Where no confusion is likely to occur, the notation for vectors, such as ψ, and quantum states, such as $|\psi\rangle$, will be used interchangeably throughout this thesis, i.e. $\psi \Leftrightarrow |\psi\rangle$.

So, $\hat{S}_{\theta_{ab}}$ is given by

$$\hat{S}_{\theta_{ab}} = \frac{\hbar}{2} \begin{pmatrix} \cos\theta_{ab} & \sin\theta_{ab} \\ \sin\theta_{ab} & -\cos\theta_{ab} \end{pmatrix}, \tag{3.7}$$

which has eigenvalues $+\hbar/2$ and $-\hbar/2$ corresponding to eigenvectors $|b_+\rangle = \binom{\cos(\theta_{ab}/2)}{\sin(\theta_{ab}/2)}$ and $|b_-\rangle = \binom{-\sin(\theta_{ab}/2)}{\cos(\theta_{ab}/2)}$ respectively. The eigenstate $|b_+\rangle$ is parallel to $\underline{b}$, i.e. has a $+ve$ component in the direction of $\underline{b}$, whereas $|b_-\rangle$ is anti-parallel with a $-ve$ component.

The overall process may now be summarised. An initial entangled state ψ is collapsed into either the state $|a_+\rangle$ or the state $|a_-\rangle$ when the spin of the electron is measured in the direction $\underline{a}$. If the electron's spin component is found to be $+ve$, corresponding to the state $|a_+\rangle$, then the subsequent measurement of the positron will leave the electron-positron system in either the state $|+\rangle_e \otimes |b_+\rangle$ or the state $|+\rangle_e \otimes |b_-\rangle$. Alternatively, if the electron's spin component is found to be $-ve$, corresponding to the state $|a_-\rangle$, then after the measurement of the positron the electron-positron system will be in either of the states $|-\rangle_e \otimes |b_+\rangle$ or $|-\rangle_e \otimes |b_-\rangle$.

With the above in mind, it is possible to rewrite $P(+\underline{a}, +\underline{b})$ as the probability of obtaining the state $|a_+\rangle$ when the electron is measured, given that before this measurement the system is in an entangled state of the form ψ, multiplied by the probability of obtaining the state $|b_+\rangle$ when the positron is subsequently measured, given that its state before this second measurement is now $|-\rangle_p$. This latter probability is

$$|\langle b_+|-\rangle_p|^2 = \left|(\cos(\theta_{ab}/2)\ ,\ \sin(\theta_{ab}/2)) \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right|^2 = \sin^2\left(\frac{\theta_{ab}}{2}\right) \tag{3.8}$$

which leads to an overall probability $P(+\underline{a}, +\underline{b}) = \frac{1}{2}\sin^2(\theta_{ab}/2)$.

By a similar argument, it can be shown that $P(+\underline{a}, +\underline{c}) = \frac{1}{2}\sin^2(\theta_{ac}/2)$ and $P(+\underline{b}, +\underline{c}) = \frac{1}{2}\sin^2(\theta_{bc}/2)$, where θ_{ac} is the angle between $\underline{a}$ and $\underline{c}$, and θ_{bc} is the angle between $\underline{b}$ and $\underline{c}$.

Now, if quantum theory is really a disguised version of classical mechanics, the probabilities derived from treating the electron-positron system according to quantum principles should obey the same constraints (3.3) as those derived from a classical treatment of the system. However, whilst the classical inequality (3.3) holds, the relation

$$\sin^2\left(\frac{\theta_{ab}}{2}\right) + \sin^2\left(\frac{\theta_{bc}}{2}\right) \geq \sin^2\left(\frac{\theta_{ac}}{2}\right) \tag{3.9}$$

formed by substituting the above quantum probabilities into (3.3) generally does not. For example, if $\underline{a}$, $\underline{b}$ and $\underline{c}$ lie in a plane with $\theta_{ab} = \pi/3$, $\theta_{bc} = \pi/3$ and $\theta_{ac} = 2\pi/3$, then (3.9) becomes $\frac{1}{4} + \frac{1}{4} \geq \frac{3}{4}$, which is clearly false. So, for quantum systems

$$P(+\underline{a}, +\underline{b}) + P(+\underline{b}, +\underline{c}) \ngeq P(+\underline{a}, +\underline{c}) \tag{3.10}$$

Thus, it may be argued that quantum and classical mechanics are fundamentally inequivalent in that they predict different results. The constraints placed by classical mechanics on a system, calculated by scientists as relationships between sets of probabilities of obtaining particular sets of results, are not present if the system is instead governed by quantum theory.

Importantly, it has also been empirically shown that such violations of the classical Bell inequalities occur in physics. Experiments with entangled pairs of photons [14] have yielded results that agree with quantum mechanics to better than 1%, but violate the Bell predictions of classical mechanics by 35%.

Summarising, the work of [9] and [14] has demonstrated that quantum theory is not equivalent to classical mechanics, but that physics obeys quantum principles. From such an viewpoint, all theories that suggest that quantum mechanics is simply a disguised theory of classical probability are ruled out, as are any theories pertaining to classical Hidden Variables. Such mechanisms will not be discussed further in this work.

The conclusion of this sub-section is that in order to describe certain physical, microscopic systems, it is quantum mechanics, and not classical, that is the correct and valid theory to use.

3.2 Quantum Cosmology

The second observation regarding the empirical nature of the Universe is the following:

There is no '*Heisenberg Cut*' in physics.

There is no rigid dividing line that segregates the quantum experiment being observed with the scientist doing the observing. There is equally no dividing line setting a scale beyond which quantum mechanics is no longer valid. Whilst most physicists readily accept that every microscopic sub-system in the Universe obeys the rules of quantum mechanics, there has never been demonstrated a definite macroscopic size or scale where quantum laws cease to be the correct theory of dynamics in favour of more fundamental classical laws.

As an example of this, it has even been shown that huge macroscopic objects such as quasars can give rise to observable quantum effects [15]. If on the line of sight between a distant quasar and the Earth is some sort of massive body, such as a galaxy, the gravitational lensing of the quasar's light induced by this body may give rise to interference patterns analogous to those arising in a Young's double-slit type device. Even if the quasar is sufficiently distant and dim such that a telescope on Earth only registers one photon

at a time, the interference fringes still arise, implying that the entire Earth-body-quasar system is behaving like a huge quantum 'Which-path' experiment.

So, if the Universe that physicists observe appears to be an enormous collection of microscopic sub-systems, i.e. is composed of protons, electrons etc., and if each of these microscopic sub-systems obeys quantum mechanics, and if there is no Heisenberg Cut directly separating these sub-systems from each other or the observer, and if the size of a system does not fundamentally affect whether it runs according to quantum laws, the conclusion drawn is that the entire Universe is itself a giant quantum system. The conjecture, therefore, is that the principles of quantum mechanics may be applied to the Universe as a whole[3].

If this conjecture is true, it should then be possible to write down a unique quantum wavefunction Ψ for the Universe that describes its large scale properties and evolution as a whole (c.f. [16][17][18][19]). This quantum state must be complicated enough to not only model a vast, intricate and expanding cosmos, but also to describe a universe that appears to be comprised of an enormous number of microscopic quantum sub-systems. Further, it must also allow physical observers, who believe themselves to be isolated classical states that are inside the Universe they are trying to understand, to experience and measure an apparently classical reality. Classical physics must therefore be a emergent phenomenon which is somehow borne from the quantum theory as an approximation on certain, presumably macroscopic, scales. The true quantum nature of reality should always be present, but will only demonstrate itself in complicated experiments designed to investigate very refined circumstances. Any formulation of the wavefunction of the Universe must somehow take account of this.

Further, every large scale characteristic of the Universe, and every physical property of every sub-system it contains, must be accounted for in any formulation of Ψ. If the wavefunction of the Universe describes *everything,* then space, time, energy, particle physics, and even semi-classical human observers must all emerge somehow from considerations of the properties of this quantum state.

It is therefore a job for physicists to attempt to discover what the Universe's state-function might be like. Now, whilst this task may appear overwhelmingly daunting, by extending the principles of standard quantum mechanics, a number of inferences can be drawn about the nature of a fully quantum universe.

[3]As an aside, note that there is also no known evidence for what could analogously be called a "Heisenberg Time" in astronomy: many cosmologists conjecture that just after the Big Bang the entire Universe should be represented by a quantum state, but no explanation is generally given as to exactly when the Universe should then *stop* being treated according to quantum principles.

The assertion proposed here is that it should not.

Firstly like all states in conventional quantum theory, the wavefunction Ψ must be a vector in a Hilbert space $\mathcal{H}$.

Secondly, given that by definition there is only one Universe, there can be no classical confusion as to which state it is in. Thus Ψ cannot be a mixed state of a classical ensemble of Universes, because such a concept is obviously contradictory. Consequently the wavefunction Ψ must always be a pure state.

Thirdly, the Hilbert space $\mathcal{H}$ containing the statevector representing the Universe must be of truly enormous dimension. One justification here is that classical physics has been ascribed to be an emergent approximation to quantum physics on certain scales, and the physical classical Universe seems to possess an almost uncountable number of degrees of freedom.

In fact, as a naïve lowest estimate of this dimension, consider the suggestion of many authors that there exists a certain minimum unit of spatial separation beyond which it is meaningless to discuss notions of classical distance. This resolutional limit is often assumed to be of the order of the Planck length, $l_P = \sqrt{\hbar G/c^3} \sim 10^{-35}$ metres, and marks the boundary of where space is assumed to no longer behave classically and continuously. Thus, given an empty universe of age $\tau_U = 15 \times 10^9$ years expanding spherically at the speed of light, c, the current number, n, of Planck volumes in the physical universe is given by

$$n = \frac{\frac{4}{3}\pi \left(c\tau_U\right)^3}{\left(l_P\right)^3} \approx 10^{184}. \tag{3.11}$$

Now, if with each of these minimum spatial volumes is associated just a single two-dimensional degree of freedom, then the total number of accessible classical states for the universe is clearly $2^{10^{184}}$. So, even in the simplest quantum model, the state vector representing the universe must have a dimension of at least $2^{10^{184}}$ if the classical degrees of freedom are expected to emerge somehow from a more fundamental quantum description.

Whilst the dimension of the Hilbert space $\mathcal{H}$ must be huge, it is still assumed in this work to be finite. This assumption is based, in part, from a desire to free the dynamics from some of the problems inherent in infinite dimensional models of physics. In quantum field theory, for example, the ultraviolet and infrared divergences occur specifically because the momentum space is unbounded. In addition, this infinite dimensional theory presents conceptual difficulties when confronted with the underlying physics: a scientist performing a calculation in quantum field theory should perhaps ask exactly what the notion of a particle of, say, infinite momentum may mean in a physical universe of bounded size and energy. This strongly echoes the ideas of Feynman [20], who questioned the validity of any infinite theory contained in a Universe of finite volume.

From this point of view, it therefore makes sense to remedy the problem at the outset by limiting the size of the Hilbert space to a finite dimension. Realistically, this should

not prove to be a problem so long as it is still sufficiently large such that every possible physically observed phenomena may be accounted for.

The fourth inference that may be drawn from an extension of the standard principles of quantum mechanics for the state of the Universe concerns its dynamics.

In the Schrödinger picture of conventional quantum theory, a given statevector ψ may be developed in two different types of way. The first way is evolution by an unitary operator $\hat{u}$, which may be thought of as a length preserving 'rotation' of the vector in its Hilbert space, i.e. $\psi \rightarrow \psi' = \hat{u}\psi$ for $|\psi| = |\psi'| = 1$. The second way is by state reduction, in which the wavefunction is 'tested' in some sense by an Hermitian operator $\hat{o}$. The initial state then 'collapses' or 'jumps' to a new state, which is one of the eigenstates of $\hat{o}$.

In fact, in the conventional, semi-classical treatment of the Universe, a physical subsystem described by quantum mechanics often develops through a series of evolutions and state reductions. Consider, for instance, a possible "day in the life" of a single electron. A free electron may be created and subsequently allowed to evolve according to the Schrödinger equation. The electron may propagate as a wave, until a later time when it is measured by some sort of apparatus and observer. As an example, if the apparatus involves a Stern-Gerlach device, the measurement process will lead to a collapse of the electron's wavefunction into one of the spin eigenstates associated with the Stern-Gerlach's orientation. Whichever of these two eigenstates the electron collapses into is then taken to represent the new state of the electron. The measurement is hence equivalent to a preparation of an electron in either a spin-up or spin-down state, in a particular direction.

The electron, now in a definite spin eigenstate, may then be allowed to evolve for another length of time until a further measurement occurs. As an example, the scientist controlling the experiment may decide that this second measurement also involves a Stern-Gerlach apparatus. Of course, if this second apparatus is orientated in the same direction as the first, the result will certainly leave the electron in the same eigenstate as before. In this case, the second measurement is equivalent to a *null* test on the electron because the state is left unchanged and no new information has been extracted from the system. Alternatively however, if the second apparatus is instead orientated at some angle to the first, then when the electron is measured it will collapse into a different spin eigenstate, with a probability dependent on the relative angle between the axes of the two Stern-Gerlach devices.

Summarising, then, in this example a state representing a free electron has evolved, before collapsing to a state with a definite spin component, which has then itself been evolved, before collapsing into another state with a different spin component. Obviously, the electron may then subsequently go on to be involved in any number of further tests.

Or course, the development of a single electron state may appear to be a particularly

specialised or contrived example. However in the real Universe, this sort of sequence goes on all the time. As an illustration, it should be recalled that whenever somebody switches on a light-bulb they are effectively starting a long chain of quantum processes, the outcome of which is the preparation of an ensemble of quantum states that propagate until their eventual measurement by the person's eye. In fact, this measurement itself usually goes on to cause many different subsequent chains.

The above process may consequently be generalised: a quantum system initially prepared by a physicist in a state ψ_n may proceed through a series of evolutions $\hat{u}_n$ and tests $\hat{o}_{n+1}$, for $n = 0, 1, 2, ...$. The unitary operators $\hat{u}_n$ that evolve the state are generally governed by the Schrödinger equation and may be of the form $\hat{u}_n = e^{-i\hat{H}_n t}$, where $\hat{H}_n$ is the Hamiltonian and t is a continuous time parameter as measured by the observer. The exact forms of the Hermitian operators $\hat{o}_{n+1}$ are chosen by the physicist depending on what she hopes to investigate, for instance in the above example by which particular component of spin is of interest. The system thus develops through a series of distinct steps: a state ψ_n may be evolved into a state $\psi_n \rightarrow \psi'_n = \hat{u}_n\psi_n$, which is tested by an operator $\hat{o}_{n+1}$, and therefore collapses into the next state ψ_{n+1} which is one of the eigenstates of $\hat{o}_{n+1}$. This new state ψ_{n+1} may then be evolved by the operator $\hat{u}_{n+1}$ to the state $\psi_{n+1} \rightarrow \psi'_{n+1} = \hat{u}_{n+1}\psi_{n+1}$, which is then tested by an operator $\hat{o}_{n+2}$, thereby collapsing it into the next state ψ_{n+2} which is one of the eigenstates of $\hat{o}_{n+2}$. And so on.

Any quantum experiment necessarily involves the concepts of state preparation, evolution, and measurement. However it is only the state reductions that are physically observed, and so it is only these collapses that can, in any real sense, be given a physical significance. This is in agreement with the conclusions of the Kochen-Specker theorem [21] (see also [22] for a review) and the results of Bell, which both demonstrate that before a quantum state is measured it cannot be said to have any physical attributes, such as a definite position or momentum. The observed properties of a state do not have pre-existing values waiting to be discovered, rather it is the actual measurement procedure and the collapse of the state that allows physicists to discuss them. This stance was summed up by Wheeler [15]: "*No elementary phenomenon is a phenomenon until it is a registered (observed) phenomenon*".

The collapse of the wavefunction necessarily involves an element of change, which in turn implies an extraction of information about the state. This, after all, is the purpose of experimentation. Certainly, for example, the measurement of a system does not decrease the physicist's knowledge of it, and it is only by performing a null test on the state of the type described earlier that the physicist's knowledge remains the same. An important

point to gain from the above example is therefore that the electron's development may, in some sense, be parameterised in terms of information extraction.

Two conclusions may be drawn from this. Firstly, because state reduction is manifestly a discrete process, the information is similarly extracted in discrete manner. It is this fact that justifies the subscript n on the state ψ_n (and hence on the operators $\hat{u}_n$ and $\hat{o}_{n+1}$), because it is possible to directly associate the state ψ_n with the result of the n^{th} collapse.

Secondly, it is noted that the 'direction' of state reduction and information extraction is equivalent to the observed 'direction' of time in physics. This follows immediately from the logic that the state ψ_{n-1}, resulting from the $(n-1)^{th}$ test represented by the operator $\hat{o}_{n-1}$, must certainly have existed after the state ψ_{n-2} but prior to the state ψ_n. Consequently, the sub-script n may also be seen as a type of discrete temporal label. Thus from the point of view of the state, time is a marker of the process of state reduction associated with information extraction. This point will be discussed to a great extent later.

By extending the standard principles of quantum theory to the Universe as a whole, the dynamics of the quantum universe are assumed to closely follow the above analogy of the dynamics of the developing electron. One important difference, however, is that any choice of test and any measurement of the Universe's state must be made by the Universe itself, and not by some external physicist. This is a consequence of the fact that if, by definition, the Universe does indeed contain everything, the conclusion is that there can be nothing 'outside'. Thus, if it is 'closed' in this way there can be no notion of any sort of external observer engaged in the process of evolving or measuring its state. The Universe must hence be the perfect example of a self-developing system.

As in the case of the electron sub-system, the development of the state Ψ of the Universe is a discrete process due to the discontinuous nature of the collapse mechanism. It is permissible, therefore, to label the state immediately after the n^{th} collapse as the n^{th} state Ψ_n. Further, it is the ability to label the state in this way that will be shown to be the origin of time in the quantum Universe. For now, however, it is noted that time is ultimately a discrete phenomenon in a universe running on quantum principles, providing perhaps a natural starting point for future theories of quantised gravity.

The quantum dynamics of the Universe is the way its state changes from $\Psi_n \rightarrow \Psi_{n+1} \rightarrow \Psi_{n+2} \rightarrow \Psi_{n+3} \rightarrow ...$ Moreover, and as with the above electron example, the mechanism governing this dynamics is, at least in principle, fairly simple.

First, note that for the sake of clarity, it is possible to imagine describing the system from the hypothetical point of view of an observer outside of the Universe, watching the state change. Although such a point of view is fundamentally unphysical, it is adopted for

convenience, and justified by the condition that the external observer does not interact with the Universe's state in any way. Thus, such a privileged witness is allowed to observe the Universe in a completely non-invasive way.

At the n^{th} stage of the Universe's development, its state may be represented by the unique vector Ψ_n. This wavefunction may then be evolved with some sort of unitary operator $\hat{U}_n$, i.e. $\Psi_n \rightarrow \Psi'_n = \hat{U}_n \Psi_n$, before being 'tested' by an Hermitian operator $\hat{\Sigma}_{n+1}$. The 'testing' process is irreversible and the state Ψ'_n collapses into one of the eigenstates Φ^i_{n+1} of $\hat{\Sigma}_{n+1}$. In fact in general, the operator $\hat{\Sigma}_{n+1}$ will possess D orthonormal eigenvectors, labelled Φ^i_{n+1} for $i = 1, ..., D$, where D is the dimension of the Hilbert space $\mathcal{H}$ of Ψ_n, $\forall n$. From this it follows that $\hat{\Sigma}_{n+1}$, and indeed $\hat{U}_n$, may both be represented by $D \times D$ matrices for all n.

The relationship between Φ^i_{n+1} and $\hat{\Sigma}_{n+1}$ obeys the usual eigenvector equation, viz.,

$$\hat{\Sigma}_{n+1} \Phi^i_{n+1} = \lambda^i \Phi^i_{n+1}, \tag{3.12}$$

where λ^i is the eigenvalue of the i^{th} eigenvector Φ^i_{n+1} of $\hat{\Sigma}_{n+1}$.

Further, given a state Ψ'_n, the probability $P(\Psi_{n+1} = \Phi^j_{n+1} | \Psi'_n)$ that the next state Ψ_{n+1} will be a particular eigenvector Φ^j_{n+1} of $\hat{\Sigma}_{n+1}$ is determined in the usual way as the square of the modulus of the probability amplitude, i.e.

$$P(\Psi_{n+1} = \Phi^j_{n+1} | \Psi'_n) = \left| \langle \Phi^j_{n+1} | \Psi'_n \rangle \right|^2. \tag{3.13}$$

The forms of the operators $\hat{U}_n$ and $\hat{\Sigma}_{n+1}$ are discussed later.

The result Φ^j_{n+1} of the test $\hat{\Sigma}_{n+1}$ is now associated with the preparation of a new state Ψ_{n+1}, which is subsequently evolved by an operator $\hat{U}_{n+1}$ to the state $\Psi'_{n+1} = \hat{U}_{n+1} \Psi_{n+1}$, before being tested by an operator $\hat{\Sigma}_{n+2}$ and collapsing to one of its D orthonormal eigenvectors Φ^i_{n+2}, $i = 1, .., D$. And so on.

Summarising, the Universe runs as an automatic process of state preparation, evolution and collapse. To this end, the Universe is envisaged to be a completely self-contained quantum automaton.

As noted earlier, if the Universe contains everything, there can be no notion of any sort of external observer engaged in the process of developing or measuring its state. At first glance, therefore, this may appear at odds with the traditional quantum mechanical tenets of state preparation and testing, and this has prompted some authors to criticise the possibility of a completely quantum universe. In fact, there are three obvious points that need addressing in any attempt to treat the Universe as a closed quantum system.

Firstly, if there are no external observers, then, as argued by Fink and Leschke [23], how can the Universe be measured? In what sense, therefore, can it be described as a quantum system?

Secondly, and again from [23], if there is only one Universe and it only 'runs' once, what is the meaning of statistically derived probabilities of the form (3.13)? In particular, by definition a description of the Universe's state must involve a description of everything contained within it. Moreover, any measurement of the state of the Universe by some sort of detection apparatus necessarily changes the detector's state. But, since this apparatus is part of the Universe, such a measurement immediately implies that the state of the Universe is itself changed during this procedure. It is consequently impossible to measure the same state of the Universe twice. So, from the point of view that quantum mechanics deals with the probability distributions of the results of repeated measurements of observables (either the same state measured a number of times, or a number of identical states each measured once) the argument of Fink and Leschke is that the rules of quantum mechanics are not applicable to the universe as a whole.

It is also noted that in conventional quantum theory, states evolve according to the time dependent Schrödinger equation. Thirdly, then, if there are no external parameters such as time, how does the Universe evolve as a quantum state?

These points will be discussed briefly here, though their explanations will become clearer throughout the course of this work, and particularly in Chapter 8. In general, the lesson learnt is that care is needed when directly applying the quantum mechanics of states in the laboratory to the special case where the state in question is the state of the entire Universe.

The standard principles of quantum mechanics were discovered by physicists based on laboratory observations of relatively tiny sub-systems of the universe, for example from the photoelectric effect induced in a small lump of metal, or the measurement of the spin of a single electron. The typical approach to an experiment involving quantum principles is to draw a dividing line between the observer and the observed: the scientist produces an isolated quantum state, allows it to evolve, and then chooses an Hermitian operator with which to test it. Whilst this is manifestly a semi-classical construction, it is normally a fairly accurate analysis because the scientist *is* sufficiently large such that classical mechanics provides a good approximation, and it is not always difficult in practice to produce a quantum state that *is* effectively isolated from the rest of the universe.

However, any semi-classical treatment can only ever be just an approximation to a reality that is fully quantum in nature. After all, recall that the quantum state under investigation can be arbitrarily large. From this point of view it is in principle possible to segregate the universe into two parts: the observer sub-system, and the sub-system comprising everything else. Given that it is possible to treat the 'everything else' sub-system as a quantum state, it seems unreasonable to expect that the Universe is really a semi-classical product of an enormous quantum sub-system containing everything apart

from a single classical observer.

Be that as it may, such an approach of an observer standing outside of the experiment being observed could be described as exo-physical. It is from this context that the usual rules of quantum mechanics were determined, including in particular those contributing to the conventional 'Measurement Problem'.

However, what this approach does not take into account is the fact that the physicists performing the experiments are themselves an integral part of the Universe they are trying to analyse. From this perspective, a laboratory experiment is actually equivalent to one part of the Universe measuring another part. Consequently, whether the true nature of the Universe is fundamentally quantum, classical or anything else, it must be an example of a system that is able to examine *itself*. This is therefore an endo-physical perspective, in which the observer is part of the system being observed.

The point is that what a physicist may conventionally believe to be an exo-physical measurement of a quantum sub-system of the Universe by an apparently external semi-classical observer, should perhaps really be viewed as an endo-physical measurement of one part of the Universe appearing to observe another part of itself. Thus, such a self-referential quantum system may not necessarily be restricted to quantum dynamics relying on external observers, because the dependence of the standard laws of quantum mechanics on external observers was only ever derived from the potentially incomplete viewpoint of exo-physical analyses of physical phenomena. These laws may therefore not be directly applicable to the Universe as whole. If the dynamics of the state should instead be described from an endo-physical point of view, the Universe must be a quantum system that relies on internal observations; there is hence neither a need nor a place for an external observer to measure and collapse the state.

Of course exactly how a quantum universe observed from the inside by endo-physical observers may give rise to internal Measurement problem type phenomena, such as emergent semi-classical physicists believing they are observing an external quantum reality, is a difficult question to be addressed. In fact, the endo-physical measurement problem is discussed more fully in Chapter 6, whilst in Chapter 8 some simple toy-models are given that describe how a simple endo-physical dynamics may be achieved.

For now, however, note that in answer to the criticism of a fully quantum universe given in [23], an analogy is drawn with the argument of Gödel [24] (see also [25][26]) that it is impossible to determine whether a given set of mathematical rules is self-consistent using just those rules alone. Whilst this may be the case, it does not imply that the rules themselves are wrong, merely that it is problematical to demonstrate their validity from the 'inside'.

Overall, if the Universe must be described by quantum principles, yet cannot support

any external observers, the conclusion must be that it is somehow able to prepare, evolve and test itself. Further, these measurements are made by different sub-systems inside the Universe, and indicate a relative change between them. This point will be discussed in due course.

Fink and Leschke's second argument is philosophically identical to asking about the meaning of the probability of obtaining a particular random result from a set of possibilities if an experiment is only ever performed once.

Consider as an example a classical coin toss experiment, noting that similar restrictions apply to any other physical situation, from atomic decays to measuring the spin of an electron. Ignoring the possibility of the coin landing on its edge, it may be generally accepted that the probability of getting a 'heads' result is equal to the probability of getting 'tails', that is $\frac{1}{2}$. This probability, however, only arises from a mathematical abstraction. To actually be empirically sure of the probability either requires the same coin to be tossed an infinite number of times, or an infinite number of coins to be tossed once. Of course, this in unphysical. In the first instance, it would take an infinite length of time to get the result. Additionally, each flip would undergo slightly alternative conditions, from different initial forces, to miniscule air currents, or even the possibility of being deflected slightly by a stray photon. It is even debateable as to what condition the coin would be in after it had been struck a million times. In the second instance, it could not be ensured that all the coins were identical or flipped under the same conditions. Equally, an infinite number of coins would require an infinite space and would possess an infinite mass, and so, according to general relativity, would curve infinite space infinitely.

Nevertheless, such an incomplete knowledge does not prevent a probability measure being placed on any result. Instead, the probability is defined relative to obtaining a particular result from a certain number of given conditions. It may be asked, for example, what the probability is of obtaining a head, given that there are two potential outcomes and that the system is not biased towards either one of them. In this sense, the probability is *defined* as $\frac{1}{2}$. The corresponding unphysical situation is consequently imagined implicitly, by assuming that *if* an infinite number of coins were tossed under identical conditions, *then* $\frac{1}{2}$of them would come up heads. If, however, a coin is only ever flipped once and gives a heads result, it is not immediately concluded that the result was deterministic, and that probability may not be used.

The same is true in quantum cosmology. In the case of the Universe represented by the state Ψ_n, the probability is defined relative to the set of D potential future states $\Psi_{n+1} = \Phi^i_{n+1}$, for $i = 1, ..., D$, that are the D normed eigenvectors of the operator $\hat{\Sigma}_{n+1}$. Again, it is possible to devise hypothetical situations in which if an infinite number of identical universes in the state Ψ_n were measured, then a fraction $|\langle\Phi^j_{n+1}|\Psi_n\rangle|^2$ of them

would give the particular result Φ^j_{n+1}, but this is just an attempt to attach empiricism onto a mathematical definition. Probability in the Universe is synonymous with potentiality; the fact that only one of these eigenstates is actually realised as the next state Ψ_{n+1} does not mean that the universe proceeds deterministically, any more than if the spin component of just one electron is measured and found to be 'up' it implies that the electron must be described using classical hidden variables. Indeed if the Universe is not a random, quantum system, the question would remain as to how it could therefore evolve deterministically given that the component quantum sub-systems of which it is comprised are clearly stochastic.

As will be expanded upon in the following chapters (particularly in Chapters 5 and 8), continuous time is taken to be a phenomenon that emerges in a fully quantum universe as its state proceeds through a long series of evolutions and collapses. Physical space, and the momenta and energies etc. of particular sub-systems, will also be shown to originate from considerations of the properties of this state Ψ, which is assumed to obey the laws of conventional quantum dynamics. However, whilst conventional quantum states in the laboratory evolve according to the continuous time Schrödinger equation in a way dependent upon their Hamiltonians, if the Universe is taken to possess no intrinsic concepts such as time or energy, exactly what role the Schrödinger equation plays in its evolution becomes an important question.

In response to this third criticism of quantum cosmology, it should be recalled that the n^{th} state Ψ_n of the Universe as discussed so far is simply defined as nothing but a vector in a Hilbert space $\mathcal{H}$. Its development is consequently only meaningful in terms of mathematical mappings of this vector, for example by norm preserving 'rotations' due to unitary evolutions, or by discontinuous jumps into another vector in $\mathcal{H}$ that is one of the eigenstates of an Hermitian operator. Care must be taken, therefore, not to attach to this state too many of the notions normally associated with emergent physical concepts, such as direct questions of how 'spatially long' this vector might be or how much 'mass' it has. In fact, exactly how the state could ultimately give rise to physics is a central theme of this thesis.

Recalling the discussion of exo-physics given above, it should be remembered that the Schrödinger equation is something scientists have discovered that appears to describe the evolution of physical quantum sub-systems. However, physical phenomena tend only to be witnessed by observers in the emergent, semi-classical regime. Consequently, the Schrödinger equation has only been defined as an emergent construct used to describe other emergent phenomena evolving in emergent time, namely, physical states in the laboratory.

So, since the continuous time Schrödinger equation was discovered in the emergent limit, it cannot automatically be expected to describe the fundamental, pre-emergent

dynamics of the state Ψ_n; its emergent definition does not necessarily imply that it has to be held as a fundamental law that describes the development of the Universe as a whole. In fact, the only constraint placed on whatever laws are chosen to evolve Ψ_n is that they must correctly reproduce every physically observed phenomenon in the semi-classical limit. The laws of emergent physics may themselves emerge from more fundamental laws governing the mathematical transformations of the state.

So, the state Ψ_n of the Universe described in this work is 'quantum' in the sense that it is a vector in a Hilbert space, and is subject to unitary transformation and to wavefunction collapse by Hermitian operators. This will be elaborated upon throughout the following.

3.3 The Stages Paradigm

The stages paradigm was proposed in [1] in an attempt to draw together the observations and conclusions of the previous two sub-sections into a mathematical framework that describes the properties and dynamical evolution of a fully quantum universe. The proposal represents a certain minimum number of parameters required to describe the development of the quantum Universe, and follows from the fact that a specification of the state Ψ_n alone cannot fully define its dynamics.

To illustrate the idea, an analogy is drawn as before with the conventional, semi-classical treatment of the single electron experiment introduced in the previous sub-section. A full description of such an experiment necessarily contains a number of features. Firstly there is the quantum state of the electron itself, represented by a vector ψ in a Hilbert space. Secondly, with the experiment is associated some sort of 'information content'. This information may, for example, include details of the Hamiltonian of the free electron, the choice of the experiment to be performed on the state (e.g. the possible orientations of the Stern-Gerlach apparatus), or even a memory of where the particular state came from or how it was prepared. Lastly a set of rules are required in order to describe exactly how the system develops, for example how the Schrödinger equation may govern the propagation of the electron as a wave, or a statement of how the inhomogeneous magnetic field of the Stern-Gerlach apparatus will perturb the Hamiltonian according to the spin of the electron. The rules are hence equivalent to the laws of physics relevant to the current situation.

As the experiment develops it progresses through a number of distinct stages. The initial stage, for example, might be defined as the one containing the newly created free electron. The next stage, then, might be defined as the period in which the electron has been measured by the first Stern-Gerlach apparatus, but has not yet encountered the

second. Finally, in the third stage of the experiment's development the electron has passed through the second apparatus also. In such a picture it is the measurement of the state of the electron that marks the end of one stage and the beginning of the next; each collapse of a state in one stage is equivalent to the preparation of the state for the next stage. This is another manifestation of the principle that only when information is actually extracted from a state can it be given any real, physical significance.

Each stage of the experiment's development is clearly associated with its own unique state, an information content and a set of rules describing the system. Moreover, some or all of these will change as the system progresses from one stage to the next. For instance, the state of the newly created electron in the first stage is clearly different from the state representing the electron in the third stage, because in the latter case the electron has been prepared in a particular spin eigenstate. Similarly the information regarding the actual choice of the next test is different from the first stage to the second, because the orientations of the Stern-Gerlach apparatuses are not the same. Equally, any information regarding the previous test is different from the first stage to the second, because the states in the first and second stages are prepared in different ways. Thus, each stage of the experiment's development may be said to be completely parameterised by the current state, information and rules of the system, and changes in these, when the wavefunction collapses, define the development of the system from one stage to the next.

By extending the above argument, the conjecture is that the Universe also progresses through a series of distinct stages, with the divide between one stage and the next occurring as the Universe's state collapses due to its self-measurement. Given that the state in each stage is necessarily different from the state contained in the previous stage, it is permissible to extend the label n defining the n^{th} state Ψ_n to the stage itself. It is hence possible to define the n^{th} stage Ω_n by

$$\Omega_n \equiv \Omega(\Psi_n,\ I_n,\ R_n) \tag{3.14}$$

that is, each stage is a function of the current state, Ψ_n, Information content, I_n, and the Rules R_n. These are explained in turn.

As described previously, the wavefunction Ψ_n is a pure state represented by a complex vector in a Hilbert space of enormous, but finite, dimension D. From the dynamics of this state is expected to emerge classical physics and all of the features in the physical Universe associated with this, including for example time, space, and particle physics. The state Ψ_n is assumed to represent the product of the sub-states of every quantum sub-system contained in the Universe (as will be expanded upon in Chapter 4). Thus a change of just one of these sub-states, for example a tiny part representing a physicist measuring a tinier part representing an electron, implies a change in the overall state of the Universe.

Consequently, the change of just one sub-state constitutes a progression of the Universe from one stage to the next, even though nearly all of the other sub-systems might appear unaffected by the jump.

In practice, it is expected that very many sub-states might change as the real Universe jumps from the state Ψ_n to the state $\Psi_{n+1} \neq \Psi_n$, corresponding to very many physical sub-systems appearing to simultaneously observe other physical sub-systems. In general, one, some, or all of the sub-systems might change as the stage develops from Ω_n to Ω_{n+1}.

I_n contains the necessary information required for the state's development. Specifically, I_n could incorporate a set of A unitary operators $\{\hat{U}_n^a : a = 1, ..., A\}$, one of which might be chosen to 'rotate' the state Ψ_n, and also a set of B potential Hermitian operators $\{\hat{O}_n^b : b = 1, ..., B\}$ that represent the different possible ways that Ψ_n could be tested; one of the set $\{\hat{O}_n^b\}$ which will hence become $\hat{\Sigma}_{n+1}$. Equivalently, then, because with each of the $\hat{O}_n^b$ is associated a basis set of D orthonormal eigenvectors, I_n also defines the set of possible next states Ψ_{n+1}. Paraphrasing, if only one of the B operators may be selected, and because each of these has D eigenstates due to the dimensionality of Ψ_n, the Information content I_n dictates that the next state Ψ_{n+1} will be one of a set of $(B \times D)$ members, though there may be a great deal of degeneracy in this set because some (but not all) of the eigenstates of $\hat{O}_n^i$ might be the same as some of the eigenstates of $\hat{O}_n^j$. Of course, until one of the operators $\hat{O}_n^i$ is chosen to be $\hat{\Sigma}_{n+1}$, and until the state reduction actually occurs, it is completely unknowable as to which of this set the subsequent state Ψ_{n+1} will be.

It is further possible that I_n may also include information about the properties of previous stages. It might, for example, contain a record of what the previous state Ψ_{n-1} was like; or possibly the two previous states Ψ_{n-1} and Ψ_{n-2}, or even the states $\Psi_{n-1}, ..., \Psi_{n-x}$ for some large x. Likewise, some sort of list might be present in I_n that details the chain of operators, $\hat{U}_{n-y}$ and $\hat{\Sigma}_{n+1-z}$ for $y, z \geq 1$, that were used as the Universe progressed through the chain of stages. In this sense, the current Information I_n may be seen as a form of memory of earlier stages, and might be used to track correlations from one stage to the next. An analogy here is with the human memory, in which the current 'state' of the brain often includes a recollection of its past 'states', or with a computer that is able to store information about past steps of a calculation for later use.

Information about the past may be used in the dynamics to enforce constraints on future states of the Universe. It might, for example, influence which of the set of B operators $\{\hat{O}_n^b\}$ is actually selected to be the next test $\hat{\Sigma}_{n+1}$. Of course, this type of development is really no different from how an experiment is often conducted in the laboratory: given that a physicist knows that she has just tested a sample with X and Y and obtained a certain state ψ_{XY}, she may decide that it must next be investigated with Z, thereby

selecting just one test out of a number of possibilities.

Using the past to influence how the present could develop into one of a set of possible futures has the potential to introduce an element of order into the dynamics of the Universe. As an illustration, imagine a universe that chooses a particular operator $\hat{\Sigma}_{n+1} = \hat{O}_n^j$ to test its state Ψ_n for the sole reason that $\hat{O}_n^j$ has eigenstates 'similar', in some sense, to the previous operator $\hat{\Sigma}_n$ that prepared Ψ_n. This would perhaps ensure that Ψ_{n+1} resembles Ψ_n to some extent, and if the process continued it might lead to a situation in which features of the universe appeared to persist from one stage to the next. If physical states did indeed develop in this manner, with the present incorporating information about the past, the mechanism might be speculated to be a root of why the real Universe appears to look so similar over certain scales.

Generalising the above, it is possible to re-parameterise the n^{th} stage of the Universe as (3.15), where $a = 1, ..., A$, $b = 1, ..., B$, and $x, y, z \geq 0$.

$$\Omega_n \equiv \Omega \left(\begin{array}{ll} \Psi_n, & [\{\hat{U}_n^a\},\ \{\hat{O}_n^b\},\ \{\{\Psi_{n-1}, ..., \Psi_{n-x}\}, \{\hat{U}_{n-1}, ..., \hat{U}_{n-y}\}, \\ & \{\hat{\Sigma}_{n+1-1}, ..., \hat{\Sigma}_{n+1-z}\}\}], \quad R_n \end{array} \right) \tag{3.15}$$

The Rules R_n are the laws dictating the dynamics obeyed by the Universe. The Rules specify that, given a state Ψ_n, it will be evolved with an operator $\hat{U}_n$ and tested with a particular operator $\hat{\Sigma}_{n+1}$. Equivalently the Rules are used to select, to act on Ψ_n, one of the A possible unitary operators $\hat{U}_n^a$, $a = 1, ..., A$, and one of the B possible Hermitian operators $\hat{O}_n^b$, $b = 1, ..., B$, out of the set of all possible operators contained by I_n.

Exactly how a particular operator is selected, i.e. what mechanism the Rules use to determine which member of the A or B possibilities is chosen, remains a difficult question for the future. Indeed whether this choice is deterministic, or itself the result of some random quantum process, is an important issue to be addressed. It is even possible that the Rules make reference to additional factors included in the Information I_n, such that perhaps the presence of a particular Ψ_{n-r}, $\hat{U}_{n-s}$ or $\hat{\Sigma}_{n+1-t}$ in I_n might lead to the selection of a particular $\hat{U}_n^a$ or $\hat{\Sigma}_{n+1}$. This possibility will be addressed many times throughout this work, and especially in Chapter 8, and may be necessary to account for many of the features present in the physically observed Universe.

In fact, it is also conceivable that the Rules R_{n-1} used to choose the operator $\hat{\Sigma}_n$ are not the same as the Rules R_n used to choose the operator $\hat{\Sigma}_{n+1}$. In other words, the Rules themselves may be subject to dynamical development according to some higher order set of "Rules of the Rules" [27], and in this case such an additional 'Meta-Rule' would also need to be incorporated into the definition (3.15) of a stage. In Chapter 8 an attempt has been made to find simple Rules that reproduce certain required features of dynamics.

All of physics is expected to emerge from the fundamental quantum picture of the Universe described in this Chapter. It is reasonable to assume, therefore, that the Rules R_n at each stage must be very carefully refined in order to produce a classical looking Universe that appears to run according to ordered and well defined laws of physics. The Rules must ensure that every phenomenon that physicists experience on the emergent scale is accounted for from the fundamental quantum level as the Universe jumps from one stage to the next. For example, if from a particular state Ψ_n appears to emerge a physical Universe of enormous spatial size that appears to be describable by $3+1$ dimensional general relativity and appears to contain very many distinct protons, electrons, neutrons etc. that have clumped together in huge lumps resembling galaxies, stars and planets, and if further, on one of these planets, plants, animals, and humans have appeared and evolved, and that some of these humans have constructed giant buildings and complicated machinery in order to measure the Universe they believe they exist in as semi-classical observers, then it is reasonable to hope that from the next state Ψ_{n+1} all of these features will also emerge, instead of, say, something totally different or even just complete disorder. Since it seems to be an observational fact that the physical Universe appears to change very little from one stage to the next (it will be shown in Chapter 4 that this is itself perhaps unexpected), it may be conjectured that the Rules, and consequently the dynamics, must be very finely tuned in order to choose an operator $\hat{\Sigma}_{n+1}$ with an eigenvector so similar to Ψ_n. In a quantum universe approximated by continuous and emergent classical laws there must be some sort of underlying 'similarity theorem' that ensures that Ψ_{n+1} is not too different from Ψ_n.

The standard laws of physics discovered in the laboratory are also presumably emergent from the Rules R_n describing the dynamics of the Universe. As an example, consider a stage of the Universe in which, at one instant, it appears that from an emergent, classical and large scale point of view, part of the state Ψ_n may be considered to describe two electron sub-systems. Further assume that, from this emergent, classical and large scale point of view, scientists have defined a measure of distance and observed that the two electrons are in close proximity[4]. Whatever the dynamics may be that actually govern the Universe on the fundamental level, they might be expected to ensure that from the next state Ψ_{n+1} would emerge a picture in which the two electrons appear slightly farther away from each other, again from the classical and large scale point of view of a scientist inside the Universe. Continuing, in the state Ψ_{n+2} following this the two electrons might be even farther apart. Thus, by observing the way in which the state of the Universe appears to change from one stage to the next, emergent physicists are able to derive emergent laws to describe emergent phenomena, such as "Like charges repel".

[4]The emergence of space and the emergence of particles are investigated in Chapters 5 and 7 respectively.

Conversely, by studying these emergent laws of physics governing the physically observed Universe, it might be able to place certain constraints of the actual Rules R_n governing the development of the underlying state from stage to stage.

With the above considerations in mind, it is possible to speculate now on the necessary sequence of events that might define the dynamical development of the Universe. Consider a fully quantum Universe, completely specified at the n^{th} step of its development by a stage $\Omega_n \equiv \Omega(\Psi_n, I_n, R_n)$. The Information content, I_n, contains a set of possible unitary operators, $\{\hat{U}_n^a : a = 1, ..., A\}$, and a set of possible Hermitian operators, $\{\hat{O}_n^b : b = 1, ..., B\}$, each of which is associated with a basis set of D orthonormal eigenvectors, $\Phi_{n+1}^{b,k}$ for $k = 1, ..., D$, as well perhaps as some sort of 'record' of previous stages.

According to the specific Rules R_n governing the Universe, and possibly making reference to the current Information I_n, one of the operators $\hat{U}_n^i$ is chosen to act on the wavefunction, and the state undergoes unitary evolution. This evolution is effectively a rotation of the D dimensional vector Ψ_n in its Hilbert space $\mathcal{H}$, viz. $\Psi_n \rightarrow \Psi'_n = \hat{U}_n^i \Psi_n$. Note however that depending on the dynamics, the chosen operator $\hat{U}_n^i$ may be equal to the identity operation $\hat{I}$ such that

$$\Psi_n \rightarrow \Psi'_n = \hat{U}_n^i \Psi_n = \hat{I}\Psi_n = \Psi_n. \tag{3.16}$$

Next the Rules select, from the set $\{\hat{O}_n^b\}$ defined in I_n, one of the Hermitian operators; say, $\hat{O}_n^j$. This is equivalent to the Universe choosing a test to perform on its state. The state consequently collapses into one of the eigenvectors of $\hat{\Sigma}_{n+1} = \hat{O}_n^j$, effectively preparing the next state Ψ_{n+1}. The probability that the new state Ψ_{n+1} will be the particular l^{th} eigenstate $\Phi_{n+1}^{j,l}$ of $\hat{\Sigma}_{n+1}$ is given by $P(\Psi_{n+1} = \Phi_{n+1}^{j,l} | \Psi'_n) = |\langle \Phi_{n+1}^{j,l} | \Psi'_n \rangle|^2$.

Details about the particular choice of operators $\hat{U}_n^i$ and $\hat{\Sigma}_{n+1} = \hat{O}_n^j$ may then be included in the new Information content I_{n+1}, which may also provide a record of the previous state Ψ_n. In fact, some of the 'old' Information content I_n may also be subsumed into the new I_{n+1}. This inclusion may be whole, $I_n \subset I_{n+1}$, partial $I_n \cap I_{n+1} \not\equiv I_n$, or even not at all $I_n \cap I_{n+1} = \emptyset$, where in the last instance the new stage could be said to contain no knowledge whatsoever of its 'history'.

In fact, the cases in which I_{n+1} does not completely encompass I_n necessarily imply an irreversible loss of information. Evidently, the 'direction' of the loss of information as the Universe develops from one stage to the next is the same as the 'direction' of time in the model, because both are based on the 'direction' of the state collapsing, i.e. from Ψ_n to Ψ_{n+1}. It is noted, moreover, that the idea of an irreversible loss of information is strongly analogous to the notion of an increasing entropy, and in this case it is recalled that the 'direction' of increasing entropy (which is equivalent to the 'direction' of the irreversible

increase in disorder of a system as it evolves) defines the arrow of time in thermodynamics. From these viewpoints, the direction of time in the Universe is seen as identical to the direction of increasing ignorance of exactly how the Universe came to have the state it has.

A second point about partial inclusions of information is the fact that no observer in the present can ever be sure of just how complete or reliable their information is regarding what past stages might have been like. Since past states can only ever be reconstructed based on whatever information about them has survived into the present, if this information survival is incomplete then the reconstruction of the past can, at best, only be partial. This conclusion reinforces the idea that physicists can only ever be truly certain of the current stage of the universe. As is consistent with the idea of Process time, only the present can be given any real existence.

The new information content I_{n+1} will contain a new set of possible operators $\{\hat{U}_{n+1}^{a'}\}$, $\{\hat{O}_{n+1}^{b'}\}$, where $a' = 1, ..., A'$ and $b' = 1, ..., B'$. The actual members of these sets may be based somehow upon the random choice of the new state Ψ_{n+1}, or on parts of previous operators or states, and will go on to provide the dynamics for the next stage. This next stage is clearly parameterised as $\Omega_{n+1} = \Omega(\Psi_{n+1}, I_{n+1}, R_{n+1})$, where the Rules R_{n+1} governing the Universe may also have changed, $R_{n+1} \not\equiv R_n$, according to any Rules of the Rules.

Overall, the Universe has developed in a discrete quantum manner from one stage Ω_n to the next Ω_{n+1}. This process is expected to continue indefinitely in a completely self-contained and automatic way. All of physics, including the dynamics of microscopic and macroscopic sub-systems evolving against a backdrop of continuous space and time in an apparently classical looking Universe, is expected to emerge from the dynamics of this self-referentially developing series of stages.

Exactly how this might occur will form the basis of the remaining chapters of this work.

4 Classicity from Quantum

Given that the physical Universe appears to look like an enormous collection of semi-classical sub-systems, yet the conclusion of the previous chapter is that it is fundamentally quantum in nature, an immediate question to be addressed is: how does apparent classical physics emerge from the fully quantum reality? If the Universe is described by a complex statevector Ψ, what properties of this state might give rise to semi-classical looking degrees of freedom?

In an attempt to understand this issue, it is beneficial to reverse part of the question and define what is meant by classicity. To this end, it is observed that classicity is in some sense synonymous with distinguishability; if a set of objects $\{A, B, C\}$ are described as classical, it implies that it is possible to make distinctions between them. These distinctions may, for example, be in terms of an observer's ability to determine that the objects have different physical properties or that they are positioned at different spatial locations.

If A, B and C can be distinguished, it follows that they may each be assumed to possess an isolated existence, and may be discussed independently of one another. From this viewpoint, classicity is therefore a way of expressing the observation that *this* object with *these* qualities is *here,* whereas *that* object with *those* properties is *there.* Certainly this is a criterion met by all macroscopic semi-classical states in physics, where for example a particular large scale apparatus is always assumed to be separate from the quantum state it is measuring, and does always have an independent existence and a well defined position.

As a consequence of the above, a state in classical mechanics representing a set of classical objects can always be separated into the distinct sub-states of which it is comprised. The same is not true in quantum theory, because the phenomenon of entanglement represents a breakdown of this ability to separate a system into independent and distinct physical sub-systems. When two (or more) quantum states become entangled they can no longer be given any independent existence, and instead it is only by taking the entire state as a whole that the system can be given any physical significance. The EPR experiment [11] provides a famous example of this.

It is, however, an important fact that a class of states exist in quantum mechanics that are not entangled. Separable states (to be defined below) represent situations in which it *is* permissible to segregate the quantum state into a set of sub-states. Further, because it is possible to develop and measure the factor sub-states of these vectors independently of each other, such sub-states may be distinguished. Thus, separable states in quantum mechanics allow physicists to discuss their constituent parts, because the factors of a separable state possess a degree of individuality. Since this is one of the requirements for

classicity, the conjecture is that it is separable states that provide a necessary starting point for the emergence of semi-classical degrees of freedom.

As a simple illustration, consider two Hilbert spaces $\mathcal{H}_\phi$ and $\mathcal{H}_\varphi$. Consider also a third Hilbert space $\mathcal{H}_{[\phi\varphi]}$ formed by taking the tensor product of $\mathcal{H}_\phi$ and $\mathcal{H}_\varphi$, i.e. $\mathcal{H}_{[\phi\varphi]} = \mathcal{H}_\phi \otimes \mathcal{H}_\varphi$. This third vector space $\mathcal{H}_{[\phi\varphi]}$ may be described as *factorisable*, with the sub-spaces $\mathcal{H}_\phi$ and $\mathcal{H}_\varphi$ being its *factors*.

Now define two states ψ_a and ψ_b,

$$\begin{aligned} |\psi_a\rangle_{\phi\varphi} &= |\phi_1\rangle_\phi \otimes |\varphi_2\rangle_\varphi \\ |\psi_b\rangle_{\phi\varphi} &= |\phi_1\rangle_\phi \otimes |\varphi_2\rangle_\varphi + |\phi_2\rangle_\phi \otimes |\varphi_1\rangle_\varphi, \end{aligned} \tag{4.1}$$

where ψ_a and ψ_b are vectors in the product space $\mathcal{H}_{[\phi\varphi]}$, i.e. $\psi_a, \psi_b \in \mathcal{H}_{[\phi\varphi]}$, but $\phi_i \in \mathcal{H}_\phi$ and $\varphi_j \in \mathcal{H}_\varphi$ for $i, j = 1, 2$.

Clearly, the state ψ_a is separable into a product of factors, one of which, ϕ_1, is in the Hilbert space $\mathcal{H}_\phi$ and the other, φ_2, is in $\mathcal{H}_\varphi$. However, if it is assumed that ϕ_1 is not a linear multiple of ϕ_2, and similarly that φ_2 is not a linear multiple of φ_1, no such separation is possible for the entangled state ψ_b.

Now, if a quantum system is prepared in the separable state ψ_a, it is possible to measure one factor of it whilst leaving the other factor unchanged. The state ψ_a may, for instance, be tested by an operator $\hat{O}$ which has an eigenstate of the form $|\chi\rangle_\phi \otimes |\varphi_2\rangle_\varphi$, where $|\chi\rangle_\phi \in \mathcal{H}_\phi$, such that the factor $|\varphi_2\rangle_\varphi \in \mathcal{H}_\varphi$ appears unaffected by this measurement. In other words, a physicist may 'ask a question' about the sub-state ϕ_1 in the factor space $\mathcal{H}_\phi$ without necessarily changing every part of the state ψ_a. It is, for example, permissible to determine whether the component of ψ_a in the Hilbert space $\mathcal{H}_\phi$ is indeed ϕ_1, without destroying ψ_a. In fact, because it is separable, it is generally possible to determine the component of ψ_a in the Hilbert space $\mathcal{H}_\phi$, without in any way affecting the component of ψ_a in the Hilbert space $\mathcal{H}_\varphi$.

However, the same is not true for the entangled state ψ_b. Any attempt to measure the component of ψ_b in either of the factor Hilbert spaces $\mathcal{H}_\phi$ or $\mathcal{H}_\phi$ destroys the entanglement, and irreversibly collapses the wavefunction of the system into a different state, i.e. into a product form.

This difference between ψ_a and ψ_b may be rephrased in terms of the role of information. For the entangled state ψ_b it is possible to learn something about the component of the vector in $\mathcal{H}_\varphi$ by performing a measurement on the component of the vector in $\mathcal{H}_\phi$. However, if during the measurement the entangled state ψ_b collapses into the product state ψ_a, no new information is gained about the factor state φ_2 in $\mathcal{H}_\varphi$ by performing a subsequent measurement on ϕ_1 in $\mathcal{H}_\phi$. In fact this will always remain the case, with the

two factors leading independent existences, unless the dynamics conspire in such a way as to re-entangle the system.

The testing of the EPR state of the electron-positron system of Section 3.1 provides a physical example of these principles. The initial entangled state $|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_e \otimes |\downarrow\rangle_p - |\downarrow\rangle_e \otimes |\uparrow\rangle_p)$ of equation (3.4) is destroyed by a measurement of the spin of the electron, and, depending on the result of this, the system after collapse may be represented by one of two possible product states $|\uparrow\rangle_e \otimes |\downarrow\rangle_p$ or $|\downarrow\rangle_e \otimes |\uparrow\rangle_p$.

Each factor of these product states consequently represents either an isolated electron or an isolated positron, with a known component of spin. Any subsequent measurement of the spin of either the electron or positron in this direction leaves this new product state unaltered (as this is simply a null test), but additionally, any further measurement in any direction involving just the positron (by using, say, an operator of the form $\hat{S}_{\theta_{ab}} = \hat{S}_z \cos\theta_{ab} + \hat{S}_x \sin\theta_{ab}$ defined previously) will not affect the state of the electron, and vice versa. Unlike the initial entangled state, the product state represents a system comprising of an electron and a positron that are isolated and independent from each other.

The conclusion of the above discussion is that a quantum state separable into a product of factors is in some sense equivalent to a system comprising of a number of distinct semi-classical sub-systems. Because it is possible to examine just one of these factors without affecting the rest of the state, these sub-states appear isolated and distinguished from each other, and can be discussed as separate from the rest of the system, exactly as required for a semi-classical description of physics to begin to emerge.

The reciprocal of this should also be true. Every sub-system that appears isolated and distinct from the others may be associated with one of the factors of the state representing the entire quantum system.

Further, by extending this argument to the case of a fully quantum Universe described by a wavefunction Ψ_n, the conjecture is that every individual, semi-classical sub-system within it is represented by a unique factor of this state.

Exactly how this may be achieved is a difficult question, and it is noted that the above statement may contain an element of idealisation. Being a factor of a state only guarantees that the sub-system it represents may be granted a degree of individuality. The individual factors still represent sub-systems governed by the laws and constraints of quantum mechanics, as is obvious, for example, for the single electron and positron factors of the earlier EPR product states, $|\uparrow\rangle_e \otimes |\downarrow\rangle_p$ and $|\downarrow\rangle_e \otimes |\uparrow\rangle_p$, which must of course be treated quantum mechanically. This, after all, is the origin of the lack of a Heisenberg Cut in the Universe.

The method for achieving 'real' classicity, in the traditional sense of the word, in large macroscopic sub-systems of the Universe is part of the difficult question of emergence to be addressed in the future. It is here that statistical theories such as decoherence may play a part, as will be discussed in Section 4.3.

Suffice to say, however, that even semi-classical, macroscopic sub-systems must need to be associated with factor sub-states of the Universe's wavefunction. The alternative, that they are actually entangled with their surroundings, would imply that they cannot be given any sense of individuality, and this would lead to the absurd and unsupported suggestion that conventional semi-classical systems are actually entangled with each other, contrary to empirical evidence. A classically distinct and isolated sub-system must be represented by a factor of the Universe's state, but if the state of the Universe can be separated it does not automatically imply that every factor may be treated according to the laws of classical mechanics, even as an approximation. Separability is a necessary condition for classicity to arise, in that it implies distinguishability, but it is unclear at this stage as to whether it is also sufficient.

It is possible that one or some of the factor sub-states may themselves be entangled within their own Hilbert sub-spaces. For example, consider a 'toy-universe' initially in the state $\Psi_0 = |Z^0\rangle$ that contains nothing but a single Z^0 boson. Also, assume that the dynamics selects a particular operator, $\hat{\Sigma}_1$, to test Ψ_0, thereby causing the universe to jump to the state $\Psi_1 = |\bar{\pi}^0\rangle \otimes |\pi^0\rangle$ representing a pion/anti-pion pair. Such a dynamics is analogous to a particle physics experiment in which the high energy boson spontaneously decays into a neutral pion and anti-pion.

If the pion itself then goes on to decay to an entangled electron/positron pair (i.e. an EPR-like state), the state of the universe, Ψ_2, after this decay may be given by $\Psi_2 = |\bar{\pi}^0\rangle \otimes |\psi\rangle$, where $|\bar{\pi}^0\rangle$ represents the sub-state of the anti-pion, and $|\psi\rangle$ the sub-state of the entangled electron and positron (3.4). Clearly, the overall state Ψ_2 is a separable product of two factors, one of which is entangled.

The current example shows how the separability of the state representing a simple system changes as it develops. In fact, if subsequently an operator $\hat{\Sigma}_3$ is selected that is equivalent to a measurement of the spin component of the electron (in a particular direction), and if the result is that it is found to be spin 'up', the next state, Ψ_3, will be of the product form $\Psi_3 = |\bar{\pi}^0\rangle \otimes |\uparrow\rangle_e \otimes |\downarrow\rangle_p$. Note that the dimensions of the Hilbert spaces of the states Ψ_0, Ψ_1, Ψ_2 and Ψ_3 must be the same, and that for example the sub-state $|\psi\rangle$ of Ψ_2 is in the same factor Hilbert space as the product of the sub-states $|\uparrow\rangle_e \otimes |\downarrow\rangle_p$ in Ψ_3. In this universe, it is evident that the separability of the system changes during the transition from Ψ_2 to Ψ_3; this will be an important feature in the following.

It is now possible to reinterpret the idea of a physical experiment from the point of view of a universe described fully by quantum mechanics. Recall that the standard semi-classical treatment of physics is to segregate the system into the subject under investigation, the various bits of apparatus, the physicist conducting the experiment, and everything else in the Universe (the 'Environment'). Indeed, it does not seem possible to perform an experiment on a quantum subject if it cannot be isolated from everything else.

This semi-classical approach can be incorporated into the quantum picture of the Universe by assuming that each of these semi-classical and distinct parts may now be represented by separate factors of the state Ψ of the Universe. This is inevitable from the viewpoint asserted in this thesis: if they are classically distinct, it follows that they cannot be entangled with each other. Hence, the state may be written as

$$|\Psi\rangle = |\psi\rangle \otimes |A\rangle \otimes |O\rangle \otimes |R\rangle \tag{4.2}$$

where $|\psi\rangle$ represents the sub-state of the subject under investigation, $|A\rangle$ the sub-state of the apparatus, $|O\rangle$ the sub-state of the observer, and $|R\rangle$ is the sub-state representing the rest of the Universe.

Of course, $|R\rangle$ will itself be a product of an enormous number of sub-states, some of which may themselves be entangled. However, for the sake of studying the tiny sub-state of interest, i.e. $|\psi\rangle$, the conventional procedure is then to ignore all of the factors of Ψ that do not contribute to the running of the experiment and focus attention on changes in $|\psi\rangle$. This is really an exo-physical approach, where the physicist falsely believes himself to be excluded from the Universe being measured, and is therefore potentially misleading, but it is a natural procedure borne from the physicists subjective experience of the 'outside' world. The 'real' situation of endo-physical measurements will be addressed in Section 6.2, and also briefly in Section 4.3.3.

As with the pion experiment described above, the separability of the state representing the Universe may change as it develops through a series of stages. Moreover, it is these changes in separability that are ultimately responsible for the generation of certain classical effects in the Universe, for example the emergence of continuous space. This will be expanded upon in the following chapters, but it is remarked here that even in the simplest quantum model hypothesised earlier, in which the Universe is represented by a state in a Hilbert space of dimension greater than $2^{10^{184}}$, the number of ways in which this state may be separated into a product of factors, some of which may or may not be entangled themselves within their factor sub-spaces, is enormous.

Summarising, the separability of a state allows a classical distinction to be made between its constituent factors. The conjecture, then, is that classicity in a fully quantum Universe emerges somehow from considerations of the separability of its state Ψ. In ad-

dition, if separability is required for classicity, and since it is an observational fact that the Universe appears to be comprised of a vast number of classically distinct sub-systems, the conclusion must be that the current state of the Universe is highly separable. It is therefore a task to investigate how this might have occurred.

4.1 Factorisation and Entanglement

When is an arbitrary state Ψ in a Hilbert space $\mathcal{H}$ separable? What rules determine whether a given vector Ψ can be written as a product of factor sub-states?

Before answering these questions, it must first be noted that the concept of a separable state necessarily implies the existence of a factorisable Hilbert space. By definition, the property that a state $\Psi \in \mathcal{H}$ is separable in the form $\Psi = \phi_1 \otimes \varphi_2$, for example, where $\phi_1 \in \mathcal{H}_1$ and $\varphi_2 \in \mathcal{H}_2$, explicitly requires that $\mathcal{H}$ can be factorised in the form $\mathcal{H} \equiv \mathcal{H}_{[12]} = \mathcal{H}_1 \otimes \mathcal{H}_2$. It is therefore a natural starting point for any discussion of the separability and entanglement properties of vectors to define what is meant by the factorisability of their vector spaces.

A Hilbert space $\mathcal{H}^{(d)} \equiv \mathcal{H}$ of dimension d is factorisable into N factors if it can be written in the tensor product form

$$\mathcal{H}^{(d)} = \mathcal{H}_1^{(d_1)} \otimes \mathcal{H}_2^{(d_2)} \otimes ... \otimes \mathcal{H}_a^{(d_a)} \otimes ... \otimes \mathcal{H}_N^{(d_N)} \tag{4.3}$$

where $\mathcal{H}_a^{(d_a)}$ for $a = 1, ..., N$ is called the a^{th} factor Hilbert space and is of dimension d_a. Clearly, $d = d_1 d_2 ... d_N$. Such a factorisation represents a particular '*split*' of the Hilbert space $\mathcal{H}^{(d)}$ into N given factors.

For convenience and clarity, note that here and elsewhere a Hilbert space $\mathcal{H}^{(d)}$ factorisable into N factors in the form of (4.3) can be written using the square bracket notation

$$\mathcal{H}^{(d)} \equiv \mathcal{H}^{(d)}_{[(1)(2)(3)...(N)]} \equiv \mathcal{H}^{(d)}_{[123...N]} \equiv \mathcal{H}^{(d)}_{[1...N]}. \tag{4.4}$$

Note also that, in general, Hilbert spaces may be referred to as 'factorisable', whereas the states they contain may be referred to as 'separable'. A Hilbert space could also be described as 'separable', but in conventional texts on vector spaces this name is taken to imply that a countable basis can be found for it; any vector in a separable Hilbert space may be written as a discrete sum of basis vectors.

It is possible now to define the factorisability, ζ, of $\mathcal{H}^{(d)}$ written in the form (4.3) as $\zeta = N/d$, that is, the ratio between the number of factors and the overall dimension of the vector space. Consequently, the case in which $d_i \in \mathbb{P}\ \forall i$, where $\mathbb{P}$ is the set of prime numbers, represents the maximum factorisability of $\mathcal{H}^{(d)}$ for a given d; such a split will be

called a '*fundamental*' or '*natural*' factorisation, and the factor Hilbert spaces will be called '*elementary*'. Obviously, for a Hilbert space of even dimension, the factorisability ζ is clearly maximised if the dimension of each factor space is two, in which instance ζ is given by $\zeta = N/2^N$.

Two dimensional Hilbert spaces are of great interest to many authors, partly because they are the simplest, and partly because of an analogy with computational physics. An orthonormal basis set for a Hilbert space $\mathcal{H}^{(2)}$ may be given by $\{|0\rangle, |1\rangle\}$, for $\langle i|j\rangle = \delta_{ij}$ with $i, j = 0, 1$, and these two vectors may be likened to any set of 'opposite' states in elementary binary logic: $|0\rangle$ may for example represent 'off', 'no', 'left-polarised', 'spin-down', or 'false', whereas $|1\rangle$ may represent the reverse, i.e. 'on', 'yes', 'right-polarised', 'spin-up' or 'true'. It is this analogy to classical 'bit' logic that earns the quantum space $\mathcal{H}^{(2)}$ the title of a qubit Hilbert space, and a vector in this space may be called a qubit state. Qubit states will be discussed a number of times throughout this thesis.

It is important to note that the left-right ordering of the factor Hilbert spaces is not taken to be significant in this work. Specifically, this implies that the factorisation (4.3) is invariant to any permutation $i \longrightarrow j_i$ of its factors $\mathcal{H}_i^{(d_i)}$, such that for example $\mathcal{H}^{(4)} = \mathcal{H}_1^{(2)} \otimes \mathcal{H}_2^{(2)} \equiv \mathcal{H}_2^{(2)} \otimes \mathcal{H}_1^{(2)}$.

Similarly, the same is taken to hold true for the states contained within these Hilbert spaces; for example, if $\phi_1 \in \mathcal{H}_1^{(2)}$ and $\varphi_2 \in \mathcal{H}_2^{(2)}$, the product state $\Psi = \phi_1 \otimes \varphi_2 \in \mathcal{H}_1^{(2)} \otimes \mathcal{H}_2^{(2)}$ is defined as equivalent to the re-ordered state $\Psi' = \varphi_2 \otimes \phi_1 \in \mathcal{H}_2^{(2)} \otimes \mathcal{H}_1^{(2)}$.

If the dimension of a Hilbert space is large, but not prime, the number of different ways in which it can be factorised might also be large.

For example, consider a four dimensional Hilbert space $\mathcal{H}^{(4)}$; the only non-trivial factorisation of $\mathcal{H}^{(4)}$ splits the Hilbert space into a product of two sub-spaces, i.e. $\mathcal{H}^{(4)} = \mathcal{H}_1^{(2)} \otimes \mathcal{H}_2^{(2)}$, where as above the sub-script is a convenient label and the super-script denotes dimension. Such a split may be called a *bi-partite* factorisation.

Alternatively consider an eight dimensional Hilbert space $\mathcal{H}^{(8)}$; this space may potentially be split into a tri-partite factorisation of three two-dimensional Hilbert spaces, or a bi-partite factorisation of one two-dimensional Hilbert space and one four-dimensional Hilbert space.

Obviously for higher dimensional cases such as $\mathcal{H}^{(24)}$, the number of ways in which the Hilbert space might be factorisable in this simple manner is even greater, schematically because $24 = 2 \times 12 = 2 \times 2 \times 6 = 2 \times 2 \times 2 \times 3 = 2 \times 4 \times 3 = 3 \times 8 = 4 \times 6$. In fact, as will be shown in Chapter 5, the actual number of ways of splitting a Hilbert spaces is much more complicated than this elementary 'dimensional' argument suggests. There are, for example, a number of different ways of factorising a 24 dimensional Hilbert space into a

product of a two dimensional factor and a twelve dimensional factor.

In a similar vein, the vectors contained in these Hilbert spaces will also possess different degrees of separability. An arbitrary vector Ψ in $\mathcal{H}^{(4)}$, for example, is either separable in the form $\Psi = \phi_1 \otimes \varphi_2$, where $\phi_1 \in \mathcal{H}_1^{(2)}$ and $\varphi_2 \in \mathcal{H}_2^{(2)}$, or not, in which case it is said to be entangled relative to the factorisation $\mathcal{H}^{(4)} = \mathcal{H}_1^{(2)} \otimes \mathcal{H}_2^{(2)}$. Similarly, for the case of an arbitrary vector Φ in $\mathcal{H}^{(8)}$, the state might either be separable into three factors, or into two factors, or into one giant entangled 'factor'. In the case in which Φ can only be separated into two factors relative to a tri-partite factorisation of $\mathcal{H}^{(8)}$, $\mathcal{H}^{(8)} = \mathcal{H}_1^{(2)} \otimes \mathcal{H}_2^{(2)} \otimes \mathcal{H}_3^{(2)}$, it is clear that the state is the product of two sub-states, one of which is entangled. Again, arbitrary states in higher dimensional Hilbert spaces might potentially be separable into products of many sub-states of differing dimension; this will be discussed more thoroughly in Chapter 5.

Although a Hilbert space $\mathcal{H}^{(d)} \equiv \mathcal{H}$ might potentially be split into the N-partite factorisation of equation (4.3), it is only whether a state is separable relative to a particular bi-partite split that is of most interest. Indeed, without loss of generality, only the separations of vectors relative to bi-partite factorisations need be investigated, and so in reality, only the possible rules governing this need be sought. This conclusion follows because it is a feature of Hilbert space mathematics that when a state is separable into a product of vectors in different factor Hilbert spaces, the factor sub-states are effectively independent. It can then be implicitly assumed that any method used to determine whether a given state Ψ is separable into two sub-states may be applied again to determine whether one of these sub-states is itself separable into a product of two sub-sub-states, because the only difference between the two cases is that the vectors investigated are of different dimensions.

In other words, any method used to separate the d dimensional vector Ψ into a product $\phi_1 \otimes \varphi_2$ of a d_1 dimensional vector, ϕ_1, and a d_2 dimensional vector, φ_2, where $d = d_1 d_2$, is effectively the same as that used to separate a d_1 dimensional vector ϕ_1 into a product $\phi_1 = \alpha_{1_a} \otimes \beta_{1_b}$ of a d_{1_a} dimensional vector, α_{1_a}, and a d_{1_b} dimensional vector, β_{1_b}, where $\alpha_{1_a} \in \mathcal{H}_{1_a}^{(d_{1_a})}$, $\beta_{12} \in \mathcal{H}_{1_b}^{(d_{1_b})}$ and $\mathcal{H}_1^{(d_1)} = \mathcal{H}_{[1_a 1_b]}^{(d_1)} = \mathcal{H}_{1_a}^{(d_{1_a})} \otimes \mathcal{H}_{1_b}^{(d_{1_b})}$.

So, a given state Ψ may be separated into a product of N factors by a process of first separating it into two factors, followed by independently separating each of these factors into two factors, followed then by independently separating each of these four factors into two factors, and so on until each of the individual factors can no longer be separated. Assuming it is known whether it is possible to separate a given vector into a product of two factors, then by repeated iteration the separation of the overall state into N sub-states can be found.

As an example, consider the tri-partite factorisation of the eight-dimensional Hilbert space $\mathcal{H}^{(8)}_{[1\ldots3]} = \mathcal{H}^{(2)}_1 \otimes \mathcal{H}^{(2)}_2 \otimes \mathcal{H}^{(2)}_3$, and also a state Φ in $\mathcal{H}^{(8)}$ that is known to be separable into three factors, i.e. can be written in the form $\Phi = \phi_1 \otimes \varphi_2 \otimes \psi_3$, where $\phi_1 \in \mathcal{H}^{(2)}_1$, $\varphi_2 \in \mathcal{H}^{(2)}_2$ and $\psi_3 \in \mathcal{H}^{(2)}_3$. It follows that Φ must also be separable into two factors,

$$\begin{aligned} \Phi &= \phi_1 \otimes (\varphi_2 \otimes \psi_3) \\ &= \phi_1 \otimes \chi_{23} \end{aligned} \tag{4.5}$$

where $\chi_{23} \equiv (\varphi_2 \otimes \psi_3)$ is an element of $\mathcal{H}^{(4)}_{[23]} \equiv \mathcal{H}^{(2)}_2 \otimes \mathcal{H}^{(2)}_3$.

This argument can be reversed. In order to show that Φ is separable into three factors, it is only necessary to first show that Φ is separable into two factors, ϕ_1 and χ_{23}, relative to the bi-partite factorisation of the Hilbert space $\mathcal{H}^{(8)} = \mathcal{H}^{(2)}_1 \otimes \mathcal{H}^{(4)}_{[23]}$, and then to show that ϕ_1 is not separable whilst χ_{23} may be written as a product of two factors, φ_2 and ψ_3, relative to the bi-partite factorisation of the Hilbert sub-space $\mathcal{H}^{(4)} = \mathcal{H}^{(2)}_2 \otimes \mathcal{H}^{(2)}_3$. It is at this point that the procedure would terminate, because the factors φ_2 and ψ_3 cannot further be separated; the state Φ can be separated into a product of no more that three factors.

Of course, in this illustration ϕ_1, φ_2 and ψ_3 cannot be separated further because they are contained in two-dimensional Hilbert spaces, but in principle the above method could be used even if they were entangled sub-states of arbitrary dimension. As an example, if it could be found that a different vector Φ' in $\mathcal{H}^{(8)}$ is separable as $\Phi' = \phi'_1 \otimes \chi'_{23}$, with $\phi'_1 \in \mathcal{H}^{(2)}_1$ and $\chi'_{23} \in \mathcal{H}^{(4)}_{[23]}$, but that χ'_{23} is entangled relative to $\mathcal{H}^{(4)}_{[23]} = \mathcal{H}^{(2)}_2 \otimes \mathcal{H}^{(2)}_3$, this result would be sufficient to prove that Φ' cannot be separated into three factors relative to this split.

As an aside, note that analogously to the fundamental splitting of the Hilbert space described earlier, a fundamental separation of a state Ψ in $\mathcal{H}^{(d)}$ may be defined as that which contains the maximum number of factors relative to a given factorisation of the Hilbert space. For example, the state Φ in $\mathcal{H}^{(8)} = \mathcal{H}^{(2)}_1 \otimes \mathcal{H}^{(2)}_2 \otimes \mathcal{H}^{(2)}_3$ written in the form $\Phi = \phi_1 \otimes \varphi_2 \otimes \psi_3$, where $\phi_1 \in \mathcal{H}^{(2)}_1$, $\varphi_2 \in \mathcal{H}^{(2)}_2$ and $\psi_3 \in \mathcal{H}^{(2)}_3$, is clearly fundamentally separated, whereas the same state written as $\Phi = \phi_1 \otimes \chi_{23}$, where χ_{23} is an element of $\mathcal{H}^{(4)}_{[23]} \equiv \mathcal{H}^{(2)}_2 \otimes \mathcal{H}^{(2)}_3$, is not. Conversely, the state Φ' written in the form $\Phi' = \phi'_1 \otimes \chi'_{23}$ defined above is fundamentally separated relative to this factorisation of $\mathcal{H}^{(8)}$.

The conclusion of the above few paragraphs is that it is only necessary to investigate whether or not a given state $\Psi \in \mathcal{H}^{(d)}$ is separable into a product $\Psi = \phi_1 \otimes \varphi_2$ of two sub-states $\phi_1 \in \mathcal{H}^{(d_1)}_1$ and $\varphi_2 \in \mathcal{H}^{(d_2)}_2$, relative to some bi-partite factorisation $\mathcal{H}^{(d)} = \mathcal{H}^{(d_1)}_1 \otimes \mathcal{H}^{(d_2)}_2$ of the d-dimensional Hilbert space. A test is hence sought to determine whether an arbitrary state can be separated into two factors, relative to such a bi-partite split.

Consider a Hilbert space $\mathcal{H} \equiv \mathcal{H}^{(d)}$ of dimension d that is factorisable into the bipartite split $\mathcal{H}^{(d)} = \mathcal{H}_1^{(d_1)} \otimes \mathcal{H}_2^{(d_2)}$, where d_a is dimension of the a^{th} factor Hilbert space, $a = 1, 2$, and $d = d_1 d_2$. It is a standard theorem of vector spaces [22] that a Hilbert space of dimension D is spanned by a set of D orthonormal basis vectors. Thus, a basis set $\mathcal{B}_a$ for the Hilbert space $\mathcal{H}_a^{(d_a)}$ may be given by

$$\mathcal{B}_a \equiv \{|i\rangle_a : i = 0, 1, ..., (d_a - 1),\ a = 1, 2\}, \tag{4.6}$$

where $\langle i|j\rangle = \delta_{ij}$. Moreover, it follows from (4.6) that an orthonormal basis set $\mathcal{B} = \mathcal{B}_{12}$ for the product Hilbert space $\mathcal{H}^{(d)}$ is given by

$$\mathcal{B} \equiv \{|i\rangle_1 \otimes |j\rangle_2 : i = 0, 1, ..., (d_1 - 1),\ j = 0, 1, ..., (d_2 - 1)\} \tag{4.7}$$

So, any vector Ψ in $\mathcal{H}$ is composed of a complex linear superposition of the members of this set, viz.

$$|\Psi\rangle = \sum\nolimits_{i=0}^{d_1-1} \sum\nolimits_{j=0}^{d_2-1} C_{ij} |i\rangle_1 \otimes |j\rangle_2 \tag{4.8}$$

where the $C_{ij} \in \mathbb{C}$ form a $d_1 \times d_2$ complex coefficient matrix.

Depending on the set of values of C_{ij} for $i = 0, 1, ..., (d_1 - 1)$ and $j = 0, 1, ..., (d_2 - 1)$, the state Ψ will be either separable or entangled relative to the factorisation $\mathcal{H}^{(d)} = \mathcal{H}_1^{(d_1)} \otimes \mathcal{H}_2^{(d_2)}$. For example, if $C_{ij} = 1$ for $i = j = 0$, but $C_{ij} = 0$ otherwise, then $|\Psi\rangle = |0\rangle_1 \otimes |0\rangle_2$, which is clearly separable.

In fact:

Theorem 4.1 *A state $|\Psi\rangle \in \mathcal{H}^{(d)}$ is separable relative to the factorisable Hilbert space basis $\mathcal{B}$ iff its coefficient matrix satisfies the 'microsingularity' condition*

$$C_{ij} C_{kl} = C_{il} C_{kj} \tag{4.9}$$

for all $0 \leqslant i, k, \leqslant (d_1 - 1)$ and $0 \leqslant j, l, \leqslant (d_2 - 1)$.

The proof of (4.9) is given below, noting that a similar result is provided by Albeverio *et al* [28] based on the idea of 'concurrency'.

Proof. $\Rightarrow$ If $C_{ij} C_{kl} = C_{il} C_{kj}$ and $|\Psi\rangle = \sum_{i=0}^{d_1-1} \sum_{j=0}^{d_2-1} C_{ij} |i\rangle_1 \otimes |j\rangle_2$:

Suppose, without loss of generality, $C_{uv} \neq 0$ for some u, v. Then multiplying (4.8) by this gives

$$\begin{aligned} C_{uv}|\Psi\rangle &= \sum\nolimits_{i=0}^{d_1-1} \sum\nolimits_{j=0}^{d_2-1} C_{uv} C_{ij} |i\rangle_1 \otimes |j\rangle_2 \\ &= \sum\nolimits_{i=0}^{d_1-1} \sum\nolimits_{j=0}^{d_2-1} C_{uj} C_{iv} |i\rangle_1 \otimes |j\rangle_2 \\ &= \sum\nolimits_{i=0}^{d_1-1} C_{iv} |i\rangle_1 \otimes \sum\nolimits_{j=0}^{d_2-1} C_{uj} |j\rangle_2 \end{aligned} \tag{4.10}$$

and the state is separable.

$\Leftarrow$ If $|\Psi\rangle$ is separable relative to $\mathcal{H}^{(d)} = \mathcal{H}_1^{(d_1)} \otimes \mathcal{H}_2^{(d_2)}$:

$$\begin{aligned} |\Psi\rangle &= \left(\sum_{i=0}^{d_1-1} a_i|i\rangle_1\right) \otimes \left(\sum_{j=0}^{d_2-1} b_j|j\rangle_2\right) \quad , \quad a_i, b_j \in \mathbb{C} \\ &= \sum_{i=0}^{d_1-1}\sum_{j=0}^{d_2-1} a_i b_j |i\rangle_1 \otimes |j\rangle_2 \end{aligned} \tag{4.11}$$

So $a_i b_j = C_{ij}$ from (4.8). It follows that

$$C_{ij}C_{kl} = a_i b_j a_k b_l = a_i b_l a_k b_j = C_{il}C_{kj} \tag{4.12}$$

■

As an example, the two qubit state $\Theta \in \mathcal{H}_{[12]}^{(4)}$ given by

$$\Theta = \alpha|0\rangle_1 \otimes |0\rangle_2 + \beta|0\rangle_1 \otimes |1\rangle_2 + \gamma|1\rangle_1 \otimes |0\rangle_2 + \delta|1\rangle_1 \otimes |1\rangle_2 \tag{4.13}$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and orthonormal basis set $\mathcal{B}_a \equiv \{|i\rangle_a : i = 0, 1\}$ for $\mathcal{H}_a^{(2)}$ and $a = 1, 2$, can be written as a separable state of the form

$$\Theta = (a|0\rangle_1 + b|1\rangle_1) \otimes (c|0\rangle_2 + d|1\rangle_2) \tag{4.14}$$

for $a, b, c, d \in \mathbb{C}$ if, and only if, $\alpha\delta = \beta\gamma$.

Note that the separability of a state Ψ in $\mathcal{H} = \mathcal{H}_1^{(d_1)} \otimes \mathcal{H}_2^{(d_2)}$ is independent of the choice of basis for the individual factor spaces $\mathcal{H}_1^{(d_1)}$ and $\mathcal{H}_2^{(d_2)}$. For example, if $\mathcal{H}_a^{(d_a)}$ has a basis $\mathcal{B}_a \equiv \{|i\rangle_a : i = 0, 1, ..., (d_a - 1)\}$ for $a = 1, 2$, the separability of Ψ is invariant to any relabelling $i \longrightarrow j_i$ of the individual elements $|i\rangle_a$. Similarly, Ψ will not be affected by any 'rotation' of the members of this basis set by local unitary operators $\hat{u}_a$, i.e. $|i\rangle_a \longrightarrow |i'\rangle_a = \hat{u}_a|i\rangle_a$.

In general, for a state Ψ to be separable relative to the bi-partite factorisation of the Hilbert space $\mathcal{H}^{(d)} = \mathcal{H}_1^{(d_1)} \otimes \mathcal{H}_2^{(d_2)}$, the number N_c of microsingularity equalities that need to be satisfied is given by

$$N_c = \frac{1}{4}[d_1(d_1 - 1)d_2(d_2 - 1)] \tag{4.15}$$

or $N_c \sim d^2/4$ for $d = d_1 d_2 \gg 1$. In addition, the set of separable states is a set of measure zero relative to the set of all possible states; the set of separable states effectively form a hypersurface in the hypervolume representing every set of values of C_{ij}.

It might be surprising, therefore, that there is any separability in the Universe at all. From the earlier 'minimum guess' that the dimensionality of the Hilbert space of the state Ψ_n of the Universe is greater than $2^{10^{184}}$, the number of microsingularity conditions that are required to ensure that Ψ_n is not entangled is at least $2^{(2\times 10^{184}-2)}$. It might therefore be expected that if a vector is chosen at random from a Hilbert space of dimension $2^{10^{184}}$,

the probability that it is separable, relative to a given bi-partite factorisation of the Hilbert space, is zero. From this argument, the probability that the statevector representing the Universe is separable might also be expected to be approximately zero. Further, if the Universe jumps through a series of states $\Psi_{n-1} \longrightarrow \Psi_n \longrightarrow \Psi_{n+1}$ as it develops, it might be expected that the Universe should almost always proceed from one entangled state to the next.

This, however, does not appear to be what is actually observed in the physical Universe. If separability is a necessary prerequisite for classicity, and given that the Universe does seem to look like a giant collection of classical objects, the state of the Universe must be highly separable. Given that it appears overwhelmingly likely that a quantum state chosen at random from the set of states of dimension $2^{10^{184}}$ is entangled, the question must remain as to how there can ever be any classicity in a Universe running according to quantum principles.

So in response to this, the conclusion must therefore be that the operator $\hat{\Sigma}_n$ used to prepare the state Ψ_n must be very carefully constrained in order to ensure that its eigenvectors are almost universally separable. Equally, the Rules R_n themselves must be very finely tuned to arrange that an operator $\hat{\Sigma}_{n+1}$ with highly separable eigenvectors is selected to form the basis for the next state Ψ_{n+1}. Quite plainly, the operators that the Universe chooses to test itself must force Ψ_n to jump from one highly separable state to the next. This is analogous to the conclusion presented at the end of Section 3.3, in which Rules are discussed that guarantee that the state Ψ_{n+1} appears so similar to Ψ_n, and is a point that will be returned to many times throughout this thesis.

An important feature of the present discussion is that states that are separable relative to a particular factorisation of a Hilbert space may be entangled relative to a different one. Consider, for example, a Hilbert space $\mathcal{H}^{(8)}$ that is the product of three qubit sub-spaces, that is $\mathcal{H}^{(8)} = \mathcal{H}^{(8)}_{[123]} = \mathcal{H}^{(2)}_1 \otimes \mathcal{H}^{(2)}_2 \otimes \mathcal{H}^{(2)}_3$.

Now define a bi-partite split of $\mathcal{H}^{(8)}$ of the form $\mathcal{H}^{(8)}_{[A3]} = \mathcal{H}^{(4)}_A \otimes \mathcal{H}^{(2)}_3$, where $\mathcal{H}^{(4)}_A = \mathcal{H}^{(4)}_{[12]} = \mathcal{H}^{(2)}_1 \otimes \mathcal{H}^{(2)}_2$, with a suitable basis set $\mathcal{B}_{A3}$ given by $\mathcal{B}_{A3} \equiv \{|ij\rangle_A \otimes |k\rangle_3 : i, j, k = 0, 1\}$ for $|ij\rangle_A \in \mathcal{H}^{(4)}_A$ and $|k\rangle_3 \in \mathcal{H}^{(2)}_3$. Consider also a state $\Phi \in \mathcal{H}^{(8)}$ defined as (4.16) for $a, b, c, d, \alpha, \beta \in \mathbb{C}$.

$$\Phi = (a|00\rangle_A + b|01\rangle_A + c|10\rangle_A + d|11\rangle_A) \otimes (\alpha|0\rangle_3 + \beta|1\rangle_3) \tag{4.16}$$

where for convenience here and in the following, the product state $|i\rangle_1 \otimes |j\rangle_2$ has been abbreviated by omitting the tensor symbol and writing

$$|i\rangle_1 \otimes |j\rangle_2 \equiv |i\rangle_1 |j\rangle_2 \equiv |ij\rangle_{12} = |ij\rangle. \tag{4.17}$$

Note that because in this contracted form the sub-script denoting the factor Hilbert space is dropped, the left-right ordering of the products must be preserved, i.e. $|ij\rangle \neq |ji\rangle$.

According to equation (4.8), Φ is equivalently specified by the coefficient matrix C_{ij} given by

$\otimes$	$\|\mathbf{0}\rangle_3$	$\|\mathbf{1}\rangle_3$
$\|\mathbf{00}\rangle_A$	$a\alpha$	$a\beta$
$\|\mathbf{01}\rangle_A$	$b\alpha$	$b\beta$
$\|\mathbf{10}\rangle_A$	$c\alpha$	$c\beta$
$\|\mathbf{11}\rangle_A$	$d\alpha$	$d\beta$

Table 4.1

where the first column and first row represent the basis vectors for the bi-partite factorisation $\mathcal{H}^{(8)} = \mathcal{H}_A^{(4)} \otimes \mathcal{H}_3^{(2)}$, and the remaining values represent the coefficients of their tensor products ($1^{st}\ column \otimes 1^{st}\ row$). Obviously Φ is separable relative to this factorisation, and the coefficient matrix clearly obeys the microsingularity condition.

Consider now a different bi-partite factorisation of the Hilbert space defined as $\mathcal{H}_{[1B]}^{(8)} = \mathcal{H}_1^{(2)} \otimes \mathcal{H}_B^{(4)}$, where $\mathcal{H}_B^{(4)} = \mathcal{H}_{[23]}^{(4)} = \mathcal{H}_2^{(2)} \otimes \mathcal{H}_3^{(2)}$. Such a factorisation is spanned by an orthonormal basis $\mathcal{B}_{1B}$ given by $\mathcal{B}_{1B} \equiv \{|i\rangle_1 \otimes |jk\rangle_B : i,j,k = 0,1\}$, where $|i\rangle_1 \in \mathcal{H}_1^{(2)}$ and $|jk\rangle_B \in \mathcal{H}_B^{(4)}$.

Now, by expanding (4.16), the vector Φ may equally be written as

$$\begin{aligned} \Phi &= a\alpha|0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 + a\beta|0\rangle_1 \otimes |0\rangle_2 \otimes |1\rangle_3 \\ &\quad + b\alpha|0\rangle_1 \otimes |1\rangle_2 \otimes |0\rangle_3 + ... + d\beta|1\rangle_1 \otimes |1\rangle_2 \otimes |1\rangle_3 \\ &= a\alpha|0\rangle_1 \otimes |00\rangle_B + a\beta|0\rangle_1 \otimes |01\rangle_B + b\alpha|0\rangle_1 \otimes |10\rangle_B + ... + d\beta|1\rangle_1 \otimes |11\rangle_B \end{aligned} \tag{4.18}$$

with the coefficient matrix

$\otimes$	$\|\mathbf{00}\rangle_B$	$\|\mathbf{01}\rangle_B$	$\|\mathbf{10}\rangle_B$	$\|\mathbf{11}\rangle_B$
$\|\mathbf{0}\rangle_1$	$a\alpha$	$a\beta$	$b\alpha$	$b\beta$
$\|\mathbf{1}\rangle_1$	$c\alpha$	$c\beta$	$d\alpha$	$d\beta$

Table 4.2

which clearly might not satisfy each of the six microsingularity equalities. Evidently, although the state Φ is separable relative to the first factorisation of $\mathcal{H}^{(8)}$, i.e. $\mathcal{H}^{(8)} = \mathcal{H}_{[A3]}^{(8)}$, it is entangled relative to the second factorisation of $\mathcal{H}^{(8)}$, i.e. $\mathcal{H}^{(8)} = \mathcal{H}_{[1B]}^{(8)}$.

This result highlights the conclusion that it is simply not enough to say that a particular state is separable, but that it must be qualified by the statement that it is separable relative to a given factorisation of the Hilbert space. More precisely, if a state $\Psi \in \mathcal{H}$ may be written in the form $\Psi = \phi \otimes \varphi$, where $\phi \in \mathcal{H}_1$ and $\varphi \in \mathcal{H}_2$ for $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, then it may be said that Ψ is separable relative to $(\mathcal{H}_1, \mathcal{H}_2)$. Alternatively, if Ψ is not separable in this way, it is said that Ψ is entangled relative to $(\mathcal{H}_1, \mathcal{H}_2)$.

The above result has an important consequence. If any state is only separable relative to a given factorisation of the Hilbert space, then the assertion that the Universe's state is

highly separable, because the Universe appears classical, is only meaningful given a certain factorisation of the Hilbert space $\mathcal{H}$ containing the Universe's state.

In order to discuss consistent physics, it might therefore be suggested that a preferred split for $\mathcal{H}$ exists, and that the Universe's state may only be described as separable, entangled, or a separable product of entangled factor sub-states, relative to this preferred factorisation. As a conjecture, the fundamental factorisation of $\mathcal{H}$, in which each factor sub-space is of prime dimension, might perhaps be a possible candidate for such a preferred split, but such a 'natural' assumption requires a great deal of future work.

4.2 Basis Sets and Operators

Whilst much of the discussion in this section has involved the properties of the states, it is important to remember that the operators themselves also play a necessary part in the dynamics. After all, it is the Hermitian operators used to test the state that provide, as their eigenvectors, the basis set of next possible states.

Recall that every operator $\hat{\Sigma}_{n+1}$ acting in a d dimensional Hilbert space $\mathcal{H} = \mathcal{H}^{(d)}$ and 'testing' a state Ψ_n (or $\Psi'_n = \hat{U}\Psi_n$, where $\hat{U}$ is unitary) is associated with a basis set $\mathfrak{B} = \{|\phi_i\rangle : i = 1, .., d\}$ of d orthonormal eigenvectors[5]. Further, as the Universe develops, its state jumps from Ψ_n to Ψ_{n+1}, and this effectively involves a process of randomly selecting one of these d orthonormal eigenvectors to be the next state Ψ_{n+1}, with the probability that a particular eigenstate $|\phi_i\rangle$ is chosen given by the usual Born rule $|\langle\phi_i|\Psi_n\rangle|^2$.

Now, each member $|\phi_i\rangle$ of $\mathfrak{B}$ could be either entangled or separable, relative to some bi-partite factorisation of $\mathcal{H} = \mathcal{H}_1^{(d_1)} \otimes \mathcal{H}_2^{(d_2)}$. In fact, the overall basis set $\mathfrak{B}$ may contain p entangled states, and consequently $q = d - p$ separable members; such a set could hence be labelled a type (p, q) basis. A basis set of type $(0, d)$ may therefore be called completely separable, whereas a type $(d, 0)$ may be described as a completely entangled basis. All other types may be called partially separable, or equivalently, partially entangled.

As an extension to the above, note that it would be necessary to introduce a third parameter, r, in order to describe basis sets that may contain states that are separable relative to a tri-partite factorisation of the Hilbert space, $\mathcal{H} = \mathcal{H}_1^{(d_1)} \otimes \mathcal{H}_2^{(d_2)} \otimes \mathcal{H}_3^{(d_3)}$ for $d = d_1 d_2 d_3$. These sets would be described as a (p, r, q) type, because they contain p

[5]Note that throughout this work, the calligraphic symbol $\mathcal{B}$ will be used to denote the particular basis set of a Hilbert space represented by $\{|0\rangle, |1\rangle, |2\rangle, ...\}$ (i.e. the 'natural' basis set), whereas the fraktur symbol $\mathfrak{B}$ will denote basis sets of orthonormal eigenvectors $\{|\phi_1\rangle, \{|\phi_2\rangle, \{|\phi_3\rangle, ...\}$ of operators. This is really just a convenience, since a basis set of eigenstates is also a basis set for the Hilbert space, and vice versa.

entangled members, q members that are separable into three factors, and r members that can be separated into just two factors, one of which a entangled relative to two of the factor Hilbert spaces.

Equivalently, a label (p, r, q) clearly indicates that the set incorporates p states that are separable into a product of one factors (totally entangled vectors), r states that are separable into a product of two factors (the partially entangled vectors), and q states that are separable into a product of three factors (the fundamentally separated vectors). Obviously, the extension generalises in a natural way, such that basis sets discussed relative to an N-partite factorisation require N parameters.

It is interesting to note that not every type of (p, q) basis set exists.

Consider a four dimensional Hilbert space factorisable into a product of two qubit factor sub-spaces, that is $\mathcal{H}^{(4)} = \mathcal{H}_1^{(2)} \otimes \mathcal{H}_2^{(2)}$. Let $\mathcal{B}_a = \{|i\rangle_a : i = 0, 1\}$ for $a = 1, 2$ be an orthonormal basis set for the factor Hilbert space $\mathcal{H}_a^{(2)}$, and turn attention to finding orthonormal basis sets spanning the total Hilbert space $\mathcal{H}^{(4)}$.

Firstly, it is possible to find basis sets of vectors that are completely separable relative to the given bi-partite factorisation of $\mathcal{H}^{(4)}$. An example of such a type $(0, 4)$ basis is $\mathfrak{B}_{(0,4)}$, defined as

$$\mathfrak{B}_{(0,4)} = \{|00\rangle,\ |01\rangle,\ |10\rangle,\ |11\rangle\}, \tag{4.19}$$

with $\langle kl|ij\rangle = \delta_{ik}\delta_{jl}$ and $|ij\rangle \equiv |i\rangle_1 \otimes |j\rangle_2$ for $i, j = 0, 1$.

Using the same notation, it is also possible to find type $(2, 2)$ basis sets for $\mathcal{H}^{(4)}$. One example, $\mathfrak{B}_{(2,2)}$, may be defined as

$$\mathfrak{B}_{(2,2)} = \left\{|00\rangle,\ |11\rangle,\ \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),\ \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\right\}. \tag{4.20}$$

Similarly, it is possible to find an example, $\mathfrak{B}_{(3,1)}$, of a type $(3, 1)$ basis,

$$\mathfrak{B}_{(3,1)} = \left\{ \begin{array}{c} |00\rangle,\ \frac{1}{\sqrt{2}}|11\rangle + \frac{1}{2}(|01\rangle + |10\rangle), \\ \frac{1}{\sqrt{2}}|11\rangle - \frac{1}{2}(|01\rangle + |10\rangle),\ \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \end{array} \right\}, \tag{4.21}$$

and an example, $\mathfrak{B}_{(4,0)}$, of a completely entangled, type $(4, 0)$ basis,

$$\mathfrak{B}_{(4,0)} = \left\{ \begin{array}{c} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),\ \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \\ \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),\ \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \end{array} \right\}. \tag{4.22}$$

However, despite the existence of type $(0, 4)$, $(2, 2)$, $(3, 1)$, and $(4, 0)$ basis sets, no example of a type $(1, 3)$ basis set can be found. This leads to the following:

Theorem 4.2 *No type* $(1, 3)$ *basis set exists for a four dimensional Hilbert space, relative to the factorisation of* $\mathcal{H}^{(4)}$ *into a product of two qubit sub-spaces.*

Proof. Let η_1, η_2, and η_3 be three orthonormal vectors in $\mathcal{H}^{(4)}$ that are separable relative to the factorisation $\mathcal{H}^{(4)} = \mathcal{H}_1^{(2)} \otimes \mathcal{H}_2^{(2)}$. Each vector η_i, $i = 1, 2, 3$, is of the form $\eta_i = \phi_i \otimes \varphi_i$, where $\phi_i \in \mathcal{H}_1^{(2)}$ and $\varphi_i \in \mathcal{H}_2^{(2)}$.

From the condition that $|\eta_i| = 1$,

$$(\langle\phi_i| \otimes \langle\varphi_i|)\,(|\phi_i\rangle \otimes |\varphi_i\rangle) = \langle\phi_i|\phi_i\rangle\langle\varphi_i|\varphi_i\rangle > 0 \tag{4.23}$$

$\Longrightarrow$ None of the factors ϕ_i or φ_i can be zero for $i = 1, 2, 3$.

Moreover, mutual orthogonality, $\langle\eta_i|\eta_j\rangle = 0$ for $i \neq j$, gives

$$\begin{aligned} \langle\phi_1|\phi_2\rangle\langle\varphi_1|\varphi_2\rangle &= 0 \\ \langle\phi_1|\phi_3\rangle\langle\varphi_1|\varphi_3\rangle &= 0 \\ \langle\phi_2|\phi_3\rangle\langle\varphi_2|\varphi_3\rangle &= 0. \end{aligned} \tag{4.24}$$

For brevity, the product $\langle\phi_i|\phi_j\rangle$ shall be defined A_{ij}, and the product $\langle\varphi_i|\varphi_j\rangle \equiv B_{ij}$, for $1 \leqslant i < j \leqslant 3$. For the above equalities $A_{ij}B_{ij} = 0$ to hold, A_{ij} and/or B_{ij} must be zero.

It is firstly evident that not all three of the A_{ij} can be zero. If this were the case, i.e. $A_{12} = A_{13} = A_{23} = 0$, then

$$A_{12} = 0 \quad \Longrightarrow \quad \langle\phi_1|\phi_2\rangle = 0, \tag{4.25}$$

which would imply, since $\phi_i \neq 0$ for $i = 1, 2, 3$, and since $\phi_i \in \mathcal{H}_1^{(2)}$ and $\mathcal{H}_1^{(2)}$ is two dimensional, that ϕ_1 and ϕ_2 form an orthogonal basis for $\mathcal{H}_1^{(2)}$. In this case, therefore,

$$\phi_3 = a\phi_1 + b\phi_2 \tag{4.26}$$

where $a, b \in \mathbb{C}$ and

$$|a|^2 + |b|^2 = 1. \tag{4.27}$$

Then,

$$\begin{aligned} A_{13} = 0 \quad &\Longrightarrow \quad \langle\phi_1|\phi_3\rangle = 0 \quad \Longrightarrow \quad a = 0 \\ A_{23} = 0 \quad &\Longrightarrow \quad \langle\phi_2|\phi_3\rangle = 0 \quad \Longrightarrow \quad b = 0. \end{aligned} \tag{4.28}$$

But (4.28) with (4.26) contradicts $\phi_i \neq 0$, leading to the conclusion that not all A_{12}, A_{13} and A_{23} can be zero. Similarly, not all three B_{ij} may be zero.

One way of satisfying the mutual orthogonality conditions, $\langle\eta_i|\eta_j\rangle = 0$ for $i \neq j$, is to assume $A_{12} = A_{13} = B_{23} = 0$ and $A_{23} \neq 0$, though by symmetry any other combination for $i < j$ and $(k \neq i)\&(l \neq j)$ of two A_{ij} and one B_{kl} being zero, or two B_{ij} and one A_{kl}, would also work.

As before, (4.25) may be used to deduce that $\phi_3 = a\phi_1 + b\phi_2$, such that ϕ_1 and ϕ_2 form an orthogonal basis for $\mathcal{H}_1^{(2)}$. From the condition $A_{13} = 0$, it is clear that $a = 0$, and so because $A_{23} \neq 0$ the conclusion is that $b \neq 0$.

Similarly to (4.25), the condition $B_{23} = 0$ with $\varphi_i \in \mathcal{H}_2^{(2)}$ implies that φ_2 and φ_3 form an orthogonal basis for $\mathcal{H}_2^{(2)}$. Hence

$$\varphi_1 = c\varphi_2 + d\varphi_3 \tag{4.29}$$

for $c, d \in \mathbb{C}$ and $|c|^2 + |d|^2 = 1$. Collecting these results gives

$$\begin{aligned} \eta_1 &= \phi_1 \otimes (c\varphi_2 + d\varphi_3) \\ \eta_2 &= \phi_2 \otimes \varphi_2 \\ \eta_3 &= b\phi_2 \otimes \varphi_3. \end{aligned} \tag{4.30}$$

which are clearly mutually orthogonal, as required.

Consider now a fourth non-zero vector $\eta_4 \in \mathcal{H}^{(4)}$. Given that ϕ_1 and ϕ_2 form an orthogonal basis for $\mathcal{H}_1^{(2)}$, and φ_2 and φ_3 form an orthogonal basis for $\mathcal{H}_2^{(2)}$, this new vector may be written as

$$\eta_4 = \alpha\phi_1 \otimes \varphi_2 + \beta\phi_1 \otimes \varphi_3 + \gamma\phi_2 \otimes \varphi_2 + \delta\phi_2 \otimes \varphi_3, \tag{4.31}$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$. Now, if η_1, η_2, η_3, and η_4 are to form an orthonormal, type $(1, 3)$ basis for $\mathcal{H}^{(4)}$, then because η_1, η_2 and η_3 are clearly separable, η_4 must be entangled relative to the bi-partite factorisation $\mathcal{H}^{(4)} = \mathcal{H}_1^{(2)} \otimes \mathcal{H}_2^{(2)}$. So, from the microsingularity condition (4.9) given earlier, the condition

$$\alpha\delta \neq \beta\gamma \tag{4.32}$$

must therefore hold for η_4. However, from orthogonality

$$\begin{aligned} \langle \eta_2 | \eta_4 \rangle &= 0 \quad \Longrightarrow \quad \gamma = 0 \\ \langle \eta_3 | \eta_4 \rangle &= 0 \quad \Longrightarrow \quad b\delta = 0, \end{aligned} \tag{4.33}$$

but since $b \neq 0$, the last equality implies δ is zero. So,

$$\alpha\delta = \beta\gamma = 0, \tag{4.34}$$

which is inconsistent with (4.9). Hence, η_4 cannot be entangled. ■

Thus, if three mutually orthogonal vectors in $\mathcal{H}^{(4)}$ are separable relative to the factorisation $\mathcal{H}^{(4)} = \mathcal{H}_1^{(2)} \otimes \mathcal{H}_2^{(2)}$, then a fourth orthogonal vector must also be separable. There can be no type $(1, 3)$ basis set for $\mathcal{H}^{(4)}$.

It is interesting as to whether such a theorem can be extended to bases in higher dimensional Hilbert spaces. Whilst no proof has been demonstrated, no type $(1,5)$ basis set has been found that spans the six dimensional Hilbert space $\mathcal{H}^{(6)}$ factorised as $\mathcal{H}^{(6)} = \mathcal{H}_1^{(3)} \otimes \mathcal{H}_2^{(2)}$. If it is actually the case that no such basis set does exist, it may lead to a conjecture that in a $d = d_1 d_2$ dimensional Hilbert space there is no type $(1, d-1)$ basis set relative to any bi-partite factorisation $\mathcal{H}^{(d)} = \mathcal{H}_1^{(d_1)} \otimes \mathcal{H}_2^{(d_2)}$. This in addition also provokes the question as to which, if any, types of basis sets (p, r, q) are forbidden relative to a tri-partite factorisation of a d dimensional Hilbert space. Consequently, which types are allowed in an N-partite split of $\mathcal{H}^{(d)}$?

The theorem described above also holds an interesting implication for physics. As discussed previously, every Hermitian operator, $\hat{\Sigma}$, in a four dimensional Hilbert space $\mathcal{H}^{(4)}$ is associated with a spectrum of four orthonormal eigenvectors. Further, these four eigenstates effectively form one possible orthonormal basis set for $\mathcal{H}^{(4)}$. So, since each basis set may be labelled as type (p, q) relative to the fundamental factorisation of the Hilbert space $\mathcal{H}^{(4)} = \mathcal{H}_1^{(2)} \otimes \mathcal{H}_2^{(2)}$, reciprocality implies that the operators themselves may also adopt this label. It is hence possible to discuss a type (p, q) Hermitian operator, $\hat{\Sigma}_{(p,q)}$, based on the separability of its eigenstates relative to this factorisation.

The conclusion of the above work is therefore that there exists no type $(1, 3)$ Hermitian operator acting on a two qubit system. There is no observable that may be represented by an operator possessing one entangled and three separable eigenstates, relative to $\mathcal{H}^{(4)} = \mathcal{H}_1^{(2)} \otimes \mathcal{H}_2^{(2)}$.

What makes this result particularly important regards the earlier problem of separability in the Universe. Even in a two qubit system, the number of separable states form a set of measure zero in comparison to the number of all possible states. So, as was remarked in the previous sub-section, the fact that separability does seem to be a common feature of physically observed quantum states is ascribed to be due to a careful choice of the operators that act upon the system. The point that can be learnt from the present discussion is that mathematics itself appears to enforce certain constraints on the way in which a system develops. For example, if a hypothetical mini-universe is imagined with a state Ψ_n existing in a Hilbert space of four dimensions, it is certain that its next state Ψ_{n+1} will not be one of the eigenstates of a type $(1, 3)$ operator. Mathematics ensures that such universes can only ever be developed with $\hat{\Sigma}_{(0,4)}$, $\hat{\Sigma}_{(2,2)}$, $\hat{\Sigma}_{(3,1)}$ or $\hat{\Sigma}_{(4,0)}$ type Hermitian operators.

Whilst two qubit universes are of, course, trivial compared to a state of dimension greater than $2^{10^{184}}$, the result highlights the assertion that the mathematics of operators places important restrictions on the development of the state. It may readily be speculated, then, on what other constraints might naturally be enforced by the operators, especially as

the dimensionality of the Hilbert space increases. Specifically, similar such constraints may ensure that the possibility of obtaining a highly separable state for the Universe is actually much more likely than might be expected. Apparent classicity may be an unavoidable and inevitable feature in a fully quantum Universe because of tight limitations fixed on its dynamics by mathematics.

4.3 Decoherence

Exactly how quantum mechanics gives way to the classical reality that scientists observe and measure has been one of the great problems of physics since the earliest days of the theory. In essence, the difficulty has been in explaining why states on the macroscopic 'everyday' scale never appear to exhibit the properties associated with quantum states. For example, large semi-classical states in the laboratory always seem to have well defined spatial locations, and are never found entangled with one other or existing in linear superpositions.

Although a number of schemes have been proposed to account for this phenomenon, by far the current most popular 'explanation' is the theory of decoherence. Since the purpose of this chapter has been to investigate some of the necessary conditions required for apparent classicity to begin to emerge from a fully quantum description of the Universe, no such study would therefore be complete without a discussion of this conventional theory.

4.3.1 The Theory of Decoherence

The main thrust of decoherence theory is that a quantum state is driven to classicality by continual interactions with its environment (see [29] [30][31], amongst others).

As an example, consider a quantum subject in the laboratory that is represented by the state ψ in a two dimensional Hilbert space $\mathcal{H}_\psi$ spanned by an arbitrary orthonormal basis $\mathcal{B}_\psi = \{|\downarrow\rangle, |\uparrow\rangle\}$. For illustration, it may be imagined that ψ represents the state of a single electron, whereas $\mathcal{B}_\psi$ represents the set of possible outcomes of a measurement of the electron's spin component in a particular direction.

Consider also the laboratory detection apparatus used to measure the electron. This is also described by a unique quantum state, and may in this simple example be represented by a vector Φ in the two-dimensional Hilbert space $\mathcal{H}_\Phi$ spanned by an orthonormal basis $\mathcal{B}_\Phi = \{|\Phi_\downarrow\rangle, |\Phi_\uparrow\rangle\}$.

Now, in order for the apparatus to behave as a detector of ψ, its state Φ must somehow be correlated with the spin states of the electron. To this end, the basis $\mathcal{B}_\Phi$ may be chosen

such that if the detector is found to be in the state $|\Phi_\downarrow\rangle$, it is taken be imply that the electron is in a spin down state, whereas if it is found to be in the state $|\Phi_\uparrow\rangle$ then the electron is assumed to be spin up. In such a system, the basis vectors $|\Phi_\downarrow\rangle$ and $|\Phi_\uparrow\rangle$ are defined as 'pointer states', and are ultimately hoped to give rise to the classical results of the measurements, i.e. what the physicist actually sees.

Let the detector initially be in the 'ground' state $|\Phi_\downarrow\rangle$. If it is to work correctly, it may be assumed that an encounter with a spin up electron induces a transition in the detector from the state $|\Phi_\downarrow\rangle$ to the state $|\Phi_\uparrow\rangle$, whereas a spin down electron leaves the apparatus' state unaffected. In other words, if the electron is initially in one of the eigenstates $|\downarrow\rangle$ or $|\uparrow\rangle$, the overall system evolves according to either (4.35) or (4.36),

$$|\quad\downarrow\quad\rangle \otimes |\Phi_\downarrow\rangle \rightarrow |\downarrow\rangle \otimes |\Phi_\downarrow\rangle \tag{4.35}$$

$$|\quad\uparrow\quad\rangle \otimes |\Phi_\downarrow\rangle \rightarrow |\uparrow\rangle \otimes |\Phi_\uparrow\rangle. \tag{4.36}$$

Such a process implicitly assumes that there is some sort of coupling between the electron and detector. This generates an interaction term in the Hamiltonian governing the system's dynamics, which leads to a unitary and deterministic evolution of the overall state into one of the products $|\downarrow\rangle \otimes |\Phi_\downarrow\rangle$ or $|\uparrow\rangle \otimes |\Phi_\uparrow\rangle$, depending on the state of the electron.

Now, the above mechanism provides the correct basis for the classically expected results if the electron is initially in one of the spin eigenstates $|\downarrow\rangle$ or $|\uparrow\rangle$. A problem arises, however, if the initial electron state is in a linear superposition of the form $\psi = \alpha|\downarrow\rangle + \beta|\uparrow\rangle$, where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$. From (4.35) and (4.36), the electron-detector system is then evolved into the state

$$(\alpha|\downarrow\rangle + \beta|\uparrow\rangle) \otimes |\Phi_\downarrow\rangle \rightarrow \alpha|\downarrow\rangle \otimes |\Phi_\downarrow\rangle + \beta|\uparrow\rangle \otimes |\Phi_\uparrow\rangle, \tag{4.37}$$

which is clearly an entangled linear superposition of two orthogonal electron-detector product states. But, such an entangled state is undesirable if it is hoped that the simple two-level apparatus may be extended to represent a classical detector, because classical objects are never seen in linear superpositions. So, if decoherence is to be an answer to the question of how classicity emerges from quantum theory, it must provide a mechanism for removing the entanglement of (4.37).

The method proposed in decoherence theory incorporates an extension of the above 'von Neumann chain' of correlated systems to an inclusion of the environment as well, which is also assumed to be a quantum state. Consider two particular states of the environment $|\Xi_\downarrow\rangle$ and $|\Xi_\uparrow\rangle$ contained in an enormous Hilbert space $\mathcal{H}_\Xi$. These two vectors are taken to be the result of an interaction between the pointer states of the detector with

its surroundings; that is, if the detector is in the state $|\Phi_\downarrow\rangle$ then the environment will be in the state $|\Xi_\downarrow\rangle$, whereas if the detector is in the state $|\Phi_\uparrow\rangle$ then the environment will be in the state $|\Xi_\uparrow\rangle$.

With this condition in place, then if the environment is initially in some ground state $|\Xi_0\rangle$, it is assumed that the detector-environment system is evolved into one of the following two correlations

$$\begin{aligned} |\Phi_\downarrow\rangle \otimes |\Xi_0\rangle &\rightarrow |\Phi_\downarrow\rangle \otimes |\Xi_\downarrow\rangle \\ |\Phi_\uparrow\rangle \otimes |\Xi_0\rangle &\rightarrow |\Phi_\uparrow\rangle \otimes |\Xi_\uparrow\rangle. \end{aligned} \tag{4.38}$$

Overall, then, an initial electron-detector-environment system $\Psi_i \equiv \psi \otimes \Phi_\downarrow \otimes \Xi_0$, where $\psi = (\alpha|\downarrow\rangle + \beta|\uparrow\rangle)$, will develop into a final entangled state Ψ_f, such that

$$|\Psi_i\rangle = |\psi\rangle \otimes |\Phi_\downarrow\rangle \otimes |\Xi_0\rangle \rightarrow \alpha|\downarrow\rangle \otimes |\Phi_\downarrow\rangle \otimes |\Xi_\downarrow\rangle + \beta|\uparrow\rangle \otimes |\Phi_\uparrow\rangle \otimes |\Xi_\uparrow\rangle = |\Psi_f\rangle. \tag{4.39}$$

If the experiment is repeated identically a large number of times, or alternatively if a large number of hypothetical identical universes are simultaneously developed in the same way, the ensemble of final states could be described in terms of the density matrix ρ defined as $\rho = |\Psi_f\rangle\langle\Psi_f|$. Clearly, then,

$$\rho = \begin{array}{c} \alpha^*\alpha|\downarrow\rangle\langle\downarrow| \otimes |\Phi_\downarrow\rangle\langle\Phi_\downarrow| \otimes |\Xi_\downarrow\rangle\langle\Xi_\downarrow| + \alpha\beta^*|\downarrow\rangle\langle\uparrow| \otimes |\Phi_\downarrow\rangle\langle\Phi_\uparrow| \otimes |\Xi_\downarrow\rangle\langle\Xi_\uparrow| \\ +\alpha^*\beta|\uparrow\rangle\langle\downarrow| \otimes |\Phi_\uparrow\rangle\langle\Phi_\downarrow| \otimes |\Xi_\uparrow\rangle\langle\Xi_\downarrow| + \beta^*\beta|\uparrow\rangle\langle\uparrow| \otimes |\Phi_\uparrow\rangle\langle\Phi_\uparrow| \otimes |\Xi_\uparrow\rangle\langle\Xi_\uparrow| \end{array}. \tag{4.40}$$

The central argument of the decoherence theorists is that if the environment is sufficiently large and possesses a large number of energy eigenstates, and if it is never carefully prepared or probed, then it may be ignored. In this case, it is possible to trace over all the states of the environment to obtain the reduced density matrix, ρ_s, of the electron-detector system of interest. Specifically,

$$\rho_s = Tr_\Xi[\rho] = \sum_\gamma \langle\Xi_\gamma|\rho|\Xi_\gamma\rangle \tag{4.41}$$

where the index γ implies a sum over every possible normalised state of the environment, including of course $|\Xi_0\rangle$, $|\Xi_\downarrow\rangle$ and $|\Xi_\uparrow\rangle$.

The result of (4.41) may be split into a sum $\rho_s = \rho_d + \rho_{od}$ of 'diagonal' elements, ρ_d, given by

$$\rho_d = \alpha^*\alpha|\downarrow\rangle\langle\downarrow| \otimes |\Phi_\downarrow\rangle\langle\Phi_\downarrow| + \beta^*\beta|\uparrow\rangle\langle\uparrow| \otimes |\Phi_\uparrow\rangle\langle\Phi_\uparrow| \tag{4.42}$$

and 'off-diagonal' elements, ρ_{od}, of the form

$$\rho_{od} = \alpha\beta^*|\downarrow\rangle\langle\uparrow| \otimes |\Phi_\downarrow\rangle\langle\Phi_\uparrow| \otimes \langle\Xi_\downarrow|\Xi_\uparrow\rangle + ... \tag{4.43}$$

In this case, the diagonal elements are equivalent to the states predicted by classical mechanics, whereas the off-diagonal elements represent the quantum coherences. Evidently, the environment has no effect on the diagonal elements, but does influence the off-diagonal terms.

However, if the environmental states are assumed to be orthonormal, $\langle \Xi_i | \Xi_j \rangle = \delta_{ij}$ for all i and j, then the off-diagonal elements clearly become zero. The resulting reduced density matrix, $\rho_s = \rho_d$, takes the form of a classical ensemble of states, with no quantum entanglement.

Overall, then, the superposed electron state $\psi = (\alpha|\downarrow\rangle + \beta|\uparrow\rangle)$ has been unitarily driven to one of its classically observed basis states $|\downarrow\rangle$ or $|\uparrow\rangle$ by an interaction with its environment, and which of these two states is now actually observed is simply a matter of classical probability. That is, when an observation is eventually made there is a probability of α^2 that the electron *is already* in the state $|\downarrow\rangle$, and a probability of β^2 that the electron is already in the state $|\uparrow\rangle$. Compare this with the pre-decoherence case $\psi = (\alpha|\downarrow\rangle + \beta|\uparrow\rangle)$, in which there is a probability of α^2 that the electron *might subsequently* be found in the state $|\downarrow\rangle$ if it is tested by some operator $\hat{B}$ with orthonormal basis $\mathcal{B}_\psi = \{|\downarrow\rangle, |\uparrow\rangle\}$, and a probability of β^2 that the electron might similarly be found in the state $|\uparrow\rangle$, but is really in neither of these states until the actual observation is made.

4.3.2 Problems with Decoherence

It is difficult to predict exactly how the theory of decoherence may fit with the paradigm proposed in this thesis. As is evident from the brief summary given above, decoherence is assumed to be a macroscopic phenomenon that would only arise from a consideration of the interactions and dynamics of an overall system of very large dimension. In this sense, decoherence may be viewed as an emergent theory that might therefore potentially be used to describe how classical physics arises as an approximation to quantum theory in the large scale limit of systems with very many degrees of freedom. From this point of view, the ideas of decoherence may well play an important part in the discussion of a quantum system represented by a state of dimension greater that $2^{10^{184}}$.

On the other hand, it is still difficult to see how decoherence theory could be applied directly to the Universe as a whole. The main point of decoherence is that a (usually microscopic) quantum system is evolved into a classical looking system by continual interactions with its external surroundings. No similar argument can be applied, however, to the case in which the quantum state in question is the Universe itself, because by definition the Universe is not contained in any sort of 'background'. In essence, there is no external environment with which the state of the Universe is able to decohere.

This observation leads, perhaps, to one of three conclusions: either $^{i)}$ decoherence is a valid theory to describe states inside the Universe, but not the overall state of the Universe itself; $^{ii)}$ the individual sub-systems of the Universe decohere each other, such that the overall state of the Universe is driven to classicity; or $^{iii)}$ decoherence is not really a fundamental theory of physics. The first of these conclusions seems a little paradoxical, and leaves the question as to where the 'line' can be drawn that specifies the validity of decoherence. The third conclusion is quite negative, though still, of course, possibly true. The remaining possibility is more interesting, and might presumably lead to a situation of the type in which quantum sub-state A is acting as the environment for quantum sub-state B, whereas the quantum sub-state B is acting as the environment for quantum sub-state C, but perhaps quantum sub-state C is acting as the environment for quantum sub-state A. Such a picture would immediately be in keeping with the assumed self-referential nature of the Universe, but a great deal of further investigation is required in order to discover how, or indeed if, such a hypothetical mechanism might work.

One problem that still exists in decoherence theory is the issue of probability. Using the electron experiment described earlier as an example, the mathematics of decoherence still provides no explanation of how one of the basis states $|\downarrow\rangle$ or $|\uparrow\rangle$ actually gets selected, and therefore why a particular one of these two is actually observed in the laboratory.

During decoherence, the interplay between an initial quantum state and its environment gives rise to a well specified interaction term in the Hamiltonian. The system then undergoes unitary evolution according to the Schrödinger equation, which forces the state into a classical looking state. The Schrödinger equation, however, is a deterministic formula, and as such the drive of the state from quantum to classical must also be deterministic. Whilst this not only gives philosophical problems, such as the possibility of a Laplacian style 'clockwork' Universe, it also raises the question as to how the state can deterministically evolve to only one out of a set of possibilities. Indeed, as remarked by Erich Joos, one of the proponents of decoherence, "*Decoherence can not explain quantum probabilities without (a) introducing a novel definition of observer systems in quantum mechanical terms (this is usually done tacitly in classical terms), and (b) postulating the required probability measure (according to the Hilbert space norm)*".

The probability measure is normally introduced into conventional quantum mechanics by the state reduction process. According to this postulate, then at the point of measurement of a quantum system, the wavefunction discontinuously jumps into one of the eigenstates of the Hermitian operator representing the observation. Moreover, it is this process that abruptly selects, irreversibly and probabilistically, the next state of the system out of a set of possibilities. Decoherence, however, contains no such mechanism, so a question must remain as to how similar selections can be made if the system is always

constrained to evolve reversibly, unitarily, and deterministically.

To give an illustration, in quantum mechanics the famous paradox of Schrödinger's Cat [32] relies on which of a set of possibilities a quantum state develops into. Adapting the earlier electron example to Schrödinger's thought experiment, it might be the case that if the electron is in the spin down state, then a gun is fired and the Cat in the sealed box is killed. If conversely the electron develops into the spin up state, then a gun is not fired and the Cat is spared.

In the conventional Copenhagen interpretation of quantum mechanics, until an observation collapses the quantum wavefunction, the state of the system is in an entangled superposition of products of a spin-down electron and a fired gun with a spin-up electron and an un-fired gun. Consequently, and taking the conclusion to absurdity, it might then be argued that the Cat is simultaneously both dead and alive. So, the question has therefore always been: at what point along the chain is the observation made? If the state reduction relies on a human observation, is the conclusion to be accepted that the Cat is able to keep one paw in both life and death until physicists decide to look inside the box?

In decoherence theory, the linear superposition is destroyed by the environment, so the electron's spin state is definitely either up or down, with the inevitable consequences. As such, the corresponding reduced density matrix for the electron-gun-Cat system only reflects an external observer's classical ignorance as to what has already happened. Paraphrasing, the 'decision' has already been made by the Universe as to what has gone on in the box, but until the physicist investigates, only classical probabilities of obtaining certain results can be discussed. This is obviously like tossing a coin: the coin definitely lands either heads or tails, but until it is uncovered it is not known which of these two possibilities has occurred.

But, in decoherence theory the question remains: how does the Universe decide whether or not the electron evolves to a spin-up or a spin-down state? How does the Universe decide if the Cat lives?

The lack of randomness is not the only problem caused by a removal of the state reduction postulate from quantum mechanics. Assuming the principle of cause and effect is to be believed, any physical state in the universe is the result of some earlier process. If further it is accepted that every system in the Universe is fundamentally quantum in nature, then every physical quantum state in the Universe must therefore be the result of some earlier quantum process.

However, if these quantum processes appear to ensure that quantum interferences are eradicated, as the decoherence paradigm suggests, it is unclear as to how any coherent quantum state might be produced in the first place. In other words, if quantum systems

are only able to develop through a process of unitary evolution, and if these evolutions effectively remove quantum coherences and superpositions, what unitary process in the decoherence paradigm can give rise to entangled states? Specifically, in the example described above, how is it ever possible to create an initially superposed state of the form $\psi = (\alpha|\downarrow\rangle + \beta|\uparrow\rangle)$ using only processes constrained to destroy such features?

Presumably the conclusion to be drawn is that either decoherence theory requires an additional mechanism in order to produce such superpositions and entanglements, or else it must be asserted that every such quantum state currently in existence has come from some sort of 'partial decohering' of an earlier state that was even more entangled and superposed. In this latter case, not only would it be necessary to specify how this partial decohering might work, but also the question would remain as to why, if the Universe has been continuously and smoothly evolving for a period of about 15 billion years, are there any quantum effects left in the current epoch at all?

Of course, if the state reduction postulate is included into the formalism, this problem is not encountered because the preparation of a superposed or entangled quantum state is simply seen as the outcome of a quantum test. Moreover, because these outcomes are associated with the eigenstates of Hermitian operators, they are not constrained to be the result of a continuous unitary process. Thus with the introduction of Hermitian operators and state reduction into the dynamics, it is possible to generate superposed entangled states, and these can then go on to be developed in subsequent ways, for example to collapse and consequently cause or prevent guns from firing.

As discussed previously, such a viewpoint forms the basis of the paradigm proposed in this thesis, in which the test $\hat{\Sigma}_n$ simultaneously collapses the 'old' state of the Universe Ψ_{n-1} to prepare and produce the 'new' state Ψ_n. In this proposal, the state of the Universe develops through a long chain consisting of a state reduction, followed by evolution, followed by a state reduction, and so on.

In addition to these theoretical difficulties, recent experiments reviewed in [33] seem to indicate that discontinuous wavefunction jumps are an observed feature of physical quantum systems. If these investigations prove conclusive, it is natural to wonder as to how such an empirical result might be reconciled by a theory of decoherence based on continuous, unitary evolution.

4.3.3 Schrodinger's Cat's Stages

As a final comment to complete this discussion, it might briefly be mentioned as to how the paradigm proposed in this work views the Schrödinger's Cat paradox, noting that a fuller and more technical account is evident from Chapter 6.

In the schematic picture given here, an initial state Ψ_n is imagined that is separable into a huge number of factors. Simplifying this, however, Ψ_n may be written in the form

$$\Psi_n = |\psi\rangle \otimes |G_u\rangle \otimes |C_l\rangle \otimes |R\rangle \tag{4.44}$$

where $|\psi\rangle \in \mathcal{H}_\psi$ represents the superposed electron state $\psi = (\alpha|\downarrow\rangle + \beta|\uparrow\rangle)$, with $|G_u\rangle \in \mathcal{H}_G$ the un-fired gun, $|C_l\rangle \in \mathcal{H}_C$ the living Cat, and $|R\rangle \in \mathcal{H}_R$ the rest of the Universe.

Obviously, Ψ_n is a vector in the total Hilbert space $\mathcal{H}_\Psi = \mathcal{H}_\psi \otimes \mathcal{H}_G \otimes \mathcal{H}_C \otimes \mathcal{H}_R$.

The next test $\hat{\Sigma}_{n+1}$ acting on Ψ_n has a basis set of orthonormal eigenvectors. If two of these eigenvectors are Φ and Θ, defined by

$$\begin{aligned} \Phi &= |\uparrow\rangle \otimes |G_u\rangle \otimes |C_l\rangle \otimes |R\rangle \\ \Theta &= |\downarrow\rangle \otimes |G_u\rangle \otimes |C_l\rangle \otimes |R\rangle \end{aligned} \tag{4.45}$$

then the next state of the Universe Ψ_{n+1} may be either Φ or Θ, with relative probabilities $|\langle\Psi_{n+1} = \Phi|\Psi_n\rangle|^2 = |\langle\uparrow|\psi\rangle|^2$ and $|\langle\Psi_{n+1} = \Theta|\Psi_n\rangle|^2 = |\langle\downarrow|\psi\rangle|^2$ respectively.

Now, assume that $\Psi_{n+1} = \Phi$. Moreover, assume also that under this circumstance, the Rules governing the Universe conspire such that the subsequent states Ψ_{n+1+j} will 'resemble' Ψ_{n+1} for a large but finite number J of further evolutions $\hat{U}_{n+1+j}$ and tests $\hat{\Sigma}_{n+1+j}$, for $J \gg 0$ and $0 \leq j \leq J$. That is, assume that these subsequent tests $\hat{\Sigma}_{n+1+j}$ for $1 \leq j \leq J$ have eigenstates that are separable in the form $\Phi_{n+1+j} = |\uparrow'\rangle \otimes |G'_u\rangle \otimes |C'_l\rangle \otimes |R'\rangle$, where $|\psi'\rangle \in \mathcal{H}_\psi$, $|G'_u\rangle \in \mathcal{H}_G, |C'_l\rangle \in \mathcal{H}_C$ and $|R'\rangle \in \mathcal{H}_R$ represent, for example, living cats and un-fired guns that have changed slightly in their *own* Hilbert spaces as the Universe has developed.

The point is that during these J developments the electron, the gun, the Cat and the rest of the Universe have not interacted with each other in any way. Specifically, the electron has not interacted with the gun, and so the Cat lives.

Alternatively, consider the case where $\Psi_{n+1} = \Theta$, and assume that the Rules now conspire so that subsequent states $\Psi_{n+1+m-1}$ resemble Ψ_{n+1} for $1 \leq m \ll J$, but that at time $(n+1+m)$ a test $\hat{\Sigma}_{n+1+m}$ is chosen which has eigenstates of the form $\Theta_{n+1+m} = |\downarrow\rangle \otimes |G_f\rangle \otimes |C'_l\rangle \otimes |R'\rangle$, where $|G_f\rangle \in \mathcal{H}_G$ represents the state of the fired gun, $|C'_l\rangle \in \mathcal{H}_C$ the living Cat that has evolved slightly and independently since its earlier state $|C_l\rangle$, and $|R'\rangle \in \mathcal{H}_R$ the rest of the Universe which has also undergone many developments during the m preceding evolutions and tests.

Moreover, if the experiment is sound, it is further assumed that this eigenstate Θ_{n+1+m} occurs with very high probability. In this case, it is further assumed that an even later time $(n+1+m+p)$, the Rules conspire to choose a test $\hat{\Sigma}_{n+1+m+p}$, for $1 \leq p \ll J$, that

has an eigenstate of the form $\Theta_{n+1+m+p} = |\downarrow\rangle \otimes |G'_f\rangle \otimes |C_d\rangle \otimes |R''\rangle$, where $|G'_f\rangle \in \mathcal{H}_G$ represents the state of the gun that has changed slightly since it was fired, $|C_d\rangle \in \mathcal{H}_C$ the Cat that has now been shot dead, and $|R''\rangle \in \mathcal{H}_R$ the rest of the Universe which has also developed further in the p evolutions and tests since it was represented by the state $|R'\rangle$. As before, assuming the experiment is consistent and the gun well aimed, it is accepted that the eigenstate $\Phi_{n+1+m+p}$ will also occur with a very high probability.

Thus, the two possible outcomes for the initial collapse of the electron sub-state from $|\psi\rangle$ to $|\uparrow\rangle$ or $|\downarrow\rangle$ lead to two different 'histories' for the Universe's development. In neither, however, is there any ambiguity in the fate of the Cat.

Obviously, the example given here is described only (highly) schematically. In reality cats and guns are complicated macroscopic states that will undergo a series of 'internal' transitions as the Universe develops, and will interact with their surroundings in a multitude of different physical ways. Indeed, it is a fundamental philosophical question regarding the nature of persistence to ask what it means to describe an object that is undergoing tiny changes from moment to moment as 'the same cat'. In fact, some of the ideas of decoherence theory may contribute an important part to this particular discussion.

In principle, however, the main point from the above treatment of the Schrödinger's Cat paradox should be evident. The conjecture is that the Universe automatically and self-referentially selects an operator $\hat{\Sigma}_{n+1}$ to test itself, and it is this self-measurement that collapses the electron sub-state into one of its basis vectors $|\downarrow\rangle$ or $|\uparrow\rangle$, without the need for a conscious observer.

Perhaps it is this combination of self-referential testing with discontinuous state reduction, and maybe even macroscopic decoherence effects, that might save the Cat's life and give it a classical identity.

5 A Quantum Origin of Space

As discussed at the beginning of the previous chapter, one qualification for the presence of classicity follows from the observation that "*this* object with *these* qualities is *here,* whereas *that* object with *those* properties is *there*". Whilst the main of the last chapter focused on the issue of when it is possible to specify 'this' or 'that' object, it did not examine how the properties of the complex vector representing the Universe might give rise to the spacetime concepts of 'here' and 'there'. This question is addressed now.

5.1 Background

When attempting to develop theories to account for the presence of space, time and matter in the Universe, physicists often adopt one of two opposing viewpoints. These methods may be described as either *bottom-up* or *top-down*, and reflect the basic difference between reductionist and holistic physics. This difference is itself perhaps predictable in a Universe containing remarkably successful principles such as quantum field theory, which exhibits both local and global features.

Many of the bottom-up approaches proceed generally from the assertion that, at its most basic level, the Universe can be represented by a vast collection of discrete events existing in some sort of mathematical space. Time and space are introduced as arising from the relations between these events, such that (classical) reality as we understand it emerges on a macroscopic scale due to the complex connections between these fundamental, microscopic entities. Wheeler was one of the earliest proponents of this idea [34], by envisaging a Universe full of a pre-geometric "dust" from which spatial degrees of freedom emerge. These 'ultimate' notions of pre-geometry have been developed more recently by Stuckey [35].

On the other hand, many of quantum cosmology's top-down approaches hold that the entire Universe should be treated as a single system. Top-down theorists often seek to write down a unique state description for the Universe, before evolving it according to a given set of laws or conditions. From this point of view, the apparent classical reality that physicists perceive is just an approximation to that part of the Universe under investigation whenever a fully quantum mechanical description can be neglected.

A selection of some of the contemporary bottom-up and top-down approaches are reviewed below in Sub-sections 5.1.1 and 5.1.2. Throughout the rest of this chapter it will then be shown how some of the general points of these two approaches might be reconciled as being different aspects of the same theory. That is, in the paradigm proposed

in this thesis, the discrete events postulated on the microscopic pre-geometric scale may be associated somehow with the factor sub-states of the single state representing the completely quantum Universe. Thus, such a viewpoint may be labelled a type of 'top-down pregeometry'. It will be argued that it is from the dynamics of these changing sub-states that familial relations may arise, and that these relations could be seen as the origin of spatial degrees of freedom in the appropriate limit.

5.1.1 Bottom-Up Approaches

One of the bottom-up theories of the Universe is the Causal Set Hypothesis [36]-[40], which states that (quoting [38]): *"...spacetime, ultimately, is discrete and ... its underlying structure is that of a locally finite, partially ordered set (a causal set)".* In this model it is postulated that classical, discrete "events" are generated at random, though it is made clear that they are not embedded into any sort of physical background space. Spacetime may then be recovered as an emergent consequence of the ordering that results from imposing certain logical relations between the members of these sets of events.

Overall, actual physical space in this paradigm manifestly consists of a causal set (or "causet") of points which yield a metric structure in the continuum limit [37]. Additionally it may be shown that the dimension of this spacetime can be a scale dependent quantity, making the model equally compatible with notions of four dimensional general relativity and higher dimensional Kaluza-Klein theories, including modern string and m-brane physics. The exact details of classical causal set theory are elaborated upon in Section 5.2.

A related idea is that of Cellular Networks (CN) [41][42], which argues that, on the microscopic scale, the geometry of space may be represented by a mesh of primordial cells or 'nodes' interacting with each other via a series of interconnecting 'bonds'. These nodes are assumed to contain physical information by way of internal state structures. The bonds themselves may be in one of a number of 'bond states', allowing the strength and types of interaction to be controlled. The evolution of the Cellular Network is similar to that of a cellular automaton in that the Universe proceeds as a giant machine, but differs in the respect that the number and type of bonds in the network may change with time. For example, one change might be that two cells unconnected in one instance may be joined by a certain type of bond in the next. The vision is of a self-organising topology that is ever changing and depends on the physics of the situation being modelled. As before, metric structures are recovered as a continuum concept.

Zizzi [43][44] continues the machine principle of Cellular Networks with the analogy

that the Universe behaves in a way similar to computational information theory, in what she defines as a "Quantum Growing Network" (QGN). The state of the Universe is postulated to be a tensor product of a vast number of elementary, two-dimensional quantum degrees of freedom (qubits) which are connected and processed by a set of quantum logic gates. Further, as time goes by, the number of qubits increases, and hence so does the dimensionality of the Universe's Hilbert space. Overall, Zizzi argues that the Quantum Growing Network system forms a 'proto-spacetime' which may give rise to physical spacetime in a manner similar to Requardt's.

In the Spin Network (or 'Spinnet') approach proposed by Penrose [45], spacetime is generated from the relations between combinations of fundamental "units", where each unit may be likened to an elementary particle that possesses no characteristics apart from total angular momentum. The units may interact with one other, and a system of interacting units may be represented by a graph. Each edge of the graph denotes a unit coming into or arising from an interaction, whereas the vertices are the interactions themselves. Penrose restricts his analysis to tri-valent graphs, which may be thought of as describing two units joining to form a third or one unit splitting into two. Note, however, that because there is no 'direction' inherent to the graphs, each is assumed to represent all of the allowed interactions between the three units. The only constraint imposed is that the vertices conserve angular momentum, such that whichever particles are chosen to be the ones 'entering' the interaction, the sum of their angular momenta must equal the sum of angular momenta of the remaining units (see [46] for a review).

Given a large number of units, a large number of graphs may be obtained. Further, if one of the edges of one graph has the same value of angular momentum as the edge of another graph, they may be joined and the two graphs connected. By continuing this process, it is possible to create a network of graphs where lines represent angular momentum carrying particles and vertices represent their interactions. Penrose shows how an emergent geometry may arise by considering this network of relations.

Markopoulou and Smolin [47] investigated the causal evolution of such spin networks by combining the Causal Set approach of chains of events with the Spin Network notion of geometry. Given the set of edges and vertices comprising a spin network, rules are suggested for generating a new set from their particular relations. In fact, a number of possible new networks may be produced by exploiting the fact that each graph may represent a number of possible interactions. If the rules are repeated a series of times, a chain of networks may be created with a definite causal structure existing between them. By considering, in the manner of Causal Sets, the sums over histories of these chains of spinnets, Markopoulou *et al* were able to generate amplitudes of transmission from an initial to a final topology. The model leads to the production of a series of timelike

surfaces, analogous to an evolving spacetime.

5.1.2 Top-Down Approaches

One search for a top-down model of the Universe has been the search for a consistent theory of quantum cosmology. On the basis that the large scale structure of the Universe is described by general relativity, some cosmologists [16] have attempted to canonically quantise the solutions of the Einstein field equations. Given canonical variables, the Lagrangian and action functional can be defined, and quantum fields can be introduced; overall a quantum state function Ψ of the Universe is generated. This method of quantum cosmology involves an investigation of the evolution of the Universe's wavefunction according to the Wheeler-deWitt equation, but is associated with the notorious "Problem of Time".

Hartle and Hawking [17] progressed quantum cosmology by adding appropriate constraint conditions to the dynamics, such that the Universe could appear to be 'created from nothing' by a manner analogous to a quantum fluctuation or tunnelling process. Further developments have also been made [48][49] by adding inflationary terms to the Lagrangian in order to account for the observed isotropy, homogeneity and flatness of the Cosmos in the current epoch. These approaches again assert that the Universe is described by a single quantum state.

Given that the Universe is observed currently to be expanding, many cosmologists extrapolate back to conclude that it must have begun from a spacetime singularity. This, however, causes problems in relativity theory because regions of very high curvature require a theory of quantum gravity, and the search for a consistent model of this has proved elusive. So, a proposed alternative to the inevitable Big Bang singularity has been the Ekpyrotic Universe model [50].

The approach begins with the hypothesis that every point in our four dimensional Universe is mapped to a point on part of a hypersurface called a "D-Brane", which may be thought of as a 'thin wall' or membrane existing in part of a higher dimensional reality. This D-brane, containing the entirety of our Universe, is separated by some sort of 'Bulk' volume from a second D-brane which may itself contain a second, 'hidden' universe.

Time had no beginning in the Ekpyrotic Universe model. In an era that conventional cosmologists may refer to as pre-Big Bang, i.e. at times greater than ~ 15 billion years ago, our 4-dimensional universe within its D-brane was cold and empty. It is postulated that at some time during this period, a light (compared to the two D-branes) 'bulk-brane' peeled away from the D-brane containing the hidden universe, and travelled across the

bulk volume towards our D-brane. When they collided, it is proposed that the bulk-brane's kinetic energy was transferred into heat and excitations of the various force and matter fields contained within our D-brane. This marked the start of what appeared to be a hot big bang in our Universe, which proceeded to expand and evolve in the way understood by standard astronomy.

The Ekpyrotic model hopes to provide a mechanism for generating the observed isotropy, flatness and homogeneity of the universe, without appealing to any artificial inflation fields, and without containing an initial singularity. Additionally, it may include an explanation for why gravity is weaker than the other three fundamental forces. Brane (and string) theory, however, is still a long way from being generally or empirically accepted, and is itself riddled with unanswered or unaddressed questions.

5.2 Classical Causal Sets

As mentioned above, a number of authors have introduced the possibility that continuous spacetime might emerge from a consideration of the relationships between the members of a causal set. In this paradigm, the Universe is envisaged to consist ultimately of an enormous number of '*events*', where each event is assumed to be a separate, discrete, mathematical object of some sort.

By definition [36], a causal set (or "causet") $\mathcal{C}$ is a locally finite, partially ordered set (or "poset") of objects $\mathcal{C} = \{x, y, ...\}$. Each member of a partially ordered set either shares, or does not share, a unique binary relationship with every other member of the set. Denoting this relationship by the symbol $\prec$, which may be seen as a type of comparison, two members x and y of a poset are hence connected as $x \prec y$ or $y \prec x$, or else x and y are said to be incomparable.

The relationship $\prec$ consequently introduces an order between the members of the set, and this is made consistent by ensuring that it is transitive (**T**) and asymmetric (**A**). In addition, it is conventionally assumed that $\prec$ is also irreflexive (**I**). So, for $x, y, z \in \mathcal{C}$ the following constraints are imposed:

$$\begin{aligned} (\mathbf{T}) \quad &: \quad x \prec y \text{ and } y \prec z \Longrightarrow x \prec z \\ (\mathbf{A}) \quad &: \quad x \prec y \Longrightarrow y \nprec x \\ (\mathbf{I}) \quad &: \quad x \nprec x \end{aligned} \tag{5.1}$$

A poset may be described as locally finite if, between any two members x and y, where $x \prec y$, there are a finite number of events a, b, c, ... such that $x \prec a \prec b \prec ... \prec y$. In other words, only a limited number of events "mediate" [51] between the event x and the event y. A causal set is defined to be such a locally finite, partially ordered set.

One method of generating a causal set is via a process of 'sequential growth' [38]. At each step of the growth process a new element is created at random, and the causal set is developed by considering the relations between this new event and those already in existence. Specifically, the new event y may either be related to each of the other events x as $x \prec y$, or else x and y are said to be unrelated. Thus the ordering of the events in the causal set is as defined by the symbol $\prec$, and it is by a succession of these orderings, i.e. the growth of the causet, that is ultimately ascribed to constitute the passage of time.

The relation $x \prec y$ is hence interpreted as the statement: "y is to the future of x". As a consequence of this interpretation, the asymmetric condition may now be seen as a removal of the possibility that the causet will contain anything resembling closed time-like curves.

The above association highlights the similarity between the relations $\prec$ in causal set theory, and the idea of lightcones in relativity. In a causal set $\mathcal{C}$, the set of elements y_i, related to an event x by the relation $x \prec y_i$, represent the causal future of x. This relationship is analogous to the volume V_X contained within the future lightcone of a point X in a theory of continuous spacetime, examples of which being general and special relativity. Conversely, an event $z \in \mathcal{C}$ that is incomparable to x may be said to be causally disconnected from x, and this is similar to the set of points outside of the lightcone of X. In classical physics, events outside of this region V_X are not affected by changes inside the lightcone, for example at X, and are hence causally independent. This places an important physical constraint on the members of $\mathcal{C}$, since continuous spacetime is eventually hoped to emerge from a causal set description.

Of course, similar associations exist for points in the past lightcone of X, and the objects y_j in $\mathcal{C}$ related to x by $y_j \prec x$.

A causal set may be represented by a Hasse diagram. Further, the set of causal sets that may be constructed from a growing number of events can be represented by a Hasse diagram of Hasse diagrams.

In each Hasse diagram, the events are shown as spots and the relations as solid lines or links between the events; emergent time runs from bottom to top, and the direction of the growth process from one causal set to the next is consequently denoted by the arrowed lines. A typical such set of diagrams is given in Figure 5.1, which contains the set of causets of less that four elements (and features as part of Fig. 1 in [38]), where each large circle represents an individual Hasse diagram, and hence a particular causal set.

In the example in hand, the initial causal set has just one event, as shown in the lowest of the large circles. The next event to be born may or may not share a temporal relation with the first; that is, it may either lie to the future of the first, or not. Thus, one of two possible causal sets may be created, as shown by the two Hasse diagrams represented by

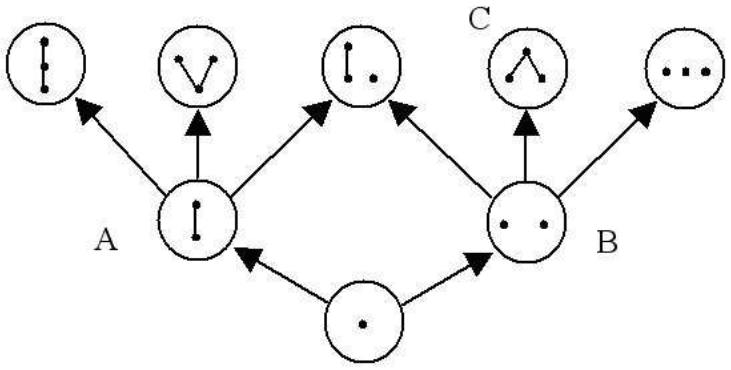

Figure 5.1: The Hasse diagram of Hasse diagrams featuring those Causal Sets containing up to three members.

the large circles labelled A (temporal relation) and B (no temporal relation). The third event to be produced may share a temporal relation with all, some, or none of the previous events, leading to the generation of five possible causal sets, and hence five possible Hasse diagrams. Figure 5.1 hence represents the sets of possible causets at three successive times.

The process may obviously be extended indefinitely to create increasingly longer causets of events, and these may be represented by a growing number of Hasse diagrams of increasing complexity.

With the above in mind, it is possible to introduce familial concepts into the sequentially growing causal set. Consider as an example a causal set $\mathcal{C}_n$with n members, $\mathcal{C}_n = \{a_1, a_2, ..., a_n\}$. Consider also a second causal set $\mathcal{C}_{n+1}$with $n+1$ members, $\mathcal{C}_{n+1} = \{a_1, a_2, ..., a_n, a_{n+1}\}$, 'grown' from $\mathcal{C}_n$ by adding the $(n+1)^{th}$ member a_{n+1}. If a_{n+1} is not in the past of any of the elements a_i, for $i = 1, ..., n$, then it is said to be a "maximal element". In this case, the causet $\mathcal{C}_{n+1}$ may be called a "child" of $\mathcal{C}_n$, which in turn may be named its "parent".

In general, a particular causet $\mathcal{C}_{n+1}$ grown from $\mathcal{C}_n$ could have a number of different topologies, because there are often very many ways of adding a maximal element a_{n+1} to $\mathcal{C}_n$ such that it is to the future of none, one, some or all of the n elements already in existence. Paraphrasing, there are a number of different Hasse diagrams that may be drawn by adding just one maximal element to $\mathcal{C}_n$, depending on 'where' it is added. The members of this set of possible causal sets are hence called "siblings" of one another, because they are all children of a common parent. Such sets of parents and children may be called "families".

This concept can be extended in the obvious way to include, for example, definitions

of grandchildren and great-grandchildren etc. As a simple illustration of these ideas, the causal sets labelled A and B in Figure 5.1 are clearly siblings, because they are children of the initial parent causet containing just one member. Similarly, the causal set C is one of the three children of B, and therefore also one of the grandchildren of the original (single member) causet.

The crux of causal set theory is that (to quote [36]): "*...a classical space-time's causal structure comes very close to determining its entire geometry*". Thus, in the large scale limit of very many events the causal sets are hoped to yield the properties of continuous spacetimes. To this end, metrics, distances and dimension should all be ready features of the topology.

It is an important feature of causal sets that the events are not taken to be embedded in any sort of physical background space. The objects themselves exist in nothing but a mathematical manifold, and it is only by taking account of the network of relations between a large number of such events that the actual geometry of the manifold, and hence the spatial relations familiar to physics, begins to emerge.

The basic methodology employed to generate space from these very large causal sets is to use the causal order of the set to determine the topology of the manifold into which it is embedded. This is converse to the standard procedure employed in continuous geometry in which the properties of the manifold and metric are used to determine the lightcones of the spacetime, and from these the causal order of events may in turn be inferred.

Concepts such as timelike geodesics and distances may be introduced into the analysis of causal sets by considering the length of paths between events [37].

Consider first a '*chain*', E, of events in a causal set, i.e. an ordered group of elements $E = \{p, q, ...\}$ in $\mathcal{C}$ in which every two elements of E are somehow related by $\prec$. By analogy with special relativity, a chain evidently possesses the causal structure of a spacetime manifold: each event $r \in E$ is either to the past or future of every other event $s \in E$. Moreover, it is possible to define the '*path length*' of a given chain between two events x and y in terms of the number of links in the chain separating these two elements.

Of course depending on the topology of $\mathcal{C}$ there may be a number of different possible chains 'linking' any two events x and y, for $x, y \in \mathcal{C}$.

However, from this observation it is possible to define a '*maximal chain*' M, where $M \subset \mathcal{C}$, as a subset of elements $M = \{a_1, a_2, ..., a_m\}$ contained in $\mathcal{C}$ such that $a_i \prec a_{i+1}$, for $1 \leq i \leq m-1$, and where there is no other element $b \in \mathcal{C}$ for which $a_i \prec b \prec a_{i+1}$. Clearly, M specifies a unique path of events between a_1 and a_m, and this is extremal in $\mathcal{C}$. Thus, it is immediately possible to define the path length of a maximal chain $a_1 \prec a_2 \prec ... \prec a_m$ in terms of the number of links between a_1 and a_m. In this case, the path length of M is

clearly given by $m - 1$.

It is from this discussion of path lengths that a notion of timelike 'distance' can arise. Given any two comparable events $x \prec ... \prec y$ in a causal set, the timelike distance $d(x, y)$ may be defined as the maximum length of path between them, i.e. the 'longest route' allowed by the topology of the causet to get from x to y.

A number of issues arise from this definition. Firstly, it implies that (timelike) distance is, at root, manifestly a counting process. As observed by [36], this is in accordance with Riemann's suggestion regarding the measurement of spatial size.

Secondly, the connection between distance and extremal chains is analogous to the use of geodesics as extremal path lengths in relativity; recall that in continuum mechanics a geodesic is defined as the extremal length between two points, and the distance between them is that length. The proposed mechanism makes use of the maximal number of objects causally separating two events, and as such the extremal distances defined in this way are associated with geodesics in continuous spacetime.

Thirdly, Brightwell *et al* [37] remark that the above definition of distance satisfies a relationship similar to the conventional 'Triangle Inequality'. For example, consider three events $x, y, z \in \mathcal{C}$ such that $x \prec y \prec z$. If the distance $d(x, z)$ is given by the path length of the maximal chain between x and z, then by definition this distance cannot be shorter than the path length between x and z via any other possible chain. Specifically, if an alternative route is via y, then this conclusion implies that $d(x, z) \geq d(x, y) + d(y, z)$, with the equality holding only when y is part of the maximal chain.

Note, however, that such a relationship differs from the standard triangle inequality of distances, given in obvious notation in the form $D(X, Y) + D(Y, Z) \geq D(X, Z)$. Moreover, it is also unclear how the above theorists would balance this 'reversed' result with the conventional case, an issue made especially pertinent by the fact that the standard version is generally taken as a pre-requisite for a metric to exist. The physical basis behind such a reversed inequality relationship needs therefore to be fully defined by the authors if it is to be used to generate metric-like structures, and careful physical and mathematical considerations are first required in order to generate relativistic spacetimes from the underlying classical causal set ideas.

The above definition of timelike distances applies to when quantifying the separation between comparable events, i.e. between those events x and y in $\mathcal{C}$ for which $x \prec ... \prec y$. For incomparable events, on the other hand, no such timelike definition is possible, because incomparable events instead share the characteristics typically exhibited by spacelike separated objects in conventional physics.

However, by exploiting this similarity between the incomparable events of causal set

theory and the causally disconnected features of spacelike separated points in continuous spacetime, it is possible to introduce an analogous definition of spacelike distances into the causal set description. Following the lead of [37], it is argued that a method of measuring spatial distances using light beams and clocks should be employed, in which the distance between two objects is determined by sending a light signal from one of these to the other and measuring the time elapsed before it is returned.

Consider a timelike geodesic G in $\mathcal{C}$, where G is defined as $a_1 \prec a_2 \prec ... \prec a_m$. From the earlier discussion, G is a maximal chain. Consider also another element $x \in \mathcal{C}$ that is not in G. If a_k is the highest member of G that is below x, then there is no other element a_i in G for which $a_k \prec a_i \prec ... \prec x$. Similarly, if a_l is the lowest member of G that is above x, then there is no other element a_j in G for which $x \prec ... \prec a_j \prec a_l$. Then, the spacelike distance $d(x, G)$ between x and G may be defined as $d(x, G) = d(a_k, a_l)/2$.

Overall, then, a measure of spacelike separation between members in $\mathcal{C}$ is recovered by considering the topology of the temporal relations over the causal set, analogous to how lightcone structures may be used in special relativity to determine spatial distances.

With the above definitions of timelike and spacelike distances in place, it is possible to begin a discussion on concepts of velocity [37]. Specifically, such velocities have meaning in terms of the ratios between average spatial distances encountered in given lengths of temporal duration. Since these spatial distances intrinsically involve concepts of geodesics and basic lightcone structure, it is here that embryonic ideas of special relativity are expected to emerge from causal set theory.

Also, once a measure of distance has been introduced into the model, it is possible to discuss concepts of 'volume' and 'area'. To this end, the (hyper)volume of the emergent spacetime may be defined in terms of numbers of events, where a certain quantity of events may specify a certain volume. As with the definition of a distance in terms of path lengths, volume is also seen here simply as a counting process. This is perhaps to be expected, since measurements of distances are in many ways nothing but measurements of the 'size' of a one dimensional volume.

Continuing this logic, the dimension, d, of the causal set may consequently be obtained in a similar way by considering average lengths of path, l, in a given volume, v. It may hence be possible to introduce relational rules of the form $v \sim l^d$, in keeping with ideas of Hausdorff dimension [52]. It is from arguments of this type that the inhomogeneous topology of causal sets may allow different physical dimensions to emerge at different locations and on different physical scales.

Whilst the classical causal set hypothesis summarised above is a promising approach to

the origin of space, a number of unresolved questions, problems and conceptual difficulties arise if it is assumed to provide a complete and consistent description of the observed Universe. These will be expressed in turn.

The first of these questions regards the physical basis behind the model: what actually *are* the postulated events that comprise the causal set? Are they to be taken as some sort of 'pregeometric particle', analogous to the momentum carrying "units" peculiar to spin networks? If this is the case, would it be possible to physically observe them, for example in a futuristic accelerator-detector experiment? Alternatively, if they are simply just mathematical objects, by what process is a physical Universe comprising of fundamental fields and forces expected to emerge?

On a related issue, what exactly *is* the physical mechanism that is responsible for the events' creation? How do these events, be they physical objects or mathematical abstracts, suddenly come into existence? Do they appear from nothing, or are they removed from some sort of giant 'reservoir' of pre-existing events before they are added to the causal set representing the Universe? If this latter supposition is correct then where is this reservoir, what is its physical basis, and what is it like? If not, and instead events just constantly appear at random, then what does this imply for physics in the Universe? Specifically, if the events are hoped to be the correct 'building-blocks' for a Universe that possesses certain physical properties and characteristics, is it to be accepted that, for example, the total energy or momentum of the Universe is increasing as more building blocks are added? If this is not the case, then a paradoxical situation occurs in which principles such as the conservation of energy, which appears fundamental for the Universe as a whole, cannot be held as fundamental for the objects representing the Universe's ultimate description.

Also, is there a physical interpretation for the apparent external time parameter used to govern *when* events are created?

A second problem with the classical causal sets produced is that they are not quantum. This is obviously not ideal if they are to form the ultimate description of a physical Universe that does contain quantum theory as a fundamental ingredient. However, the objection here is not that the model has simply not yet been extended to the case where, for example, the events are quantum variables instead of classical objects. Rather, as it stands the classical causal set description goes against some of the principles present at the very heart of quantum theory.

As an illustration of why this is so, consider three particular events $x, y, z \in \mathcal{C}$ related by $x \prec y \prec z$. This relational statement seems to imply that the events x, y and z each possess an existence independent of each other and of everything else in the causal set. In other words, in classical causal sets every event is granted just as much physical significance

as every other; in $\mathcal{C}$ the events x, y, z all exist to exactly the same extent. Once created an event exists forever, such that for example z is always in the future of y 'for all time', and x is always in the past of y. Indeed, in order for two events to be directly compared as $y \prec z$, it must presumably be accepted that both y and z exist, at least in some sense. The binary comparisons $\prec$ are therefore taken to relate pre-existing relationships between the events in the causet, and as such are assumed to reflect pre-existing attributes of the objects.

Such a viewpoint is manifestly a classical 'Block Universe' approach. This is perhaps why classical causal sets are expected to generate continuous spacetime general relativity so successfully in the emergent limit, because relativity's overall vision is of a Universe existing in a $3+1$ dimensional arena in which the temporal parameter is given an equally 'eternal' dimensional footing as the other three.

However, as suggested in Section 2.2, it may be partly because conventional general relativity relies on Block Universe models of physics that is preventing its unification with quantum theory.

According to the conclusions of the Kochen-Specker theorem, the results of the work of Bell discussed in Chapter 3, and the view of Bohr that the quantum analogues of classical values (such as position and momentum) do not possess any reality independent of observation, then it is the Process time approach that is necessary for a consistent description of quantum mechanics. Assuming, then, that quantum theory is a foundational feature of the Universe, if the event x exists in the past of the event y, and the event z exists in the future of the event y, then x, y and z cannot be given equivalent existence. Only one of these, i.e. the 'present' event, can have any physical existence, and even this does not exist in the sense traditionally assumed by classical physics.

In quantum theory, only the present can be known with any degree of certainty. It is not possible to discuss *the* future, because no such concept physically exists, but only *potential* futures in terms of conditional probabilities. Similarly, the past only has significance in terms of what observers in the present can recall about where they came from.

It is unclear how this conclusion may be incorporated into a classical causal set description of physics relying on the equivalence of the existence of x, y and z across time.

A further criticism of causal sets (from the point of view of quantum theory) comes from an interpretation of what the Hasse diagrams actually represent. In the growth process from a parent Hasse diagram to one of its child causets, a new event may be incorporated that is to the future of two (or more) incomparable events. The problem associated with this is that without any sort of external agent building the causet, how does this event 'know' that it is to the future of these incomparable events, given that no information can be exchanged between them? Without a god-like observer, how are

the temporal relations $\prec$ decided? Since time is a phenomenon expected to emerge from the model, what is the mechanism for deciding how one event is related to another, so that time can indeed emerge? Indeed, given that the whole point of causal set theory is that the events are not embedded in any sort of physical background spacetime, how is any locational memory contained in the dynamics at all? In the diagram labelled C in Figure 5.1, for example, it is not clear how the event created last 'knows' which way it is related to the other two: if the first and second events exist independently of one another, how can they communicate in such a way to 'inform' the third of its temporal position? Is it necessary to postulate some sort of external source of information, so far ignored in classical causal set theory, that stores the location of each of the events?

On a related note, since the addition of a maximal element to a parent causet may give rise to very many possible child sets, how is it that just one of these new causets actually gets selected to form the basis for the single reality experienced by the Universe?

Overall, it is argued that the ideas of classical causal sets provide a good starting point for a discussion of the emergence of space. They do, however, lack a firm physical basis, and it is unclear how they could be 'quantised,' at least directly.

But, as will be shown in the remainder of this chapter, it is possible to naturally generate structures resembling causal sets from the fully quantum description of the Universe proposed in this thesis. Moreover, the 'objects' forming these structures will be shown to have identifiable physical grounding, and so the quantum causal sets introduced in the following are not restricted by the problems inherent in their classical counterparts. Thus, they are ascribed to potentially address the issue of how continuous space and time may emerge in a fully quantum universe.

Further, it will be shown that many of the Hasse diagrams generated in classical causal set theory may also be recreated in the proposed quantum model. However, whilst it may be mathematically possible to produce any configuration of elements in a classical Hasse diagram, it is argued that not all types are permissible in physics. So, in the proposed model only those parts of the Hasse diagrams that are allowed by quantum mechanics, and are hence physically meaningful, are generated.

5.3 Splits and Partitions

In classical causal set theory, continuous spacetime is generated from the relations between collections of classical objects. Since the intention is now to investigate how similar relations might arise from a quantum perspective, an obvious starting point is to examine how the classical objects of the classical theory might have analogues existing as features of the quantum paradigm.

As discussed in Chapter 4, classicity is associated with separability of the quantum state Ψ_n representing the Universe. In a state that is separable relative to a given factorisation of the total Hilbert space, each factor sub-state may be considered classically isolated from every other sub-state, in the sense that a measurement of a factor sub-state contained in one particular factor Hilbert space does not necessarily affect any of the other factor sub-states contained in other factor Hilbert spaces. This is unlike the case of entangled states, because their individual components cannot be measured without destroying the entire state. Therefore, as concluded previously, factor sub-states may consequently be given a form of classical identity.

It is asserted, then, that it is the factors of separable states that may be associated with the classical 'events' of conventional causal set theory, and it is hence from the relations between these factors that physical spacetime might be generated. The alternative, that continuous space instead emerges from considerations of entangled states, is contrary to what would be expected based on empirical observations: entangled states in quantum theory exhibit characteristics of spatial non-locality. Physical space is in many ways a classical construct, as expected from the observation that *this* object is *here*, relative to *that* object *there*.

Thus, the conjecture that will be discussed in the following is how structures analogous to those occurring in classical causal sets might arise naturally by considering the way in which the state of the Universe changes its separability as it develops through a series of stages, i.e. from Ψ_n to Ψ_{n+1} to Ψ_{n+2} ... and so on.

Since classical causal set theory operates in the regime of large numbers of events, and that in the proposed paradigm these classical events are assumed to be analogous to the factors of the overall quantum state, it is expected that quantum causal sets will require highly separable states in order to yield a picture of continuous spacetime. It is therefore necessary to go beyond the simple bi- and tri-partite factorisations of the total Hilbert space discussed previously, and generalise to the case in which large numbers of factors may be present. So, before a full discussion of quantum causal sets can properly begin, it is necessary to introduce a precise notation in order to describe highly separable states in highly factorisable Hilbert spaces.

As before, the situation of interest contains a state Ψ in a Hilbert space $\mathcal{H}$ of finite dimension. In anticipation of a discussion of quantum computation in subsequent chapters of this work, it is alternatively possible to call such a Hilbert space a 'quantum register'. Factor Hilbert sub-spaces of $\mathcal{H}$ may hence be labelled 'subregisters'.

If $\mathcal{H} \equiv \mathcal{H}_{[1...N]}$ is defined as a Hilbert space that may be written as a product of N

subregisters, then $\mathcal{H}_{[1...N]}$ is clearly given by the tensor product

$$\mathcal{H}_{[1...N]} \equiv \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes ... \otimes \mathcal{H}_N \tag{5.2}$$

where $\mathcal{H}_i$ is called the i^{th} factor Hilbert space or quantum subregister, and $1 \leq i \leq N$. By choice, the notation $\mathcal{H}_{[1...N]}$ will generally be used to imply the Hilbert space's fundamental factorisation, that is, each factor Hilbert space $\mathcal{H}_i$ is an 'elementary subregister' of prime dimension, d_i. This choice will henceforth be assumed from now on, unless stated otherwise.

Note that in order for $\mathcal{H}$ to contain the state Ψ_n representing the Universe, its dimension d must be huge. The number of factors N may therefore be in principle very large, with the condition that $d = \prod_{i=1}^{N} d_i$.

As before, it is remarked that the ordering of the factor spaces is not important in the above use of the tensor product symbol. The mathematics is invariant to any rearrangement of the individual subregisters, such that for example $\mathcal{H}_1 \otimes \mathcal{H}_2 \equiv \mathcal{H}_2 \otimes \mathcal{H}_1$ etc. Indeed, if this were not the case the problem would arise that there are no obvious physical criteria for suggesting why some factors spaces should either be placed 'further away' than others from a particular subregister, or be given any special position in the tensor product ordering. In other words, in the factorisable register $\mathcal{H}_{[1..3]} \equiv \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ it is meaningless to say that $\mathcal{H}_1$ is 'nearer' to $\mathcal{H}_2$ than it is to $\mathcal{H}_3$ simply because of the way the tensor product is written; $\mathcal{H}_{[1..3]}$ may equally well be expanded as $\mathcal{H}_{[1..3]} \equiv \mathcal{H}_1 \otimes \mathcal{H}_3 \otimes \mathcal{H}_2$. The factor Hilbert spaces are just vector spaces, and should therefore not be thought of as embedded in any sort of physical background space with any pre-existing distance or locational relationships.

As before, this property is taken to hold for the states in the Hilbert spaces as well, and is an important feature of their non-locality. After all, a state such as Φ in $\mathcal{H}_{[1..3]} \equiv \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ may be separable in the form $\Phi = \phi_2 \otimes \varphi_{13}$, where $\phi_2 \in \mathcal{H}_2$ with $\varphi_{13} \in \mathcal{H}_{[13]} \equiv \mathcal{H}_1 \otimes \mathcal{H}_3$, but φ_{13} might be entangled relative to $\mathcal{H}_{[13]}$. In other words, the entanglements can 'stretch across' factor Hilbert spaces.

In general, factorisable Hilbert spaces $\mathcal{H}_{[1...N]}$ in quantum mechanics may contain states that are completely separable, completely entangled, or a separable product of factors, at least one of which is entangled relative to the fundamental split of the overall Hilbert space into its elementary subregisters. It is therefore convenient to define a notation in order to describe what 'type' of fundamental separation an arbitrary state in $\mathcal{H}_{[1..N]}$ may have.

Consider first a Hilbert space $\mathcal{H}_{[12]}$ factorisable into two subregisters, $\mathcal{H}_{[12]} = \mathcal{H}_1 \otimes \mathcal{H}_2$. By axiom, the overall space $\mathcal{H}_{[12]}$ defines the total set of vectors contained in $\mathcal{H}_1 \otimes \mathcal{H}_2$.

The *separation* $\mathcal{H}_{12}$, then, is defined as the subset of states contained in $\mathcal{H}_{[12]}$ that are separable relative to $\mathcal{H}_{[12]} = \mathcal{H}_1 \otimes \mathcal{H}_2$. That is,

$$\mathcal{H}_{12} \equiv \{|\phi\rangle_1 \otimes |\varphi\rangle_2 : |\phi\rangle_1 \in \mathcal{H}_1 \ , \ |\varphi\rangle_2 \in \mathcal{H}_2\}. \tag{5.3}$$

Note that as mentioned in Chapter 4, the subset $\mathcal{H}_{12}$ is a set of measure zero relative to the set $\mathcal{H}_{[12]}$.

For obvious reasons, $\mathcal{H}_{12}$ may be labelled a 'rank-2' separation, and this definition may be extended in a natural way. Specifically, the rank-k separation $\mathcal{H}_{i_1 i_2 ... i_k}$ is defined as the subset of vectors contained in the Hilbert space $\mathcal{H}_{[i_1 i_2 ... i_k]} = \mathcal{H}_{i_1} \otimes \mathcal{H}_{i_2} \otimes ... \otimes \mathcal{H}_{i_k}$ that are separable into k factors, i.e.

$$\mathcal{H}_{i_1 i_2 ... i_k} \equiv \{|\psi_1\rangle_{i_1} \otimes |\psi_2\rangle_{i_2} \otimes ... \otimes |\psi_k\rangle_{i_k} : |\psi_a\rangle_{i_a} \in \mathcal{H}_{i_a} \ , \ 1 \leq a \leq k\}. \tag{5.4}$$

For convenience, it is also possible to allow the zero vector $\mathbf{0}$ to be a member of $\mathcal{H}_{i_1 i_2 ... i_k}$, because this vector can always be written in the form

$$\mathbf{0} = \mathbf{0}_{i_1} \otimes \mathbf{0}_{i_2} \otimes ... \otimes \mathbf{0}_{i_k}, \tag{5.5}$$

where $\mathbf{0}_{i_a}$ is the zero vector in $\mathcal{H}_{i_a}$. Note, however, that since any vector multiplied by zero is zero, then although $\mathbf{0}$ could at first glance also appear entangled, for example

$$\mathbf{0} = (\chi_{i_1} \otimes \mathbf{0}_{i_2} \otimes ... \otimes \mathbf{0}_{i_k}) + (\mathbf{0}_{i_1} \otimes \chi_{i_2} \otimes ... \otimes \mathbf{0}_{i_k}) + ... \tag{5.6}$$

where $\chi_{i_a} \in \mathcal{H}_{i_a}$, it could always be rewritten in the form

$$\begin{aligned} \mathbf{0} &= (\mathbf{0}_{i_1} \otimes \mathbf{0}_{i_2} \otimes ... \otimes \mathbf{0}_{i_k}) + (\mathbf{0}_{i_1} \otimes \mathbf{0}_{i_2} \otimes ... \otimes \mathbf{0}_{i_k}) + ... \\ &= C(\mathbf{0}_{i_1} \otimes \mathbf{0}_{i_2} \otimes ... \otimes \mathbf{0}_{i_k}) = (\mathbf{0}_{i_1} \otimes \mathbf{0}_{i_2} \otimes ... \otimes \mathbf{0}_{i_k}) \end{aligned} \tag{5.7}$$

where C is a constant, so is in fact separable. In other words, the zero vector never contributes to entanglements in a non-trivial way.

From the above discussion of separations, the convention is adopted from now on that lower indices on Hilbert spaces denote the subset of $\mathcal{H}$ containing separable states, i.e. the separations, whereas lower indices within square brackets on Hilbert spaces denote, as before, the overall set of states, i.e. the tensor product of subregisters.

This leaves free the use of upper indices for a discussion of the *entanglements*, which may be defined in terms of the complements of the separations. For example, in the simplest case in which the Hilbert space $\mathcal{H}_{[12]}$ is factorisable into two subregisters, $\mathcal{H}_{[12]} = \mathcal{H}_1 \otimes \mathcal{H}_2$, the rank-2 entanglement $\mathcal{H}^{12}$ is defined as the subset of vectors in $\mathcal{H}_{[12]}$ that are entangled relative to this split. Moreover, since every state in $\mathcal{H}_{[12]}$ is either separable into two factors or else completely entangled, $\mathcal{H}^{12}$ contains all the states that are not separable, and so may be defined as

$$\mathcal{H}^{12} = \mathcal{H}_{[12]} - \mathcal{H}_{12} \tag{5.8}$$

such that $\mathcal{H}_{[12]} = \mathcal{H}_{12} \cup \mathcal{H}^{12}$. By definition, $\mathcal{H}_{12} \cap \mathcal{H}^{12} = \emptyset$.

Before the above ideas can be generalised to higher order entanglements it is necessary to introduce the concept of *separation product*. Consider a Hilbert space $\mathcal{H}_{[1\ldots N]}$ factorisable in the form $\mathcal{H}_{[1\ldots N]} = \mathcal{H}_a \otimes \mathcal{H}_b$; clearly, if $N > 2$ then this bi-partite factorisation is not fundamental and the dimension of at least one of $\mathcal{H}_a$ or $\mathcal{H}_b$ is not prime. If $\mathcal{H}'_a$ and $\mathcal{H}'_b$ are arbitrary subsets of $\mathcal{H}_a$ and $\mathcal{H}_b$ respectively, then the separation product $\mathcal{H}'_a \bullet \mathcal{H}'_b$ is defined as the subset of states in $\mathcal{H}_{[1\ldots N]}$ that may be written as a product of factors, one of which is contained in $\mathcal{H}'_a$ and the other in $\mathcal{H}'_b$. Thus, $\mathcal{H}'_a \bullet \mathcal{H}'_b$ defines the subset

$$\mathcal{H}'_a \bullet \mathcal{H}'_b \equiv \{|\phi\rangle'_a \otimes |\varphi\rangle'_b : |\phi\rangle'_a \in \mathcal{H}'_a \subseteq \mathcal{H}_a \ , \ |\varphi\rangle'_b \in \mathcal{H}'_b \subseteq \mathcal{H}_b\}. \tag{5.9}$$

It is clear that the separation $\mathcal{H}_{12}$ described previously is just a simple case of this, i.e. $\mathcal{H}_{12} = \mathcal{H}_1 \bullet \mathcal{H}_2$.

The separation product is associative, commutative and cumulative, i.e.

$$\begin{aligned} (\mathcal{H}_i \bullet \mathcal{H}_j) \bullet \mathcal{H}_k &= \mathcal{H}_i \bullet (\mathcal{H}_j \bullet \mathcal{H}_k) \\ (\mathcal{H}_i \bullet \mathcal{H}_j) \bullet \mathcal{H}_k &= (\mathcal{H}_j \bullet \mathcal{H}_i) \bullet \mathcal{H}_k \\ (\mathcal{H}_i \bullet \mathcal{H}_j) \bullet \mathcal{H}_k &= \mathcal{H}_{ij} \bullet \mathcal{H}_k = \mathcal{H}_{ijk}. \end{aligned} \tag{5.10}$$

The definition of separation product can be used to specify subsets of the total Hilbert space that contain a product of factors, one or more of which is entangled. As an example, the separation product $\mathcal{H}^{ij} \bullet \mathcal{H}_k$ labels the subset of $\mathcal{H}_{[ijk]}$ containing states that are a product of two factors, one of which is entangled relative to $\mathcal{H}_i \otimes \mathcal{H}_j$. Specifically,

$$\mathcal{H}^{ij} \bullet \mathcal{H}_k \equiv \{|\phi\rangle \otimes |\varphi\rangle : |\phi\rangle \in \mathcal{H}^{ij} \ , \ |\varphi\rangle \in \mathcal{H}_k\}. \tag{5.11}$$

This definition can be extended to higher orders in the obvious way, such that for example $\mathcal{H}^{ij} \bullet \mathcal{H}_k \bullet \mathcal{H}_l \bullet \mathcal{H}^{mnp}$ identifies the subset of states in $\mathcal{H}_{[ijklmnp]}$ separable as

$$\mathcal{H}^{ij} \bullet \mathcal{H}_k \bullet \mathcal{H}_l \bullet \mathcal{H}^{mnp} \equiv \{|\phi\rangle \otimes |\varphi\rangle \otimes |\chi\rangle \otimes |\psi\rangle\} \tag{5.12}$$

where $|\phi\rangle \in \mathcal{H}^{ij}$, $|\varphi\rangle \in \mathcal{H}_k$, $|\chi\rangle \in \mathcal{H}_l$ and $|\psi\rangle \in \mathcal{H}^{mnp}$. Note that the indices of the entanglements are also commutative, such that for example $\mathcal{H}^{mnp} = \mathcal{H}^{npm}$, as expected from the property of a tensor product Hilbert space that its subregisters are not in any definite or particular order.

Of course, the associativity of the entanglements follows directly from the associativity of the separation product, for example if $\mathcal{H}^{ab} \equiv \mathcal{H}_X$ etc., then

$$(\mathcal{H}^{ab} \bullet \mathcal{H}^{cd}) \bullet \mathcal{H}^{ef} = (\mathcal{H}_X \bullet \mathcal{H}_Y) \bullet \mathcal{H}_Z = \mathcal{H}_X \bullet (\mathcal{H}_Y \bullet \mathcal{H}_Z) = \mathcal{H}^{ab} \bullet (\mathcal{H}^{cd} \bullet \mathcal{H}^{ef}). \tag{5.13}$$

Similar illustrations can be used to demonstrate the commutivity of the entanglements.

To simplify complicated expressions such as $\mathcal{H}^{ij} \bullet \mathcal{H}_k \bullet \mathcal{H}_l \bullet \mathcal{H}^{mnp}$, a single symbol $\mathcal{H}^{ij\bullet mnp}_{k\bullet l}$ may be employed where the use of sub-scripts and super-scripts indicates the

separations and entanglements. This symbol can itself be further simplified by making use of the cumulative property of the separation, i.e. $\mathcal{H}_a \bullet \mathcal{H}_b = \mathcal{H}_{a\bullet b} = \mathcal{H}_{ab}$.

Note, however, that no such cumulativity property directly exists for the entanglements. By way of an illustration of this, consider the observation that a state Φ in an entanglement such as $\mathcal{H}^{abcd}$ cannot, by definition, be separated into a product of entangled states of the form $\theta \otimes \eta$, where θ is in the entanglement $\mathcal{H}^{ab}$ and η is in the entanglement $\mathcal{H}^{cd}$; a vector that is separable into a product of two entangled factors is not the same as a vector that is separable into one giant entangled state. In other words, $\mathcal{H}^{abcd} \neq \mathcal{H}^{ab} \bullet \mathcal{H}^{cd}$ $(= \mathcal{H}^{ab\bullet cd})$, even though $\mathcal{H}_{abcd} = \mathcal{H}_{ab\bullet cd} = \mathcal{H}_{ab} \bullet \mathcal{H}_{cd} = \mathcal{H}_a \bullet \mathcal{H}_b \bullet \mathcal{H}_c \bullet \mathcal{H}_d$.

Overall, separation products such as $\mathcal{H}^{ij} \bullet \mathcal{H}_k \bullet \mathcal{H}_l \bullet \mathcal{H}^{mnp}$ may consequently be written in a number of alternative ways; for example

$$\begin{aligned} \mathcal{H}^{ij} \bullet \mathcal{H}_k \bullet \mathcal{H}_l \bullet \mathcal{H}^{mnp} &= \mathcal{H}^{ij} \bullet \mathcal{H}^{mnp} \bullet \mathcal{H}_l \bullet \mathcal{H}_k = \mathcal{H}^{ji}_{l} \bullet \mathcal{H}^{mpn}_{k} \\ &= \mathcal{H}^{ij}_{k\bullet l} \bullet \mathcal{H}^{mnp} = \mathcal{H}^{ij\bullet mnp}_{k\bullet l} \\ &= \mathcal{H}^{ij\bullet mnp}_{kl} \neq \mathcal{H}^{ijmnp}_{kl}. \end{aligned} \tag{5.14}$$

It is now possible to define rank-k entanglements in terms of their complements. Starting with the rank-3 entanglement in the total Hilbert space $\mathcal{H}_{[123]} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$, it is immediately noted that $\mathcal{H}^{123}$ is not simply given by $\mathcal{H}^{123} = \mathcal{H}_{[123]} - \mathcal{H}_{123}$. Rather the sets of states that are separable into a product of two factors, one of which is entangled, must also be included. Thus, $\mathcal{H}^{123}$ is given by

$$\mathcal{H}^{123} = \mathcal{H}_{[123]} - \mathcal{H}_{123} \cup \mathcal{H}^{23}_1 \cup \mathcal{H}^{13}_2 \cup \mathcal{H}^{12}_3. \tag{5.15}$$

Similarly, the rank-4 entanglement $\mathcal{H}^{1234}$ in $\mathcal{H}_{[1234]}$ is equal to

$$\begin{aligned} \mathcal{H}^{1234} = \mathcal{H}_{[1234]} &- \mathcal{H}_{1234} \cup \mathcal{H}^{34}_{12} \cup \mathcal{H}^{24}_{13} \cup \mathcal{H}^{23}_{14} \cup \mathcal{H}^{14}_{23} \cup \mathcal{H}^{13}_{24} \cup \mathcal{H}^{12}_{34} \\ &\cup \mathcal{H}^{234}_1 \cup \mathcal{H}^{134}_2 \cup \mathcal{H}^{124}_3 \cup \mathcal{H}^{123}_4 \cup \mathcal{H}^{12\bullet 34} \cup \mathcal{H}^{13\bullet 24} \cup \mathcal{H}^{14\bullet 23}. \end{aligned} \tag{5.16}$$

Rank-k entanglements can clearly be defined in similar ways, though their expressions rapidly become more complicated as k increases.

Equations such as (5.15) and (5.16) can be rearranged such that the overall register is decomposed into a union of disjoint separations and entanglements, for example $\mathcal{H}_{[123]} = \mathcal{H}_{123} \cup \mathcal{H}^{23}_1 \cup \mathcal{H}^{13}_2 \cup \mathcal{H}^{12}_3 \cup \mathcal{H}^{123}$. Making use of the language familiar to set theory, such a decomposition of a Hilbert space $\mathcal{H}_{[1\ldots N]}$ may be called its 'lattice of partitions' with each subset being called a 'partition'. In general, each partition is a separation product of separations and entanglements of various ranks, with the condition that the total number of indices equals the overall number of subregisters. In addition each subscript index, and each group of superscript indices, specifies one 'block' of the partition, such that for example the partition $\mathcal{H}^{23\bullet 56}_{14}$ contains four blocks, denoted by '1', '23', '4' and '56'. The

union of all partitions of $\mathcal{H}_{[1...N]}$ may equivalently be called the '(natural) partitioning' of the Hilbert space.

It is important to realise that none of the partitions in $\mathcal{H}_{[1...N]}$ are vector spaces themselves. This conclusion follows from two reasons. Firstly, the zero vector has been defined to be a member of the separation $\mathcal{H}_{1...N}$, so only this partition could potentially be a vector space. Secondly none of the partitions are closed under arbitrary transformations of the vectors they contain. Given a vector X contained in one partition, it is always possible to add a second vector x to X such that the new vector $Y = X + x$ is contained in a different partition. Similarly it is possible to find unitary transformations $\hat{U}$ that 'rotate' X into the vector $Z = \hat{U}X$, where Z is also a member of a different partition from X. Of course, all four vectors x, X, Y and Z are members of $\mathcal{H}_{[1...N]}$, which *is* a vector space.

The use of upper and lower indices on the symbol $\mathcal{H}$ to denote subsets of $\mathcal{H}_{[1...N]}$ containing various separations and entanglements can be extended to the vectors contained within these partitions. For example, the vector $\Phi_{kl}^{ij\bullet mnp}$ is taken to be a member of the partition $\mathcal{H}_{kl}^{ij\bullet mnp}$, and implies that Φ can be separated into four factors of the form

$$\Phi_{kl}^{ij\bullet mnp} = |\phi\rangle^{ij} \otimes |\phi'\rangle_k \otimes |\phi''\rangle_l \otimes |\phi'''\rangle^{mnp} \tag{5.17}$$

where $|\phi\rangle^{ij} \in \mathcal{H}^{ij}$, $|\phi'\rangle_k \in \mathcal{H}_k$, $|\phi''\rangle_l \in \mathcal{H}_l$ and $|\phi'''\rangle^{mnp} \in \mathcal{H}^{mnp}$. Obviously $\Phi_{kl}^{ij\bullet mnp} \in \mathcal{H}_{[i...p]}$ and $\Phi_{kl}^{ij\bullet mnp} \in \mathcal{H}_{kl}^{ij\bullet mnp}$ because $\mathcal{H}_{kl}^{ij\bullet mnp} \subset \mathcal{H}_{[i...p]}$.

Care is needed when applying this notation, however, because sub-scripts used in this thesis, and elsewhere, are often context dependent. For example Ψ_{12} might denote a state in the separation $\mathcal{H}_{12}$, or an arbitrary state in $\mathcal{H}_{[12]}$, or even the state in the twelfth stage Ω_{12} defined as $\Omega_{12} \equiv \Omega(\Psi_{12}, I_{12}, R_{12})$. The same goes for super-scripts, where the notation Φ^{12} might perhaps alternatively label a vector in the entanglement $\mathcal{H}^{12}$, an arbitrary state Φ in a twelve dimensional Hilbert space $\mathcal{H}^{12}_{[1...N]}$, or maybe even one out of E possible eigenvectors Φ^a of some Hermitian operator $\hat{O}$ for $1 \leq a \leq E$ where $E \geq 12$.

For any given vector $\Psi \in \mathcal{H}_{[1...N]}$, it is possible to determine which partition it is in by a repeated application of the microsingularity test (4.9) introduced in Section 4.1. For example, to show that a state $\Theta \in \mathcal{H}_{[1...3]}$ is completely entangled, i.e. can be written in the form Θ^{123} in the partition $\mathcal{H}^{123}$, it must be confirmed that Θ is not in $\mathcal{H}_{123}$, $\mathcal{H}_1^{23}$, $\mathcal{H}_2^{13}$ or $\mathcal{H}_3^{12}$. This is turn can be proved by demonstrating that Θ is not separable relative to any of the three bi-partite factorisations of the total Hilbert space, i.e. $\mathcal{H}_1 \otimes \mathcal{H}_{[23]}$, $\mathcal{H}_2 \otimes \mathcal{H}_{[13]}$ and $\mathcal{H}_3 \otimes \mathcal{H}_{[12]}$, because if this is true it also immediately follows that Θ is not separable relative to the tri-partite factorisation $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$.

From earlier discussions, any state $\Theta \in \mathcal{H}_{[1...3]}$ can be expanded in the form

$$\Theta = \sum\nolimits_{i=0}^{d_1-1} \sum\nolimits_{j=0}^{d_2-1} \sum\nolimits_{k=0}^{d_3-1} C_{ijk} |i\rangle_1 \otimes |j\rangle_2 \otimes |k\rangle_3 \tag{5.18}$$

where the Hilbert space $\mathcal{H}_a$ of dimension d_a is spanned by the orthonormal basis $\mathcal{B}_a = \{|b\rangle_a : 0 \leq b \leq d_a - 1\}$ for $a = 1, 2, 3$ and the $C_{ijk} \in \mathbb{C}$ are complex coefficients. The microsingularity condition can now be used to determine the separability of Θ relative to each of the three bi-partite factorisations of $\mathcal{H}_{[1...3]}$.

For example, to confirm whether Θ is separable relative to $\mathcal{H}_1 \otimes \mathcal{H}_{[23]}$, equation (5.18) should be rewritten as

$$\Theta = \sum\nolimits_{i=0}^{d_1-1} \sum\nolimits_{x=0}^{(d_2 d_3)-1} K_{ix} |i\rangle_1 \otimes |x\rangle_{23} \tag{5.19}$$

where $\mathcal{B}_{23} = \{|x\rangle_{23} : 0 \leq x \leq (d_2 d_3 - 1)\}$ forms an orthonormal basis for $\mathcal{H}_{[23]}$, with for example $|0\rangle_{23} = |0\rangle_2 \otimes |0\rangle_3$, $|1\rangle_{23} = |0\rangle_2 \otimes |1\rangle_3$, ..., up to $|d_2 d_3 - 1\rangle_{23} = |d_2 - 1\rangle_2 \otimes |d_3 - 1\rangle_3$. The general term $|x\rangle_{23}$ in this basis may be given by $|x\rangle_{23} = |j\rangle_2 \otimes |k\rangle_3$ when j is the integer part of the quotient $x/(d_3)$ and k is the remainder. The coefficients K_{ix} are obtained from C_{ijk} in the same way, such that for example K_{i1} is equivalent to C_{i01}.

Now, if $K_{ix} K_{yz} = K_{iz} K_{yx}$ for all $0 \leq i, y \leq (d_1 - 1)$ and $0 \leq x, z \leq (d_2 d_3 - 1)$, then Ψ is separable relative to $\mathcal{H}_1 \otimes \mathcal{H}_{[23]}$. If this is not the case, Θ is entangled relative to $\mathcal{H}_1 \otimes \mathcal{H}_{[23]}$, and if the same method shows that Θ is also entangled relative to $\mathcal{H}_2 \otimes \mathcal{H}_{[13]}$ and $\mathcal{H}_3 \otimes \mathcal{H}_{[12]}$, it can be concluded that $\Theta \in \mathcal{H}^{123}$.

Similar procedures can be employed to determine which particular partition of the lattice of $\mathcal{H}_{[1...N]}$ any given vector $\Psi \in \mathcal{H}_{[1...N]}$ is in, though the corresponding number of microsingularity tests that need to be performed increases greatly with N.

The state $\Phi_{kl}^{ij \bullet mnp} \in \mathcal{H}_{kl}^{ij \bullet mnp}$ in (5.17) is an example of a vector that is a separable product of factors, two of which are entangled relative to the fundamental splitting of the overall Hilbert space $\mathcal{H}_{[i...p]}$ into its seven subregisters. In general, however, if an arbitrary state contained within a Hilbert space $\mathcal{H}_{[1...N]}$ is chosen at random, there are very many ways in which it might potentially be separated into F factors, where $1 \leq F \leq N$, because there are in general many different partitions comprising of F blocks. For example, the state $\Theta_l^{ip \bullet kn \bullet jm}$ in $\mathcal{H}_{[i...p]}$ is also separated into four factors, but in a completely different manner from $\Phi_{kl}^{ij \bullet mnp}$.

Of course, if $F = 1$ then the state is completely entangled, whereas if $F = N$ it is completely separable, but for all other values of F the state is separated into a product of factors, at least one of which is entangled. Further, the number of ways in which an arbitrary state may potentially be separated into F factors increases rapidly as the number, N, of subregisters in the fundamental factorisation of the Hilbert space increases.

For example, in a Hilbert space $\mathcal{H}_{[1]}$ of prime dimension, which is therefore fundamentally split into just one subregister, every state can obviously only be separated into one factor. States in a Hilbert space $\mathcal{H}_{[12]}$ that is fundamentally split into two subregisters

$\mathcal{H}_{[12]} = \mathcal{H}_1 \otimes \mathcal{H}_2$, however, are either entangled relative to this split, or else they are separable into two factors; it can be said that there are two possible 'types' of state separations in $\mathcal{H}_{[12]}$.

In a Hilbert space $\mathcal{H}_{[123]}$ fundamentally split into three subregisters $\mathcal{H}_{[123]} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$, though, a state is either completely entangled relative to this split, or it is completely separable into three factors, or else it is separable into one of the forms $\alpha_1 \otimes \beta_{23}$, $\lambda_2 \otimes \mu_{13}$ or $\phi_3 \otimes \varphi_{12}$, where $\alpha_1 \in \mathcal{H}_1$, $\lambda_2 \in \mathcal{H}_2$ and $\phi_3 \in \mathcal{H}_3$, with β_{23}, μ_{13} and φ_{12} being sub-states that are entangled relative to $\mathcal{H}_2 \otimes \mathcal{H}_3$, $\mathcal{H}_1 \otimes \mathcal{H}_3$ and $\mathcal{H}_1 \otimes \mathcal{H}_2$ respectively. Given an arbitrary state in $\mathcal{H}_{[123]}$, there are clearly five different types of way in which it might be separable relative to $\mathcal{H}_{[123]}$: one of these types will have one factor, three types will have two factors, and one will have three factors. Equivalently, every state in $\mathcal{H}_{[123]}$ is in one of the five partitions that comprise the partitioning of the total Hilbert space, $\mathcal{H}_{[123]} = \mathcal{H}_{123} \cup \mathcal{H}_1^{23} \cup \mathcal{H}_2^{13} \cup \mathcal{H}_3^{12} \cup \mathcal{H}^{123}$.

In fact, it can be shown that in Hilbert spaces $\mathcal{H}_{[1\ldots4]}$ fundamentally split into four subregisters, there are 15 different types of way in which a given state might possibly be separated, whereas Hilbert spaces of the form $\mathcal{H}_{[1\ldots5]}$ allow the possibility of 52 different types of separation. This number grows to 203 for $\mathcal{H}_{[1\ldots6]}$.

Generally, if h_N is defined as the number of ways in which an arbitrary state in $\mathcal{H}_{[1\ldots N]}$ might possibly be separated, then this number is given by the iterative formula

$$h_N = \sum_{i=0}^{N-1} C_i^{N-1} h_{(N-1)-i} \tag{5.20}$$

where C_b^a is the combination function, $C_b^a = a!/[(a-b)!b!]$, and the initial condition $h_0 = 1$ follows from the assumption that there is only one way of separating a state contained in zero Hilbert spaces[6]. The above relation also specifies the number, h_N, of partitions comprising the lattice of $\mathcal{H}_{[1\ldots N]}$, as expected from the fact that every state in $\mathcal{H}_{[1\ldots N]}$ is in exactly one of the Hilbert space's partitions, and that it is always possible to find an example of a state in $\mathcal{H}_{[1\ldots N]}$ that is a member of a given partition.

Equation (5.20) effectively generates the list of Bell numbers used in combinatorics to number the set of partitions of a set of size N, and is equivalently given by Dobinski's formula (see [53] for an illustration of these points).

An intuitive proof of (5.20) is given from the following. Consider a factorisable Hilbert space $\mathcal{H}_{[1\ldots N]}$. Every state in $\mathcal{H}_{[1\ldots N]}$ will be associated with its own fundamental separation, i.e. a way or writing the state into the maximum possible number of factors relative to $\mathcal{H}_{[1\ldots N]}$, because each state is in one, and only one, partition of $\mathcal{H}_{[1\ldots N]}$.

[6]Compare the generally accepted result $0! = 1$. If this argument appears *ad hoc,* h_N may equally be defined as $h_N = 1 + \sum_{i=0}^{N-2} C_i^{N-1} h_{(N-1)-i}$ without loss of generality.

Now assume that every state in $\mathcal{H}_{[1...N]}$ is fundamentally separable into one of h_N possible types, where h_N is not yet known and the relation (5.20) is not assumed. For example, it was shown earlier that every vector in $\mathcal{H}_{[1...3]}$ is fundamentally separable into one out of five possible types.

Clearly, this number h_N of possible types is given by the sum of the number of ways that vectors in $\mathcal{H}_{[1...N]}$ might be separable into just one factor, plus the number of ways that vectors might be separable into just two factors, plus the number of ways that vectors might be separable into just three factors, plus..., plus the number of ways that vectors might be separable into just N factors. Thus, h_N is also the total number of possible partitions in the lattice of $\mathcal{H}_{[1...N]}$, or equivalently the total number of types of vector that exist in $\mathcal{H}_{[1...N]}$.

Of course, there is only one type of way in which vectors in $\mathcal{H}_{[1...N]}$ may be fundamentally separated into one factor, and only one type of way in which vectors in $\mathcal{H}_{[1...N]}$ may be fundamentally separated into N factors.

Because $\mathcal{H}_{[1...N]}$ is of fixed dimension, every vector it contains must have a component in every subregister $\mathcal{H}_i$ of $\mathcal{H}_{[1...N]}$, for $1 \leq i \leq N$. Therefore, every vector in $\mathcal{H}_{[1...N]}$ must consequently have a component in the subregister $\mathcal{H}_1$, and this component will be in one of the F factors of the overall state (whatever F may be). Further, whichever sub-state of the overall product it is in, the component in $\mathcal{H}_1$ will either be in a factor of the state on its own, or entangled with a component from just one other subregister, or entangled with the components from two other subregisters, or..., or entangled with the components from each of the $N-1$ other subregisters (in which case $F = 1$).

In other words, a given state Φ in $\mathcal{H}_{[1...N]}$ might be fundamentally separable as

$$\Phi = X_1 \otimes Y_{[2...N]} \tag{5.21}$$

or

$$\Phi = X^{1i} \otimes Y_{[2...(i-1)(i+1)...N]} \tag{5.22}$$

or

$$\Phi = X^{1ij} \otimes Y_{[2...(i-1)(i+1)...(j-1)(j+1)...N]} \tag{5.23}$$

or... etc., for $1 < i, j, k, ... \leq N$ and $i \neq j \neq k \neq ...$. Here $X_1 \in \mathcal{H}_1$, but $Y_{[2...N]} \in \mathcal{H}_{[2...N]}$ is any vector (completely entangled, completely separable, or a separable product of entangled factors) in $\mathcal{H}_{[2...N]}$. Similarly $X^{1i} \in \mathcal{H}^{1i}$ and $X^{1ij} \in \mathcal{H}^{1ij}$, but $Y_{[2...(i-1)(i+1)...N]} \in \mathcal{H}_{[2...(i-1)(i+1)...N]}$ and $Y_{[2...(i-1)(i+1)...(j-1)(j+1)...N]} \in \mathcal{H}_{[2...(i-1)(i+1)...(j-1)(j+1)...N]}$ are arbitrary vectors that also may or may not be separable.

The summation proceeds as follows.

If the component in $\mathcal{H}_1$ of a state is in a factor sub-state on its own, i.e. is not entangled with anything, there are $(N-1)$ components of the state left 'free', corresponding to the

remaining $(N-1)$ subregisters $\mathcal{H}_2$, $\mathcal{H}_3$, ..., $\mathcal{H}_N$. This remaining part of the state is a vector in $\mathcal{H}_{[2...N]}$, and so by assumption this may be separated into one of h_{N-1} different ways. So, there are h_{N-1} different ways in which states in $\mathcal{H}_{[1...N]}$ might be separated in the form $X_1 \otimes Y_{[2...N]}$.

Now, there are precisely $C_1^{N-1} = (N-1)$ ways of selecting just one component of a vector in $\mathcal{H}_{[2...N]}$, i.e. $(N-1)$ ways of choosing just one of the components in one of the remaining $(N-1)$ factor spaces $\mathcal{H}_i$ for $2 \leq i \leq N$. There are hence $(N-1)$ different types of factor of the form X^{1i} for states in $\mathcal{H}_{[1...N]}$, such that the component in $\mathcal{H}_1$ is entangled with the component in $\mathcal{H}_i$. Further, each of these ways leaves a remaining vector in $\mathcal{H}_{[2...(i-1)(i+1)...N]}$, with $(N-2)$ 'free' components, and this vector may itself be separated into one of h_{N-2} different ways. So, overall there are $(N-1)h_{N-2}$ different ways of separating states in $\mathcal{H}_{[1...N]}$ in the form $X^{1i} \otimes Y_{[2...(i-1)(i+1)...N]}$.

Continuing, there are C_2^{N-1} ways of selecting two components of a vector in $\mathcal{H}_{[2...N]}$, such that one is in the subregister $\mathcal{H}_i$ and the other is in the subregister $\mathcal{H}_j$, for $2 \leq i, j \leq N$ and $i \neq j$. There are hence C_2^{N-1} different types of factor of the form X^{1ij} for states in $\mathcal{H}_{[1...N]}$, such that the component in $\mathcal{H}_1$ is entangled with just two of the other components. This leaves a remaining vector in $\mathcal{H}_{[2...(i-1)(i+1)...(j-1)(j+1)...N]}$, which has $(N-3)$ free components, and this could be separable in one of h_{N-3} different ways. So overall there are $C_2^{N-1}h_{N-3}$ different ways of separating states in $\mathcal{H}_{[1...N]}$ in the form $X^{1ij} \otimes Y_{[2...(i-1)(i+1)...(j-1)(j+1)...N]}$.

This analysis can be continued. In general, there are C_x^{N-1} ways of selecting x components of a vector in $\mathcal{H}_{[2...N]}$, such that the a_b^{th} component is in the a_b^{th} subregister $\mathcal{H}_{a_b}$, for $0 \leq x \leq (N-1)$, whilst $2 \leq a_b \leq N$ and $b = 1, 2, ..., x$, with, of course, no two components being in the same subregister. There are hence C_x^{N-1} different types of factor of the form $X^{1a_1a_2...a_x}$, such that the component in $\mathcal{H}_1$ is entangled with x of the other components. This leaves a remaining vector which has $(N-1-x)$ 'free' components, and this vector will be separable in one of h_{N-1-x} different ways. So, there are $C_x^{N-1}h_{N-1-x}$ different ways of separating states in $\mathcal{H}_{[1...N]}$ into a product of factors, one of which is $X^{1a_1a_2...a_x}$.

Overall, the total number h_N of ways in which arbitrary vectors in $\mathcal{H}_{[1...N]}$ might potentially be separated into a product of factors is given by the exhaustive sum of the number of ways in which vectors in $\mathcal{H}_{[1...N]}$ might be separated such that their component in $\mathcal{H}_1$ is in a factor sub-state on its own, added to the number of ways in which vectors in $\mathcal{H}_{[1...N]}$ might be separated such that there is a factor containing the component in $\mathcal{H}_1$entangled with a component from one other subregister, added to the number of ways in which vectors in $\mathcal{H}_{[1...N]}$ might be separated such that there is a factor containing the component in $\mathcal{H}_1$entangled with the components from two other subregisters, and so on to the addition of the number of ways in which vectors in $\mathcal{H}_{[1...N]}$ might be separated such

that there is a factor containing the component in $\mathcal{H}_1$entangled with the components from every other subregister.

From the above, this gives

$$h_N = h_{N-1} + (N-1)h_{N-2} + C_2^{N-1}h_{N-3} + ... + C_x^{N-1}h_{N-1-x} + ... + C_{N-1}^{N-1}h_{N-N} \quad (5.24)$$

where the last term is equal to unity because there is only one way of separating a state into one entangled factor.

Clearly, then, it follows that h_N is given by (5.20).

As discussed earlier, the first few values for h_N are $h_0 = 1$, $h_1 = 1$, $h_2 = 2$, $h_3 = 5$, $h_4 = 15$, $h_5 = 52$, $h_6 = 203$, such that h_N evidently grows quickly for even relatively low values of N. Indeed, note that even for a five qubit system, the number of ways its state may be separable is greater than the dimension of its Hilbert space: a five qubit Hilbert space $\mathcal{H}_{[1...5]}^{(32)}$ of 32 dimensions contains $h_5 = 52$ partitions.

For the case in which N is of the order 10^{184}, the value of h_N is expected to be truly enormous. So, for a Universe represented by a state of dimension greater than $2^{10^{184}}$, the number of partitions contained in the lattice of its Hilbert space $\mathcal{H}_{[1...10^{184}]}$ is clearly very large. This should consequently provide an incredibly rich structure, with a tremendous number of different ways in which the state of the Universe might potentially separate.

As will be shown, this provides a wide scope for the Universe's dynamics.

5.4 Probability Amplitudes and Quantum Causal Sets

Now that a notation has been introduced to cope with large dimensional Hilbert spaces, it is possible to examine how a causal set structure might arise from a fully quantum description of physics.

From the discussion that the Universe may always be represented by a state Ψ_n in a Hilbert space $\mathcal{H}$ of enormous dimension, and from the outcome of the previous section that any vector in a given Hilbert space is always contained within one, and only one, of the partitions of this space, the conclusion must be that the Universe's state is always in one of the partitions of $\mathcal{H}$. The state Ψ_n is separable in a specific way, and is always a product of between 1 and N factors, where N is the number of subregisters in the fundamental factorisation of $\mathcal{H}$.

As the wavefunction of the Universe develops from one state Ψ_n to the next Ψ_{n+1}, its pattern of separability might change. That is, if the state Ψ_n may be fundamentally separated into a product of F_n factors, $1 \leq F_n \leq N$, the state Ψ_{n+1} may be separable into F_{n+1} factors, where F_n is not necessarily equal to F_{n+1}. In fact, even if $F_n = F_{n+1}$ the

states Ψ_n and Ψ_{n+1} may have completely different patterns of separability, since there are in general many different partitions comprising of F blocks. It is this changing pattern of separability that will be shown to be the origin of family structures in the quantum universe, and hence the beginning of a discussion of quantum causal sets.

As conjectured at the start of Section 5.3, the individual factors of the state of the Universe may be analogous to the events of classical causal set theory. For example, the growth of the events in classical causets satisfies 'internal temporality', in the sense that every new event is born either to the future of, or unrelated to, every other event; no event is created to the past of already existing events. The same is true in the present model, because the next potential state Ψ_{n+1} is an outcome (i.e. one of the eigenvectors) of a test on the 'current' state Ψ_n, and so any factor of Ψ_{n+1} cannot in any way be thought of as in the past of any of the factors of Ψ_n.

It is important to reiterate, however, that the quantum and classical models are not completely congruent. For example, as has been discussed previously the relation $x \prec y \prec z$ between three classical events has no *direct* equivalent in the quantum theory. After all, consider three consecutive states Ψ_{n-1}, Ψ_n and Ψ_{n+1}: whilst Ψ_{n+1} may indeed be one of the possible outcomes of a test on Ψ_n (which is itself one of the outcomes of a test on Ψ_{n-1}), the successive states Ψ_{n-1}, Ψ_n and Ψ_{n+1} cannot be granted equivalent degrees of existence according to the Kochen-Specker theorem, and so cannot be directly compared. Compared to the current state Ψ_n only *potential* future states Ψ_{n+1} can be discussed.

One similarity that does still occur between the classical and quantum cases is the notion of 'links', which are defined as being irreducible relations. In the classical theory described in Section 5.2, for example, two events x and y are linked if $x \prec y$ and there is no other event z such that $x \prec z \prec y$, or if $y \prec x$ and there is no other event z' such that $y \prec z' \prec x$. Analogously, in the proposed quantum scenario the states Ψ_{n-1} and Ψ_n could immediately be described as 'linked', because by definition there is no intermediate state between them.

A further similarity arises from the classical causal set concepts of families: related notions are also present in the quantum case, based, in fact, on the factorisability of the probability amplitude. To demonstrate this, consider the inner product $\langle \Psi_{n+1} = \Phi | \Psi_n \rangle$ between the current state Ψ_n and one of the next potential states $\Psi_{n+1} = \Phi$, where Φ is one of the eigenvectors of some operator $\hat{\Sigma}_{n+1}$. The states Ψ_n and Ψ_{n+1} are each contained within particular partitions of the total Hilbert space $\mathcal{H}_{[1\ldots N]}$, where as before $\mathcal{H}_{[1\ldots N]}$ is assumed factorisable into N subregisters. Now, because the factors of one state can only take inner products with factors of another state if they lie in the same factor space of some split of the total Hilbert space (where these factor spaces are not necessarily elementary),

then, depending on the details of the partitions containing Ψ_n and Ψ_{n+1}, the probability amplitude may be separable into a number, r, of factors.

Paraphrasing, if $\psi \in \mathcal{H}_{[\psi]}$ is a factor of $\Psi_n \in \mathcal{H}_{[1...N]}$, where $\mathcal{H}_{[\psi]}$ is one of the factors of some split of $\mathcal{H}_{[1...N]}$ and need not be of prime dimension, and if $\phi \in \mathcal{H}_{[\phi]}$ is a factor of Ψ_{n+1}, where $\mathcal{H}_{[\phi]}$ is one of the factors of some split of $\mathcal{H}_{[1...N]}$ and also need not be of prime dimension, then $\langle\phi|\psi\rangle$ will contribute a factor to the overall probability amplitude $\langle\Psi_{n+1}|\Psi_n\rangle$ iff $\mathcal{H}_{[\psi]} = \mathcal{H}_{[\phi]}$.

This leads to the definition of a '*family*': ψ and ϕ, in successive states Ψ_n and Ψ_{n+1} respectively, constitute a family if $\langle\phi|\psi\rangle$ is a factor of $\langle\Psi_{n+1}|\Psi_n\rangle$ and if $\langle\phi|\psi\rangle$ cannot itself be factorised further.

The above observation can be generalised, such that ψ might be a product of A factors, $\psi = \psi_1 \otimes \psi_2 \otimes ... \otimes \psi_A$, and ϕ might be a product of B factors, $\phi = \phi_1 \otimes \phi_2 \otimes ... \otimes \phi_B$, where A is not necessarily equal to B. In this case, the definition of the family encompasses the factors of which ψ and ϕ are a product.

Suppose now that the state of the Universe $\Psi_n \in \mathcal{H}_{[1...N]}$ is separable into k factors, i.e. $\Psi_n = \psi_{a_1} \otimes \psi_{a_2} \otimes ... \otimes \psi_{a_k}$, where the individual factors ψ_{a_i}, for $1 \leq i \leq k$, may, or may not, themselves be entangled relative to the fundamental factorisation of $\mathcal{H}_{[1...N]}$. Each factor ψ_{a_i} is in its own factor Hilbert space $\mathcal{H}_{[a_i]}$, and this may itself be a product of elementary subregisters with the condition that $\mathcal{H}_{[1...N]} = \mathcal{H}_{[a_1]} \otimes \mathcal{H}_{[a_2]} \otimes ... \otimes \mathcal{H}_{[a_k]}$. For simplicity, it may also be assumed that the factor sub-states are normalised within their own factor Hilbert spaces, i.e. $\langle\psi_{a_i}|\psi_{a_i}\rangle = 1$.

Consider now the next test of the Universe, $\hat{\Sigma}_{n+1}$. This test has d orthonormal eigenvectors, where d is the dimension of the Hilbert space $\mathcal{H}_{[1....N]}$. Of course, if each elementary subregister of $\mathcal{H}_{[1...N]}$ is a qubit sub-space then clearly $d = 2^N$. If Φ is one of these d eigenvectors, then the conditional probability $P(\Psi_{n+1} = \Phi|\Psi_n, \hat{\Sigma}_{n+1})$ that the next state Ψ_{n+1} of the Universe is Φ, given a test $\hat{\Sigma}_{n+1}$, is given by the usual Born probability rule $|\langle\Phi|\Psi_n\rangle|^2$.

Suppose that Φ is separable into l factors, i.e. $\Phi = \phi_{b_1} \otimes \phi_{b_2} \otimes ... \otimes \phi_{b_l}$, each of which is also contained in its own factor Hilbert space $\mathcal{H}_{[b_j]}$, for $1 \leq j \leq l$, with $\prod_{j=1}^{l} \otimes \mathcal{H}_{[b_j]} = \mathcal{H}_{[1...N]}$ and $\langle\phi_{b_j}|\phi_{b_j}\rangle = 1$. Now, depending on the particular partitions of $\mathcal{H}_{[1...N]}$ in which Ψ_n and Ψ_{n+1} are members, that is, depending on how the various Hilbert spaces $\mathcal{H}_{[a_i]}$ and $\mathcal{H}_{[b_j]}$ 'overlap' with one another, the probability amplitude $\langle\Phi|\Psi_n\rangle$ may be separable into a product of factors. In other words,

$$P(\Psi_{n+1} = \Phi|\Psi_n, \hat{\Sigma}_{n+1}) = P_1 P_2 ... P_r \tag{5.25}$$

where the overall probability is factorisable into r factors P_s, for $1 \leq s \leq r$, and each factor can be interpreted as a conditional transition probability within a particular family.

Assuming that (5.25) represents the 'fundamental factorisation' of the probability $P(\Psi_{n+1} = \Phi|\Psi_n, \hat{\Sigma}_{n+1})$, then r represents the maximum number of factors associated with the transition amplitude, and is constrained by $r \leq \min(k, l)$. Thus, in this case there are r families involved in the transition of the state from Ψ_n to $\Psi_{n+1} = \Phi$. Further, because each factor of the transition amplitude involves a distinct portion of the overall set of quantum subregisters comprising the total Hilbert space, the complete set of factors P_s specifies a particular r-partite split of $\mathcal{H}_{[1...N]}$.

Summarising, then, leads to the following definition. For the quantum transition from the state Ψ_n to a potential state Ψ_{n+1}, both of which are vectors in a Hilbert space $\mathcal{H}_{[1...N]}$ factorisable into at least two subregisters, $N \geq 2$, the number of families involved is defined as equal to the number of factors in the fundamental factorisation of the probability amplitude $\langle\Psi_{n+1}|\Psi_n\rangle$, as determined from which particular partitions of $\mathcal{H}_{[1...N]}$ the states Ψ_n and Ψ_{n+1} are in.

Analogous to classical causal sets, once a family has been identified it is possible to define concepts such as *parents, children* and *siblings*. Specifically, in a given family transition $\langle\phi_{b_j}|\psi_{a_i}\rangle$, where ψ_{a_i} is a factor of Ψ_n and ϕ_{b_j} is a factor of Ψ_{n+1}, the sub-state ψ_{a_i} may be called the 'parent' of ϕ_{b_j}, which is in turn its 'child'. Further, if ψ_{a_i} is itself a product of X factors, $\psi_{a_i} = \alpha_{a_{i\_1}} \otimes \alpha_{a_{i\_2}} \otimes ... \otimes \alpha_{a_{1\_X}}$, and if ϕ_{b_j} is a product of Y factors $\phi_{b_j} = \beta_{b_{j\_1}} \otimes \beta_{b_{j\_2}} \otimes ... \otimes \beta_{b_{j\_Y}}$, then each factor $\alpha_{a_{1\_x}}$ for $1 \leq x \leq X$ is a parent of each factor $\beta_{b_{j\_y}}$ for $1 \leq y \leq Y$, which are its children or 'offspring'. Similarly, every factor $\beta_{b_{j\_y}}$ is a sibling of every other factor $\beta_{b_{j\_z}}$, for $1 \leq y, z \leq Y$ and $y \neq z$, because they share a parent. Note, however, that the fact that the β's are siblings does not imply that the α's must also be siblings. Which, if any, of the factors of ψ_{a_i} are siblings of each other depends entirely on the factorisation of the transition amplitude $\langle\Psi_n|\Psi_{n-1}\rangle$, as will be discussed shortly.

First, however, consider as an illustration of these ideas a Hilbert space $\mathcal{H}_{[1...8]}$ factorisable into eight elementary subregisters, $\mathcal{H}_{[1...8]} = \mathcal{H}_1 \otimes ... \otimes \mathcal{H}_8$. Consider also the 'current' state $\Psi_n \in \mathcal{H}_{[1...8]}$ and one of the potential next states Ψ_{n+1}, defined as $\Psi_n = \Theta_{123}^{456\bullet 78}$ and $\Psi_{n+1} = \Phi_{145}^{23\bullet 678}$ respectively. Clearly, Ψ_n is in the partition $\mathcal{H}_{123}^{456\bullet 78}$, whereas $\Psi_{n+1} \in \mathcal{H}_{145}^{23\bullet 678}$, and the states may be written in the forms

$$\begin{aligned} |\Psi_n\rangle &= \Theta_{123}^{456\bullet 78} = |\Theta_1\rangle \otimes |\Theta_2\rangle \otimes |\Theta_3\rangle \otimes |\Theta^{456}\rangle \otimes |\Theta^{78}\rangle \\ |\Psi_{n+1}\rangle &= \Phi_{145}^{23\bullet 678} = |\Phi_1\rangle \otimes |\Phi^{23}\rangle \otimes |\Phi_4\rangle \otimes |\Phi_5\rangle \otimes |\Phi^{678}\rangle \end{aligned} \tag{5.26}$$

in obvious notation. Note that here, and in the following, the usual Hilbert space subscripts on the ket factors have been omitted to avoid potentially confusing clashes in the products

of amplitudes; specifically, then, it is implicitly assumed that

$$|\Theta^{456}\rangle \equiv |\Theta^{456}\rangle_{456} \in \mathcal{H}_{[456]} \tag{5.27}$$

and $|\Phi_1\rangle \in \mathcal{H}_1$, etc.

Therefore, the transition amplitude $\langle\Psi_{n+1}|\Psi_n\rangle$ takes the form $\langle\Phi_{145}^{23\bullet 678}|\Theta_{123}^{456\bullet 78}\rangle$, which is fundamentally factorised as

$$\begin{aligned}\langle\Phi_{145}^{23\bullet 678}|\Theta_{123}^{456\bullet 78}\rangle &= \langle\Phi_1|\Theta_1\rangle\langle\Phi^{23}|\Theta_{23}\rangle\langle\Phi_{45}^{678}|\Theta^{456\bullet 78}\rangle \\ &= \langle\Phi_1|\Theta_1\rangle\ \langle\Phi^{23}|(|\Theta_2\rangle\otimes|\Theta_3\rangle) \\ &\quad\times(\langle\Phi_4|\otimes\langle\Phi_5|\otimes\langle\Phi^{678}|)(|\Theta^{456}\rangle\otimes|\Theta^{78}\rangle).\end{aligned} \tag{5.28}$$

So, the probability $P = |\langle\Psi_{n+1} = \Phi_{145}^{23\bullet 678}|\Psi_n = \Theta_{123}^{456\bullet 78}\rangle|^2$ can be fundamentally factorised in the form $P = P_1P_2P_3$ where $P_1 = |\langle\Phi_1|\Theta_1\rangle|^2$, $P_2 = \left|\langle\Phi^{23}|\Theta_{23}\rangle\right|^2$ and $P_3 = \left|\langle\Phi_{45}^{678}|\Theta^{456\bullet 78}\rangle\right|^2$. Further, it is evident that $\Psi_n = \Theta_{123}^{456\bullet 78}$ has $k = 5$ factors, $\Psi_{n+1} = \Phi_{145}^{23\bullet 678}$ has $l = 5$ factors, and the probability P has $r = 3$ factors, which clearly satisfies the relation $r \leq \min(k, l)$.

Moreover, in this transition, Θ_1 is the (single) parent of Φ_1, which has no siblings. The factor Φ^{23} also has no siblings, and is the child of its parents, namely Θ_2 and Θ_3. Lastly, the factors Θ^{456} and Θ^{78} are the parents of Φ_4, Φ_5, and Φ^{678}, which are siblings of one another.

Just as the sets of events generated in classical causal sets can be depicted by Hasse diagrams, so too can the family structures produced by the quantum transitions also be represented pictorially. The convention adopted is that every possible factor state present in a transition amplitude is drawn as a large circle, whilst each factor of the relevant r-partite split of the Hilbert space is denoted by a small circle. These two types of circle are labelled in the obvious way, with, for example, a small circle labelled as $[isy...b]$ denoting the factor $\mathcal{H}_{[isy...b]}$ of the total Hilbert space.

The 'time' parameter, n, is assigned to run upwards in the diagrams, such that the large circles representing the ket vectors of the transition amplitude are below the large circles that represent the bra vectors. In addition, the large circles are linked to the small circles in a way that depends on how their factor states are contained in the factor Hilbert spaces of the r-partite split. Specifically, with links drawn as arrows the convention becomes that those arrows pointing towards the bottom of a small circle run from a set of parent factor states, whilst those arrows coming from the top of this small circle point to their corresponding set of children.

With these conventions adopted, the transition from the state $\Psi_n = \Theta_{123}^{456\bullet 78}$ to the state $\Psi_{n+1} = \Phi_{145}^{23\bullet 678}$ can be depicted by the diagram shown in Figure 5.2.

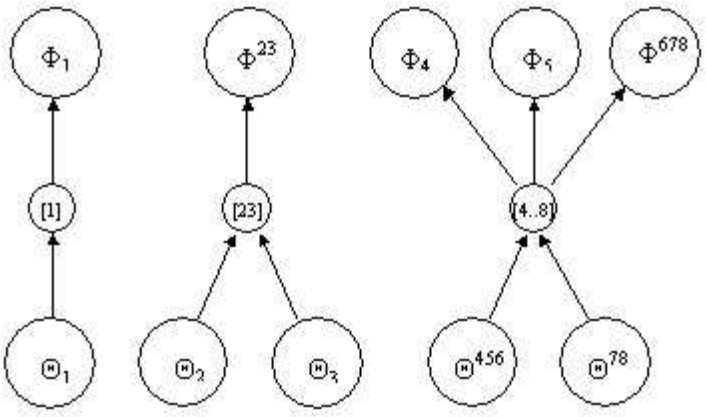

Figure 5.2: The family structures present in the transition amplitude from $\Psi_n = \Theta_{123}^{456\bullet 78}$ to $\Psi_{n+1} = \Phi_{145}^{23\bullet 678}$.

In general, familial relations will be generated by every transition amplitude, and so as the state of the Universe develops through many transitions, $\Psi_n \to \Psi_{n+1} \to \Psi_{n+2} \to ...$ an extended network of families will begin to emerge. In addition, definitions of grandparents, grandchildren, cousins etc. will become apparent, as will identifications of great-grandparents, great-great-grandparents, and so on. For example, if A is factor of Ψ_n, B is factor of Ψ_{n+1} and C is factor of Ψ_{n+2}, and if A is a parent of B and B is a parent of C, then A is necessarily a grandparent of C. Further, as the state develops, individual families may merge with other families, or may even remain isolated from all others over a large number of jumps.

Exactly what happens will depend on the specific dynamics that govern the system, as will be discussed later.

The existence of familial relations extending through a number of transitions gives rise to causal set relationships, with the associated concepts of lightcones and volume measures. To demonstrate this observation, consider as an illustrative example a model universe represented by a state in a Hilbert space fundamentally factorised into six quantum subregisters, $\mathcal{H}_{[1...6]}$. Consider also a possible sequence in the universe's development, in which five successive states Ψ_0, Ψ_1, ..., Ψ_4 have the following form

$$\begin{aligned} \Psi_0 &= \varphi^{123456} \quad ; \quad \Psi_1 = \psi_1^{23\bullet 456} \quad ; \quad \Psi_2 = \theta_{16}^{24\bullet 35} \\ \Psi_3 &= \eta_4^{12\bullet 356} \quad ; \quad \Psi_4 = \chi^{12\bullet 34\bullet 56}. \end{aligned} \tag{5.29}$$

Note the inevitable notational clash here: in this example, subscripts on the capital Greek letters (e.g. Ψ_n) will denote temporal ordering, whereas subscipts on lower case Greek letters will denote separable factors.

With the probability amplitudes given in the usual way, for example

$$\langle \Psi_{n+2} = \theta_{16}^{24\bullet 35} | \Psi_{n+1} = \psi_1^{23\bullet 456} \rangle = \langle \theta_1 | \psi_1 \rangle \langle \theta_6^{24\bullet 35} | \psi^{23\bullet 456} \rangle \tag{5.30}$$

etc., the above sequence of states can be represented by the diagram given in Figure 5.3.

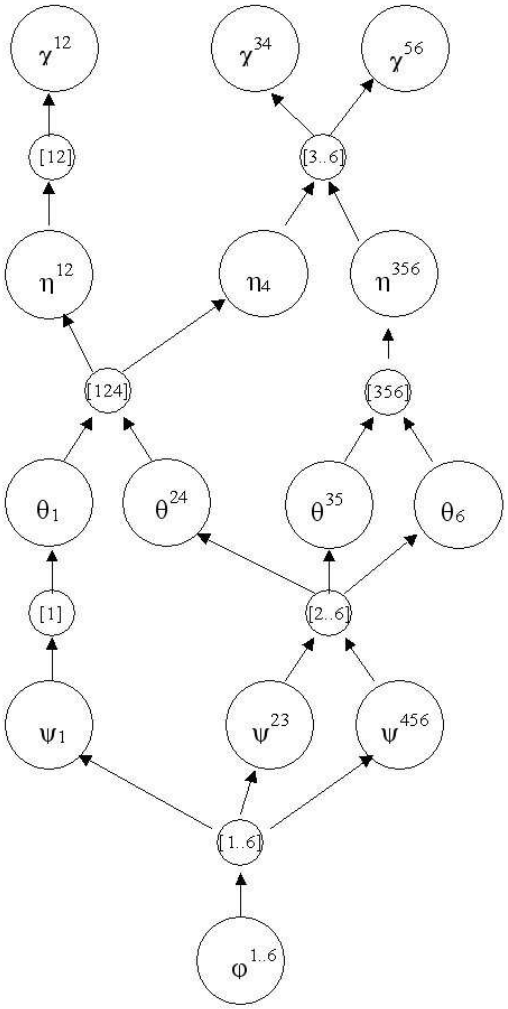

Figure 5.3: One possible network of families produced as a toy-universe develops over five jumps.

The universe of this example begins in an initial entangled state $\Psi_0 = \varphi^{123456}$. Since it is argued throughout this thesis that separability is a necessary prerequisite for classicity, then at time $n = 0$ the universe cannot be given any classical attributes. In fact from entangled states of the form $\Psi_0 = \varphi^{123456}$, no notions of internal observers, apparatus, or systems under investigation will be able to emerge. Further, since it has also been argued that the appearance of space relies on the existence of classicity, then an initial entangled

state cannot contain any sort of spatial relationships.

Separability, and hence the possibility of classicity, occurs in the next state $\Psi_1 = \psi_1^{23\bullet 456}$, which may be written as a product of three factors. Of course, it is still not possible to define measures of distance at this stage, that is to say that the factor $|\psi_1\rangle$ is so many units away from the factor $|\psi^{456}\rangle$, because these factors are nothing but 'pregeometric vectors' in a Hilbert space, whereas physical space is a phenomenon that is only expected to emerge by considerations of the relationships between large numbers of such factors over very many transitions. Likewise, there is no immediately obvious definition of volume on this pregeometric level. However, as with classical causal sets, embryonic notions of volume may be estimated by a process of counting; for the quantum causal sets proposed here, measurements of volume are expected to relate somehow to the number of factors present in the current state of the universe. As a first approach, it is assumed that more separable states will generate greater emergent volumes than less separable states, but it is still unclear at this stage of research exactly how such a programme should proceed.

During the transition from $\Psi_1 = \psi_1^{23\bullet 456}$ to $\Psi_2 = \theta_{16}^{24\bullet 35}$, the factor containing the component of the state in the Hilbert space $\mathcal{H}_1$ does not change relative to the partition structure of the total Hilbert space. In other words, although the state jumps from one partition of $\mathcal{H}_{[1\ldots6]}$ to another during the transition, both Ψ_1 and Ψ_2 have a factor in the same block, i.e. $\mathcal{H}_1$. Consequently the component of the state in the factor Hilbert space $\mathcal{H}_1$ changes with no 'interaction' with any other component in $\mathcal{H}_{[2\ldots6]}$, and this may be physically interpreted as the universe appearing to split into two distinct sub-universes, neither of which influences the other. Although highly speculative, this may be the sort of mechanism required to describe the behaviour of black holes in a fully quantum universe of very many subregisters, in which an entire region of emergent spacetime appears cut off from everything else.

A further point can be made if θ_1 happens to be the same as ψ_1. In this case, that factor would appear to have been 'frozen' in time, whilst the rest of the universe evolved around it. Such a freezing is a result of a *local null test,* defined in general as an operator $\hat{O}_{n+1}$ with eigenvectors of the form $\Phi = \alpha_a \otimes \gamma_b$ testing the separable state $\Theta_n = \alpha_a \otimes \beta_b$, where $\Phi, \Theta_n \in \mathcal{H}_{[ab]}$, $\alpha_a \in \mathcal{H}_{[a]}$, $\beta_b, \gamma_b \in \mathcal{H}_{[b]}$ and $\mathcal{H}_{[a]}$ and $\mathcal{H}_{[b]}$ need not be fundamentally factorised. Local null tests are often observed in physics, for example when a spin-$\frac{1}{2}$particle prepared via the spin-up channel of a Stern-Gerlach apparatus is passed through an identically orientated Stern-Gerlach device; as discussed in Chapters 3 and 6, in this type of situation no new information is acquired about the state by repeating such a test.

Note that *global null tests* could also be a feature of the dynamics, defined in general as an operator $\hat{O}'_{n+1}$ with an eigenvector Θ testing the state $\Psi_n = \Theta$. Such a global null test leaves the entire state unchanged, and is therefore not physically 'noticeable'.

The appearance of local null tests, i.e. the persistence of some factors of the state over a sequence of jumps, has a number of consequences for the quantum causal sets. Firstly it introduces a concept of endo-time into the dynamics, that is, the property that over a series of transitions, different factors of the state will 'experience' different durations, where time is defined in terms of change. In other words, whilst one factor could change a times as the state Ψ_n develops to Ψ_{n+m}, a different factor may change b times, where $m \geq a, b$. Consequently this endo-time is non-integrable, because the number of physically significant jumps that one particular factor experiences as the Universe develops from the state Ψ_n to the state Ψ_{n+m} depends on the chain of intermediate states Ψ_{n+1}, Ψ_{n+2}, ..., Ψ_{n+m-1}. This will be expanded upon in Chapter 8, but for now note that endo-time is a 'route dependent' concept, analogous to the use of proper time in relativity. Further, since isolated, classical-looking observers will ultimately be associated with different groups of factors of the Universe's state, the possibility that these factors may experience different durations of time might account for one of the origins of different inertial frames of reference in emergent relativity, in which different observers witness different passages of time.

Unlike the physically unobserved (and therefore fictitious) exo-time parameter, n, endo-time is not necessarily absolute. There is no reason to assume that any one factor has any more claim to be experiencing the 'real' time than any other. For example if, again, $\theta_1 = \psi_1$ in Figure 5.3 then ψ_1 could in principle be regarded as simultaneous with θ^{24}, θ^{35}, and θ_6, or instead simultaneous with ψ^{23} and ψ^{456}. Thus, the occurrence of local null tests also provides a basis for an eventual discussion of different planes of simultaneity in emergent relativity.

Once a notion of endo-time has been introduced it is possible to discuss timelike distances in a manner analogous to classical causal sets. Generally speaking, the timelike distance between a factor and one of its 'ancestors' is related to the number of intermediate factors in its family structure. For example, the factors ψ^{456} and θ_6 may be described as separated by one 'time step' or one 'tick' of the 'Universe's quantum clock' (or '*q-tick*' [54]), whereas the factors φ^{123456} and θ_6 are separated by two. Likewise, the factors ψ_1 and η^{12} are also separated by two time steps, because there is one intermediate factor θ_1. However, if again it was the case that $\theta_1 = \psi_1$ due to a local null test, then in this instance the factors ψ_1 and η^{12} would instead be described as separated by only one q-tick as there are now no physically distinguishable intermediate factors.

As before, this highlights the fact that endo-time is a concept that depends on a particular endo-observer's route: if $\theta_1 = \psi_1$ then the timelike distance between φ^{123456} and χ^{34} is three from the point of view of an observer 'following' the θ_1 path, but is four from the point of view of an observer associated with the alternative paths via θ^{24}, θ^{35} or θ_6.

A naïve concept of lightcone structure can also be gathered from the above example represented by Figure 5.3. Consider as an illustration the factor θ_1 of the state Ψ_2 and the factor η^{356} of the state Ψ_3. If these factors were simply associated with classical events, i.e. taken to just be the 'objects' of the classical causal set theory described in Section 5.2, they would be described as incomparable. That is, there would be no relation of the form $\prec$ linking the events θ_1 and η^{356} as $\theta_1 \prec \eta^{356}$. The conclusion is that η^{356} is out of the lightcone of θ_1, and is hence not in its causal future, and so any change in θ_1 could not be expected to influence the event η^{356}.

This type of lightcone structure is also potentially present in the quantum causal sets introduced here. It might be possible, for instance, to discuss whether counterfactual changes in the factors of one state affect particular factors of later states, simply by a consideration of how the state of Universe changes from being in one partition to the next as it develops. For example, in the current toy-universe model it appears that a counterfactual change in the factor $|\theta_1\rangle$ of Ψ_2 will not affect the factor $|\eta^{356}\rangle$ in Ψ_3 because they are in completely different blocks of the partition. In other words, because $|\theta_1\rangle$ is not a parent of $|\eta^{356}\rangle$, a change in $|\theta_1\rangle$ may be expected to leave $|\eta^{356}\rangle$ invariant. So, by considering how counterfactual changes in one factor of the universe's state might influence factors in subsequent states, an embryonic concept analogous to lightcone structure is introduced at the pregeometric level. Moreover, once such a notion is established, it is possible to discuss features such as geodesics and spacelike distance, and ultimately therefore also emergent spacetime, exactly as in the case of classical causal sets.

Note, however, that this line of thinking may be missing an essential point. In a fully quantum universe with no external observers, the development of the state Ψ_n is achieved by Hermitian operators of the form $\hat{\Sigma}_{n+1}$ chosen self-referentially by the Universe itself, as discussed further in Chapter 8. In other words, there is expected to be some sort of feedback mechanism in which the current stage somehow affects which test $\hat{\Sigma}_{n+1}$ is used next. This immediately leads to a serious problem for the counterfactual argument given above, because any change in just one factor of a state Ψ_n might lead to a completely different next test $\hat{\Sigma}'_{n+1}$, and this may have a completely different set of eigenvectors.

In the case of the example at hand, the state Ψ_2 with the factor θ_1 is assumed to give rise (somehow) to the selection of an operator $\hat{\Sigma}_3$ which has an eigenvector Ψ_3 that has a factor η^{356}. If this selection of $\hat{\Sigma}_3$ does not depend on Ψ_2, then the above simple description of lightcone structure in terms of counterfactual changes is appropriate. However if instead the choice of $\hat{\Sigma}_3$ does indeed depend on Ψ_2, then changing Ψ_2 may affect $\hat{\Sigma}_3$. So in this case, if the state Ψ_2 instead had a factor θ'_1, the next test chosen by the universe might be $\hat{\Sigma}'_3$, and this alternative operator might have a completely different set of eigenvectors, with perhaps none of them possessing η^{356} as a factor. In fact, even a small change from

θ_1 to θ_1' in Ψ_2 might lead to a next state Ψ_3 that is completely entangled.

Clearly then, in a self-referential Universe developed according to a choice of operator based upon the current state, an additional mechanism must be involved in order to ensure that the emergent lightcone structure and Einstein locality observed in physics is generated. This mechanism will be shown to involve the operators themselves.

5.5 Factorisation and Entanglement of Operators

The previous section showed where a discussion of causal set structure might begin to emerge from a changing quantum state description of the Universe. What has not been addressed, however, is how such patterns could arise in the first place, that is, how and why the separability of the state can change from one jump to the next.

Since any state is an eigenvector of an Hermitian quantum operator, it is these tests that must ultimately be responsible for the way in which the Universe might develop over a series of collapses. Specifically in fact, the set of eigenvectors belonging to the operator that is chosen to test the Universe will determine how separable the next potential state will be. For example, if the rules governing the dynamics dictate that, for all n, an operator $\hat{\Sigma}_{n+1}$ is chosen that has only entangled eigenvectors, then the state Ψ_{n+1} of the Universe will always be entangled and there is no chance that the type of causal set structure described in the previous section will ever arise. For this reason, the types of possible Hermitian operator used to develop the Universe must consequently also be examined.

Up until now, only the separation and entanglement properties of the states have been investigated. In this section, however, it will be demonstrated that the operators themselves may also be separable or entangled. Further, these properties will be shown to also generate structures analogous to those of causal sets, and this will have far reaching consequences for the states.

The set of Hermitian operators $\mathbb{H}(\mathcal{H}^{(D)})$ of order D is a D^2-dimensional, real vector space [55]. In general then, every Hermitian operator $\hat{O} \in \mathbb{H}(\mathcal{H}^{(D)})$ acting on a state in a D dimensional Hilbert space $\mathcal{H}^{(D)}$ can be constructed from linear combinations of the D^2 independent elements that span this real vector space [22]. These D^2 fundamental 'building blocks' will be called a *skeleton set,* $\mathcal{S}$, of operators, and are the operators' equivalent in $\mathbb{H}(\mathcal{H}^{(D)})$ of the set $\mathcal{B}$ of basis vectors used to construct arbitrary states in $\mathcal{H}^{(D)}$.

Specifically, if $\mathcal{S} \equiv \{\hat{\sigma}^\lambda : \lambda = 0, 1, .., (D^2-1)\}$ is defined as the skeleton set of operators for a D dimensional Hilbert space $\mathcal{H}^{(D)}$, then any Hermitian operator $\hat{O}$ acting upon states

in $\mathcal{H}^{(D)}$ may be written in the form

$$\hat{O} = a_0\hat{\sigma}^0 + a_1\hat{\sigma}^1 + ... + a_{D^2-1}\hat{\sigma}^{D^2-1} = \sum_{\lambda=0}^{D^2-1} a_\lambda \hat{\sigma}^\lambda \tag{5.31}$$

where a_λ is a real parameter.

Consider as an illustration a single qubit quantum register, i.e. a two-dimensional Hilbert space $\mathcal{H}_A^{(2)} \equiv \mathcal{H}_A$ labelled by the subscript A, with the superscript '(2)' that indicates dimension being now implicitly assumed and hence dropped. Any Hermitian operator $\hat{o} \in \mathbb{H}(\mathcal{H}_A)$ acting on a single qubit state in $\mathcal{H}_A$ is composed of a linear sum of the $2^2 = 4$ members of the skeleton set $\mathcal{S}_A$ of $\mathbb{H}(\mathcal{H}_A)$, defined as $\mathcal{S}_A \equiv \{\hat{\sigma}_A^\mu : \mu = 0, 1, 2, 3\}$, where the $\{\hat{\sigma}_A^\mu\}$ may be associated with the three Pauli spin operators and the identity operator $\hat{I} = \hat{\sigma}_A^0$, as may be readily verified.

So, the skeleton operators in $\mathcal{S}_A$ are taken to obey the algebraic relations

$$\begin{aligned} \hat{\sigma}_A^i\hat{\sigma}_A^j &= \delta_{ij}\hat{\sigma}_A^0 + i\epsilon_{ijk}\hat{\sigma}_A^k \\ \hat{\sigma}_A^i\hat{\sigma}_A^0 &= \hat{\sigma}_A^0\hat{\sigma}_A^i = \hat{\sigma}_A^i \end{aligned} \tag{5.32}$$

where $i, j, k = 1, 2, 3$, but $i = \sqrt{-1}$ when it is not used as an index. Here and below, the Einstein summation convention is assumed over lower case Latin and Greek indices, and the Levi-Civita tensor is defined in the usual way:

$$\epsilon_{ijk} = \left\{ \begin{array}{ll} 0 & \text{for } i = j, k \text{ or } j = k \\ +1 & \text{for } ijk,\ kij,\ \text{or } jki \\ -1 & \text{for } ikj,\ jik,\ \text{or } kji \end{array} \right\}. \tag{5.33}$$

A more compact way of writing the relations (5.32) is

$$\hat{\sigma}_A^\mu\hat{\sigma}_A^\upsilon = C_\omega^{\mu\upsilon}\hat{\sigma}_A^\omega \tag{5.34}$$

where $\mu, \upsilon, \omega = 0, 1, 2, 3$ and the coefficients $C_\omega^{\mu\upsilon}$ are given by

$$\begin{aligned} C_\omega^{0\upsilon} &= C_\omega^{\upsilon 0} = \delta_{\upsilon\omega} \\ C_0^{ij} &= \delta_{ij} \quad , \quad C_k^{ij} = i\epsilon_{ijk}. \end{aligned} \tag{5.35}$$

It is possible to obtain a matrix form for the operators $\hat{\sigma}_A^\mu$. Consider an orthonormal basis set $\mathcal{B}_A$ of vectors spanning $\mathcal{H}_A$, defined as $\mathcal{B}_A = \{|0\rangle_A, |1\rangle_A\}$. A representation of the operators $\hat{\sigma}_A^\mu$ in this basis may be given by

$$\hat{\sigma}_A^\mu = \sum_{m,n=0}^{1} [\sigma_A^\mu]_{(m+1)(n+1)}|m\rangle\langle n|, \tag{5.36}$$

where $m, n = 0, 1$ and $[\sigma_A^\mu]_{(m+1)(n+1)}$ is the value in the $(m+1)^{th}$ row of the $(n+1)^{th}$ column of a 2×2 matrix $[\sigma_A^\mu]$. As might be expected, one possible such set of matrices may be defined in the standard way of Pauli:

$$\begin{aligned} [\sigma_A^0] &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad [\sigma_A^1] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ [\sigma_A^2] &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad [\sigma_A^3] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \end{aligned} \tag{5.37}$$

which clearly satisfy (5.34).

In addition, it can further be shown that the skeleton set of operators $\mathcal{S}_A = \{\hat{\sigma}_A^\mu\}$ may be associated with the identity ($\mu = 0$) and the generators ($\mu = i = 1, 2, 3$) of the group $SU(2)$. Consequently, any special unitary operator $\hat{U}$ acting on $\mathcal{H}_A$ may be written in the form

$$\hat{U} = \exp\left[i \sum_{\mu=0}^{3} u_\mu \sigma_A^\mu \right] \tag{5.38}$$

where the $u_\mu \in \mathbb{R}$ are real parameters.

The above arguments can be extended to Hilbert spaces factorisable into more than one qubit subregister. Consider a Hilbert space $\mathcal{H}_{[1...N]}$ formed from the tensor product of N qubit factor spaces, $\mathcal{H}_{[1...N]} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_N$. An orthonormal basis $\mathcal{B}_a$ for the a^{th} factor space $\mathcal{H}_a$ ($= \mathcal{H}_a^{(2)}$ as before) may be defined as $\mathcal{B}_a = \{|0\rangle_a, |1\rangle_a\}$, where ${}_a\langle n|m\rangle_a = \delta_{mn}$ for $m, n = 0, 1$.

The skeleton set $\mathcal{S}_N$ for the total Hilbert space $\mathcal{H}_{[1...N]}$ can be generated by taking the tensor products of the skeleton operators of the individual qubit spaces, i.e.

$$\mathcal{S}_N \equiv \{\hat{\sigma}_1^{\mu_1} \otimes \hat{\sigma}_2^{\mu_2} \otimes ... \otimes \hat{\sigma}_N^{\mu_N} : \mu_a = 0, 1, 2, 3 \text{ for } a = 1, 2, ..., N\}, \tag{5.39}$$

which is clearly a set containing $4^N = (2^N)^2 = D^2$ members.

Note that for convenience the skeleton set $\mathcal{S}_N$ may also be rewritten as $\mathcal{S}_N \equiv \{\hat{\sigma}_{1..N}^\eta : \eta = 0, 1, ..., (4^N - 1)\}$, with the first member $\hat{\sigma}_{1..N}^0 = \hat{\sigma}_1^0 \otimes \hat{\sigma}_2^0 \otimes ... \otimes \hat{\sigma}_N^0$ etc.

The set $\mathcal{S}_N$ forms a basis for the real vector space $\mathbb{H}(\mathcal{H}_{[1...N]})$ of Hermitian operators in $\mathcal{H}_{[1...N]}$. Any operator $\hat{A} \in \mathbb{H}(\mathcal{H}_{[1...N]})$ can be written as a linear sum[7] of the members of $\mathcal{S}_N$

$$\begin{aligned} \hat{A} &= \sum_{\mu_1=0}^{3} \sum_{\mu_2=0}^{3} ... \sum_{\mu_N=0}^{3} A_{\mu_1\mu_2...\mu_N} \hat{\sigma}_1^{\mu_1} \otimes \hat{\sigma}_2^{\mu_2} \otimes ... \otimes \hat{\sigma}_N^{\mu_N} \\ &= A_{00...0}(\hat{\sigma}_1^0 \otimes \hat{\sigma}_2^0 \otimes ... \otimes \hat{\sigma}_N^0) + A_{10...0}(\hat{\sigma}_1^1 \otimes \hat{\sigma}_2^0 \otimes ... \otimes \hat{\sigma}_N^0) + ... \\ &\quad ... + A_{33...3}(\hat{\sigma}_1^3 \otimes \hat{\sigma}_2^3 \otimes ... \otimes \hat{\sigma}_N^3) \end{aligned} \tag{5.40}$$

[7] For explicitness, the Einstein summation convention has been replaced in this expression by the 'sum' signs.

where the coefficients $A_{\mu_1\mu_2...\mu_N}$ are all real, as required for Hermicity.

Consider now a second operator $\hat{B} \in \mathbb{H}(\mathcal{H}_{[1...N]})$, such that it is possible to in turn define a third operator $\hat{X}$ as the multiplicative product $\hat{X} \equiv \hat{A}\hat{B}$. Assuming again that the Einstein convention is adopted on repeated Greek indices, $\hat{A}$ and $\hat{B}$ may be written as $\hat{A} = A_{\mu_1\mu_2...\mu_N}\hat{\sigma}_1^{\mu_1} \otimes \hat{\sigma}_2^{\mu_2} \otimes ... \otimes \hat{\sigma}_N^{\mu_N}$ and $\hat{B} = B_{\upsilon_1\upsilon_2...\upsilon_N}\hat{\sigma}_1^{\upsilon_1} \otimes \hat{\sigma}_2^{\upsilon_2} \otimes ... \otimes \hat{\sigma}_N^{\upsilon_N}$ for $0 \leq \mu_r \leq 3$ and $0 \leq \upsilon_s \leq 3$ with $r, s = 1, 2, ..., N$.

So, the product $\hat{X}$ is given by

$$\begin{aligned} \hat{X} &= (A_{\mu_1\mu_2...\mu_N}\hat{\sigma}_1^{\mu_1} \otimes \hat{\sigma}_2^{\mu_2} \otimes ... \otimes \hat{\sigma}_N^{\mu_N})(B_{\upsilon_1\upsilon_2...\upsilon_N}\hat{\sigma}_1^{\upsilon_1} \otimes \hat{\sigma}_2^{\upsilon_2} \otimes ... \otimes \hat{\sigma}_N^{\upsilon_N}) \quad (5.41) \\ &= A_{\mu_1\mu_2...\mu_N}B_{\upsilon_1\upsilon_2...\upsilon_N}(\hat{\sigma}_1^{\mu_1} \otimes \hat{\sigma}_2^{\mu_2} \otimes ... \otimes \hat{\sigma}_N^{\mu_N})(\hat{\sigma}_1^{\upsilon_1} \otimes \hat{\sigma}_2^{\upsilon_2} \otimes ... \otimes \hat{\sigma}_N^{\upsilon_N}) \\ &= A_{\mu_1\mu_2...\mu_N}B_{\upsilon_1\upsilon_2...\upsilon_N}C_{\omega_1}^{\mu_1\upsilon_1}C_{\omega_2}^{\mu_2\upsilon_2}...C_{\omega_N}^{\mu_N\upsilon_N}\hat{\sigma}_1^{\omega_1} \otimes \hat{\sigma}_2^{\omega_2} \otimes ... \otimes \hat{\sigma}_N^{\omega_N} \end{aligned}$$

from (5.34). The coefficients $A_{\mu_1\mu_2...\mu_N}B_{\upsilon_1\upsilon_2...\upsilon_N}C_{\omega_1}^{\mu_1\upsilon_1}C_{\omega_2}^{\mu_2\upsilon_2}...C_{\omega_N}^{\mu_N\upsilon_N}$ are just products of real parameters, so this last line may be rewritten in the form

$$\hat{X} = X_{\omega_1\omega_2...\omega_N}\hat{\sigma}_1^{\omega_1} \otimes \hat{\sigma}_2^{\omega_2} \otimes ... \otimes \hat{\sigma}_N^{\omega_N} \quad (5.42)$$

where $X_{\omega_1\omega_2...\omega_N} \in \mathbb{R}$ for $\omega_t = 0, 1, 2, 3$ and $t = 1, 2, ..., N$. The product operator $\hat{X}$ is a linear sum of the members of $\mathcal{S}_N$ with real coefficients, and so is clearly a member of $\mathbb{H}(\mathcal{H}_{[1...N]})$. The set $\mathbb{H}(\mathcal{H}_{[1...N]})$ is hence confirmed closed under the multiplication rule, as expected for a vector space, and is an algebra over the real number field.

Consider again the Hermitian operator $\hat{A} \in \mathbb{H}(\mathcal{H}_{[1...N]})$ defined as $\hat{A} = A_{\mu_1\mu_2...\mu_N}\hat{\sigma}_1^{\mu_1} \otimes \hat{\sigma}_2^{\mu_2} \otimes ... \otimes \hat{\sigma}_N^{\mu_N}$. Depending on the actual values of the coefficients $A_{\mu_1\mu_2...\mu_N}$, this operator may, or may not, factorise relative to the skeleton set associated with some particular split of the total Hilbert space $\mathcal{H}_{[1...N]}$. For instance, if $\mathcal{H}_{[1...N]}$ can be factorised into the bi-partite split $\mathcal{H}_{[1...N]} = \mathcal{H}_V \otimes \mathcal{H}_W$, where $\mathcal{H}_V$ and $\mathcal{H}_W$ need not be of prime dimension, it may be the case that $\hat{A}$ can be written in the form $\hat{A} = \hat{V} \otimes \hat{W}$, where $\hat{V}$ is an Hermitian operator acting in the factor sub-space $\mathcal{H}_V$ and $\hat{W}$ is an Hermitian operator acting in the factor sub-space $\mathcal{H}_W$.

As an example, in the factorisable two qubit Hilbert space $\mathcal{H}_{[12]} = \mathcal{H}_1 \otimes \mathcal{H}_2$, the space of Hermitian operators $\mathbb{H}(\mathcal{H}_{[12]})$ is spanned by the skeleton set $\mathcal{S}_{12}$ defined as $\mathcal{S}_{12} = \{\hat{\sigma}_1^{\mu_1} \otimes \hat{\sigma}_2^{\mu_2} : \mu_1, \mu_2 = 0, 1, 2, 3\}$. Clearly then, an Hermitian operator $\hat{E} \in \mathbb{H}(\mathcal{H}_{[12]})$ of the form

$$\hat{E} = \frac{1}{2}(3\hat{\sigma}_1^1 \otimes \hat{\sigma}_2^1 + \hat{\sigma}_1^2 \otimes \hat{\sigma}_2^2) \quad (5.43)$$

is entangled, whereas an Hermitian operator $\hat{F} \in \mathbb{H}(\mathcal{H}_{[12]})$ of the form

$$\hat{F} = 3\hat{\sigma}_1^0 \otimes \hat{\sigma}_2^1 - \hat{\sigma}_1^3 \otimes \hat{\sigma}_2^1 = (3\hat{\sigma}_1^0 - \hat{\sigma}_1^3) \otimes \hat{\sigma}_2^1 \quad (5.44)$$

is factorisable relative to $\mathcal{S}_{12}$ in $\mathcal{H}_{[12]} = \mathcal{H}_1 \otimes \mathcal{H}_2$.

Whether or not an arbitrary Hermitian operator $\hat{O}$ in $\mathbb{H}(\mathcal{H}_{[1...N]})$ is factorisable in the form $\hat{O} = \hat{Y} \otimes \hat{Z}$, relative to the skeleton set of a particular bi-partite split of the total Hilbert space $\mathcal{H}_{[1...N]} = \mathcal{H}_Y \otimes \mathcal{H}_Z$, may be determined in a manner that is similar to the microsingularity test given in Chapter 4 used to discover whether an arbitrary state in $\mathcal{H}_{[1...N]}$ is separable relative to $\mathcal{H}_{[1...N]} = \mathcal{H}_Y \otimes \mathcal{H}_Z$.

Theorem 5.1 *An arbitrary Hermitian operator $\hat{O} \in \mathbb{H}(\mathcal{H}_{[1...N]})$ is factorisable in the form $\hat{O} = \hat{Y} \otimes \hat{Z}$, relative to the skeleton set $\mathcal{S}_{YZ} = \{\hat{\sigma}_Y^\alpha \otimes \hat{\sigma}_Z^\beta\}$ of a particular bi-partite split of the total Hilbert space $\mathcal{H}_{[1...N]} = \mathcal{H}_Y \otimes \mathcal{H}_Z$ such that $\hat{Y} \in \mathbb{H}(\mathcal{H}_Y)$ and $\hat{Z} \in \mathbb{H}(\mathcal{H}_Z)$ if, for $\hat{O} = C_{\alpha\beta}\hat{\sigma}_Y^\alpha \otimes \hat{\sigma}_Z^\beta$,*

$$C_{\alpha\beta}C_{\gamma\delta} = C_{\alpha\delta}C_{\gamma\beta} \tag{5.45}$$

for all values of the indices $\alpha, \gamma = 0, ..., (D_Y - 1)$ and $\beta, \delta = 0, ..., (D_Z - 1)$, where $C_{\alpha\beta}$ is a real coefficient, $\hat{\sigma}_Y^\alpha$ is a skeleton operator for $\mathbb{H}(\mathcal{H}_Y)$, $\hat{\sigma}_Z^\beta$ is a skeleton operator for $\mathbb{H}(\mathcal{H}_Z)$, and D_Y and D_Z are the dimensions of $\mathbb{H}(\mathcal{H}_Y)$ and $\mathbb{H}(\mathcal{H}_Z)$ respectively.

This may be shown as follows:

Proof. $\Rightarrow$ Suppose that the coefficients of the operator $\hat{O} = C_{\alpha\beta}\hat{\sigma}_Y^\alpha \otimes \hat{\sigma}_Z^\beta$ satisfy the microsingularity condition (5.45), and without loss of generality assume that $\hat{O}$ is not the zero operator. This implies that at least one coefficient $C_{\alpha\beta}$ must be non-zero for $\alpha = 0, ..., (D_Y - 1)$ and $\beta = 0, ..., (D_Z - 1)$. Further, since any $C_{\gamma\delta}$ is just a real number, the product $C_{\gamma\delta}\hat{O}$ is just a scalar multiple of $\hat{O}$, so

$$\begin{aligned} \hat{O} &= C_{\alpha\beta}\hat{\sigma}_Y^\alpha \otimes \hat{\sigma}_Z^\beta \\ C_{\gamma\delta}\hat{O} &= C_{\gamma\delta}C_{\alpha\beta}\hat{\sigma}_Y^\alpha \otimes \hat{\sigma}_Z^\beta \\ &= C_{\alpha\delta}C_{\gamma\beta}\hat{\sigma}_Y^\alpha \otimes \hat{\sigma}_Z^\beta \\ &= (C_{\alpha\delta}\hat{\sigma}_Y^\alpha) \otimes (C_{\gamma\beta}\hat{\sigma}_Z^\beta). \end{aligned} \tag{5.46}$$

Clearly, then, $\hat{O}$ is factorisable with respect to $\mathcal{S}_{YZ}$.

$\Leftarrow$ If $\hat{O} \in \mathbb{H}(\mathcal{H}_{[1...N]})$ factorises relative to $\mathcal{S}_{YZ}$ then

$$\hat{O} = (y_\alpha\hat{\sigma}_Y^\alpha) \otimes (z_\beta\hat{\sigma}_Z^\beta) \tag{5.47}$$

where y_α and z_β are real parameters. So

$$\hat{O} = y_\mu z_\upsilon \hat{\sigma}_Y^\mu \otimes \hat{\sigma}_Z^\upsilon. \tag{5.48}$$

Taking $C_{\alpha\beta} = y_\alpha z_\beta$, and similarly $C_{\gamma\delta} = y_\gamma z_\delta$, the microsingularity condition (5.45) is clearly satisfied because

$$C_{\alpha\beta}C_{\gamma\delta} = y_\alpha z_\beta y_\gamma z_\delta = y_\alpha z_\delta y_\gamma z_\beta = C_{\alpha\delta}C_{\gamma\beta}. \tag{5.49}$$

■

Just as the states can be described as existing in certain entanglements or separations of the Hilbert space, the operators testing them can also be placed into similar such sets. Thus, the 'partition structure' of the operators may also be discussed. Again, the use of upper and lower indices may be adopted in the obvious way, and the symbol "•" may be used to denote separable products. For example, in this notation the operator $\hat{O}_a^{bc\bullet def}$ acting on states in $\mathcal{H}_{[a...f]}$ is assumed fundamentally factorisable into three sub-operators of the form

$$\hat{O}_a^{bc\bullet def} = \hat{A}_a \otimes \hat{A}^{bc} \otimes \hat{A}^{def} \tag{5.50}$$

where $\hat{A}_a$ acts on states in $\mathcal{H}_a$, $\hat{A}^{bc}$ is an entangled sub-operator acting in $\mathcal{H}_{[bc]}$, and $\hat{A}^{def}$ is an entangled sub-operator relative to the skeleton set of $\mathcal{H}_{[def]}$. Moreover, $\hat{O}_a^{bc\bullet def}$ is a member of the set $\mathbb{H}(\mathcal{H}_{[a...f]})_a^{bc\bullet def}$ of Hermitian operators in $\mathcal{H}_{[a...f]}$ that are fundamentally factorisable into three factors relative to the skeleton set of the split $\mathcal{H}_{[a...f]} = \mathcal{H}_{[a]} \otimes \mathcal{H}_{[bc]} \otimes \mathcal{H}_{[d...f]}$.

Whilst Hermitian operators may be factorisable or entangled, it should be noted that not every type of entangled or factorisable operator is necessarily Hermitian. This is an important point, since it is only Hermitian operators that are responsible for physically realisable observables, and only the eigenvectors of Hermitian operators that make up the physically realisable states.

Moreover, it should further be noted that not every type of Hermitian operator can validly be used to test the quantum state of the Universe. This follows because with every Hermitian operator is associated a set of eigenvalues, each of which implies a corresponding eigenvector. However, if two (or more) of these eigenvalues are the same, their eigenvectors are not uniquely determined. This is a standard result of linear algebra [56].

In the paradigm proposed in this thesis, the quantum state of the Universe Ψ_n in its D dimensional Hilbert space is developed by collapsing into one of the eigenvectors of an Hermitian operator $\hat{\Sigma}_{n+1}$. In this mechanism, the operator $\hat{\Sigma}_{n+1}$ is assumed to uniquely provide a complete, orthonormal set of D eigenvectors, Φ_{n+1}^i for $i = 1, 2, ..., D$, which effectively produces a preferred basis for the next set of potential states Ψ_{n+1}. It is therefore necessary that this set of eigenvectors has members that are not only distinguishable, but are also well defined and specific.

This conclusion is partly because if two eigenvectors have the same eigenvalue they cannot be distinguished by any sort of measuring apparatus, since it is generally the eigenvalues that are actually recorded (c.f. energy eigenvalues in conventional laboratory physics). So, since the jump from the state Ψ_n to the next state Ψ_{n+1} has been ascribed to be parameterised in terms of information acquisition, any such uncertainty as to what state

this information implies would cause problems for an interpretation of how the Universe is developing.

Additionally, if the eigenvectors of $\hat{\Sigma}_{n+1}$ are not uniquely specified, problems arise involving the identification of the members of the set of potential next states. This, in turn, could lead to an ambiguity regarding what (pure) state the Universe is actually in.

As a very simple illustration of the importance of these ideas, consider a universe consisting of just a single qutrit, i.e. one represented by a state in a three dimensional Hilbert space $\mathcal{H}^{(3)}$ spanned by the orthonormal basis $\mathcal{B}^{(3)} = \{|i\rangle : i = 0, 1, 2\}$. Consider also an operator of the form $\hat{P} = |0\rangle\langle 0|$, denoted in this representation by the matrix

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.51}$$

It can readily be shown that the states $\Theta_1 = |0\rangle$, $\Theta_2 = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$, and $\Theta_3 = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$ are three orthonormal eigenvectors of $\hat{P}$, with eigenvalues $\mu_1 = 1$, $\mu_2 = 0$ and $\mu_3 = 0$ respectively, because they satisfy $\hat{P}\Theta_i = \mu_i\Theta_i$ and $\langle\Theta_j|\Theta_i\rangle = \delta_{ij}$ for $i, j = 1, 2, 3$. These states therefore form an orthonormal basis $\mathfrak{B}^{(3)}$ for $\hat{P}$. But it can also be shown that there is no uniqueness in this construction of $\mathfrak{B}^{(3)}$, because any other set of states $|0\rangle$, $(\alpha|1\rangle + \beta|2\rangle)$, and $(\beta^*|1\rangle - \alpha^*|2\rangle)$ for any other values of $\alpha, \beta \in \mathbb{C}$ also comprise an orthogonal basis set $\mathfrak{B}^{(3)\prime}$ of eigenstates. There is hence an inherent 'ambiguity' in the eigenvectors of $\hat{P}$, making it an example of the type of Hermitian operator that cannot be used in the development of the state of the universe. In order to develop the universe from the state Ψ_n to Ψ_{n+1}, a unique basis set $\mathfrak{B}_{n+1}$ must be specified.

A suggestion at this point might be to introduce additional ansatz into the dynamics in order to overcome the above types of problem. In conventional quantum mechanics, for example, operators with degenerate eigenvalues (and hence continuous spectra of eigenstates) may be dealt with by an inclusion of Lüders' generalised projection postulate [57][31], and additional procedures may be employed to select a unique preferred basis set of eigenstates from the infinite set of possibilities possessed by the degenerate operator. For instance, it may be suggested that upon testing, the system selects a basis containing the eigenvector that is 'nearest' to the initial state. Paraphrasing such a possibility: if a quantum object represented by the state ψ is tested by the degenerate operator $\hat{o}$, then according to this 'selection mechanism' it may be taken to collapse to a member of a basis set of eigenstates of $\hat{o}$ that contains the particular eigenvector φ for which the value of $|\langle\varphi|\psi\rangle|^2$ is greatest.

Now, it is not clear at this stage exactly how such suggestions could affect, or be incorporated into, the dynamics proposed in this thesis for the developing quantum uni-

verse. For a start, Lüders' postulate was originally phrased in terms of density matrices, whereas such an approach is not the direct focus of this thesis on the grounds that ensemble descriptions of the Universe are considered unphysical, as discussed in Chapter 3. Moreover, Lüders' idea of generalised projections was also developed from an exo-physical perspective for quantum systems in the laboratory, and it is not always obvious whether any such concepts may be directly applicable to the case where the state describes the entire universe.

As for the above 'selection mechanism', it is not immediately clear how the other members of the required orthogonal basis set of vectors may be self-referentially selected during the procedure; recall that in a $D > 2$ dimensional Hilbert space, there are an infinite number of $(D-1)$ mutually orthogonal vectors that are also orthogonal to any given state φ. Additionally, it could be expected that there is a high propensity for such a mechanism to result in null tests on the universe, and these are not ascribed to play a role in the dynamics of the proposed paradigm. Elaborating on this last point, the 'nearest' eigenstate could be the same as the initial state, and this would potentially lead to trivial dynamics. As an example of this possibility, if a single qutrit toy-universe is initially in a state $\psi = (\alpha|1\rangle + \beta|2\rangle)$, the suggested mechanism could imply that the degenerate operator $\hat{P}$ described in (5.51) could leave the universe in the same state $\varphi = (\alpha|1\rangle + \beta|2\rangle)$.

Two more technical points concerning incorporating the above suggestions into the quantum universe dynamics are also appropriate. Firstly, even if additional mechanisms are postulated in conventional quantum mechanics that select a unique basis set from a continuous spectra of possibilities, they do not necessarily have a place in the work described here. After all, in the paradigm proposed in this thesis it is the orthogonal basis sets that have prime importance in the quantum developments, and not the equivalence class of operators that they imply. In other words, and by reversing the overall argument, since by definition the dynamics governing the transition from state Ψ_n to Ψ_{n+1} requires the specification of a unique basis set $\mathfrak{B}_{n+1}$ $(\ni \Psi_{n+1})$, which *then in turn* implies the specification of an equivalence class of Hermitian operators, the question of what happens if the eigenvalues of the operator $\hat{\Sigma}_{n+1}$ are degenerate does not automatically arise. By definition, the operator $\hat{\Sigma}_{n+1}$ is not being used to generate a unique basis set $\mathfrak{B}_{n+1}$; the unique orthonormal basis set $\mathfrak{B}_{n+1}$ is instead used to imply the operator $\hat{\Sigma}_{n+1}$.

Secondly, if the dynamics were to dictate that the current state Ψ_n is used to select a particular basis set $\mathfrak{B}_{n+1}$ from the degenerate operator $\hat{\Sigma}_{n+1}$ (which is identical to arguing: "if the dynamics were to dictate that the current state Ψ_n is used to select a particular basis set $\mathfrak{B}_{n+1}$ from the infinite set of possible basis sets of eigenstates of the degenerate operator $\hat{\Sigma}_{n+1}$"), then the question would remain: "why does the dynamics bother to define the infinite set in the first place if only one member $\mathfrak{B}_{n+1}$ is deterministically

picked?". What is the point in defining a degenerate operator, and hence an infinite number of possible basis sets, if additional constraints are then required to select just one of these bases? Why not instead just define a particular unique basis set $\mathfrak{B}_{n+1}$, and then consider the equivalence class of operators that this implies?

Overall, whilst none of the above issues explicitly forbid the use of generalised measurements within the framework aimed at in this work, it is equally evident that their inclusion into the quantum universe dynamics would require careful attention. Thus, the question of how or whether the tests of the universe may be allowed to possess degenerate eigenvalues is left as an area of investigation for the future, and the overall constraint of only accepting non-degenerate operators $\hat{\Sigma}_{n+1}$ is enforced for simplicity by definition to avoid possible such considerations.

Summarising, in the paradigm proposed here, the orthonormal basis set $\mathfrak{B}_{n+1}$ must be uniquely specified if it is to be used in the development of the state Ψ_n of the universe. Thus, the D eigenvectors Φ^i_{n+1}, $i = 1, ..., D$, of an operator $\hat{\Sigma}_{n+1}$ that form $\mathfrak{B}_{n+1}$ must also be uniquely specified. To ensure this, the constraint is therefore assumed that only operators $\hat{\Sigma}_{n+1}$ with D non-degenerate and real eigenvalues may be used to test the universe.

This conclusion leads to a definition of three different types of Hermitian operator, useful in the following:

i) A *Degenerate* operator is an Hermitian operator with at least two identical eigenvalues;

ii) A *Weak* operator is an Hermitian operator which is either degenerate, or at least one of its eigenvalues is zero;

iii) A *Strong* operator is an Hermitian operator which is not weak, i.e. all of its eigenvalues are different and none are zero.

It immediately follows that projection operators are weak, as is evident from, for example, (5.51).

The necessary distinction between Strong and Weak operators will become apparent when tensor products of operators are considered; it will be shown later that products of weak operators are in general insufficient to determine a preferred basis for the developing state, whereas products of strong operators may be used. Products of strong operators may thus be associated with the physical tests of the state of the Universe.

Note that for an operator $\hat{\Sigma}_{n+1}$ satisfying the eigenvector equation $\left|(\hat{\Sigma}_{n+1} - \lambda_i \hat{I})\right| = 0$, with eigenvalue $\lambda_i \in \mathbb{R}$, $i = 1, ..., D$ and the identity $\hat{I}$, the actual, absolute values of λ_i

are not important. What *is* important is the basis set of eigenvectors they represent, and, specifically for the case of strong operators, that these eigenvectors are all different. Indeed, given an arbitrary operator $\hat{O}$ with eigenvalues $\{a, b, c, ...\}$, it is possible to find a second arbitrary operator $\hat{O}'$ with eigenvalues $\{k'a, k'b, k'c, ...\}$, where $k' \in \mathbb{R}$ is a real, non-zero constant, that has the same spectrum of eigenvectors as $\hat{O}$. Moreover, the alternative operator $\hat{O}''$ defined as $\hat{O}'' = \hat{O} + k''\hat{I}$ also has the same eigenvectors as $\hat{O}$, even though its eigenvalues $\{(a + k''), (b + k''), (c + k''), ...\}$ are 'shifted' from those of $\hat{O}$ by a constant amount k''.

Conversely, note that two different strong operators may have the same set of eigenvalues but different eigenvectors, the Pauli matrices being a good example.

Of course, these principles are familiar to many physics experiments, where, for example, energy and momentum eigenvalues often only have relative significance. Moreover, the three operators $\hat{O}$, $\hat{O}'$ and $\hat{O}''$ would be physically indistinct, in the sense that if the Universe collapsed to a particular state Ψ_{n+1} that was one of the members of this set of eigenvectors, an observer would be unable to determine whether it was $\hat{O}$, $\hat{O}'$ or $\hat{O}''$ that was used to test Ψ_n.

Summarising, although their actual values are unimportant, it is necessary that the members of the set of eigenvalues are all different if distinctions are to be made between the members of the corresponding set of eigenstates.

In addition to the ideas of 'weak' and 'strong' operators, a further definition useful in the following is the '*pairwise-product*'. Consider a set $X \equiv \{x_1, x_2, ..., x_M\}$ with M members, and the set $Y \equiv \{y_1, y_2, ..., y_N\}$ with N members. The pairwise-product XY of the sets X and Y is defined as the set of all the products $XY \equiv \{x_i y_j : i = 1, ..., M, j = 1, ..., N\}$. Clearly, XY is a set containing $M \times N$ members.

As an extension, the *pairwise tensor product* of two sets can similarly be defined in an obvious way.

Return now to the issue of the separability of operators. Consider a Hilbert space $\mathcal{H}_{[12]}$ factorisable into two subspaces, $\mathcal{H}_{[12]} = \mathcal{H}_1 \otimes \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are of dimensions d_1 and d_2 respectively, which need not be prime. Consider also the Hermitian operators $\hat{A}_1 \in \mathbb{H}(\mathcal{H}_1)$ and $\hat{B}_2 \in \mathbb{H}(\mathcal{H}_2)$, such that the product operator $\hat{O}_{12} = \hat{A}_1 \otimes \hat{B}_2$ is a factorisable member of $\mathbb{H}(\mathcal{H}_{[12]})$, i.e. $\hat{O}_{12} \in \mathbb{H}(\mathcal{H}_{[12]})_{12}$. In addition, let the set of eigenvalues V_A of $\hat{A}_1$ be $V_A = \{a_1, a_2, ..., a_{d_1}\}$ and the set of eigenvalues V_B of $\hat{B}_2$ be $V_B = \{b_1, b_2, ..., b_{d_2}\}$.

The set of eigenvalues V_{AB} of the product operator $\hat{O}_{12}$ is given by the pairwise-product

$V_{AB} = V_A V_B$ of the sets of eigenvalues of the operators $\hat{A}_1$ and $\hat{B}_2$. So, V_{AB} is

$$\begin{aligned} V_{AB} &= \{a_1, a_2, ..., a_{d_1}\}\{b_1, b_2, ..., b_{d_2}\} \\ &= \{a_1 b_1, a_1 b_2, ..., a_1 b_{d_2}, a_2 b_1, a_2 b_2, ..., a_{d_1} b_{d_2}\} \\ &= \{a_i b_j : i = 1, ..., d_1 \ , \ j = 1, ..., d_2\}. \end{aligned} \tag{5.52}$$

Then, the following conclusions hold:

Theorem 5.2 *If either of $\hat{A}_1$ or $\hat{B}_2$ is weak, then the product $\hat{O}_{12}$ has a degenerate set of eigenvalues, and is hence also weak.*

Proof. Without loss of generality, let $\hat{A}_1$ be weak. Then, at least one member a_x of V_A is zero, or else two members a_x and a_y are equal.

If a_x is zero, then the d_2 members of V_{AB} of the form $a_x b_j$ for $j = 1, ..., d_2$ are also zero. Hence, $\hat{O}_{12}$ has d_2 degenerate eigenvalues, and is consequently a weak operator.

Alternatively, if $a_x = a_y$ then $a_x b_j = a_y b_j$ for all $j = 1, ..., d_2$, which means that V_{AB} contains d_2 sets of degenerate 'pairs'. Hence, $\hat{O}_{12}$ is weak. ■

The above theorem is logically equivalent to the statement: *only if $\hat{A}_1$ and $\hat{B}_2$ are both strong might the operator $\hat{O}_{12} = \hat{A}_1 \otimes \hat{B}_2$ be strong.* Alternatively, *if $\hat{O}_{12} = \hat{A}_1 \otimes \hat{B}_2$ is strong, then $\hat{A}_1$ and $\hat{B}_2$ must both be strong.*

However:

Theorem 5.3 *If $\hat{A}_1$ and $\hat{B}_2$ are both strong, then the product $\hat{O}_{12}$ may be either weak or strong, depending on the actual eigenvalues of $\hat{A}_1$ and $\hat{B}_2$.*

Proof. The proof is obvious. Whether or not the members of

$$V_{AB} = \{a_1 b_1, a_1 b_2, ..., a_1 b_{d_2}, a_2 b_1, a_2 b_2, ..., a_{d_1} b_{d_2}\} \tag{5.53}$$

are degenerate (noting that none can be zero if $\hat{A}_1$ and $\hat{B}_2$ are strong) clearly depends on the specific values of each of $a_1, a_2, ..., a_{d_1}$ and $b_1, b_2, ..., b_{d_2}$. ■

For example, let $d_1 = d_2 = 2$ in order to consider a Hilbert space $\mathcal{H}_{[12]}$ factorisable into two qubit subregisters, $\mathcal{H}_{[12]} = \mathcal{H}_1 \otimes \mathcal{H}_2$, and the skeleton set of operators $\mathcal{S}_{12} = \{\hat{\sigma}_1^{\mu_1} \otimes \hat{\sigma}_2^{\mu_2} : \mu_1, \mu_2 = 0, 1, 2, 3\}$ where $\hat{\sigma}_1^{\mu_1}$ and $\hat{\sigma}_2^{\mu_2}$ are analogous to the Pauli operators. Then:

1. Consider also an operator $\hat{o}$ defined as $\hat{o} \equiv \hat{\sigma}_1^1 \otimes \hat{\sigma}_2^2$. The skeleton operator $\hat{\sigma}_1^1$ is a strong Hermitian operator, with eigenvalues $+1$ and -1; similarly, the skeleton operator $\hat{\sigma}_2^2$ is also a strong Hermitian operator, and also has eigenvalues $+1$ and -1. Thus, the four eigenvalues of the product operator $\hat{o}$ are the products: $(1) \times (1) = 1$, $(1) \times (-1) = -1$, $(-1) \times (1) = -1$ and $(-1) \times (-1) = 1$. So, $\hat{o}$ clearly has degenerate eigenvalues, and is hence a weak operator that is the product of strong operators.

2. Consider instead the Hermitian operator $\hat{F}$ defined as $\hat{F} \equiv (3\hat{\sigma}_1^0 - \hat{\sigma}_1^3) \otimes \hat{\sigma}_2^1$ in (5.44). The strong Hermitian operator $(3\hat{\sigma}_1^0 - \hat{\sigma}_1^3)$ has eigenvalues of 2 and 4, whereas the strong Hermitian operator $\hat{\sigma}_2^1$ has eigenvalues of $+1$ and -1, so the four eigenvalues of the product operator $\hat{F}$ are: $(2) \times (1) = 2$, $(2) \times (-1) = -2$, $(4) \times (1) = 4$ and $(4) \times (-1) = -4$. Clearly, $\hat{F}$ has non-degenerate and non-zero eigenvalues, and is hence a strong operator that is the product of strong operators.

Note that the result of '1.' generalises to higher dimensional cases: every element of the skeleton set (5.39) associated with an n-qubit register is weak for $n > 1$. The members of n-qubit skeleton sets are Hermitian, but have degenerate eigenvalues.

For clarity, the conclusions of the above two theorems have been summed up below. If W and S denote weak and strong operators respectively, the following truth table is generated where the first row denotes the 'status' of $\hat{A}_1$, the first column denotes the status of $\hat{B}_2$ operator, and the remaining values denote the status of the resulting product operator $\hat{O}_{12} = \hat{A}_1 \otimes \hat{B}_2$:

–	$\hat{\mathbf{A}}_1$ **is W**	$\hat{\mathbf{A}}_1$ **is S**
$\hat{\mathbf{B}}_2$ **is W**	W	W
$\hat{\mathbf{B}}_2$ **is S**	W	S or W ?

Table 5.1

The results of the previous two theorems extend to operators that are the products of more than two factors. The generalisation of the first theorem implies that if an operator $\hat{O}_{1...M}$ is a product of M factor operators $\hat{O}_{1...M} = \hat{o}_1 \otimes \hat{o}_2 \otimes ...\hat{o}_M$, then every factor $\hat{o}_i$ must be strong if $\hat{O}_{1...M}$ is strong. This follows because operators of the type $\hat{O}_{1...M}$ can always arbitrarily be rewritten as a product of just two factors: the factor representing a particular $\hat{o}_i$ and the factor containing every other operator $\hat{o}_j$ for $1 \leq j \leq M$ and $j \neq i$. So, if any of the $\hat{o}_i$ are weak, the pair-wise product of the eigenvalues of these two factors contains either degeneracy or zeroes, and hence $\hat{O}_{1...M}$ must also be weak.

Likewise, the extension of the second of the above theorems follows naturally, since the spectrum of eigenvalues of an operator will always depend on the set of the products of the eigenvalues of its factors.

Attention is now turned to the eigenstates of the operators themselves:

Theorem 5.4 *All the eigenstates of a strong, factorisable operator are separable.*

Proof. Without loss of generality, consider a strong operator $\hat{O}_{12}$ factorisable into two factor operators, $\hat{O}_{12} = \hat{A}_1 \otimes \hat{B}_2$. From the earlier theorem, the factors $\hat{A}_1$ and $\hat{B}_2$ must also both be strong operators.

As before, let the eigenvalues of $\hat{A}_1$ be $\{a_1, a_2, ..., a_{d_1}\}$. Each eigenvalue a_i corresponds to a particular normalised eigenvector $|a_i\rangle_1$, such that the overall set of eigenvectors for $i = 1, ..., d_1$ forms an orthogonal basis set $\mathfrak{B}_1$ of states spanning the d_1 dimensional factor Hilbert space $\mathcal{H}_1$, i.e. $\mathfrak{B}_1 = \{|a_i\rangle_1 : i = 1, ..., d_1\}$.

Similarly, if the eigenvalues of $\hat{B}_2$ are $\{b_1, b_2, ..., b_{d_2}\}$, then each eigenvalue b_j corresponds to a particular normalised eigenvector $|b_j\rangle_2$, and this set of eigenvectors for $j = 1, ..., d_2$ forms an orthogonal basis set $\mathfrak{B}_2$ of states spanning the d_2 dimensional factor Hilbert space $\mathcal{H}_2$, i.e. $\mathfrak{B}_2 = \{|b_j\rangle_2 : j = 1, ..., d_2\}$.

Consider now the pairwise (tensor) product of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ defined as the set $\{|a_i\rangle_1 \otimes |b_j\rangle_2 : i = 1, ..., d_1 \ j = 1, ..., d_2\}$. Clearly, this set has $d_1 d_2$ members.

Now consider one of the members ψ of this set, $\psi = |a_x\rangle_1 \otimes |b_y\rangle_2$. Evidently, ψ is separable, and is a member of the partition $\psi \in \mathcal{H}_{12} = (\mathcal{H}_1 \bullet \mathcal{H}_2) \subset \mathcal{H}_{[12]}$. Moreover, ψ is an eigenstate of $\hat{O}_{12}$ because:

$$\begin{aligned} \hat{O}_{12}\psi &= \hat{O}_{12}(|a_x\rangle_1 \otimes |b_y\rangle_2) \\ &= [\hat{A}_1 \otimes \hat{B}_2](|a_x\rangle_1 \otimes |b_y\rangle_2) \\ &= [\hat{A}_1|a_x\rangle_1] \otimes [\hat{B}_2|b_y\rangle_2] \\ &= a_x|a_x\rangle_1 \otimes b_y|b_y\rangle_2 = o_{xy}|a_x\rangle_1 \otimes |b_y\rangle_2 \end{aligned} \tag{5.54}$$

where $o_{xy} = a_x b_y \in \mathbb{R}^+$.

Similarly, every other member of the set $\{|a_i\rangle_1 \otimes |b_j\rangle_2 : i = 1, ..., d_1 \ j = 1, ..., d_2\}$ is an eigenstate of $\hat{O}_{12}$, and is also a member of the separation $\mathcal{H}_{12}$.

However, because $\hat{O}_{12}$ is a strong operator acting on states in a $d_1 d_2$ dimensional Hilbert space, it has precisely $d_1 d_2$ independent eigenstates. Since there are $d_1 d_2$ independent eigenstates of $\hat{O}_{12}$ in the set $\{|a_i\rangle_1 \otimes |b_j\rangle_2 : i = 1, ..., d_1 \ j = 1, ..., d_2\}$, this set must be an exhaustive, orthonormal basis $\mathfrak{B}_{12}$ for $\hat{O}_{12}$.

Hence, every eigenstate of the strong, factorisable operator $\hat{O}_{12}$ is separable.

The proof extends to strong, separable operators of higher degrees of factorisation in the obvious way. ■

In the context of the proposed paradigm that only strong (Hermitian) operators are used in the development of the Universe's state, the above theorem can be rephrased as: separable tests only have separable outcomes.

An important consequence of this is that entangled states cannot be the outcome of separable operators. Paraphrasing: *entangled states can only be the outcome of entangled operators.*The converse, however, is not true: entangled operators can have entangled eigenstates, but they can also have separable eigenstates.

So, overall the 'Golden Rule' is that:

No operator can have more factors than any of its eigenstates,
but an eigenstate can have more factors than its operator.

As illustrations of these ideas, consider again a two qubit toy-universe, represented by a state in a four dimensional Hilbert space $\mathcal{H}^{(4)}_{[12]} = \mathcal{H}^{(2)}_1 \otimes \mathcal{H}^{(2)}_2$ spanned by the vector basis $\mathcal{B}_{12} = \{|i\rangle_1 \otimes |j\rangle_2 \equiv |ij\rangle : i, j = 0, 1\}$, with the usual skeleton set $\mathcal{S}_{12} = \{\hat{\sigma}^{\mu_1}_1 \otimes \hat{\sigma}^{\mu_2}_2 : \mu_1, \mu_2 = 0, 1, 2, 3\}$ for the operators in $\mathbb{H}(\mathcal{H}^{(4)}_{[12]})$. The operator

$$\hat{F} = (3\hat{\sigma}^0_1 - \hat{\sigma}^3_1) \otimes \hat{\sigma}^1_2 \tag{5.55}$$

is factorisable, and has four separable eigenstates: $\frac{1}{\sqrt{2}}(|01\rangle + |00\rangle)$, $\frac{1}{\sqrt{2}}(|01\rangle - |00\rangle)$, $\frac{1}{\sqrt{2}}(|11\rangle + |10\rangle)$, and $\frac{1}{\sqrt{2}}(|11\rangle - |10\rangle)$. Conversely, the operator

$$\hat{E} = \frac{1}{2}(3\hat{\sigma}^1_1 \otimes \hat{\sigma}^1_2 + \hat{\sigma}^2_1 \otimes \hat{\sigma}^2_2) \tag{5.56}$$

is entangled and has four entangled eigenstates: $\frac{1}{\sqrt{2}}\left(|11\rangle + |00\rangle\right)$, $\frac{1}{\sqrt{2}}(|11\rangle - |00\rangle)$, $\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$ and $\frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$. However, the operator

$$\hat{M} = \hat{\sigma}^1_1 \otimes \hat{\sigma}^1_2 + \hat{\sigma}^2_1 \otimes \hat{\sigma}^2_2 + (\hat{\sigma}^0_1 \otimes \hat{\sigma}^3_2)/2 + (\hat{\sigma}^3_1 \otimes \hat{\sigma}^0_2)/2 \tag{5.57}$$

is also entangled but has a mixture of separable $|00\rangle$, $|11\rangle$ and entangled $\frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$, $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ eigenstates.

It is interesting to note here that although a set of eigenstates may look relatively 'simple', the operator they come from may still be a non-trivial combination of skeleton operators. This observation is perhaps a reminder of how much more complicated the set of skeleton operators is compared to the set of states; recall that a Hilbert space $\mathcal{H}^{(D)}$ of dimension D is spanned by a basis set of D independent vectors, whereas the corresponding space of Hermitian operators $\mathbb{H}(\mathcal{H}^{(D)})$ is parameterised by a skeleton set containing D^2 members. In the present case, although operators in two qubit Hilbert spaces $\mathcal{H}^{(4)}_{[12]}$ only have four eigenstates, they will nevertheless comprise of linear sums of up to sixteen basis operators. It is perhaps not surprising, then, that the structure of the operators is considerably 'richer' than that of the corresponding sets of eigenstates.

Just as the factors of the states can be represented pictorially in ways analogous to the Hasse diagrams of classical causal set theory, the operators of which they are outcomes can too. In the chosen convention, emergent time is taken to run upwards again, and every factor of an operator will be denoted by a square; so, an operator associated with k factors may be represented by k squares in a row. Arrows pointing into a square come from the group of factor states that are tested by the factor operator it represents, whereas arrows leaving a square point to the set of outcome factor states of this factor operator.

As an example, a graphical representation of the theorem that 'separable tests only have separable outcomes' is given in Figure 5.4. Figure 5.4a shows a separable operator $\hat{O}_{12}$ producing an entangled outcome ψ^{12}, which is a forbidden process. In Figure 5.4b, however, an entangled operator $\hat{O}^{12}$ is producing a separable outcome ψ_{12}, whilst in Figure 5.4c an entangled operator $\hat{O}^{12}$ is producing an entangled outcome ψ^{12}, and these processes are allowed.

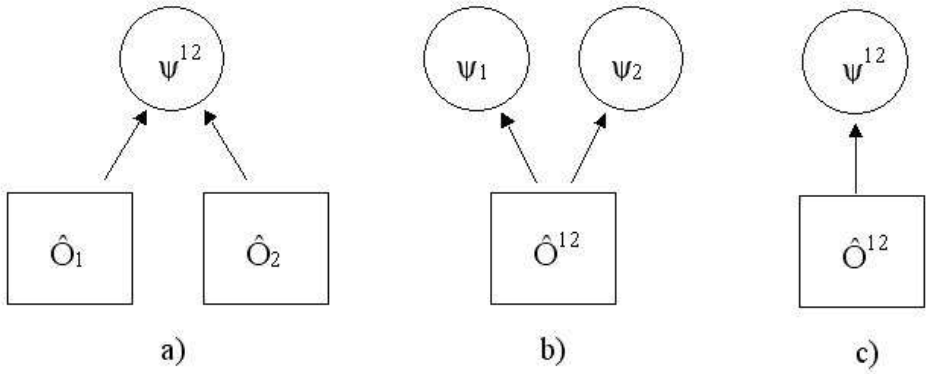

Figure 5.4: Pictorial representation of the relationship between operators and factor states. The process described in Figure a) is forbidden, whereas those of Figures b) and c) are allowed.

The ideas and theorems of this section place important mathematical constraints on the operators used in the development of the Universe. In a fully quantum Universe represented by a unique state Ψ_n, which is an eigenstate of an operator $\hat{\Sigma}_n$ in an enormous and fundamentally factorised Hilbert space $\mathcal{H}_{[1...N]}$, if Ψ_n is a separable product of factor sub-states, some of which may themselves be entangled relative to $\mathcal{H}_{[1...N]}$, then the conclusion must be that the individual factors of the operator $\hat{\Sigma}_n$ associated with the entangled factors of Ψ_n cannot themselves be factorised any further within the factor Hilbert spaces containing these entangled factor states. This result will lead to important consequences for the generation of quantum causal sets, as discussed in the next section.

5.6 Einstein Locality and Quantum Causal Sets

Section 5.4 indicated where relationships analogous to those of classical causal set theory may arise from a consideration of the changes in separability of the quantum state of the Universe. So, given that Section 5.5 showed that the operators responsible for developing the Universe's state may also exhibit properties of entanglement and separability, it might therefore be expected that these operators will also generate causal set structures. This

implies the existence of two different types of causal set in a fully quantum picture of the Universe, and these may in turn lead to different results in the large scale limit when considering aspects of emergent physical spacetime. This discussion will be the focus of the present section.

Before elaborating on this point, however, it should be stressed that any parallels between tests and states should not necessarily be assumed too automatically. After all, vectors and operators are mathematically very different. For example, the states are members of D-dimensional, complex Hilbert spaces $\mathcal{H}^{(D)}$, whereas the corresponding tests that act upon them exist in D^2-dimensional, real vector spaces $\mathbb{H}(\mathcal{H}^{(D)})$. Similarly, a bra state $|\Psi\rangle$ may be represented by a column vector with D elements, whilst the operators may be represented by self-adjoint $D \times D$ square matrices.

A further difference is evident from an examination of the product structure of the vector spaces. For two states $\Psi, \Phi \in \mathcal{H}^{(D)}$, it is possible to define an inner product of the form $\langle\Phi|\Psi\rangle \in \mathbb{C}$, which is interpreted in the proposed paradigm as the probability amplitude for the Universe to develop from the state Ψ to the state Φ. Conversely, no such inner product is defined for two operators $\hat{A}, \hat{B} \in \mathbb{H}(\mathcal{H}^{(D)})$, and there is hence no analogous physical interpretation. However, it is possible to define a third operator $\hat{C} \in \mathbb{H}(\mathcal{H}^{(D)})$ as the product $\hat{C} \equiv \hat{A}\hat{B}$, even though this type of transformation has no equivalent in the space of states. Indeed for vectors, the product $\Psi\Phi$ is meaningless.

There are also more obvious differences between vectors and operators regarding what they physically represent in quantum theory. The vectors represent the states of actual quantum systems, and so every phenomenon that is associated with wavefunctions in the laboratory has also to be applicable for the vectors. Thus, the vectors may be expected to exist in complex linear superpositions, and may appear to exhibit non-local and non-classical correlations that are at odds with emergent views of relativity and general covariance.

On the other hand, the operators are assigned to represent the observables of quantum theory, and these tend to have classical analogues that obey Einstein locality and causality: tests separated by spacelike distances do not affect one another. In fact, the canonical quantisation procedure successfully employed in conventional quantum mechanics is a process by which classical variables are directly replaced with their quantum operator counterparts. It should not, then, perhaps be too surprising that the resultant quantum operators therefore appear to obey classical laws of dynamics. An example here is that operators associated with emergent observables separated by spacelike distances tend to commute.

This point is very much the stance of Peres: quantum mechanics as such does not

strictly have to satisfy covariance in every respect, but its physical observables do [58].

A physical illustration of this type of argument is evident in quantum field theory. Local observables such a energy and momentum density operators satisfy microscopic causality, because their commutators vanish at spacelike intervals, but the local quantum fields out of which they are constructed need not commute at such separations [59]. In other words, Einstein locality must always hold for the physical observables, but it need not for the quantum states themselves. Of course, this may in turn be because the states are never directly 'experienced' *per se*, whereas it was only ever by experiencing physical observables that the (emergent) theory of relativity was discovered.

The differences highlighted above between states and operators should manifest themselves in the type of causal sets they produce. Specifically, whatever type of structure arises from the state's causal set may be expected to exhibit characteristics of non-locality, whereas whatever type of structure arises from the operator's causal set might conversely be expected to obey Einstein locality. Indeed, if this were not the case it would be necessary to explain how these features of empirical physics otherwise emerge in the observed Universe if they are not present on the underlying pregeometric level.

As discussed a number of times so far in this work, the dynamics proposed for the Universe is that its state $\Psi_n \in \mathcal{H}^{(D)}_{[1..N]}$ is developed by collapsing into one of the D orthonormal eigenvectors Φ^i, $i = 1, ..., D$, of an Hermitian operator $\hat{\Sigma}_{n+1} \in \mathbb{H}(\mathcal{H}^{(D)}_{[1..N]})$. Further, the conditional probability $P(\Psi_{n+1} = \Phi^i | \Psi_n, \hat{\Sigma}_{n+1})$ that the next state Ψ_{n+1} will be the i^{th} eigenvector of $\hat{\Sigma}_{n+1}$, given that the Universe is initially in a state Ψ_n and is tested with an operator $\hat{\Sigma}_{n+1}$, is determined by the usual rule of Born:

$$P(\Psi_{n+1} = \Phi^i | \Psi_n, \hat{\Sigma}_{n+1}) = \left| \langle \Phi^i | \Psi_n \rangle \right|^2 . \tag{5.58}$$

The above probability of the Universe collapsing from a state Ψ_n to one of the eigenstates Φ^i of an operator $\hat{\Sigma}_{n+1}$ may be associated with the concept of entropy. Recall that the Shannon entropy, S, attached to a particular probability distribution $\{p_1, p_2, ..., p_M\}$ is given by

$$S \equiv -\sum_{r=1}^{M} p_r \ln p_r, \tag{5.59}$$

and is a reflection of a physicist's ignorance of the result prior to a test that has M outcomes of weighted probability [22].

So, the Shannon entropy associated with the Universe jumping into one out of a set of D possible outcomes Φ^i of a given test $\hat{\Sigma}_{n+1}$, each with probability $P^i = P(\Psi_{n+1} =$

$\Phi^i|\Psi_n, \hat{\Sigma}_{n+1})$, is given by

$$\begin{aligned} S &= -\sum_{i=1}^{D} P^i \ln P^i \\ &= -\sum_{i=1}^{D} \left|\langle \Phi^i|\Psi_n\rangle\right|^2 \ln \left|\langle \Phi^i|\Psi_n\rangle\right|^2 . \end{aligned} \tag{5.60}$$

Note that this is a classical entropy result, as expected because state reduction processes do not permit quantum interference terms.

Equation (5.58) provides the correct probability for obtaining a particular next state $\Psi_{n+1} = \Phi^i$ as the result of a particular test $\hat{\Sigma}_{n+1}$, and equation (5.60) looks at the corresponding entropy associated with the set of potential outcomes of this measurement. What these relations do not do, however, is provide an answer as to why the test $\hat{\Sigma}_{n+1}$ was ever used in the first place; they say nothing about the Universe's actual selection of this particular operator. This is perhaps unsettling, because without specifying which operator $\hat{\Sigma}_{n+1}$ is chosen to test the Universe, the probability amplitude $\langle \Psi_{n+1} = \Phi^i|\Psi_n\rangle$ is meaningless. Without specifying $\hat{\Sigma}_{n+1}$ it is quite possible that a different operator $\hat{\Sigma}'_{n+1}$ could be used, and this alternative test may not even have Φ^i as an eigenstate. In this case it would then be pointless to ask about the relative probability of the next state Ψ_{n+1} being Φ^i.

Although the issue is discussed to some extent in Chapter 8, at present there is no known mechanism for understanding how or why the Universe selects a particular operator $\hat{\Sigma}_{n+1}$ to test itself, a point that is summed up by the statement: '*Only some of the rules [of the Universe] are currently understood; we can calculate answers to quantum questions, but we do not know why those questions have been asked in the first place*' [1]. It is asserted, then, that any measure of the entropy associated with the Universe developing through a series of states should take this additional ignorance into account.

To this end, recall the conjecture of Chapter 3 that the Universe may be completely parameterised by a unique stage Ω_n defined as $\Omega_n \equiv \Omega(\Psi_n, I_n, R_n)$. Moreover, recall that the current 'information content' I_n was taken to contain the set of possible operators $\{\hat{O}_n^b : b = 1, ..., B\}$ that might each provide a basis for the next potential state Ψ_{n+1} of the Universe in the next stage Ω_{n+1}. Then, $\hat{\Sigma}_{n+1}$ will be one of B possibilities, which may be labelled $\hat{\Sigma}_{n+1}^b (\equiv \hat{O}_n^b)$. If it may be assumed that there exists a certain probability $P^b = P(\hat{O}_n^b|\Omega_n)$ that a particular operator $\hat{O}_n^b$ is chosen by the Universe at time $n+1$ to be $\hat{\Sigma}_{n+1}$, then $\sum_{b=1}^{B} P^b = 1$, noting that possibly $B = 1$ if the operators are selected deterministically.

Thus, if the Universe is initially in the stage Ω_n, the probability $P^{(b,i)}$ that it will be tested by an operator $\hat{\Sigma}_{n+1} = \hat{\Sigma}_{n+1}^b = \hat{O}_n^b$ and will then subsequently jump from the state

Ψ_n to a particular state $\Psi_{n+1} = \Phi^{b,i}$, which is referred to as the i^{th} eigenvector of the operator $\hat{\Sigma}^b_{n+1}$, is given by

$$P^{(b,i)} = P^b P^i = P(\hat{\Sigma}^b_{n+1}|\Omega_n) \left|\langle \Phi^{b,i}|\Psi_n\rangle\right|^2 \tag{5.61}$$

where $\sum_{i=1}^{D} \left|\langle \Phi^{b,i}|\Psi_n\rangle\right|^2 = 1$ as expected.

Further, the Shannon entropy that may be associated with this jump is given by

$$\begin{aligned} S^{(1)} &= -\sum_{b=1}^{B}\sum_{i=1}^{D} P^{(b,i)} \ln P^{(b,i)} \\ &= -\sum_{b=1}^{B}\sum_{i=1}^{D} P(\hat{\Sigma}^b_{n+1}|\Omega_n) \left|\langle \Phi^{b,i}_{n+1}|\Psi_n\rangle\right|^2 \\ &\quad \times \left\{ \ln P(\hat{\Sigma}^b_{n+1}|\Omega_n) + \ln \left|\langle \Phi^{b,i}_{n+1}|\Psi_n\rangle\right|^2 \right\} \\ &= -\sum_{b=1}^{B} P(\hat{\Sigma}^b_{n+1}|\Omega_n) \ln P(\hat{\Sigma}^b_{n+1}|\Omega_n) \\ &\quad -\sum_{b=1}^{B} P(\hat{\Sigma}^b_{n+1}|\Omega_n) \sum_{i=1}^{D} \left|\langle \Phi^{b,i}|\Psi_n\rangle\right|^2 \ln \left|\langle \Phi^{b,i}|\Psi_n\rangle\right|^2 \\ &= S^t + \sum_{b=1}^{B} P(\hat{\Sigma}^b_{n+1}|\Omega_n) S^b \end{aligned} \tag{5.62}$$

where $S^t \equiv -\sum_{b=1}^{B} P(\hat{\Sigma}^b_{n+1}|\Omega_n) \ln P(\hat{\Sigma}^b_{n+1}|\Omega_n)$ is the entropy associated with the selection of the test, $S \equiv -\sum_{i=1}^{D} \left|\langle \Phi^{b,i}|\Psi_n\rangle\right|^2 \ln \left|\langle \Phi^{b,i}|\Psi_n\rangle\right|^2$ is the entropy associated with the collapse from the state Ψ_n to one of the set of possible eigenvectors of this test, and the superscript (1) is used to denote that $S^{(1)}$ is defined over one jump. Thus, the entropy (5.62) reflects the ignorance associated with how the Universe might develop from the current stage Ω_n to a potential stage Ω_{n+1}.

It is possible to extend these ideas to gain an appreciation of the entropy associated with the Universe prior to it developing over a series of jumps. Define $P^{b_{n+1}}_{(b_n,i_n)} = P(\hat{\Sigma}^{b_{n+1}}_{n+1}|\Omega^{b_n,i_n}_n)$ as the probability that an operator $\hat{\Sigma}^{b_{n+1}}_{n+1}$ will be chosen from a set $\{\hat{\Sigma}^{b_{n+1}}_{n+1} : b_{n+1} = 1, ..., B_{n+1}\}$ of B_{n+1} possibilities, given that the Universe is currently in the stage $\Omega^{b_n,i_n}_n = \{\Psi^{b_n,i_n}_n, I^{b_n,i_n}_n, R^{b_n,i_n}_n\}$ where the superscript (b_b, i_n) implies, for example, that the state Ψ^{b_n,i_n}_n is one of the D outcomes $\Psi^{b_n,i_n}_n = \Phi^{b_n,i_n}$, for $i = 1, ..., D$, of one of B_n possible tests $\hat{\Sigma}^{b_n}_n$ contained in the previous stage $\Omega^{b_{n-1}}_{n-1}$.

Similarly, the variable

$$\begin{aligned} P^{i_{n+1}}_{(b_n,i_n)} &\equiv P(\Psi^{b_{n+1},i_{n+1}}_{n+1} = \Phi^{b_{n+1},i_{n+1}}|\Psi^{b_n,i_n}_n, \hat{\Sigma}^{b_{n+1}}_{n+1}) \\ &= \left|\langle \Phi^{b_{n+1},i_{n+1}}|\Psi_n\rangle\right|^2 \end{aligned} \tag{5.63}$$

is defined as the probability that the outcome of this chosen test $\hat{\Sigma}_{n+1}^{b_{n+1}}$ is $\Psi_{n+1}^{b_{n+1},i_{n+1}} = \Phi^{b_{n+1},i_{n+1}}$.

Overall then,

$$P_{(b_n,i_n)}^{(b_{n+1},i_{n+1})} = P(\hat{\Sigma}_{n+1}^{b_{n+1}}|\Omega_n^{b_n,i_n})P(\Psi_{n+1}^{b_{n+1},i_{n+1}} = \Phi^{b_{n+1},i_{n+1}}|\Psi_n^{b_n,i_n},\hat{\Sigma}_{n+1}^{b_{n+1}}) \tag{5.64}$$

is defined as the probability that, given an initial stage $\Omega_n^{b_n,i_n}$, the next test will be $\hat{\Sigma}_{n+1}^{b_{n+1}}$ and the resulting next state will be the eigenvector $\Psi_{n+1}^{b_{n+1},i_{n+1}} = \Phi^{b_{n+1},i_{n+1}}$ of $\hat{\Sigma}_{n+1}^{b_{n+1}}$.

Since all the probabilities are classical due to the nature of the state reduction process, it is possible to define chains of jumps in terms of products of probabilities. Thus,

$$P_{(b_n,i_n),(b_{n+1},i_{n+1}),...,(b_{n+m-1},i_{n+m-1})}^{(b_{n+1},i_{n+1})(b_{n+2},i_{n+2}),...,(b_{n+m},i_{n+m})} = P_{(b_n,i_n)}^{(b_{n+1},i_{n+1})}P_{(b_{n+1},i_{n+1})}^{(b_{n+2},i_{n+2})}...P_{(b_{n+m-1},i_{n+m-1})}^{(b_{n+m},i_{n+m})} \tag{5.65}$$

is defined as the probability that the Universe will jump from the state $\Psi_n^{b_n,i_n} = \Phi^{b_n,i_n}$ to the state $\Psi_{n+1}^{b_{n+1},i_{n+1}} = \Phi^{b_{n+1},i_{n+1}}$ via the test $\hat{\Sigma}_{n+1}^{b_{n+1}}$, and that this new state will jump to the state $\Psi_{n+2}^{b_{n+2},i_{n+2}} = \Phi^{b_{n+2},i_{n+2}}$ via the test $\hat{\Sigma}_{n+2}^{b_{n+2}}$, and so on until the state $\Psi_{n+m-1}^{b_{n+m-1},i_{n+m-1}} = \Phi^{b_{n+m-1},i_{n+m-1}}$ finally jumps to the state $\Psi_{n+m}^{b_{n+m},i_{n+m}} = \Phi^{b_{n+m},i_{n+m}}$ via the test $\hat{\Sigma}_{n+m}^{b_{n+m}}$.

Using this notation, the entropy $S^{(1)}$ given in (5.62) may be rewritten as $S^{(1)} = -\sum_{b_{n+1}=1}^{B_{n+1}}\sum_{i_{n+1}=1}^{D}P_{(b_n,i_n)}^{(b_{n+1},i_{n+1})}\ln P_{(b_n,i_n)}^{(b_{n+1},i_{n+1})}$. Similarly, the entropy $S^{(2)}$ over two jumps may be given by

$$\begin{aligned} S^{(2)} &= -\sum_{b_{n+1}=1}^{B_{n+1}}\sum_{i_{n+1}=1}^{D}\sum_{b_{n+2}=1}^{B_{n+2}}\sum_{i_{n+2}=1}^{D} \\ &\quad \left[P_{(b_n,i_n)}^{(b_{n+1},i_{n+1})}P_{(b_{n+1},i_{n+1})}^{(b_{n+2},i_{n+2})}\left\{\ln(P_{(b_n,i_n)}^{(b_{n+1},i_{n+1})}P_{(b_{n+1},i_{n+1})}^{(b_{n+2},i_{n+2})})\right\}\right] \end{aligned} \tag{5.66}$$

such that overall, the m jump entropy $S^{(m)}$ may hence be defined as

$$\begin{aligned} S^{(m)} &= -\sum_{b_{n+1}=1}^{B_{n+1}}\sum_{i_{n+1}=1}^{D}\sum_{b_{n+2}=1}^{B_{n+2}}\sum_{i_{n+2}=1}^{D}\cdots\sum_{b_{n+m}=1}^{B_{n+m}}\sum_{i_{n+m}=1}^{D} \\ &\quad \left[\begin{array}{c} P_{(b_n,i_n)}^{(b_{n+1},i_{n+1})}P_{(b_{n+1},i_{n+1})}^{(b_{n+2},i_{n+2})}...P_{(b_{n+m-1},i_{n+m-1})}^{(b_{n+m},i_{n+m})}\times \\ \left\{\ln P_{(b_n,i_n)}^{(b_{n+1},i_{n+1})} + \ln P_{(b_{n+1},i_{n+1})}^{(b_{n+2},i_{n+2})} + ... + \ln P_{(b_{n+m-1},i_{n+m-1})}^{(b_{n+m},i_{n+m})}\right\} \end{array}\right] \\ &= -\sum_{b_{n+1}=1}^{B_{n+1}}\sum_{i_{n+1}=1}^{D}\cdots\sum_{b_{n+m}=1}^{B_{n+m}}\sum_{i_{n+m}=1}^{D} \\ &\quad \left[\begin{array}{c} P(\hat{\Sigma}_{n+1}^{b_{n+1}}|\Omega_n^{b_n,i_n})\left|\langle\Phi^{b_{n+1},i_{n+1}}|\Psi^{b_n,i_n}\rangle\right|^2\times ... \\ ... \times P(\hat{\Sigma}_{n+m}^{b_{n+m}}|\Omega_{n+m-1}^{b_{n+m-1},i_{n+m-1}})\left|\langle\Phi^{b_{n+m},i_{n+m}}|\Psi^{b_{n+m-1},i_{n+m-1}}\rangle\right|^2 \\ \times\left\{\begin{array}{c}\ln P(\hat{\Sigma}_{n+1}^{b_{n+1}}|\Omega_n^{b_n,i_n}) \\ +\ln\left|\langle\Phi^{b_{n+1},i_{n+1}}|\Psi^{b_n,i_n}\rangle\right|^2 + ... \\ ... + \ln P(\hat{\Sigma}_{n+m}^{b_{n+m}}|\Omega_{n+m-1}^{b_{n+m-1},i_{n+m-1}}) \\ +\ln\left|\langle\Phi^{b_{n+m},i_{n+m}}|\Psi^{b_{n+m-1},i_{n+m-1}}\rangle\right|^2\end{array}\right\}\end{array}\right]. \end{aligned} \tag{5.67}$$

Since there is at present no way of knowing how large B_n actually is, or how its value changes with n, the number of potential next states may be literally gigantic. Moreover,

the scope of possible 'futures' for the Universe will clearly increase rapidly over even a relatively small number of jumps, especially when it is considered how large the dimension D of the Hilbert space is likely to be, and hence how large the set of orthogonal eigenstates is for each operator.

It is therefore obvious that the number of possible causal sets that may be produced over a chain of jumps is also gigantic. This point is analogous to the Hasse diagram of Hasse diagrams presented in [38] that are generated by examining the collection of possible classical causal sets that can be grown by adding one new event to an existing set. In fact, over the m jumps from $\Psi_n^{b_n,i_n}$ to $\Psi_{n+m}^{b_{n+m},i_{n+m}}$ in the above case, there will in principle be a whole 'tree' of M different possible causets produced, where

$$M \leq D^m \times (B_{n+1} \times B_{n+2} \times B_{n+3} \times ... \times B_{n+m}). \tag{5.68}$$

Note that the inequality reflects the fact that there may be some degeneracy in this set of M members, because two operators $\hat{\Sigma}_{n+x}^{b_{n+x}}$ and $\hat{\Sigma}_{n+x}^{c_{n+x}}$ for $b_{n+x}, c_{n+x} = 1, ..., B_{n+x}$ and $1 \leq x \leq m$ may have y eigenvectors in common, $0 \leq y < D$.

Although the probabilities $P_{(b_{n+x},i_{n+x})}^{i_{n+x+1}}$ for particular state transitions from Ψ_{n+x} to $\Phi^{b_{n+x+1},i_{n+x+1}}$ given a specific operator $\hat{\Sigma}_{n+x+1}^{b_{n+x+1}}$ are evaluated by the Born rule $P_{(b_{n+x},i_{n+x})}^{i_{n+x+1}} = \left|\langle \Phi^{b_{n+x+1},i_{n+x+1}} | \Psi_{n+x} \rangle\right|^2$, as mentioned earlier there is no similar rule known for specifying the probabilities $P_{(b_{n+x},i_{n+x})}^{b_{n+x+1}} = P(\hat{\Sigma}_{n+x+1}^{b_{n+x+1}} | \Omega_{n+x}^{b_{n+x},i_{n+x}})$ of choosing this particular operator $\hat{\Sigma}_{n+x+1}^{b_{n+x+1}}$ from a set of B_{n+x+1} possibilities.

Of course this selection could actually be deterministic, so there is in fact no choice, and this would give rise to a semi-clockwork Universe in which quantum state reduction provides the only randomness. In such a Universe it would always be possible to predict, with certainty, in advance which test the Universe will choose to test itself with x stages into the future, assuming that this deterministic rule is known.

Alternatively perhaps, in a Universe free of external observers the choice of next test may depend somehow on the current state that the Universe is in. As will be discussed in Chapter 8, the way in which such self-referential Universes might be developed after x jumps may not be knowable until it has developed through the $x-1$ preceding stages. This is possibly how (at least some of) the dynamics of the physical Universe works, because human scientists, themselves just groups of factors of the state of the Universe, do appear to be able to exert some sort of influence on how this state they exist as part of actually gets tested, because they do seem able to prod and probe those factors that represent their surroundings.

However, even if the physical Universe does develop according to a type of self-referential mechanism, exactly how its next operator might be controlled by the current state is still unknown.

It is here that an appeal is made to empirical physics. Since it appears to be the case that the Universe is highly classical, and hence highly separable, whichever method is used by the Universe to select its next test seems to be constrained to choose an operator that possesses a highly separable set of eigenvectors. Since current thinking also indicates that the Universe has, on average, changed very slowly over the last 10 or so billion years, however the Universe actually selects its next operator must ensure that the test chosen has an outcome that it almost identical to the present state. In addition, given that it seems an experimentally verified fact that physical observables in the Universe are constrained by Einstein locality, it can also be assumed that whatever mechanism the Universe uses to select the next operator to test itself with, the physically observed outcomes of this operator must also obey the principles of relativity.

Rephrasing this last point, since the operators are expected to correspond to physical observables in the emergent limit, their results must eventually correspond to the outcomes of their classical counterparts. Similarly, and reversing this line of thinking, if physicists are able to quantise particular classical variables to get the quantum operator equivalents, the resulting quantum operators may still ultimately be expected to obey some of the classical laws. For example, if classical variables are always forced to obey Einstein locality, and if these variables can be directly quantised to produce operators that yield accurate physical results, it may be fair to assume that, in general, quantum operators in the Universe are also forced to obey Einstein locality. So, their observed outcomes will not permit features such as superluminal communication. In other words, if observed physics is limited by Einstein locality, the operators representing these observables may be too.

Thus, however the Universe selects its quantum operators, the choice made will ultimately be expected to give the results familiar to classical experiments. Moreover, since Einstein locality is an important fact of classical physics, this feature must therefore somehow be reflected by the operators. So, one way to guarantee this condition would be to argue that only those operators that are constrained by relativistic relationships are allowed to be chosen. In other words, any operator selected by the Universe must have a set of factor operators that do not violate classical causal laws.

If the above conjectures are correct, they might then suggest that the causal set structures generated by the changing operators create a pattern of Einstein locality, in terms of their arrangements of factorisation and entanglement. Further, since the conclusion of the previous section was that separable operators can only have separable outcomes, this pattern of operators would in turn produce an arrangement of factor states that also frequently share relationships obeying Einstein locality. And, since it is the states that actually constitute physical reality, the observed relativity in the physical Universe may hence be seen to be a consequence of a causal set formed from operators constrained to

obey Einstein locality. Only under certain specially contrived circumstances, such as those occurring in EPR experiments, would the true quantum non-locality of the states become apparent.

It is quite possible that the Universe could choose a series of operators to test itself with that produces a causal set structure that changes very little from one stage to the next. Indeed, all sorts of patterns of separations and entanglements could be present in the set, with many different types of local or global relationships appearing to emerge over a chain of jumps, and even the possibility of particular groups or families of factors existing semi-permanently. Such a series of patterns is analogous to those produced in automata such as Conway's "Game of Life" [60], and could ultimately be responsible for all the observed features of the physical Universe, including, for example, apparent persistence, space, dimension, particle physics, and even semi-classical looking endo-observers who are made up of groups of factors seeming to persist in a nearly unchanged way over very many jumps. Of course, underlying all of this structure would still be the counting procedures used to estimate the number of jumps (giving rise to an emergent local temporal parameter) and estimations of familial relationships (which give rise to emergent spatial relationships).

As an example, consider a classical cellular automaton in which the values in the individual cells depend somehow on 'nextdoor neighbour' interactions. Such a dynamics may give rise to zones of causal influence, in which cells outside of this zone are unable to influence cells inside it, and vice versa. It is possible that the operators testing the Universe could also adopt a dynamics that depends on the interaction with 'neighbours', analogous to such classical cellular automata, where neighbouring factors are defined in terms of the familial relationships formed by the outcome states. For example, one way of defining two factors of the operator as neighbours might be if their respective outcome factor eigenstates share a 'parent' factor state. Omitting the exact details, the overall point is that if the patterns of separations produced by the operator's causal set are somehow forced to look, to some extent, like a type of cellular automaton whose cells' values change according to nextdoor neighbour interactions, the effect might be a case in which the resulting states will possess patterns of separability that incorporate these causal zones. Further, such zones may strongly resemble the lightcone structures of relativity, and might produce a set of observed outcomes that are fully consistent with Einstein causality.

The overall conclusion of this section is that there are two different types of causal set present in a fully quantum Universe. The first is generated by the changing separability of the operators used to test the Universe. Whilst it is not known how this set is produced, it does seem to give rise to observables that respect Einstein locality and everything this entails, such as a maximum speed for the propagation of physical signals.

The second causal set is generated by the changing separability of the state. This set incorporates all of the features associated with quantum states in conventional physics, and can, for example, support the non-local correlations and apparently superluminal transmission of information familiar in EPR type experiments. In fact, since such correlations do not respect Einstein locality, it might be taken as a further indication that there is an underlying quantum and pregeometric structure lurking beneath the classical and continuous Lorentzian spacetime manifold. However, the observation that most of empirical science appears to follow classical physics does seem to indicate that it is only under rare and special circumstances that the true 'quantum' nature of the states becomes blatantly apparent. Indeed, scientists generally have to try very hard in order to prepare a factor of the Universe's state that is entangled, for instance, and even harder to keep it that way. The repeated efforts of computer scientists to build a working quantum computer is a good example of this.

The conclusion, then, is that since the states are ultimately the outcome of the operators, and since in a self-referential Universe the choice of operator may depend somehow on the current state, there must be a very careful interplay between the two different causal sets in order to produce the type of Universe that physicists actually observe.

5.7 Physical Examples

The objective of this chapter has been to investigate the types of mechanism inherent in a quantum Universe that may ultimately be responsible for the existence of spatial relationships. Whilst there is still a very long way to go before the details are understood of exactly how the deep and intricate theory of General Relativity could emerge from the fully quantum picture, it is still possible to schematically describe how the current line of thinking might fit into a number of physical situations. To this end, in this final section a number of physically motivated examples are discussed in terms of the connections between the states representing them and the operators used in their development.

5.7.1 The Quantum Big Bang

The physical Universe is very large. However, given that it also currently appears to be expanding, the conventional conclusion is that it was once very much smaller than it is today. In fact, by observing the acceleration of its increase in volume[8], cosmologists have

[8]Depending upon different particular models and metrics used to describe the large scale structure of the Cosmos.

extrapolated backwards in time and concluded that the Universe must once have had no size at all [61]. Further, by measuring the light emitted from far off clusters of stars in order to determine when they were formed [62], astronomers have managed to establish that the Universe had no size at a time between about 10 and 20 billion years ago. This is the traditional Big Bang scenario, and is often taken to imply the very beginning of time, physics, existence and reality.

In the paradigm proposed in this work, physical space is a marker of separability of the Universe's state. Thus, the observations of the above Standard Model of cosmology may be interpreted here as a wavefunction that is today highly separable, but was considerably less so in the past. Moreover, if there was a time at which the Universe could be attributed with no physical size, there could then have been no spatial relationships existing, and hence by the presented arguments its state must consequently have been completely entangled.

Classical general relativistic cosmology asserts that time must have began at the Big Bang, because without space there is no spacetime and hence no time. However this is not a necessary conclusion of the present work, and in the Stages paradigm it is conjectured that time had no beginning. After all, physical time is seen as an emergent phenomenon appearing as a complex vector jumps from one state to the next in its enormous Hilbert space (as elaborated upon further in Chapter 8). It is hence quite possible that the state and the Hilbert space can be conjectured as existing eternally, assuming such a phrase can be used to describe something existing 'outside' of physical time, removing from the dynamics the uncomfortable view of conventional physics that the Universe suddenly appeared out of nothing and 'no-when'.

During an era that might be referred to as pre-Big Bang (i.e. beyond the time cosmologists have extrapolated a zero size Universe), the Universe's state would have been completely entangled, from the point of view of the proposed paradigm. In fact its state may have remained entangled for a large number of jumps, during which period no classical structures, including space, could have emerged. From an alternative perspective, whichever operators were used to develop the Universe through this chain of entangled states must themselves have been completely entangled, because separable operators cannot have entangled outcomes.

Consider, however, a case in which the Universe (somehow) eventually chooses an operator to test itself that has separable outcomes, and further that the Universe ends up jumping into one of these separable states. This may at first glance appear unlikely, given the discussion of Chapter 4 that separable states form a set of measure zero relative to the set of all states, but is not impossible in a Universe that may have already remained entangled for a 'near-infinite' number of jumps. Besides this, since it is an empirical fact

that the Universe is large, classical looking and separable, it can be concluded that at some point it must have stopped being entangled.

After this collapse to a separable state, the Rules governing the way the operators are chosen may have selected another operator that is also separable, and the Universe would then have jumped to another separable state. In fact, this new state could have even more factors than the previous one. If this procedure is repeated a number of times, a situation might arise in which the state of the Universe is monotonically becoming more separable as it develops, and this could eventually give rise to the observed expansion of continuous space. Overall, the selection of a series of separable operators drives the Universe to develop through a series of separable states, and hence to the possible emergence of spatial relationships.

The initial jump from a fully entangled state to a separable one could now be called the 'Quantum Big Bang', and this may ultimately be what physicists are really extrapolating back to when then examine the Universe's past and conclude that it once had no spatial size. However, unlike the Standard Model scenario, the presented description of the Universe's development has the desirable feature that there is no initial singularity at the Quantum Big Bang, and so is not associated with any of the accompanying problems of quantum gravity. In fact, this line of thinking once again reinforces the idea that simply quantising space and gravity is the wrong direction to proceed. Rather, the proposal is that space should perhaps be seen as something that is meaningless without quantum relations.

In addition to the removal of the singularity problem, the suggested dynamics for the development of the Universe may also provide an origin for the observed homogeneity and isotropy of the Universe. Just after the Quantum Big Bang, the individual factors of the Universe's state could still be highly entangled within their respective factor Hilbert spaces, which may themselves be of enormous dimension. Since entangled states exhibit the properties of non-local correlations, when the entangled factor states eventually develop into separable products of factor states (that are themselves contained in the factor Hilbert spaces of the larger factor Hilbert space containing the entangled factor state), these new factors may end up having similar 'properties', even though they may now appear to be large, emergent spatial distances apart. In other words, the non-local correlations of entangled factors just after the Quantum Big Bang may potentially help to solve the Horizon problem of cosmology.

Note that it is, in fact, entirely possible that before the chain of entangled states present in the pre-Big Bang era, there could have been whole cycles of expansion (i.e. increasing separability) and contraction (i.e. decreasing separability) back to a 'Quantum Big

Crunch' of renewed total entanglement. Indeed, there could also have been any number of 'false starts' in which the Universe chose a series of separable operators, before suddenly choosing an entangled operator and jumping back to a completely entangled state. However, if no information regarding these has survived into the present era, perhaps because no information can be encoded as relations between factors when a state is fully entangled, there is no way of knowing about them. A return to full entanglement represents a return to no familial relationships within the state, and since these are what might ultimately constitute physical phenomena, all that scientists can ever look at is what has happened since the last Quantum Big Bang.

Consider as an example of these ideas a universe represented by a state Ψ_n contained within a Hilbert space $\mathcal{H}_{[1...N]}$ of dimension $D = 2^{2^M}$ consisting of a vast number $N = 2^M$ of qubit subregisters, where $M \in \mathbb{Z}^+$. With $\mathcal{H}_{[1...N]}$ is associated, in the usual way, a basis set $\mathcal{B}_{[1...N]}$ of orthonormal vectors and a skeleton set $\mathcal{S}_N$ of operators. Further, for all n the state Ψ_n is one of the D eigenstates of some Hermitian operator $\hat{\Sigma}_n$, where $\hat{\Sigma}_n \in \mathbb{H}(\mathcal{H}_{[1..N]})$.

By defining $n = 1$ as the Quantum Big Bang, then, if the above discussion is true, states Ψ_n for $n < 1$ are fully entangled relative to $\mathcal{B}_{[1...N]}$, and are hence the results of operators that are fully entangled. Conversely, states Ψ_n for $n \geq 1$ are separable into a number of factors.

Now, suppose that the Rules of the universe dictate that for $0 \leq n \leq M-1$ the operator $\hat{\Sigma}_{n+1}$ has twice as many factors as $\hat{\Sigma}_n$ according to the scheme:

$$\begin{aligned}
\hat{\Sigma}_0 &= \hat{A}^{1...2^M} \\
\hat{\Sigma}_1 &= \hat{A}^{1...2^{M-1}} \otimes \hat{A}^{(2^{M-1}+1)...2^M} \\
\hat{\Sigma}_2 &= \hat{A}^{1...2^{M-2}} \otimes \hat{A}^{(2^{M-2}+1)...2\times 2^{M-2}} \\
&\quad \otimes \hat{A}^{(2\times 2^{M-2}+1)...3\times 2^{M-2}} \otimes \hat{A}^{(3\times 2^{M-2}+1)...2^M} \\
&\quad \vdots \\
\hat{\Sigma}_M &= \hat{A}_1 \otimes \hat{A}_2 \otimes \hat{A}_3 \otimes ... \otimes \hat{A}_M
\end{aligned} \tag{5.69}$$

where, for example, $\hat{A}^{1...2^{M-1}}$ is an Hermitian operator in $\mathbb{H}(\mathcal{H}_{[1...N]})^{1...2^{M-1}}$ such that $\hat{\Sigma}_1 \in \mathbb{H}(\mathcal{H}_{[1...N]})^{(1...2^{M-1})\bullet((2^{M-1}+1)...2^M)} \subset \mathbb{H}(\mathcal{H}_{[1..N]})$. Further, any eigenstate of $\hat{A}^{1...2^{M-1}}$ is in $\mathcal{H}_{[1..2^{M-1}]}$, and duly contributes at least one factor to the next state Ψ_1.

Since separable operators only have separable outcomes, it is certain that Ψ_1 will have at least two factors, whereas Ψ_2 cannot have less than four factors, and Ψ_3 must have at least eight factors, and so on up to Ψ_M which is separable into M factors. So, whatever the operators $\hat{\Sigma}_{n+1}$ actually are, the resulting state Ψ_{n+1} of this universe may be expected to have more factors than the previous state Ψ_n, for $0 \leq n \leq M-1$; certainly, if $M \gg 1$

it may be the case that 'on average' the number of factors of the state Ψ_n could possibly increase roughly monotonically with $0 \leq n \leq M$.

Moreover, since separability has already been shown to be a necessary prerequisite for spatial relationships, this type of development with deterministically chosen operators may provide a basic starting point for a discussion on the expansion of space.

As a simple illustration of this last point, consider the case where $M = 2$, such that $N = 4$, $D = 16$, and the Hilbert space is denoted by $\mathcal{H}_{[1...4]}$. The operators for $n = 0, 1, 2$ are then of the form: $\hat{\Sigma}_0 = \hat{A}^{1...4}$, $\hat{\Sigma}_1 = \hat{A}^{12} \otimes \hat{A}^{34}$, and $\hat{\Sigma}_2 = \hat{A}_1 \otimes \hat{A}_2 \otimes \hat{A}_3 \otimes \hat{A}_4$.

A corresponding set of states in the development of this universe could therefore be

$$\begin{aligned} \Psi_0 &= \varphi^{1234} \quad , \quad \Psi_1 = \theta^{12 \bullet 34} \equiv \theta^{12} \otimes \theta^{34} \\ \Psi_2 &= \psi_{1234} \equiv \psi_1 \otimes \psi_2 \otimes \psi_3 \otimes \psi_4 \end{aligned} \tag{5.70}$$

where $\varphi^{1234} \in \mathcal{H}^{1234}$, $\theta^{12 \bullet 34} (\equiv \theta^{12} \otimes \theta^{34}) \in \mathcal{H}^{12 \bullet 34}$ and $\psi_{1234} \in \mathcal{H}_{1234}$. In this case, the changing separability of the state would consequently lead to the type of causal set structure illustrated in Figure 5.5.

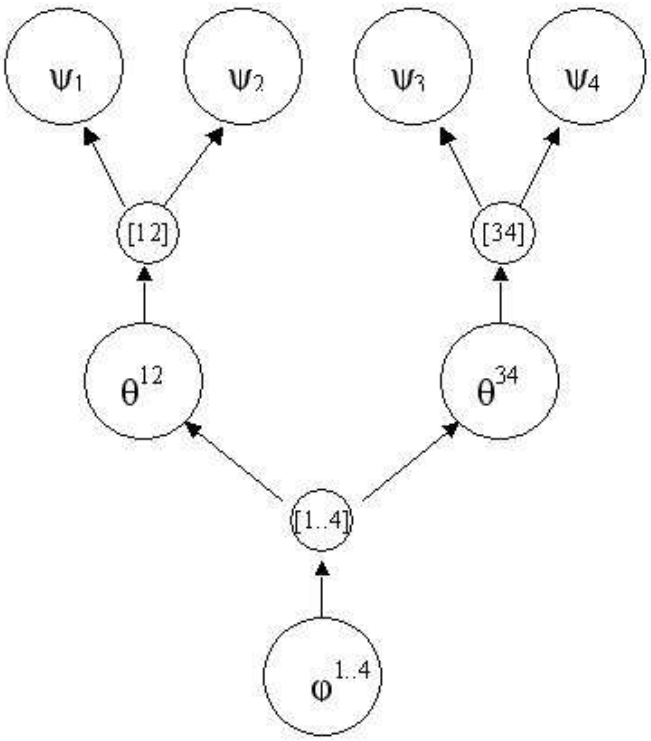

Figure 5.5: Causal set structure for the state of an expanding universe of four qubits.

As discussed previously in Section 5.4, such a universe with a deterministic choice of operator readily permits a discussion of embryonic lightcone structure, and so ultimately

also concepts of distance and metrics. In this sense, the states ψ_3 and ψ_4 are 'outside' of the causal future of θ^{12} because a counterfactual change in θ^{12} will not influence either ψ_3 or ψ_4.

Note that of the above scheme is not, of course, the only mechanism that could be used to model an expanding universe. There could instead be a type of 'feedback' mechanism, in which the choice of next operator is influenced by how separable or entangled the current state of the universe is. Alternatively, there could be a mechanism in which, for a finite series of jumps, an operator $\hat{\Sigma}_{n+1}$ is selected that has exponentially many more factors than the previous test $\hat{\Sigma}_n$. This latter type of process could cause the state Ψ_{n+1} to have exponentially more factors than the state Ψ_n, and this could lead to a period of rapid expansion analogous to the era of inflation postulated [48] in the Standard Model of cosmology.

5.7.2 EPR Paradoxes

As discussed in Chapter 3 the non-local consequences of quantum entanglement appear to cause problems for the theory of relativity, because the latter places physics in a background 'arena' of classical and continuous spacetime. For example, recall the EPR experiment featured earlier involving an entangled electron and positron. If the electron is measured first and found to be in a spin-up state then the positron will consequently be found to be in a spin-down state, and vice versa. Further, the standard priciples of quantum mechanics (as verified by, amongst others, the Bell inequality) argue that before the first measurement both the electron and positron may be thought of as existing in both spin states simultaneously. Relativity's problem with this can then be summed up by the question: if the electron detector is x metres away from the positron detector, and if the positron's spin is measured t seconds after the electron's spin is measured, then how can any physical signal 'inform' the second particle that, say, the electron has been found in a spin up state such that the positron must consequently be found to be spin down, if $x/t > c$ where c is the velocity of light? In other words, the measurement of a particle at one location appears to be influenced by a measurement of a particle at a different location, even though these two events are not in causal contact.

In fact, by setting up the system so that $x \gg 1$ and $t \ll 1$ it has been experimentally shown [63] that if the correlations were arranged by a signal travelling physically from one particle to the other, this signal would require a velocity of at least $10^4 c$, and this conclusion appears to be contradict special relativity which asserts that nothing can travel faster than the speed of light.

However, in the paradigm proposed in this work the EPR paradox is not a problem at all. From the presented viewpoint there is no background space over which correlations have to cross, and the measurement of the electron and positron are only spacelike separated from an emergent point of view. From the point of view of the proposed fully quantum approach, the entangled electron-postitron state, the two detectors, the physicists and everything else are just associated with factors of the state representing the Universe, and so it is not correct to say that when the positron is measured it is fundamentally x metres away from where the electron was measured. On the pregeometric quantum level the electron and positron are nothing but factors of a vector in a Hilbert space, and physical spatial relationships are meaningless here.

As a schematic illustration of how an EPR type experiment might proceed in a fully quantum Universe, consider the following chain of stages in the state's development. Note first, however, that as with the Schrödinger's cat discussion of Section 4.3.3 the example below is really just a highly simplified overview; in reality detectors (and the physicists observing them) are incredibly complicated sets of factors, constantly undergoing many different types of internal developments and interactions with their surroundings.

Let a particular split of the Hilbert space $\mathcal{H}$ of the Universe be of the form

$$\mathcal{H} = \mathcal{H}_e \otimes \mathcal{H}_p \otimes \mathcal{H}_E \otimes \mathcal{H}_P \otimes \mathcal{H}_U \tag{5.71}$$

where $\mathcal{H}_e$ represents the factor Hilbert space of an electron, $\mathcal{H}_p$ represents the factor Hilbert space of a positron, $\mathcal{H}_E$ represents the factor Hilbert space of an electron detector, $\mathcal{H}_P$ represents the factor Hilbert space of a positron detector, and $\mathcal{H}_U$ represents the factor Hilbert space containing everything else in the Universe. Note that none of these five factor Hilbert spaces need be of prime dimension.

Consider now an operator $\hat{\Sigma}_n$ factorisable in the form:

$$\hat{\Sigma}_n = \hat{A}^{ep} \otimes \hat{A}_E \otimes \hat{A}_P \otimes \hat{A}_U \tag{5.72}$$

where, for example, $\hat{A}^{ep} \in \mathbb{H}(\mathcal{H})^{ep}$, with the entanglement $\mathcal{H}^{ep} \subset \mathcal{H}_{[ep]}$, and $\hat{\Sigma}_n \in \mathbb{H}(\mathcal{H})^{ep}_{EPU}$. Obviously, this separable operator $\hat{\Sigma}_n$ will have separable eigenstates. So, assume that the resulting next state of the Universe turns out to be of the form:

$$\Psi_n = |\psi\rangle^{ep} \otimes |D\rangle_E \otimes |D\rangle_P \otimes |R\rangle_U \tag{5.73}$$

where $|\psi\rangle^{ep} \in \mathcal{H}^{ep}$, $|D\rangle_E \in \mathcal{H}_E$, $|D\rangle_P \in \mathcal{H}_P$ and $|R\rangle \in \mathcal{H}_U$, and hence $\Psi_n \in \mathcal{H}^{ep}_{EPU} \subset \mathcal{H}$, etc.

In a Universe represented by a state Ψ_n, the factor $|\psi\rangle^{ep}$ may be interpreted as the initial entangled electron-positron sub-state, with $|D\rangle_E$ the initial state of the electron detector and $|D\rangle_P$ the initial state of the positron detector. Of course, some of these

factor states may also be separable relative to a more fundamental split of their respective factor Hilbert spaces, and some of the factors of the operators may also be factorised further. Indeed, the factor $|R\rangle_U$ representing the combined sub-states of everything else in the Universe is presumably separable into very many factors in order to account for all of these other parts, but for clarity this issue is ignored here.

Overall, the operator $\hat{\Sigma}_n$ and the subsequent collapse into the state Ψ_n are equivalent to the preparation of a Universe containing an entangled electron-positron pair.

Assume now that the Rules governing the Universe conspire in such as way as to choose an operator $\hat{\Sigma}_{n+1}$ to test Ψ_n with, defined as

$$\hat{\Sigma}_{n+1} = \hat{A}_p \otimes \hat{A}^{Ee} \otimes \hat{A}_P \otimes \hat{A}_U, \tag{5.74}$$

and further that this test collapses the Universe into the state Ψ_{n+1} defined as

$$\Psi_{n+1} = |\downarrow\rangle_p \otimes |u\rangle^{Ee} \otimes |D\rangle_P \otimes |R'\rangle_U. \tag{5.75}$$

Now, in Ψ_{n+1} the factor $|u\rangle^{Ee}$ is interpreted as an entangled sub-state between a spin-up electron and an electron detector. Similarly, $|\downarrow\rangle_p$ may be interpreted as a factor of the Universe representing a spin-down positron. Note however that the positron detector is still in its initial condition $|D\rangle_P$: the factor operator $\hat{A}_P$ of $\hat{\Sigma}_{n+1}$ is effectively behaving as a local null test in $\mathcal{H}_P$ because it was also a factor of $\hat{\Sigma}_n$. The factor $|R'\rangle_U \in \mathcal{H}_U$ is interpreted as the part of the Universe that has nothing to do with the electron-positron-detector system developing in its own way, and is again ignored.

Suppose further that the Rules now conspire to choose an operator $\hat{\Sigma}_{n+2}$ of the form

$$\hat{\Sigma}_{n+2} = \hat{A}^{Ee} \otimes \hat{A}^{Pp} \otimes \hat{A}_U \tag{5.76}$$

and that the Universe subsequently collapses to the state Ψ_{n+2} defined as

$$\Psi_{n+2} = |u\rangle^{Ee} \otimes |d\rangle^{Pp} \otimes |R''\rangle. \tag{5.77}$$

In this case, $|d\rangle^{Pp}$ might be interpreted as a correlated sub-state between a spin-down positron and a positron detector.

The sequence of states Ψ_n, Ψ_{n+1} and Ψ_{n+2} offers a schematic picture of how a fully quantum Universe might view an EPR type experiment involving the preparation of an initial entangled electron-positron pair, through to the measurement of the electron, and then followed by the measurement of the positron, noting that the issue of the actual relationship between entanglement, changes of partition and endophysical measurements will be addressed properly in the next chapter.

In the emergent limit, $|D\rangle_E$ is taken to represent that part of the Universe associated with an electron detector. Moreover, in this limit the factor $\hat{A}^{Ee}$ of the operator $\hat{\Sigma}_{n+1}$ is

associated with the 'interaction' between the electron detector and the component of the entangled electron-positron pair in the electron's Hilbert space $\mathcal{H}_e$. The factor $\hat{A}^{Ee}$ is hence the pregeometric equivalent of a detector physically testing the spin of the electron, and is therefore analogous to one of the 'usual' Hermitian operators familiar to conventional physics experiments in which an isolated semi-classical apparatus measures an isolated system described by quantum mechanics. The difference between the current work and that of familiar physics is that these single, isolated experiments of conventional physics are taken in the larger context of the whole Universe being developed at once, instead of just a tiny part of it. As has been discussed previously, this difference arises from the acknowledgement that because the Universe is *everything,* any change in one part of it, no matter how small, necessarily implies a change in the state of the whole.

The sub-state $|u\rangle^{Ee}$ may be seen as the outcome of this test $\hat{A}^{Ee}$, and would ultimately correspond in the emergent limit to the physical result of the interaction between an entangled electron and a detector. Thus, the factor $|u\rangle^{Ee}$ is taken to be the result of this measurement, and in this case represents the situation in which the detector finds the electron to be spin up.

By the argument of Section 3.1, any measurement of the entangled electron automatically collapses the state of the positron, in this instance into a spin down factor $|\downarrow\rangle_p$. Consequently, then, the overall development of the state from $... \otimes |\downarrow\rangle_p \otimes |D\rangle_P \otimes ...$ to $... \otimes |d\rangle^{Pp} \otimes ...$ could be interpreted in the emergent limit as a semi-classical detector measuring the positron's spin with a test $\hat{A}^{Pp}$ to give the result $|d\rangle^{Pp}$. Thus, the detector duly finds the positron to be spin down.

Of course, many other tests $\hat{\Sigma}_{n+2}$ could have been selected by the Universe to develop Ψ_{n+1}, just as semi-classical scientists appear able to choose many alternative ways of measuring a quantum sub-system. For example, a particular factor $\hat{B}^{Pp}$ of an alternative operator $\hat{\Sigma}'_{n+2}$ could represent the spin of the positron being measured along a completely different axis, or it could even imply a test being performed that may have nothing to do with spin at all. However, an important constraint is that if the Universe is in the state Ψ_{n+1}, and if it tests itself with an operator $\hat{\Sigma}_{n+2}$ containing a factor $\hat{A}^{Pp}$ that, in the emergent limit, measures the component of spin of the positron in the same emergent direction as the component of spin of the electron was measured in, only those states Ψ_{n+2} containing a factor representing a spin-down positron result will have a non-zero probability of occurring.

Now consider the familial relationships present in the causal sets produced from the network of earlier states Ψ_{n-m}, for $m = 1, 2, ...$, and relating to what is going on in the rest of the Universe. The result might be that in the emergent limit one factor

$|D\rangle_E$ (corresponding to the factor operator $\hat{A}_D$) of the state Ψ_n representing the electron detector seems to be located at one point in emergent space, whilst another factor $|D\rangle_P$ that represents the positron detector (and corresponding to the factor operator $\hat{A}_P$) seems to be located at another point in emergent space. Moreover, the subsequent factors $|u\rangle^{Ee}$ and $|d\rangle^{Pp}$ may also appear to have definite locations in the emergent limit.

The point is that in this emergent limit, it might therefore appear that the results of the measurements of the electron and positron are correlated across emergent spacelike distances, apparently defying relativity. However, this conflict is resolved by noting that it is only a problem on the emergent scale: on the 'true' quantum level such locational descriptions are meaningless, and so theories of emergent physics such as Lorentz covariance cannot be applied there. In this quantum picture the entire experiment is seen as nothing but a change in the separability of the vector representing the state of the Universe as it jumps from being in one partition to another. There are hence no contradictions to superluminality conditions because velocity is not defined on this pregeometric level. From this point of view there is no paradox in EPR.

5.7.3 Superluminal Correlation

The following simple example illustrates how even a small difference between two consecutive operators can lead to large consequences for the resulting two consecutive states.

Consider a Hilbert space $\mathcal{H}_{[1...2N]}$ factorisable into $2N$ qubit subregisters. Consider further the n^{th} operator $\hat{\Sigma}_n$, which happens to be factorisable into two entangled suboperators, $\hat{\Sigma}_n = \hat{A}^{1...N} \otimes \hat{B}^{(N+1)...2N}$, where $\hat{A}^{1...N} \in \mathbb{H}(\mathcal{H}_{[1...N]})^{1...N}$ and $\hat{B}^{(N+1)...2N} \in \mathbb{H}(\mathcal{H}_{[(N+1)...2N]})^{(N+1)...2N}$.

Suppose also that the particular eigenstate of $\hat{\Sigma}_n$ that becomes the next state Ψ_n is of the form $\Psi_n = \psi^{1...N} \otimes \psi^{(N+1)...2N}$, such that clearly $\Psi_n \in \mathcal{H}^{(1...N)\bullet((N+1)...2N)}$ with $\psi^{1...N} \in \mathcal{H}^{1...N}$ and $\psi^{(N+1)...2N} \in \mathcal{H}^{(N+1)...2N}$. Evidently, each factor is entangled relative to its factor subspace, that is, each is entangled relative to half of the overall quantum register.

Now consider the next operator $\hat{\Sigma}_{n+1}$, and suppose that the rules governing the universe dictate that this is also a product of entangled operators, but of the form $\hat{\Sigma}_{n+1} = \hat{C}^{1...(N+1)} \otimes \hat{D}^{(N+2)...2N}$. Roughly speaking, in this type of development it may be envisaged that the $(N+1)^{th}$ qubit has 'gone over' from one factor of the operator $\hat{\Sigma}_n$ to the other in the selection of $\hat{\Sigma}_{n+1}$; the factor $\hat{A}$ has 'gained' a qubit from the factor $\hat{B}$ as they 'became' $\hat{C}$ and $\hat{D}$ respectively. So overall the way the operators $\hat{\Sigma}_n$ and $\hat{\Sigma}_{n+1}$ factorise only differs by one qubit, and if $N \gg 1$ it may therefore be said that $\hat{\Sigma}_n$ and $\hat{\Sigma}_{n+1}$ appear highly similar from this factorisation point of view.

However, given operators $\hat{\Sigma}_n$ and $\hat{\Sigma}_{n+1}$ of this form, then by the discussion of Section 5.4 for any eigenstate Θ of $\hat{\Sigma}_{n+1}$ the probability amplitude $\langle \Psi_{n+1} = \Theta | \Psi_n \rangle$ may not factorise. Thus, the conclusion is that by making what appears to be a very small change from the perspective of the operators, the family structure of the state's causal set could be destroyed. Moreover, for a Universe with a very large number of quantum subregisters, although this one qubit change in the operators may appear almost insignificant, it could end up having far reaching consequences across the entire state. Indeed, since family structure will ultimately account for the presence of spatial relationships, even small, local changes in the operator structure could give rise to an emergent situation that appeared to support superluminal correlations. This again highlights one of the important differences between states and operators: even by making a small change in the operator structure that might appear consistent with Einstein locality and emergent theories of relativity, enormous changes in the factors of the state could result which might eventually lead to an apparent violation of these principles.

5.7.4 Persistence

As is readily apparent from observing the nature of the Universe, some physical objects appear to persist over time. A single atom, for example, is often assumed to be identical from one instant to the next if it is not interacting with anything, and even macroscopic states such as humans tend to believe that they continue to be the 'same' person for a number of years.

Because time in the proposed paradigm is a concept that is expected to emerge as the state of the Universe develops through a series of stages, the existence of persistence is therefore equivalent to the observation that some features of the state appear to 'survive' relatively unchanged from one jump to the next. Moreover, because it is generally classical objects that are observed to possess this property of longevity, the concept of persistence may be seen as evidence that particular factors of the Universe's can sometimes remain approximately unaltered as it develops.

Now, the appearance of classical features in the Universe has previously been shown to be a result of the separability of its state. The observation that there is any persistence at all may therefore seem surprising. After all, when arguments of microsingularity are taken into account, as well as the fact that separable states are contained in sets of measure zero, it appears apparently 'inevitable' that the Universe *should* jump from one completely entangled state to another.

However as has been discussed a number of times in this work, the assertion that the state jumps from one highly separable vector Ψ_n to the next Ψ_{n+1} is ascribed to be due to

the Rules that govern the Universe's dynamics very carefully selecting the operators $\hat{\Sigma}_n$ and $\hat{\Sigma}_{n+1}$. Further to this, since persistence is clearly a ready feature of empirical science, it is possible to argue that the Rules may also be confined to only choose those operators that have outcomes that are similar, in some sense, to the current state.

One way of achieving this result is to consider the earlier conclusion that the separability of a state may be dictated by the factorisability of the operator of which it is an eigenvector.

Consider a Universe represented by a state in a Hilbert space $\mathcal{H}_{[1...N]}$. Further, assume that the Rules conspire in such a way that the n^{th} Hermitian operator $\hat{\Sigma}_n \in \mathbb{H}(\mathcal{H}_{[1...N]})$ used to develop the Universe is of the form

$$\hat{\Sigma}_n = \hat{A}_{a_1} \otimes \hat{A}_{a_2} \otimes ... \otimes \hat{A}_{a_k} \tag{5.78}$$

where $\hat{A}_{a_i} \in \mathbb{H}(\mathcal{H}_{[a_i]})$ and $\mathcal{H}_{[a_i]}$ need not be of prime dimension. By the conclusion of Section 5.5, whatever eigenvector of $\hat{\Sigma}_n$ becomes the next state Ψ_n will therefore have at least k factors. So, Ψ_n will be of the form

$$\Psi_n = \psi_{a_1} \otimes \psi_{a_2} \otimes ... \otimes \psi_{a_k} \tag{5.79}$$

where $\psi_{a_i} \in \mathcal{H}_{[a_i]}$, noting that ψ_{a_i} may itself be a product of factors, some of which may be entangled relative to the fundamental factorisation of $\mathcal{H}_{[a_i]}$.

Consider now the next test of the Universe $\hat{\Sigma}_{n+1}$, and assume that the Rules dictate that it is also factorisable into k sub-operators. Further, assume that the Rules also specify that each of the k sub-operators of $\hat{\Sigma}_{n+1}$ acts in the same factor Hilbert space as one of the k individual factors of $\hat{\Sigma}_n$. In other words, $\hat{\Sigma}_{n+1}$ is constrained to possess the same sort of 'partition structure' as $\hat{\Sigma}_n$, and may hence be of the form:

$$\hat{\Sigma}_{n+1} = \hat{B}_{a_1} \otimes \hat{B}_{a_2} \otimes ... \otimes \hat{B}_{a_k} \tag{5.80}$$

where $\hat{B}_{a_i} \in \mathbb{H}(\mathcal{H}_{[a_i]})$. Now, as before any eigenvector of $\hat{\Sigma}_{n+1}$ must also have no less than k factors, so whatever the next state Ψ_{n+1} of the Universe actually is, it clearly has to be of the form

$$\Psi_{n+1} = \phi_{a_1} \otimes \phi_{a_2} \otimes ... \otimes \phi_{a_k} \tag{5.81}$$

where $\phi_{a_i} \in \mathcal{H}_{[a_i]}$, noting that ϕ_{a_i} may also be a product of (possibly entangled) factors.

The point is that in this type of development, the state Ψ_{n+1} has a very similar structure to the previous state Ψ_n in terms of which partitions of $\mathcal{H}_{[1...N]}$ they are members of. Consequently, the factor ψ_{a_i} of Ψ_n may be thought of as developing into the factor ϕ_{a_i} of Ψ_{n+1} without 'interacting' with any of the other factors. Thus, this could be an embryonic form of 'semi-persistence' of the sub-state in the factor Hilbert space $\mathcal{H}_{[a_i]}$.

Moreover, because each sub-state ψ_{a_i} could actually be a group of factors itself, this mechanism allows the possibility for macroscopic sets of sub-states to survive relatively unchanged from one jump to the next. Although of course clearly only a schematic model here, the persistence and apparently isolated nature of semi-classical objects such as apparatus, laboratories and physicists, each of which is associated with large groups of factors, may ultimately be a consequence of the relationships between factorisable operators and separable states.

A more definite form of persistence would be evident if the Rules instead selected the alternative operator $\hat{\Sigma}'_{n+1}$, defined as

$$\hat{\Sigma}'_{n+1} = \hat{B}_{a_1} \otimes \hat{B}_{a_2} \otimes ... \otimes \hat{B}_{a_{j-1}} \otimes \hat{A}_{a_j} \otimes \hat{B}_{a_{j+1}} \otimes ... \otimes \hat{B}_{a_k}. \tag{5.82}$$

Any next state Ψ'_{n+1} resulting from an eigenvector of $\hat{\Sigma}'_{n+1}$ is of the form

$$\Psi'_{n+1} = \phi_{a_1} \otimes \phi_{a_2} \otimes ... \otimes \phi_{a_{j-1}} \otimes \psi_{a_j} \otimes \phi_{a_{j+1}} \otimes ... \otimes \phi_{a_k} \tag{5.83}$$

which clearly has a factor $\psi_{a_j} \in \mathcal{H}_{[a_j]}$. So, the factor in $\mathcal{H}_{[a_j]}$ of both Ψ_n and Ψ'_{n+1} is ψ_{a_j}, such that the sub-operator $\hat{A}_{a_j}$ of $\hat{\Sigma}'_{n+1}$ is acting as a local null test. Thus, the factor ψ_{a_j} has clearly remained unchanged in the development of the Universe from Ψ_n to Ψ'_{n+1}. From the point of view of the other factors ϕ_{a_i} of Ψ'_{n+1}, the factor ψ_{a_j} can therefore be said to have persisted during this jump.

Of course, by carefully choosing the subsequent operators $\hat{\Sigma}_{n+2}$, $\hat{\Sigma}_{n+3}$, $\hat{\Sigma}_{n+4}$, ... the rules could readily ensure that particular factors persist over many more developments of the state.

5.7.5 Position and Dimension

In this final example it will be shown how positional relationships might be encoded in terms of factorisation and entanglement. Additionally, it will also be shown that such relations may also afford a natural inclusion of the properties of dimension. A simple illustration of these ideas will first be given, followed then by a generalisation to more complicated examples.

Consider a cubic lattice in three spatial dimensions. In fact for simplicity consider the smallest such lattice, that is, a single cube formed from only eight points, where one point is on each of the eight corners of the cube. Clearly, each 'edge' of the cube implies the minimum separation between two adjacent sites, and may be associated with a length of 1 unit.

Now, in order to discuss positional relationships within the cube it is necessary to define a suitable set of axes. This can be achieved by arbitrarily selecting any one of the corner sites to be an 'origin', and then using the direction of the three corners that are adjacent to this origin to specify a set of three orthogonal, Cartesian axes. One such choice is illustrated in Figure 5.6, where it is now possible to label the positions of the sites according to this set of axes; for example the origin site is denoted $(0, 0, 0)$, whereas the site furthest (i.e. $\sqrt{3}$ units) from the origin is $(1, 1, 1)$.

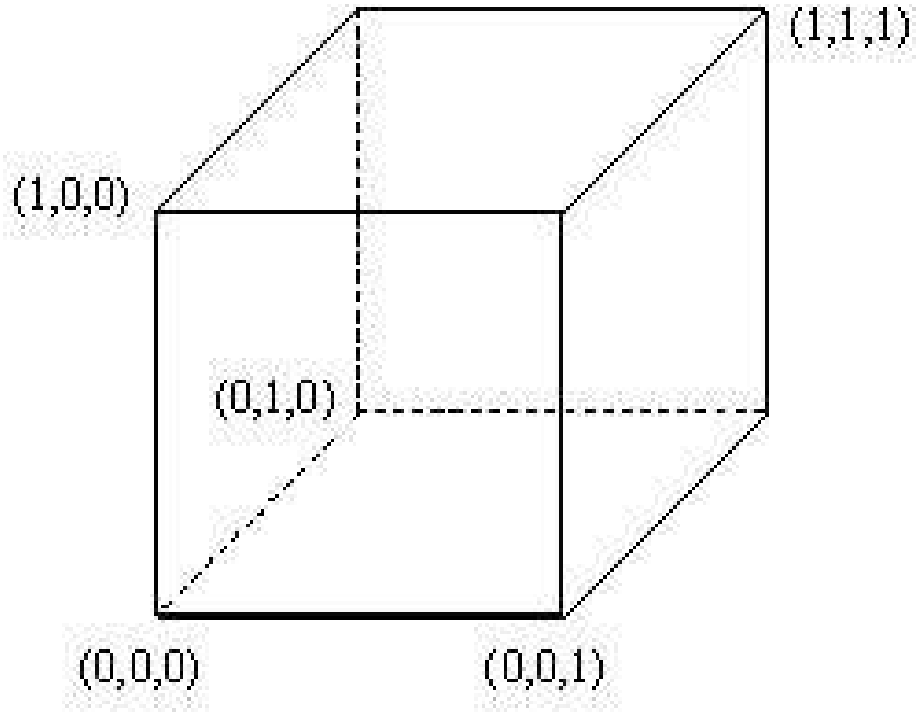

Figure 5.6: The cubic lattice formed from eight points, with the corners labelled according to a set of orthogonal axes through the origin $(0, 0, 0)$.

It is possible to label each of the corners with a unique integer a, where $a = 0, 1, ..., 7$. One way of achieving would be to take the coordinates (x, y, z) of the corner as the coefficients in a series expansion of the powers of two, such that a may be given by the rule

$$a = x(2^0) + y(2^1) + z(2^2). \tag{5.84}$$

For example, the corner positioned at $(0, 0, 0)$ corresponds to $a = 0$, whereas the corner $(0, 1, 0)$ may be denoted by $a = 2$, whilst $(1, 1, 1)$ corresponds to $a = 7$ etc. Indeed, by relabelling the a^{th} site as b, where $b = a + 1$ such that $1 \leq b \leq 8$, the eight sites can be numbered cardinally according to their positions on the lattice.

Consider now a model universe represented by a state in a Hilbert space $\mathcal{H}_{[1...N]}$, and an Hermitian operator $\hat{o}_n \in \mathbb{H}(\mathcal{H}_{[1...N]})$ used to develop this state. Suppose that $\hat{o}_n$ is chosen according to the rule that it can be fundamentally factorised into eight sub-operators. Suppose further that the rules dictate that each factor of $\hat{o}_n$ is itself entangled relative to the skeleton sets of c subregisters, where $1 \leq c \leq 8$, and that no two factors of $\hat{o}_n$ may be entangled relative to the same number of subregisters. Thus, one of the factor operators acts on states in just one quantum subregister, whereas another factor of $\hat{o}_n$ is an entangled operator acting on states contained in the space of two subregisters, and so on, up to the eighth factor that is an entangled operator acting upon states contained in the space of eight subregisters. Evidently, the total number of subregisters N in $\mathcal{H}_{[1...N]}$ is given by $N = 1 + 2 + ... + 8 = 36$, and one possible such operator $\hat{o}_n$ may be defined as

$$\hat{o}_n = \hat{A}_1 \otimes \hat{A}^{23} \otimes \hat{A}^{456} \otimes \hat{A}^{7...(10)} \otimes \hat{A}^{(11)...(15)} \otimes \hat{A}^{(16)...(21)} \otimes \hat{A}^{(22)...(28)} \otimes \hat{A}^{(29)...(36)} \quad (5.85)$$

where $\hat{A}^{i...j} \in \mathbb{H}(\mathcal{H}_{[i...j]})^{i...j} \subset \mathbb{H}(\mathcal{H}_{[i...j]})$ and $\mathcal{H}_{[i...j]} \subset \mathcal{H}_{[1...N]}$.

It is hence possible to assign a unique number c with each of the factors of $\hat{o}_n$ in terms of the number of subregisters over which it is entangled. Moreover, since the discussion of the previous paragraph showed that it is also possible to associate positions on a lattice with numbers, the individual factors of $\hat{o}_n$ can conversely each be associated with a sort of 'position'. In other words, the factor $\hat{A}_1$ of $\hat{o}_n$ may be labelled by the number $c = 1$ and so may, in some sense, be associated with the position $(0, 0, 0)$. Similarly the factor $\hat{A}^{456}$ corresponds to the number $c = 3$ and so may be denoted as $(0, 1, 0)$, whereas $\hat{A}^{(29)...(36)}$ may be denoted by the number $c = 8$ and may hence be associated with $(1, 1, 1)$.

Further, any eigenvector of $\hat{o}_n$ must be separable into at least eight factors, and so these outcomes would therefore also follow the pattern of spatial positioning affiliated with the operator. Hence whichever eigenvector the universe collapses into, the factors of this next state ψ_n must share some of the 'locational information' of the factors of the operator $\hat{o}_n$. In other words, whatever factor $\alpha \in \mathcal{H}_{[i...j]}$ of the next state $\psi_n = ... \otimes \alpha \otimes ...$ is the result of a factor $\hat{A}^{i...j}$ of $\hat{o}_n$, the location of $\hat{A}^{i...j}$ on the lattice can also be used to describe the corresponding position of the factor α of the state.

Summarising, although the underlying structure is just a single cube, the rules selecting the operator $\hat{o}_n$ imply that its factors, and those of its eigenvectors, may be discussed in terms of the position of the lattice's sites. Number and position are interchangeable, and according to the rules, so are factor and number.

In order to generalise these ideas, consider a (very large) prime number p and a positive integer d. By analogy with the base $p = 2$ expansion of a given above, the p-adic expansion [64] of any non-negative integer $P < p^d$ to base p is given by

$$P = i_0(p^0) + i_1(p^1) + i_2(p^2) + ... + i_{d-1}(p^{d-1}) \quad (5.86)$$

where $i_j \in \mathbb{Z}^*$ is a non-negative integer that is less that p, and the subscript j identifies which power of p a particular i is a coefficient of. Thus, any integer $0 \leq P \leq (p^d - 1)$ can be uniquely specified by a series of coefficients i_j, for $j = 0, 1, ..., (d-1)$, and a prime number p.

Consider now a d-dimensional lattice of points, where this lattice may be imagined to be a d-dimensional 'cube' with edges of 'length' $(p-1)$ units. As before, by picking one corner of the lattice as the origin $(0, 0, ..., 0)$, the position of each site can be associated with a specific number between 0 and $(p^d - 1)$. For example, the origin $(0, 0, ..., 0)$ may be associated with the number zero, whereas the number $(p^d - 1)$ is related to the position $(p-1, p-1, ..., p-1)$, and so on. So by continuing the analogy of the single cube example of earlier, every integer P in the appropriate range may be thought of as mapped to a unique site on a lattice with coordinates $(i_0, i_1, ..., i_{d-1})$. As before, these numbers may be granted cardinality by relabelling them as $P' = 1, 2, ..., p^d$ defined as $P' = P + 1$.

Now consider a universe represented by a state in $\mathcal{H}_{[1...N]}$, and a particular Hermitian operator $\hat{O}_n \in \mathbb{H}(\mathcal{H}_{[1...N]})$ factorisable as before into a product of p^d sub-operators. Assuming again the rules are such that each factor sub-operator of $\hat{O}_n$ is entangled relative to the skeleton sets associated with between one and p^d subregisters, and that no two factors of $\hat{O}_n$ are entangled relative to the same number of subregisters, one possible form of $\hat{O}_n$ is given by

$$\hat{O}_n = \hat{A}_1 \otimes \hat{A}^{23} \otimes \hat{A}^{456} \otimes ... \otimes \hat{A}^{M...N}. \tag{5.87}$$

Clearly, the total number N of subregisters required for such a prescription is given by the arithmetic progression

$$\begin{aligned} N &= 1 + 2 + 3 + ... + p^d \\ &= p^d(p^d + 1)/2 \end{aligned} \tag{5.88}$$

and because the 'last' factor $\hat{A}^{M...N}$ of $\hat{O}_n$ is entangled relative to p^d subregisters, M is given by $M = p^d(p^d - 1)/2$. So, $\hat{O}_n \in \mathbb{H}(\mathcal{H}_{[1...p^d(p^d+1)/2]})_1^{23 \bullet 456 \cdot ... \bullet (M...N)} \subset \mathbb{H}(\mathcal{H}_{[1...N]})$, with $\hat{A}^{M...N} \in \mathbb{H}(\mathcal{H})^{(p^d(p^d-1)/2)...(p^d(p^d+1)/2)}$ etc.

As before, by assigning each factor of $\hat{O}_n$ a unique number according to how many subregisters it is entangled relative to, and by associating each of these numbers with a coordinate, the factors of $\hat{O}_n$ may be associated with 'positions' in a lattice. Thus, the factor $\hat{A}_1$ may be assigned the number 1 and so may be associated with the coordinate $(0, 0, ..., 0)$, whereas the factor $\hat{A}^{23}$ may similarly be associated with $(1, 0, ..., 0)$, whilst $\hat{A}^{456}$ may be associated[9] with $(2, 0, ..., 0)$, and so on, until $\hat{A}^{M...N}$ is may be associated with $(p-1, p-1, ..., p-1)$.

[9] Assuming $p \geq 3$. Otherwise, say if $p = 2$, $\hat{A}^{456}$ will be associated with $(0, 1, 0, ..., 0)$ etc.

Likewise, the eigenstates of $\hat{O}_n$ will also share the separability properties of this operator, and so their factors may similarly also be considered in terms of these simple positional relationships.

It is easy to see how dynamics may be incorporated into this type of model. As long as the rules governing what type of operators are allowed remain the same, the lattice structure associated with the operators' factors will be preserved. For instance, the dynamics may permit permutations of the form $r \longrightarrow s_r$ in the structure of the operators, where r, s label different subregisters, and this is equivalent to exchanging the subregisters over which a factor of the operator is entangled relative to. For example, if the permutation was such that the next operator $\hat{O}_{n+1}$ is of the form

$$\hat{O}_{n+1} = \hat{B}_3 \otimes \hat{B}^{(1)(12)} \otimes \hat{B}^{(7)(8)(14)} \otimes \hat{B}^{(2)(4)(5)(27)} \otimes ... \tag{5.89}$$

the result might be a set of eigenvectors whose factors are completely different from those of the previous state, but are all still constrained by the same type of lattice structure.

It is important to reiterate that the lattice formed from the operator's factors does not exist in any sort of background space. The positional relationships, and hence the corresponding measure of dimension, are simply a consequence of the way the operator $\hat{O}_n$ factorises in terms of the skeleton set of basis operators spanning the subregisters of the total Hilbert space, and this is itself just a result of whatever rules dictate the dynamics. As always throughout this work, physical space is not seen as absolute but as a marker of distinction between objects in an underlying mathematical structure.

So as a final remark it should be noted that the present discussion of position should not be taken too rigidly. As has been a central conclusion of this chapter, the relationships between the pregeometric quantum register and the eventual emergence of physical space is a subtle one that requires a great deal of future work before it is completely understood. Indeed, even in the above example it is observed that a given number P may potentially be expanded in many different ways, because many different bases p could be chosen, and this would lead to a set of alternative lattices of different dimensions.

Of course, whether or not this last comment has any physical meaning is an important question to be faced, and might perhaps imply that either: the current example is too 'naïve' to describe proper physics; or that it is missing an important constraint that forces every lattice in the 'real' Universe to be three dimensional; or even that it might possibly allow the occurrence of multi-dimensional ideas such as Kaluza-Klein and string theory. What is clear, however, is that in a fully quantum Universe with no external observers, the Universe must somehow organise itself in such a way so that internal, semi-classical observers are able to experience a reality with near-continuous spatial relationships. In

such a Universe, the underlying quantum structure must somehow be responsible for sophisticated theories such as relativity and four dimensional spacetime to emerge.

6 Quantum Registers as Quantum Computers

In the previous chapter, it was discussed how spatial degrees of freedom might begin to emerge if the causal set structure of operators obeys relationships analogous to those of a classical cellular automaton. Moreover, starting from the premise that there are no external observers dictating its development, a central theme of this thesis is that the Universe is acting as a giant, self-regulating quantum automaton. From these viewpoints, the development of its state is envisaged to be like an enormous quantum calculation, such that overall the Universe may be thought of as behaving like an enormous quantum computer. This conjecture is discussed now.

The present chapter is split into two parts. In Section 6.1 it is shown how simple quantum computational methods may be applied to a system consisting of a finite number of Hilbert space subregisters. Since in the proposed paradigm a state is considered that exists in a large, but finite, number of such quantum subregisters, it is argued that these principles are equally applicable to the case where the system is the Universe itself.

Because computation is often seen as synonymous with data manipulation, in Section 6.2 the role of information change and exchange is defined for quantum systems. It is then discussed how endo-physical scientists might obtain 'answers' for the Universe's quantum calculation, and how these answers might be interpreted.

6.1 Computing with CNOT

In this section it will be demonstrated how operators can be used to perform computations in simple quantum systems of qubits. Specifically, in fact, the example of the Controlled-NOT (or CNOT) operator will be examined. It must be noted, however, that such computations are not just simply mathematical exercises; instead, they will be compared with the actual, physical results of classical computations, namely by a formulation of the Bell inequality. Some of the implications of this comparison will then be discussed.

Overall, the work described here will serve as a preparation for the following chapter in which it will be shown how, by treating the state with quantum computational methods, particle field theoretic concepts may arise in a fully quantum Universe.

Before quantum computational principles can be applied to a system of quantum subregisters, though, it is useful to review some of the ideas of classical computation. Specifically, it will be beneficial to illustrate how the classical analogue of the quantum CNOT operator, namely the CNOT logic gate, may be employed in classical computation. This

issue is hence addressed first, noting that a more thorough background description of both classical and quantum computation is provided in Appendix A.

6.1.1 Classical Computation

Broadly speaking, conventional classical computation involves processing the 'values' contained at a sequence of 'sites', where each site will definitely take one, and only one, value from a finite set of possibilities. In fact in general, classical computation can actually be achieved by a particular manipulation of a finite set of *bits*, each of which is in one of two possible states. The workings of modern, digital microelectronic computers are an example of this. Moreover, because of their binary property, conventional logic may be applied to these bits, and their states may consequently be labelled 'true' or 'false', or perhaps 'on' or 'off', or even '0' or '1'. Equally, the processing of these bits may be accomplished by the use of binary logic gates.

Classical computations generally involve three parts: there is the specification of the Input, usually given in the form of a string of bits of which each has a particular value; there is the computation itself, which involves the processing of these bits according to a particular set of gates in a certain order; and there is the Output, which is the result of the computation, and is also usually given in terms of a string of bit values.

Any sequence of 0's and 1's, and consequently any string of bits, denotes a unique binary number. So from this perspective, a classical computation involving the transformation of an input series of bit values into an output sequence may be interpreted as a calculation being performed on an initial number to generate an 'answer'. This answer is also a number, and may itself go on to be processed in subsequent computations.

Note how this could easily be seen as analogous to the quantum Universe, in which an initial state Ψ_n is developed into the next state Ψ_{n+1} by some particular combination of unitary and Hermitian operators.

Just as the particular choice of quantum operators dictates the way the Universe is developed, it is the transformations that determine how a certain sequence of bits is processed in a computation. It is consequently the particular choice of logic gates that define which particular classical computation is performed on the input. As with operators in quantum mechanics, a number of different types of operation are also possible here.

As an illustration, consider a classical system consisting of just two bits X and Y. Each bit may take one of two values, such that X may have the value x and Y may have the value y, where $x, y = 0, 1$. Thus the state S of the system may be denoted by the pair $S = (x, y)$, and this will clearly be one of four distinct possibilities.

A *local* operation may be defined as any operation that acts on the bits individually and independently, with no reference to the values of any of the other bits. Examples in the two bit system are the operations U and V, defined as

$$\begin{aligned} U(x,y) &= (x \oplus x, y) = (0, y) \\ V(x,y) &= (x, y \oplus 1) \end{aligned} \tag{6.1}$$

where the symbol $\oplus$ denotes addition modulo two, i.e.

$$0 \oplus 0 = 0 \quad , \quad 0 \oplus 1 = 1 \quad , \quad 1 \oplus 0 = 1 \quad , \quad 1 \oplus 1 = 0. \tag{6.2}$$

Conversely, a *global* operation is one that acts on the whole state S of the system. In these operations, the way one of the two bits is processed depends on the value of the other bit.

In fact, note that for all systems with more than one bit it is generally possible to consider *non-local* operations, that is, those which act on two or more bits. In such operations, the way a particular bit is developed may be affected by the value of at least one other bit. Of course, clearly in a two bit system every non-local operation is also global, but for systems of more than two bits the global operations are just special, extreme cases of non-local operations.

An example of such a global operation acting on the two bit system X and Y is the CNOT logic gate, C, defined as

$$C(x,y) = (x, x \oplus y). \tag{6.3}$$

For obvious reasons from (6.3), in the above use of CNOT it is possible to describe X as the 'donor' bit and Y as the 'acceptor' bit.

The above gate may be thought of as a type of 'question and answer' operation. Processing a state $S = (x, y)$ with C may be viewed as equivalent to asking a question of the value x of the bit X, and registering the answer with a response in the value y of the bit Y.

The CNOT computation is reversible. That is, in this case there exists an inverse operation C^{-1} defined as

$$C^{-1}(x, x \oplus y) = (x, y) \tag{6.4}$$

such that $C^{-1}C(x,y) = (x,y)$; in fact, clearly $C = C^{-1}$. Analogous to the gate C, the inverse operation C^{-1} may then be interpreted as the statement: "given a particular result, what was the question of which it is an answer?". In this case, the answer is $(x, x \oplus y)$ and the question is (x, y).

Note that the inverse gate C^{-1} is different from the 'conjugate' operation $\widetilde{C}$ defined as

$$\widetilde{C}(x, y) = (x \oplus y, y) \tag{6.5}$$

which reverses the role of acceptor and donor.

From this last gate $\widetilde{C}$ it is possible to build a 'transpose gate' C^T that swaps the values of the bits X and Y, i.e. $C^T(x, y) = (y, x)$. With the above descriptions of C and $\widetilde{C}$ this can be achieved by writing C^T as $C^T = \widetilde{C}C\widetilde{C}$.

The two bit CNOT gate is a global operation when acting on a system containing just two bits, but merely a non-local operation for an N bit system if $N > 2$. Specifically, given a string of N bits $Z_1, Z_2, ..., Z_N$ in the state $S = (z_1, z_2, ..., z_N)$, where $z_a = 0, 1$ for $a = 1, 2, ...N$, the CNOT gate $C_{(i,j)}$ may be defined as

$$C_{(i,j)}(z_1, z_2, ..., z_i, ..., z_{j-1}, z_j, z_{j+1}, ..., z_N) = (z_1, z_2, ..., z_i, ..., z_{j-1}, z_i \oplus z_j, z_{j+1}, ..., z_N). \tag{6.6}$$

This last definition will be useful later.

6.1.2 Quantum Computation

A classical computation involving operations performed on a classical state has immediate parallels with the way operators in quantum mechanics act on quantum states. This latter process may therefore naturally be called a quantum computation. Furthermore, a classical computation involving operations performed on a series of bits is itself analogous to the way operators in quantum mechanics can act on products of qubits. Indeed, just as a classical bit is defined as some sort of 'entity' that can take one of two possible values, a qubit is defined relative to a basis comprising of two different (orthogonal) states. However, whilst classical bits are restricted to always have one value or the other, the states of quantum bits can exist as complex linear superpositions of their basis vectors.

Consider a two dimensional (qubit) Hilbert space, $\mathcal{H}_a^{(2)}$, where the super-script may again be assumed and hence dropped, and the sub-script denotes that this space belongs to the a^{th} qubit, in preparation for the later discussion of many qubit systems. Assume also that $\mathcal{H}_a$ is spanned by the orthonormal basis set $\mathcal{B}_a$ defined as $\mathcal{B}_a \equiv \{|0\rangle_a, |1\rangle_a\}$, where ${}_a\langle i|j\rangle_a = \delta_{ij}$ for $i, j = 0, 1$, and note that these elements may be represented by column vectors of the form $|0\rangle_a \equiv \binom{1}{0}_a$ and $|1\rangle_a \equiv \binom{0}{1}_a$.

Define now the projection operators $\hat{P}_a^0$ and $\hat{P}_a^1$ acting on the a^{th} space as

$$\hat{P}_a^0 \equiv |0\rangle_{aa}\langle 0| \quad , \quad \hat{P}_a^1 \equiv |1\rangle_{aa}\langle 1| \tag{6.7}$$

and the 'transition' operators $\hat{Q}_a$ and $\hat{Q}_a^\dagger$ as

$$\hat{Q}_a \equiv |0\rangle_{aa}\langle 1| \quad , \quad \hat{Q}_a^\dagger \equiv |1\rangle_{aa}\langle 0|. \tag{6.8}$$

The application of these four operators to a qubit may be interpreted in particular ways. For example, the projection operator $\hat{P}_a^0$ may be thought of as equivalent to the question: "is the a^{th} qubit in the state $|0\rangle$?"; a similar question is appropriate for $\hat{P}_a^1$. The transition operators $\hat{Q}$ and $\hat{Q}^\dagger$ are analogous to the ladder operators of field theory: $\hat{Q}_a$ may be thought of as an operator which, when applied to a qubit in Hilbert space $\mathcal{H}_a$, 'annihilates' the state $|1\rangle$ and 'replaces' it with a state $|0\rangle$. Conversely, $\hat{Q}_a^\dagger$ may be considered as an operator that 'destroys' $|0\rangle$ and 'creates' $|1\rangle$. These connections are explored in Chapter 7.

Every operator acting on states in the qubit Hilbert space $\mathcal{H}_a$ can be built from complex sums of the four operators $\hat{P}_a^0$, $\hat{P}_a^1$, $\hat{Q}_a$ and $\hat{Q}_a^\dagger$. So, if $\hat{A}_a$ is an arbitrary operator acting in $\mathcal{H}_a$ it may be written

$$\hat{A}_a = A_a^1\hat{P}_a^0 + A_a^2\hat{P}_a^1 + A_a^3\hat{Q}_a + A_a^4\hat{Q}_a^\dagger \tag{6.9}$$

where $A_a^1, A_a^2, A_a^3, A_a^4 \in \mathbb{C}$, or alternatively $\hat{A}_a = (A_a^1, A_a^2, A_a^3, A_a^4)$ for brevity. Similarly, the Hermitian conjugate operator $\hat{A}_a^*$ may be given by $\hat{A}_a^* = (A_a^{1*}, A_a^{2*}, A_a^{4*}, A_a^{3*})$.

As an example, the identity and Pauli operators, $\hat{\sigma}_a^\mu$ for $\mu = 0, 1, 2, 3$, can be defined as

$$\begin{aligned} \hat{\sigma}_a^0 &= \hat{P}_a^0 + \hat{P}_a^1 \quad , \quad \hat{\sigma}_a^1 = \hat{Q}_a + \hat{Q}_a^\dagger \\ \hat{\sigma}_a^2 &= -i(\hat{Q}_a - \hat{Q}_a^\dagger) \quad , \quad \hat{\sigma}_a^3 = \hat{P}_a^0 - \hat{P}_a^1 \end{aligned} \tag{6.10}$$

or equally

$$\begin{aligned} \hat{\sigma}_a^0 &= (1,1,0,0) \quad , \quad \hat{\sigma}_a^1 = (0,0,1,1) \\ \hat{\sigma}_a^2 &= (0,0,-i,i) \quad , \quad \hat{\sigma}_a^3 = (1,-1,0,0) \end{aligned} \tag{6.11}$$

and these clearly satisfy the 'standard' algebra (5.32), and the representation (5.37), as given in Chapter 5.

It is possible to define products of operators in the above notation. As an illustration, consider two operators $\hat{A}_a$ and $\hat{B}_a$ defined as $\hat{A}_a = (A_a^1, A_a^2, A_a^3, A_a^4)$ and $\hat{B}_a = (B_a^1, B_a^2, B_a^3, B_a^4)$. The product $\hat{A}_a\hat{B}_a$ is then given by

$$\hat{A}_a\hat{B}_a = ([A_a^1B_a^1 + A_a^3B_a^4], [A_a^2B_a^2 + A_a^4B_a^3], [A_a^1B_a^3 + A_a^3B_a^2], [A_a^2B_a^4 + A_a^4B_a^1]). \tag{6.12}$$

The product algebra of the operators in (6.7) and (6.8) is summarised in Table 6.1. Specifically, the product XY is read as the member X of the first column multiplied by the member Y of the first row; for example, $\hat{P}^0\hat{Q} = \hat{Q}$, whereas $\hat{Q}\hat{P}^0 = 0$.

–	$\hat{\mathbf{P}}^0$	$\hat{\mathbf{P}}^1$	$\hat{\mathbf{Q}}$	$\hat{\mathbf{Q}}^\dagger$
$\hat{\mathbf{P}}^0$	$\hat{P}^0$	0	$\hat{Q}$	0
$\hat{\mathbf{P}}^1$	0	$\hat{P}^1$	0	$\hat{Q}^+$
$\hat{\mathbf{Q}}$	0	$\hat{Q}$	0	$\hat{P}^0$
$\hat{\mathbf{Q}}^+$	$\hat{Q}^+$	0	$\hat{P}^1$	0

Table 6.1

The results of the products of these operators also comment on the role of information in quantum processes. The idempotency of the projection products, $\hat{P}^0\hat{P}^0 = \hat{P}^0$ and $\hat{P}^1\hat{P}^1 = \hat{P}^1$, for example, may be interpreted as the observation that once a 'question' has been asked of a quantum system, no new information can be extracted by asking the same question again. This reflects the deeper principle of quantum mechanics that once a state has collapsed into one of the eigenvectors of a particular operator, testing the system a second time with the same operator reproduces the same result.

On the other hand, note that the transition operators $\hat{Q}_a$ and $\hat{Q}_a^\dagger$ obey the fermionic algebra in the sense that their squares vanish, $(\hat{Q}_a)^2 = (\hat{Q}_a^\dagger)^2 = 0$. As with their association with quantum field operators, this property will be useful in Chapter 7. For now, however, note that any operator $\hat{A}$ that satisfies the rule $(\hat{A})^{p+1} = 0$ may be called a 'parafermionic operator of order p', following the language of [65] and [66], where $p \in \mathbb{Z}^*$ is the lowest integer required for this rule to be true. Thus, $\hat{Q}_a$ and $\hat{Q}_a^\dagger$ may be labelled parafermions of order 1.

It is important to note that the construction of Table 6.1 does not rely on any use of group theory. The sixteen entries in the table come directly from the logic induced by taking the inner products of the basis vectors from which the four individual operators $\hat{P}_a^0$, $\hat{P}_a^1$, $\hat{Q}_a$ and $\hat{Q}_a^\dagger$ are defined. Conversely, in fact, the rotational symmetry of the $SU(2)$ group can be shown to be preserved as a natural feature of the underlying qubit perspective.

To demonstrate this, define the '*Transformation*' operators $\hat{T}_a^{ij}$ acting on the space $\mathcal{H}_a$ as $\hat{T}_a^{ij} = |i\rangle_{aa}\langle j|$, where $i, j = 0, 1$. Clearly, $\hat{T}_a^{ij}$ is one of four possible operators corresponding to the two projection and two transition operators defined above in (6.7) and (6.8). That is,

$$\begin{aligned} \hat{T}_a^{00} &= \hat{P}_a^0 \quad , \quad \hat{T}_a^{01} = \hat{Q}_a \\ \hat{T}_a^{10} &= \hat{Q}_a^\dagger \quad , \quad \hat{T}_a^{11} = \hat{P}_a^1. \end{aligned} \tag{6.13}$$

So, Table 6.1 can be summarised for $i, j, k, l = 0, 1$ by the relation

$$\hat{T}_a^{ij}\hat{T}_a^{kl} = \delta_{jk}\hat{T}_a^{il}. \tag{6.14}$$

Consider now a unitary operator $\hat{U}(\boldsymbol{\theta})$ defined as

$$\hat{U}(\boldsymbol{\theta}) \equiv \exp\left(i\sum_{j=1}^{3}\theta_j\hat{\sigma}_a^j\right) \tag{6.15}$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$, $\theta_j \in \mathbb{R}$ and the $\hat{\sigma}_a^j$ are generators of the group $SU(2)$. The set of operators $\hat{U}(\boldsymbol{\theta})$ form an $SU(2)$ rotation group acting on states in $\mathcal{H}_a$.

Define now a new transformation operator $(\hat{T}_a^{ij})'$ as

$$(\hat{T}_a^{ij})' \equiv \hat{U}^*(\boldsymbol{\theta})\hat{T}_a^{ij}\hat{U}(\boldsymbol{\theta}) \tag{6.16}$$

with $\hat{U}^*(\boldsymbol{\theta})\hat{U}(\boldsymbol{\theta}) = \hat{I}_a$ where $\hat{I}_a$ is the identity in $\mathcal{H}_a$. This new operator satisfies the product rule $(\hat{T}_a^{ij})'(\hat{T}_a^{kl})' = \delta_{jk}(\hat{T}_a^{il})'$, so it may be concluded that (6.14) is invariant to rotations of the operators $\hat{T}_a^{ij}$ under the group $SU(2)$.

The above conclusion might be surprising, given that spatial relationships were not used in any way in the construction of (6.14) or Table 6.1, and indicates why the language of spin may be used in discussions of qubits (e.g. spin up, spin down etc.). However, it should be reiterated that the qubits should not be thought of as fundamentally possessing any sort of physical spin, and the rotations are really nothing but mathematical transformations; physical spin is expected to eventually appear in the emergent limit once space, directions, and frames of reference have been established.

It is, though, still encouraging perhaps to note that spin relations automatically seem to emerge as a natural feature of the algebraic rules inherent in the underlying qubit structure.

Just as it is possible to perform computations on classical bit systems using a reversible two-bit CNOT gate, it is possible to define a quantum CNOT operator also. In actual fact this possibility turns out to be of fundamental importance, because it can be shown that any qubit quantum computation can be performed using just local unitary operators and the CNOT gate alone [67].

As with the classical case, the quantum CNOT asks a 'question' of one qubit and registers the response with a second.

Consider a four dimensional Hilbert space $\mathcal{H}_{[ab]}^{(4)}$, factorisable into two qubit subregisters in the form $\mathcal{H}_{[ab]}^{(4)} = \mathcal{H}_a^{(2)} \otimes \mathcal{H}_b^{(2)}$, where from now on the dimensional super-scripts shall again be omitted. A suitable, separable orthonormal basis $\mathcal{B}_{ab}$ for $\mathcal{H}_{[ab]}$ may be defined in the usual way as $\mathcal{B}_{ab} \equiv \{|00\rangle_{ab}, |01\rangle_{ab}, |10\rangle_{ab}, |11\rangle_{ab}\}$.

The quantum CNOT operator $\hat{C}_{(a,b)}$ acting on the qubits a and b may now be defined as

$$\begin{aligned} \hat{C}_{(a,b)} &= \hat{P}_a^0 \otimes \hat{\sigma}_b^0 + \hat{P}_a^1 \otimes \hat{\sigma}_b^1 \\ &= \hat{P}_a^0 \otimes (\hat{P}_b^0 + \hat{P}_b^1) + \hat{P}_a^1 \otimes (\hat{Q}_a + \hat{Q}_a^\dagger) \end{aligned} \tag{6.17}$$

with $\hat{\sigma}_b^0$ the identity, and $\hat{\sigma}_b^1$ the usual 'first' Pauli operator, acting on qubit b. Note that in the representation $\mathcal{B}_{ab}$, the CNOT operator may equally be written as the matrix

$$C_{(a,b)} = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \end{pmatrix}. \tag{6.18}$$

CNOT is a unitary operator, and might hence be used to evolve the state of the two qubit system. To illustrate how, assume that the initial state Ψ_i of the system is in one of the four, separable basis states of $\mathcal{B}_{ab}$. The final state $\Psi_f \equiv \hat{C}_{(a,b)}\Psi_i$ is then given by Table 6.2

$\boldsymbol{\Psi}_i$	$\|00\rangle$	$\|01\rangle$	$\|10\rangle$	$\|11\rangle$
$\boldsymbol{\Psi}_f \equiv \hat{\mathbf{C}}_{(a,b)}\boldsymbol{\Psi}_i$	$\|00\rangle$	$\|01\rangle$	$\|11\rangle$	$\|10\rangle$

Table 6.2

From Table 6.2 the result of an application of $\hat{C}_{(a,b)}$ to the system is apparent: if qubit a is in the state $|0\rangle$ then the state of qubit b remains the same, whereas if qubit a is in the state $|1\rangle$ then the application of $\hat{C}_{(a,b)}$ 'flips' the state of qubit b either from $|0\rangle \rightarrow |1\rangle$ or $|1\rangle \rightarrow |0\rangle$.

Care is needed, however, before physically interpreting exactly how $\hat{C}_{(a,b)}$ is working. From one perspective it appears that an application of $\hat{C}_{(a,b)}$ to $\boldsymbol{\Psi}_i$ does not affect qubit a. The operator $\hat{C}_{(a,b)}$ appears to be non-invasively determining the state of qubit a, and then consequently registering the result with qubit b. This, however, is a very classical viewpoint, and is therefore not the best way to proceed. From a quantum perspective it must be noted that any determination of the state of qubit a necessarily involves an extraction of information, and from quantum principles it should therefore be expected that such an extraction will irreversibly alter the state of the system.

It is this conflict between attempted non-invasive, classical information extraction and the truly invasive, quantum CNOT that forms the basis of the following example of how quantum computational methods could be applied to quantum register systems. In the next sub-section it is shown how a variation of the Bell inequalities can arise from a classical computation acting on a classical bit system. Then by contrast, in the following sub-section similar Bell inequalities will be derived for a quantum computation acting on a product of qubits. However, it is shown further that once the outcome of a quantum computation is obtained, the state collapses to a classical-looking system, and as such any potential Bell violations are irretrievably lost. So, not only should the following sub-sections provide a physically applicable demonstration of how quantum computational

principles can be used for systems consisting of quantum subregisters, but it will also illustrate how in a Universe ultimately envisaged to behave as an enormous quantum computer, the conclusion must be reinforced that the extraction of information during measurement inevitably comes with a cost.

6.1.3 Classical Calculations and the Bell Inequality

Consider a single classical bit that, at any time, can either take the value 0 or the value 1. Physically, such a bit could in principle represent any classical bi-level system, for example the off-on states of a digital switch, or a random number generator programmed to provide one of two possible outputs. Let the value of the bit at time T be labelled x_T, and for simplicity further assume that the temporal parameter changes discretely such that $T = 0, 1, 2, ...$, as is apparent in the case of the 'clock' of a modern microprocessor (c.f. the parameter "n" used to denote different states Ψ_n and stages Ω_n etc.). Moreover, because it is a classical system that is being discussed, it is reasonable to conclude that at any particular time T the bit is definitely in a unique state $x_T = 0$ or $x_T = 1$ whether or not it is actually observed; in classical mechanics every dynamical variable may be said to possess 'independent existence', irrespective of attempts to measure it.

Consider now the dynamics, of which there are of course many different types, that govern the bit's development from its value x_T at time T to the value x_{T+1} at time $T+1$. Generally speaking, any such development implies some sort of function F that, when applied to the bit at time T, provides the value x_{T+1} at time $T+1$. It is consequently possible to represent this mapping by the operation $x_{T+1} = F(x_T)$.

One of the simplest set of dynamics could incorporate a deterministic rule of the form $x_{T+1} = x_T$, such that the value of the bit remains constant for all time. In this situation, after N timesteps the 'history' of the bit's value will either be $(N+1)$ lots of 0's or else $(N+1)$ lots of 1's, depending on its value at the initial time. Alternatively, another simple deterministic rule might be that $x_{T+1} = (x_T \oplus 1)$, where $\oplus$ denoted addition modulo 2, in which case the history of the bit's value will be of the form $... \rightarrow 0 \rightarrow 1 \rightarrow 0 \rightarrow 1 \rightarrow ...$

Consider instead, however, a dynamics based on randomness, such that at each time step there is a unique probability that the bit will have the value 0 and a unique probability that the bit will take the value 1, where these two probabilities sum to unity as expected. The probability $P(x_{T+1} = 0)$ that the value of the bit at time $T+1$ is 0 could be irrespective of the value x_T of the bit at time T, as could the probability $P(x_{T+1} = 1)$ that the value of the bit at time $T+1$ is 1. On the other hand, it is conversely possible to consider a case where the dynamics might be based upon conditional probabilities, incorporating for

example the probability $P(x_{T+1} = 0|x_T)$ that the value of the bit at time $T + 1$ is 0, or similarly the conditional probability $P(x_{T+1} = 1|x_T)$ that the value of the bit at time $T + 1$ is 1, *given* that its value at time T was x_T.

Overall, then, it is possible to postulate a 'Random Operation' R that acts on a bit of value x_T, at time T, to give a value x_{T+1}, at time $T + 1$, with a particular conditional probability; that is $R(x_T) = x_{T+1}$, where x_{T+1} occurs with probability $P(x_{T+1}|x_T)$.

Assume now that at time $T = 0$ the bit has the initial value $x_0 = 0$, and consider such a dynamics based upon conditional probabilities. By assuming conservation of probability, if the conditional probability $P(x_1 = 0|x_0 = 0)$ is denoted

$$P(x_1 = 0|x_0 = 0) = a \tag{6.19}$$

then the probability $P(x_1 = 1|x_0 = 0)$ is given by

$$P(x_1 = 1|x_0 = 0) = \bar{a} \equiv 1 - a. \tag{6.20}$$

It is similarly possible to consider the probabilities of obtaining particular 'chains' of results. If the conditional probabilities $P(x_2 = 0|x_1 = 0)$ and $P(x_2 = 0|x_1 = 1)$ are given by

$$\begin{aligned} P(x_2 &= 0|x_1 = 0) = b \\ P(x_2 &= 0|x_1 = 1) = c \end{aligned} \tag{6.21}$$

then it immediately follows that

$$\begin{aligned} P(x_2 &= 1|x_1 = 0) = \bar{b} \\ P(x_2 &= 1|x_1 = 1) = \bar{c} \end{aligned} \tag{6.22}$$

so

$$\begin{aligned} P(x_2 &= 0 \;\&\; x_1 = 0|x_0 = 0) = P(x_1 = 0|x_0 = 0)P(x_2 = 0|x_1 = 0) = ab \\ P(x_2 &= 1 \;\&\; x_1 = 0|x_0 = 0) = P(x_1 = 0|x_0 = 0)P(x_2 = 1|x_1 = 0) = a\bar{b} \\ P(x_2 &= 0 \;\&\; x_1 = 1|x_0 = 0) = P(x_1 = 1|x_0 = 0)P(x_2 = 0|x_1 = 1) = \bar{a}c \\ P(x_2 &= 1 \;\&\; x_1 = 1|x_0 = 0) = P(x_1 = 1|x_0 = 0)P(x_2 = 1|x_1 = 1) = \bar{a}\bar{c} \end{aligned} \tag{6.23}$$

with as before, and in the following, a 'bar' over any variable α implies $\bar{\alpha} = 1 - \alpha$, and noting that

$$\left(\begin{array}{c} P(x_2 = 0 \;\&\; x_1 = 0|x_0 = 0) + P(x_2 = 1 \;\&\; x_1 = 0|x_0 = 0) \\ +P(x_2 = 0 \;\&\; x_1 = 1|x_0 = 0) + P(x_2 = 1 \;\&\; x_1 = 1|x_0 = 0) \end{array} \right) = 1 \tag{6.24}$$

as expected. In words, a composite probability of the form $P(x_2 = 1 \ \& \ x_1 = 0|x_0 = 0)$ clearly represents the combined likelihood of the bit having the values $x_0 = 0$, $x_1 = 0$ and $x_2 = 1$ at the times $T = 0, 1$ and 2 respectively.

As an additional convenience, it is possible to denote consecutive results as a string of 0's and 1's, with time running from left to right. As an illustration, the sequence of results $x_0 = 0$, $x_1 = 0$ and $x_2 = 1$ is labelled in this notation by the string 001, and from (6.23) occurs with probability $P(001) = a\bar{b}$. Note that, perhaps rather confusingly, $P(001) \equiv P(x_2 = 1 \ \& \ x_1 = 0|x_0 = 0)$; as will be obvious from the quantum case discussed later, the reason for this order reversal comes from a desire to consider probabilities $P(x_{T+1}|x_T)$ as analogous to quantum probability amplitudes of the form $|\langle x_{T+1}|x_T\rangle|$, and sequences $x_0x_1x_2...$ as analogous to qubit states of the form $|x_0\rangle \otimes |x_1\rangle \otimes |x_2\rangle \otimes ...$

It is important to specify exactly what is meant by 'probability' in this example. To this end, the probabilities are taken to imply here that if a very large number N of identical bits were all in the same initial state $x_0 = 0$, and if they were all subject to these same dynamics, then at time $T = 1$ a number Na would be expected to have the value $x_1 = 0$, and so $N(1 - a)$ would consequently have the value $x_1 = 1$. Alternatively, the probabilities may equally be viewed as the frequencies of obtaining particular 'histories' if the experiment was performed very many times N. So after two time steps, Nab of the experiments would be expected to have resulted in the history 000, whereas $Na\bar{b}$ of the experiments would be expected to have resulted in the history 001, whilst $N\bar{a}c$ of the experiments would be expected to have resulted in the history 010, and $N\bar{a}\bar{c}$ of the experiments would be expected to have resulted in the history 011.

The above process may be continued indefinitely. Clearly, with each time step the number of different possible sequences of results doubles, such that at time T the bit will have experienced one of 2^T possible histories, each with a specific probability of occurring.

Consider, however, just the set of possible histories for a bit developing from time $T = 0$ to time $T = 4$. By defining the additional probabilities

$$\begin{aligned} P(0000) &= abd \quad , \quad P(0010) = a\bar{b}e \quad , \quad P(0100) = \bar{a}cf \quad , \quad P(0110) = \bar{a}\bar{c}g \qquad (6.25) \\ P(00000) &= abdh \quad , \quad P(00010) = abd\bar{i} \quad , \quad P(00100) = a\bar{b}ej \quad , \quad P(00110) = a\bar{b}\bar{e}k \\ P(01000) &= \bar{a}cfl \quad , \quad P(01010) = \bar{a}c\bar{f}m \quad , \quad P(01100) = \bar{a}\bar{c}gn \quad , \quad P(01110) = \bar{a}\bar{c}go \end{aligned}$$

the individual histories can be illustrated as the 'branches' in Figure 6.1, where time runs downwards. At each of the 2^T individual 'forks' occurring at time T, the value at the fork represents the probability that the bit at time T will have the history $0...x_{T-2}x_{T-1}x_T$, whereas the value at the bottom of the left hand branch of this fork denotes the probability that the bit at time $T + 1$ will have the history $0...x_{T-2}x_{T-1}x_T0$, whilst the value at the

bottom of the right hand branch of this fork denotes the probability that the bit at time $T+1$ will have the history $0...x_{T-2}x_{T-1}x_T1$.

So, for example, if N bits are developed under identical conditions the diagram shows that $Na\bar{b}\bar{e}$ of them would be expected to have undergone the history 0011 (i.e. [left][right][right]) at time $T=3$, whereas $N\bar{a}c\bar{f}\bar{m}$ would be expected to have undergone the history 01011 (i.e. [right][left][right][right]) at time $T=4$, etc.

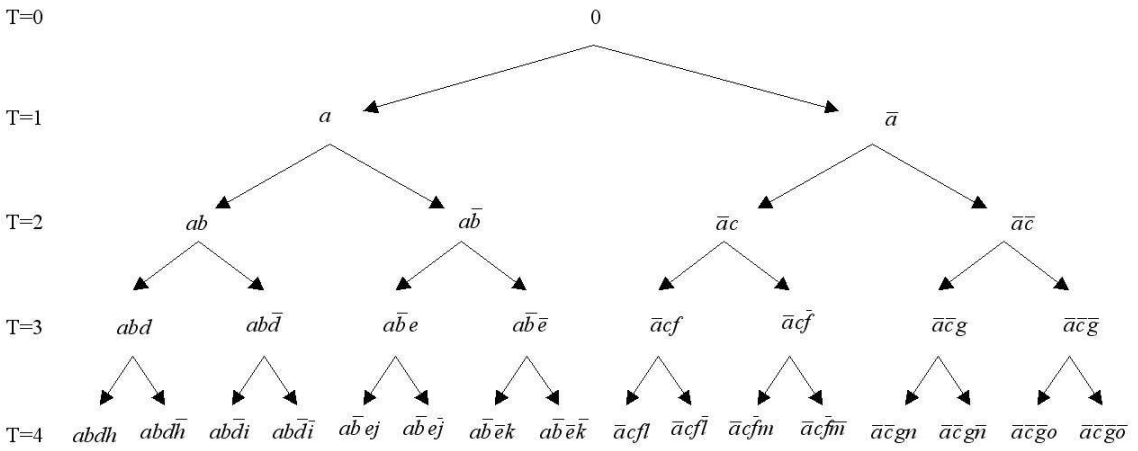

Figure 6.1: Classical probability 'tree' for a developing bit.

In this model, any information regarding the value of the bit at time $T-1$ is irretrievably lost at time T. Without a form of 'memory' recording the bit's development there is no way of reconstructing any of the histories featured, and hence there is no way of concluding, for example, at time $T>2$ that the bit had the value x_1 at time $T=1$, and had the value x_2 at time $T=2$, etc. All that can be known at time T is that the bit has either the value $x_T=0$ or 1 *now*.

To overcome this it is therefore desirable to incorporate a type of 'information store' into the model. Consider, as a possible method of achieving such an effect, an M bit system labelled at time T by the string $L_T=[x_0x_1x_2...x_{M-1}]_T$, where $x_i=0,1$ denotes the value of the i^{th} bit, for $i=0,...,(M-1)$, labelled by left-right position $i+1$. Moreover, assume that at initial time $T=0$ all the bits have the value 0, such that $L_0=[0_00_10_2...0_{M-1}]_0$.

Note the change in the use of the sub-script on the variable x_i here; the symbol x_i is now used to denote the value x of the i^{th} bit, and not a value at time i, as it was previously. Instead, the sub-script T on L_T and outside the square brackets $[...]_T$ takes the place of the temporal parameter. Thus, the expression $L_T=[x_0x_1x_2...x_{M-1}]_T$ denotes a string $x_0x_1x_2...x_{M-1}$ of M bits of respective values x_0, x_1, x_2, ..., x_{M-1} (for $x_i=0,1$) at time T, labelled by L_T. So for example if $L_3=[1_00_11_2...0_{M-1}]_3$, it implies that at time $T=3$

the 0^{th} bit currently has the value 1, whilst the 1^{st} bit has the value 0, whereas the 2^{nd} bit has the value 1, and so on.

In actual fact, this distinction will not matter much in the following, since the i^{th} bit will be used below to encode information at time i; the underlying change in sub-script nomenclature is, however, nevertheless apparent.

Let the zeroth bit, $i = 0$, be associated with the single developing bit discussed so far in this section. So, in this case the previous dynamics is equivalent to the operation

$$L_{T+1} = R(L_T) = R([x_0x_1x_2...x_{M-1}]_T) = R([x_00_10_2...0_{M-1}]_T) = [x_00_10_2...0_{M-1}]_{T+1} \tag{6.26}$$

such that for example the string $L_1 = [0_00_10_2...0_{M-1}]_1$ occurs with probability a, whereas the string $L_1 = [1_00_10_2...0_{M-1}]_1$ occurs with probability $\bar{a}$. Further, the probability that the particular string $L_2 = [1_00_10_2...0_{M-1}]_2$ occurs at time 2, and that the previous string was $L_1 = [0_00_10_2...0_{M-1}]_1$, is clearly $a\bar{b}$. And so on.

To incorporate a 'memory' into the system, the rule is now introduced that at time T the T^{th} bit is assigned the current value of the zeroth bit, where $T \leq (M-1)$. This result is accomplished by an operation $C_{(0,T)}$, where

$$C_{(0,T)}(L_T) = C_{(0,T)}([x_0x_1x_2...x_{T-1}x_Tx_{T+1}...x_{M-1}]_T) = [x_0x_1x_2...x_{T-1}x_0x_{T+1}...x_{M-1}]_T \tag{6.27}$$

so that the value of the T^{th} bit is therefore providing a record of the value of the zeroth bit at time T.

The rule is obviously based on the relations {if $x_0 = 0$ at time T then $x_T \rightarrow 0$} and {if $x_0 = 1$ at time T then $x_T \rightarrow 1$}, and given that every bit initially has the value 0 it is clear to see that $C_{(0,T)}$ is just the CNOT operator of Section 6.1.1.

Overall, then, the development of this system from time T to $T+1$ follows the procedure $L_{T+1} = C_{(0,T+1)}(R(L_T))$.

In order to 'remember' the four time steps illustrated in Figure 6.1, a model involving at least five bits is required. At initial time $T = 0$ the system is described by the string $L_0 = [0_00_10_20_30_4]_0$, and as it develops the bits labelled $1, 2, 3$ and 4 record the history of bit 0 at those respective times. As an example, given a large number N of such systems it is expected that at time $T = 4$ a fraction $N\bar{a}cf\bar{l}$ will have the string $L_4 = [1_01_10_20_31_4]_4$, because the probability $P(01001)$ that the value of bit 0 develops from 0 to 1 to 0 to 0 to 1 over the four times steps is given by $\bar{a}cf\bar{l}$. Similarly, the probability $P(01101)$ that the zeroth bit had the values $0, 1, 1, 0$ and 1 at times $T = 0, 1, 2, 3$ and 4 respectively is $\bar{a}\bar{c}g\bar{n}$, and so if N five bit systems were developed, $N\bar{a}\bar{c}g\bar{n}$ of them would be expected to have the configuration $L_4 = [1_01_11_20_31_4]_4$ at time $T = 4$.

With the probabilities of obtaining specific histories of results defined in Figure 6.1, it

is possible to examine the probability of obtaining a particular type of result. For example, a physicist might be interested in the probability p_α that at time $T = 3$ the bit, irrespective of its history, has the value 0, and this is readily given by the sum $p_\alpha = abd + a\bar{b}e + \bar{a}cf + \bar{a}\bar{c}g$. Alternatively, it might be of concern to discuss the probability p_β that at time $T = 4$ the bit has the value 1, given that at time $T = 3$ it also had the value 1, and this is clearly $p_\beta = abd\bar{i} + a\bar{b}\bar{e}\bar{k} + \bar{a}cf\bar{m} + \bar{a}\bar{c}\bar{g}\bar{o}$.

One particular probability of interest is the correlation K_{xy}, defined as the probability that the bit had the same value at times $T = x$ and $T = y$ minus the probability that the bit had different values at times $T = x$ and $T = y$. Thus for an experiment performed N times, K_{xy} is equally defined as the number of histories for which the bit had different, anti-correlated values at times x and y, subtracted from the number of histories for which the bit had the same, correlated value at times x and y, all divided by N. As an example, the correlation K_{12} is given by

$$\begin{aligned} K_{12} &= (ab + \bar{a}\bar{c}) - (a\bar{b} + \bar{a}c) \\ &= ab + (1-a)(1-c) - a(1-b) - (1-a)c \\ &= 1 - 2a - 2c + 2ab + 2ac. \end{aligned} \tag{6.28}$$

It is further possible to define 'multiple correlations'. One such possibility, K, is given by

$$K = K_{12} + K_{23} + K_{34} - K_{14} \tag{6.29}$$

which reduces to

$$\begin{aligned} K &= 2 - 4(c+g) + 4(ac - ae + cg + ag - ak + gn) \\ &\quad +4(abe - acg - abi + abk + aek + cfl - agn - cgn) \\ &\quad +4(abdi - abek - acfl + acgn) \end{aligned} \tag{6.30}$$

given the above probabilities. Moreover, because each individual probability $a, b, ..., o$ is $0 \leq a, b, ..., o \leq 1$ it can readily be shown that K satisfies the inequality

$$-2 \leq K \leq 2. \tag{6.31}$$

The importance of (6.31) is that it is the temporal equivalent of the spacelike classical Bell inequality, as given in Section 3.1, for a bit classically developing over four time steps. It represents the relations between unequal-time correlations induced in a single bit system, where at each time the bit has a definite and independently existing value (0 or 1) irrespective of whether or not it is actually measured. Moreover, because every classical bit must follow one of these 16 possible histories between times $T = 1$ and 4, the correlations of any bit that obeys classical equations of dynamics must necessarily satisfy this inequality.

Furthermore, the unequal-time correlations of a single qubit would also have to obey this temporal Bell inequality *if* it were to be argued that its 'quantum' dynamics ultimately depended upon any sort of classical hidden variables, or if the value of the qubit at any time could somehow be known without disturbing the system. This issue will be discussed now.

6.1.4 Quantum Calculations and the Bell Inequality.

In this final part of Section 6.1, the ideas of the previous three subsections will be drawn together to show how the quantum CNOT operator can be used to perform a particular quantum computation on a set of qubits. Moreover, because the CNOT operator can be applied to the qubits a number of times in succession, the possibility of comparing correlations between results at unequal times might be expected to arise, just as in the case of the classical calculation described above. Thus, a 'quantised' version of the model introduced in 6.1.3 is presented, and this leads to a generation of relations analogous to the classical temporal Bell inequality, which in turn allows the implications and validity of the suggested quantum method to be explored.

Note that an alternative analysis of this was introduced in [68]. It is felt, however, that approach lacked clarity, and so in this final sub-section an improvement and reinterpretation of the issues advocated by Kim *et al* is sought. Important conclusions may then be drawn about the nature of quantum computation, and these will consequently provide insight into some of the limitations inherent in any attempt to treat the Universe as a giant quantum computer.

In the following work, subscripts are generally used to label qubits, with the exception that the subscript on Ψ_T is used to denote a (discrete) temporal parameter, $T = 0, 1, 2, ...$ As a further comment on this, note that T is taken below as a time parameter that is external to the qubit system being discussed; thus, T is the 'usual' time of conventional exophysics, as experienced by an observer who is not part of the system under investigation. This last point will be an important factor in the later interpretation of the model.

Consider a five qubit system represented at time T by a state Ψ_T in a Hilbert space $\mathcal{H}^{(32)}_{[0...4]}$ spanned by an orthonormal basis $\mathcal{B}_{0...4} \equiv \{|i\rangle_0 \otimes |j\rangle_1 \otimes |k\rangle_2 \otimes |l\rangle_3 \otimes |m\rangle_4 : i, j, k, l, m = 0, 1\}$. Assume that at initial time $T = 0$ every qubit is in the 'down' state $|0\rangle$, such that the overall state may be written

$$|\Psi_0\rangle = |0\rangle_0 \otimes |0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 \otimes |0\rangle_4 = |00000\rangle \tag{6.32}$$

where the qubit subscripts may be omitted in favour of left-right positioning for brevity.

Consider also an unitary operator $\hat{S}_0$ acting locally on qubit 0, defined as

$$\hat{S}_0 \equiv \exp\left(-\frac{i}{2}\alpha\hat{\sigma}_0^1\right) \tag{6.33}$$

where $\alpha \in \mathbb{R}$ is a small real parameter and $\hat{\sigma}_0^a$ is the a^{th} Pauli operator $\hat{\sigma}^a$ acting in subregister $\mathcal{H}_0$ for $a = 1, 2, 3$.

Note that a unitary operator $\hat{S}_0$ acting locally on qubit 0 is equivalent to the operator $\hat{S}_0^G$ acting globally on the entire state Ψ, where $\hat{S}_0^G$ is defined by

$$\hat{S}_0^G \equiv \hat{S}_0 \otimes \hat{I}_1 \otimes \hat{I}_2 \otimes \hat{I}_3 \otimes \hat{I}_4 \tag{6.34}$$

with $\hat{I}_t$ the identity operator in Hilbert space $\mathcal{H}_t$ for $t = 0, 1, 2, 3, 4$. This equivalence will be discussed in Chapter 7, but for now note that the two forms of the operator are used interchangeably.

The exponential (6.33) can be expanded, to obtain

$$\begin{aligned}
\hat{S}_0 &= \sum_{n=0}^{\infty}\left(-\frac{i}{2}\alpha\right)^n \frac{(\hat{\sigma}_0^1)^n}{n!} \qquad (6.35)\\
&= \sum_{n=0}^{\infty}\left(-\frac{i}{2}\alpha\right)^{2n} \frac{(\hat{\sigma}_0^1)^{2n}}{(2n)!} + \sum_{n=0}^{\infty}\left(-\frac{i}{2}\alpha\right)^{2n+1} \frac{(\hat{\sigma}_0^1)^{2n+1}}{(2n+1)!}\\
&= (\hat{\sigma}_0^0)\sum_{n=0}^{\infty}\frac{1}{(2n)!}\left(-\frac{i}{2}\alpha\right)^{2n} + (\hat{\sigma}_0^1)\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(-\frac{i}{2}\alpha\right)^{2n+1}\\
&= \cos\left(\frac{\alpha}{2}\right)\hat{\sigma}_0^0 - i\sin\left(\frac{\alpha}{2}\right)\hat{\sigma}_0^1.
\end{aligned}$$

Consider also the generalised definition of the two-qubit, quantum CNOT operator $\hat{C}_{(r,s)}$ that acts on subregisters r and s

$$\hat{C}_{(r,s)} \equiv \hat{P}_r^0 \otimes \hat{\sigma}_s^0 + \hat{P}_r^1 \otimes \hat{\sigma}_s^1 \tag{6.36}$$

with $\hat{P}_r^z = |z\rangle_{rr}\langle z|$ for $z = 0, 1$. Again, this operator can also be extended to one that acts globally by taking a suitable product with $\hat{I}_t$ for $t = 0, 1, ..., 4$ where $t \neq r, s$.

In the example presented in this sub-section, and for reasons to become apparent, the procedure performed by the quantum computation involves an application of the operator $\hat{S}_0$ to the state $|\Psi_T\rangle$, followed by the operator $\hat{C}_{(r,s)}$. Moreover, in the chosen dynamics attention is restricted to the case where $r = 0$ and $s = T+1$, such that the CNOT operator may be written $\hat{C}_{(0,T+1)}$. Thus given a state $|\Psi_T\rangle$, the state $|\Psi_{T+1}\rangle$ at time $T + 1$ is

$$|\Psi_{T+1}\rangle = \hat{C}_{(0,T+1)}\hat{S}_0|\Psi_T\rangle. \tag{6.37}$$

The state $|\Psi_{T+1}\rangle$ may itself then be evolved with $\hat{S}_0$ and $\hat{C}_{(0,T+2)}$ to generate the later state $|\Psi_{T+2}\rangle$. So overall, given the initial condition of Ψ_0 at $T = 0$, the state Ψ_T at time T will be

$$|\Psi_T\rangle = (\hat{C}_{(0,T)}\hat{S}_0)(\hat{C}_{(0,T-1)}\hat{S}_0)...(\hat{C}_{(0,2)}\hat{S}_0)(\hat{C}_{(0,1)}\hat{S}_0)|\Psi_0\rangle = \hat{U}_T|\Psi_0\rangle \tag{6.38}$$

where $\hat{U}_T \equiv (\hat{C}_{(0,T)}\hat{S}_0)(\hat{C}_{(0,T-1)}\hat{S}_0)...(\hat{C}_{(0,2)}\hat{S}_0)(\hat{C}_{(0,1)}\hat{S}_0)$ is also a unitary operator.

Given the above general algorithm, the specific development of the state of the five qubit system under investigation proceeds as follows. If the initial state Ψ_0 is $|00000\rangle$, then

$$\begin{aligned} \hat{S}_0|00000\rangle &= \left(\cos\left(\frac{\alpha}{2}\right)\hat{\sigma}_0^0 - i\sin\left(\frac{\alpha}{2}\right)\hat{\sigma}_0^1\right)\otimes|00000\rangle_{01234} \\ &= \left(\cos\left(\frac{\alpha}{2}\right)|0\rangle_0 - i\sin\left(\frac{\alpha}{2}\right)|1\rangle_0\right)\otimes|0000\rangle_{1234} \end{aligned} \tag{6.39}$$

and this result is subsequently 'registered' with qubit 1 by the CNOT operator acting in the prescribed way, such that the later state $|\Psi_1\rangle = \hat{C}_{(0,1)}\hat{S}_0|00000\rangle$ is given by

$$\begin{aligned} |\Psi_1\rangle &= \left[\hat{P}_0^0\otimes\hat{\sigma}_1^0 + \hat{P}_0^1\otimes\hat{\sigma}_1^1\right]\left(\cos\left(\frac{\alpha}{2}\right)|0\rangle_0\otimes|0\rangle_1 - i\sin\left(\frac{\alpha}{2}\right)|1\rangle_0\otimes|0\rangle_1\right)\otimes|000\rangle_{234} \\ &= \left(\cos\left(\frac{\alpha}{2}\right)|0\rangle_0\otimes|0\rangle_1 - i\sin\left(\frac{\alpha}{2}\right)|1\rangle_0\otimes|1\rangle_1\right)\otimes|000\rangle_{234}. \end{aligned} \tag{6.40}$$

Continuing,

$$\begin{aligned} \hat{S}_0|\Psi_1\rangle &= \left[\cos\left(\frac{\alpha}{2}\right)\hat{\sigma}_0^0 - i\sin\left(\frac{\alpha}{2}\right)\hat{\sigma}_0^1\right]\left(\cos\left(\frac{\alpha}{2}\right)|00\rangle_{01} - i\sin\left(\frac{\alpha}{2}\right)|11\rangle_{01}\right)\otimes|000\rangle_{234} \\ &= \begin{pmatrix} \cos^2\left(\frac{\alpha}{2}\right)|00\rangle_{01} - i\cos\left(\frac{\alpha}{2}\right)\sin\left(\frac{\alpha}{2}\right)|11\rangle_{01} \\ -i\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)|10\rangle_{01} - \sin^2\left(\frac{\alpha}{2}\right)|01\rangle_{01} \end{pmatrix}\otimes|000\rangle_{234} \end{aligned} \tag{6.41}$$

and so the subsequent state $|\Psi_2\rangle = \hat{C}_{(0,2)}\hat{S}_0|\Psi_1\rangle$ is given by

$$\begin{aligned} |\Psi_2\rangle &= \left[\hat{P}_0^0\otimes\hat{\sigma}_2^0 + \hat{P}_0^1\otimes\hat{\sigma}_2^1\right]\begin{pmatrix} \cos^2\left(\frac{\alpha}{2}\right)|000\rangle_{012} - i\cos\left(\frac{\alpha}{2}\right)\sin\left(\frac{\alpha}{2}\right)|110\rangle_{012} \\ -i\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)|100\rangle_{012} - \sin^2\left(\frac{\alpha}{2}\right)|010\rangle_{012} \end{pmatrix}\otimes|00\rangle_{34} \\ &= \begin{pmatrix} \cos^2\left(\frac{\alpha}{2}\right)|000\rangle_{012} - i\cos\left(\frac{\alpha}{2}\right)\sin\left(\frac{\alpha}{2}\right)|111\rangle_{012} \\ -i\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)|101\rangle_{012} - \sin^2\left(\frac{\alpha}{2}\right)|010\rangle_{012} \end{pmatrix}\otimes|00\rangle_{34}. \end{aligned} \tag{6.42}$$

The states $|\Psi_3\rangle$ and $|\Psi_4\rangle$ at times $T = 3$ and 4 can generated in a similar way, with the results

$$|\Psi_3\rangle = \begin{pmatrix} \cos^3\left(\frac{\alpha}{2}\right)|0000\rangle_{0123} - i\cos^2\left(\frac{\alpha}{2}\right)\sin\left(\frac{\alpha}{2}\right)|1111\rangle_{0123} \\ -i\sin\left(\frac{\alpha}{2}\right)\cos^2\left(\frac{\alpha}{2}\right)|1011\rangle_{0123} - \cos\left(\frac{\alpha}{2}\right)\sin^2\left(\frac{\alpha}{2}\right)|0100\rangle_{0123} \\ -i\sin\left(\frac{\alpha}{2}\right)\cos^2\left(\frac{\alpha}{2}\right)|1001\rangle_{0123} - \cos\left(\frac{\alpha}{2}\right)\sin^2\left(\frac{\alpha}{2}\right)|0110\rangle_{0123} \\ -\sin^2\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)|0010\rangle_{0123} + i\sin^3\left(\frac{\alpha}{2}\right)|1101\rangle_{0123} \end{pmatrix}\otimes|0\rangle_4 \tag{6.43}$$

and

$$|\Psi_4\rangle = \begin{pmatrix} \cos^4\left(\frac{\alpha}{2}\right)|00000\rangle_{01234} - i\cos^3\left(\frac{\alpha}{2}\right)\sin\left(\frac{\alpha}{2}\right)|11111\rangle_{01234} \\ -i\sin\left(\frac{\alpha}{2}\right)\cos^3\left(\frac{\alpha}{2}\right)|10111\rangle_{01234} - \cos^2\left(\frac{\alpha}{2}\right)\sin^2\left(\frac{\alpha}{2}\right)|01000\rangle_{01234} \\ -i\sin\left(\frac{\alpha}{2}\right)\cos^3\left(\frac{\alpha}{2}\right)|10011\rangle_{01234} - \cos^2\left(\frac{\alpha}{2}\right)\sin^2\left(\frac{\alpha}{2}\right)|01100\rangle_{01234} \\ -\sin^2\left(\frac{\alpha}{2}\right)\cos^2\left(\frac{\alpha}{2}\right)|00100\rangle_{01234} + i\cos\left(\frac{\alpha}{2}\right)\sin^3\left(\frac{\alpha}{2}\right)|11011\rangle_{01234} \\ -i\sin\left(\frac{\alpha}{2}\right)\cos^3\left(\frac{\alpha}{2}\right)|10001\rangle_{01234} - \cos^2\left(\frac{\alpha}{2}\right)\sin^2\left(\frac{\alpha}{2}\right)|01110\rangle_{01234} \\ -\sin^2\left(\frac{\alpha}{2}\right)\cos^2\left(\frac{\alpha}{2}\right)|00110\rangle_{01234} + i\cos\left(\frac{\alpha}{2}\right)\sin^3\left(\frac{\alpha}{2}\right)|11001\rangle_{01234} \\ -\sin^2\left(\frac{\alpha}{2}\right)\cos^2\left(\frac{\alpha}{2}\right)|00010\rangle_{01234} + i\cos\left(\frac{\alpha}{2}\right)\sin^3\left(\frac{\alpha}{2}\right)|11101\rangle_{01234} \\ +i\sin^3\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)|10101\rangle_{01234} + \sin^4\left(\frac{\alpha}{2}\right)|01010\rangle_{01234} \end{pmatrix}. \quad (6.44)$$

For the purposes of illustration, it is possible to associate a classical interpretation to the action of the sequence $(\hat{C}_{(0,1)}\hat{S}_0)$, $(\hat{C}_{(0,2)}\hat{S}_0), ...$ and the consequent development of the system. Firstly, the operator $\hat{S}_0$ may be thought of as one that locally 'rotates' the state of qubit 0 independently of the other four qubits. Then, during the development of the system from time T to $T+1$ it is assumed that the CNOT operator somehow 'examines' the state of qubit 0 before 'extracting' this information and registering it with qubit $T+1$. Moreover, in the current example this registration appears analogous to a 'copying' procedure, because each qubit is initially in the state $|0\rangle$ and CNOT operates according to the rules $\hat{C}_{(a,b)}(|0\rangle_a \otimes |0\rangle_b) \rightarrow |0\rangle_a \otimes |0\rangle_b$ and $\hat{C}_{(a,b)}(|1\rangle_a \otimes |0\rangle_b) \rightarrow |1\rangle_a \otimes |1\rangle_b$.

Thus, the action of the operators may be thought of as producing a 'wave of information' that sweeps through the system, moving in time along the chain of qubits. During the evolution from Ψ_T to Ψ_{T+1} only the 0^{th} and $(T+1)^{th}$ qubits appear affected, and once this classical looking 'wave' has 'passed' the $(T+1)^{th}$ qubit its sub-state is never again altered by the dynamics. So, the qubit $T+1$ appears to serve as a permanent record of the sub-state of qubit 0 after the application of $\hat{S}$ to Ψ_T.

Because the spins of the qubits $1, 2, 3$ and 4 appear to contain information about the state of qubit 0 at times $T=1$, $T=2$, $T=3$ and $T=4$ respectively, it might be natural to expect that these four qubits could be interrogated in order to learn about the 'history' of the development of qubit 0. In fact, because once information is encoded into these qubits it is assumed permanent, the individual spins of these four correlated qubits $1, 2, 3$ and 4 in the final state Ψ_4 might be expected to contain a record of the whole history of qubit 0's development. In other words, it may be hoped that by asking a specific question about the state of the T^{th} qubit of Ψ_4, insight might be gained into the state of qubit 0 at time T, where $1 \leq T \leq 4$.

Furthermore, in fact, and following the lead of [68], one possible such insight might involve the quantum correlation Q_{xy} defined as

$$Q_{xy} = \langle\Psi_4|\hat{\sigma}_x^3\hat{\sigma}_y^3|\Psi_4\rangle \quad (6.45)$$

which is the quantum analogue of the classical correlation K_{xy} defined in Section 6.1.3, and where $x, y = 1, 2, 3, 4$ indicate the four 'memory' qubits. Again $\hat{\sigma}_x^3$ is a Pauli operator acting locally on qubit space x, and may therefore as before also be associated with an equivalent global operator by extending it in the obvious way.

The validity of this 'insight' is investigated now.

In the context of (6.45), the operator $\hat{\sigma}_x^3$ may be interpreted as an object that 'asks a question' of the spin of qubit x in Ψ_4 whilst doing nothing to any of the other qubits: if the x^{th} qubit is in the spin-down state $|0\rangle_x = \binom{1}{0}_x$ the operation $\hat{\sigma}_x^3|0\rangle_x = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)_x \binom{1}{0}_x$ gives a result $|0\rangle_x$, whereas if the x^{th} qubit is in the spin-up state $|1\rangle_x = \binom{0}{1}_x$ then $\hat{\sigma}_x^3|1\rangle_x$ gives a result $-|1\rangle_x$.

So, the correlation Q_{xy} sums the amplitudes of the terms in $|\Psi_4\rangle$ for which the x^{th} and y^{th} qubits have the same spin state, and subtracts from this the amplitudes of terms in $|\Psi_4\rangle$ for which the x^{th} and y^{th} qubits have opposite spin states. Moreover, due to the sequential procedure followed in the generation of Ψ_4, the suggestion is then that Q_{xy} may be thought of as the correlation between the state of qubit 0 at the times x and y. It is this suggestion that is now examined.

First, though, the correlations Q_{xy} must be evaluated. In the present representation $Q_{12} = \langle\Psi_4|\hat{\sigma}_1^3\hat{\sigma}_2^3|\Psi_4\rangle$ is given by

$$\begin{aligned} Q_{12} \;=\; & \cos^4\left(\frac{\alpha}{2}\right)\cos^4\left(\frac{\alpha}{2}\right)\left[{}_{01234}\langle 00000|\hat{\sigma}_1^3\hat{\sigma}_2^3|00000\rangle_{01234}\right] \\ & + \cos^4\left(\frac{\alpha}{2}\right)\left(-i\cos^3\left(\frac{\alpha}{2}\right)\sin\left(\frac{\alpha}{2}\right)\right)\left[{}_{01234}\langle 00000|\hat{\sigma}_1^3\hat{\sigma}_2^3|11111\rangle_{01234}\right] + \ldots \\ & + \left(+i\cos^3\left(\frac{\alpha}{2}\right)\sin\left(\frac{\alpha}{2}\right)\right)\left(-i\cos^3\left(\frac{\alpha}{2}\right)\sin\left(\frac{\alpha}{2}\right)\right)\left[{}_{01234}\langle 11111|\hat{\sigma}_1^3\hat{\sigma}_2^3|11111\rangle_{01234}\right] + \\ & \ldots + \sin^4\left(\frac{\alpha}{2}\right)\sin^4\left(\frac{\alpha}{2}\right)\left[{}_{01234}\langle 01010|\hat{\sigma}_1^3\hat{\sigma}_2^3|01010\rangle_{01234}\right] \end{aligned} \tag{6.46}$$

which, by using the orthonormality relation

$${}_{01234}\langle i'j'k'l'm'|ijklm\rangle_{01234} = \delta_{ii'}\delta_{jj'}\delta_{kk'}\delta_{ll'}\delta_{mm'} \tag{6.47}$$

becomes

$$\begin{aligned} Q_{12} &= \cos^8\left(\frac{\alpha}{2}\right) + \cos^6\left(\frac{\alpha}{2}\right)\sin^2\left(\frac{\alpha}{2}\right) - \sin^2\left(\frac{\alpha}{2}\right)\cos^6\left(\frac{\alpha}{2}\right) - \cos^4\left(\frac{\alpha}{2}\right)\sin^4\left(\frac{\alpha}{2}\right) \\ &\quad + \sin^2\left(\frac{\alpha}{2}\right)\cos^6\left(\frac{\alpha}{2}\right) + \cos^4\left(\frac{\alpha}{2}\right)\sin^4\left(\frac{\alpha}{2}\right) - \sin^4\left(\frac{\alpha}{2}\right)\cos^4\left(\frac{\alpha}{2}\right) \\ &\quad - \cos^2\left(\frac{\alpha}{2}\right)\sin^6\left(\frac{\alpha}{2}\right) + \sin^2\left(\frac{\alpha}{2}\right)\cos^6\left(\frac{\alpha}{2}\right) + \sin^4\left(\frac{\alpha}{2}\right)\cos^4\left(\frac{\alpha}{2}\right) \\ &\quad - \sin^4\left(\frac{\alpha}{2}\right)\cos^4\left(\frac{\alpha}{2}\right) - \cos^2\left(\frac{\alpha}{2}\right)\sin^6\left(\frac{\alpha}{2}\right) + \sin^4\left(\frac{\alpha}{2}\right)\cos^4\left(\frac{\alpha}{2}\right) \\ &\quad + \cos^2\left(\frac{\alpha}{2}\right)\sin^6\left(\frac{\alpha}{2}\right) - \sin^6\left(\frac{\alpha}{2}\right)\cos^2\left(\frac{\alpha}{2}\right) - \sin^8\left(\frac{\alpha}{2}\right) \\ &= \cos^8\left(\frac{\alpha}{2}\right) + 2\sin^2\left(\frac{\alpha}{2}\right)\cos^6\left(\frac{\alpha}{2}\right) - 2\cos^2\left(\frac{\alpha}{2}\right)\sin^6\left(\frac{\alpha}{2}\right) - \sin^8\left(\frac{\alpha}{2}\right) \\ &= \cos(\alpha). \end{aligned} \tag{6.48}$$

Similarly, it can be shown that

$$\begin{aligned} Q_{23} &= \cos(\alpha) \\ Q_{34} &= \cos(\alpha) \end{aligned} \tag{6.49}$$

but

$$Q_{14} = \cos^3(\alpha). \tag{6.50}$$

As with the classical result K of Section 6.1.3, it is possible to define a multiple correlation Q in the manner

$$Q = Q_{12} + Q_{23} + Q_{34} - Q_{14}. \tag{6.51}$$

such that in the present case

$$Q = 3\cos(\alpha) - \cos^3(\alpha). \tag{6.52}$$

However, by differentiating it can readily be shown that

$$-2 \leq Q \leq 2 \tag{6.53}$$

for all α, exactly as in the case of the classical Bell inequality. Thus, it appears that the qubit system is obeying classical rules of dynamics, which initially suggests that something has gone wrong in the analysis: if the qubits are to obey quantum dynamics, they might be expected to violate the Bell inequality for at least some values of α.

The problem with the above process is that at first glance the operator $\hat{C}_{(0,T)}$ seems to be behaving like an information extraction process. Every time a CNOT operator is used

it seems to imply a modification of the state such that parts of it appear to be 'storing' information regarding the current state of qubit 0. An attempt is then made to access this store at some later time.

This can lead to interpretational difficulties in quantum mechanics. In classical mechanics it is perfectly reasonable to discuss a system of individual bits, each of which possesses a definite value at all times. Moreover, the bits possess these values whether or not they are actually observed. So, in a classical dynamics it is possible to consider the type of model described in Section 6.1.3 in which at any time T the 'zeroth' bit has a certain and specific value, and this value is unambiguously and non-invasively copied by the bit T without affecting anything else.

In quantum theory, however, the same is not true. Firstly, there can be no analogous copying procedure in quantum mechanics. The No-Cloning theorem [69] demonstrates that there is no general unitary operator $\hat{u}$ that maps an arbitrary initial product state of the form $\psi = |A\rangle_i \otimes |B\rangle_j$ into a final product state $\psi' = |A\rangle_i \otimes |A\rangle_j$, where $|X\rangle_a \in \mathcal{H}_a$ for $a = i, j$, such that $\psi' = \hat{u}\psi$. Even though this is not a direct limitation in the current case because the CNOT operator does not in general preserve separability, and is therefore not actually trying to evolve states in this forbidden way, it is evident in this respect that the classical CNOT operation and its quantum operator counterpart are not completely equivalent in their action.

Secondly, when a system becomes entangled (a phenomenon unique to quantum mechanics) it is no longer valid to discuss the components of the entanglement in different factor Hilbert spaces as having any sort of individual existence, independent of one another. Just as in the EPR situation of Chapter 3 and the discussion of separability in Chapter 4, the introduction of entanglement automatically and directly implies a breakdown of the ability to state that "this object with these properties is here". From this point of view it is therefore incorrect to say that qubit 0, which is initially a factor substate of the completely separable state Ψ_0, has any properties on its own, i.e. is either independently up or down, at the times $T = 1, 2, 3, 4$ when it is entangled with the other qubits.

Thirdly, and perhaps more fundamentally, it is also not correct to assume that the system has any physical properties at all independent of observation, and so it is inaccurate to argue that the individual qubits are in any definite state before the measurement. In other words, if the system is not measured at time T it is not only impossible to say which state it is in at this time, but also to say that this state actually exists in any sort of physical sense. It cannot therefore be said that the system is undergoing any particular 'history' or classical 'trajectory'. In fact, the fourth state Ψ_4 may be seen as an entangled linear superposition of every potential classical 'history', and it is only when

a measurement is finally performed that the system is forced to collapse into a particular configuration of qubit sub-states. Moreover if the test is of a certain sort, for example $\hat{\sigma}_y^3$, it may be natural at this point to falsely conclude that the resulting product of sub-states indicates a particular single-valued history for the system (and hence qubit 0), because this is what would be inducted in the world of classical physics familiar to scientists.

Strictly, in fact, it is actually misleading to even use the word 'history', and this point leads to an important comment on the role of time in quantum dynamics. Recall that, normally, quantum probabilities are used to discuss potential futures. In the present case, however, it might appear that an attempt is being made to discuss a potential past, and this is contrary to the usual assertion that the past is a definite and unique, well defined classical construct.

But, the resolution of this contradiction is to note that the potential pasts discussed in the superposition of 'histories' in Ψ_4 instead really form the basis for a set of potential futures. Further, in quantum mechanics these potential futures are themselves only defined relative to the eigenstates of whichever Hermitian operator is actually used to test the state. More accurately, then, the discussion regarding the system's 'history' should perhaps be replaced by the question: "if an operator $\hat{\sigma}_y^3$ is chosen that has eigenvectors $|00000\rangle_{01234}$, $|10000\rangle_{01234}$, $|01000\rangle_{01234}$, ..., $|11111\rangle_{01234}$ what is the probability of projecting Ψ_4 into one of these possible future states?".

Indeed, this point would be clarified further if, instead of $\hat{\sigma}_y^3$, the state Ψ_4 was tested by an operator that only had entangled eigenstates; in this instance, none of the separable product states $|ijklm\rangle_{01234}$ could be an outcome, and so no confusion would occur by associating the result of this test with an apparent classical history for the system.

The issue can be addressed further. Because of the earlier discussions that only observed states can be considered physical, it is not strictly meaningful to consider the state having any physical reality whatsoever between the preparation of the state Ψ_0 at time 0 and the measurement of Ψ_4 after time 4. The quantum system is not proceeding through a definite sequence of states $\Psi_0 \rightarrow \Psi_1 \rightarrow \Psi_2 \rightarrow \Psi_3 \rightarrow \Psi_4$, as would be expected in a semi-classical model evolving along a specific trajectory; instead, only initially prepared states Ψ_0 and measured outcomes $\langle\Psi_4|\hat{\sigma}_x^3\hat{\sigma}_y^3|\Psi_4\rangle$ are physically relevant.

In other words, from the point of view of the state there are not four individual steps existing between Ψ_0 and Ψ_4, and this conclusion may be highlighted by rewriting $\Psi_4 = \hat{U}_4\Psi_0$ as in equation (6.38) and noting that, because $\hat{U}_4 \equiv \hat{C}_{(0,4)}\hat{S}_0\hat{C}_{(0,3)}\hat{S}_0\hat{C}_{(0,2)}\hat{S}_0\hat{C}_{(0,1)}\hat{S}_0$ is just a unitary operator, the state Ψ_4 is really only 'one evolution' away from the state Ψ_0.

In fact, an immediate analogy may be drawn here to the quantum universe model

proposed throughout this thesis: two states Ψ_n and Ψ_{n+1} of the Universe are deemed successive if there is only one collapse 'separating' them. The states Ψ_0 and Ψ_4 of the present model, however, are not separated by any collapses; Ψ_4 is simply an evolved version of Ψ_0, and as such it is not true to say that the system has undergone four distinct developments.

It might perhaps be more honest, then, to relabel Ψ_4 as Ψ_0', that is, an evolved version of Ψ_0. The 'temporal parameter' 4 on Ψ_4 should really only be seen as a marker of the external processes occurring during the unitary Schrödinger evolution from $\Psi_0 \rightarrow \Psi_0'$, for $\Psi_0' \equiv \Psi_4 = \hat{U}_4\Psi_0$, such that the state in question can only be said to have a 'history' between $T = 0$ and $T = 4$ relative to the development of the rest of the Universe during this time. This 'external history' occurs because an external scientist, who is assumed isolated from the state, has physically applied the operator $\hat{S}_0$, followed by the operator $\hat{C}_{(0,1)}$, then $\hat{S}_0$ again, then $\hat{C}_{(0,2)}$, and so on until she applies $\hat{C}_{(0,4)}$. Indeed, recall that T was originally defined just as an external time parameter, and is hence only valid relative to the observer who can remember 'doing something' in the interval during which the state was evolving, and thereby noticing that, relative to other external processes, four time steps (or '(q-) ticks' of the observer's clock [54]) appeared to elapse between the preparation of the state Ψ_0 and the eventual measurement of $\langle\Psi_4|\hat{\sigma}_x^3\hat{\sigma}_y^3|\Psi_4\rangle$. This point reinforces the discussions in Chapters 5 and 8 that physical time in the quantum Universe is not absolute but contextual, and should only be discussed relative to change and the 'path' taken by endo-observers.

Summarising, from the 'internal' point of view of the isolated quantum system, the state Ψ_0 develops to $\Psi_4 \equiv \Psi_0'$ in one 'rotation', whereas from an external viewpoint of an observer developing and interacting with her surroundings in her own personal time T the process appears to occur in four distinct steps. Moreover, it is by falsely granting the external time parameter T an unphysical, internal significance that may be seen to form an origin of the current difficulties.

The problem is additionally complicated in the present situation by misinterpreting the result of $\langle\Psi_4|\hat{\sigma}_x^3\hat{\sigma}_y^3|\Psi_4\rangle$ as seeming to indicate not only that the system followed a particular classical path, but also as to what this path was. Specifically, the inherent error is to assume that even if the state could physically be discussed between external times 0 and 4, the result of $\langle\Psi_4|\hat{\sigma}_x^3\hat{\sigma}_y^3|\Psi_4\rangle$ would actually provide insight into what it was doing. In this case, the mistake lies in assuming that both the operators $\hat{\sigma}_x^3$ and $\hat{\sigma}_y^3$ are measuring the same state Ψ_4. This, however, is not true, because Ψ_4 cannot be measured non-invasively: the measurement of Ψ_4 by the operator $\hat{\sigma}_y^3$ collapses the entangled state

into one of the 16 superposed states[10] given in equation (6.44), but by doing so destroys all of the quantum interferences exhibited by Ψ_4. Thus the system is projected into a classical-looking product of qubit sub-states.

Any subsequent measurement with an operator $\hat{\sigma}_x^3$ then produces with certainty either an up result or a down result, because qubit x is no longer in an entangled superposition of the two. So, contrary to what might be hoped, this further investigation of the system by the operator $\hat{\sigma}_x^3$ is not asking a question about what the state of qubit x was *before* the measurement of Ψ_4 by $\hat{\sigma}_y^3$, but is asking about what the state of qubit x is *afterwards*. Similarly, and from above, any such further investigation by $\hat{\sigma}_x^3$ is not asking a question about what the state of qubit x (or equivalently qubit 0) *was* at an earlier (external) time $T = x$, but is asking about what the state of qubit x *is now*.

As an illustration of this, if $y = 4$ and $\hat{\sigma}_4^3$ finds qubit 4 to be up, and thereby collapses the system into, say, the state Φ given by $\Phi \equiv |11001\rangle_{01234}$, the subsequent measurement of Φ by $\hat{\sigma}_1^3$ seems at first glance to indicate that qubit 1 was definitely in an up state at external time $T = 1$. This, however, is not the correct analysis: at external time $T = 1$ qubit 1 was really in an entangled linear superposition of both up and down states, assuming of course that it is possible to give any existence at all to the state at a time when it was not measured. In short, the mistake is then to conclude that $|11001\rangle_{01234}$ represents the history of the system over an external time span $1 \leq T \leq 4$, and not just the outcome eigenstate of a particular test.

Overall, therefore, the correlation $Q_{xy} = \langle\Psi_4|\hat{\sigma}_x^3\hat{\sigma}_y^3|\Psi_4\rangle$ should not be viewed as asking about how many possible 'histories' of the state between external times $T = 0$ and 4 shared certain characteristics, but is asking about how many from a set of sixteen classical looking eigenstates of the form $|ijklm\rangle_{01234}$ share them.

It is hardly surprising, then, that by ascribing to a state such as $\Phi \equiv |11001\rangle_{01234}$ the semi-classical status of representing what actually happened, i.e. the perspective of a single-valued classical reality for the state's history, the correlations Q do not violate the classical Bell inequality. In this sense the above method is just a form of 'dressed-up' classical probability, with statistical correlations compared between states that have undergone well defined histories. In effect, the classical probabilities a, ab, $a\bar{b}\bar{e}$, ... etc. of Section 6.1.3 have been replaced by sines and cosines, such that for example the probability $P(01100) = \bar{a}\bar{c}gn$ that the bits have the final configuration 01100 in the classical case is directly equivalent to the quantum case where there is a probability $\cos^4\left(\frac{\alpha}{2}\right)\sin^4\left(\frac{\alpha}{2}\right)$ that the state collapses to $|01100\rangle_{01234}$.

As an aside, note that it is of course always possible to examine an ensemble of identi-

[10] Given Ψ_4 and an operator with 32 orthonormal eigenstates of the form $|ijklm\rangle_{01234}$ for $i, j, k, l, m = 0, 1$, there are 16 non-zero amplitudes $\langle ijklm|\Psi_4\rangle$, and hence 16 possible next states.

cally prepared and evolved states Ψ_4 and consider a density matrix of possible eigenstates, but it must be stressed that this is still just a classical, probabilistic result due to the nature of the collapse process, and as such would hence still be expected to obey Bell relations in the corresponding classical way.

Summarising, the problem of the initial analysis of this system was two-fold. Firstly, it is incorrect to assume that $\hat{\sigma}_x^3$ and $\hat{\sigma}_y^3$ are both measuring the state Ψ_4, because neither acts non-invasively. Secondly, it is wrong to apply an external time parameter T internally to the state. So, any outcome resulting from these should not be thought of as containing information about the 'historical development' of the state between $1 \leq T \leq 4$.

It is beneficial to rephrase this synopsis in the context of the quantum Universe. In the fully quantum reality proposed in this thesis, the above type of 'experiment' would ultimately have to be viewed on the emergent level as one group of factors of the Universe's state (representing a scientist) appearing to prepare, evolve and test another group of factors (the five qubit system), even though this perspective was not specified *per se*. Moreover, and as discussed in Chapter 5 and elsewhere, each of these groups of factors would be capable of experiencing their own passages of endo-time, relative to their own internal transitions and changes, as the Universe jumps from one state to the next.

Of course, in the current case the variable T was defined as the endo-time of the observer. The problem then arose because this time parameter T was taken to be absolute and universal, even though it is only relative, endo-times that can be given any actual physical significance. After all, recall that the endo-time of an observer is the exo-time of the observed, and vice versa. T cannot therefore also be taken to be the endo-time for the qubit system, and it is by incorrectly doing so that results in a misinterpretation of (6.45).

Overall, since the development of the state of the Universe is ultimately taken to be responsible for the development of the sub-states of everything in it (including the observer and the qubits), it must be assumed to change in a very special way if the observer appears to experience four distinct time steps between the preparation of the qubits and their measurement, whilst the qubit system itself appears to experience none. So, apart from the described comments and constraints on the role of time in this type of quantum computation, a general conclusion of the above discussion should therefore be that care is clearly needed when attempting to analyse the sort of endo-physical experiment presented here. It is this issue that is the focus of the next section.

6.2 Information Flow in the Quantum Universe

In many ways, computation may be described as the manipulation of *information*. After all, classical computer science generically involves the encoding of some sort of physical input into a series of 'symbols', the meaning of which is only valid relative to the information regarding what they actually represent. Furthermore, during an actual computation these symbols are processed to generate a final sequence, and the information contained in this can itself then be decoded to describe the properties of a physical output. Of course, it does not matter what form these symbols take: 0's and 1's, low-high voltages, squares and circles, offs or ons; what is important is the information content they represent. In fact, it is worth noting here that even a hard drive, the largest part of a modern personal computer, is specifically designed for the storage of data that is not related in any obvious way to the physical input it represents that was entered into the machine.

In other words, it may in some ways be imagined that in computational procedures the 'properties' of the physical input are directly translated into information, for example as a specific series of 0's and 1's, and then it is the information itself that is actually processed.

For a Universe running as an enormous quantum computation, the same ideas might be expected to be true. In fact, the central theme of Chapters 4 and 5 was to demonstrate how physical concepts such as classical identity and spatial location might be *encoded* as certain features of the quantum state representing the Universe, in this case in terms of separability and the ensuing relationships between the various factors. Indeed, the possibility of encoding space through informational methods should perhaps not be surprising; after all recall that space and positional relations may themselves be envisaged as a type of information storage process: a spatial separation between two objects is ultimately equivalent to the information that they are semi-classical, individual and not in contact.

So, in the sense proposed in the previous chapters, the separability of the state may be said to comprise a part of the information it intrinsically carries. Moreover, due to this relationship a change in separability from one jump to the next consequently implies a change in information. Thus, by defining the separability of the state as part of its information content, it is evident that some of this information can be used (i.e. 'decoded') to deduce particular physical properties, such as a quantum origin of space.

It might ultimately be expected, then, that any observed changes in physical systems involve changes in the information carried by the state. In fact, by reversing this argument due to the assumed 'primacy' of the quantum state, it might equally be expected that changes in the information contained in the state could result in physical changes occurring on the emergent level. Going further, since in the paradigm proposed in this thesis the

passage of time and the jump from the state Ψ_n to Ψ_{n+1} is parameterised in terms of the acquisition of information, a change in the information content of the state might be considered essential for any suggested dynamics of the Universe.

As a highly schematic example of this, it could be imagined that a particular change in the state's information content from one jump to the next could, somehow, eventually result in one collection of factors of the Universe's state (representing a human observer) being led to believe that another group of factors of the Universe's state (representing particle A) has reacted with a third group of factors (representing particle B). Although the details are left deliberately vague here, the point is that by a change in the information carried by the state of the Universe, perhaps involving a change in its separability, it might on the emergent level appear to the semi-classical observer that an experiment has been performed between particles A and B, or even perhaps that particles A and B collided.

The purpose of this section is to examine what it actually means to say that the information content of a state has changed. The issues will be examined as to what the necessary conditions for this are, what it implies, and how this relates to physics. The nature of physical experimentation and endo-physical measurement in the quantum Universe will then be explored.

6.2.1 Types of Transformation

Consider a procedure $\mathbf{\Pi}$ that relates two states Θ and Φ, both of which are contained in the Hilbert space $\mathcal{H}$. As will be explained below in a variety of contexts, the relationship here is defined such that the state Φ is the result of the procedure $\mathbf{\Pi}$ being performed on the state Θ; this procedure could perhaps involve a state reduction process, or even be some sort of mapping of the form $\mathbf{\Pi} : \Theta \rightarrow \Theta' = \Phi$. In fact, since information changing procedures are taken to provide the basis for dynamical development of the quantum Universe, in cases incorporating information change it might be possible to view Θ as Ψ_n and Φ as Ψ_{n+1}, as will be discussed later.

The issue of current interest is now to determine when a given procedure $\mathbf{\Pi}$ may be said to result in a change in the information carried by the state. In short, for what types of process $\mathbf{\Pi}$ is the outcome Φ noticeably and physically different from Θ?

Before explaining what is meant to say that information has changed in a quantum system, it is perhaps easier to first demonstrate what it means to say that it has not. Consider, for example, a null test on the state Θ, defined previously as an operator $\hat{N}$ with an eigenvector Φ where $\Phi \equiv \Theta$, such that $\hat{N}\Theta = \lambda\Theta$ with λ an eigenvalue. In this case, the procedure $\hat{N}$ gives an outcome Φ that is exactly the same as the initial state Θ, such that

this type of test leads to no overall change: the resulting state Φ is indistinguishable from the initial state Θ, and the action of the test is as if nothing has happened. Clearly, then, the development of a quantum system with these sorts of null tests is effectively trivial, because only differences can be physically observed. Consequently, and as discussed in Chapter 3, a quantum universe in the state Ψ_n may be developed any number of times by operators $\hat{\Sigma}$ that possess the eigenvector Ψ_n, but it is only when an operator $\hat{\Sigma}'$ is encountered which does not have this eigenvector that the universe jumps to a different state. Under such circumstances, it is now possible to label whichever eigenvector the universe happens to jump to as the new state, Ψ_{n+1}.

Of course, it is also possible to discuss *local* null tests. If Θ is separable in the form $\Theta \equiv \alpha \otimes \beta$, where $\alpha \in \mathcal{H}_\alpha$, $\beta \in \mathcal{H}_\beta$ and $\Theta \in \mathcal{H}_{[\alpha\beta]} \equiv \mathcal{H}$, then an operator $\hat{N}_\beta$ may be said to be local null test on β if it has eigenvectors of the form $\Phi = \gamma \otimes \beta$, where $\gamma \in \mathcal{H}_\alpha$. In this case, a development from Θ to Φ leaves the factor β invariant even though the sub-state in Hilbert space $\mathcal{H}_\alpha$ has changed.

As discussed in Chapter 5, local null tests play an important role in the dynamics of the quantum Universe, for example as the origin of a 'route-dependent endo-time' experienced by different factors of the developing state. Further, in this instance $\Theta \neq \Phi$ so it might be expected that the information carried by the states may have changed in some respect. This assertion is to be investigated.

In global null tests the information contained in the outcome state $\Phi(= \Theta)$ is obviously always the same as that carried by the initial state Θ, whereas in local null tests the information carried by the 'before' and 'after' states may be different. This type of comparison leads naturally to a discussion of 'passive' and 'active' transformations, both of which will be defined below. As will become evident, a global null test provides a trivial example of a passive transformation, whereas local null tests may conversely cause active transformations.

Passive and active transformations are defined [70] by the statement: "*An active transformation is one which actually changes the physical state of a system and makes sense even in the absence of a coordinate system, whereas a passive transformation is merely a change in the coordinate system of no physical significance.*" Furthermore, on the grounds that every transformation either changes the physical state of a system, or else it does not, the definition of passive transformations is revised in the following to include every transformation that is not active. Thus, passive transformations are taken to be effectively synonymous with 'non-active' transformations, and hence every transformation is assumed to be either passive or active.

So, after a passive transformation from one state of a system to another, no observable changes have occurred. Any apparent differences in 'before' and 'after' states are

merely superficial, and may only arise due to a change in the way the system is being described. Alternatively, since generally in physics, and especially in the endo-physical approach advocated in this thesis, only relative differences between the parts of a system are measurable, an example of a passive transformation is therefore one in which every part of the system is altered in exactly the same way.

Cases of such transformations include:

- The relabelling of the vacuum ground state in quantum field theory by the addition of a constant term to the energy eigenvalues. This effectively forms the basis of the renormalisation program, and is 'valid' (in some senses) because only relative differences are measurable in the laboratory;

- The addition of a constant term a to two numbers x and y under the subtraction operation. That is, if $x \rightarrow x' = x + a$ and $y \rightarrow y' = y + a$, then $x - y = x' - y' = (x + a) - (y + a)$;

- The rotation of an entire space through some angle, such that no changes occur in the relative positional relationships between any of the objects inside this space. Such a transformation is unobservable from the perspective of an observer contained entirely within the space;

- A change in the coordinate system of, say, a 3-dimensional space from Cartesian to cylindrical axes. Indeed, much of general relativity is based upon this type of invariance;

- A change in basis for a Hilbert space. For example, if $\mathcal{B}_a = \{|i\rangle_a : i = 0, 1, ..., d_a\}$ is an orthonormal basis for Hilbert space $\mathcal{H}_a$, and if $\mathcal{B}_b = \{|j\rangle_b : j = 0, 1, ..., d_b\}$ is an orthonormal basis for Hilbert space $\mathcal{H}_b$, then a state $\psi \in \mathcal{H}_{[ab]}$ given by $\psi = \sum_{x=0}^{d_a} \sum_{y=0}^{d_b} C_{xy} |x\rangle_a \otimes |y\rangle_b$ with complex coefficient matrix C_{xy} is invariant to any relabelling $\mathcal{B}_a \rightarrow \mathcal{B}'_a = \{|i'\rangle_a : i' = 0, 1, ..., d_a\}$ and $\mathcal{B}_b \rightarrow \mathcal{B}'_b = \{|j'\rangle_a : j' = 0, 1, ..., d_a\}$ of the individual subregister bases, where $|i'\rangle_a \neq |i\rangle_a$ and $|j'\rangle_b \neq |j\rangle_b$;

and so on. The point is that although a mathematical change may appear to have occurred during a passive transformation, there are no intrinsic physical consequences. In short, many passive transformations may be realised or removed simply by relabelling the 'axes'.

Conversely, an active transformation is one for which differences do become apparent when 'before' and 'after' states are compared. In other words, if Φ cannot simply be rewritten as a relabelled version of Θ, then it must be an active transformation that relates Θ to Φ.

In fact, in active transformations different parts of the system may actually change relative to each other. In this sense, then, it is active transformations that are seen to occur in real physics experiments, because in these situations the physicist notices that the state under investigation has changed relative to herself (which she often believes has not changed). A good example here is the measurement of an entangled EPR state and its subsequent collapse into a product of factors: physically these 'before' and 'after' states are completely different, and this fact is observable.

As discussed earlier, the separability of a state may be described as being part of the 'information' it contains. So, a change in separability of a state must therefore result in a change in this information. As a consequence of this, another example of an active transformation between Θ and Φ is one for which these two states lie in different partitions of the total Hilbert space $\mathcal{H}$. In this case, Θ and Φ must both be separable in different ways, and so by the above description must duly represent different information contents. As an illustration, it is evident that the EPR experiment mentioned above satisfies this condition.

Active and passive transformations may readily be seen in the context of the quantum Universe. For instance, an unitary operator $\hat{U}$ acting globally on every element of the Hilbert space $\mathcal{H}$ leaves all inner products between these elements invariant. That is, if $\Psi'_n = \hat{U}\Psi_n$ and $\Psi'_{n+1} = \hat{U}\Psi_{n+1}$, then $\langle\Psi_{n+1}|\Psi_n\rangle = \langle\Psi'_{n+1}|\Psi'_n\rangle$, and such a transformation could not physically be detected. On the other hand, because (by assumption) $\Psi_{n+1} \neq \Psi_n$ then the jump from state Ψ_n to the state Ψ_{n+1} must be regarded as a physically realisable, active transformation. Active and passive transformations consequently play different roles in dynamics.

However, given that in an endo-physical description of reality both the 'experiment' and the 'experimentee' are seen as different parts of the same quantum Universe, what is really of interest in such a picture are the relative changes occurring between the sub-states of this system. Further, it is evident that such real physical results, where one system is observed to change relative to another, must not be 'explainable away' simply by a passive relabelling of axes, or by a global transformation of every part of the Universe in the same manner. These changes are physically observed, and so because a passive transformation leads to no physical change in a system, the conclusion is then that passive transformations cannot be responsible for changes in the information contained in the state of the quantum Universe as it develops. On the contrary, because active transformations do result in a change of the state of a system, it is these that are conjectured to ultimately form the basis for information change and exchange, and the nature of endo-physical observation.

It is possible to further investigate what is meant by a passive change of information

in quantum mechanics by using the concept of local transformations. In fact, because by definition a transformation that is not passive must be active, a converse exploration is equally demonstrated.

As described above, a simple relabelling of the basis implies a passive transformation. In such relabellings there are no non-trivial differences in the state of the system before and after the relabelling, and hence no observable changes. The information contained by the state is therefore invariant to such a change. Furthermore, because of the asserted link between information content and separability, in passive transformations the partition containing the state is not expected to change: separable states remain separable, and entangled factors remain entangled, etc.

Reversing these lines of argument provides a definition for passive transformation. Consider a Hilbert space $\mathcal{H}_{[1...N]}$ factorisable into N subregisters of prime dimension, and further assume that the states Θ and Φ are contained in this space. Consider also a basis set $\mathcal{B}_A$ of vectors spanning $\mathcal{H}_{[1...N]}$, and assume that each member of $\mathcal{B}_A$ is completely separable relative to the N subregisters of $\mathcal{H}_{[1...N]}$. By defining

$$\mathcal{B}_{A,m} = \{|i_m\rangle_m : i_m = 0, 1, ..., (d_m - 1)\} \tag{6.54}$$

as an orthonormal basis for factor Hilbert space $\mathcal{H}_m$ of dimension d_m, where $m = 1, 2, ..., N$, the set $\mathcal{B}_A$ may be defined as

$$\mathcal{B}_A \equiv \{|i_1\rangle_1 \otimes |i_2\rangle_2 \otimes ... \otimes |i_N\rangle_N : i_m = 0, 1, ..., (d_m - 1); m = 1, 2, ..., N\}. \tag{6.55}$$

Of course, this construction is not unique. It is equally possible to define a different orthonormal basis set $\mathcal{B}_{B,m}$ for the factor Hilbert space $\mathcal{H}_m$ as $\mathcal{B}_{B,m} = \{|i'_m\rangle_m : i'_m = 0, 1, ..., (d_m - 1)\}$, and similarly define another completely separable basis set $\mathcal{B}_B$ of vectors spanning $\mathcal{H}_{[1...N]}$ as

$$\mathcal{B}_B \equiv \{|i'_1\rangle_1 \otimes |i'_2\rangle_2 \otimes ... \otimes |i'_N\rangle_N : i'_m = 0, 1, ..., (d_m - 1); m = 1, 2, ..., N\}. \tag{6.56}$$

Consider now a state Θ, defined relative to $\mathcal{B}_A$ as

$$\Theta = \sum_{i_1=0}^{(d_1-1)} \sum_{i_2=0}^{(d_2-1)} ... \sum_{i_N=0}^{(d_N-1)} C_{i_1 i_2 ... i_N} |i_1\rangle_1 \otimes |i_2\rangle_2 \otimes ... \otimes |i_N\rangle_N, \tag{6.57}$$

and a state Φ defined relative to $\mathcal{B}_B$ as

$$\Phi = \sum_{i'_1=0}^{(d_1-1)} \sum_{i'_2=0}^{(d_2-1)} ... \sum_{i'_N=0}^{(d_N-1)} C_{i'_1 i'_2 ... i'_N} |i'_1\rangle_1 \otimes |i'_2\rangle_2 \otimes ... \otimes |i'_N\rangle_N \tag{6.58}$$

where the $C_{i_1 i_2 ... i_N}$ and $C_{i'_1 i'_2 ... i'_N}$ give rise to complex coefficient matrices.

It is desirable to examine whether Θ and Φ are intrinsically different vectors, or whether Φ is instead just a 'relabelled' version of Θ in an alternative basis. To this end, if it is

possible to rotate (in a strictly mathematical sense) the N individual components $|i_m\rangle_m \in \mathcal{H}_m$ of Θ into the components $|i'_m\rangle_m$ to give a new state Θ', then it is evident that Θ and Φ are equivalent if $\Theta' = \Phi$.

Similarly, if the basis set $\mathcal{B}_{A,m}$ is transformed into the basis set $\mathcal{B}_{B,m}$ for each m, such that $\mathcal{B}_A$ is therefore transformed into $\mathcal{B}_B$, and if Θ is rewritten in this new basis and called Θ', then the states Θ and Φ are again mathematically equivalent if $\Theta' = \Phi$. In both cases Φ is just a different version of Θ, but expressed in an alternative basis. In such a circumstance it may then be said that Θ can be passively transformed into Φ.

This observation can be stated more precisely by noting that the individual rotations can be achieved by the use of unitary operators acting locally. If the unitary operator $\hat{U}_L$ is defined as a product of local unitary operators, $\hat{U}_L = \hat{u}_1 \otimes \hat{u}_2 \otimes ... \otimes \hat{u}_N$ where $\hat{u}_m$ is a unitary operator acting locally in factor Hilbert space $\mathcal{H}_m$, then the state Φ is just a relabelled version of Θ if

$$\Theta' \equiv \hat{U}_L\Theta = (\hat{u}_1 \otimes \hat{u}_2 \otimes ... \otimes \hat{u}_N)\Theta = \Phi. \tag{6.59}$$

Paraphrasing, Θ and Φ are equivalent if there exists a locally acting, unitary operator $\hat{U}_L$ relating them in this way.

As an aside, note that by definition any unitary transformation $\hat{U}$ acting locally in the individual factors of $\mathcal{H}_{[1...N]}$ can be written in the factorisable form $\hat{U} = \hat{u}_1 \otimes \hat{u}_2 \otimes ... \otimes \hat{u}_N$, where $\hat{u}_m$ is an unitary operator in $\mathcal{H}_m$ for $m = 1, 2, ..., N$. Equally, if it acts locally the operator $\hat{U}$ must necessarily be factorisable. Moreover, because it is acting locally on every subregister it must be completely or fundamentally factorised, i.e. factorisable into N factors.

The above results may be compared to global transformations, that is, those in which the *overall* basis sets $\mathcal{B}_A$ and $\mathcal{B}_B$ are rotated instead of their *individual* subregister basis sets $\mathcal{B}_{A,m}$ and $\mathcal{B}_{B,m}$, or equivalently those for which the overall state Θ is transformed 'at once' instead of its components in $\mathcal{H}_m$ being transformed individually. It is always mathematically possible to find a unitary operator $\hat{U}_G$ that transforms a given state Θ globally into any other state Φ, such that $\Phi = \hat{U}_G\Theta$. However, such a transformation may not go unnoticed: for example, Θ could be completely separable whilst Φ could be completely entangled, and these such states are fundamentally and physically different. Physically, separations and entanglements are completely different entities.

The same is not true for states that can be transformed locally into each other. If a state Θ can be locally transformed into the state Φ, then not only are these two states mathematically equivalent but they are also physically indistinct. In this case, the physical information contained in the state has remained unaffected by the rotation because

such states can be interchanged simply by a suitable and trivial relabelling of the basis. Paraphrasing, because a local unitary transformation just leads to a rotation or relabelling of the basis, it is unobservable in physics, and it may hence be described as passive. Conversely, recall that it is the active transformations, i.e. the processes that cause relative changes, that are of interest to endo-physics.

As an example of these ideas, consider a four dimensional Hilbert space $\mathcal{H}_{[12]}$ factorisable as $\mathcal{H}_{[12]} = \mathcal{H}_1 \otimes \mathcal{H}_2$, and spanned by the orthonormal basis $\mathcal{B}_{12} = \{|i\rangle_1 \otimes |j\rangle_2 : i, j = 0, 1\}$. Consider also the separable states $\mu = |0\rangle_1 \otimes |1\rangle_2$ and $\chi = (a|0\rangle_1 + b|1\rangle_1) \otimes (c|0\rangle_2 + d|1\rangle_2)$ for $a, b, c, d \in \mathbb{C}$, as well as the entangled state $\omega = |0\rangle_1 \otimes |0\rangle_2 + |1\rangle_1 \otimes |1\rangle_2$. It is always possible to find unitary transformations $\hat{U}_G$ and $\hat{U}'_G$ that act globally on one of the states μ, χ or ω, and not locally on its individual components, that transform it into one of the others. That is, it is always possible to find global, unitary operators of the form $\hat{U}_G$ and $\hat{U}'_G$ that satisfy $\mu = \hat{U}_G\chi = \hat{U}'_G\omega$.

It is also possible to find a (factorisable) local unitary transformation $\hat{U}''_L = \hat{u}_1 \otimes \hat{u}_2$ that relates μ to χ, where $\hat{u}_1$ and $\hat{u}_2$ are unitary operators acting locally on the components of μ and χ in the subregister Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. That is, there exists an operator $\hat{U}''_L$ such that

$$\begin{aligned} \hat{U}''_L\mu &= [\hat{u}_1 \otimes \hat{u}_2](|0\rangle_1 \otimes |1\rangle_2) = [\hat{u}_1|0\rangle_1] \otimes [\hat{u}_2|1\rangle_2] \\ &= (a|0\rangle_1 + b|1\rangle_1) \otimes (c|0\rangle_2 + d|1\rangle_2) = \chi. \end{aligned} \tag{6.60}$$

However it is not possible to find a factorisable, unitary operator of the form $\hat{U}'''_L = \hat{v}_1 \otimes \hat{v}_2$ that transforms ω to μ, where $\hat{v}_1$ and $\hat{v}_2$ are local unitary operators in $\mathcal{H}_1$ and $\mathcal{H}_2$. That is, there is no unitary operator $\hat{U}'''_L$ acting locally such that $\mu = \hat{U}'''_L\omega = (\hat{v}_1 \otimes \hat{v}_2)\omega$. Equally, the separable state χ cannot be transformed into the entangled state ω by unitary operators acting locally in the individual subregisters.

It seems that the outcome of a unitary operator acting locally on a state is in the same partition as the original. This observation leads to the following theorem.

Theorem 6.1 *Separations and entanglements are preserved by unitary transformations acting locally in the individual subregisters.*

Proof. The theorem is proved first for the separations, and this is then used for the entanglements.

- **Separations:**

Consider a Hilbert space $\mathcal{H}_{[1...N]}$ factorisable in N subregisters $\mathcal{H}_m$ of prime dimension. Consider also a unitary operator $\hat{U}_{1...N}$ acting locally in the individual subregisters $\mathcal{H}_m$. Then, by definition, $\hat{U}_{1...N}$ must be completely factorisable, such that $\hat{U}_{1...N} \equiv \hat{u}_1 \otimes \hat{u}_2 \otimes ... \otimes \hat{u}_N$.

Consider further a completely separable state $\varphi_{1...N} \in \mathcal{H}_{1...N} \subset \mathcal{H}_{[1...N]}$ written as $\varphi_{1...N} \equiv \varphi_1 \otimes \varphi_2 \otimes ... \otimes \varphi_N$ where $\varphi_m \in \mathcal{H}_m$. The evolution of $\varphi_{1...N}$ by $\hat{U}_{1...N}$ is given by

$$\begin{aligned} \hat{U}_{1...N}\varphi_{1...N} &= (\hat{u}_1 \otimes \hat{u}_2 \otimes ... \otimes \hat{u}_N)(\varphi_1 \otimes \varphi_2 \otimes ... \otimes \varphi_N) \\ &= (\hat{u}_1\varphi_1) \otimes (\hat{u}_2\varphi_2) \otimes ... \otimes (\hat{u}_N\varphi_N) \\ &= \varphi'_1 \otimes \varphi'_2 \otimes ... \otimes \varphi'_N \end{aligned} \tag{6.61}$$

where $\varphi'_m \equiv \hat{u}_m\varphi_m \in \mathcal{H}_m$. Clearly the outcome $\varphi'_1 \otimes \varphi'_2 \otimes ... \otimes \varphi'_N$ is also a member of the separation $\mathcal{H}_{1...N}$, as required for the theorem.

- **Entanglements:**

The proof here is by contradiction. Consider now a completely entangled state $\psi^{1...N} \in \mathcal{H}^{1...N} \subset \mathcal{H}_{[1...N]}$, and consider a hypothetical unitary operator $\hat{U}_{1...N}$ that, when acting locally upon $\psi^{1...N}$, evolves it into a state ψ' that is not in the partition $\mathcal{H}^{1...N}$. Because $\psi' \notin \mathcal{H}^{1...N}$ it follows that ψ' must be separable into at least two factors, and so must be a member of the complement set $\psi' \in (\mathcal{H}_{[1...N]} - \mathcal{H}^{1...N})$.

In this case, it would be possible to write

$$\hat{U}_{1...N}\psi^{1...N} = \psi' = \alpha \otimes \beta \tag{6.62}$$

with $\alpha \in \mathcal{H}_\alpha \equiv \mathcal{H}_{[1...M]}$ and $\beta \in \mathcal{H}_\beta \equiv \mathcal{H}_{[(M+1)...N]}$, where for simplification a suitable relabelling of the subregisters has been performed to adopt an ascending order, and it is noted that α and β may themselves be separable further relative to their factor Hilbert spaces.

Now, because $\hat{U}_{1...N}$ is unitary it has a unique inverse denoted by $(\hat{U}_{1...N})^{-1}$, such that $(\hat{U}_{1...N})^{-1}(\hat{U}_{1...N}) = \hat{I}_{[1...N]}$ where $\hat{I}_{[1...N]}$ is the identity operator in $\mathcal{H}_{[1...N]}$. To discover the form of $(\hat{U}_{1...N})^{-1}$, observe that if it may be written as a completely factorisable product

$$(\hat{U}_{1...N})^{-1} = \hat{u}_1^{-1} \otimes \hat{u}_2^{-1} \otimes ... \otimes \hat{u}_N^{-1} \tag{6.63}$$

then

$$\begin{aligned} (\hat{U}_{1...N})^{-1}(\hat{U}_{1...N}) &= \left(\hat{u}_1^{-1} \otimes \hat{u}_2^{-1} \otimes ... \otimes \hat{u}_N^{-1}\right)(\hat{u}_1 \otimes \hat{u}_2 \otimes ... \otimes \hat{u}_N) \\ &= \left(\hat{u}_1^{-1}\hat{u}_1\right) \otimes \left(\hat{u}_2^{-1}\hat{u}_2\right) \otimes ... \otimes \left(\hat{u}_N^{-1}\hat{u}_N\right) \\ &= \left(\hat{I}_1\right) \otimes \left(\hat{I}_2\right) \otimes ... \otimes \left(\hat{I}_N\right) = \hat{I}_{1...N} \end{aligned} \tag{6.64}$$

where $\hat{I}_m$ is the identity operator for $\mathcal{H}_m$. Moreover, because by definition there can only be one unitary operator $(\hat{U}_{1\ldots N})^{-1}$ satisfying $(\hat{U}_{1\ldots N})^{-1}(\hat{U}_{1\ldots N}) = \hat{I}_{[1\ldots N]}$ (i.e. $\hat{U}_{1\ldots N}$ can only have one inverse), the inverse of $\hat{U}_{1\ldots N}$ must be this completely factorisable operator.

Because $(\hat{U}_{1\ldots N})^{-1}$ is completely factorisable, it must also factorise in the form

$$(\hat{U}_{1\ldots N})^{-1} = (\hat{U}_{1\ldots M})^{-1} \otimes (\hat{U}_{(M+1)\ldots N})^{-1} = (\hat{U}_\alpha)^{-1} \otimes (\hat{U}_\beta)^{-1}, \tag{6.65}$$

with $\hat{U}_\alpha$ and $\hat{U}_\beta$ acting locally in Hilbert spaces $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$ respectively. So, applying $(\hat{U}_{1\ldots N})^{-1}$ to ψ' gives

$$\begin{aligned} (\hat{U}_{1\ldots N})^{-1}\psi' &= \left((\hat{U}_\alpha)^{-1} \otimes (\hat{U}_\beta)^{-1}\right)(\alpha \otimes \beta) \\ &= \left((\hat{U}_\alpha)^{-1}\alpha\right) \otimes \left((\hat{U}_\beta)^{-1}\beta\right). \end{aligned} \tag{6.66}$$

However, $(\hat{U}_\alpha)^{-1}$ and $(\hat{U}_\beta)^{-1}$ are themselves just unitary operators, so $(\hat{U}_\alpha)^{-1}\alpha = \alpha'$ and $(\hat{U}_\beta)^{-1}\beta = \beta'$, where $\alpha' \in \mathcal{H}_\alpha \equiv \mathcal{H}_{[1\ldots M]}$ and $\beta' \in \mathcal{H}_\beta \equiv \mathcal{H}_{[(M+1)\ldots N]}$. So

$$(\hat{U}_{1\ldots N})^{-1}\psi' = \alpha' \otimes \beta' \tag{6.67}$$

but note that also

$$(\hat{U}_{1\ldots N})^{-1}\psi' = (\hat{U}_{1\ldots N})^{-1}[\hat{U}_{1\ldots N}\psi^{1\ldots N}] = \psi^{1\ldots N} \tag{6.68}$$

such that equating leads to

$$\psi^{1\ldots N} = \alpha' \otimes \beta' \tag{6.69}$$

which is clearly a contradiction. Thus, the equation $\hat{U}_{1\ldots N}\psi^{1\ldots N} = \psi' = \alpha \otimes \beta$ cannot be true, so the conclusion must be that local unitary operators $\hat{U}_{1\ldots N}$ preserve entanglements: $\hat{U}_{1\ldots N}\psi^{1\ldots N} = \psi''$, where $\psi'' \in \mathcal{H}^{1\ldots N}$.

Note that the proof is readily extended to states that are a separable product of entangled sub-states, because the above argument is equally true for each of the individual factors. ■

Thus, unitary operators acting locally on the components of a state in its fundamental subregisters do not affect its partition. Conversely, partition changing processes may not be accomplished by local unitary operators, and so cannot be 'removed' by a simple relabelling of the basis.

It is consequently now possible to specify what is meant by information change, as discussed in the next sub-section.

6.2.2 Information Change and Exchange

Attention is now turned to transformations between the states of the quantum Universe. For what potential transitions from Ψ_n to a next possible state is there an inherent and intrinsic change of information? Further, in the context of the 'measurement problem', for what transitions from Ψ_n to Ψ_{n+1} is there an actual exchange of information between different parts of the state?

From the work of the previous sub-section, it is now possible to argue that the transition from Θ and Φ implies a change of information if these two states are fundamentally different. The question then becomes: when might a difference be described as fundamental, to which an answer may rely on whether or not it can be 'transformed away' by a simple relabelling of the basis. Specifically, because a basis relabelling may be accomplished by the use of local unitary operators, it is evident that Θ and Φ are indistinct if Θ may be locally transformed into Φ.

Going further, it has been demonstrated that unitary operators acting locally on a state do not affect its partition. Thus, if Θ and Φ are in different partitions it is clear that Θ cannot be transformed into Φ by the use of local unitary operators. Moreover, this implies that Θ may not then be rewritten as Φ simply by a passive relabelling of the basis. Consequently, such a partition changing transition must imply a change in the information carried by the state, and this point is reinforced by recalling that because a state's separability is related to its information content, any change in partition necessarily results in a change of information.

Summarising, partition changing processes necessarily imply an information change, and hence represent active transformations. Conversely, it might automatically be expected that all partition preserving transformations are accordingly passive.

Care is needed, however, when applying this rule to the situation where the state is not just an arbitrary vector in an abstract mathematical space, but is instead representing an actual physical system. The following examples demonstrate this point.

For instance, does the change from the state Θ defined as $\Theta \equiv |0\rangle_1 \otimes |1\rangle_2 - |1\rangle_1 \otimes |0\rangle_2$ to the state $\Phi \equiv |0\rangle_1 \otimes |1\rangle_2 + |1\rangle_1 \otimes |0\rangle_2$ constitute a change in information? It can be achieved by the local unitary operator $\hat{\sigma}_1^3 \otimes \hat{I}_2$, such that $\Phi = [\hat{\sigma}_1^3 \otimes \hat{I}_2]\Theta$ where $\hat{\sigma}_1^3$ is a Pauli operator in $\mathcal{H}_1$ and $\hat{I}_2$ is the identity in $\mathcal{H}_2$, and so may be thought of simply as a rotation of the state of qubit 1, or alternatively as a relabelling of the $|1\rangle_1$ basis of $\mathcal{H}_2$ to $-|1\rangle_1$. Clearly, Θ and Φ are in the same partition here, but in this circumstance are the 'before' and 'after' entangled states fundamentally the same?

Indeed, could such an apparently passive transformation ever be physically allowed in nature? Say, for example, that $\mathcal{H}_1$ represents the Hilbert space of an electron and $\mathcal{H}_2$ the

Hilbert space of a positron, with $|0\rangle$ representing a spin down state and $|1\rangle$ a spin up (in some direction). On the subregister level the two states Θ and Φ are related by a local unitary change of basis, but if the wavefunction is instead ascribed to represent entangled up and down electrons and positrons the result of the above sort of change would be an anti-symmetric state becoming a symmetric one, and these are physically different. Is this then meant to imply that transformations that are defined as mathematically passive could potentially lead to observable physical consequences?

As a second example of this type of problem, consider a state $\Psi_n \in \mathcal{H}_{[12R]}$ of the universe prepared such that

$$\Psi_n = [|0\rangle_1 \otimes |0\rangle_2 - |1\rangle_1 \otimes |1\rangle_2] \otimes |R\rangle_R \tag{6.70}$$

where $\mathcal{H}_1$ and $\mathcal{H}_2$ are qubit subregisters spanned by orthonormal bases $\{|0\rangle_1, |1\rangle_1\}$ and $\{|0\rangle_2, |1\rangle_2\}$, and R represents the rest of the universe in Hilbert sub-space $\mathcal{H}_R$, presumably itself of enormous dimension and a product of very many factors. Obviously, qubits 1 and 2 are in a correlated state here, so an arbitrary measurement of qubit 1 with the result $|i\rangle_1$, for $i = 0, 1$, projects qubit 2 into the same state $|i\rangle_2$, and vice versa.

Assume now that the next state Ψ_{n+1} of the universe turns out to be

$$\Psi_{n+1} = |0\rangle_1 \otimes |r\rangle_{2R} \tag{6.71}$$

where $|r\rangle_{2R}$ is some element of $\mathcal{H}_{[2R]}$. On the emergent level, and as discussed later in Section 6.2.3, the jump from Ψ_n to Ψ_{n+1} might correspond to an apparent endo-physical measurement of qubit 2 by an 'apparatus' contained in $|R\rangle_R$, with the result that it must have been found in a down state, $|0\rangle_2$, because qubit 1 has been left in the state $|0\rangle_1$. So, in this case it would be expected that if qubit 1 is measured next it will also be found to be spin down, $|0\rangle_1$.

However, if local transformations are always unobservable, it is possible to find a local unitary operator $\hat{u}_1$ that results in an effective relabelling of the basis of $\mathcal{H}_1$ as $|0\rangle_1 \to |1\rangle_1$ and $|1\rangle_1 \to |0\rangle_1$, i.e. $\hat{u}_1\{|0\rangle_1, |1\rangle_1\} = \{|1\rangle_1, |0\rangle_1\}$. In other words, applying this rotation to Ψ_{n+1} gives Ψ'_{n+1}, where

$$\Psi_{n+1} \to \Psi'_{n+1} = [\hat{u}_1 \otimes \hat{I}_{2R}]\Psi_{n+1} = [\hat{u}_1 \otimes \hat{I}_{2R}](|0\rangle_1 \otimes |r\rangle_{2R}) = |1\rangle_1 \otimes |r\rangle_{2R}. \tag{6.72}$$

and $\hat{I}_{2R}$ is the identity in $\mathcal{H}_{[2R]}$. Now when it is observed by emergent apparatus in $\mathcal{H}_{[2R]}$, qubit 1 will be found in the up state $|1\rangle_1$. This appears to violate what would be expected from an initially correlated state, such that an apparent passive transformation has again led to an observed physical discrepancy.

In fact, this type of example can be given even greater significance. Consider a state $\Psi_n = |\nu_L\rangle_\nu \otimes |R'\rangle_{R'}$, where $|\nu_L\rangle_\nu \in \mathcal{H}_\nu$ represents a (left-handed) neutrino, $|R'\rangle_{R'} \in \mathcal{H}_{R'}$

represents the rest of the Universe, and $\Psi_n \in \mathcal{H}_{[\nu R']}$. In this case it might appear possible to passively transform the sub-state of Ψ_n in $\mathcal{H}_\nu$ with a local unitary operator $\hat{u}_\nu$ such that

$$\Psi_n \rightarrow \Psi'_n = [\hat{u}_\nu \otimes \hat{I}_{R'}]\Psi_n = [\hat{u}_\nu \otimes \hat{I}_{R'}]|\nu_L\rangle_\nu \otimes |R'\rangle_{R'} = |\nu_R\rangle_\nu \otimes |R'\rangle_{R'} \tag{6.73}$$

where $|\nu_R\rangle_\nu$ represents the state of a right-handed neutrino, with $\hat{I}_{R'}$ the identity in $\mathcal{H}_{R'}$. However, right-handed neutrinos are thought not to exist in nature[11], so the local, passive, non-partition changing and unobservable unitary transformation of the type $|\nu_L\rangle \rightarrow |\nu_R\rangle$ has ultimately lead to unphysical consequences.

So, how is it that apparently passive, local relabellings of basis can be reconciled with such drastic resultant changes in the physical properties of the quantum system?

Firstly, one suggestion might be perhaps that the Universe forbids certain unitary processes. It may not be valid, for example, to rotate just some basis sets and not others; perhaps the bases of qubits 1 or 2 in the second example may not be relabelled without also relabelling the basis of R in a similar way. In fact, as discussed in previous chapters, the suggestion here would be that this is potentially another case of the Universe being highly selective when deciding which operators it chooses to develop itself with. This conclusion, however, is not a sufficient argument. Ignoring the fact that changing every basis set in the same way is really just equivalent to a global transformation, it must also be noted that the local unitary transformations discussed above are seen as mathematical relabellings and not necessarily as direct physical evolutions of the system. In other words, although it might be possible to impose the constraint that, say, $|\nu_L\rangle$ cannot be *evolved* into $|\nu_R\rangle$ (e.g. by any sort of Schrödinger dynamics) it may not equally be imposed that $|\nu_L\rangle$ cannot be *relabelled* as $|\nu_R\rangle$.

A second potential argument could then be to conject that physics on the subregister level might obey different or additional constraints from that in the emergent limit of real particles, for example electrons or neutrinos. This solution fails, however, because all physical characteristics are expected to emerge from the proposed pregeometric description, and not the other way around, and so any considerations or invariances on the pregeometric level might also be expected to exist at its large scale limit. At the very least, if this argument was correct a good reason would be required to explain why such pregeometric equivalencies disappear on emergent scales.

Alternatively, it might perhaps be argued that although local unitary operations lead to mathematical invariances, the physical characteristics exhibited by a particular (sub-) state, for example its asymmetry or left-handedness, are actually bestowed upon it by external influences. A neutrino may be left handed, for example, only relative to a frame

[11] Or at least, no right handed neutrinos have ever been observed.

of reference defined on the emergent scale by R, and as such this type of description may be meaningless on the pregeometric, sub-register level. Whilst this 'lack of properties' explanation may be very much in the spirit of quantum theory, it still misses the essential point that relative to these external frames an observable change in the state does actually seem to be caused by an apparent, passive (local) transformation.

So, a better conclusion from the observation that the above types of passive relabellings are not seen in nature might then be to suggest that the Universe selects its own preferred basis, such that all ups and downs and rights and lefts are defined relative to this. In this case, local rotations would represent a physical change in the information carried by the state, because the new, transformed state could be compared to this absolute basis. Thus, such rotations would be observable, and are hence inequivalent to simple passive relabellings.

Further, in fact, a more endo-physical suggestion might be that the preferred basis need not actually be defined by the entire universe, but by an internal endo-observer. In other words, a preferred basis for a sub-state under investigation may emerge by considering the basis of the sub-state(s) representing the observer.

Consider, for instance, the second example of above. The initial measurement of qubit 2 by the apparatus may be seen as defining a preferred up-down 'axis', because in order to find qubit 2 to be either up or down it is necessary to specify what these 'directions' are relative to. Moreover, in order to initially know that qubits 1 and 2 have been prepared in the correlated way of above, their bases must also be correlated. Thus, the 'fixing' of the basis for qubit 2 also necessarily fixes qubit 1's basis, such that any subsequent investigation of qubit 1 can only be valid relative to the preferred 'direction' defined by the measurement of qubit 2. So, when qubit 2 is found to be 'down', an up-down axis is automatically defined, and it is this axis that must then be used for qubit 1 if a consistent description of the system is to be used.

Of course, any other basis could have been chosen before the measurement of qubit 2. However, once a 'preferred axis' has been chosen it must remain fixed, and must also be used for qubit 1 if previously correlated sub-states are to be compared.

A similar argument applies for the case of the neutrinos. In order to contest that a neutrino state has been prepared as left-handed, a preferred left-right basis must first be agreed upon.

Summarising, then, by finding qubit 2 to be down, i.e. $|0\rangle_2$, relative to the basis $\{|0\rangle_2, |1\rangle_2\}$, the corresponding basis $\{|0\rangle_1, |1\rangle_1\}$ immediately becomes 'preferred' for qubit 1. Any subsequent rotation of qubit 1 (say from $|0\rangle_1$ to $|1\rangle_1$) now changes its state relative to this chosen basis, and as such would imply an active transformation. It is not surprising, then, that this active transformation leads to a difference in the information attributed

to the state. In fact, taking this point to absurdity for the sake of clarity, once qubit 2 has been found to be 'down' in some basis, it could obviously not itself then be passively relabelled as 'up' relative to the same axis.

Note that none of this is saying that $|0\rangle_1$ is the only possible outcome of a $measurement$ on qubit 1. What it does mean, however, is that prior to such a measurement the initial state of qubit 1 must be $|0\rangle_1$, and not an arbitrarily relabelled version.

Overall, then, it is the state collapse processes, i.e. the measurements, that cause information to be 'extracted' from the system and hence a preferred basis to be defined. Before a measurement it is possible to arbitrarily relabel 0's and 1's (i.e. $\Psi_n \rightarrow \Psi'_n$), but after information has been extracted and the state has irreversibly changed it is then too late to consider further changes of the basis. Paraphrasing, after a measurement the 'direction' of the basis becomes fixed, so any additional rotation would be a transformation relative to this chosen direction. Moreover, because comparisons are now possible, this sort of rotation is no longer un-observable, and so any such 'relabelling' of the basis sets becomes an active process.

Of course, the differences caused by rotating a sub-state relative to the preferred basis of the observer could themselves be passively removed by transforming this basis in the same way as the sub-state. However, this then becomes effectively equivalent to a global transformation, and is therefore not relevant to the present discussion.

Thus, it is the state collapse mechanism that prohibits passive transformations from leading to observable physical consequences, and conversely prevents actual changes in individual factors from being passively 'transformed away'. Moreover, this conclusion again highlights the fact that it is the state reduction postulate of quantum mechanics that introduces non-trivial dynamics into a system, and thereby parameterises physical changes in terms of information acquisition.

From the above discussions, it is evident that real, physical information change is a concept that is meaningful relative to the comparison of states against the same basis. This is perhaps not too surprising: after all, the ability to physically compare objects is a fundamental prerequisite to any discussion involving change. Moreover, in a fully quantum Universe this basis is defined by an internal, endo-physical 'observer' engaged in a process of apparent, emergent measurements. In fact, these two points represent the very essence of the measurement problem: in quantum theory a state under investigation changes when it interacts with an observer. It is important, then, to specify what it really means to talk of an endo-physical measurement.

Of course, from an exo-physical perspective it is always valid to discuss the measurement of an isolated state by an external observer. From the endo-physical perspective of

a fully quantum Universe, however, what is generally of issue is how the entire state of the Universe changes in such a way that it appears (on emergent scales) like one of its sub-states has measured a second. In other words, what is more of interest in an endo-physical discussion of measurement is not whether information has *changed* during a transition from Ψ_n to Ψ_{n+1}, but whether information has been *exchanged.* Paraphrasing, of more concern than information change, where separate factors may be taken to change individually and independently of one other, is information exchange, in which the *relationship* between different parts of the state is altered.

It is possible to formally define what is meant by a change in information, and what is meant by an exchange of information. As is evident from earlier, a change in information occurs during a transition between Θ and Φ if these two states are fundamentally different. Moreover, this difference must not just simply be mathematical, but must also take account of certain physical constraints, such as prior measurements giving rise to preferred bases.

However, within this set of information changing transitions are the information exchanging processes, defined below as those procedures in which parts of the state appear to interact with each other. Specifically, if Θ and Φ are both contained in the Hilbert space $\mathcal{H}_{[1\ldots N]}$, then during a transition from Θ to Φ the component of the state in subregister $\mathcal{H}_i$ may be defined as having exchanged information with the component of the state in $\mathcal{H}_j$ if the 'relationship' between these two components has changed. As an example, if these two components were perhaps separate in Θ but are entangled with one another in Φ, it is evident that the relationship between them has been altered. Note, then, that an exchange of information necessarily implies a change in information, but a change in information may not necessarily have to imply an exchange.

What is ideally sought, therefore, is a test of whether two particular sub-states appear to have exchanged information with each other during a particular transition. In other words, does the relationship between the components of the state in factor Hilbert spaces $\mathcal{H}_i$ and $\mathcal{H}_j$ change during the jump from Ψ_n to Ψ_{n+1}? Thus, do the components appear to 'interact' with each other in any way? Indeed, could it ultimately be possible to consider the component in $\mathcal{H}_i$ as 'measuring' the component in $\mathcal{H}_j$ during the transition, at least in an emergent sense?

To begin to answer these questions, recall that the dynamics of the quantum Universe relies on the principle that, given an Hermitian operator $\hat{\Sigma}_{n+1}$, whichever eigenvector is selected automatically becomes the initial state for the subsequent transition involving $\hat{\Sigma}_{n+2}$. That is, the development of the Universe is viewed as a giant and autonomous process of quantum testing and retesting. Moreover, because it is assumed that the actual outcome Ψ_{n+1} of a test is necessarily different from the previous state Ψ_n, the operator

$\hat{\Sigma}_{n+1}$ used in the Universe's development must induce a change in information, and so must represent an active transformation. Further, because of the lack of an external agent deciding upon a preferred basis for the quantum Universe, any change of information within the system cannot arise simply from a relabelling of the 'axes'. This is another reason why the transition from Ψ_n to Ψ_{n+1} cannot therefore be a consequence of any sort of passive transformation.

The previous sub-section demonstrated that unitary operators acting locally in the individual subregisters have unobservable consequences, and that these transformations preserve the separation and entanglement properties of the state. The opposite is also considered: any change in partition of the state of the Universe must imply an active transformation because it cannot be removed by a local relabelling of the basis. Consequently, a change of partition results in a change in the information carried by the state. Of course, this last point might be expected immediately: given that the information content of a state has already been related to the ways its separates, any change in partition must automatically imply a change in information.

This can be presented formally. Consider two successive states Ψ_n and Ψ_{n+1} in the Hilbert space $\mathcal{H}_{[1\ldots N]}$. The state Ψ_n will have F_n factors and lies in the partition $\mathcal{P}_n$, which has F_n blocks, where $1 \leq F_n \leq N$. Similarly the state Ψ_{n+1} will have F_{n+1} factors and lies in the partition $\mathcal{P}_{n+1}$ with F_{n+1} blocks, noting that F_{n+1} is not necessarily equal to F_n, and that even if $F_n = F_{n+1}$ the partition $\mathcal{P}_{n+1}$ is not necessarily the same as $\mathcal{P}_n$ (e.g. $\mathcal{H}_1^{234} \not\equiv \mathcal{H}^{12\bullet 34}$, but both have two blocks).

From the conclusions of the earlier discussions, it is now possible to conject that:

- Information has been exchanged during the transition from Ψ_n to Ψ_{n+1} if $\mathcal{P}_n \neq \mathcal{P}_{n+1}$.

The converse is also true: an exchange of information implies a change in partition. The above conjecture follows from the very definition of a partition; if the state Ψ_n is in a different partition from the state Ψ_{n+1}, it means that at least two components of the state have changed their block during the transition. Moreover, it must then be the case that at least one of the components of Ψ_n, in a particular subregister $\mathcal{H}_i$, must have changed its relationship (i.e. its entanglement) with at least one other component, in a different subregister $\mathcal{H}_j$, when the state became Ψ_{n+1}.

To rephrase this conjecture, consider the probability amplitude $P = \langle \Psi_{n+1} | \Psi_n \rangle$. As shown in Section 5.4 this probability amplitude will have F_P factors, where $F_P \leq \min(F_n, F_{n+1})$. For example, if Ψ_n has two factors whilst Ψ_{n+1} is fully entangled, then $F_n = 2$ and $F_{n+1} = 1$, such that $F_P = 1$.

Now, by the above argument, information exchange has occurred in the transition from Ψ_n to Ψ_{n+1} if

$$F_P < F_n \quad \text{or} \quad F_P < F_{n+1}. \tag{6.74}$$

In other words, if the number of factors of the probability amplitude is less than the number of factors of either the initial or final states, then Ψ_n and Ψ_{n+1} are in different partitions, and the transition is an information exchanging process.

Of course, this condition is immediately satisfied if $F_n \neq F_{n+1}$.

To go further, consider the general form of P, fundamentally factorised as

$$P = \langle \Psi_{n+1}^{(1)} | \Psi_n^{(1)} \rangle \langle \Psi_{n+1}^{(2)} | \Psi_n^{(2)} \rangle ... \langle \Psi_{n+1}^{(F_P)} | \Psi_n^{(F_P)} \rangle \tag{6.75}$$

where $\Psi_n^{(p)}$ and $\Psi_{n+1}^{(p)}$ may themselves be products of $k_n^{(p)}$ and $k_{n+1}^{(p)}$ factors respectively, and $p = 1, 2, ..., F_P$ for $k_n^{(p)}, k_{n+1}^{(p)} \in \mathbb{Z}^+$ and $\sum_{p=1}^{F_p} k_n^{(p)} = F_n$, but $k_n^{(p)}$ is not necessarily equal to $k_{n+1}^{(p)}$.

Moreover each factor of $\Psi_n^{(p)}$ is in some block of $\mathcal{P}_n$, whereas each factor of $\Psi_{n+1}^{(p)}$ is in some block of $\mathcal{P}_{n+1}$, with the proviso that for P to factorise in the above way the two sub-states $\Psi_n^{(p)}$ and $\Psi_{n+1}^{(p)}$ must be contained in exactly the same set of subregisters, such that $\Psi_n^{(p)}, \Psi_{n+1}^{(p)} \in \mathcal{H}_{[p]}$ for $\prod_{p=1}^{F_P} (\otimes \mathcal{H}_{[p]}) = \mathcal{H}_{[1...N]}$.

From this, it is now possible to assert that during the jump from Ψ_n to Ψ_{n+1}:

- The component of the state in factor Hilbert space $\mathcal{H}_i$ exchanges information with the component in factor Hilbert space $\mathcal{H}_j$, for $i \neq j$, if the components of Ψ_n in $\mathcal{H}_i$ and $\mathcal{H}_j$ are in the same block B but the components of Ψ_{n+1} in $\mathcal{H}_i$ and $\mathcal{H}_j$ are not in B, *or* if the components of Ψ_{n+1} in $\mathcal{H}_i$ and $\mathcal{H}_j$ are in the same block B' but the components of Ψ_n in $\mathcal{H}_i$ and $\mathcal{H}_j$ are not in B'.

Clearly, this statement is equivalent to the condition that:

- The component of the state in factor Hilbert space $\mathcal{H}_i$ exchanges information with the component in factor Hilbert space $\mathcal{H}_j$, for $i \neq j$, if the components of Ψ_n in $\mathcal{H}_i$ and $\mathcal{H}_j$ are in the same block B of $\mathcal{P}_n$ but $\mathcal{P}_{n+1}$ does not possess this block, *or* if the components of Ψ_{n+1} in $\mathcal{H}_i$ and $\mathcal{H}_j$ are in the same block B' of $\mathcal{P}_{n+1}$ but $\mathcal{P}_n$ does not possess this block.

As an illustration, consider a three qubit universe in the Hilbert space $\mathcal{H}_{[123]}$ spanned by the orthonormal basis

$$\mathcal{B}_{123} = \{ |a\rangle_1 \otimes |b\rangle_2 \otimes |c\rangle_3 = |abc\rangle_{123} : a, b, c = 0, 1 \}. \tag{6.76}$$

If two consecutive states ψ_n and ψ_{n+1} turn out to be

$$\begin{aligned} \psi_n &= |000\rangle_{123} = |0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 \\ \psi_{n+1} &= |000\rangle_{123} + |011\rangle_{123} = |0\rangle_1 \otimes (|00\rangle_{23} + |11\rangle_{23}) \end{aligned} \tag{6.77}$$

it is evident that $\psi_n \in \mathcal{H}_{123}$ whilst $\psi_{n+1} \in \mathcal{H}_1^{23}$. Clearly, the component of ψ_n in factor Hilbert space $\mathcal{H}_1$ is in the same block as the component of ψ_{n+1} in this subregister, and so may not be said to exchange information with any of the other components during the transition from ψ_n to ψ_{n+1}. On the other hand, the components of ψ_n in factor Hilbert spaces $\mathcal{H}_2$ and $\mathcal{H}_3$ are in different blocks from the components of ψ_{n+1} in $\mathcal{H}_2$ and $\mathcal{H}_3$, and clearly the relationship between these components changes during the jump. In this case, it may be said that information is exchanged between qubits 2 and 3 during the transition from ψ_n to ψ_{n+1}.

6.2.3 The Ideal Physics Experiment

Discussions of information exchange during the development of a system lead naturally onto questions of the measurement problem in quantum mechanics and the nature of endo-physical experimentation. After all, the acquisition of information is the very purpose of measurement.

The concept of measurement is generally well understood in exo-physics, with a famous exception being the problem of state reduction in quantum mechanics and the corresponding conflicts of interpretation regarding what this actually means. In exo-physics, discussions often involve large, semi-classical observers surrounded by an even larger environment, who are observing the isolated and microscopic quantum state under investigation. Moreover, during this process the observers and environment are often assumed to be unaffected, or at least any changes in them are taken to be insignificant.

These perspectives, however, are incompatible with the notion of endo-physics, and are therefore also ultimately incompatible with the nature of a physical Universe incorporating quantum theory as the fundamental ingredient. As has been remarked previously, in a fully quantum treatment of the Universe the system under investigation, the observer and every element of the environment should really just be viewed as factors or groups of factors of the single state representing all of physical reality. An endo-physical experiment consequently involves one part of the Universe's state appearing to measure another part of it. Moreover, this also implies that the state itself must be developing in a highly organised way if emergent, internal scientists are to gain the illusion that they are independent, classical and isolated observers who can investigate and develop their surroundings with apparent free will.

The question of endo-physical measurement is an enormous subject, undoubtedly worthy of a research programme in its own right. It is therefore not attempted here to provide a complete and self-contained study of how this process actually works. After all, recall that there is currently no mechanism known for determining how and why the Universe selects which operators it uses to test itself with, so a theory of how emergent, internal physicists gain the impression that they can decide how the Universe around them develops must be even further away.

What can be described, however, are the essential points that this unknown theory should incorporate, or alternatively what conditions such a mechanism might be expected to satisfy. In other words, in the following the necessary features and constraints for endo-physical experimentation are discussed.

Consider the sort of experiment performed by physicists everyday. Such a situation necessarily incorporates at least two parts: there must be a 'subject' to be measured, and there must be some sort of 'device' to do the measuring[12]. Furthermore, before any measurement takes place these two parts must have some sort of independent existence, and so must be classically isolated from each other.

Now, in the context of the paradigm proposed in this thesis, the above conditions are achieved by recalling that in the quantum Universe every classically isolated physical system is associated with a factor of the Universe's state. Thus, in order for any physical experiment to occur it is required that the Universe's state must be separable into at least two factors, one of which is ultimately taken to represent the 'subject' and the other is taken to represent the 'device'.

Consider the Universe at 'time' n, represented by a state Ψ_n contained in a Hilbert space $\mathcal{H}_{[1...N]}$ factorisable into N subregisters $\mathcal{H}_m$, for $m = 1, 2, ...N$. Since all of physical reality is expected to emerge from this fundamental state description, it may be assumed that some 'portion' of Ψ_n accounts for the subject, some of it accounts for the device, and the rest accounts for everything else in the Universe.

Thus, without loss of generality assume that the factor of Ψ_n representing the subject is contained in subregisters 1 to x. That is, assume that it is the components of Ψ_n in $\mathcal{H}_{[1...x]}$ that are (somehow) responsible for the physical appearance of the subject on emergent scales. Similarly, assume that the device emerges from some sort of consideration of those components of Ψ_n contained in the subregisters $(x + 1)$ to y, where again the exact mechanism of this origination lies in the realm of an unknown theory of emergence.

[12]Of course, for the sake of the present argument it does not matter what either the 'device' or the 'subject' actually are. For example, in different contexts the generic word 'device' could be taken to imply a large piece of equipment, or a single 'pointer-state', or even the eye of a human observer; it is the fact that a physical experiment necessarily contains both a subject and a device that is important.

This leaves subregisters $(y+1)$ to N to account for the emergence of everything else in the Universe, noting that in the above the subregisters have been arbitrarily labelled in ascending order for clarity. Of course, generally in physics experiments the subjects are much smaller than the devices that measure them, which are in turn dwarfed by the scale of everything else in the Universe. It may therefore naturally be expected that $x \ll y \ll N$, but this is not essential.

The above division of $\mathcal{H}_{[1\ldots N]}$ into subject, device and everything else denotes a particular split of the Hilbert space. That is,

$$\mathcal{H}_{[1\ldots N]} = \mathcal{H}_{[1\ldots x]} \otimes \mathcal{H}_{[(x+1)\ldots y]} \otimes \mathcal{H}_{[(y+1)\ldots N]}. \tag{6.78}$$

The present discussion is concerned with the relationships occurring in a single, isolated endo-measurement between a 'subject' and a 'device'. It is therefore possible to further simplify the above situation by considering a 'toy-universe' containing nothing but these two features, that is, one for which $y = N$. Such a universe defines the alternative split

$$\mathcal{H}_{[1\ldots N]} = \mathcal{H}_{[1\ldots y]} = \mathcal{H}_{[1\ldots x]} \otimes \mathcal{H}_{[(x+1)\ldots y]} = \mathcal{H}_{[A]} \otimes \mathcal{H}_{[B]} \tag{6.79}$$

where the sub-scripts A and B are adopted for brevity to denote respective subject and device factor Hilbert spaces.

Now, in order to consider a classically distinct subject and device, argued as above to be essential pre-conditions to any discussion of experiments, the state ψ_n of the toy-universe must be separable relative to the split (6.79). Thus, if $|\phi_n\rangle$ represents the sub-state of the subject at time n, and $|\Lambda_n\rangle$ the sub-state of the device at time n, then clearly $|\phi_n\rangle \in \mathcal{H}_{[1\ldots x]} \equiv \mathcal{H}_{[A]}$ and $|\Lambda_n\rangle \in \mathcal{H}_{[(x+1)\ldots y]} \equiv \mathcal{H}_{[B]}$, such that

$$\psi_n = |\phi_n\rangle_{[1\ldots x]} \otimes |\Lambda_n\rangle_{[(x+1)\ldots y]}. \tag{6.80}$$

Clearly, ψ_n is in the separation given by

$$\psi_n \in \mathcal{H}_{[1\ldots x]\bullet[(x+1)\ldots y]} = \mathcal{H}_{AB}. \tag{6.81}$$

It is important that no mention has been made so far as to the actual nature of either the 'subject' or 'device'. Indeed, $|\phi_n\rangle_{[1\ldots x]}$ could itself be separable, entangled, or a separable product of entangled factors relative to its factor Hilbert space $\mathcal{H}_{[1\ldots x]}$, and $|\Lambda_n\rangle_{[(x+1)\ldots y]}$ could similarly be in any of the partitions of $\mathcal{H}_{[(x+1)\ldots y]}$, recalling that each of the factors $\mathcal{H}_{[1\ldots x]}$ and $\mathcal{H}_{[(x+1)\ldots y]}$ are themselves vector spaces. In fact, in the 'real world' case where $y \neq N$ it is almost taken for granted that the sub-state in $\mathcal{H}_{[(y+1)\ldots N]}$ is highly separable if it is expected to represent everything else in a semi-classical looking Universe.

Consider now the next state of the universe, ψ_{n+1}, which is one of the eigenvectors of some Hermitian operator $\hat{\Sigma}_{n+1}$. There are, of course, a number of different forms that ψ_{n+1} could take, and it could potentially be in any one of the many partitions of $\mathcal{H}_{[1\ldots y]}$.

For example, this subsequent state could also be in the separation $\mathcal{H}_{AB}$, such that it may be of the form

$$\psi_{n+1} = |\phi_{n+1}\rangle_{[1...x]} \otimes |\Lambda_{n+1}\rangle_{[(x+1)...y]}, \tag{6.82}$$

with $|\phi_{n+1}\rangle_{[A]} \neq |\phi_n\rangle_{[A]}$, or $|\Lambda_{n+1}\rangle_{[B]} \neq |\Lambda_n\rangle_{[B]}$ if, by axiom, $\hat{\Sigma}_{n+1}$ is assumed not to be a null test. In this case, and from the conclusions of the previous sub-section, no information may be said to have been exchanged between the components in Hilbert sub-space $\mathcal{H}_{[1...x]}$ and the components in $\mathcal{H}_{[(x+1)...y]}$ during the transition, because the state did not change in separability relative to these: $\psi_n \in \mathcal{H}_{AB}\ and\ \psi_{n+1} \in \mathcal{H}_{AB}$.

So from the point of view of the separation $\mathcal{H}_{AB}$, the sub-state in $\mathcal{H}_{[1...x]}$ and the sub-state in $\mathcal{H}_{[(x+1)...y]}$ are developing independently of one another, with no sort of influence or interaction occurring between them. To all intents and purposes, during this jump from ψ_n to ψ_{n+1} the factor in $\mathcal{H}_{[1...x]}$ and the factor $\mathcal{H}_{[(x+1)...y]}$ would be developing like distinct 'mini-universes' separate from each other, though care is needed not to take this interpretation too far because the operator $\hat{\Sigma}_{n+1}$ is still acting across the entire Hilbert space, $\mathcal{H}_{[1...N]}$.

In this case, it may be convenient to represent the transition $\psi_n \rightarrow \psi_{n+1}$ as

$$\{|\phi_n\rangle_{[1...x]} \rightarrow |\phi_{n+1}\rangle_{[1...x]}\} \otimes \{|\Lambda_n\rangle_{[(x+1)...y]} \rightarrow |\Lambda_{n+1}\rangle_{[(x+1)...y]}\}. \tag{6.83}$$

Of course, the preservation of the separability of the state relative to $\mathcal{H}_{[AB]}$ does not automatically imply that the factors themselves have developed in trivial ways. After all, recall that $\mathcal{H}_{[1...x]}$ and $\mathcal{H}_{[(x+1)...y]}$ may each be a product of very many subregisters, and this gives rise to the possibility of many different types of internal transitions within these individual spaces. For example, $|\phi_n\rangle_{[A]}$ may be completely separable relative to $\mathcal{H}_{[1...x]}$, whereas $|\phi_{n+1}\rangle_{[A]}$ may be completely entangled, assuming that $x > 1$.

In fact, it is this type of possibility that provides the most manifest difference between information change and information exchange: the information content of the factors of the state in sub-spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ may have changed during the transition from ψ_n to ψ_{n+1}, even though no information was exchanged between these two sub-states. Of course, due to the internal transitions, information may still potentially have been exchanged between any of the factors of $|\phi_n\rangle_{[1...x]}$, and, similarly, also between any of the individual components of $|\Lambda_n\rangle_{[(x+1)...y]}$.

Indeed, these points may be applied to the context of the quantum universe by observing that both the 'subject' and 'device' described in the present discussion may be arbitrarily large, and might therefore incorporate many different levels of sub-subjects and sub-devices within themselves. For instance, the 'subject' could be a large 'black-box' containing an electron of unknown spin, a loaded gun, and a Cat, etc. Moreover, this is what generally happens in laboratory physics, where a sample is often prepared and left

to undergo many different 'internal interactions' before it is eventually measured by an apparatus at some later time; an experiment in the field of chemistry is a good example of this.

Continuing, because the factors in each of these sub-spaces are effectively developing like independent mini-universes, then if this isolation remains for many more transitions they might also begin to develop their own internal causal set type relationships, as apparent from (6.83). This could, in turn, give rise to concepts such as internal measures of space and differing notions of endo-time.

Assume instead, however, that the jump from ψ_n to ψ_{n+1} represents the pregeometric equivalent of a device measuring a subject. Indeed, given that scientists do seem to be able to perform experiments, and that these scientists and their equipment are fundamentally just sub-states of the quantum Universe, there must be some sort of origin for this emergent effect.

Now, although it may be difficult at this stage to say exactly how such a pregeometric experiment occurs, by appealing to the consequences of actual physical measurements it is possible to make inferences about their microscopic counterparts. For example, an experiment necessarily involves an extraction of information, because the purpose of a measurement is ultimately to obtain information about the subject under investigation. Thus, the information content of the sub-state representing the device must necessarily change during the measurement: its information afterwards must be different from its information before, because it must incorporate the newly acquired information regarding the measured subject.

Likewise, and for two similar reasons, the information content of the sub-state representing the subject must also change. Firstly, because there are no non-invasive measurements in quantum physics, any observation automatically affects the sub-state being observed. The only exceptions to this rule are null tests, and these are considered unobservable. Secondly, and encompassing the first point, the symmetry of the situation implies an equivalence between observer and observed. From the point of view of a fully quantum Universe, both the observer and the system under investigation are just factors of the overall state, and so it is not really valid to say 'who' is actually doing the measuring, nor are there any real grounds to make such a choice. So, if a device is measuring a subject, symmetry implies that the subject is equally measuring the device. In fact, in the emergent limit it is only ever possible to discuss a physical observer performing a measurement on a subject (instead of vice versa) because observer states are often taken to be very much larger that the systems under investigation. Thus, if these 'observer' states do not change very much during the process, it may be valid to make the approximation of a 'constant' observer measuring a changing quantum system. This point is discussed

later.

The conclusion of the above discussion is therefore that an endo-physical measurement relies on an exchange of information between the device and the subject. Moreover, and by the results of the previous sub-sections, for an exchange of this type to occur it is necessary for the relationship between one of the components of the state in $\mathcal{H}_{[1\ldots x]}$ to change its relationship with one of the state's components in $\mathcal{H}_{[(x+1)\ldots y]}$ during the jump from ψ_n to ψ_{n+1}. Furthermore, this in turn implies that the state ψ_{n+1} must be entangled relative to the split $\mathcal{H}_A \otimes \mathcal{H}_B$, and hence a member of $\mathcal{H}^{AB}$.

Of course, exactly how this partition change physically affects the state is a greater question. Indeed, the resolution of this issue involves the actual choice of the operators $\hat{\Sigma}_{n+1}$ themselves, and this requires a knowledge of exactly how the self-referential nature of the Universe's development might work (a point that is discussed in Chapter 8). Elaborating on this, presumably the sub-state of the device must be changed in a way that depends on the sub-state of the subject if the jump from ψ_n to ψ_{n+1} is to represent the type of measurement familiar to experimental physics. Indeed, if this were not the case the process could hardly be called a measurement at all, because no useful information would have been extracted about the subject by the device.

Similarly, and by the symmetry of the situation, the sub-state of the subject must also be changed during the transition from ψ_n to ψ_{n+1} in a way that depends on the sub-state of the device. This point is also echoed in empirical science, where the state a subject is projected into upon measurement may depend very much on the object that was measuring it; for example, in an experiment involving the measurement of an electron's spin, the electron is projected into a spin-state that depends on the orientation of the Stern-Gerlach apparatus. Such 'feed-back' mechanisms, however, are beyond the scope of the present discussion.

What can be concluded, though, is still an important point regarding the nature of experimentation in the quantum universe:

- *An endo-physical measurement necessarily implies an exchange of information between the subject and the device. Further, this necessarily implies a partition change involving some of those components of the state representing the subject and some of those components of the state from which the (emergent) description of the device arises.*

Specifically, if $\psi_n \in \mathcal{H}_{AB}$, then a subject sub-state in $\mathcal{H}_A$ is 'measured' by a device sub-state in $\mathcal{H}_B$ during the transition from ψ_n to ψ_{n+1} iff $\psi_{n+1} \in \mathcal{H}^{AB}$. Clearly, for such a measurement the partition $\mathcal{P}_n$ of $\mathcal{H}_{[1\ldots y]}$ containing ψ_n cannot be equal to the partition $\mathcal{P}_{n+1}$ of $\mathcal{H}_{[1\ldots y]}$ containing ψ_{n+1}.

It is this point that justifies the earlier simplification of discussing a 'toy-universe' containing just a subject and a device. If the original case is again considered, i.e. when $y \neq N$ and the Universe's Hilbert space $\mathcal{H}_{[1\ldots N]}$ is split as (6.78) allowing the state $\Psi_n \in \mathcal{H}_{[1\ldots N]}$ to have a 'rest of the Universe' factor $|r_n\rangle_{[(y+1)\ldots N]}$ in $\mathcal{H}_{[(y+1)\ldots N]}$, then as long as $|r_n\rangle_{[(y+1)\ldots N]}$ does not interact or exchange information with any of the components of the state in $\mathcal{H}_A$ or $\mathcal{H}_B$, then this sub-state is effectively existing in its own isolated mini-universe. Paraphrasing, if the development of the components of the state in spaces $\mathcal{H}_A$ or $\mathcal{H}_B$ is restricted such that they can only interact with other components in $\mathcal{H}_A$ or $\mathcal{H}_B$, then these are also effectively existing as a mini-universe separate from the components of the state in $\mathcal{H}_{[(y+1)\ldots N]}$.

In fact, if $\mathcal{H}_{[AB]}$ is itself suitably factorisable into sets of sub-spaces and sub-sub-spaces, it is additionally possible that within this mini-universe whole levels of sub-measurements could simultaneously occur as it develops from one state to the next. This type of process would thus be equivalent to various sub-devices measuring sub-subjects, and sub-sub-devices measuring sub-sub-subjects, etc., each of which is contained in its own factor sub-space of $\mathcal{H}_{[AB]}$. As before, a strong parallel is drawn here with the Schrödinger's Cat paradox, where within the Hilbert space of the 'black-box' containing the Cat, the gun, and the electron, numerous levels of endo-measurement could be occurring.

These points may be stated more formally: because $\mathcal{H}_{[AB]} = \mathcal{H}_A \otimes \mathcal{H}_B$ it just a vector space in its own right it can in some sense be treated as an independent entity. It is then always possible to tensor product $\mathcal{H}_{[AB]}$ with additional Hilbert spaces without affecting the physics as long as sub-states contained in $\mathcal{H}_{[AB]}$ do not become entangled with components in these new spaces. Moreover, because any factors of $\mathcal{H}_A$ and $\mathcal{H}_B$ are themselves also vector spaces, each of these may too be granted an independent existence.

This again reinforces the point that it is acceptable to consider just a 'subject and device' toy-universe without loss of generality.

The caveat to this discussion involves the operators. In the quantum Universe, the entire system is tested by a global, Hermitian operator $\hat{\Sigma}_{n+1}$ acting self-referentially according to the current stage. It is this fact that may prevent the separate mini-universes from being real, physical, isolated and independent universes, because the overall choice of operator affecting the sub-state in one mini-universe may be influenced by the sub-state of another mini-universe. In other words, the operator $\hat{\Sigma}_{n+1}$, which is obviously responsible for developing sub-states in $\mathcal{H}_{[1\ldots y]}$, may be dependent on the sub-state contained in $\mathcal{H}_{[(y+1)\ldots N]}$. This point is addressed again by Chapter 8, but in the present discussion involving just the *principles* of an ideal physics experiment such a technicality is not too drastic.

Note that a jump from one state to the next could contain many different 'isolated

universes' if $N \gg y$. Indeed, if the probability amplitude $\langle \Psi_{n+1} | \Psi_n \rangle$ factorises into F_P factors, then each of these is effectively representing a separate mini-universe during that transition. So, every factor of $\langle \Psi_{n+1} | \Psi_n \rangle$ that contains an initial product of factors of Ψ_n entangling with one another during the jump to Ψ_{n+1} implies an endo-physical measurement occurring between these factors of the initial product. It is hence possible that a jump from one state to the next may permit many different sets of device sub-states independently and simultaneously appearing to measure their own subject sub-states.

Information exchanging partition changes need not actually be too dramatic, a point that can be illustrated when the above spaces $\mathcal{H}_{[A]}$ and $\mathcal{H}_{[B]}$ are written as $\mathcal{H}_{[1...x]}$ and $\mathcal{H}_{[(x+1)...y]}$, with $y = N$ again for simplicity. For example, assume that $|\phi_n\rangle_{[1...x]}$ is completely separable relative to $\mathcal{H}_{[1...x]}$, i.e. $|\phi_n\rangle_{[1...x]} = |\phi_n\rangle_{1...x}$, and $|\Lambda_n\rangle_{[(x+1)...y]}$ is completely separable relative to $\mathcal{H}_{[(x+1)...y]}$, such that ψ_n is in the partition $\mathcal{P}_n \equiv \mathcal{H}_{1...y}$. Assume further that the next state of the Universe ψ_{n+1} is given by

$$\psi_{n+1} = |\varphi_{n+1}\rangle_{[1...(i-1)(i+1)...x]} \otimes |\Upsilon_{n+1}\rangle_{[i(x+1)...y]} \tag{6.84}$$

where $|\varphi_{n+1}\rangle_{[1...(i-1)(i+1)...x]}$ is completely separable relative to $\mathcal{H}_{[1...(i-1)(i+1)...x]}$, i.e.

$$|\varphi_{n+1}\rangle_{[1...(i-1)(i+1)...x]} \in \mathcal{H}_{1...(i-1)(i+1)...x}. \tag{6.85}$$

Finally, assume that the component of ψ_{n+1} in $\mathcal{H}_i$ is entangled with just one other component, namely the component of ψ_{n+1} in $\mathcal{H}_j$, where $(x+1) \leq j \leq y$, and that $|\Upsilon_{n+1}\rangle_{[i(x+1)...y]}$ is completely separable relative to $\mathcal{H}_{[i(x+1)...y]}$ apart from this one entangled factor. Evidently, $|\Upsilon_{n+1}\rangle_{[i(x+1)...y]}$ has $(y-x)$ factors. Then, ψ_{n+1} is in the partition $\mathcal{P}_{n+1}$ given by

$$\mathcal{P}_{n+1} \equiv \mathcal{H}^{ij}_{1...(i-1)(i+1)...(j-1)(j+1)...y} \tag{6.86}$$

and may be written

$$\psi_{n+1} = |\varphi_{n+1}\rangle_{1...(i-1)(i+1)...x} \otimes |\Upsilon_{n+1}\rangle^{ij}_{(x+1)...(j-1)(j+1)...y} \tag{6.87}$$

Clearly, the jump from ψ_n to ψ_{n+1} represents an information exchanging process between the factor of ψ_n in $\mathcal{H}_{[A]} \equiv \mathcal{H}_{[1...x]}$ and the factor of ψ_n in $\mathcal{H}_{[B]} \equiv \mathcal{H}_{[(x+1)...y]}$, because the state has changed its separability relative to $\mathcal{H}_{[AB]}$. That is, $\psi_n \in \mathcal{H}_{AB}$ whereas $\psi_{n+1} \in \mathcal{H}^{AB}$, such that $\mathcal{P}_n \neq \mathcal{P}_{n+1}$. This conclusion, however, is despite the fact that nearly all of the components of the state representing the subject (in the individual subregisters $\mathcal{H}_m$, for $1 \leq m \leq x$) and nearly all of the components of the state representing the device (in the individual subregisters $\mathcal{H}_m$, for $(x+1) \leq m \leq y$) did not interact with anything or each other.

The above points regarding different sets and levels of subjects and devices in the quantum Universe call for the definition of an endo-physical measurement to be refined. Consider a Universe in a Hilbert space $\mathcal{H}_{[1...N]}$, where N is large such that states in $\mathcal{H}_{[1...N]}$ may be highly separable. Consider also a particular k-partite split $\mathcal{H}_{[K_1K_2...K_k]}$ of $\mathcal{H}_{[1...N]}$ as

$$\mathcal{H}_{[1...N]} \equiv \mathcal{H}_{[K_1K_2...K_k]} = \mathcal{H}_{[K_1]} \otimes \mathcal{H}_{[K_2]} \otimes ... \otimes \mathcal{H}_{[K_k]} \tag{6.88}$$

where the k factor spaces of the split need not be fundamentally factorised relative to the individual subregisters, and so need not be of prime dimension; the $(K_a)^{th}$ factor space $\mathcal{H}_{[K_a]}$ could be the tensor product of a number of elementary subregisters, such that for example $\mathcal{H}_{[K_1]} = \mathcal{H}_6 \otimes \mathcal{H}_{14} \otimes \mathcal{H}_{23}$, $\mathcal{H}_{[K_2]} = \mathcal{H}_4 \otimes \mathcal{H}_{10}$, etc.

Assume now that the n^{th} state Ψ_n of the Universe has a factor in the product sub-space $\mathcal{H}_{[K_XK_Y]}$ of $\mathcal{H}_{[1...N]}$, where $1 \leq X, Y \leq k$, and moreover that this factor is separable relative to $\mathcal{H}_{[K_X]} \otimes \mathcal{H}_{[K_Y]}$. Immediately, this implies that Ψ_n must have a factor in the sub-space $\mathcal{H}_{[K_X]}$ and a factor in the sub-space $\mathcal{H}_{[K_Y]}$. Without loss of generality, the factor in $\mathcal{H}_{[K_X]}$ may be called the 'device' whereas the factor in $\mathcal{H}_{[K_Y]}$ may be called the 'subject', though of course a vice versa description would be equally true, and exactly how well these labelled sub-states represent actual physical and macroscopic objects is a question for a theory of emergence.

Now, from an extension of the earlier definition given for a toy-universe containing just an isolated device measuring an isolated subject, the factor of Ψ_n in sub-space $\mathcal{H}_{[K_X]}$ may be said to 'measure' (i.e. exchange information with) the factor of Ψ_n in sub-space $\mathcal{H}_{[K_Y]}$ during the transition from Ψ_n to Ψ_{n+1} if there exists a factor of Ψ_{n+1} that is entangled relative to *at least* $\mathcal{H}_{[K_X]} \otimes \mathcal{H}_{[K_Y]}$. In other words, for such a measurement the state Ψ_{n+1} must either have a factor that is entangled relative to $\mathcal{H}_{[K_X]} \otimes \mathcal{H}_{[K_Y]}$, or else a factor that is entangled relative to the larger tensor product sub-space $\mathcal{H}_{[K_X]} \otimes \mathcal{H}_{[K_Y]} \otimes \mathcal{H}_{[C]}$, where $\mathcal{H}_{[C]}$ is an arbitrary factor Hilbert space of dimension

$$2 \leq \dim(\mathcal{H}_{[C]}) \leq \dim(\mathcal{H}_{[1...N]}) \div \left[\dim(\mathcal{H}_{[K_X]}) \times \dim(\mathcal{H}_{[K_Y]})\right] \tag{6.89}$$

in some given split of $\mathcal{H}_{[1...N]}$ that includes $\mathcal{H}_{[K_X]}$ and $\mathcal{H}_{[K_Y]}$ as sub-spaces. Clearly, $\mathcal{H}_{[C]}$ must satisfy

$$\mathcal{H}_{[C]} \cap \mathcal{H}_{[K_X]} = \mathcal{H}_{[C]} \cap \mathcal{H}_{[K_Y]} = \emptyset. \tag{6.90}$$

The above assertion defines a necessary condition for an endo-physical measurement to occur between any two given factors of a state during a transition, whether or not these factors may be further separated relative to a more fundamental splitting of the Hilbert space. Paraphrasing this definition:

- *Given a particular split* **S** *of* $\mathcal{H}_{[1...N]}$*, two factors* X *and* Y *of a state* Ψ_n *that is*

separable relative to **S** *exchange information during a jump from* Ψ_n *to* Ψ_{n+1} *if these factors become entangled relative to this original split.*

This definition will be particularly important when discussing measurements on the emergent level. Consider a state that is fundamentally separated[13] relative to $\mathcal{H}_{[1\ldots N]}$; clearly, the above definition will apply to endo-physical measurements occurring between each of these factors. However, as shown in Chapter 4, given a fundamentally separated state, its factors can often be 'grouped' into larger factors such that it can be re-written as a separable product of these new sub-states relative to an alternative, less fundamental split of the Hilbert space (i.e. a split that has less than N factor spaces). For example, a state $\phi_{123} \in \mathcal{H}_{[1\ldots3]}$ defined as $\phi_{123} = \phi_1 \otimes \phi_2 \otimes \phi_3$ with three factors may be re-written as the state $\phi_{A3} = \phi_A \otimes \phi_3$ with two factors, where $\phi_A \equiv \phi_1 \otimes \phi_2 \in \mathcal{H}_{[A]} \equiv \mathcal{H}_{[12]}$, relative to the split $\mathcal{H}_{[A3]}$.

These large factors may then play an important role in the theory of emergence, where it may be convenient to consider a particular group of (more fundamental) sub-states as a single entity in order to describe a certain macroscopic 'object'. The point is that the above assertion is always applicable when discussing whether or not an endo-physical measurement has occurred between two given factor sub-states, regardless of which particular split they are being defined separate relative to. Thus for the simple example in $\mathcal{H}_{[1\ldots3]}$, the rule could be applied to define a potential measurement occurring between ϕ_1 and ϕ_2, or between ϕ_1 and ϕ_3, or between ϕ_2 and ϕ_3, or, crucially, also between ϕ_A and ϕ_3, depending upon which particular situation is of current interest.

In other words, it has been possible to define information exchange on any given 'level' of separation, and the method is equally valid both on the most fundamental level, i.e. between components in the N individual subregisters, and on more 'macroscopic' scales between factors of Ψ_n that may themselves be separable relative to a more fundamental split of the Hilbert space. Of course, exactly which level or split is the most appropriate in a given situation to describe an endo-physical measurement in an actual physical, laboratory experiment depends entirely on how these macroscopic objects emerge from the underlying pregeometric structure.

Thus, it is evident that an endo-physical measurement occurring on one level may not necessarily occur on another. Equivalently, an exchange occurring between factors defined relative to one particular split of the Hilbert space does not necessarily occur between every factor defined relative to a different split. This point resonates strongly with the

[13]Recall from Chapter 4 that a state may be said to be written in a 'fundamentally separated' form if it cannot be separated further relative to the fundamental factorisation of the Hilbert space (that is, the split of $\mathcal{H}_{[1\ldots N]}$ with N sub-spaces).

Schrödinger's Cat paradox, and is equally apparent in the second example of above where the completely separable state (6.84) jumped to the 'nearly' completely separable state (6.87).

As concluded above, in order to argue that any endo-physical measurements have occurred during a transition from an arbitrary initial state to a final one, there must be a change of partition[14]. However, different degrees of fundamental separability of different initial and final states lead to different degrees of partition change, in which $\mathcal{P}_{n+1}$ bears different degrees of 'resemblance' to $\mathcal{P}_n$, and this inevitably leads to different degrees of information exchange. For instance, given an initial state $\Theta_{1\ldots N} \in \mathcal{H}_{[1\ldots N]}$ that is completely separable and in the partition $\mathcal{H}_{1\ldots N}$, it is intrinsically obvious that a next state $\Phi^{12}_{3\ldots N}$ in $\mathcal{H}^{12}_{3\ldots N}$ is 'more similar' to $\Theta_{1\ldots N}$ than an alternative next state $\tilde{\Phi}^{1\ldots N}$ in $\mathcal{H}^{1\ldots N}$, and so less information might be expected to be exchanged during a transition from $\Theta_{1\ldots N}$ to $\Phi^{12}_{3\ldots N}$ than from $\Theta_{1\ldots N}$ to $\tilde{\Phi}^{1\ldots N}$.

It is therefore useful to define the concept of '*partition overlap*' that attempts to account for how 'congruent' $\mathcal{P}_{n+1}$ is to $\mathcal{P}_n$, and is related to the factorisability of the probability amplitude discussed previously. Specifically, this partition overlap may then help to quantify just 'how much' information is exchanged during a particular transition from one state to a next, as shown below.

Clearly, partition overlaps will be appropriate for discussions involving information exchanges occurring on the most fundamental level (i.e. between components of the state in the individual subregisters themselves), because partitions are defined relative to the fundamental factorisation of the Hilbert space $\mathcal{H}_{[1\ldots N]}$ into its N subregisters, and not just an arbitrary 'higher order' split. Of course, this is the most natural stance to adopt anyway, because it is changes occurring on the most fundamental, non-emergent level that are assumed ultimately responsible for changes on every other scale.

However, a definition similar to the partition overlap given below could easily be formulated for any other given split if the 'partitions' are replaced by '*split-partitions*', which are defined in an obvious manner in terms of the number of different ways an arbitrary state could potentially be separable relative to a particular split of the Hilbert space.

As an illustrative example of this last idea, the split $\mathcal{H}_{[A3]} \equiv \mathcal{H}_{[A]} \otimes \mathcal{H}_{[3]}$ of the Hilbert space $\mathcal{H}_{[123]}$, where $\mathcal{H}_{[A]} \equiv \mathcal{H}_{[12]}$, has two split-partitions $\mathcal{H}_{A3}$ and $\mathcal{H}^{A3}$, even though the overall space $\mathcal{H}_{[123]}$ has five partitions: $\mathcal{H}_{123}$, $\mathcal{H}^{12}_{3}$, $\mathcal{H}^{13}_{2}$, $\mathcal{H}^{23}_{1}$ and $\mathcal{H}^{123}$.

[14]Noting that similar reverse statements are not necessarily true: a change in partition does not automatically imply that every arbitrary pair of factors of the state must have exchanged information during the jump. To judge whether a particular 'device' has measured a particular 'subject', changes in the separability of the state relative to the given split are of issue.

Similarly, the split $\mathcal{H}_{[AB5]} \equiv \mathcal{H}_{[A]} \otimes \mathcal{H}_{[B]} \otimes \mathcal{H}_{[5]}$ of the Hilbert space $\mathcal{H}_{[1...5]}$, where $\mathcal{H}_{[A]} \equiv \mathcal{H}_{[12]}$ and $\mathcal{H}_{[B]} \equiv \mathcal{H}_{[34]}$, clearly has five split-partitions, $\mathcal{H}_{AB5}$, $\mathcal{H}_A^{B5}$, $\mathcal{H}_B^{A5}$, $\mathcal{H}_5^{AB}$ and $\mathcal{H}^{AB5}$, even though the alternative split $\mathcal{H}_{[A345]}$ has 15 split-partitions, whilst equation (5.20) of Section 5.3 showed that $\mathcal{H}_{[1...5]}$ itself has 52 partitions.

In fact, the concept of split-partitions is evidently more general than that of partitions: a partition is just a special case of a split-partition where the split in question is the fundamental factorisation of the Hilbert space $\mathcal{H}_{[1...N]}$ into its N subregister sub-spaces. Ultimately, 'split-partition overlaps' could therefore be employed to compare how 'similar' one state is to the next relative to the same split, just as partition overlaps will be shown below to provide a comparison of how similar one partition is to another relative to the fundamental factorisation of the Hilbert space. Again, this concept may be useful from the perspective of emergence, and for discussions of how much information is exchanged during a particular transition from the point of view of sets of 'macroscopic' sub-states that are themselves further separable relative to a more fundamental split of the Hilbert space.

Consider a Hilbert space $\mathcal{H}_{[1...N]}$, and a partition $\mathcal{P}_\alpha$ of this with a blocks $B_\alpha^{(1)}, B_\alpha^{(2)}, ..., B_\alpha^{(a)}$. Consider also a second partition $\mathcal{P}_\beta$ of $\mathcal{H}_{[1...N]}$ which has b blocks $B_\beta^{(1)}, ..., B_\beta^{(b)}$. Now, the block $B_\alpha^{(S)}$, for $S = 1, 2, ...a$, implies that every state in the partition $\mathcal{P}_\alpha$ of $\mathcal{H}_{[1...N]}$ possesses a factor that is entangled relative to the sub-space $\mathcal{H}^{(S)}$ defined as the product of the $N(S)$ subregisters:

$$\mathcal{H}^{(S)} \equiv \mathcal{H}_{S_1} \otimes \mathcal{H}_{S_2} \otimes ... \otimes \mathcal{H}_{S_{N(S)}} = \mathcal{H}_{[S_1 S_2 ... S_{N(S)}]} \subset \mathcal{H}_{[1...N]} \tag{6.91}$$

where each S_s is uniquely one of the set $\{1, 2, ..., N\}$ for all $s = 1, 2, ..., N(S)$ and $S = 1, 2, ..., a$. Clearly, the partition $\mathcal{P}_\alpha$ necessarily defines some split of $\mathcal{H}_{[1...N]}$, such that

$$\mathcal{H}_{[1...N]} \equiv \prod_{S=1}^{a} (\otimes \mathcal{H}^{(S)}) = \prod_{S=1}^{a} \prod_{s=1}^{N(S)} (\otimes \mathcal{H}_{S_s}) \tag{6.92}$$

with the obvious condition

$$N = N(1) + N(2) + ... + N(a). \tag{6.93}$$

Similarly, the block $B_\beta^{(T)}$ for $T = 1, 2, ..., b$ implies that every state in the partition $\mathcal{P}_\beta$ of $\mathcal{H}_{[1...N]}$ possesses a factor that is entangled relative to the sub-space $\mathcal{H}^{(T)}$ defined as the product of $N(T)$ subregisters, i.e.

$$\mathcal{H}^{(T)} \equiv \mathcal{H}_{T_1} \otimes \mathcal{H}_{T_2} \otimes ... \otimes \mathcal{H}_{T_{N(T)}} \tag{6.94}$$

where each T_t is uniquely one of the set $\{1, 2, ..., N\}$ for all $t = 1, 2, ..., N(T)$ and $T = 1, 2, ..., b$, and this defines another split of $\mathcal{H}_{[1...N]}$.

Consider now a function $F(B_\alpha^{(S)}, B_\beta^{(T)})$ that effectively compares block $B_\alpha^{(S)}$ to block $B_\beta^{(T)}$, with the result that $F(B_\alpha^{(S)}, B_\beta^{(T)}) = 1$ if $B_\alpha^{(S)} = B_\beta^{(T)}$, but $F(B_\alpha^{(S)}, B_\beta^{(T)}) = 0$ otherwise. The equality $B_\alpha^{(S)} = B_\beta^{(T)}$ is taken to hold only if there is a one-to-one equivalence between the subregisters $\mathcal{H}_{S_s}$ in $B_\alpha^{(S)}$ and the subregisters $\mathcal{H}_{T_t}$ in $B_\beta^{(T)}$; that is, $B_\alpha^{(S)} = B_\beta^{(T)}$ if for each $s = 1, 2, ..., N(S)$ there is one, and only one, t such that

$$\mathcal{H}_{S_s} = \mathcal{H}_{T_t} \tag{6.95}$$

for $t = 1, 2, ..., N(T)$.

Then, the partition overlap $\mathcal{P}(\alpha, \beta)$ between $\mathcal{P}_\alpha$ and $\mathcal{P}_\beta$ may be defined as

$$\mathcal{P}(\alpha, \beta) = \sum_{S=1}^{a} \sum_{T=1}^{b} \frac{F(B_\alpha^S, B_\beta^T)}{\max(a, b)} \tag{6.96}$$

with normalising factor $\max(a, b)$. Clearly, $\mathcal{P}(\alpha, \beta) = \mathcal{P}(\beta, \alpha)$, as would be expected from symmetry.

As an example, consider three states λ, μ and ω in $\mathcal{H}_{[1...4]}$ that are elements of the separations $\lambda \in \mathcal{H}_{1234}$, $\mu \in \mathcal{H}_{12}^{34}$ and $\omega \in \mathcal{H}_4^{123}$. Evidently, the partition $\mathcal{P}_\lambda$ of $\mathcal{H}_{[1...4]}$ of which λ is a member contains four blocks: $1, 2, 3$ and 4. Similarly, $\mathcal{P}_\mu$ has three blocks, $1, 2$, and 34, whilst $\mathcal{P}_\omega$ has only two blocks, 123 and 4. In this case, then, the partition overlap $\mathcal{P}(\lambda, \mu)$ between $\mathcal{P}_\lambda$ and $\mathcal{P}_\mu$ is clearly $(1 + 1)/4 = \frac{1}{2}$, whilst $\mathcal{P}(\lambda, \omega) = 1/4$ and $\mathcal{P}(\mu, \omega) = (0/3) = 0$.

This idea can now be incorporated into the discussion of the quantum universe. To this end, it is asserted that more information is exchanged during a transition from $\Psi_n \in \mathcal{P}_n \subset \mathcal{H}_{[1...N]}$ to $\Psi_{n+1} \in \mathcal{P}_{n+1} \subset \mathcal{H}_{[1...N]}$ if the partition overlap $\mathcal{P}(n, n+1)$ is small than if the partition overlap is large. This follows immediately from the observation that in order for $\mathcal{P}(n, n + 1)$ to be large, most of the components of the state must not have changed their block during the jump from Ψ_n to Ψ_{n+1}, and so have not exchanged information. Thus, the case where $\mathcal{P}(n, n+1) = 0$ represents maximum information exchange, whereas when $\mathcal{P}(n, n + 1) = 1$ no information has been exchanged. Note that for $\mathcal{P}(n, n + 1) = 1$, a necessary but insufficient condition is that $a = b$.

So, it is expected that a jump from a state contained in $\mathcal{H}_{1234}$ to a state contained in $\mathcal{H}_{12}^{34}$ results in less information being exchanged than an alternative jump to a state contained in $\mathcal{H}_4^{123}$.

It is now obvious how the partitions used in the definition of the partition overlap could easily be replaced by split-partitions to provide an analogous definition for a *split-partition overlap*. For example, given a particular split $\mathcal{H}_{[ABC]}$ of $\mathcal{H}_{[1...N]}$ and two of its split-partitions $\mathcal{H}_{ABC}$ and $\mathcal{H}_A^{BC}$, the split-partition overlap between $\mathcal{H}_{ABC}$ and $\mathcal{H}_A^{BC}$ is clearly $(1/3)$; this is despite the fact that each of $\mathcal{H}_A$, $\mathcal{H}_B$ and $\mathcal{H}_C$ may itself be factorisable

into very many subregisters relative to the fundamental factorisation $\mathcal{H}_{[1...N]}$. Of course, and as before, which particular split is of interest depends very much on the 'level' required to describe a given emergent, physical situation.

The idea of partition overlap is related to the statement that, generally, the more factorisable a transition amplitude $\langle \Psi_{n+1} | \Psi_n \rangle$ is, the less information is exchanged.

The above points lead to the question as to when a state such as $|\varphi_{n+1}\rangle$ can really still be said to represent just a 'later version' of the same 'subject' as $|\phi_n\rangle$ if these factors are in different Hilbert sub-spaces of different dimensions. In other words, how similar is the physical object that emerges from $|\varphi_{n+1}\rangle$ to the object that emerged from the earlier factor $|\phi_n\rangle$? An analogous comment arises concerning the similarity of the 'device's' states at different times, such as $|\Lambda_n\rangle_{[(x+1)...y]}$ and $|\Upsilon_{n+1}\rangle_{[i(x+1)...y]}$; can both of these be said to represent the 'before' and 'after' configurations of the same piece of physical apparatus?

Furthermore, it is also an important question to ask about just how much 'pregeometric information' (i.e. component information) needs to be exchanged to constitute the sort of real, physical measurements occurring in laboratories. For instance, is the one component exchange of (6.87) enough, or are more exchanges required for the device to really 'learn' about the subject? Indeed, is it alternatively necessary for the subject and device to become completely entangled with each other in order for a physical measurement to take place? This last point is presumably not the case, since in real physical experiments the device at least generally seems to possess a classical identity after the interaction, and this alternative would ultimately lead to all sorts of Schrödinger's Cat type paradoxes. The issue still remains, however, as to how much entanglement is either 'allowed' or required.

Of course, it is in practice very difficult to say exactly how 'similar' one sub-state physically is to another. Indeed, even if differences could easily be quantified, any resulting argument would then have to rely on knowing exactly how classical objects emerge from the underlying pregeometric description, and a theory of this has not yet been completed. In other words, only once it is understood how the factor state $|\phi_n\rangle$ gives rise to a physical description of the 'subject' can it be compared to whatever semi-classical object emerges from a similar treatment of $|\varphi_{n+1}\rangle$.

That said, it is a natural speculation to suggest that for large macroscopic objects represented by sub-states (or groups of sub-states, depending on the split being discussed) in factor Hilbert spaces of very large dimension, the 'addition or subtraction of just one or two components' may not be expected to affect their emergent appearance too much, and this justifies the earlier argument of why it is usually reasonable to accept the approximation that a large semi-classical observer often seems unaffected by an observation. Conversely, for microscopic sub-states in factor Hilbert spaces of very low dimension, the 'loss of one or two components' might be much more severe, and may lead to an object

that looks completely different on emergent scales. These, however, are just heuristic arguments, and a great deal of work on the issues of emergent and persistence is required in order to fully justify them.

The ideas of the above discussions may now be summarised. When a factor remains in the same block during a jump from one state to the next, it has not exchanged information with any of the others. Such factors are effectively de-coupled and isolated from everything else, and so appear to develop independently. It is only when the partition of the state changes in such a way that the relationship between a component of it contained in one factor changes with respect to a component of the state contained in another factor that an exchange of information occurs between them. Then, these two factors may be said to have interacted during the transition. This type of partition changing process is thus viewed as the pregeometric origin of an endo-physical measurement of one particular factor by another.

On the emergent scale, of course, actual physical measurements are highly complex sets of events. Real experiments involving real devices extracting real information from real subjects may well take place over very many jumps of the state of the Universe, and may incorporate devices with perhaps very many constituent parts each experiencing their own passages of internal endo-time and giving rise to whole different levels of sub-measurements. In fact, particle detections in high energy collider physics provide the perfect example of this point. However, if quantum mechanics does indeed hold for a consistent picture of physical reality, such emergent experiments should fundamentally result from chains and sets of endo-measurements occurring on the pregeometric scale, and as such might ultimately be hoped to be governed by the principles discussed in this section.

As a final remark, it is worth commenting on the roles of the operators in the above types of development. In the type of development that was discussed first, where both ψ_n and ψ_{n+1} were in the same separation $\mathcal{H}_{AB}$, the factorisability of the operator $\hat{\Sigma}_{n+1}$ is not important. This is because both factorisable and entangled operators can have separable outcomes, as discussed in Chapter 5. The only circumstance where the factorisability of $\hat{\Sigma}_{n+1}$ would matter is if, somehow and for some reason, it is known in advance that ψ_{n+1} *must* be in the same separation as ψ_n; in this case either a factorisable operator would have to chosen, or a particular entangled operator would have to be selected that has only separable outcomes.

For the second type of development, however, where ψ_n and ψ_{n+1} were in different separations and ψ_{n+1} was entangled relative to the split $\mathcal{H}_{[AB]}$, the operator $\hat{\Sigma}_{n+1}$ must also be entangled because factorisable operators can only have separable outcomes. In

other words, for the outcome of $\hat{\Sigma}_{n+1}$ to be in the entanglement $\mathcal{H}^{AB}$, i.e. entangled relative to $\mathcal{H}_A \otimes \mathcal{H}_B$, this operator must also be entangled relative to $\mathcal{H}_A \otimes \mathcal{H}_B$. This observation then leads to the result that for a measurement to occur between a 'subject' and a 'device', i.e. for previously separate factors to become entangled, the operator must be entangled.

These last points may be placed in context by remembering that operators are used in conventional quantum mechanics to denote physical tests. Moreover, these physical tests are themselves often also associated with sets of physical apparatus. Recall, however, that in a Universe free from external observers, and as discussed more fully in Chapter 8, its development depends upon operators self-referentially chosen according to the current stage. Thus, since every physical 'object' is expected to emerge from the underlying pregeometric description, and because human physicists do appear to be able to construct sets of apparatus in order to measure things, it must be the case that groups of factors representing devices are somehow able to influence the Universe's decision about which particular operator is chosen to test the state. In other words, the presence of a given set of factors in Ψ_n may result in a certain choice of operator $\hat{\Sigma}_{n+1}$, and so the existence of a particular emergent device and subject may consequently lead to a particular 'action' being taken by the Universe. So, in the quantum Universe, groups of factors representing a physical apparatus may also hence be labelled by the action of the particular operator they induce.

Of course as noted before, exactly how and why particular operators are chosen to develop the Universe's state is an interesting question for the future. How this decision might be sufficiently self-referential to give the impression that physical devices are measuring physical subjects, however, will be addressed later.

7 Quantum Field Theory from Quantum Computation

A central theme of this thesis is to investigate how the semi-classical picture of physics familiar to science may begin to arise from a fundamental quantum state description. Specifically, one matter of particular interest is the question of when it might be possible to argue that "this object with these properties is here". Now, two thirds of this issue have already been addressed: Chapter 4 discussed when it is possible to describe something as a distinct and independent looking object, whereas Chapter 5 investigated the concept of spatial location. It is therefore time to examine the remainder of these three points, namely, how a state represented by a vector in a Hilbert space may give rise to objects with particular physical properties.

Clearly, the idea of a 'property' is very vague, and the word is often used in science to describe almost any number of the physical characteristics exhibited by a semi-classical object, for example its size, or shape, or weight, or appearance, or odour. However, ever since the philosophy of Democritus [71], a reductionist viewpoint has generally been accepted in which each of these qualities is ultimately a feature resulting from a more fundamental picture of reality, such that every macroscopic object comprises of enormous numbers of microscopic 'indivisible elements'. Furthermore, it is the different ways that these individual entities interact and group together that are expected to eventually account for the types of phenomena observed in the everyday world.

Of course, over time this picture has been refined, and it is now known that Democritus' "atoms" should really be associated with elementary particles of given mass, charge, colour, spin etc. Going further, these particles are themselves in turn associated with the various excitation modes of quantum field theory (QFT), and are hence directly determined by the laws, symmetries and formalisms of the equations governing this. Thus, in conventional physics it is the theory of quantised fields that is often ultimately taken to provide an explanation as for why a particular object has the properties it does.

The objective attempted in this present chapter is to go one step further. Because the notion of operators and states in a Hilbert space is taken in this thesis to be *the* fundamental description of physical reality, if it may be shown how quantum field theory might emerge from such a picture, it might consequently be argued that semi-classical properties will also equally arise as a natural continuation, just as in the conventional case. To this end, it is the 'link' between quantum field theory and the model proposed in this thesis that is now explored.

The mechanism suggested to achieve this result follows naturally from the work of the preceding Chapter 6. That is, a treatment of the Universe with the principles of

quantum computation will be shown to reproduce the desired field theoretic concepts. Of course, such an approach may not be too surprising; after all, given the suggestion that the Universe is running as a giant quantum automaton, the application of quantum computational procedures might in some sense naturally be expected to account for every physical effect.

In fact, the emergence of QFT from the type of quantum computation discussed in this thesis is not just desirable for the completeness and consistency of the proposed quantum Universe paradigm. Further to this hope, it might also be expected that such an analysis could give rise to a slightly modified (and hence improved) version of quantum field theory that is free from some of problems inherent in the traditional case. Indeed, this might be hoped for immediately: given that the Hilbert space of the quantum Universe is taken to be very large but finite, and consists of a discrete set of subregisters, the problems associated with infinite dimensional and continuous theories may be expected not to arise. As an example of this, it could be hoped that in the finite case the infra-red and ultra-violet divergences may not occur. This, too, would clearly be an additional success for the model.

Note that the idea of generating particle field theories from this type of analysis is not completely new or unconventional. Feynman, for example, envisaged a description of quantum field theory resulting from quantum computation [72], and Bjorken and Drell similarly demonstrated how QFT may be derived from a set of objects, each of which is positioned at a unique and well specified site [59] (and referenced therein to [73]-[75]). This latter model will be seen to have strong analogies with the method presented in the following. More recently, Wu and Lidar [76] explored the algebraic relationships existing between qubits and parafermions, and Deutsch [77] discussed a qubit field theory embedded in a fixed background spacetime.

7.1 Preliminary Structure

The proposal starts from the premise that the usual quantum field theory familiar to physicists is really an effective, emergent view of a more fundamental mechanism at work. The overall approach will be to use to the rules and principles of quantum computation to construct a model of QFT from the basic, underlying structure of operators and statevectors in a factorisable Hilbert space.

Consider the Hilbert space $\mathcal{H}_{[1...N]}$ factorisable into N subregisters $\mathcal{H}_m$ of prime dimension, for $m = 1, 2, ..., N$. As always, the sub-script m is used merely as a convenient label; the overall Hilbert space is assumed invariant to any left-right re-positioning of the

subregister spaces, and hence m it is not in any way meant to denote physical location. This last point should perhaps be emphasised by remembering that positions and distances have only been defined on the emergent scale, due in fact to considerations of causal set relationships as discussed in Chapter 5, and may be further reinforced by observing that *if* the N 'sites' were taken to be directly equivalent to physical locations, it would be difficult to envisage how the three dimensional Minkowski space of physics could be translated into the one dimensional lattice that would result. These issues will be important to recall later.

The model suggested below may be greatly simplified by assuming that each subregister $\mathcal{H}_m$ is two dimensional, such that the overall space $\mathcal{H}_{[1...N]}$ is a product of N qubit subspaces. In this instance, it is possible to define an orthonormal basis $\mathcal{B}_m$ for $\mathcal{H}_m$ in the usual way as

$$\mathcal{B}_m \equiv \{|0\rangle_m, |1\rangle_m\} \tag{7.1}$$

where, as before, $|0\rangle_m$ may be referred to as 'down' and $|1\rangle_m$ as 'up'. Thus, a basis $\mathcal{B}_{1...N}$ for $\mathcal{H}_{[1...N]}$ may be defined as

$$\mathcal{B}_{1...N} \equiv \{|i_1\rangle_1 \otimes |i_2\rangle_2 \otimes ... \otimes |i_N\rangle_N : i_m = 0, 1; m = 1, 2, ...N\} \tag{7.2}$$

where the representation $|0\rangle_m \equiv \binom{1}{0}_m$ and $|1\rangle_m \equiv \binom{0}{1}_m$ could also be adopted.

Recall now the work of Section 6.1, where equation (6.13) defined the 'Transformation' operator $\hat{T}_m^{ij}$ acting in the space $\mathcal{H}_m$ as $\hat{T}_m^{ij} = |i\rangle_{mm}\langle j|$, for $i, j = 0, 1$. As was discussed, any operator acting locally in $\mathcal{H}_m$ may be built up of complex sums of these four operators. It is beneficial now to enlarge this definition to describe 'extended transformation operators', which have the same resulting effects as their local counterparts, but act globally in the whole space $\mathcal{H}_{[1...N]}$. As alluded to in Section 6.1, this extension may be accomplished by taking the tensor product of the local transformation operator $\hat{T}_m^{ij}$ with $N-1$ identity operators $\hat{I}_{m'}$ in $\mathcal{H}_{m'}$ where $m, m' = 1, 2, ..., N$ but $m \neq m'$. So, the extended transformation operator $\hat{\mathcal{T}}_m^{ij}$ that has the same effect as the local transformation $\hat{T}_m^{ij}$ is defined as

$$\hat{\mathcal{T}}_m^{ij} = \hat{I}_1 \otimes \hat{I}_2 \otimes ... \otimes \hat{I}_{m-1} \otimes \hat{T}_m^{ij} \otimes \hat{I}_{m+1} \otimes ... \otimes \hat{I}_N. \tag{7.3}$$

Clearly, these extended operators satisfy the product rule

$$\hat{\mathcal{T}}_m^{ij}\hat{\mathcal{T}}_m^{kl} = \delta_{jk}\hat{\mathcal{T}}_m^{il} \tag{7.4}$$

and the commutation relation

$$[\hat{\mathcal{T}}_r^{ij}, \hat{\mathcal{T}}_s^{kl}] = 0 \quad , \quad r \neq s. \tag{7.5}$$

Note that by using these types of transformation operator, the model of quantum field theory to be presented encodes ideas such as information and logic from the outset, as discussed in Section 6.1.

Just as arbitrary local operators can be constructed from local transformation operators, the extended transformation operators may be used to generate arbitrary operators that act globally on the whole state. Crucially, however, this also includes the construction of arbitrary global operators that appear to act locally on a particular sub-space. For example, the 'extended Pauli operators', written $\underline{\hat{\sigma}}^a_m$ for $a = 1, 2, 3$, may be given by

$$\begin{aligned} \underline{\hat{\sigma}}^1_m &\equiv \hat{I}_1 \otimes \hat{I}_2 \otimes ... \otimes \hat{I}_{m-1} \otimes \hat{\sigma}^1_m \otimes \hat{I}_{m+1} \otimes ... \otimes \hat{I}_N \\ &= \hat{I}_1 \otimes \hat{I}_2 \otimes ... \otimes \hat{I}_{m-1} \otimes \left[\hat{T}^{01}_m + \hat{T}^{10}_m\right] \otimes \hat{I}_{m+1} \otimes ... \otimes \hat{I}_N \\ &= \hat{\mathcal{T}}^{01}_m + \hat{\mathcal{T}}^{10}_m \end{aligned} \tag{7.6}$$

and similarly

$$\begin{aligned} \underline{\hat{\sigma}}^2_m &\equiv -i\hat{\mathcal{T}}^{01}_m + i\hat{\mathcal{T}}^{10}_m \\ \underline{\hat{\sigma}}^3_m &\equiv \hat{\mathcal{T}}^{00}_m - \hat{\mathcal{T}}^{11}_m. \end{aligned} \tag{7.7}$$

where these *appear* to act only upon components of the state in the sub-space $\mathcal{H}_m$ of $\mathcal{H}_{[1...N]}$.

The three extended operators $\underline{\hat{\sigma}}^a_m$ obey an algebra analogous to their local equivalents, $\hat{\sigma}^a_m$. Assuming the Einstein summation convention over c only, then

$$\underline{\hat{\sigma}}^a_r \underline{\hat{\sigma}}^b_s = \delta_{rs}\left(\delta_{ab}\underline{\hat{\sigma}}^0_r + i\epsilon_{abc}\underline{\hat{\sigma}}^c_r\right) + (1 - \delta_{rs})\,\underline{\hat{\sigma}}^b_s \underline{\hat{\sigma}}^a_r \tag{7.8}$$

for $a, b, c = 1, 2, 3$ and $r, s = 1, 2, ..., N$, with the extended identity operator $\underline{\hat{\sigma}}^0_m$ defined as

$$\underline{\hat{\sigma}}^0_m \equiv \hat{\sigma}^0_1 \otimes \hat{\sigma}^0_2 \otimes ... \otimes \hat{\sigma}^0_N \equiv \hat{I}_1 \otimes \hat{I}_2 \otimes ... \otimes \hat{I}_N \equiv \hat{I}_{1...N} \tag{7.9}$$

and where the first set of bracketed terms in (7.8) account for local products when $\hat{\sigma}^a_r$ and $\hat{\sigma}^b_s$ act in the same sub-register, and the second set is due to the commutation relation (7.5) for local operators acting in different sub-registers. Moreover, it also follows that

$$\underline{\hat{\sigma}}^a_r \underline{\hat{\sigma}}^0_s = \underline{\hat{\sigma}}^0_s \underline{\hat{\sigma}}^a_r = \underline{\hat{\sigma}}^a_r \tag{7.10}$$

noting how these results compare with the 'usual' local Pauli algebra of (5.32), as given in the standard literature [12].

As an aside, note that the extended Pauli operators may be used to demonstrate the group symmetry of the extended transformation operators. Consider a unitary operator $\hat{U}(\boldsymbol{\theta})$ defined as

$$\hat{U}(\boldsymbol{\theta}) \equiv \exp\left(i\theta^1_m \underline{\hat{\sigma}}^1_m + i\theta^2_m \underline{\hat{\sigma}}^2_m + i\theta^3_m \underline{\hat{\sigma}}^3_m\right) \tag{7.11}$$

that appears to act locally in the m^{th} Hilbert sub-space $\mathcal{H}_m$, where $\theta^1_m, \theta^2_m, \theta^2_m \in \mathbb{R}$. Then, the algebra of the extended transformation operators is invariant to 'rotations' under the $SU(2)$ group, such that the operator $(\hat{\mathcal{T}}^{ij}_m)'$ defined as

$$(\hat{\mathcal{T}}^{ij}_m)' \equiv \hat{U}^*(\boldsymbol{\theta})\hat{\mathcal{T}}^{ij}_m\hat{U}(\boldsymbol{\theta}) \tag{7.12}$$

where $\hat{U}^*(\boldsymbol{\theta})$ is the conjugate transpose of $\hat{U}(\boldsymbol{\theta})$, obeys

$$(\hat{\mathcal{T}}_m^{ij})'(\hat{\mathcal{T}}_m^{kl})' = \delta_{jk}(\hat{\mathcal{T}}_m^{il})' \tag{7.13}$$

as expected.

It is here that a discussion of the algebraic structure necessary for field theory can begin. Consider the extended transformation operators $\hat{\mathcal{T}}_m^{10}$ and $\hat{\mathcal{T}}_m^{01}$. If the 'ground state' in $\mathcal{H}_{[1...N]}$, written as $|\underline{0}\rangle$, is chosen to be

$$|\underline{0}\rangle = |\underline{0}\rangle_{1...N} = |00...0\rangle_{1...N} = |0\rangle_1 \otimes |0\rangle_2 \otimes ... \otimes |0\rangle_N \tag{7.14}$$

then the operator $\hat{\mathcal{T}}_m^{10}$ acting on $|\underline{0}\rangle$ results in the transition of the state of the m^{th} qubit (i.e. the component of $|\underline{0}\rangle$ in $\mathcal{H}_m$) of $\mathcal{H}_{[1...N]}$ from $|0\rangle_m$ to $|1\rangle_m$. That is

$$\hat{\mathcal{T}}_m^{10}|\underline{0}\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes ... \otimes |0\rangle_{m-1} \otimes |1\rangle_m \otimes |0\rangle_{m+1} \otimes ... \otimes |0\rangle_N. \tag{7.15}$$

This transformation may be reversed by the operator $\hat{\mathcal{T}}_m^{01}$. That is, if the m^{th} qubit is in the state $|1\rangle_m$, then $\hat{\mathcal{T}}_m^{01}$ leads to a transition of this to $|0\rangle_m$, such that

$$\hat{\mathcal{T}}_m^{10}\left(|0\rangle_1 \otimes |0\rangle_2 \otimes ... \otimes |0\rangle_{m-1} \otimes |1\rangle_m \otimes |0\rangle_{m+1} \otimes ... \otimes |0\rangle_N\right) = |\underline{0}\rangle. \tag{7.16}$$

The operator $\hat{\mathcal{T}}_m^{10}$ can be applied in products that effectively act on different sub-registers; viz,

$$\begin{aligned} \hat{\mathcal{T}}_r^{10}\hat{\mathcal{T}}_s^{10}|\underline{0}\rangle \quad = \quad & |0\rangle_1 \otimes |0\rangle_2 \otimes ... \otimes |0\rangle_{r-1} \otimes |1\rangle_r \otimes |0\rangle_{r+1} \otimes ... \\ & ... \otimes |0\rangle_{s-1} \otimes |1\rangle_s \otimes |0\rangle_{s+1} \otimes ... \otimes |0\rangle_N \end{aligned} \tag{7.17}$$

for $r \neq s$, noting that

$$\hat{\mathcal{T}}_r^{10}\hat{\mathcal{T}}_r^{10}|\underline{0}\rangle = 0 \tag{7.18}$$

as expected from an analogy with the local transformation operators, and as discussed in Section 6.1. Similar results apply in obvious ways for higher order products of $\hat{\mathcal{T}}_r^{10}\hat{\mathcal{T}}_s^{10}\hat{\mathcal{T}}_t^{10}...$, for products of the 'opposite' operators $\hat{\mathcal{T}}_m^{01}$, or for various 'mixtures' of the $\hat{\mathcal{T}}_r^{10}$'s and $\hat{\mathcal{T}}_s^{01}$'s.

The last result (7.18) follows naturally from the rule that if the operator $\hat{\mathcal{T}}_m^{10}$ is applied to a qubit that is in the state $|1\rangle_m$, then the outcome is 0. Similarly, if the operator $\hat{\mathcal{T}}_m^{01}$ is applied to a qubit that is in the state $|0\rangle_m$, then the outcome of this is 0. Clearly, then,

$$\hat{\mathcal{T}}_m^{01}|\underline{0}\rangle = 0. \tag{7.19}$$

The operators $\hat{\mathcal{T}}_m^{10}$ and $\hat{\mathcal{T}}_m^{01}$ are in some sense analogous to the creation and annihilation ladder operators of quantum field theory. Specifically, $\hat{\mathcal{T}}_m^{10}$ may be interpreted as an

operator that creates a $|1\rangle_m$ state from the 'vacuum' $|\underline{0}\rangle$, whereas $\hat{\mathcal{T}}_m^{01}$ destroys this to return the ground state.

In addition to this comparison, $\hat{\mathcal{T}}_m^{10}$ and $\hat{\mathcal{T}}_m^{01}$ are seen (7.5) to obey some of the commutation relations familiar to bosonic ladder operators. It might be suggested, therefore, that they could hence be used to construct a pregeometric theory that might reproduce the properties of physical bosons in the emergent limit. However, a problem with this hypothesis is that conventional bosonic theories permit 'multi-occupation number states' [78]. That is, given a conventional, bosonic creation operator $\hat{a}_B^\dagger$ acting on a conventional vacuum ground state $\varphi_0 = |0\rangle$, then

$$\hat{a}_B^\dagger|0\rangle = |1_B\rangle \tag{7.20}$$

produces the single boson particle state $|1_B\rangle$, whilst

$$\hat{a}_B^\dagger\hat{a}_B^\dagger|0\rangle = \hat{a}_B|1_B\rangle = |2_B\rangle \tag{7.21}$$

yields the two boson particle state, and so on.

Conversely, an immediate difficulty that would be faced in attempts to reconstruct bosonic field theory from the transformation operators of above is that although $\hat{\mathcal{T}}_m^{10}|\underline{0}\rangle \neq 0$, a second application of $\hat{\mathcal{T}}_m^{10}$ to this new state gives $\hat{\mathcal{T}}_m^{10}\hat{\mathcal{T}}_m^{10}|\underline{0}\rangle = 0$. So, even though $\hat{\mathcal{T}}_m^{10}|\underline{0}\rangle$ is not being directly interpreted here as a single particle state *per se*, it is still difficult to see how multiple particle states could ultimately be generated in this manner if the underlying qubit operator algebra is so contrary to that employed to describe bosons.

However, note that the relationship $\hat{\mathcal{T}}_m^{10}\hat{\mathcal{T}}_m^{10}|\underline{0}\rangle = 0$ is instead similar to the Exclusion Principle condition used in standard quantum field theory for particles obeying fermi-dirac statistics. Thus, it is this connection that is now explored, and an attempt is made to recover fermionic field theory from the pregeometric framework.

To this end, consider first the non-local operator η_m defined as

$$\begin{aligned} \eta_m &\equiv \underline{\hat{\sigma}}_1^3\underline{\hat{\sigma}}_2^3...\underline{\hat{\sigma}}_{m-1}^3\hat{I}_m \quad , \quad 2 \leq m \leq N \\ &= \hat{\sigma}_1^3 \otimes \hat{\sigma}_2^3 \otimes ... \otimes \hat{\sigma}_{m-1}^3 \otimes \hat{I}_m \otimes \hat{I}_{m+1} \otimes ... \otimes \hat{I}_N \end{aligned} \tag{7.22}$$

with $\eta_1 \equiv \hat{I}_1 \otimes \hat{I}_2 \otimes ... \otimes \hat{I}_N$.

Now, consider the operator $\hat{\alpha}_m$ defined as

$$\hat{\alpha}_m = \eta_m\hat{\mathcal{T}}_m^{01} \tag{7.23}$$

and its adjoint (i.e. Hermitian conjugate) operator $\hat{\alpha}_m^\dagger$

$$\hat{\alpha}_m^\dagger = \eta_m\hat{\mathcal{T}}_m^{10}. \tag{7.24}$$

Clearly,

$$\hat{\alpha}_m|\underline{0}\rangle = 0 \tag{7.25}$$

and

$$\hat{\alpha}_m\left(|0\rangle_1 \otimes |0\rangle_2 \otimes ... \otimes |0\rangle_{m-1} \otimes |1\rangle_m \otimes |0\rangle_{m+1} \otimes ... \otimes |0\rangle_N\right) = |\underline{0}\rangle \tag{7.26}$$

whilst

$$\hat{\alpha}_m^\dagger|\underline{0}\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes ... \otimes |0\rangle_{m-1} \otimes |1\rangle_m \otimes |0\rangle_{m+1} \otimes ... \otimes |0\rangle_N \tag{7.27}$$

and

$$\hat{\alpha}_m^\dagger\hat{\alpha}_m^\dagger|0\rangle = 0. \tag{7.28}$$

Moreover, due to the presence of the operators $\underline{\hat{\sigma}}_1^3$, $\underline{\hat{\sigma}}_2^3$, ..., $\underline{\hat{\sigma}}_{m-1}^3$, and hence unlike the transformation operators $\hat{\mathcal{T}}_m^{01}$ and $\hat{\mathcal{T}}_m^{10}$, the operators $\hat{\alpha}_m$ and $\hat{\alpha}_m^\dagger$ obey anti-commutation relations

$$\begin{aligned}\{\hat{\alpha}_r, \hat{\alpha}_s\} &= 0 \\ \{\hat{\alpha}_r^\dagger, \hat{\alpha}_s^\dagger\} &= 0 \\ \{\hat{\alpha}_r, \hat{\alpha}_s^\dagger\} &= \delta_{rs}\hat{I}_{1...N}\end{aligned} \tag{7.29}$$

as may be readily demonstrated.

Proof. A proof of the first of these is presented as follows. Given

$$\{\hat{\alpha}_r, \hat{\alpha}_s\} = \hat{\alpha}_r\hat{\alpha}_s + \hat{\alpha}_s\hat{\alpha}_r = \eta_r\hat{\mathcal{T}}_r^{01}\eta_s\hat{\mathcal{T}}_s^{01} + \eta_s\hat{\mathcal{T}}_s^{01}\eta_r\hat{\mathcal{T}}_r^{01} \tag{7.30}$$

then expanding produces

$$\begin{aligned}&= \left(\hat{\sigma}_1^3\hat{\sigma}_2^3...\hat{\sigma}_{r-1}^3\hat{I}_r...\hat{I}_N\right)\left(\hat{I}_1\hat{I}_2...\hat{T}_r^{01}...\hat{I}_N\right)\left(\hat{\sigma}_1^3\hat{\sigma}_2^3...\hat{\sigma}_{s-1}^3\hat{I}_s...\hat{I}_N\right)\left(\hat{I}_1\hat{I}_2...\hat{T}_s^{01}...\hat{I}_N\right) \\ &\quad + \left(\hat{\sigma}_1^3\hat{\sigma}_2^3...\hat{\sigma}_{s-1}^3\hat{I}_s...\hat{I}_N\right)\left(\hat{I}_1\hat{I}_2...\hat{T}_s^{01}...\hat{I}_N\right)\left(\hat{\sigma}_1^3\hat{\sigma}_2^3...\hat{\sigma}_{r-1}^3\hat{I}_r...\hat{I}_N\right)\left(\hat{I}_1\hat{I}_2...\hat{T}_r^{01}...\hat{I}_N\right) \\ &= \left(\hat{\sigma}_1^3\hat{\sigma}_2^3...\hat{\sigma}_{r-1}^3\hat{T}_r^{01}...\hat{I}_N\right)\left(\hat{\sigma}_1^3\hat{\sigma}_2^3...\hat{\sigma}_{s-1}^3\hat{T}_s^{01}...\hat{I}_N\right) \\ &\quad + \left(\hat{\sigma}_1^3\hat{\sigma}_2^3...\hat{\sigma}_{s-1}^3\hat{T}_s^{01}...\hat{I}_N\right)\left(\hat{\sigma}_1^3\hat{\sigma}_2^3...\hat{\sigma}_{r-1}^3\hat{T}_r^{01}...\hat{I}_N\right)\end{aligned} \tag{7.31}$$

where tensor product symbols have been omitted. Assuming, without loss of generality, that $r < s$ gives

$$\begin{aligned}&= \left(\hat{\sigma}_1^3\hat{\sigma}_1^3\right)\left(\hat{\sigma}_2^3\hat{\sigma}_2^3\right)...\left(\hat{\sigma}_{r-1}^3\hat{\sigma}_{r-1}^3\right)\left(\hat{T}_r^{01}\hat{\sigma}_r^3\right)\left(\hat{I}_{r+1}\hat{\sigma}_{r+1}^3\right)...\left(\hat{I}_s\hat{T}_s^{01}\right)...\left(\hat{I}_N\hat{I}_N\right) \\ &\quad + \left(\hat{\sigma}_1^3\hat{\sigma}_1^3\right)\left(\hat{\sigma}_2^3\hat{\sigma}_2^3\right)...\left(\hat{\sigma}_{r-1}^3\hat{\sigma}_{r-1}^3\right)\left(\hat{\sigma}_r^3\hat{T}_r^{01}\right)\left(\hat{\sigma}_{r+1}^3\hat{I}_{r+1}\right)...\left(\hat{T}_s^{01}\hat{I}_s\right)...\left(\hat{I}_N\hat{I}_N\right).\end{aligned} \tag{7.32}$$

Now, from the algebra (5.32) of the Pauli operators, $\hat{\sigma}_m^3\hat{\sigma}_m^3 = \hat{\sigma}_m^0 = \hat{I}_m$, it is evident that

$$\begin{aligned}\{\hat{\alpha}_r, \hat{\alpha}_s\} &= \hat{I}_1\hat{I}_2...\left(\hat{T}_r^{01}\hat{\sigma}_r^3\right)\left(\hat{\sigma}_{r+1}^3\right)...\left(\hat{T}_s^{01}\right)\hat{I}_{s+1}...\hat{I}_N \\ &\quad + \hat{I}_1\hat{I}_2...\left(\hat{\sigma}_r^3\hat{T}_r^{01}\right)\left(\hat{\sigma}_{r+1}^3\right)...\left(\hat{T}_s^{01}\right)\hat{I}_{s+1}...\hat{I}_N \\ &= \hat{I}_1\hat{I}_2...\left(\hat{T}_r^{01}\hat{\sigma}_r^3 + \hat{\sigma}_r^3\hat{T}_r^{01}\right)\left(\hat{\sigma}_{r+1}^3\right)...\left(\hat{T}_s^{01}\right)\hat{I}_{s+1}...\hat{I}_N.\end{aligned} \tag{7.33}$$

Recall, however, that (like all local operators) the Pauli operators can be written as a complex sum of local transformation operators. So, by using (6.10), i.e. $\hat{\sigma}_r^3 = \hat{T}_r^{00} - \hat{T}_r^{11}$, it is evident that

$$\begin{aligned} \left(\hat{T}_r^{01}\hat{\sigma}_r^3 + \hat{\sigma}_r^3\hat{T}_r^{01}\right) &= \{\hat{T}_r^{01}, \hat{\sigma}_r^3\} \\ &= \left(\hat{T}_r^{01}[\hat{T}_r^{00} - \hat{T}_r^{11}] + [\hat{T}_r^{00} - \hat{T}_r^{11}]\hat{T}_r^{01}\right) \\ &= -\hat{T}_r^{01} + \hat{T}_r^{01} = 0 \end{aligned} \tag{7.34}$$

where the last line follows from the usual $\hat{T}_r^{ij}$ algebra (6.14). So, substituting in gives

$$\{\hat{\alpha}_r, \hat{\alpha}_s\} = 0 \tag{7.35}$$

as expected.

Clearly, the proof holds also for $r \geq s$.

Moreover, the relations $\{\hat{\alpha}_r^\dagger, \hat{\alpha}_s^\dagger\} = 0$ and $\{\hat{\alpha}_r, \hat{\alpha}_s^\dagger\} = \delta_{rs}\hat{I}_r$ can be readily verified via analogous methods. ■

Due to their similarities to conventional theory, the fermionic-looking operators $\hat{\alpha}_m^\dagger$ and $\hat{\alpha}_m$ will be called pregeometric (fermionic) creation and annihilation operators respectively, or equivalently, qubit ladder operators. The extent of this similarity will be investigated fully in due course.

As an aside, however, note that the result (7.28) can also be given by the relation

$$\left(\hat{\alpha}_m^\dagger\right)^D |0\rangle \quad \begin{Bmatrix} \neq 0 & , & D = 1 \\ = 0 & , & D > 1 \end{Bmatrix} \tag{7.36}$$

in anticipation of higher order generalisations in the future; the algebra (7.28) obeys parafermionic statistics of order 1 [65][66].

Just as the transformation operators were invariant under $SU(2)$ rotations, so too are the pregeometric creation and annihilation operators. Specifically, using the rotation operator $\hat{U}(\boldsymbol{\theta})$ given in (7.11), then the operators $(\hat{\alpha}_m^\dagger)'$ and $(\hat{\alpha}_m)'$ defined as

$$\begin{aligned} (\hat{\alpha}_m^\dagger)' &\equiv \hat{U}^*(\boldsymbol{\theta})\hat{\alpha}_m^\dagger\hat{U}(\boldsymbol{\theta}) \\ (\hat{\alpha}_m)' &\equiv \hat{U}^*(\boldsymbol{\theta})\hat{\alpha}_m\hat{U}(\boldsymbol{\theta}) \end{aligned} \tag{7.37}$$

also obey the fermionic algebra. That is

$$\begin{aligned} \{(\hat{\alpha}_r)', (\hat{\alpha}_s)'\} &= \{(\hat{\alpha}_r^\dagger)', (\hat{\alpha}_s^\dagger)'\} = 0 \\ \{(\hat{\alpha}_r)', (\hat{\alpha}_s^\dagger)'\} &= \delta_{rs}\hat{I}_{1\ldots N}. \end{aligned} \tag{7.38}$$

This result may lead to important consequences for the gauge symmetry of the emergent theory.

As with separable states and factorisable Hilbert spaces, it is evident that the pregeometric creation and annihilation operators are also invariant to any left-right re-positioning of their factors. This, of course, is because it is always assumed that the factor sub-operator with subscript t still acts in the sub-register $\mathcal{H}_t$, for $1 \leq t \leq N$, such that the imposed left-right ordering of the equations is merely typographical. So, $\hat{\sigma}_a^3$ always acts in $\mathcal{H}_a$, $\hat{I}_b$ always acts in $\mathcal{H}_b$, $\hat{T}_c^{10}$ always acts in $\mathcal{H}_c$, and so on, such that a sub-operator's location within the tensor product is immaterial. As an example, the pregeometric creation operator

$$\hat{\alpha}_r^\dagger = \hat{\sigma}_1^3 \hat{\sigma}_2^3 ... \hat{\sigma}_{14}^3 \hat{T}_{15}^{10} \hat{I}_{16} ... \hat{I}_{28} \tag{7.39}$$

may be rewritten as

$$\hat{\alpha}_r^\dagger = \hat{I}_{23} \hat{I}_{17} \hat{\sigma}_{14}^3 \hat{\sigma}_1^3 \hat{I}_{28} \hat{\sigma}_6^3 \hat{T}_{15}^{10} \hat{I}_{18} ... \tag{7.40}$$

without affecting the anti-commutation algebra.

7.2 Dirac Field Theory

It is now possible to begin to construct quantum field theories from the basic principles described above. Specifically, attention will be focused on the emergence of Dirac theory from the underlying pregeometric structure, because this field is often taken to be one of the most basic (and hence important) ingredients of elementary particle physics. Indeed, it is even possible to describe many boson species in terms of groups of fermions obeying the Dirac theory; the pion of particle phenomenology, and the existence of Cooper pairs in superconductivity provide good physical examples of this point.

So, in this section a description of spin-$\frac{1}{2}$ fermions will be given.

For convenience, the two types of particle and two types of anti-particle associated with the field's excitations will be referred to below as spin-up and spin-down electrons and positrons. In should be noted, however, that this is merely for linguistic advantage, and in principle the presented analysis is not restricted to any particular particle species.

In order to justify the enormous simplification involved in considering just a single, isolated Dirac field, recall the types of situation in physics in which such circumstances are generally encountered. In conventional particle accelerator experiments, for example, scientists often go to great lengths to construct apparatus that effectively 'shuts an area off' from the rest of the Universe, such that the region inside the collider can be treated as an isolated system in which only a few basic fields are present.

Now, in a fully quantum Universe, this sort of experimental arrangement is taken to arise in the large-scale limit when the state Ψ_n is sufficiently and suitably separable so that its factors give rise to such an emergent, semi-classical picture. In this case, various factors

and groups of factors may be used to represent the detector, the particles it contains, the physicist, the laboratory, and, indeed, everything else. Moreover, in fact, the operators chosen to develop the Universe are assumed to be carefully and self-referentially chosen such that this semi-classical description appears to persist over a number of jumps, as has been discussed previously.

So, it should therefore be reasonable to meaningfully discuss parts of the Universe that seem to contain nothing but isolated, fundamental fields, because this is what scientists tend to be able to do in real, physical experiments. Furthermore, since it has been conjectured that there is a strong link between the 'parts' of the Universe and the factors of its state, it is equally reasonable to assert the possibility of discussing factors that 'contain' just the particles inside the detector. Indeed, by rephrasing this argument for the perspective advocated in this thesis, it might be possible to generate isolated quantum fields out of a consideration of the pregeometric 'properties' of a particular part of the Universe. Specifically, and in the language of the previous chapter, the information content of such regions might somehow be expected to include notions of quantum fields, though exactly how this might be achieved is what is to be examined below.

Of course, the factors that represent the insides of particle detectors will also possess many other types of features. For instance, they will have a well defined location in emergent physical space because of their familial relationships with other factors (c.f. Chapter 5), and since the separability of the Universe may change as it jumps from one state to the next, it may be possible to discuss 'observers' appearing to measure the substate representing the colliding particles (c.f. Chapter 6). Ultimately, then, it should be possible to envisage a typical particle physics experiment from a pregeometric point of view, where isolated particles appear to collide and be scattered, before being measured by various components, detectors and scientists.

For now, however, just the isolated particle fields shall be discussed, with the implicit assumption being made that any such procedures could also eventually be applied to more 'complicated' situations.

So consider just that factor φ, of the state $\Psi \in \mathcal{H}_{[1\ldots N]}$ of the Universe, that represents the 'inside' of the collider. Thus in the following, the label φ will be used to denote the part of the Universe's state Ψ responsible for a description of everything of interest that occurs inside the detector during a collision event. By re-labelling the subregisters of the overall Hilbert space $\mathcal{H}_{[1\ldots N]}$ of Ψ in a convenient way, the factor φ may be said to be contained in a factor Hilbert sub-space $\mathcal{H}_{[1\ldots N']}$, where $\mathcal{H}_{[1\ldots N']} \subset \mathcal{H}_{[1\ldots N]}$. Of course, φ may or may not itself be highly separable relative to $\mathcal{H}_{[1\ldots N']}$. Clearly, the remaining subspace $\mathcal{H}_{[(N'+1)\ldots N]}$ contains factors and groups of factors that represent the Physicist, P, the Apparatus, A, and the Rest of the Universe, R.

Overall, the goal therefore becomes to investigate the circumstances in which the Hilbert space $\mathcal{H}_{[1...N']}$ may be described as 'containing' a single, isolated Dirac field.

In conventional quantum field theory, and in particular in the S-Matrix approach to particle scattering [79], it is assumed that what actually occurs during the collision may be represented by a type of 'Black Box'; only the initial 'In' particle state, $|\psi_{in}\rangle$, and final 'Out' particle state, $|\psi_{out}\rangle$, are of interest to physicists. Moreover, in the Heisenberg picture of dynamics traditionally used in quantum field theory, it is asserted that the initially prepared state $|\psi_{in}\rangle$ is effectively 'frozen in time' until its later measurement by an observer, at which point it is collapsed into $|\psi_{out}\rangle$. Thus, a typical particle physics experiment proceeds by the scientist preparing an initial a-particle In state, before measuring it at some time later time with some sort of Hermitian operator (representing an observable), thereby collapsing it into a final b-particle Out state. Consequently, the time evolution of the system is enforced by transforming the Observables, in a way that depends, in fact, upon the time experienced by the physicist between the initial preparation of $|\psi_{in}\rangle$ and its eventual measurement. Amplitudes between initial and particular final states may therefore be considered.

Now, in order to recreate standard physics from the pregeometric perspective aimed at in this thesis, the above type of setup must be reproducible in the quantum Universe paradigm. So, the principles employed in conventional particle theory are used to guide the present analysis.

To this end, consider an initial state Ψ_n defined as

$$\Psi_n = |\varphi_{in}\rangle \otimes |P_i\rangle \otimes |A_i\rangle \otimes |R_i\rangle \tag{7.41}$$

where $|\varphi_{in}\rangle \in \mathcal{H}_{[1...N']}$ represents the initially prepared sub-state of the particles in the detector prior to the collision (i.e. before anything has happened), and $|P_i\rangle, |A_i\rangle$ and $|R_i\rangle$ the initial sub-states of the Physicist, Apparatus and Rest of Universe respectively.

Moreover, assume that the series of jumps from $\Psi_n \longrightarrow \Psi_{n+1} \longrightarrow ... \longrightarrow \Psi_{n+n'}$, where $n' \gg n$, represent, on the emergent level, a observer-apparatus-environment system performing a particle collision experiment. In this case, $\Psi_{n+n'}$ may be taken to be of the approximate form

$$\Psi_{n+n'} = |\varphi_{out}\rangle \otimes |P_f\rangle \otimes |A_f\rangle \otimes |R_f\rangle \tag{7.42}$$

where $|\varphi_{out}\rangle$ represents the final sub-state of the particles in the detector after the collision, and $|P_f\rangle, |A_f\rangle$ and $|R_f\rangle$ the respective final sub-states of the Physicist, Apparatus and Rest of Universe. Of course, the operator $\hat{\Sigma}_{n+n'}$, of which $\Psi_{n+n'}$ is an eigenstate, must be carefully defined such that the Physicist factor in $\Psi_{n+n'-1}$ believes herself to be choosing (with apparent free-will) a particular laboratory test to measure the particle sub-state with; this general issue is discussed more fully in Chapter 8.

By attempting to keep the dynamics congruent to the situation familiar to conventional physics, a number of inferences may be drawn about what the above sequence of states might be like. Firstly, because the sub-state representing the particles is assumed 'frozen in time' between its preparation as part of Ψ_n and measurement as part of $\Psi_{n+n'}$, whatever the operators in the sequence $\hat{\Sigma}_{n+1} \longrightarrow \hat{\Sigma}_{n+1} \longrightarrow ...\hat{\Sigma}_{n+n'-1}$ might actually be, each must be taken to result in a null test on the factor of the universe in $\mathcal{H}_{[1...N']}$. In such a case, φ_{in} would consequently appear unchanged during this period. Moreover, and in the language of Chapter 6, no information would therefore be exchanged between the components of the state in $\mathcal{H}_{[1...N']}$ and the components of the state in $\mathcal{H}_{[(N'+1)...N]}$ during this time; as desired, the sub-state in $\mathcal{H}_{[1...N']}$ representing the inside of the detector is effectively isolated from the remainder of the Universe. Of course, the Physicist, Apparatus and Rest of Universe do interact, entangle and exchange information with one another throughout this duration.

Overall, therefore, it is asserted that the Universe jumps through a series of states of the form

$$\begin{aligned} \Psi_{n+1} &= |\varphi_{in}\rangle \otimes |P'\rangle \otimes |A'\rangle \otimes |R'\rangle \\ \Psi_{n+2} &= |\varphi_{in}\rangle \otimes |P''\rangle \otimes |A''\rangle \otimes |R''\rangle \\ &\vdots \\ \Psi_{n+n'-1} &= |\varphi_{in}\rangle \otimes |P'''^{...\prime}\rangle \otimes |A'''^{...\prime}\rangle \otimes |R'''^{...\prime}\rangle \end{aligned} \tag{7.43}$$

where P'', for example, is a factor representing a physicist that has evolved and developed since initial 'time' n.

As expected, the operators $\hat{\Sigma}_{n+1}, \hat{\Sigma}_{n+2}, ..., \hat{\Sigma}_{n+n'-1}$ must be very carefully constrained, defined and selected, in order to ensure that this type of pattern occurs.

A second feature that may be concluded about the above system by drawing parallels to the laboratory case is that the factors of the Universe representing the physicist will bestow certain properties upon φ. For example, because they know it represents what is going on inside a physical machine, they may assume it occupies a particular volume, or represents a certain set of conditions. Consider just this volume: if φ is ultimately taken to represent the state inside a physical detector, whatever spatial degrees of freedom emerge from a causal set description of the operators acting in its Hilbert sub-space must match the observed spatial properties of the object (i.e. the detector) appearing to physically contain it.

Moreover, recall from Section 5.7.5 that, under certain conditions, particular sub-operators acting locally in particular subregisters may be mapped to positions situated on a three dimensional lattice; of course, this was not too say that the subregisters are

actually located at these sites *per se*, but that in the large scale limit such a description may be effective. Such an observation then places an important constraint on the size of the Hilbert sub-space $\mathcal{H}_{[1...N']}$ containing φ: if the inside of a detector is modestly assumed to occupy a volume of one cubic metre, $\mathcal{H}_{[1...N']}$ must be sufficiently large such that a causal set description can give rise to one cubic metre's 'worth' of volume.

In fact going further, and following the estimation of Section 3.2 for the minimum number of degrees of freedom of the quantum Universe (in which continuous spatial resolutions were assumed valid down to at least distance scales of the order of the Planck length, l_P), if the minimum number n_{m^3} of discrete 'points' in a cubic metre is given by

$$n_{m^3} = \left(\frac{1}{l_P}\right)^3 \sim \left(10^{+35}\right)^3 = 10^{105} \tag{7.44}$$

and if each of these n_{m^3} points is associated with just a single, two-dimensional qubit degree of freedom, then the dimension of $\mathcal{H}_{[1...N']}$ must be at least $2^{10^{105}}$.

In actual fact, the model of field theory presented in the following will be simplified by restricting the discussion to dynamics occurring in a one dimensional volume. Thus, from this viewpoint it is assumed that if an external scientist considers φ, she would conclude that it is of emergent length L because it is contained in a one-dimensional detector with a 'known' internal volume of L. This simplification will be justified later by restricting the field theory analysis to one dimension of momentum, and noting that it is hoped that an extension to D dimensional space could be achieved merely by using a formalism of greater complexity. For now, however, note that by using such an assumption, the number of spatial points discussed is given by

$$n_{m^1} = 10^{35} \tag{7.45}$$

where this value is still sufficiently enormous such that a causal set approximation of continuous space is expected to be valid.

The physicist factor will also be able to make statements regarding the duration of the sub-state $|\varphi_{in}\rangle$. For example, by recording various changes in her surroundings as the Universe developed from Ψ_n to Ψ_{n+1} to... to $\Psi_{n+n'}$, she may be able to argue that a certain quantity of (emergent) time elapsed between the preparation of φ_{in} in Ψ_n and the measurement of φ_{out} as part of $\Psi_{n+n'}$. In fact, such a determination is vital: measures of time are necessary for many quantum field calculations to be performed and probability amplitudes to be evaluated, because these are generally dependent on the extent of dynamical unitary evolutions.

It is possible to estimate a value for the length of time perceived by the physicist between Ψ_n and $\Psi_{n+n'}$ by again appealing to what actually occurs in laboratory collider

experiments. If the colliding particles are assumed to be travelling at the speed of light, c, and are assumed to interact only when in the cubic metre enclosed inside the detector, then the physicist could conclude that they interact for a duration of $(1/c) \sim 10^{-8}$ seconds. In other words, if the interaction is taken to endure between jumps n and $n + n'$, then this period will last approximately 10^{-8} seconds according to the clock of the emergent observer who had defined c and the metre (and hence also the size of the detector).

Moreover, if it is also assumed that physicists are able to resolve a continuous temporal parameter down to at least durations of the order of the Planck time t_P, then an extrapolation indicates that the time taken for the Universe to jump discretely from a state Ψ_n to the next state Ψ_{n+1} could not be greater than this value. So, from the external point of view of the emergent observer, the number N_J of jumps during which the Universe will possess a factor φ_{in} is given by

$$N_J \equiv (n + n' - 1) - n \geq \left(\frac{1}{ct_P}\right) \sim \left(\frac{1}{(10^8)(10^{-43})}\right) = 10^{35} \tag{7.46}$$

which may again be expected to be large enough to generate the type of causal set structures required.

Attention may now be turned to the actual field theory. An immediate question is: given that physical particle states are conventionally taken to be governed by annihilation and creation operators, and also the assumption of the fundamental, underlying pregeometric structure advocated in this thesis, then is there a mechanism by which these conventional operators could emerge from a consideration of their pregeometric counterparts? In other words, given a pregeometric creation operator of the form $\hat{\alpha}_r^\dagger$, how might it be possible to relate this to the operator $\hat{c}^\dagger(p, \kappa)$ that creates a physical electron with momentum p and spin κ, or to the operator $\hat{d}^\dagger(p', \kappa')$ that creates a physical positron with momentum p' and spin κ'? Complicated functions of such operators could then ultimately be used to relate the physical In state $|\varphi_{in}\rangle$ to the final Out state $|\varphi_{out}\rangle$, as is typically done in the Heisenberg picture approach to particle collision experiments.

To begin to answer this, recall that it is well known that momentum and position space variables may be related to one another in quantum theory by the use of Fourier transforms [80]. This relationship follows from their reciprocal dependence in the uncertainty principle, and is often exploited in conventional quantum field theory. So, if it is possible to discuss ladder operators in position space, it is equally possible to discuss ladder operators in momentum space by using Fourier methods.

Now, since an origin of spatial position has already be suggested for the quantum Universe, it might be possible to use this as a starting point in order to obtain ladder operators that are a function of momentum. Thus, it may be possible to employ Fourier transform methods to generate momentum space relations from the positional degrees of

freedom that emerge from the causal set structure. It is this type of procedure that is consequently investigated below.

However, because space is assumed to be discrete and finite in the paradigm envisaged in this thesis, the conventional Fourier transform must be replaced with a discretised version. This then places an additional constraint on the proposal: the standard results of the continuous theory must still emerge in the large scale limit if the model suggested is to accurately represent the observed physics. This issue will be discussed in due course.

Given that the model in hand has been restricted to a single spatial dimension, assume that a causal set analysis of the local operators in the sub-space $\mathcal{H}_{[1...N']}$ 'containing' the Dirac field concludes that the region of space that ultimately emerges from it is of length L. Moreover, assume that this length may be associated with a discrete, one-dimensional lattice consisting of $(2M+1)$ points, where M is very large, such that $(2M+1) \geq n_{m^1} = 10^{35}$. Finally, assume that $\mathcal{H}_{[1...N']}$ may be factorised into at least $(2M+1)$ subspaces, such that $N' \geq (2M+1) \geq n_{m^1}$. Then, it may be argued that each local operator acting in each of these factor sub-spaces of $\mathcal{H}_{[1...N']}$ could somehow map to one of these $(2M+1)$ possible 'positions' or 'sites', analogously to the mechanism presented in Section 5.7.5 for relating sub-operators to the vertices of a three dimensional lattice. Equivalently, each subregister of $\mathcal{H}_{[1...N']}$ may be asserted to somehow correspond to a certain one of the $(2M+1)$ sites.

Now, for the sake of simplicity, assume that $\mathcal{H}_{[1...N']}$ is a tensor product of just $(2M+1)$ sub-spaces (this assumption will be enlarged later). In this case, it might be possible to relate each local sub-operator (or subspace) in a one-to-one way to a unique position along the one-dimensional lattice consisting of $(2M+1)$ points. Thus overall, and from the approximate point of view of emergence, it might therefore be possible to imagine that each of the $(2M+1)$ subregisters (or local sub-operator acting in it) in $\mathcal{H}_{[1...N']}$ may in some sense be considered to exist at a definite location along this line of emergent length L.

(To illustrate the proposed perspective further, a slightly different analysis could instead be schematically discussed. Assume that the Hilbert sub-space $\mathcal{H}_{[1...N']}$ containing the particles is a tensor product of at least $(2M+1)$ subregisters, where M is very large, and that a causal set analysis concludes that the region of space that ultimately emerges from it (by considering the sub-states and sub-operators acting in it) is of length L. Then, given that the model has been restricted to a single spatial dimension, it may be argued from the approximate point of view of emergence that each subregister of $\mathcal{H}_{[1...N']}$ is 'responsible', somehow, for one of $(2M+1)$ possible 'positions' or 'sites'. Equivalently, the particular sub-operator that acts locally in a particular subregister of $\mathcal{H}_{[1...N']}$ may be asserted to correspond to a certain one of these $(2M+1)$ sites. Reversing this argument, then,

each local sub-operator (or subregister) may effectively be mapped to a unique position along a one-dimensional lattice consisting of $(2M+1)$ points, similarly to the mechanism of Section 5.7.5., and it is therefore possible to imagine that each subregister (or local sub-operator) in $\mathcal{H}_{[1...N']}$ may in some sense be considered to exist at a definite location when this lattice is associated with a continuous one-dimensional volume of length L.)

Note that the presented approach differs greatly from the work of Deutsch [77], in which qubit subregisters are instead actually embedded into a fixed and independently existing background space whose nature or origin is not further questioned or justified. Note also that the pregeometric ladder operators discussed in the previous section act locally in particular subregister spaces; this observation will be crucial in the following.

Overall, a vision is imagined in which each subregister of $\mathcal{H}_{[1...N']}$ is effectively placed at a unique site along a locus of $(2M+1)$ points that correspond to a spatial length of L in the continuum limit. Each sub-operator (for example, the pregeometric ladder operators) acting locally in one of these subregisters may therefore also be envisaged to act at a definite one of these $(2M+1)$ positions.

From the above, it is possible to define a length scale X as the average, effective distance between these subregisters. If the overall length arising out of $\mathcal{H}_{[1...N']}$ is taken to be L, and because there are precisely $2M$ intervals between $(2M+1)$ points, then

$$X = \left(\frac{L}{2M}\right) \tag{7.47}$$

noting that X is defined only by extrapolating backwards from the overall length experienced on the emergent scale and is meaningless on the actual pregeometric level.

Paraphrasing these ideas: from an emergent perspective the vision is of a one dimensional lattice of length L comprising of $(2M+1)$ points separated by a distance of X, but on the pregeometric level the $(2M+1)$ subregisters are still just factors of the overall Hilbert space and are not embedded into any sort of spatial background. As with the causal set discussions of Chapter 5, it is the overall network of relationships between operators and states that leads to the generation of effective concepts such as distance on the macroscopic scale.

Note that the above considerations immediately imply a splitting of the Hilbert space $\mathcal{H}_{[1...N']}$ containing φ, which may be denoted as

$$\begin{aligned} \mathcal{H}_{[1...N']} &= \mathcal{H}_{[\varphi_{(-M)}]} \otimes \mathcal{H}_{[\varphi_{(-M+1)}]} \otimes ... \otimes \mathcal{H}_{[\varphi_{(M)}]} \\ &= \prod_{R=-M}^{M} \otimes \mathcal{H}_{[\varphi_{(R)}]}. \end{aligned} \tag{7.48}$$

where R runs from $-M$ to M for reasons that will become apparent. Moreover, from this split a suitable relabelling of the subregisters may be performed in the obvious way

for visual convenience, such that from a given emergent perspective the $(-M)^{th}$ sub-space $\mathcal{H}_{[\varphi_{(-M)}]}$ of $\mathcal{H}_{[1...N']}$ may be thought of as being located at the 'left-most' position, whereas sub-operators acting locally in the $(+M)^{th}$ subregister $\mathcal{H}_{[\varphi_{(M)}]}$ of $\mathcal{H}_{[1...N']}$ may be thought of as being responsible for the 'right-most' position, with the remaining sub-spaces positioned at the obvious locations in between.

Now consider the components of the state in each of these $(2M+1)$ subregisters. The nature of this set of components may be thought of as defining a type of field across the one dimensional lattice. Moreover, concepts of information can be applied in this context, such that the information contained in the R^{th} factor space can be seen as analogous to the value of the field at spatial position R.

Assuming that this field is not trivially constant, the minimum dimension of each subregister $\mathcal{H}_{[\varphi_{(R)}]}$ is evidently two. Defining, then, the basis vectors for each subregister in the usual way as $|0\rangle$ and $|1\rangle$, it implies that at each of the locations it is possible to assign a 'value' for the field in terms of these bases. This information will play an important part in describing the physical properties of the particle system, as shown later.

However, matters are straight away seen as being considerably more involved than a simple two dimensional scenario permits. After all, a single Dirac field allows the existence of four distinct particle species (both spin-up and spin-down electrons and positrons), so each of the $(2M+1)$ factor sub-spaces of $\mathcal{H}_{[1...N']}$ will be required to contain information regarding four different types of particle at that site if such a field is ultimately to be modelled. Thus, in order to consistently describe all of these possibilities at least eight degrees of freedom will be required for each of the $(2M+1)$ factors of $\mathcal{H}_{[1...N']}$. These may in turn be grouped into four sets, each corresponding to a sub-space of $\mathcal{H}_{[\varphi_{(R)}]}$ relating to a given particle species.

The simplest possible model for the Dirac field may consequently be achieved by associating four qubits to each of the $(2M+1)$ factor spaces, one for each particle type. In this case

$$N' = 4(2M+1) = 4 \times 10^{35} \tag{7.49}$$

and it is possible to consider a further split of each of the $(2M+1)$ sub-spaces of $\mathcal{H}_{[1...N']}$ into the products of subregisters they comprise. That is, the sub-space $\mathcal{H}_{[\varphi_{(R)}]}$ may be written

$$\mathcal{H}_{[\varphi_{(R)}]} = \mathcal{H}_{R_{(1)}} \otimes \mathcal{H}_{R_{(2)}} \otimes \mathcal{H}_{R_{(3)}} \otimes \mathcal{H}_{R_{(4)}} \tag{7.50}$$

such that

$$\mathcal{H}_{[1...N']} = \prod_{R=-M}^{M} \prod_{z=1}^{4} \otimes \mathcal{H}_{R_{(z)}}. \tag{7.51}$$

Evidently, the dimension of the Hilbert space $\mathcal{H}_{[1...N']}$ is $2^{4\times 10^{35}}$.

As desired, each of the four qubits in each $\mathcal{H}_{[\varphi_{(R)}]}$ is to be associated with one particle type. So, by arbitrarily relabelling the subregisters for convenience, the sub-register $\mathcal{H}_{R_{(1)}}$ will be used in the following to contain information about the spin-down electron, whereas $\mathcal{H}_{R_{(2)}}$ is chosen to be the space containing information for a spin-up electron, whilst $\mathcal{H}_{R_{(3)}}$ regards the spin-down positron, and $\mathcal{H}_{R_{(4)}}$ is the space to be used for the construction of the spin-up positron, for all $R = -M, ..., M$.

Moreover, by choosing an orthonormal $\mathcal{B}_{R_{(z)}}$ basis for $\mathcal{H}_{R_{(z)}}$ to be $\mathcal{B}_{R_{(z)}} \equiv \{|0\rangle_{R_{(z)}}, |1\rangle_{R_{(z)}}\}$ it is possible to specify the component of φ in a particular one of the $4(2M+1)$ sub-spaces of $\mathcal{H}_{[1...N']}$.

The pregeometric ladder operators discussed in the previous section may be employed to manipulate the components of the state $\varphi \in \mathcal{H}_{[1...N']}$ at various emergent positions.

For example, by defining the completely separable vector $|\underline{0}\rangle_{1...N'}$ as the 'vacuum' or 'ground' state in $\mathcal{H}_{[1...N']}$, the operation

$$\hat{\alpha}^{\dagger}_{R_{(z)}}|\underline{0}\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes ... \otimes |0\rangle_{R_{(z-1)}} \otimes |1\rangle_{R_{(z)}} \otimes |0\rangle_{R_{(z+1)}} \otimes ... \otimes |0\rangle_N \tag{7.52}$$

evidently changes the component of the state $\varphi = |\underline{0}\rangle$ in $\mathcal{H}_{R_{(z)}}$ from $|0\rangle_{R_{(z)}}$ to $|1\rangle_{R_{(z)}}$, whilst the operator $\hat{\alpha}_{R_{(z)}}$ reverses this 'occupation' of subregister[15] $\mathcal{H}_{R_{(z)}}$. Equivalently, it may be possible to imagine that the operation $\hat{\alpha}^{\dagger}_{R_{(z)}}|\underline{0}\rangle$ creates a particular 'pregeometric particle' at location $R_{(z)}$ from the vacuum, whereas the operator $\hat{\alpha}_{R_{(z)}}$ annihilates this result. So, in this case a component of the form $|1\rangle_{R_{(z)}}$ is being chosen to relate to the existence of a particular 'pregeometric particle' at a given position $R_{(z)}$, whereas the value $|0\rangle_{R_{(z)}}$ is implying its absence; the similarities to the type of binary logic discussed in quantum computation are obvious.

Overall, it is the information contained in the individual factor spaces that will be used to control the physics of the situation.

Of course,

$$\begin{aligned} \hat{\alpha}^{\dagger}_{R_{(z)}}\hat{\alpha}^{\dagger}_{R_{(z)}}|\underline{0}\rangle &= 0 \\ \hat{\alpha}_{R_{(z)}}|\underline{0}\rangle &= 0 \end{aligned} \tag{7.53}$$

as expected.

The introduction of ladder operators to the current analysis is actually more fundamental than the above simple manipulation of the components of the state seems to suggest. After all, recall that the second quantisation procedure of conventional field theory involves re-writing the field as a field operator, before this is then typically expressed in terms of annihilation and creation operators. Thus in the present case, it is expected

[15] With the notation of (7.52) defined 'cyclically', such that $R_{(0)} \equiv (R-1)_{(4)}$ and $R_{(5)} \equiv (R+1)_{(1)}$.

that pregeometric field operators acting across the space of interest could ultimately be described in terms of the pregeometric ladder operators.

The overall aim is therefore to investigate how such a 'pregeometric field' could relate to the physical particle field used in conventional quantum theory. In particular, it is considered how 'real' Dirac particles in momentum space might emerge from the simple logic of the pregeometric framework. The issue effectively becomes one of relating the momentum space ladder operators of familiar physics to the pregeometric annihilation and creation operators that act at definite positions along the emergent spatial lattice. Concepts normally expressed in conventional particle physics as functions of momentum space ladder operators, such as momentum space Hamiltonian or field operators, may then be recovered from the underlying pregeometric framework by substituting the usual momentum space annihilation and creation operators for their pregeometric definitions, thereby re-writing the Hamiltonian or field in terms of these pregeometric ladder operators.

In fact, it is in principle possible to conjecture many different ways of bridging the gap between the pregeometric picture and emergent momentum space. However, due to the conventional relationship that exists between momentum and position in terms of Fourier analysis, it is this type of method that presents itself as a natural candidate to provide such a mechanism here. It is this that is now proposed.

Consider the standard Fourier Transform [81] of an arbitrary function $f(\tau)$ with the variable ω

$$f(\tau) = \int_{-\infty}^{\infty} [F(\omega) \exp(i\tau\omega)] \; d\omega \tag{7.54}$$

where $F(\omega)$ is an amplitude of value

$$F(\omega) = \left(\frac{1}{2\pi}\right) \int_{-\infty}^{\infty} [f(\tau) \exp(-i\tau\omega)] \; d\tau. \tag{7.55}$$

The present goal is to write the ladder operators in momentum space as a Fourier transform of terms in pregeometric space. Thus, by temporarily introducing the continuous position variable x, then a 'first guess' for the form of an annihilation operator $\hat{a}(p, \kappa)$ of known spin κ may initially be given by

$$\hat{a}(p, \kappa) = \int_{-\infty}^{\infty} [F(x) \exp(ipx)] dx \tag{7.56}$$

with modifications required as follows:

- Firstly, the parameter x must be discretised if it is to fit into the proposed paradigm, because space in the quantum Universe is assumed ultimately to be non-continuous. Assuming from above that the minimum spatial resolution of the volume is X, then x is given by

$$x = mX \quad , \quad m = 0, \pm 1, \pm 2, ... \tag{7.57}$$

and similarly

$$F(x) = F(mX) = F_m \tag{7.58}$$

such that $\hat{a}(p, \kappa)$ is given by a sum of discrete terms instead of as a continuous integral.

- Next, the limits need to be constrained. Since the method is being used to describe a finite model (i.e. a part of a finite dimensional Quantum Universe) of length L with a finite number of possible positions, the Fourier sum must also be finite. Specifically, since the integral is normally performed across all space, the sum in this case must be taken over all the positions for which the system is defined. This is hence bounded by the number of 'sites' contained in $\mathcal{H}_{[1...N']}$, namely $(2M + 1)$. So, by preserving symmetry around the origin, the minimum and maximum positions are given by

$$x_{\min} = -L/2 = (-M)(X) \quad , \quad x_{\max} = +L/2 = (+M)(X) \tag{7.59}$$

  from which it follows that $-M \leq m \leq M$ as expected. Clearly, the index m is equivalent to the index R used above to label the $(2M + 1)$ factors. Moreover, for $M \gg 1$ it follows that $x_{\max} \gg 1$ and $x_{\min} \ll -1$, such that the infinite integral is well approximated by the sum.

- Finally, the amplitudes $F_m = F_R$ are hoped to be functions of the pregeometric ladder operators, since this is after all the point of the current work. Taking proportionality here gives

$$F_R = A_p \hat{\alpha}(R) \tag{7.60}$$

  where A_p is a type of 'form function' to be investigated, and the pregeometric annihilation operator has been temporarily written as $\hat{\alpha}(R)$ to indicate that it is a function of R.

Overall, $\hat{a}(p, \kappa)$ becomes of the form

$$\hat{a}(p, \kappa) = \sum_{R=-M}^{M} A_p \hat{\alpha}(R) \exp(iRpX) . \tag{7.61}$$

The argument of the exponential in equation (7.61) introduces a periodicity into the analysis; specifically,

$$(\exp(ipX))^R = (\exp(i[pX \pm 2j\pi]))^R \quad , \quad j = 0, 1, 2... \tag{7.62}$$

Moreover, defining the resulting maximum positive momentum as $p_{\max}$, and assuming symmetry about the lowest value $p = 0$ such that $|p| = |-p|$, it implies that $p_{\max}$ is in practice bounded by the half-period

$$|p_{\max}| X = \pi \tag{7.63}$$

so $p_{\max} = \pi/X$. Note that this relationship indicates how the largest momentum is related to the reciprocal of the smallest spatial resolution, as analogous to the uncertainty principle. Importantly, observe also that since $M \gg 1$, it follows that $X \ll 1$, so $p_{\max} \gg 1$ as required.

The energy E_p of the mode of momentum p may now be defined in the conventional way. It follows, then, that this too is bounded, such that

$$(E_p)_{\max} \equiv \left(p_{\max}^2 c^2 + \mu^2 c^4\right)^{1/2} \tag{7.64}$$

where μ is the particle's mass and c is the speed of light. Again, it is remarked that $(E_p)_{\max}$ is clearly very large.

The operator $\hat{\alpha}(R)$ of equation (7.61) is assumed to be one of four possible types because it could act on one of four possible qubits for each R. Thus, equation (7.61) must be refined further to account for the four different sorts of particle that could be annihilated by the generic operator $\hat{a}(p, \kappa)$. If $\hat{c}(p, \kappa)$ is defined as the annihilation operator for electrons of momentum p and spin κ, for $\kappa = 1, 2$ where $\kappa = 1$ represents spin-down and $\kappa = 2$ represents spin-up, and if $\hat{d}(p, \kappa)$ conversely annihilates positrons of momentum p and spin κ, then from the discussion following (7.51) and the relation (7.61) it is evident that

$$\hat{c}(p, \kappa) = \sum_{R=-M}^{M} A_p \hat{\alpha}_{4(M+R)+\kappa} \exp\left(iRpX\right) \tag{7.65}$$

and

$$\hat{d}(p, \kappa) = \sum_{R=-M}^{M} A_p \hat{\alpha}_{4(M+R)+2+\kappa} \exp\left(iRpX\right). \tag{7.66}$$

The corresponding creation operators may be defined in an analogous way as the Hermitian conjugates of the annihilation operators; viz.

$$\hat{c}^\dagger(p, \kappa) = \sum_{R=-M}^{M} A_p^* \hat{\alpha}_{4(M+R)+\kappa}^\dagger \exp\left(-iRpX\right) \tag{7.67}$$

and

$$\hat{d}^\dagger(p, \kappa) = \sum_{R=-M}^{M} A_p^* \hat{\alpha}_{4(M+R)+2+\kappa}^\dagger \exp\left(-iRpX\right). \tag{7.68}$$

For the conclusions defined in (7.65)-(7.68) to be accepted as physically valid, they must be able to reproduce the standard results obtained for conventional fermionic ladder operators. For example, the anti-commutation relations of momentum space annihilation and creation operators must be obeyed, such that all the anti-commutators vanish apart from the results $\{\hat{c}(p, \kappa), \hat{c}^\dagger(p', \kappa')\}$ and $\{\hat{d}(p, \kappa), \hat{d}^\dagger(p', \kappa')\}$ which should give

$$\{\hat{c}(p, \kappa), \hat{c}^\dagger(p', \kappa')\} = \{\hat{d}(p, \kappa), \hat{d}^\dagger(p', \kappa')\} = \delta_{pp'}\delta_{\kappa\kappa'}. \tag{7.69}$$

The vanishing terms are clearly satisfied: for the results defined in (7.65)-(7.68)

$$\begin{aligned}
\{\hat{c}(p,\kappa),\hat{c}(p',\kappa')\} &= \{\hat{c}^{\dagger}(p,\kappa),\hat{c}^{\dagger}(p',\kappa')\}=0 \\
\{\hat{d}(p,\kappa),\hat{d}(p',\kappa')\} &= \{\hat{d}^{\dagger}(p,\kappa),\hat{d}^{\dagger}(p',\kappa')\}=0 \\
\{\hat{c}(p,\kappa),\hat{d}(p',\kappa')\} &= \{\hat{c}(p,\kappa),\hat{d}^{\dagger}(p',\kappa')\}=0 \\
\{\hat{c}^{\dagger}(p,\kappa),\hat{d}(p',\kappa')\} &= \{\hat{c}^{\dagger}(p,\kappa),\hat{d}^{\dagger}(p',\kappa')\}=0
\end{aligned} \tag{7.70}$$

which follow immediately from the relations (7.29) for the pregeometric ladder operators.

Considering instead the relations $\{\hat{c}(p,\kappa),\hat{c}^{\dagger}(p',\kappa')\}$ and $\{\hat{d}(p,\kappa),\hat{d}^{\dagger}(p',\kappa')\}$, gives

$$\begin{aligned}
\{\hat{c}(p,\kappa),\hat{c}^{\dagger}(p',\kappa')\} &= \sum_{R=-M}^{M} A_p\hat{\alpha}_{4(M+R)+\kappa}\exp\left(iRpX\right)\sum_{S=-M}^{M} A^{*}_{p'}\hat{\alpha}^{\dagger}_{4(M+S)+\kappa'}\exp\left(-iSp'X\right) \\
&\quad+\sum_{S=-M}^{M} A^{*}_{p'}\hat{\alpha}^{\dagger}_{4(M+S)+\kappa'}\exp\left(-iSp'X\right)\sum_{R=-M}^{M} A_p\hat{\alpha}_{4(M+R)+\kappa}\exp\left(iRpX\right) \\
&= \sum_{R=-M}^{M}\sum_{S=-M}^{M} A_pA^{*}_{p'}\exp\left(i(Rp-Sp')X\right)\left[\begin{array}{c}\hat{\alpha}_{4(M+R)+\kappa}\hat{\alpha}^{\dagger}_{4(M+S)+\kappa'} \\ +\hat{\alpha}^{\dagger}_{4(M+S)+\kappa'}\hat{\alpha}_{4(M+R)+\kappa}\end{array}\right] \\
&= \sum_{R=-M}^{M}\sum_{S=-M}^{M} A_pA^{*}_{p'}\{\hat{\alpha}_{4(M+R)+\kappa},\hat{\alpha}^{\dagger}_{4(M+S)+\kappa'}\}\exp\left(i(Rp-Sp')X\right)
\end{aligned} \tag{7.71}$$

So

$$\begin{aligned}
\{\hat{c}(p,\kappa),\hat{c}^{\dagger}(p',\kappa')\} &= \sum_{R=-M}^{M}\sum_{S=-M}^{M} A_pA^{*}_{p'}\delta_{RS}\delta_{\kappa\kappa'}\exp\left(i(Rp-Sp')X\right) \\
&= \sum_{R=-M}^{M} A_pA^{*}_{p'}\delta_{\kappa\kappa'}\exp\left(iR(p-p')X\right)
\end{aligned} \tag{7.72}$$

with a similar result for $\{\hat{d}(p,\kappa),\hat{d}^{\dagger}(p',\kappa')\}$.

Now, consider the Fourier expansion of the continuous space Dirac delta function $\delta(p)$ of period 2∞, defined in the usual way as [81][82]

$$\delta(p)\equiv\left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty}(1)\exp\left(-ipx\right)dx. \tag{7.73}$$

By making the same type of approximation as before, i.e. associating the function with a large, but finite, period of at least $2\left(\frac{\pi}{X}\right)$ defined over a discretised background space of emergent length $L=2MX$, the expression (7.73) may be truncated, and re-written as

$$\delta_p\equiv\left(\frac{1}{2\pi}\right)\sum_{R=-M}^{M} C_R\exp\left(-iRpX\right). \tag{7.74}$$

Multiplying both sides by $e^{iR'pX}$ and integrating over all momentum $|p| \leq p_{\max}$ gives

$$\begin{aligned}\int_{-\pi/X}^{\pi/X} \delta_p \exp\left(iR'pX\right)\ dp &\equiv \left(\frac{1}{2\pi}\right) \sum_{R=-M}^{M} C_R \int_{-\pi/X}^{\pi/X} \exp\left(-i(R-R')pX\right)\ dp \\ e^0 &= \left(\frac{1}{2\pi}\right) \sum_{R=-M}^{M} C_R \delta_{RR'} \left(\frac{2\pi}{X}\right) \\ 1 &= C_{R'} \left(\frac{1}{X}\right) \end{aligned} \tag{7.75}$$

so $C_R = X$ such that $\sum_{R=-M}^{M} \exp\left(-iRpX\right) = \left(\frac{2\pi}{X}\right)\delta_p$ from (7.74).

Clearly then

$$\sum_{R=-M}^{M} \exp\left(-iR(p-p')X\right) = \left(\frac{2\pi}{X}\right)\delta_{pp'}. \tag{7.76}$$

and substituting this into (7.72) gives

$$\begin{aligned}\{\hat{c}(p,\kappa), \hat{c}^\dagger(p',\kappa')\} &= \sum_{R=-M}^{M} A_p A_{p'}^* \delta_{\kappa\kappa'} \left(\frac{2\pi}{X}\right)\delta_{p'p} \\ &= \left(\frac{2\pi}{X}\right)(A_p A_{p'}^*)\delta_{\kappa\kappa'}\delta_{pp'} \end{aligned} \tag{7.77}$$

which is equal in form to the usual anti-commutation algebra (7.69). So, from a comparison of (7.69) and (7.77) it follows that the expressions are equal if $A_p \in \mathbb{R}$ may be defined as

$$A_p = \sqrt{\frac{X}{2\pi}}. \tag{7.78}$$

Collecting these solutions defines the momentum space annihilation and creation operators in terms of pregeometric ladder operators. For electrons these are

$$\begin{aligned}\hat{c}(p,\kappa) &= \sqrt{\frac{X}{2\pi}} \sum_{R=-M}^{M} \hat{\alpha}_{4(M+R)+\kappa} \exp\left(iRpX\right) \\ \hat{c}^\dagger(p,\kappa) &= \sqrt{\frac{X}{2\pi}} \sum_{R=-M}^{M} \hat{\alpha}^\dagger_{4(M+R)+\kappa} \exp\left(-iRpX\right) \end{aligned} \tag{7.79}$$

whilst for positrons the results are

$$\begin{aligned}\hat{d}(p,\kappa) &= \sqrt{\frac{X}{2\pi}} \sum_{R=-M}^{M} \hat{\alpha}_{4(M+R)+2+\kappa} \exp\left(iRpX\right) \\ \hat{d}^\dagger(p,\kappa) &= \sqrt{\frac{X}{2\pi}} \sum_{R=-M}^{M} \hat{\alpha}^\dagger_{4(M+R)+2+\kappa} \exp\left(-iRpX\right). \end{aligned} \tag{7.80}$$

These expressions may be substituted into the standard equations of fermionic field theory, to give, for example, the momentum space field operators. More importantly, perhaps, they may also be used to construct the actual observables familiar to the conventional theory, as discussed next.

7.2.1 The Hamiltonian

The results of above can be used to formulate a version of the Hamiltonian in terms of pregeometric ladder operators. Such a formulation is important, as is the derivation of the momentum and charge operators discussed later, because it is operators like these that form the basis for actual observables in physics.

Of course, such operators require careful interpretation from the perspective of the quantum Universe paradigm proposed in this thesis. Specifically, the operators of below are assumed to be associated with the part of the operator $\hat{\Sigma}_{n+n'}$ (which is used to develop the entire Universe) that appears to test the sub-state $|\varphi_{in}\rangle$ of $\Psi_{n+n'-1}$. Thus, the Hamiltonian, momentum and charge operators discussed below are expected to ultimately be represented by different factors of different possible tests $\hat{\Sigma}_{n+n'}$.

Moreover, given that the quantum Universe is taken to be completely self-contained and autonomous, the operators it self-referentially chooses must be very carefully controlled if emergent endo-observers are to gain the impression that they can detect electrons and positrons in the medley of ways familiar to physicists. This again is emphatic of the point that a quantum state cannot really be said to exist independently of the tests used to observe it, and consequently that different choices of test (e.g. energy or charge) lead to different 'experiences' of physical reality by emergent endo-observers.

Consider the conventional Hamiltonian operator $\hat{H}$ for the free-field theory of spin-$\frac{1}{2}$ fermions, defined [59] in three dimensional momentum space as

$$\hat{H} = \sum_{\kappa} \int_{-\infty}^{\infty} E_p \left[\hat{c}^{\dagger}(\mathbf{p},\kappa)\hat{c}(\mathbf{p},\kappa) + \hat{d}^{\dagger}(\mathbf{p},\kappa)\hat{d}(\mathbf{p},\kappa) \right] \, d^3p \tag{7.81}$$

noting that this equation has been derived in Appendix B for completeness. Here, E_p is the energy of the particle, $\mathbf{p}$ is its momentum 3-vector, and the sum is over both spin states $\kappa = 1, 2$.

Now, noting that in the present chapter the momentum $\mathbf{p}$ has been restricted to a one dimensional variable p, it is possible to rewrite the conventional Hamiltonian in terms of the pregeometric operators defined in (7.79) and (7.80). So,

$$\begin{aligned} \hat{H} \;\; = \;\; & \sum_{\kappa} \int_{-\pi/X}^{\pi/X} E_p \sum_{R=-M}^{M} \sum_{S=-M}^{M} \left(\frac{X}{2\pi} \right) \\ & \times \left[\begin{array}{c} \hat{\alpha}^{\dagger}_{4(M+R)+\kappa} \exp\left(-iRpX\right) \hat{\alpha}_{4(M+S)+\kappa} \exp\left(iSpX\right) \\ +\hat{\alpha}^{\dagger}_{4(M+R)+2+\kappa} \exp\left(-iRpX\right) \hat{\alpha}_{4(M+S)+2+\kappa} \exp\left(iSpX\right) \end{array} \right] \, dp \end{aligned} \tag{7.82}$$

with the obvious imposition that the integral limits $\pm\infty$ have been constrained to $\pm\pi/X$

as before. Recalling that the energy is defined as $E_p = (p^2c^2 + \mu^2c^4)^{1/2}$, it follows that

$$\begin{aligned}\hat{H} &= \left(\frac{X}{2\pi}\right)\sum_{\kappa}\sum_{R=-M}^{M}\sum_{S=-M}^{M}\int_{-\pi/X}^{\pi/X}(p^2c^2+\mu^2c^4)^{1/2}e^{i(S-R)pX} \\ &\times\begin{bmatrix}\hat{\alpha}^{\dagger}_{4(M+R)+\kappa}\hat{\alpha}_{4(M+S)+\kappa} \\ +\hat{\alpha}^{\dagger}_{4(M+R)+2+\kappa}\hat{\alpha}_{4(M+S)+2+\kappa}\end{bmatrix}\,dp.\end{aligned} \tag{7.83}$$

Erdélyi *et al* [83] list no known explicit solution for this integral, suggesting that the Hamiltonian may only be evaluated as a numerical approximation. Whilst on the surface this may appear unsatisfactory, it does evidence the fact that the Hamiltonian is a highly non-trivial function of pregeometric variables, as might perhaps be expected for an operator defined in the emergent limit.

7.2.2 The Momentum Operator

Just as for the Hamiltonian, it is also possible to write the emergent momentum operator in terms of the pregeometric ladder operators.

Recall the conventional momentum operator $\hat{\mathbf{P}}$, defined [59] as

$$\hat{\mathbf{P}} = \sum_{\kappa}\int_{-\infty}^{\infty}\mathbf{p}\left[\hat{c}^{\dagger}(\mathbf{p},\kappa)\hat{c}(\mathbf{p},\kappa)+\hat{d}^{\dagger}(\mathbf{p},\kappa)\hat{d}(\mathbf{p},\kappa)\right]\,d^3p \tag{7.84}$$

and derived also in Appendix B. By restricting again the analysis to one finite dimension, and substituting in the relations (7.79) and (7.80), the momentum operator becomes

$$\hat{\mathbf{P}} = \left(\frac{X}{2\pi}\right)\sum_{\kappa}\sum_{R=-M}^{M}\sum_{S=-M}^{M}\begin{bmatrix}\hat{\alpha}^{\dagger}_{4(M+R)+\kappa}\hat{\alpha}_{4(M+S)+\kappa} \\ +\hat{\alpha}^{\dagger}_{4(M+R)+2+\kappa}\hat{\alpha}_{4(M+S)+2+\kappa}\end{bmatrix}\int_{-\pi/X}^{\pi/X}pe^{i(S-R)pX}\,dp. \tag{7.85}$$

In order to solve the integral, it will prove useful to separate this last expression into a 'diagonal' part for which $R = S$ and an off-diagonal part for which $R \neq S$. Thus

$$\hat{\mathbf{P}} = \hat{\mathbf{P}}_D + \hat{\mathbf{P}}_{OD} \tag{7.86}$$

where

$$\hat{\mathbf{P}}_D = \left(\frac{X}{2\pi}\right)\sum_{\kappa}\sum_{R=-M}^{M}\begin{bmatrix}\hat{\alpha}^{\dagger}_{4(M+R)+\kappa}\hat{\alpha}_{4(M+R)+\kappa} \\ +\hat{\alpha}^{\dagger}_{4(M+R)+2+\kappa}\hat{\alpha}_{4(M+R)+2+\kappa}\end{bmatrix}\int_{-\pi/X}^{\pi/X}p\,dp \tag{7.87}$$

and

$$\begin{aligned}\hat{\mathbf{P}}_{OD} &= \left(\frac{X}{2\pi}\right)\sum_{\kappa}\sum_{R=-M}^{M}\sum_{S=-M}^{M}(1-\delta_{RS}) \\ &\times\begin{bmatrix}\hat{\alpha}^{\dagger}_{4(M+R)+\kappa}\hat{\alpha}_{4(M+S)+\kappa} \\ +\hat{\alpha}^{\dagger}_{4(M+R)+2+\kappa}\hat{\alpha}_{4(M+S)+2+\kappa}\end{bmatrix}\int_{-\pi/X}^{\pi/X}pe^{i(S-R)pX}\,dp.\end{aligned} \tag{7.88}$$

The integral in (7.87) gives

$$\int_{-\pi/X}^{\pi/X} p\ dp = \left(\frac{\pi^2}{2X^2} - \frac{\pi^2}{2X^2}\right) = 0 \tag{7.89}$$

such that

$$\hat{\mathbf{P}}_D = 0. \tag{7.90}$$

Turning now to the off-diagonal case, and defining the integral in (7.88) to be $\Lambda_{R\neq S}$, it follows that

$$\begin{aligned}
\Lambda_{R\neq S} &= \int_{-\pi/X}^{\pi/X} p e^{i(S-R)pX}\ dp \quad , \quad R \neq S \qquad (7.91)\\
&= \left(\frac{1}{(S-R)^2X^2}\right)\left\{\begin{array}{c} i\pi(R-S)\left\{e^{i\pi(S-R)} + e^{-i\pi(S-R)}\right\} \\ +\left\{e^{i\pi(S-R)} - e^{-i\pi(S-R)}\right\} \end{array}\right\}\\
&= \left(\frac{2i}{(S-R)^2X^2}\right)\left\{\sin(\pi(S-R)) + \pi(R-S)\cos(\pi(S-R))\right\}.
\end{aligned}$$

However, because $R, S \in \mathbb{Z}$ and $R \neq S$

$$\sin(\pi(S-R)) = 0 \quad , \quad \forall R, S \tag{7.92}$$

and

$$\cos(\pi(S-R)) = (-1)^{(S-R)} \quad , \quad \forall R, S. \tag{7.93}$$

So finally

$$\begin{aligned}
\hat{\mathbf{P}}_{OD} &= \left(\frac{-i}{X}\right)\sum_{\kappa}\sum_{R=-M}^{M}\sum_{S=-M}^{M}(1-\delta_{RS})\left(\frac{(-1)^{(S-R)}}{(S-R)}\right) \qquad (7.94)\\
&\quad \times\left[\begin{array}{c} \hat{\alpha}^{\dagger}_{4(M+R)+\kappa}\hat{\alpha}_{4(M+S)+\kappa} \\ +\hat{\alpha}^{\dagger}_{4(M+R)+2+\kappa}\hat{\alpha}_{4(M+S)+2+\kappa} \end{array}\right].
\end{aligned}$$

Overall,

$$\begin{aligned}
\hat{\mathbf{P}} &= \left(\frac{1}{X}\right)\sum_{\kappa}\sum_{R=-M}^{M}\sum_{S=-M}^{M}\left[\begin{array}{c} \hat{\alpha}^{\dagger}_{4(M+R)+\kappa}\hat{\alpha}_{4(M+S)+\kappa} \\ +\hat{\alpha}^{\dagger}_{4(M+R)+2+\kappa}\hat{\alpha}_{4(M+S)+2+\kappa} \end{array}\right] \qquad (7.95)\\
&\quad \times\left(\frac{-i(-1)^{(S-R)}}{(S-R)}\right)(1-\delta_{RS}).
\end{aligned}$$

Again, note that this emergent construct is also a complicated function of basic pre-geometric logic operators.

7.2.3 Charge

To conclude this section, it is shown how the conventional charge operator may also be written as a function of pregeometric ladder operators.

Recall the operator $\hat{Q}$, defined as

$$\hat{Q} = q\int_{-\infty}^{\infty} d^3p \sum_{\kappa=1}^{2} \left[\hat{c}^\dagger(\mathbf{p},\kappa)\hat{c}(\mathbf{p},\kappa) - \hat{d}^\dagger(\mathbf{p},\kappa)\hat{d}(\mathbf{p},\kappa)\right] \tag{7.96}$$

and again derived in Appendix B. Here, q is an arbitrary scalar constant, but of course for 'real' electrons and positrons it is known to have the value $q = -e \sim -1.602 \times 10^{-19}$ [84].

By once again considering only a finite, one dimensional volume, and substituting in the relations (7.79) and (7.80), it follows that $\hat{Q}$ becomes

$$\hat{Q} = \left(\frac{X}{2\pi}\right) q \sum_{\kappa} \sum_{R=-M}^{M} \sum_{S=-M}^{M} \begin{bmatrix} \hat{\alpha}^\dagger_{4(M+R)+\kappa}\hat{\alpha}_{4(M+S)+\kappa} \\ -\hat{\alpha}^\dagger_{4(M+R)+2+\kappa}\hat{\alpha}_{4(M+S)+2+\kappa} \end{bmatrix} \int_{-\pi/X}^{\pi/X} e^{i(S-R)pX}\, dp. \tag{7.97}$$

Solving the integral gives

$$\begin{aligned} \hat{Q} &= \left(\frac{Xq}{2\pi}\right) \sum_{\kappa} \sum_{R=-M}^{M} \sum_{S=-M}^{M} \begin{bmatrix} \hat{\alpha}^\dagger_{4(M+R)+\kappa}\hat{\alpha}_{4(M+S)+\kappa} \\ -\hat{\alpha}^\dagger_{4(M+R)+2+\kappa}\hat{\alpha}_{4(M+S)+2+\kappa} \end{bmatrix} \\ &\quad \times \left(\frac{1}{i(S-R)X}\right)(e^{i(S-R)\pi} - e^{-i(S-R)\pi}) \\ &= \left(\frac{q}{\pi}\right) \sum_{\kappa} \sum_{R=-M}^{M} \sum_{S=-M}^{M} \begin{bmatrix} \hat{\alpha}^\dagger_{4(M+R)+\kappa}\hat{\alpha}_{4(M+S)+\kappa} \\ -\hat{\alpha}^\dagger_{4(M+R)+2+\kappa}\hat{\alpha}_{4(M+S)+2+\kappa} \end{bmatrix} \left(\frac{\sin((S-R)\pi)}{(S-R)}\right). \end{aligned} \tag{7.98}$$

Now, when $(S-R) \neq 0$, the relation

$$\left(\frac{\sin((S-R)\pi)}{(S-R)}\right) = 0 \tag{7.99}$$

holds for all S, R. However, when $(S-R) = 0$ it becomes

$$\lim_{(S-R)=0} \left(\frac{\sin((S-R)\pi)}{(S-R)}\right) = \pi \tag{7.100}$$

as may be readily verified by Maclaurin expansion.

Clearly, the expression only has non-zero values when $R = S$. So, substituting this result into (7.98) gives the final expression for the charge operator

$$\hat{Q} = q \sum_{\kappa} \sum_{R=-M}^{M} \begin{bmatrix} \hat{\alpha}^\dagger_{4(M+R)+\kappa}\hat{\alpha}_{4(M+R)+\kappa} \\ -\hat{\alpha}^\dagger_{4(M+R)+2+\kappa}\hat{\alpha}_{4(M+R)+2+\kappa} \end{bmatrix}. \tag{7.101}$$

7.3 Field Theory and CNOT

Instead of the pregeometric ladder operators, it is both possible and useful to write the expressions for the above Hamiltonian, momentum and charge in terms of a set of more conventional quantum operators acting on the qubits. One such set comprises of extended local unitary operators and the extended CNOT operator defined[16] as

$$\begin{aligned} \hat{C}_{(a,b)} &= \hat{P}_a^0 \otimes \hat{\sigma}_b^0 + \hat{P}_a^1 \otimes \hat{\sigma}_b^1 \\ &= \hat{P}_a^0 \otimes (\hat{P}_b^0 + \hat{P}_b^1) + \hat{P}_a^1 \otimes (\hat{Q}_a + \hat{Q}_a^\dagger) \end{aligned} \tag{7.102}$$

where the tensor product of identity operators $\hat{I}_c$, $c \neq a, b$ and $c = 1, 2, ..., N'$, in the extension has been omitted for brevity. Of course, this definition could easily be extended even further, such that the tensor product is also taken with the identity operators acting in every other sub-space of the Universe's total Hilbert space $\mathcal{H}_{[1...N]} \supset \mathcal{H}_{[1...N']}$, but this is an unnecessary amendment here.

As has been discussed previously, $\hat{C}_{(a,b)}$ acts on every qubit in the Hilbert space $\mathcal{H}_{[1...N']}$, but only changes the value of the qubit sub-state in sub-space $\mathcal{H}_b$ depending on the value of the qubit sub-state in sub-space $\mathcal{H}_a$.

The motivation for choosing this particular set of operators is two-fold. Firstly, such a possibility provides an immediate bridge between the work of this chapter and the discussions of quantum computation in Chapter 6. Secondly, and perhaps more importantly, this choice follows directly from the suggestion of Feynman [20] that all of physics could in principle originate from quantum computation, and then from the work of Barenco *et al* [67] that every qubit quantum computation may be achieved by the use of local unitary operations and the CNOT gate. Thus by amalgamating this second idea into the model proposed currently, the result is demonstrated that fermionic field theory in momentum space can be obtained from these standard quantum computational operators acting on qubits defined on the pregeometric level. Feynman's prediction is hence confirmed.

In order to convert the equations for the Hamiltonian, momentum and charge operators into this chosen set of operators, the goal would be to express the sums of products of pregeometric ladder operators present in their constructions in terms of local unitary operators and CNOT. By way of a demonstration of how this can be achieved, consider a typical such sum of products given by

$$\Pi_{RS} = \hat{\alpha}^\dagger_{4(M+R)+\kappa} \hat{\alpha}_{4(M+S)+\kappa} + \hat{\alpha}^\dagger_{4(M+R)+2+\kappa} \hat{\alpha}_{4(M+S)+2+\kappa}. \tag{7.103}$$

For convenience in this example, it is possible to restrict attention to just the first of these products. Furthermore, it is also possible to re-label the sub-registers featured, and

[16] Repeated here from equation (6.17).

define the product Π_{rs} as

$$\Pi_{rs} = \hat{\alpha}_r^\dagger \hat{\alpha}_s \tag{7.104}$$

where $r = 4(M+R)+\kappa$ and $s = 4(M+S)+\kappa$. Writing out the expression in full extended notation, this product becomes

$$\begin{aligned} \Pi_{rs} &= \left(\hat{\sigma}_1^3\hat{\sigma}_2^3...\hat{\sigma}_{r-1}^3\hat{T}_r^{10}\hat{I}_{r+1}...\hat{I}_N\right)\left(\hat{\sigma}_1^3\hat{\sigma}_2^3...\hat{\sigma}_{s-1}^3\hat{T}_s^{01}\hat{I}_{s+1}...\hat{I}_N\right) \\ &= \hat{I}_1\hat{I}_2...\hat{I}_{r-1}(\hat{T}_r^{10}\hat{\sigma}_r^3)\hat{\sigma}_{r+1}^3...\hat{\sigma}_{s-1}^3\hat{T}_s^{01}\hat{I}_{s+1}...\hat{I}_N \end{aligned} \tag{7.105}$$

using $\hat{\sigma}_t^0 \equiv \hat{I}_t$ with the usual $SU(2)$ product algebra (5.32), and assuming $r < s$ without loss of generality. So,

$$\begin{aligned} \Pi_{rs} &= (\hat{T}_r^{10}\hat{\sigma}_r^3)\hat{T}_s^{01}\hat{I}_1\hat{I}_2...\hat{I}_{r-1}\hat{\sigma}_{r+1}^3...\hat{\sigma}_{s-1}^3\hat{I}_{s+1}...\hat{I}_N \\ &= \hat{T}_r^{10}\hat{T}_s^{01}\hat{I}_1\hat{I}_2...\hat{I}_{r-1}\hat{\sigma}_{r+1}^3...\hat{\sigma}_{s-1}^3\hat{I}_{s+1}...\hat{I}_N \end{aligned} \tag{7.106}$$

using the assumed 'rearrangement' property of the tensor product and recalling the definition (6.10) that $\hat{\sigma}_r^3 = \hat{T}_r^{00} - \hat{T}_r^{11}$.

Now consider just the product of local transformation operators, $\hat{T}_r^{10}\hat{T}_s^{01}$. The question becomes: what combinations of local unitary operators and CNOT gates will give $\hat{T}_r^{10}\hat{T}_s^{01}$ as a result? To begin to answer this, consider the product of CNOT operators $\hat{C}_{(r,s)}\hat{C}_{(s,r)}$ given by

$$\begin{aligned} \hat{C}_{(r,s)}\hat{C}_{(s,r)} &= \left(\hat{P}_r^0\hat{P}_s^0 + \hat{P}_r^0\hat{P}_s^1 + \hat{P}_r^1\hat{Q}_s + \hat{P}_r^1\hat{Q}_s^\dagger\right) \\ &\quad \times \left(\hat{P}_r^0\hat{P}_s^0 + \hat{P}_r^1\hat{P}_s^0 + \hat{Q}_r\hat{P}_s^1 + \hat{Q}_r^\dagger\hat{P}_s^1\right) \end{aligned} \tag{7.107}$$

where tensor product symbols are omitted and the notations $\hat{T}^{00} \equiv \hat{P}^0$, $\hat{T}^{11} \equiv \hat{P}^1$, $\hat{T}^{01} \equiv \hat{Q}$ and $\hat{T}^{10} \equiv \hat{Q}^\dagger$ used in Section 6.1 have been adopted instead for clarity. By recalling the algebra of transformation operators (6.14), this becomes

$$\hat{C}_{(r,s)}\hat{C}_{(s,r)} = \hat{P}_r^0\hat{P}_s^0 + \hat{Q}_r\hat{P}_s^1 + \hat{Q}_r^\dagger\hat{Q}_s + \hat{P}_r^1\hat{Q}_s^\dagger. \tag{7.108}$$

It is possible to multiply the product $\hat{C}_{(r,s)}\hat{C}_{(s,r)}$ with unitary operators that act locally upon the individual qubits in sub-spaces $\mathcal{H}_r$ and $\mathcal{H}_s$. Three such operators are $\hat{\sigma}_r^3 \otimes \hat{\sigma}_s^0$, $\hat{\sigma}_r^0 \otimes \hat{\sigma}_s^3$ and $\hat{\sigma}_r^3 \otimes \hat{\sigma}_s^3$, and these lead to the results

$$\begin{aligned} \left(\hat{\sigma}_r^3 \otimes \hat{\sigma}_s^0\right)\left[\hat{C}_{(r,s)}\hat{C}_{(s,r)}\right] &= \left(\hat{P}_r^0\hat{I}_s - \hat{P}_r^1\hat{I}_s\right)\left[\hat{P}_r^0\hat{P}_s^0 + \hat{Q}_r\hat{P}_s^1 + \hat{Q}_r^\dagger\hat{Q}_s + \hat{P}_r^1\hat{Q}_s^\dagger\right] \\ &= \hat{P}_r^0\hat{P}_s^0 + \hat{Q}_r\hat{P}_s^1 - \hat{Q}_r^\dagger\hat{Q}_s - \hat{P}_r^1\hat{Q}_s^\dagger \end{aligned} \tag{7.109}$$

and

$$\begin{aligned} \left(\hat{\sigma}_r^0 \otimes \hat{\sigma}_s^3\right)\left[\hat{C}_{(r,s)}\hat{C}_{(s,r)}\right] &= \left(\hat{I}_r\hat{P}_s^0 - \hat{I}_r\hat{P}_s^1\right)\left[\hat{P}_r^0\hat{P}_s^0 + \hat{Q}_r\hat{P}_s^1 + \hat{Q}_r^\dagger\hat{Q}_s + \hat{P}_r^1\hat{Q}_s^\dagger\right] \\ &= \hat{P}_r^0\hat{P}_s^0 - \hat{Q}_r\hat{P}_s^1 + \hat{Q}_r^\dagger\hat{Q}_s - \hat{P}_r^1\hat{Q}_s^\dagger \end{aligned} \tag{7.110}$$

and

$$\begin{aligned}\left(\hat{\sigma}_r^3 \otimes \hat{\sigma}_s^3\right)\left[\hat{C}_{(r,s)}\hat{C}_{(s,r)}\right] &= \left(\hat{P}_r^0\hat{P}_s^0 - \hat{P}_r^0\hat{P}_s^1 - \hat{P}_r^1\hat{P}_s^0 + \hat{P}_r^1\hat{P}_s^1\right) \\ &\quad \times \left[\hat{P}_r^0\hat{P}_s^0 + \hat{Q}_r\hat{P}_s^1 + \hat{Q}_r^\dagger\hat{Q}_s + \hat{P}_r^1\hat{Q}_s^\dagger\right] \\ &= \hat{P}_r^0\hat{P}_s^0 - \hat{Q}_r\hat{P}_s^1 - \hat{Q}_r^\dagger\hat{Q}_s + \hat{P}_r^1\hat{Q}_s^\dagger. \end{aligned} \tag{7.111}$$

Now, adding (7.108) to (7.110) and then subtracting (7.109) and (7.111) gives $4\hat{Q}_r^\dagger\hat{Q}_s$, so it is evident that

$$\begin{aligned}\hat{Q}_r^\dagger\hat{Q}_s &= \frac{1}{4}\left(1 - \hat{\sigma}_r^3\otimes\hat{\sigma}_s^0 + \hat{\sigma}_r^0\otimes\hat{\sigma}_s^3 - \hat{\sigma}_r^3\otimes\hat{\sigma}_s^3\right)\hat{C}_{(r,s)}\hat{C}_{(s,r)} \\ &= \frac{1}{4}\left[\left(\hat{\sigma}_r^0 - \hat{\sigma}_r^3\right)\otimes\left(\hat{\sigma}_s^0 + \hat{\sigma}_s^3\right)\right]\hat{C}_{(r,s)}\hat{C}_{(s,r)}. \end{aligned} \tag{7.112}$$

Substituting this into (7.106) gives

$$\Pi_{rs} = \frac{1}{4}\left[\left(\hat{\sigma}_r^0 - \hat{\sigma}_r^3\right)\otimes\left(\hat{\sigma}_s^0 + \hat{\sigma}_s^3\right)\right]\hat{C}_{(r,s)}\hat{C}_{(s,r)}\hat{\sigma}_1^0\hat{\sigma}_2^0...\hat{\sigma}_{r-1}^0\hat{\sigma}_{r+1}^3...\hat{\sigma}_{s-1}^3\hat{\sigma}_{s+1}^0...\hat{\sigma}_N^0 \tag{7.113}$$

with the usual interchangeability between $\hat{\sigma}_m^0$ and $\hat{I}_m$.

Similarly, products of local transformation operators of the form $\hat{T}_r^{01}\hat{T}_s^{10} \equiv \hat{Q}_r\hat{Q}_s^\dagger$ may be obtained from alternative products of CNOT and Pauli operators. Viz, from

$$\begin{aligned}\hat{C}_{(s,r)}\hat{C}_{(r,s)} &= \left(\hat{P}_r^0\hat{P}_s^0 + \hat{P}_r^1\hat{P}_s^0 + \hat{Q}_r\hat{P}_s^1 + \hat{Q}_r^\dagger\hat{P}_s^1\right) \\ &\quad \times \left(\hat{P}_r^0\hat{P}_s^0 + \hat{P}_r^0\hat{P}_s^1 + \hat{P}_r^1\hat{Q}_s + \hat{P}_r^1\hat{Q}_s^\dagger\right) \\ &= \hat{P}_r^0\hat{P}_s^0 + \hat{P}_r^1\hat{Q}_s + \hat{Q}_r\hat{Q}_s^\dagger + \hat{Q}_r^\dagger\hat{P}_s^1 \end{aligned} \tag{7.114}$$

it follows that

$$\left(\hat{\sigma}_r^3\otimes\hat{\sigma}_s^0\right)\left[\hat{C}_{(s,r)}\hat{C}_{(r,s)}\right] = \hat{P}_r^0\hat{P}_s^0 - \hat{P}_r^1\hat{Q}_s + \hat{Q}_r\hat{Q}_s^\dagger - \hat{Q}_r^\dagger\hat{P}_s^1 \tag{7.115}$$

with

$$\left(\hat{\sigma}_r^0\otimes\hat{\sigma}_s^3\right)\left[\hat{C}_{(s,r)}\hat{C}_{(r,s)}\right] = \hat{P}_r^0\hat{P}_s^0 + \hat{P}_r^1\hat{Q}_s - \hat{Q}_r\hat{Q}_s^\dagger - \hat{Q}_r^\dagger\hat{P}_s^1 \tag{7.116}$$

and

$$\left(\hat{\sigma}_r^3\otimes\hat{\sigma}_s^3\right)\left[\hat{C}_{(s,r)}\hat{C}_{(r,s)}\right] = \hat{P}_r^0\hat{P}_s^0 - \hat{P}_r^1\hat{Q}_s - \hat{Q}_r\hat{Q}_s^\dagger + \hat{Q}_r^\dagger\hat{P}_s^1 \tag{7.117}$$

so

$$\begin{aligned}\hat{Q}_r\hat{Q}_s^\dagger &= \frac{1}{4}\left[1 + \hat{\sigma}_r^3\otimes\hat{\sigma}_s^0 - \hat{\sigma}_r^0\otimes\hat{\sigma}_s^3 - \hat{\sigma}_r^3\otimes\hat{\sigma}_s^3\right]\hat{C}_{(s,r)}\hat{C}_{(r,s)} \\ &= \frac{1}{4}\left[\left(\hat{\sigma}_r^0 + \hat{\sigma}_r^3\right)\otimes\left(\hat{\sigma}_s^0 - \hat{\sigma}_s^3\right)\right]\hat{C}_{(s,r)}\hat{C}_{(r,s)}. \end{aligned} \tag{7.118}$$

With results such as these, it is easy to see how it is possible to write the Hamiltonian, momentum and charge operators in terms of local unitary operators and the two-qubit CNOT gate, as expected and desired. Thus, the emergence of physics from universal quantum computation acting upon pregeometric qubit structure is shown.

7.4 Discussion

The aim of this chapter has been to demonstrate how quantum field theoretic descriptions of physical particles might begin to emerge from the underlying pregeometric structure proposed in this thesis. Whilst some success may therefore be claimed from the point of view of fermions, due to a derivation of emergent ladder, Hamiltonian, momentum and charge operators for particles obeying the Dirac equation, a number of issues still remain and the overall programme behind this work is far from being complete. These points are highlighted now.

Firstly, the observation is made that elementary particles are objects possessing more than just spin. The fundamental fermions currently understood in the Standard Model, namely quarks and leptons, are known to possess an array of different physical characteristics in addition to angular momentum, examples being colour, flavour, chirality, etc. A truly complete theory of matter would therefore have to explain how each of these degrees of freedom emerge from the sub-register picture. Indeed, this problem was stated in its large-scale entirety by the original brief of the chapter: how can the pregeometric description advocated in this thesis account for the enormous variety of 'properties' exhibited by classical-looking objects?

An obvious direction to proceed therefore involves an extension of the presented discussion of the Dirac field to multi-field theories incorporating, say, colour and flavour degrees of freedom.

Exactly how this should best be accomplished remains a question for the future, but it is however possible to speculate that the principles and types of methods used in the previous section would not be wholly inappropriate in an implementation of alternative fields. After all, historically the theory of colour gauge symmetry grew from a foundation based on an initial study of the Dirac field, so it might be expected that any future pregeometric description of hadrons could equally be derived from a set of underlying logic operations, just as the Dirac field was shown to be in the present work. Of course, this is a viewpoint prejudiced by tradition, but a consideration of this type of argument might at least provide a reasonable starting point for more advanced models, or alternatively perhaps suggest the need for a novel approach to the problem.

Following on from the above point regarding additional degrees of freedom, it is noted that one other important property of an elementary particle is its rest mass. In a fully quantum Universe this too would be expected to have a pregeometric origin, an issue that is presumably related to how a Higgs-type mechanism could emerge from the described sub-register picture. However, this extension is actually more pertinent than the questions

of the origin of colour or flavour, because the mass of the particle was implicitly assumed in the formalism of the above discussion, specifically as the parameter μ introduced to arrive at the Hamiltonian. So, without explicitly knowing the mechanism for generating mass in the quantum Universe, a question then arises as to whether it is valid to make such an insertion.

The introduction of μ, however, may be justified on two grounds. Firstly, physicists often ignore various degrees of freedom when discussing particular effects. The conventional Dirac equation, for example, provides a perfectly good description of spin-$\frac{1}{2}$ particles of a given mass, even though the actual mechanism that produces this mass is neglected[17]. It seems reasonable, then, that a similar approximation is equally valid in a pregeometric discussion, certainly at least as a first step towards a more complete picture of field theory.

Secondly, the mass term μ was only used anyway to formulate the Hamiltonian. So, even if its introduction does involve an element of 'cheating', the results found for the ladder, momentum and charge operators still provide a valid description of fermionic objects.

Nevertheless, for a consistent and thorough understanding of physical reality, an account of the pregeometric origin of mass is a further necessary direction to take.

In addition, it is interesting to speculate how or whether any such hypothetical mechanism might influence the types of causal set structure exhibited by the state of the Universe as it changes its separability through a sequence of jumps. The answer to this question would itself provide useful insight into the origin of general relativity and the apparent curvature of space by mass in the quantum Universe.

Perhaps the most obvious extension to an understanding of how fermionic particles emerge from the sub-register picture involves asking the question of how bosons might also.

This, however, immediately presents a difficulty. To demonstrate why, consider a conventional fermionic ladder operator $\hat{a}_F^\dagger(p)$ that creates a particle of momentum p from the vacuum $|0\rangle$, to give the single particle state

$$\hat{a}_F^\dagger(p)|0\rangle = |1_F^{(p)}\rangle \tag{7.119}$$

where the actual spin of the particle is temporarily ignored. As expected, $\hat{a}_F^\dagger(p)$ satisfies the usual relationship

$$\hat{a}_F^\dagger(p)\hat{a}_F^\dagger(p)|0\rangle = 0 \tag{7.120}$$

with

$$\hat{a}_F^\dagger(p')\hat{a}_F^\dagger(p)|0\rangle = (1-\delta_{pp'})|1_F^{(p')}1_F^{(p)}\rangle \tag{7.121}$$

[17] And indeed was totally unknown at the time of Dirac. Perhaps this is further support for the argument that the development of quantum field theory in terms of pregeometry could follow an 'historical' route.

and so on. The point is that the maximum occupancy of a given particle state is one, as predicted for objects obeying Fermi-Dirac statistics. Furthermore, these relations imply that the operator $\hat{a}_F^\dagger(p)$ may be seen, in some ways, as being analogous to a transition operator between the qubit states $|0\rangle_m$ and $|1\rangle_m$, because these too follow the product algebra of fermions. Of course, it was this type of association that formed the basis for the work of the previous section.

Consider instead, however, a conventional bosonic ladder operator $\hat{a}_B^\dagger(p)$ that creates a boson of momentum p from the vacuum $|0\rangle$, such that

$$\hat{a}_B^\dagger(p)|0\rangle = |1_B^{(p)}\rangle \tag{7.122}$$

where $|1_B^{(p)}\rangle$ represents a single boson particle state.

Now, because $\hat{a}_B^\dagger(p)$ governs the creation of bosons, it is taken to obey the relationships

$$\begin{aligned} \hat{a}_B^\dagger(p)\hat{a}_B^\dagger(p)|0\rangle &= \hat{a}_B^\dagger(p)|1_B^{(p)}\rangle = |2_B^{(p)}\rangle \\ \hat{a}_B^\dagger(p)\hat{a}_B^\dagger(p)\hat{a}_B^\dagger(p)|0\rangle &= |3_B^{(p)}\rangle \\ \vdots &= \vdots \\ (\hat{a}_B^\dagger(p))^n|0\rangle &= |n_B^{(p)}\rangle \end{aligned} \tag{7.123}$$

that is, the theory allows multi-occupancy of states: the state $|n_B^{(p)}\rangle$ contains n identical bosons. Moreover, the statistics are assumed to be valid for all n up to $n = \infty$.

But, it is difficult to envisage how such multi-occupancy states could be incorporated into the binary 'on/off' logic of qubits. Not only that, but it is also difficult to imagine how infinite occupancies could arise at all from any pregeometric structure based on finite dimensional Hilbert spaces. Ultimately, then, these two comments might perhaps lead to the criticism that only half of particle physics could ever emerge from the subregister model suggested.

Whilst the origin of bosons in the quantum Universe remains an unresolved issue, a number of points may be made against the above conclusion, and which should therefore provide a guide for future research. For example, it is noted that in the Standard Model of particle physics the four fundamental bosons may not so much be interpreted as observable entities, but should perhaps instead be viewed in terms of representations of interactions. In other words, it is remarked that these bosons are not necessarily directly observed *per se*, but that it is only their effect on the fermions comprising physical matter that is actually seen. Thus, the apparent existence of bosonic particles in conventional physics could perhaps be viewed simply as an artifact of a misunderstanding of how fermions interact with one another, based itself on the mistake of assuming that fermionic particles are representative of the most fundamental level of reality. Real, physical bosonic particles

such as photons, gluons and gravitons may not actually exist, or at least not in the sense that they are conventionally assumed to.

So, the suggestion here is that a discussion of bosons may therefore naturally be postponed until the questions of Section 6.2 surrounding pregeometric interactions, reactions and measurements are better understood. In short, the argument is that it may never be possible to understand the ultimate nature of the fundamental forces and the apparent existence of bosons simply by devising ever more complicated particle theories and experiments (as conventional physics attempts to do), but will instead only be resolved when a fuller cognition of the issues present at the very heart of quantum mechanics' measurement problem has been achieved. Of course, an admission is also made here that an element of "sweeping under the carpet" may appear to be present in this argument.

Alternatively, another potential way around the highlighted problem might be to relax the 'infinity condition' of bosonic occupancy. As has been done consistently throughout this work, infinities have often been removed in the quantum Universe paradigm by assuming finite, but very large, degrees of freedom. This may then be the solution for the present case involving bosons, and is reinforced by finite energy arguments in favour of restricting the number of particles of given energy-momentum to non-infinite values. Paraphrasing this last point, and echoing the words of Feynman [20], the question is again asked as to whether it is really physically possible to have an infinite number of positive energy particles in a Universe of finite size and mass.

So from this perspective, and as speculated upon by Lu and Widar [76], it might be suggested that bosons should instead really obey a rule of the form

$$(\hat{a}_B^\dagger(p))^q|0\rangle = |q_B^{(p)}\rangle \tag{7.124}$$

but

$$(\hat{a}_B^\dagger(p))^{q+1}|0\rangle = 0 \tag{7.125}$$

where $q \gg 1$. In the language of Green [65] and Greenberg [66], bosons would then be viewed as parafermions of order q, and would be governed by ladder operators of the form

$$\hat{a}_B^\dagger(p)|n\rangle = |(n+1)_B^{(p)}\rangle \tag{7.126}$$

for $0 \leq n \leq (q-1)$, and

$$\hat{a}_B(p)|n\rangle = |(n-1)_B^{(p)}\rangle \tag{7.127}$$

for $1 \leq n \leq q$, with

$$\hat{a}_B(p)|0\rangle = 0 \tag{7.128}$$

In principle, as long as q is sufficiently large such that the maximum occupancy $|q_B^{(p)}\rangle$ is never actually reached physically, the mechanism would, to all intents and purposes,

appear equivalent to the conventional theory of bosons that allows any number of particles to be in the same state.

Of course, this solution is still highly speculative, and a great deal of work is required to investigate whether the usual laws applying to bosons may be extended to such high-order parafermions.

Emergent operators possessing the relations inherent to (7.126) and (7.127) may not actually be able to arise from a qubit subregister structure, especially if the procedure described in Section 7.2 is used. To illustrate why this might be the case, recall that qubit Hilbert (sub-)spaces are spanned by just two orthogonal basis states, and can hence only exhibit two possible transition operators relating them. Furthermore, these two possible transition operators were taken above to be the ultimate origin of ladder operators in momentum space that governed physical particle states that could only exist in either one of two possible occupancies (i.e. fermions). Thus, it is difficult to see how this result could be generalised to account for emergent ladder operators that cause apparent multi-occupancy of states, if these emergent operators are a direct consequence of pregeometric transition operators confined to act only in two-level spaces.

Consequently, in order to incorporate higher order occupancies it might be necessary to remove the constraint of using two-level, qubit Hilbert sub-spaces and instead consider subregisters of higher dimension. Of course, this removal may be justified anyway by recalling that qubit spaces were only ever employed in the first place because they provided the simplest starting point for the ensuing discussion, and not because of any physical constraints.

So, assuming that the m^{th} subregister $\mathcal{H}_m^{(d)}$ of the Hilbert space $\mathcal{H}_\varphi$ containing the field φ is not a qubit space but is of dimension d, then $\mathcal{H}_m^{(d)}$ may be spanned by an orthonormal basis set $\mathcal{B}_m^{(d)}$ defined as $\mathcal{B}_m^{(d)} \equiv \{|i\rangle_m : i = 0, 1, ..., (d-1)\}$. In this case, it is now possible to define transformation operators between the bases of $\mathcal{H}_m^{(d)}$ in an analogous manner to that used for qubits. That is, an operator of the form

$$[\hat{T}^{(d)}]_m^{(i+1)i} = |i+1\rangle_{mm}\langle i| \tag{7.129}$$

acting on the state $|i\rangle_m \in \mathcal{H}_m^{(d)}$ changes it to

$$[\hat{T}^{(d)}]_m^{(i+1)i}|i\rangle_m = |i+1\rangle_m \tag{7.130}$$

for $i = 0, 1, ..., (d-2)$, whereas an operator of the form

$$[\hat{T}^{(d)}]_m^{i(i+1)} = |i\rangle_{mm}\langle i+1| \tag{7.131}$$

acting on the state $|i+1\rangle_m \in \mathcal{H}_m^{(d)}$ changes it to

$$[\hat{T}^{(d)}]_m^{i(i+1)}|i+1\rangle_m = |i\rangle_m. \tag{7.132}$$

Of course, due to the orthogonality of the bases, any operator $[\hat{T}^{(d)}]_m^{kj}$ acting on the basis state $|i\rangle_m$ gives

$$[\hat{T}^{(d)}]_m^{kj}|i\rangle_m = 0 \tag{7.133}$$

unless $j = i$. So, it is therefore useful to define 'general ladder operators' such as $\hat{A}_m^{(d)}$ and $\hat{A}_m^{(d)\dagger}$ acting in the space $\mathcal{H}_m^{(d)}$; assuming that the current state contained in $\mathcal{H}_m^{(d)}$ is in one of the d basis states $|i\rangle_m$, then the operator $\hat{A}_m^{(d)}$ given by

$$\hat{A}_m^{(d)} \equiv [\hat{T}^{(d)}]_m^{01} + [\hat{T}^{(d)}]_m^{12} + ... + [\hat{T}^{(d)}]_m^{(d-2)(d-1)} \tag{7.134}$$

'lowers' it to $|i-1\rangle_m$, for $i \geq 1$, whereas the general operator $\hat{A}_m^{(d)\dagger}$ of the form

$$\hat{A}_m^{(d)\dagger} \equiv [\hat{T}^{(d)}]_m^{10} + [\hat{T}^{(d)}]_m^{21} + ... + [\hat{T}^{(d)}]_m^{(d-1)(d-2)} \tag{7.135}$$

may be used to 'raise' it from this state $|i\rangle_m$ to $|i+1\rangle_m$, for $i \leq (d-2)$. Clearly, $\hat{A}_m^{(d)\dagger}$ is behaving analogously to the pregeometric creation operator defined previously for qubits, whilst the generalisation $\hat{A}_m^{(d)}$ is evidently related to the qubit annihilation operator.

Such higher dimensional subregisters and ladder operators may play an important part in accounting for the large number of fields, both fermionic and bosonic, familiar in the emergent world of high-energy physics.

As a final comment to this chapter, it is observed that the treatment given in Section 7.2 is for a free-field theory. However, almost all 'interesting' physics, and certainly that occurring in a real collider experiment, arises out of the interactions existing between particles. Moreover, it is also worth recalling that is the actual results of the interactions, and hence ultimately the observations of the particles, that are considered the only physically real phenomena in quantum mechanics. It is therefore an important question to ask exactly how an interacting field theory could emerge from the underlying structure of the proposed paradigm.

Of course, this issue involves a huge research program in its own right. From the pregeometric point of view advocated in this thesis, the question involves the arduous task of describing the mechanism by which information may be exchanged and extracted from the emergent field theory picture, and hence relies upon a full integration of the principles introduced in this and the preceding chapter. Putting this in perspective, the question could equally be phrased as analogous to how the measurement problem of standard quantum mechanics may be reconciled with the awesome formalism of conventional quantum field theory.

This point may be continued. One eventual goal of future research would be to apply the types of ideas presented here to describe real collider physics experiments from the point of view of the underlying pregeometric structure. So, further to the above question

regarding the incorporation of the work of the previous chapter into the results of the current one, such an investigation would then require non-trivial extensions in order to explain how:

1. large collections of pregeometric particles organise themselves into the 'elementary' particles familiar to the Standard Model;

2. how these elementary particles interact and organise themselves to form atoms and molecules;

3. how these then accumulate into large, semi-classical looking objects, some of which are called human beings;

4. and how these scientists can then manipulate other large collections of emergent objects such that they resemble apparatus, laboratories, isolated particles etc., and then perform experiments with them.

Point "4." itself incorporates an enormous number of different questions, such as how humans gain the illusion of free-will, and hence why particular experiments are actually performed. This is particularly important in the context of a collider physics experiment, because certain particles may only appear to exist when they are being looked for. Indeed, whilst this last comment may appear meta-physical, it should be recalled that it is a central tenet of the principles of quantum measurement theory, and follows naturally from the conclusion of Wheeler [15] that no phenomenon can really be said to exist independently of observation.

The comments "1." to "4." also highlight the importance of the role of observables in the above dynamics: after all, it is these that are responsible for the resulting states. As before, the conclusion is again that these operators must be selected very carefully if large, separate groups of factors can form and persist over enormous numbers of jumps, such that classical-looking collider experiments may be carried out on the emergent scale.

Furthermore, and as has been discussed previously, recall that from an endo-physical point of view the process of a scientist performing an experiment on a subject is equivalent to one part of the Universe's state appearing to perform an experiment on another part of it. Moreover, in a fully quantum Universe free from external influences, this implies a very careful and intricate interplay between the state representing the scientists and subjects, and the Hermitian operator used to develop it. How this type of mechanism might work is therefore also an important question.

It is this interplay that is the focus of the final chapter.

8 The Developing Quantum Universe

Three of the major conclusions that have played an important role in the paradigm proposed in this thesis are the following:

1. The Universe is a fully quantum system, and as such may be represented by a pure state Ψ_n in a Hilbert space $\mathcal{H}^{(D)}$ of enormous dimension D, where this vector is one of the D orthonormal eigenstates of the Hermitian operator $\hat{\Sigma}_n$ chosen by the dynamics;

2. The Universe is developed by testing its state with the Hermitian operator $\hat{\Sigma}_{n+1}$. It consequently jumps to the new state Ψ_{n+1};

3. The Universe is 'closed' in the sense that it is self-contained, because by definition the Universe contains everything. Thus, there cannot be any notion of an external observer.

In conventional quantum mechanics, however, these points might appear somewhat contradictory. In laboratory quantum mechanics, a wavefunction (for example a single, free electron) may be developed by testing it with a given Hermitian operator, which, in general, collapses the state into one of its eigenvectors. Moreover, these Hermitian operators represent the observables of the system, and as such denote the various physical tests that a scientist may perform on the quantum subject.

Now, in conventional quantum mechanics the choice of the test is usually made by the physicist, and is hence often assumed to be a result of the free-will processes of a semi-classical observer. So, and perhaps most importantly, these processes are generally assumed to be occurring externally to the quantum system under investigation.

Herein lies the problem. If the quantum Universe does not possess observers external to it, yet is governed by state reduction dynamics according to Hermitian operators $\hat{\Sigma}_{n+1}$, the question remains as to how these tests are actually chosen.

In other words, if there are no external agents to decide the Universe's fate, and if the existence of the states Ψ_n, Ψ_{n+1}, $\Psi_{n+2}, ...$ may be explained as being the results of the operators $\hat{\Sigma}_n$, $\hat{\Sigma}_{n+1}$, $\hat{\Sigma}_{n+2}, ...$, what mechanism accounts for, governs, or explains the choice of the these tests? Specifically, if the actual state Ψ_n of the Universe depends (albeit stochastically) upon the choice of test, how is this operator $\hat{\Sigma}_n$ actually chosen? Ultimately, then, in the case where the quantum state encompasses the entire Universe, what takes the place of the external observer familiar to conventional quantum mechanics? Thus, how may the traditional exo-observer be replaced by any sort of endo-physical equivalent?

It is these issues that are addressed in this chapter.

Before beginning to see how these questions might be answered, however, a number of points should be noted.

Firstly, if the entire Universe is described by quantum principles, yet cannot support any sort of external observers, the conclusion must be that it is somehow able to prepare, evolve and test itself. Moreover, the result of this test could then be associated with the preparation of a new state, which could itself subsequently be evolved and tested, in an automatic process. Thus, and as has been discussed a number of times throughout this work, the Universe could ultimately be envisaged to be a giant, self-developing *quantum automaton*.

Overall, then, the suggestion is that the Universe itself somehow chooses a sequence of tests $\hat{\Sigma}_n$, $\hat{\Sigma}_{n+1}$, $\hat{\Sigma}_{n+2}$, ... to test its state with.

Secondly, and as has also been discussed a number of times, the exact way the Universe develops appears to be highly constrained. Equivalently, therefore, this statement implies that whatever mechanism is actually responsible for choosing the operator used for the state's development, it must itself be highly constrained. The existence of continuous looking space, the nature of particle physics experiments, and the apparent persistence of semi-classical objects all support this conclusion.

However, the actual nature of these observations suggests something else about whatever mechanism is responsible for choosing operators. Namely, since the Universe appears to look so very similar from one stage to the next, part of the mechanism must be responsible for ensuring that this is the case. For example, the mechanism might be such that the Universe examines, somehow, its current state Ψ_n, and then only chooses operators $\hat{\Sigma}_{n+1}$ that have an eigenstate that is 'almost identical' to Ψ_n in some sense. Thus, and by the usual rules of probability amplitudes, it would then be highly likely that the next state Ψ_{n+1} is virtually the same as Ψ_n, especially when the enormous dimensionality of the Hilbert space of Ψ_n is taken into account.

Continuing, if this process repeats for all n, the overall result would therefore be of a Universe that appears to change only very slightly from one jump to the next, such that over a large number of steps it could appear to evolve smoothly and continuously, just as appears to be the situation in physics.

Such a mechanism would thus be an example of a '*self-referential*' system, because the dynamics are such that the Universe examines itself before deciding upon a choice of test. Specifically, in the above example, the selection of $\hat{\Sigma}_{n+1}$ depends somehow upon the state Ψ_n.

In fact, the suggestion that the Universe might actually develop in this self-referential manner is reinforced further by the fact that human physicists, who are themselves just parts of the Universe's state, do appear to be able to manipulate and test the sub-states around them. In physics, the next state of the Universe does seem to depend on the current state, because scientists do appear able to perform physical experiments, and consequently 'determine' future states of reality (within the limits set by quantum probability, of course). In essence, because the initial wavefunction of a subject, and the outcome of any laboratory quantum experiment upon it, are both viewed in the proposed paradigm as just factors of states of the Universe, physicists must actually be changing the overall state of the entire Universe whenever they perform an experiment.

Furthermore, in fact, the physicist, the apparatus, and the laboratory should themselves all really be viewed simply as groups of factors of the Universe's state. So, because the factor that represents a piece of physical apparatus does appear to be able to examine the sub-state representing the sample under investigation, such an experiment should really be interpreted as one part of the Universe's state being developed according to another part of it. The conclusion must therefore be that the real, physical Universe is developing self-referentially.

As an aside and an illustration, note that an example of a self-referential system is a modern computer. Many computers possess software that is able to run diagnostic checks upon themselves; the software may examine the internal state of the machine, and 'decide' upon a course of action depending on which particular configuration the computer is currently in. This decision may be based entirely upon the current state of the computer according to the 'rules' of action programmed into the software.

In fact, in the context of the current work a better example may involve a quantum computer, but the point remains the same. To continue the parallel drawn throughout his work, the development of the Universe is therefore again viewed as an enormous quantum computation, such that overall the Universe is envisaged to be an enormous, self-referential, quantum automaton.

The structure of this chapter is as follows. Firstly, definitions will be given for some of the different types of way in which a quantum system free from external agents may be able to decide upon a choice of operator. For the reasons given above, particular attention will then be given to the self-referential mechanisms in which the development of the universe depends somehow upon its current state. The different sorts of such 'state self-referential' mechanisms will then be discussed, and examples given in order to demonstrate and explain some of the issues surrounding the dynamics that result.

Many of the examples will take the form of 'toy-universe' models that are represented by states in Hilbert spaces of low dimension, with the general assumption being made that the reduction in complexity of these models from the real Universe[18] has not been made at the expense of discarding any overriding physical principles. This is analogous to how similar such low dimensional examples have been used throughout this thesis to illustrate various points when it has been necessary to not become overwhelmed by the enormous number of degrees of freedom inherent in the real, physical situation.

8.1 Types of Development

The dynamics of the developing Universe are, at least in principle, fairly simple.

At the n^{th} stage of the Universe's development, its state may be represented by the unique vector Ψ_n. This wavefunction may be 'tested' by some non-degenerate Hermitian operator $\hat{\Sigma}_{n+1}$, or might first be 'evolved' with some sort of unitary operator $\hat{U}_n$, i.e. $\Psi_n \rightarrow \Psi'_n = \hat{U}_n \Psi_n$, which is equivalent to a 'rotation' of the vector within its Hilbert space, before then being 'tested' by $\hat{\Sigma}_{n+1}$. Moreover, this testing process is irreversible, and the state Ψ_n (or Ψ'_n) consequently collapses to a next state Ψ_{n+1}, which is one of the eigenstates of $\hat{\Sigma}_{n+1}$. Thus, the development of the universe is a discontinuous process, with jumps from one state to the next occurring due to state reduction. The exact forms of the operators $\hat{U}_n$ and $\hat{\Sigma}_{n+1}$ are dictated by the Rules governing the system.

In the following, various sets of Rules are investigated, where the aim is to generate a dynamics that develops the Universe in an automatic way. A goal of this chapter is therefore to examine some of the different mechanisms by which a decision regarding the choice of next operator $\hat{\Sigma}_{n+1}$ may be made, without appealing to any sort of external observer with free-will.

Now, in the paradigm proposed in this thesis, a Stage Ω_n of the Universe is parameterised by (amongst others) a state Ψ_n, the operator $\hat{\Sigma}_n$ of which Ψ_n is an eigenstate, and the Rules R_n governing the dynamics. Of course, the presence of an operator $\hat{\Sigma}_n$ equivalently implies the presence of a unique basis set $\mathfrak{B}_n^{(D)}$ of orthonormal vectors[19], which spans the Hilbert space $\mathcal{H}^{(D)}$ of Ψ_n and comprises of the D eigenstates of $\hat{\Sigma}_n$.

The general question of this chapter then becomes: how can some, all, or none of these parameters be used to decide what happens next? In other words, given a state Ψ_n and an

[18] In general in the following, a capitalised 'Universe' will indicate The Real, physical Universe, whereas a lower case 'universe' will imply the low-dimensional, toy-model case. The context of the word should nevertheless make it clear anyway which U/universe is being discussed.

[19] Noting that the reverse of this is not strictly true: a given basis set $\mathfrak{B}_n^{(D)}$ does not specify a unique operator $\hat{\Sigma}_n$, but instead describes an entire equivalence class.

operator $\hat{\Sigma}_n$, what Rules R_n could govern the mechanism for generating Ψ_{n+1}? In short, what are the different types of mechanism that could relate the present state Ψ_n to the next state Ψ_{n+1}?

In fact, it turns out that there are a number of different types of Rule that could be used to determine a state's development. One possible such list of types of mechanism is now introduced, with its members described in turn. The list is written in order of increasing (intuitive) complexity:

Type 0: *Free Will.* The state is developed by an operator chosen by an external observer.

Type I: *Evolution.* Ψ_n is mapped to Ψ_{n+1} by continuous evolution, say by an unitary function U, such that $\Psi_n \rightarrow \Psi_{n+1} = \Psi'_n = U\,\Psi_n$.

Type II: *Deterministic Choice.* The Rules R_n select an operator $\hat{\Sigma}_{n+1}$ by some sort of deterministic algorithm without reference to the current state or last operator. Thus, the choice of operator $\hat{\Sigma}_{n+1}$ does not depend in any way upon Ψ_n or $\hat{\Sigma}_n$.

Type IIa: *Probabilistic Choice.* The Rules R_n select an operator $\hat{\Sigma}_{n+1}$ according to some sort of probabilistic algorithm, without reference to the current state or last operator. Thus, $\hat{\Sigma}_{n+1} = \hat{A}$ with probability P_A, whilst $\hat{\Sigma}_{n+1} = \hat{B}$ with probability P_B, etc., where the various probabilities sum to unity.

Type III: *State Self-Referential.* The Rules dictate that the algorithm used to generate the next operator $\hat{\Sigma}_{n+1}$ (or equivalently $\mathfrak{B}_{n+1}$) refers, somehow, to the vector Ψ_n. Thus, the choice of the next operator depends on the current state.

Type IIIa: *Probabilistic State Self-Referential.* As with Type III, the Rules still dictate that the algorithm used to generate the next operator $\hat{\Sigma}_{n+1}$ (or equivalently $\mathfrak{B}_{n+1}$) refers to the current state Ψ_n, but there is now an element of probability in the algorithm. For instance, a given state Ψ_n could imply a probabilistic choice of one of a number of possible operators, or alternatively, the way in which the next operator $\hat{\Sigma}_{n+1}$ depends on the state Ψ_n could be probabilistic. As a schematic example of this last possibility, given two 'functions' f and g defined somehow by the Rules, it could follow that $\hat{\Sigma}_{n+1} = f(\Psi_n)$ with probability P_f, whereas $\hat{\Sigma}_{n+1} = g(\Psi_n)$ with probability P_g, with $P_f + P_g = 1$.

Type IV: *Basis Self-Referential.* The next operator $\hat{\Sigma}_{n+1}$ is generated in a way that makes reference to the current basis, such that in effect $\hat{\Sigma}_{n+1}$ depends somehow on $\hat{\Sigma}_n$ (or equivalently $\mathfrak{B}_{n+1}$ depends somehow on $\mathfrak{B}_n$).

Type IVa: *Probabilistic Basis Self-Referential.* The choice of the next operator depends probabilistically on the current basis. As analogously to the Type III case, a given basis $\mathfrak{B}_n$ could imply a number of potential bases $\mathfrak{B}_{n+1}$, or there could alternatively perhaps be a number of different ways in which the basis $\mathfrak{B}_{n+1}$ depends on the basis $\mathfrak{B}_n$, with the actual choice in both circumstances made probabilistically.

Type V: *Fully Self-Referential.* The next operator $\hat{\Sigma}_{n+1}$ is generated in a way that makes reference to the current basis $\mathfrak{B}_n$ *and* whichever of its eigenvectors was actually chosen to be the current state Ψ_n, Thus, the choice of the next operator depends deterministically on both Ψ_n and $\mathfrak{B}_n$.

Type Va: *Probabilistic Fully Self-Referential.* The choice of the next operator depends probabilistically on the current basis *and* whichever of its eigenvectors was actually chosen to be the current state Ψ_n. As before, this probabilistic algorithm could take a number of forms, one schematic example of which would be if $\hat{\Sigma}_{n+1} = F_n(\Psi_n, \hat{\Sigma}_n)$ with probability P_F whereas $\hat{\Sigma}_{n+1} = G_n(\Psi_n, \hat{\Sigma}_n)$ with probability P_G, where F_n and G_n are 'functions' defined in the Rules, and $P_F + P_G = 1$.

Of course, further Types are possible if more parameters are specified. For example, if the current Stage Ω_n also contains information regarding previous states Ψ_{n-1}, $\Psi_{n-2}, ...$ it would additionally be possible to consider higher order 'Historic' Rules, such as an 'Historic Type III mechanism' in which the choice of operator $\hat{\Sigma}_{n+1}$ somehow depends on Ψ_n *and* Ψ_{n-1} *and*... etc. Variants on this theme may also be imagined, in obvious ways, for this and other Types by including or excluding particular permutations of the parameters.

This chapter focuses primarily on Type III and $IIIa$ mechanisms, for the reasons given now.

Firstly, any Type 0 mechanism is clearly unsatisfactory for the development of a Universe free from external observers. It is also wrong from the epistemological viewpoint, in which every phenomenon, including free will, is ultimately hoped to be explainable from a consistent set of underlying physical laws.

The Type I mechanisms are effectively the same as decoherence and Many Worlds type dynamics. They would hence be accompanied by all the problems and contortions associated with these. So, since such difficulties are hoped to be avoided, not least for the reasons outlined in Section 4.3, the Type I mechanisms will be neglected in the following discussions.

Type II and IIa mechanisms are indicative of the current state of play in conventional quantum mechanics, and acknowledge the fact that although physicists are able to obtain

accurate answers to quantum questions (via probability amplitudes etc.), they cannot say why these questions were posed in the first place. In the language of causal sets, the causal set generated over a series of stages by operators chosen according to Type II Rules would be completely independent of the causal set generated by the states over the same period. Thus, in Type II and IIa models it is difficult to see how the very finely tuned constraints discussed throughout this thesis could be achieved, because the mechanisms do not possess any of the 'feed-back' processes that appear necessary for the emergence of classical looking physics.

Lastly, although Types IV, IVa, V and Va do, in principle, describe possible Rules for the development of the physical Universe, they are also anticipated to give rise to mechanisms that are considerably more complicated than the Type III cases. In short, the dynamics of systems based upon Rules in which the next operator $\hat{\Sigma}_{n+1}$ depends upon the current state Ψ_n and/or the current basis $\mathfrak{B}_n$ are expected to be far richer than those in which the next operator depends only upon the current state, and as such a full investigation of the Types IV, IVa, V and Va will be left as a necessary avenue of inquiry for future research.

Type III and $IIIa$ Rules hence provide a suitable starting point, useful to investigate some of the ways in which the quantum Universe could develop. The central focus of this chapter is therefore to investigate operator selection mechanisms that permit the Universe to develop self-referentially according to the current state.

To this end, the description of a Type III mechanism may now be expanded. Specifically, if the quantum state of the Universe after n jumps is represented by Ψ_n, then a state self-referential mechanism is defined as one in which if Ψ_n has one particular 'property' then it is tested with one particular operator (say, $\hat{\Sigma}_{n+1} = \hat{X}_{n+1}$), but if it has another 'property' it is instead tested with a different operator (i.e. by $\hat{\Sigma}_{n+1} = \hat{Y}_{n+1}$). And so on, including of course what happens if it instead has a third, or a fourth, or a..., particular 'property'. Of course, the Rules must be sufficiently well defined so that whatever the current state Ψ_n may be, or whatever 'properties' it might have, it will definitely lead to a certain and unique selection of the next operator $\hat{\Sigma}_{n+1}$, for all n.

Thus, in such state self-referential mechanisms, the way in which the universe develops depends entirely upon which state it is in. Clearly, this is analogous to the type of computer described earlier that is able to 'examine' its state in order to decide upon the next course of action. Again, the physical Universe may be compared in this fashion to a type of self-examining quantum computer.

In fact, it is even possible to imagine extending the above definitions, and consider a universe in which if Ψ_n has one particular 'property' then it is *evolved* in one way (say,

by $\hat{U}_n = \hat{u}_n^A$), but if it has another property it is instead evolved in a second way (i.e. by $\hat{U}_n = \hat{u}_n^B$). So, in these universes it is the next unitary operator $\hat{U}_n$ that depends on the current state Ψ_n. Such a dynamics could be defined as obeying Type III^* Rules[20], and is an issue discussed further in Section 8.5. The evolved state $\Psi'_n = \hat{U}_n \Psi_n$ could then be tested by an operator $\hat{\Sigma}_{n+1}$, itself chosen by a particular Type of Rule.

In the general case, the choice of both the next unitary operator $\hat{U}_n$ and the next test $\hat{\Sigma}_{n+1}$ may depend on their own individual Types of Rule.

The Type III and $IIIa$ categories of mechanism can in fact themselves be further subdivided, because there are very many different sorts of way in which the actual choice of operator $\hat{\Sigma}_{n+1}$ could depend on Ψ_n.

One important subdivision involves the question as to whether the operators are defined independently of the states, or whether they are somehow 'created' by them. In the first instance, it is possible to imagine a dynamics associated with a large, fixed set of possible operators defined at the outset, with the decision about which member of this set is actually chosen to be $\hat{\Sigma}_{n+1}$ made in reference to the state Ψ_n. In the second scenario, no such pre-existing set of potential operators is present, and the state Ψ_n is assumed to somehow 'generate' the operator $\hat{\Sigma}_{n+1}$ itself according to whatever Rules R_n govern the universe's dynamics. In other words, in the former case the state Ψ_n may be said to 'pick' an operator $\hat{\Sigma}_{n+1}$ from a pre-existing list, whereas in the latter circumstance the state Ψ_n is directly responsible for 'creating' the operator $\hat{\Sigma}_{n+1}$.

Clearly, the difference between these two sorts of Rule is most manifest when Ψ_n is unspecified: in the first sort of dynamics it cannot now be said which operator $\hat{\Sigma}_{n+1}$ will be chosen out of the list of known possibilities, whilst in the second case there is now simply no way of knowing what $\hat{\Sigma}_{n+1}$ might be like at all.

For obvious reasons, these two different sorts of Type III mechanism may be labelled *List-Sort* and *Generated-Sort* respectively, and are discussed more fully in the next few sections.

8.2 List-Sort Dynamics

Consider a universe represented at 'time' n by a state Ψ_n in a Hilbert space $\mathcal{H}$ of enormous dimension D. As has been asserted previously, it is assumed that the next state Ψ_{n+1} will be one of the D orthonormal eigenstates of whichever Hermitian operator $\hat{\Sigma}_{n+1}$ is chosen by the Rules R_n to develop the system. The question is: what will this operator actually be?

[20] Similar 'starred' extensions are readily imagined for other Types of Rule.

In List-Sort dynamics, it is conjectured that along with the overall Hilbert space defining the system, a universe is also provided with an enormous List $\mathfrak{L}$ of different Hermitian operators

$$\mathfrak{L} \equiv \{\hat{B}_1, \hat{B}_2, ..., \hat{B}_l\} = \{\hat{B}_i : i = 1, 2, ..., l\} = \{\hat{B}_i\} \tag{8.1}$$

where l may be called the 'List Length'.

Then, the central principle behind the development of this universe according to Type III List-Sort dynamics is that a particular state Ψ_n leads to the selection of one, and only one, operator from the List $\mathfrak{L}$. This chosen operator is consequently identified with $\hat{\Sigma}_{n+1}$, and can hence be used to test the state Ψ_n. The next state Ψ_{n+1} will thus be one of the eigenvectors of this operator[21].

In fact, each operator $\hat{B}_i$ in $\mathfrak{L}$ is actually a member of an equivalence class of Hermitian operators, which could be written $\hat{B}_i \equiv \{\hat{B}_i^1, \hat{B}_i^2, \hat{B}_i^3, ...\}$, with identical eigenvectors but different eigenvalues. Moreover, this equivalence class may be used to specify a unique basis $\mathfrak{B}_i$ containing a set of D orthonormal vectors spanning the Hilbert space $\mathcal{H}$, such that $\mathfrak{B}_i$ may be given by

$$\mathfrak{B}_i \equiv \{\Theta_i^a : a = 1, 2, ..., D\} \tag{8.2}$$

with $\langle \Theta_i^a | \Theta_i^b \rangle = \delta_{ab}$, where the set $\{\Theta_i^a\}$ are the eigenvectors of $\hat{B}_i$.

Effectively, then, the List $\mathfrak{L}$ also implies a List of different possible basis sets

$$\mathfrak{L} \equiv \{\mathfrak{B}_1, \mathfrak{B}_2, ..., \mathfrak{B}_l\} = \{\mathfrak{B}_i : i = 1, 2, ..., l\} = \{\mathfrak{B}_i\}. \tag{8.3}$$

such that bases $\mathfrak{B}_i$ and operators $\hat{B}_i$ may be used interchangeably in the following, depending on context.

So, the state Ψ_n of a universe developing according to List-Sort dynamics implies the selection of one, and only one, basis set from the List $\mathfrak{L}$, and this in turn implies the selection of a unique equivalence class of operators. The next state Ψ_{n+1} will hence be one of the elements of whichever basis set was chosen from the List.

Of course, exactly which basis set is chosen from the List depends on the actual Rules R_n that govern the universe's dynamics, and the current state Ψ_n. However, once a basis set has been picked, it automatically fixes the set of possible next states. So, if the Rules conspire such that a particular state $\Psi_n = \Phi_A$ leads to a selection of the basis $\mathfrak{B}_A$ from

[21] As an aside, note that other Types of List-Sort dynamics are permitted. A Type II List-Sort mechanism, for example, could choose the next operator $\hat{\Sigma}_{n+1}$ from a List according to some sort of deterministic algorithm that does not refer in any way to the current state Ψ_n.

Type III List-Sort dynamics could similarly be expected to have their analogies in universes developing according to Type IV, IVa, V and Va mechanisms.

$\mathfrak{L}$, then it is assumed that Ψ_n will be tested by the operator $\hat{\Sigma}_{n+1} = \hat{B}_A$, so that the next state Ψ_{n+1} will be one of the vectors $\{\Theta_A^a\}$.

Furthermore, whichever one of these eigenvectors will actually be realised is given in the usual, stochastic way of the Born probability rule, such that the probability P_{n+1}^a that the next state of the universe is $\Psi_{n+1} = \Theta_A^a$ is clearly

$$P_{n+1}^a = |\langle \Psi_{n+1} = \Theta_A^a | \Psi_n \rangle|^2 . \tag{8.4}$$

Note that there is no form of 'compound' probability involved here: the selection of the basis $\mathfrak{B}_A$ is assumed to be fully deterministic given the state $\Psi_n = \Phi_A$, such that under this circumstance, $\hat{B}_A$ is chosen from the List $\mathfrak{L}$ with probability 1. This rule will be modified later.

As discussed, it is hoped in this section to investigate state self-referential Rules, that is, mechanisms in which the actual selection of a particular basis from the List $\{\mathfrak{B}_i\}$ depends upon the current state Ψ_n.

To this end, it is possible to schematically envisage a situation in which if Ψ_n has one particular 'property', then the basis set $\mathfrak{B}_A$ is chosen from the List, whereas if Ψ_n has instead a different such 'property', then the basis set $\mathfrak{B}_B$ is alternatively chosen from the List, whilst if Ψ_n has neither of these but has instead a third particular 'property', then the basis set $\mathfrak{B}_C$ is chosen from the List, and so on.

Clearly, the Rules must be defined such that whichever state the universe is in, one, and only one, operator is picked from the List. Effectively, then, this implies that the various 'properties' made reference to by the Rules must be mutually exclusive: that is, if the state Ψ_n has a property A, it cannot also have the properties $B, C, ...$, and so there is therefore no ambiguity in which of $\mathfrak{B}_A, \mathfrak{B}_B, \mathfrak{B}_C, ...$ is picked from the List to give $\hat{\Sigma}_{n+1}$. Moreover, the set of possible properties made reference to must also be exhaustive, such that whatever Ψ_n may be, it will definitely have one (and only one) of the properties $A, B, C....$, so that the Rules can definitely select one of $\mathfrak{B}_A, \mathfrak{B}_B, \mathfrak{B}_C, ...$

The question dominating the dynamics them becomes one concerning which particular 'property' the Rules R_n deem important. Again, of course, this depends entirely on how these Rules are defined, and many different types of Rule are possible.

The above general definition of a Type III List-Sort mechanism is perhaps best demonstrated by illustration. As a simple example of these ideas, consider a universe represented by a state Ψ_n and governed by List-Sort dynamics according to the List $\mathfrak{L} = \{\mathfrak{B}_1, \mathfrak{B}_2, ..., \mathfrak{B}_l\}$. Assume also that the Hilbert space $\mathcal{H}$ of this universe may be written as a product of l sub-registers

$$\mathcal{H} \equiv \mathcal{H}_{[12...l]} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_l. \tag{8.5}$$

Then, the state Ψ_n is separable into F_n factors relative to this fundamental factorisation, where F_n is clearly an integer between 1 and l.

Now, the dynamics of this toy-universe could be such that it is governed by the simple, algorithmic rule

- Given a state Ψ_n separable into F_n factors, the basis set $\mathfrak{B}_{F_n}$ is chosen from the list $\mathfrak{L}$.

So, Ψ_n is tested by the Hermitian operator $\hat{\Sigma}_{n+1} = \hat{B}_{F_n}$, such that the next state Ψ_{n+1} is one of the eigenvectors $\{\Theta^a_{F_n} : a = 1, ..., D\}$, with individual probabilities summing as

$$\sum_{a=1}^{D} |\langle \Theta^a_{F_n} | \Psi_n \rangle|^2 = 1. \tag{8.6}$$

Thus, the development of this toy-universe proceeds in a manner that depends on a particular 'property' of its state. In this instance, the 'property' in question regards the universe's separability. Specifically, if Ψ_n has F_n factors relative to the fundamental factorisation of $\mathcal{H}_{[12...l]}$, the Rules conspire such that the state is tested by the $(F_n)^{th}$ operator $\hat{B}_{F_n}$ from the List $\mathfrak{L} = \{\hat{B}_1, \hat{B}_2, ..., \hat{B}_l\}$. Evidently, and as required, the particular properties of the states referred to by the Rules are mutually exclusive, because if Ψ_n has $F_n = A$ factors it cannot also have $F_n = B$ factors, where $A \neq B$; they are also exhaustive, because no matter what Ψ_n is, it will definitely possess between 1 and l factors.

Now, whichever eigenvector of $\hat{B}_{F_n}$ becomes the new state Ψ_{n+1} will possess F_{n+1} factors, where $1 \leq F_{n+1} \leq l$. The Rules hence dictate that the $(F_{n+1})^{th}$ test $\hat{B}_{F_{n+1}}$ is selected from the List $\mathfrak{L}$ to become $\hat{\Sigma}_{n+2}$, and the universe collapses to a state Ψ_{n+2} which is one of the eigenvectors of this chosen operator.

Clearly, the procedure may be iterated indefinitely.

As an aside, note that if each operator $\hat{B}_{F_n}$ in the above List is itself factorisable into F_n sub-operators, the described Rules could give rise to a universe in which the state Ψ_{n+1} may have the same number of factors as the state Ψ_n (or, at least, no fewer: $F_{n+1} \geq F_n$; recall from Chapter 5 that factorisable operators only have separable eigenvectors), exactly as required for both apparent persistence and a quantum causal set description of classical spacetime to begin to arise.

Furthermore, the particular List-Sort mechanism discussed here is also attractive from the point of view of generating the types of pregeometric lightcone structure necessary for the emergence of classical spacetime. Recall from Section 5.4 that such basic causal relationships are expected to arise from considering how the individual factors of Ψ_{n+1} are affected by counterfactual changes in the factors of Ψ_n. In the present case, moreover, a separability preserving operator $\hat{\Sigma}_{n+1}$, that is selected solely according to the number of

factors of Ψ_n, could permit such counterfactual arguments to be applied, if, for example, $\hat{B}_{i+1}$ is factorisable relative to the same split $\mathcal{H}_{[x_1 x_2 ... x_i]}$ of the Hilbert space as $\hat{B}_i$ for all $i = 1, ..., (l-1)$ and $\left(\prod_{y=1}^{i} \otimes \mathcal{H}_{x_y}\right) = \mathcal{H}_{[12...l]}$.

Thus, List-Sort Rules might effectively be used to generate potential lightcone-like causal relationships between the factors of successive states and operators, and could therefore possibly be employed to develop a universe containing classical-looking space-time.

Inflationary scenarios could also potentially be accounted for by modifying the Rules in similar, suitable ways.

The above illustrative model is just one very simple way of obtaining List-Sort dynamics; many alternative Rules are, of course, possible. In fact, although the state of a universe in a D dimensional Hilbert space governed by a List of List Length l will always be one of only $D \times l$ eigenvectors, there are in general very many ways in which the various different 'properties' of the members of this set could be used to specify particular choices of next operator.

As a second illustration, then, it could be imagined that if these $D \times l$ different vectors are arbitrarily labelled $\{\Phi_j : j = 1, 2, ...(D \times l)\}$, a universe could be considered that follows the Rules

- If $\Psi_n \in \{\Phi_1, \Phi_2, ..., \Phi_{a_1}\}$, then choose the operator $\hat{B}_1$ from the List $\mathfrak{L} = \{\hat{B}_1, \hat{B}_2, ..., \hat{B}_l\}$ to be $\hat{\Sigma}_{n+1}$;
- Alternatively, if $\Psi_n \in \{\Phi_{a_1+1}, \Phi_{a_1+2}, ..., \Phi_{a_2}\}$, then instead choose the operator $\hat{B}_2$ from the List $\mathfrak{L}$ to be $\hat{\Sigma}_{n+1}$;
- $\vdots$
- And so on, up to the case where if $\Psi_n \in \{\Phi_{a_{l-1}+1}, \Phi_{a_{l-1}+2}, ..., \Phi_{a_l}\}$, the operator $\hat{B}_l$ is chosen from the List $\mathfrak{L}$ to be $\hat{\Sigma}_{n+1}$;

where the vectors have arbitrarily re-labelled in ascending order for simplicity in this example, and where

$$a_1 + a_2 + ... + a_l = D \times l. \tag{8.7}$$

As before and as required, the choice of next operator $\hat{\Sigma}_{n+1}$ depends on the current state Ψ_n. Also as before, the algorithm may be repeated indefinitely, because $\Psi_{n+1} \in \{\Phi_j : j = 1, 2, ...(D \times l)\}$ for all n.

Two points are immediately obvious from this second example. First, it is evident that more that one state can lead to the same choice of operator. In fact, this observation was

also paralleled in the first example, because if $F_n \neq 1$ or $F_n \neq l$, the state Ψ_n could be in a number of different partitions, each with F_n factors, and each resulting in the same operator $\hat{B}_{F_n}$. The general conclusion is therefore that although a given state Ψ_n must result in the definite choice of a specific operator $\hat{\Sigma}_{n+1}$, the reverse need not be true: an operator $\hat{\Sigma}_{n+1}$ could have been chosen due to the universe being in any one of a variety of different states.

Secondly, it is apparent that the same List $\mathfrak{L}$ was appropriate in both examples, even though the 'property' of the state that is of interest to the Rules was completely different in the two cases. This highlights the overall point that is the actual Rules relating the state Ψ_n to the operator $\hat{\Sigma}_{n+1}$ that are of most importance in List-Sort mechanisms.

The List-Sort mechanisms discussed up until now have been manifestly deterministic, at least as far as the choice of operators is concerned: a given state Ψ_n results in the selection of a unique operator $\hat{\Sigma}_{n+1}$ from the List, because the universe is obeying Type III List-Sort Rules. These types of mechanism, however, may be extended to *Probabilistic List-Sort* dynamics (governed by Type $IIIa$ List-Sort Rules) in a straightforward manner.

One way of achieving this would be to relax the 'uniqueness' condition of the basis $\mathfrak{B}_i$ chosen by the Rules in reference to Ψ_n. Specifically, given a state Ψ_n, then instead of this deterministically implying the definite selection of a single basis $\mathfrak{B}_i$ from the list $\mathfrak{L}$, it could instead be used to imply the selection of a set of possible bases $\mathfrak{B}_i$, $\mathfrak{B}_j$, $\mathfrak{B}_k$, ... from $\mathfrak{L}$, with probabilities p_i, p_j, p_k, ... respectively. Of course, in this case the condition must hold that

$$p_i + p_j + p_k + ... = 1. \tag{8.8}$$

In other words, a given state (or more specifically, perhaps, a given 'property' of a given state) could give rise to a number of potential next tests $\hat{B}_i, \hat{B}_j, \hat{B}_k, ...$, but which of these operators actually becomes $\hat{\Sigma}_{n+1}$ is determined randomly. Thus, given a state Ψ_n, the next test $\hat{\Sigma}_{n+1}$ could be the operator $\hat{B}_i$ with probability p_i, or the operator $\hat{B}_j$ with probability p_j, and so on.

Equally, then, the probability that the next state Ψ_{n+1} will be any one of the set of eigenstates $\{\Theta_i^a\}$ of $\hat{B}_i$ is given by p_i, whereas the probability it will instead be any one of the set of eigenstates $\{\Theta_j^a\}$ of $\hat{B}_j$ is given by p_j, etc.

The probability of choosing a particular operator compounds with the standard quantum mechanical probability governing the state collapse mechanism to give the overall probability that the universe will jump to a particular next state. Thus, given a state Ψ_n, the probability P_{n+1}^a that the next state Ψ_{n+1} will be a particular eigenstate Θ_A^a of $\hat{B}_A$ is the product of the probability that the operator $\hat{B}_A$ is chosen, with the probability that

the state will then collapse to this eigenvector; viz,

$$P^a_{n+1} = p_A \left|\langle \Theta^a_A | \Psi_n \rangle\right|^2 \tag{8.9}$$

noting that a measure of entropy could be applied in this type of situation, as analogous to Section 5.6.

Actually, the probability that the next state Ψ_{n+1} will be a particular vector may in fact be much higher than this individual result, because it is possible that a given eigenstate may be a member of more than one of the potential basis sets. Of course, this point is a mere technicality in the present discussion, and does not affect the general principles being suggested; it would only be important if just 'before' and 'after' states were being considered, with the 'route' (i.e. the operator) taken by the state being ignored.

As a final comment, note that exactly how the universe is provided with the particular List $\mathfrak{L}$ containing the particular members $\{\mathfrak{B}_i\}$ is unexplained. Indeed, it is perhaps unexplainable, and in a universe developing according to List-Sort Rules, the List $\mathfrak{L}$ may simply have to be accepted along with the laws of physics as just one of the necessary pre-requisites for dynamics to occur. Paraphrasing, it would be as if the universe needs to be provided with an enormous 'data-bank' of possible operators at the outset, just as a (quantum) computer must be provided with all of the logic gates necessary for its functioning in order for it to work.

In fact, this quantum computational analogy may be continued: the Rules R_n governing the universe could similarly be associated with the algorithm or program governing the steps taken during a computation, whilst the state Ψ_n may be related to the current internal configuration of the machine. Again, the universe is viewed as a giant, self-referential quantum automaton.

Of course, if the actual Universe develops according to List-Sort dynamics, it would be a task for physicists to attempt to discover what its List is, and hence what the total set of possible operators available to the physical Universe actually are. *Why* it has whatever List it has, however, is perhaps a question for either the Anthropic Principle or for philosophy.

8.3 Examples of List-Sort Dynamics

The aim of this section is to provide a set of examples that illustrate how toy-universes obeying List-Sort Rules might develop. For reasons of simplicity, attention will be restricted to low dimensional systems, with the usual assumption being made that the underlying principles are not entirely unrelated to those existing in more complicated situations. Specifically, a two qubit model is discussed.

Consider a universe represented by a state in a four dimensional Hilbert space $\mathcal{H}$ fundamentally factorised into two qubit subregisters, i.e.

$$\mathcal{H} \equiv \mathcal{H}^{(4)}_{[12]} = \mathcal{H}^{(2)}_1 \otimes \mathcal{H}^{(2)}_2 \tag{8.10}$$

where dimensional superscripts shall, again, generally be dropped from now on.

Define also an orthonormal basis for $\mathcal{H}_a$, where $a = 1, 2$, as $\mathcal{B}_a = \{|0\rangle_a, |1\rangle_a\}$, such that any state Ψ of the two qubit universe may be written in the form

$$\Psi = \sum_{ij} C_{ij} |i\rangle_1 \otimes |j\rangle_2 \quad , \quad i, j = 0, 1 \tag{8.11}$$

where the four coefficients $C_{ij} \in \mathbb{C}$ produce a 2×2 matrix.

As before, and when no confusion is likely to occur, the subscripts denoting subregisters 1 and 2 will be omitted in the following in favour of the convention that vectors to the left of the tensor product represent the state of qubit 1 in $\mathcal{H}_1$, whereas vectors to the right of the tensor product represent the state of qubit 2 in $\mathcal{H}_2$. Further, the additional simplification may be made that the tensor product is always implied, so that its symbol is consequently omitted: $|i\rangle_1 \otimes |j\rangle_2 \equiv |i\rangle \otimes |j\rangle \equiv |i\rangle|j\rangle \equiv |ij\rangle$.

For a universe represented by a vector in a factorisable Hilbert space, it is possible that its state may be entangled. Moreover, and as has been discussed previously, the issue of entanglement versus separability is of fundamental importance in any discussion of quantum theory, and has played a central role in this thesis.

In Chapter 4 the issue of separability versus entanglement was introduced, with a goal being to classify the different ways an arbitrary vector could be entangled or separable relative to a given factorisation of its Hilbert space. In the current situation, however, any state in $\mathcal{H}^{(4)}_{12}$ is either fully entangled or fully separable, because it can only possess either 1 or 2 factors. This simplification will be useful in the following.

Consider an arbitrary set $\mathfrak{B}$ of orthonormal vectors that forms a basis for a given Hilbert space. Specifically, in fact, it is a standard theorem of vector algebra that for a Hilbert space of dimension D, each basis set of such vectors contains D elements.

Now, the elements of these basis sets will each possess a certain degree of separability or entanglement relative to the fundamental factorisation of the Hilbert space, and this may be classified by referring to the 'type' of the basis. For example, and returning to the current four dimensional case of a two qubit system, a basis set $\mathfrak{B}_{(p,q)}$ can be said to be of type (p, q) if it contains p entangled and q separable elements, where $p + q = 4$. Furthermore, these $p + q$ vectors may be associated with the $p + q$ orthogonal eigenstates of a set of operators acting upon the two qubit universe, and so to define the basis $\mathfrak{B}_{(p,q)}$ also defines the equivalence class of operators $\hat{B}_{(p,q)}$.

Of course, the above analysis is highly simplistic, and may neglect a number of important points. Indeed, it was shown in Section 4.2 that only type $(0,4)$, $(2,2)$, $(3,1)$, and $(4,0)$ bases can be found to span the Hilbert space $\mathcal{H}^{(4)}_{[12]}$, and that no example of a type $(1,3)$ basis set can exist.

Nevertheless, from these elementary ideas it is now possible to begin to construct toy-model universes that develop according to state self-referential List-Sort Rules, as is shown by the following examples.

8.3.1 Example I

Consider as above a two qubit universe represented by a state Ψ_n in the factorisable Hilbert space $\mathcal{H}_{[12]}$. Consider further the particular $(2,2)$ type basis set $\mathfrak{B}_{(2,2)}$ of orthonormal vectors described by

$$\mathfrak{B}_{(2,2)} = \left\{ |00\rangle \ , \ |11\rangle \ , \ \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right) \ , \ \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \right\} \tag{8.12}$$

and also the type $(4,0)$ basis set $\mathfrak{B}_{(4,0)}$ defined by

$$\mathfrak{B}_{(4,0)} = \left\{ \begin{array}{c} \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \ , \ \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \ , \\ \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right) \ , \ \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \end{array} \right\}. \tag{8.13}$$

For convenience and brevity, $\mathfrak{B}_{(2,2)}$ may be written as $\mathfrak{B}_{(2,2)} = \{a, b, c, d\}$, where the order as defined in (8.12) is preserved such that $c = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ etc. Similarly, $\mathfrak{B}_{(4,0)}$ may be written as $\mathfrak{B}_{(4,0)} = \{e, f, g, h\}$, where for example $f = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$.

The basis set $\mathfrak{B}_{(2,2)}$ corresponds to the set of orthonormal eigenvectors of an (equivalence class of) operator $\hat{B}_{(2,2)}$ of the form given in (8.14), where A, B, C, D are real, non-degenerate and non-zero eigenvalues

$$\begin{aligned} \hat{B}_{(2,2)} \quad = \quad & A|00\rangle\langle 00| + B|11\rangle\langle 11| + \frac{C}{2}(|01\rangle + |10\rangle)(\langle 01| + \langle 10|) \\ & + \frac{D}{2}(|01\rangle - |10\rangle)(\langle 01| - \langle 10|) \end{aligned} \tag{8.14}$$

with a similar construction of $\hat{B}_{(4,0)}$ from $\mathfrak{B}_{(4,0)}$.

Now, in the language of the List-Sort dynamics, it is possible to define a List $\mathfrak{L}_X$ of list length 2 as

$$\mathfrak{L}_X \equiv \{\mathfrak{B}_{(2,2)}, \mathfrak{B}_{(4,0)}\} \tag{8.15}$$

or equivalently, $\mathfrak{L}_X \equiv \{\hat{B}_{(2,2)}, \hat{B}_{(4,0)}\}$.

Moreover, and for the sake of example, it is also possible to define the set of Rules governing the development of this toy-universe such that they make reference to the above List. A dynamics based upon a Type III List-Sort mechanism is thus defined.

On possible such mechanism is the following. Suppose the universe develops according to the Rule that

- If the state Ψ_n is separable, then it is tested by an operator whose eigenstates form the basis $\mathfrak{B}_{(4,0)}$;
- But if instead Ψ_n is entangled, it is alternatively tested by an operator whose eigenstates form the basis $\mathfrak{B}_{(2,2)}$.

Then, the development of the universe involves an operator chosen from the List $\mathfrak{L}_X$ in a manner that depends upon a 'property' of the current state. Specifically, if Ψ_n is separable then an operator $\hat{B}_{(4,0)}$ is chosen from the List to be $\hat{\Sigma}_{n+1}$ and the state Ψ_{n+1} will be one of the eigenstates $\{e, f, g, h\}$, whereas if Ψ_n is entangled then an operator $\hat{B}_{(2,2)}$ is instead picked from the List and the state Ψ_{n+1} will be one of the vectors $\{a, b, c, d\}$. Note that these two properties are mutually exclusive and exhaustive, as required: every state is either separable or entangled.

Of course, *which* of the four eigenstates is actually chosen in each case depends entirely on the random nature of the quantum collapse process.

As an illustration of how such a model could develop, consider a universe in an initial state Ψ_0 given by $\Psi_0 = |00\rangle = a$. This state is separable, so by following the Rules, the next state Ψ_1 will be one of the eigenvectors of an operator $\hat{\Sigma}_1 = \hat{B}_{(4,0)}$, such that

$$\hat{B}_{(4,0)}|\Psi_1\rangle = \lambda^{4,0}|\Psi_1\rangle. \tag{8.16}$$

Ψ_1 will be one of the states e, f, g, or h, with corresponding eigenvalues $\lambda_e^{4,0}$, $\lambda_f^{4,0}$, $\lambda_g^{4,0}$, or $\lambda_h^{4,0}$ respectively, where the actual values of the $\lambda^{4,0}$ need play no further part in the discussion, save to say that they are real, non-degenerate and non-zero (c.f. the discussion of Strong operators in Chapter 5).

As in conventional quantum mechanics, the probability $P(\Psi_1, \Psi_0)$ of jumping from the state Ψ_0 to a potential state Ψ_1 is given by the square of the amplitude, that is

$$P(\Psi_1, \Psi_0) = |\langle\Psi_1|\Psi_0\rangle|^2 . \tag{8.17}$$

So, from an examination of (8.12) and (8.13), the relationship (8.17) clearly leads to the amplitudes: $\langle e|a\rangle = \langle f|a\rangle = 1/\sqrt{2}$ and $\langle g|a\rangle = \langle h|a\rangle = 0$. Thus, if the initial state of the universe is $\Psi_0 = a = |00\rangle$ then the subsequent state Ψ_1 will be either

$$\Psi_1 = \Psi_1^e = e = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \tag{8.18}$$

or

$$\Psi_1 = \Psi_1^f = f = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \tag{8.19}$$

with equal probabilities of $\frac{1}{2}$.

Now, both of the states Ψ_1^e and Ψ_1^f are entangled, so no matter what happens, Ψ_1 will be entangled. Consequently, and according to the defined algorithm, the Rules next pick out the basis set $\mathfrak{B}_{(2,2)}$ from the list $\mathfrak{L}_X$, such that the successive state Ψ_2 will be one of the eigenvectors of an operator $\hat{\Sigma}_2 = \hat{B}_{(2,2)}$; viz.,

$$\hat{B}_{(2,2)}|\Psi_2\rangle = \lambda^{2,2}|\Psi_2\rangle. \tag{8.20}$$

Thus, Ψ_2 will be one of the states a, b, c, or d, with corresponding eigenvalues $\lambda_a^{2,2}$, $\lambda_b^{2,2}$, $\lambda_c^{2,2}$ or $\lambda_d^{2,2}$, which are again ignored.

This time, the relevant amplitudes in the transition from Ψ_1 to Ψ_2 are $^{i)}$ $\langle a|e\rangle = \langle b|e\rangle = 1/\sqrt{2}$ with $\langle c|e\rangle = \langle d|e\rangle = 0$, or $^{ii)}$ $\langle a|f\rangle = \langle b|f\rangle = 1/\sqrt{2}$ with $\langle c|f\rangle = \langle d|f\rangle = 0$, depending on whether Ψ_1 is e or f.

So, if the state Ψ_1 of the universe after the transition $\Psi_0 \rightarrow \Psi_1$ was measured and found to be Ψ_1^e, then the next state will either be $\Psi_2 = \Psi_2^{e,a} = a = |00\rangle$ or alternatively $\Psi_2 = \Psi_2^{e,b} = b = |11\rangle$, each with equal probability of $\frac{1}{2}$.

However, if instead the collapse from Ψ_0 to Ψ_1 had left the state at time $n = 1$ as Ψ_1^f, then the subsequent state will be either $\Psi_2 = \Psi_2^{f,a} = a = |00\rangle$ or $\Psi_2 = \Psi_2^{f,b} = b = |11\rangle$, again each with equal probability of $\frac{1}{2}$.

So even after two jumps, the random nature of the quantum state reduction and the chosen Rules for the List-Sort mechanism have led to four different 'histories' for the development of the state from $\Psi_0 \rightarrow \Psi_1 \rightarrow \Psi_2$. Namely, the four possible 'routes' are either $a \rightarrow e \rightarrow a$, or $a \rightarrow e \rightarrow b$ or $a \rightarrow f \rightarrow a$ or $a \rightarrow f \rightarrow b$.

Although it may appear trivial in this case, note that the probability of going from $\Psi_1 = e$ to $\Psi_2 = a$ is $\frac{1}{2}$, and not $\frac{1}{4}$ as would be the case if the sum

$$\langle a|e\rangle + \langle b|e\rangle + \langle a|f\rangle + \langle b|f\rangle \tag{8.21}$$

had to be normed to unity. This is because although the transition from one state to the next makes use of quantum probability amplitudes, once a jump has happened it is possible to say with certainty which state the system is in. In other words, if the state Ψ_1 is measured and found to be $\Psi_1 = \Psi_1^e = e$, it is no longer valid to discuss the probability of jumping from the alternative state $\Psi_1 = \Psi_1^f = f$ to any possible future state Ψ_2, because Ψ_1^f does not exist.

In fact, the ability to describe the state with certainty is a fundamental difference between Schrödinger evolution and state reduction, and arguably leads to the single valued nature of reality. Once state reduction has occurred, the universe is in a unique state Ψ_n^x, and it is therefore meaningless to discuss the probability of transition from any other state

Ψ_n^y. This is a central principle of quantum theory, which holds that although Ψ_n^x and Ψ_n^y may both have been potential futures of the state Ψ_{n-1}, once state reduction has selected the state Ψ_n^x, no other state Ψ_n^y can be said to exist. Over a series of jumps it is therefore necessary to discuss the *classical* probability that the universe will jump from one state to another, and then from that new, now 'known' state to the next.

This type of reasoning was at the heart of the discussion of Section 6.1.4 concerning the qubit Bell inequalities.

The difference between a dynamics based on state reduction and one acting without it may be highlighted by appealing again to the above example. If $\Psi_0 = a$ and also $\Psi_2 = a$ then the quantum probability $P(\Psi_2, \Psi_0)$ of jumping directly from Ψ_2 to Ψ_0 (if this were allowed by the Rules) would be $|\langle\Psi_2|\Psi_0\rangle|^2 = 1$, which is effectively equivalent to a null test from the point of view of the universe. If, however, it is specified that the universe develops from Ψ_0 to Ψ_1 to Ψ_2, where $\Psi_1 = \Psi_1^e$ and so $\Psi_2 = a = \Psi_2^{e,a}$, then the probability $P(\Psi_2^{e,a}, \Psi_1^e, \Psi_0)$ is instead given by

$$P(\Psi_2^{e,a}, \Psi_1^e, \Psi_0) = |\langle\Psi_2^{e,a}|\Psi_1^e\rangle|^2 \, |\langle\Psi_1^e|\Psi_0\rangle|^2 = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \tag{8.22}$$

and this indicates that information has been extracted from the system. As discussed previously, the state reduction mechanism is essential in order to associate the subscript on Ψ_n with a temporal-like parameter.

Generalising, the probability $P(\Psi_{n+N}, \Psi_{n+N-1}, ..., \Psi_n)$ of the universe developing from the state Ψ_n to a given state Ψ_{n+N} via a series of *specified* intermediate states $\Psi_{n+1,}$ Ψ_{n+2}, ..., Ψ_{n+N-1} is given by the classical product of the squared moduli of the N appropriate quantum probability amplitudes, such that

$$P(\Psi_{n+N}, \Psi_{n+N-1}, ..., \Psi_n) = |\langle\Psi_{n+N}|\Psi_{n+N-1}\rangle|^2 \, |\langle\Psi_{n+N-1}|\Psi_{n+N-2}\rangle|^2 ... \, |\langle\Psi_{n+1}|\Psi_n\rangle|^2 . \tag{8.23}$$

Note that this generalisation may be related to the concept of entropy discussed in Section 5.6, where a measure of entropy is associated with different sets of possible futures of a quantum system.

Note also that the inclusion of state reduction shows the inherent asymmetry and irreversibility of time. Given that the universe is in a state $\Psi_{n,}$ it is reasonable to ask the question: what is the probability that the universe will jump to a specific state $\Psi_{n+1,}$ and will then jump to another specified state $\Psi_{n+2,}$ and so on through a chain of specified states up to Ψ_{n+N}? However, the reverse question is different. If it is known that the universe is in a state Ψ_{n+N}, then the probability that it jumped from the previous state Ψ_{n+N-1} is 1, assuming that no information has been lost during the transition such that Ψ_{n+N-1} is also known. It is meaningless to ask in this context about the probability of arriving

at the current state from a given alternative past, because state reduction ensures that only one past actually occurred. Although it may not always be possible in the present to retrodict with certainty what the past actually *was*, a physicist can be sure that a unique past *did* occur. This is different from discussions of the future, because it is never possible to predict what *will* happen, but only what *might* happen as a potential outcome of a potential quantum test. Again, strong parallels are drawn here with the discussion of the qubit Bell inequalities in Section 6.1.4.

Returning now to the example in hand, the universe described by this model would continue to be tested by the operators $\hat{B}_{(2,2)}$ or $\hat{B}_{(4,0)}$ according to whether its state is respectively entangled or separable.

In fact, ignoring the 'route' by which it got there, it can easily be shown that after $2n$ steps, for $n \in \mathbb{N}$, the universe is in either the state $\Psi_{2n} = a = |00\rangle$ or $\Psi_{2n} = b = |11\rangle$, each with probability $\frac{1}{2}$, whereas after $2n-1$ steps the universe is in either the state $\Psi_{2n-1} = e = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ or $\Psi_{2n-1} = f = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$, also each with probability $\frac{1}{2}$. The system will 'oscillate' between having a state that is separable and one that is entangled, though which particular separable (i.e. a or b) or entangled (i.e. e or f) state it is actually in depends on the random nature of quantum state reduction.

It is perhaps surprising to note that a different choice of initial condition, Ψ_0, would not drastically affect the subsequent development of the model. To illustrate why, observe that, according to the Rules governing the dynamics, if the universe were ever to collapse to any of the vectors c, d, g or h, it would remain in that state from then on, because these states are eigenvectors of both operators. So, if the initial state was an arbitrary normalised vector of the form

$$\Psi_0 = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \tag{8.24}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, it is possible to conclude that after a period of n steps the universe would either be following the above pattern of $...[a/b] \longrightarrow [e/f] \longrightarrow [a/b]...$, where "$[a/b]$" denotes "$a$ or b" etc., or else it would be 'stuck' in one of the states c $(= g)$ or d $(= h)$. Of course, exactly which course of action has the highest propensity for occurring depends on the actual values of the complex coefficients $\alpha, \beta, \gamma, \delta$, because these determine whether Ψ_0 is entangled or separable, and the nature of the potential probability amplitudes $\langle\Psi_1|\Psi_0\rangle$.

Actually, the details of the above 'extension' are in fact considered fairly unimportant anyway. After all, it is the principles behind the mechanisms investigated in this section that are of interest, specifically those concerning the question of how a universe might develop according to List-Sort Rules. Thus, in the remaining examples it is to be recalled that a different choice of initial condition would not add anything significant to the discussion, and so the possibility of choosing alternative states as Ψ_0 is omitted.

8.3.2 Example II

A toy-universe model that is perhaps more interesting than that of Example I could add to the List $\mathfrak{L}_X$ a type $(0,4)$ basis set $\mathfrak{B}_{(0,4)}$, where this completely separable basis is defined as

$$\mathfrak{B}_{(0,4)} = \left\{ \begin{array}{c} \frac{1}{2}\left(|00\rangle + |10\rangle + |01\rangle + |11\rangle\right) \ , \ \frac{1}{2}(|00\rangle + |10\rangle - |01\rangle - |11\rangle) \\ \frac{1}{\sqrt{2}}\left(|00\rangle - |10\rangle\right) \ , \ \frac{1}{\sqrt{2}}(|01\rangle - |11\rangle) \end{array} \right\} \tag{8.25}$$

Analogously to before, where the bases $\mathfrak{B}_{(2,2)}$ and $\mathfrak{B}_{(4,0)}$ were redefined as $\{a,b,c,d\}$ and $\{e,f,g,h\}$ respectively, the elements of $\mathfrak{B}_{(0,4)}$ written in the above order may be labelled as $\mathfrak{B}_{(0,4)} = \{j,k,l,m\}$ for simplicity, with for example $l = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes |0\rangle$.

Clearly, the addition of $\mathfrak{B}_{(0,4)}$ to $\mathfrak{L}_X$ defines a new List $\mathfrak{L}_Y$, such that

$$\mathfrak{L}_Y \equiv \{\mathfrak{B}_{(2,2)}, \mathfrak{B}_{(4,0)}, \mathfrak{B}_{(0,4)}\} \tag{8.26}$$

or equivalently $\mathfrak{L}_Y \equiv \{\hat{B}_{(2,2)}, \hat{B}_{(4,0)}, \hat{B}_{(0,4)}\}$, where evidently $\mathfrak{L}_Y \supset \mathfrak{L}_X$.

In the mechanism proposed now, it is supposed that the universe follows the Rule

- If the state Ψ_n is separable then the basis $\mathfrak{B}_{(2,2)} = \{a,b,c,d\}$ is picked from the List $\mathfrak{L}_Y$, and the next operator $\hat{\Sigma}_{n+1}$ to test the state will be $\hat{B}_{(2,2)}$;
- Whereas if Ψ_n is entangled then the basis $\mathfrak{B}_{(0,4)} = \{j,k,l,m\}$ is instead chosen from $\mathfrak{L}_Y$, and the next state Ψ_{n+1} is one of the eigenstates of an alternative operator $\hat{\Sigma}_{n+1} = \hat{B}_{(0,4)}$.

So, and as desired, a particular 'property' of the state Ψ_n of the universe (again, its separability) is being used to select a particular operator from the List $\mathfrak{L}_Y$ to become $\hat{\Sigma}_{n+1}$; the universe is hence governed by Type III List-Sort dynamics.

As as aside, note that there is no general constraint in the List-Sort method for the Rules to use *every* operator contained in the specified List. Indeed, the current mechanism could equally be realised with the shorter List $\mathfrak{L}_Z$ defined as $\mathfrak{L}_Z \equiv \{\mathfrak{B}_{(2,2)}, \mathfrak{B}_{(0,4)}\}$, where $\mathfrak{L}_Z \subset \mathfrak{L}_Y$.

However, if the actual physical Universe runs according to List-Sort Rules, a question would remain in this case as to why any operator should be included on its List if it could never be used. In general, then, such 'non-essential' operators should perhaps best be removed for efficiency.

That said, in the present example, the inclusion of the non-used $\mathfrak{B}_{(4,0)}$ to the List does not make any real difference, and so the longer List $\mathfrak{L}_Y$ will be retained; this is in preparation for Examples III and IV, where the whole of $\mathfrak{L}_Y$ will be employed.

At any time n the state of the universe in question is constrained to be one of the eight possible vectors: a, b, c, d, j, k, l, m. Furthermore, which of these it actually *is* at time n determines the basis set at time $n+1$, according to the above Rules.

In addition, and as in Example I, the likelihood of transition from each of these states to the next depends on the probability amplitudes. Specifically, the probability $|\langle y|x\rangle|^2$ of transition from the state $\Psi_n = |x\rangle$ to a potential state $\Psi_{n+1} = |y\rangle$ is given as the element in column x, row y of Table 8.1

*	**a**	**b**	**c**	**d**	**j**	**k**	**l**	**m**
a	1	0	-	-	1/4	1/4	1/2	0
b	0	1	-	-	1/4	1/4	0	1/2
c	0	0	-	-	1/2	0	1/4	1/4
d	0	0	-	-	0	1/2	1/4	1/4
j	-	-	1/2	0	-	-	-	-
k	-	-	0	1/2	-	-	-	-
l	-	-	1/4	1/4	-	-	-	-
m	-	-	1/4	1/4	-	-	-	-

Table 8.1

noting that transitions which are forbidden by the Rules, for example a separable state jumping to one of the members of $\mathfrak{B}_{(0,4)}$, are indicated by a dash.

Moreover, note that the probabilities are normed in such a way that

$$\sum_i |\langle i|x\rangle|^2 = 1 \quad , \quad i, x = a, b, c, d, j, k, l, m \tag{8.27}$$

because if the universe is in a state $\Psi_n = |x\rangle$ it must certainly be able to jump to *something*.

By way of example, let the initial state Ψ_0 of the universe be $\Psi_0 = c = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. This state is entangled, so according to the Rules the next state Ψ_1 will be one of the members of the basis set $\mathfrak{B}_{(0,4)} = \{j, k, l, m\}$, and will hence be an eigenstate of the equation

$$\hat{B}_{(0,4)}|\Psi_1\rangle = \lambda^{0,4}|\Psi_1\rangle \tag{8.28}$$

where $\lambda^{0,4}$ is an eigenvalue that is subsequently ignored.

It is possible to develop the state of this toy-universe model over a number of steps, just as it was in Example I. So, if $\Psi_0 = c$, then Ψ_1 must be either j, l or m because the probability that the universe will collapse to the state $\Psi_1 = k$ is zero.

Moreover, by applying the same logic, and from the results given in Table 8.1, it follows that if $\Psi_1 = j$ then Ψ_2 must be either a, b or c. Alternatively, if instead $\Psi_1 = l$, it implies that Ψ_2 must be either a, c or d, whereas if $\Psi_1 = m$ it implies that Ψ_2 must be either b, c or d.

This process may be continued to generate a set of possible transitions from Ψ_2 to Ψ_3, and then from Ψ_3 to Ψ_4, etc., until a 'family tree' of different possible chains of states are created.

Of course, quantitatively some states or patterns are more likely to occur than others due to the list of probabilities given in Table 8.1. For example, once the universe has jumped into the state a, then according to the Rules it will remain in this state forever.

It is easy to write a computer program that will iterate this two qubit universe over N steps according to the specified Rules. Furthermore, a number of questions can then be asked of the system's development. For example, what is the probability that after $N = 3$ jumps the universe will have proceeded through the history $\Psi_0 = c$, $\Psi_1 = m$, $\Psi_2 = d$, $\Psi_3 = k$? (Answer: 1/32). Alternatively, what is the probability that after $N = 57$ jumps the universe is in the state j? (Answer: $\sim 9.3132 \times 10^{-10}$).

One interesting question is: what is the probability that after N steps the universe is in an entangled state, given that $\Psi_0 = c$? Paraphrasing, what is the probability that $\Psi_N = c$ or $\Psi_N = d$? The result of this is shown in Figure 8.1, where the x-axis is n and the y-axis is the probability $P(\Psi_n = [c/d])$.

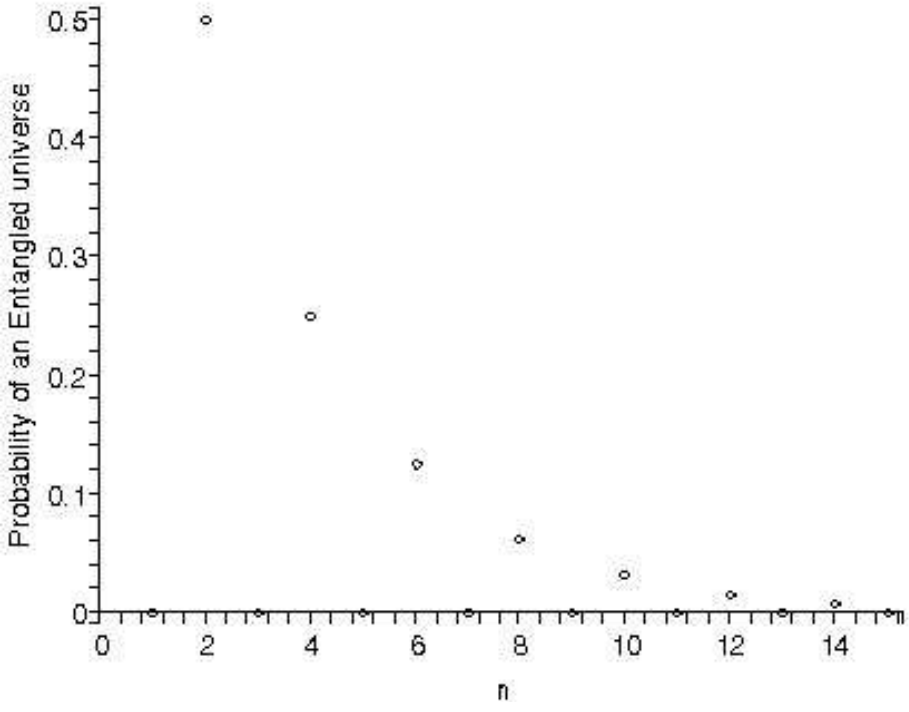

Figure 8.1: Probability, $P(\Psi_n = [c/d])$, of an entangled universe after n steps.

As is evident from the graph, the outcome of the presented Rules governing a universe

in an initial state $\Psi_0 = c$ is similar to the result of the previous toy-model universe (for the case when Ψ_0 equalled a), in the respect that the wavefunction Ψ_n will definitely be separable at periodic intervals: in this example, Ψ_n will always be separable when n is odd, whereas in Example I the state was always separable when n was even. The major difference between these two examples, however, is that in the current case the probability of getting an entangled state at the remaining times decreases as the number of jumps, n, increases, whereas in Example I this probability remained at unity.

This model therefore has an important physical interpretation. The situation here is of a universe that begins in an initial entangled state, but is driven to a greater likelihood of separability as it develops. Obvious comparisons can hence consequently be made with the discussions of Chapters 3, 4 and 5 in which it was suggested how the actual Universe may have developed in an analogous manner, from an initial entangled state at the quantum Big Bang, to one that now appears to possess an enormous amount of semi-classical looking separability and persistence.

8.3.3 Example III

The dynamics of the models in sub-sections 8.3.1 and 8.3.2 could be described as 'semi-deterministic'. That is, although the quantum reduction of the state into one of the eigenvectors of the operator is a stochastic process, if it is known which state Ψ_n the universe is in, it is always possible to say with certainty what the next test $\hat{\Sigma}_{n+1}$ will be. The models in the previous two sub-sections provided examples of deterministic List-Sort dynamics, that is, Type III Rules.

It is, however, possible to consider a development mechanism based on probabilistic List-Sort, Type $IIIa$ dynamics, as alluded to in Section 8.2. Under such circumstances, a given state Ψ_n may imply a number of potential 'candidates' to become the next operator $\hat{\Sigma}_{n+1}$, but which is actually chosen depends on some kind of random factor. Thus, each potential operator is associated with a particular probability of being chosen, given the presence of a certain state.

Of course, there are many forms that these various operator probabilities could take. Firstly, for example, they could simply be fixed 'weighting factors', where each operator on the list is associated with a fixed probability of being chosen, given a particular property of the state (for instance, whether it is entangled or separable). Secondly, however, they could instead involve a Rule in which these probabilities themselves are a function of the current state, as will be explained later. Thirdly, even, the choice of operator probability could actually depend somehow on some sort of higher order quantum process that would be in need of definition.

For the time being, attention will be restricted to the simplest possible type of situation, and a model will be considered in which the 'random factor' is constant. In the following case of Example IV in Sub-section 8.3.4, however, this factor will instead be a variable that depends upon which particular state the universe is currently in.

Thus for the example at hand, an elementary dynamics is suggested in which if the state of the universe has one particular 'property' then it is developed in one way, but if it has another 'property' then it will instead be developed in one of two potential ways, though which of these ways is actually chosen is a random process. Specifically, the probability that it will be tested by the first of the two potential operators is defined to be a constant, Q, whereas the probability that the next state will be one of the eigenstates of the alternative potential operator is given by $(1-Q)$, where $0 \leq Q \leq 1$.

Consider as before a two qubit system, and consider again the above List $\mathfrak{L}_Y$, defined as $\mathfrak{L}_Y \equiv \{\mathfrak{B}_{(2,2)}, \mathfrak{B}_{(4,0)}, \mathfrak{B}_{(0,4)}\}$. Suppose further that the universe follows probabilistic List-Sort dynamics, and is governed by the Rules

- If the state of the universe is entangled, then the basis $\mathfrak{B}_{(0,4)}$ is picked from the list $\mathfrak{L}_Y$, and Ψ_n is tested by an operator $\hat{\Sigma}_{n+1} = \hat{B}_{(0,4)}$, such that the next state Ψ_{n+1} is one of the members of $\mathfrak{B}_{(0,4)}$;

- However, if Ψ_n is separable there is a probability Q that the operator $\hat{B}_{(2,2)}$ will be chosen from the list to be $\hat{\Sigma}_{n+1}$, but also a probability $(1-Q)$ that $\hat{B}_{(4,0)}$ will be selected instead. Thus if Ψ_n is separable, there is a probability Q that the next state Ψ_{n+1} will be one of the elements of $\mathfrak{B}_{(2,2)}$, but a probability $(1-Q)$ that it will instead be one of the elements of $\mathfrak{B}_{(4,0)}$.

Here $\mathfrak{B}_{(2,2)} = \{a, b, c, d\}$, $\mathfrak{B}_{(4,0)} = \{e, f, g, h\}$ and $\mathfrak{B}_{(0,4)} = \{j, k, l, m\}$ are as defined previously, and $0 \leq Q \leq 1$. Clearly, such a universe is governed by a Type $IIIa$ mechanism.

As in the previous examples, the two qubit universe described here will develop in an automatic way, with its state jumping from one vector to the next according to the Rules that govern the model's dynamics.

Also as in the previous examples, the individual probability amplitudes will play a crucial role in determining the propensity for a given state Ψ_n to jump to a particular future state Ψ_{n+1}, within, of course, the boundaries set by the Rules. In fact, it is easy to generalise Table 8.1 for the 144 probabilities given by $|\langle y|x\rangle|^2$ for $\Psi_n = |x\rangle$, $\Psi_{n+1} = |y\rangle$ and $x, y = a, b, c, ..., m$, noting again that many of the transitions would be 'dashed' because they are forbidden in the current mechanism.

To illustrate the type of dynamics proposed here, assume that as in Example II, the two qubit universe may be prepared in the initial state $\Psi_0 = c = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$, where

it is again noted that alternative initial conditions would not give rise to significantly different or interesting outcomes.

Clearly, $\Psi_0 = c$ is entangled, so the Rules dictate that the next operator $\hat{\Sigma}_1$ will be $\hat{B}_{(0,4)}$, such that the next state Ψ_1 will be either j, k, l or m with relative probabilities of 1/2, 0, 1/4 and 1/4 respectively.

Now, if Ψ_1 turns out to be $\Psi_1 = j$, it is evident that the universe has collapsed to a separable state (actually, the same would be true whether it had collapsed to l or m, but that is beside the point). So, according to the Rules, the next operator $\hat{\Sigma}_2$ to test the state will either be $\hat{\Sigma}_2 = \hat{B}_{(2,2)}$ with probability Q, or else $\hat{\Sigma}_2 = \hat{B}_{(4,0)}$ with probability $(1 - Q)$. Overall, then, the next state Ψ_2 will be one of eight possibilities: it will be one of the vectors a, b, c, d, e, f, g, h with relative probabilities given in Table 8.2.

$\boldsymbol{\Psi}_2$	a	b	c	d	e	f	g	h
Prob. $P(\boldsymbol{\Psi}_2, \boldsymbol{\Psi}_1 = j)$	$Q/4$	$Q/4$	$Q/2$	0	$(1-Q)/2$	0	$(1-Q)/2$	0

Table 8.2

Of course, similar tables would be generated for the probabilities of jumping to a particular state Ψ_2 from the alternative states $\Psi_1 = l$ or $\Psi_1 = m$. In these instants, the same set $\{a, ..., h\}$ of eight possible vectors would be present, because l and m are both separable and would hence both imply that $\hat{\Sigma}_2 = \hat{B}_{(2,2)}$ with probability Q or $\hat{\Sigma}_2 = \hat{B}_{(4,0)}$ with probability $(1-Q)$, but the various quantum probability amplitudes that result would now be different.

As in the previous examples, the above model could be developed through an arbitrary number of steps to give rise to complicated 'trees' of possible histories for the system, each with a particular probability of occurring. Also as before, various questions can be asked regarding the possible nature of the system after a given number of jumps.

The number Q is seen as a free parameter in the model. Two particular situations, however, are the special cases when $Q = 0$ and $Q = 1$. For $Q = 0$, the development proceeds in a similar way to that experienced by the system described in Example I, that is, a universe with a state that oscillates between being certainly entangled and certainly separable.

In the converse case of $Q = 1$, however, the model is instead identical to the system described in Example II, that is, a universe with a state that it increasingly more likely to be separable as it develops.

But, a more novel situation occurs for $0 < Q < 1$. In these cases the probability that the universe is separable or entangled after n jumps tends to some fixed value as n becomes large. Additionally, unlike for $Q = 1$, in which for $n = odd$ the state is always separable and it is only the $n = even$ states whose probability of being entangled is driven to zero, for $0 < Q < 1$ the probability that the state is entangled (or separable) tends to the same

fixed value for both odd and even values of n; at any 'time' n there is always a possibility that the state could be entangled. In such a universe, the likelihood of the n^{th} state being separable for $n \gg 1$ is approximately the same as the likelihood of the $(n-1)^{th}$ state also being separable. This point is illustrated in Figure 8.2, which is a plot of the probability (y-axis) of getting an entangled state after n jumps (x-axis) for $Q = 1/2$.

Such a model may have an important physical interpretation in terms of discussions regarding the emergence of persistence.

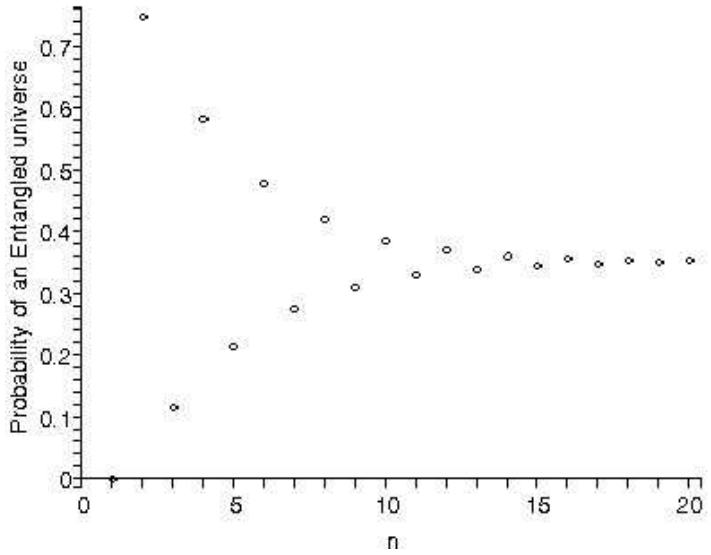

Figure 8.2: Probability of an entangled universe after n steps for $Q = 1/2$.

Of course, it is not in principle difficult to determine what the probability of obtaining an entangled state tends to as n becomes large, for the given initial condition, value of Q, and set of Rules. It may also be interesting to consider the rate at which the probability tends to this fixed value. In fact, since from the above graph the convergence to this value appears 'smooth' (in some sense), it is not impossible to suggest that the probability of obtaining an entangled universe might begin to be approximated by a continuous function of n, particularly as n increases. In short, it might be possible to fit a continuous curve to the above data, and the equation of this curve might play an important role in discussions regarding the emergence of continuous physics from the underlying and discrete pregeometric structure.

The same argument may also be true for the results indicated by Figure 8.1 of Example II: a continuous 'decay' curve could be fitted to the results when n is even, and the equation of this curve might be useful in approximating average properties of the system.

Continuing this train of thought, and by considering the various probabilities that the

universe will be in a particular state $\Psi_n \in \{a, ..., m\}$ after n steps, it is noted that it is possible to describe the likely 'trajectories' or 'histories' of the system between times 0 to n. In other words, by considering the probabilities of obtaining various 'histories' $\Psi_0 \rightarrow \Psi_1 \rightarrow ... \rightarrow \Psi_n$, a discrete and probabilistic 'equation of motion' could be determined for the development of the universe. Moreover, it might then be the case that this too could be approximated using continuous looking laws and functions of n. In this case, therefore, the discrete process of jumps of the system would, in some sense, effectively begin to be described by continuous equations, exactly as required for the emergence of continuous physics.

It is intriguing to speculate on the potential links between this type of analysis of the presented models, and the types of dynamics discussed in models of quantum stochastic calculus (e.g. [85]).

8.3.4 Example IV

Example III is a probabilistic List-Sort universe that develops in a way that depends on a fixed probability: the choice of operator $\hat{B}_{(2,2)}$ or $\hat{B}_{(4,0)}$ is influenced by the value of Q, and Q remains constant throughout. A natural extension to this type of mechanism is therefore to allow the probability of using $\hat{B}_{(2,2)}$ or $\hat{B}_{(4,0)}$ to depend on the current state.

It is important to clarify the difference between these two types of mechanism. In the 'fixed' case, the Rules select a set of potential operators from the List based upon a particular 'property' of the state, and there is then a fixed probability as to which of these operators is actually used. So, in the case of Example III, the probability of picking a given operator from the List depended only on a 'property' of the state: if the state was separable, then there was a probability Q of choosing the operator $\hat{B}_{(2,2)}$, but a probability $(1 - Q)$ of instead choosing the operator $\hat{B}_{(4,0)}$. Moreover, these probabilities were independent of what the details of the state actually were: all that was important was whether it was entangled or separable, because this was the only property used in the selection process.

In the type of mechanism proposed in this sub-section, however, although the Rules are such that a particular 'property' of the state is still used to select a set of potential operators, which operator from this set actually gets chosen does now depend on the precise details of the state. In other words, the probability of picking a particular operator from the set of potentials is not a fixed number defined at the outset, but is instead a variable defined as a particular function of the state. Thus, and as will be shown below, in the type of mechanism proposed here it is not sufficient to simply say whether the state Ψ_n is entangled or separable in order to determine the propensity of using a particular next

operator; it is also necessary to know exactly what the state is in order to determine the probability of what the next operator $\hat{\Sigma}_{n+1}$ will be.

The general case of this type of idea is therefore the following. Given a state Ψ_n in a D dimensional Hilbert space $\mathcal{H}$ with a List of bases $\mathfrak{L}$ defined as $\mathfrak{L} \equiv \{\mathfrak{B}_a : a = 1, 2, ..., l\}$, the Rules could pick out a set of potential operators $\{\hat{B}_i,\ \hat{B}_j,\ \hat{B}_k, ...\}$ to be the next test $\hat{\Sigma}_{n+1}$, with respective probabilities $p_i,\ p_j,\ p_k, ...$ However, unlike in the previous, fixed probability case for which $p_i,\ p_j,\ p_k, ...$ were constants, a mechanism is now considered where

$$p_i = f_i(\Psi_n) \quad , \quad p_j = f_j(\Psi_n) \quad , \quad \ldots \tag{8.29}$$

with the actual functions $\{f_i,\ f_j, ...\}$ defined in the Rules governing the dynamics, and where

$$f_i(\Psi_n) + f_j(\Psi_n) + ... = 1. \tag{8.30}$$

The idea is perhaps best illustrated by example.

Consider again a two qubit universe, and the basis sets $\mathfrak{B}_{(2,2)} = \{a, b, c, d\}$, $\mathfrak{B}_{(4,0)} = \{e, f, g, h\}$ and $\mathfrak{B}_{(0,4)} = \{j, k, l, m\}$ defining the List $\mathfrak{L}_Y$ as before. In this illustration, the Rules governing the system are chosen to be analogous to those used in Example III; viz.

- If Ψ_n is entangled then the basis $\mathfrak{B}_{(0,4)}$ is picked from the list $\mathfrak{L}_Y$, and the next state Ψ_{n+1} is one of the eigenstates of $\hat{B}_{(0,4)}$;
- However, if Ψ_n is separable then there is a probability R that the basis $\mathfrak{B}_{(2,2)}$ is picked from the list $\mathfrak{L}_Y$, such that the next state Ψ_{n+1} is one of the eigenstates of $\hat{B}_{(2,2)}$, but a probability $(1 - R)$ that the basis $\mathfrak{B}_{(4,0)}$ is instead picked from the list $\mathfrak{L}_Y$, such that the next state Ψ_{n+1} is one of the eigenstates of $\hat{B}_{(4,0)}$.

In this case, however, R is not a constant, but is a function of Ψ_n. Specifically, R could be defined in this example by

$$R = |\langle \Psi_n | X \rangle|^2 = |\langle X | \Psi_n \rangle|^2 \tag{8.31}$$

where X is some fixed 'reference' vector that is normed such that $0 \leq R \leq 1$.

Clearly, the value of R depends upon which state the universe is currently in, thereby making the dynamics strongly self-referential. As a consequence, it turns out that some separable states Ψ_n are more likely to be tested by the operator $\hat{B}_{(2,2)}$ whilst others are more likely to be tested by $\hat{B}_{(4,0)}$, depending of course on the magnitude of the inner product of Ψ_n with X.

For the sake of this illustration, X could arbitrarily be chosen as

$$X = \frac{1}{2}\left(|00\rangle + |10\rangle + |01\rangle + |11\rangle\right) \tag{8.32}$$

so that in fact $X = j$. Thus, R will be given by one of the square amplitudes $R = |\langle y|X = j\rangle|^2$ for $y = a, b, j, k, l, m$, noting that $\{a, b, j, k, l, m\}$ are the only separable states, and hence the only states of relevance here. Clearly, the values of $\langle a|j\rangle$ and $\langle b|j\rangle$ may be readily extracted from Table 8.2 (by putting $Q = 1$), whilst $\langle j|j\rangle = 1$ with $\langle k|j\rangle = \langle l|j\rangle = \langle m|j\rangle = 0$, and this highlights the above point that some states are considerably more likely to be tested by, say, $\hat{B}_{(2,2)}$ than others.

So as an example, if the universe is known to be in the separable state $\Psi_n = a$, the Rules dictate that the next operator $\hat{\Sigma}_{n+1}$ will be $\hat{B}_{(2,2)}$ with probability $|\langle a|j\rangle|^2$, but will be $\hat{B}_{(4,0)}$ with probability $(1 - |\langle a|j\rangle|^2)$.

Furthermore, given the state $\Psi_n = a$, the 'compound' probability $P(e, a)$ that the next state Ψ_{n+1} will be the element e of the basis set $\mathfrak{B}_{(4,0)}$ is clearly given by

$$\begin{aligned} P(e,a) &= (1 - |\langle a|j\rangle|^2)|\langle e|a\rangle|^2 \\ &= (3/4)\ (1/2)\ = 3/8 \end{aligned} \tag{8.33}$$

which is just the product of the probability that a will be tested by the operator $\hat{B}_{(4,0)}$ multiplied by the probability that the outcome of this test will be e.

Of course, if Ψ_{n+1} is indeed the entangled state e, then the next test will be $\hat{\Sigma}_{n+2} = \hat{B}_{(0,4)}$, and the subsequent state Ψ_{n+2} will be one of the elements $\{j, k, l, m\}$ of the basis set $\mathfrak{B}_{(0,4)}$. Evidently, and as in previous models, it is easy to continue this process indefinitely and generate a 'tree' of sets of possible 'histories' from a given initial state. It is also possible to ask questions of the system, such as the probability of obtaining a certain Ψ_N at time N, or whether the universe at time M is likely to be entangled or separable.

As with Examples II and III, the model presented here also has an important physical interpretation. In the system described in this sub-section, a dynamics is presented in which the presence of certain states leads to a greater propensity that the universe will be developed by a particular operator. In other words, some states are more likely to be tested by certain operators than others.

This, however, is generally what occurs in the real Universe. Given a state that is separable into a number of particular factors (representing an apparatus, a subject, a physicist etc.), it is often possible to predict what the next operator may be like, because scientists are generally able to set up certain experiments in the laboratory, and represent them by Hermitian operators. Furthermore, the presence of a particular set of initial sub-states does generally seem to make some choices of test considerably more likely than others. As an example, if a Stern-Gerlach machine and an electron are present as factors

of Ψ_n, it might be expected that the Universe will select an operator that appears to represent a spin-measurement; indeed, this course of action certainly seems more likely than an alternative choice of operator being made, where, perhaps, the next state Ψ_{n+1} of the universe appears to contain the results of some sort of position measuring experiment.

Thus, the point is that certain initial conditions, i.e. certain states Ψ_n, do appear to constrain the Universe to develop in certain ways; a particular state Ψ_n does seem to make a particular test more likely than others.

These issues themselves lead onto a general philosophical point. Conventional quantum mechanics generally deals with statements of the form: "if a given quantum system is tested in a certain way, what is the probability that a certain outcome will be measured?". However, this view ignores the more fundamental question that should perhaps be asked first: "what is the probability that a physicist will choose to apply that particular test to the system anyway?".

Such a question is presumably an important feature in a fully quantum universe. As has been discussed previously, if the Universe contains everything, there can be no external agent acting as "The Physicist" deciding which test to apply to its state at any particular time. The choice of operator acting upon the wavefunction must therefore be a result of something going on inside the Universe. Further, assuming that human physicists are themselves quantum systems (or at least are comprised of quantum systems), they must be subject to quantum laws and are the outcomes of quantum tests. Thus, any 'decision' they appear to make, regarding the selection of a particular operator to test their surroundings with, is the result of an earlier quantum process[22]. This conclusion was very much the stance of Feynman [72].

So, given that the Universe is represented by a quantum state, and that this state is one of the eigenvectors of a quantum operator, physicists are left with the question of why this particular operator was selected. Exactly how this selection mechanism might work, and whether it is based on a deterministic algorithm or the stochastic result of quantum probability, are interesting questions seldom addressed in a science normally concerned with predicting the answers to specific, well defined questions. In a self-referential, quantum Universe featuring endo-physical observers, however, they must be unavoidable considerations.

8.4 Generated-Sort Dynamics

In Type III List-Sort dynamics, the universe possesses an enormous set $\mathfrak{L}$ of 'pre-ordained' operators $\{\hat{B}_1, \hat{B}_2, ..., \hat{B}_l\}$, and the Rules R_n select just one of these to be the next test

[22]It is intended here to ignore arguably metaphysical notions that involve free-will or consciousness.

$\hat{\Sigma}_{n+1}$ based on the 'properties' of the current state Ψ_n.

A converse to this sort of mechanism would therefore be one in which there is no pre-existing set of operators waiting to be picked to test the state. Under circumstances such as these, a dynamics could be imagined in which the next operator $\hat{\Sigma}_{n+1}$ is somehow 'created' at time n in a manner that is based entirely upon the current state. In other words, in these scenarios the next operator $\hat{\Sigma}_{n+1}$ is not selected from an already existing List, but is instead generated from Ψ_n according to the Rules governing the universe in question.

Thus, the operator $\hat{\Sigma}_{n+1}$ could be taken to be some sort of function f_n of the current state, and it would be possible to write

$$\hat{\Sigma}_{n+1} = f_n(\Psi_n) \tag{8.34}$$

or equivalently

$$\mathfrak{B}_{n+1} = f_n(\Psi_n) \tag{8.35}$$

where $\mathfrak{B}_{n+1} \equiv \{\Phi^1_{n+1}, \Phi^2_{n+1}, ..., \Phi^D_{n+1}\}$ for a D dimensional Hilbert space, and $\langle \Phi^i_{n+1} | \Phi^j_{n+1} \rangle = \delta_{ij}$ for $i, j = 1, 2, ...D$.

Such a mechanism may be called a '*Generated-Sort*' dynamics.

Generated-Sort dynamics could lead to a 'phase space' of possible states that is much larger than that available in List-Sort dynamics, where the phase space is defined in terms of the number of *different* states the universe could exist in over all time n. In particular, in Generated-Sort dynamics this set of different states could be unbounded, whereas in List-Sort dynamics the total set of possible states will always be constrained according to the size of the List.

To justify this last point, note that a finite List $\mathfrak{L}$ of operators acting over a finite Hilbert space implies a finite number of possible states. Specifically, given that every possible Hermitian operator in a Hilbert space of dimension D possesses D orthogonal eigenstates, then if the number l of possible equivalent classes of operators contained in $\mathfrak{L}$ is finite (i.e. $\mathfrak{L}$ is of List Length l), there can be no more than $D \times l$ different states for the universe, such that the universe's phase space is of 'size' $D \times l$. In other words, as the universe develops, its state will always be one of these $D \times l$ different possibilities. Of course, quantum stochastics does still ensure that it is generally impossible to say in advance *which* of this set the n^{th} state will be.

In fact, this point is evident from the examples of Section 8.3: for any two qubit model with the List $\mathfrak{L}_X$ of List Length 2, the universe could only ever be in one out of no more than eight possible states, $\{a, b, ..., h\}$.

In Generated-Sort dynamics, however, the set of different possible states could potentially be limitless, depending of course on the exact details of the function $f_n(\Psi_n)$.

Specifically, the set of different possible futures for a state obeying Generated-Sort dynamics could, in principle, increase exponentially with n.

There are two obvious ways that may be introduced in order to achieve a Generated-Sort dynamics according to the Rule $\mathfrak{B}_{n+1} = f_n(\Psi_n)$. These may be called

1. Basis Method;

2. One-to-Many Method;

and are each described in turn.

In the Basis Method, the individual elements $\{\Phi^1_{n+1}, \Phi^2_{n+1}, ..., \Phi^D_{n+1}\}$ of $\mathfrak{B}_{n+1}$ are different functions of the state Ψ_n. Thus, in this mechanism it is assumed that (for each n) the function f_n really implies a set of D 'sub-functions'

$$f_n = \{f_n^{(1)}, f_n^{(2)}, ..., f_n^{(D)}\} \tag{8.36}$$

such that the basis set $\mathfrak{B}_{n+1}$ is given according to the Rule

$$\mathfrak{B}_{n+1} = f_n(\Psi_n) = \{f_n^{(1)}(\Psi_n), f_n^{(2)}(\Psi_n), ..., f_n^{(D)}(\Psi_n)\} \tag{8.37}$$

where clearly

$$\Phi^i_{n+1} = f_n^{(i)}(\Psi_n) \quad , \quad i = 1, 2, ..., D \tag{8.38}$$

with the constraints that the sub-functions $\{f_n^{(i)}\}$ are defined so that $f_n^{(i)}(\Psi_n)$ is orthogonal to $f_n^{(j)}(\Psi_n)$ for all $i \neq j$, $\left|f_n^{(i)}(\Psi_n)\right| = 1$, and $f_n^{(i)}(\Psi_n) \neq \Psi_n$. Clearly, each $f_n^{(i)}$ is a $1 \rightarrow 1$ function that maps a given state Ψ_n to a different, unique vector Φ^i_{n+1}.

Conversely, in the One-to-Many Method it is assumed that f_n is instead defined as some sort of $1 \rightarrow D$ function that maps the state Ψ_n into D different, orthogonal vectors. These D states are then taken to form the basis set $\mathfrak{B}_{n+1}$.

The exact mechanics and viability of these two potential methods is explored in the following sub-sections.

Note first, however, that a Rule of the form $\mathfrak{B}_{n+1} = f_n(\Psi_n)$ is manifestly deterministic: given a state Ψ_n, it is assumed that the function f_n is used to generate a unique basis set $\mathfrak{B}_{n+1}$. The Generated-Sort mechanisms could, though, be extended to a probabilistic (Type $IIIa$) dynamics in the obvious way, by re-writing the Rule for generating $\mathfrak{B}_{n+1}$ as

$$\mathfrak{B}_{n+1} = \left\{ \begin{array}{c} f_n(\Psi_n) \text{ with Probability } P_f \\ g_n(\Psi_n) \text{ with Probability } P_g \\ \vdots \end{array} \right\} \tag{8.39}$$

where the probabilities $P_f, P_g, ...$ of using the various functions f_n, $g_n, ...$ sum as

$$P_f + P_g + ... = 1. \tag{8.40}$$

As with their analogies in List-Sort dynamics, the forms of the above probabilities could themselves be fixed or variable, depending of course on the Rules governing the system.

However, a modification from a deterministic (Type III) Rule to a probabilistic (Type $IIIa$) one does not significantly add to the discussion presented in this section. The possibility of the above extension will therefore be taken for granted from now on, and will hence not be explored further.

8.4.1 The Basis Method

As indicated above, in the Basis Method the next basis set $\mathfrak{B}_{n+1} = \{\Phi^1_{n+1}, \Phi^2_{n+1}, ..., \Phi^D_{n+1}\}$ of orthogonal eigenstates is generated from the vector Ψ_n according to the rule $\Phi^i_{n+1} = f^{(i)}_n(\Psi_n)$ for $i = 1, ..., D$. The question then becomes: what sort of functions $f^{(i)}_n$ are able to give rise to such a mechanism?

To begin to answer this, note that each $f^{(i)}_n$ is necessarily a function that maps a vector Ψ_n in $\mathcal{H}^{(D)}$ uniquely into another vector Φ^i_{n+1} in $\mathcal{H}^{(D)}$. It is hence possible to associate with $f^{(i)}_n$ a unitary operator $\hat{U}^{(i)}_n$ that achieves the same end, that is

$$\hat{U}^{(i)}_n \Psi_n = \Phi^i_{n+1}. \tag{8.41}$$

Thus, the procedure that creates the basis set $\mathfrak{B}_{n+1}$ from the state Ψ_n may be performed by defining a set of unitary operators, $\{\hat{U}^{(i)}_n : i = 1, ...D\}$.

Now, the actual forms of these unitary operators are generally seen as free parameters in the model, defined, perhaps, by whatever Rules govern the system. But, because the basis $\mathfrak{B}_{n+1}$ must contain a set of orthonormal vectors, that is

$$\langle \Phi^i_{n+1} | \Phi^j_{n+1} \rangle = \delta_{ij} \quad , \quad i, j = 1, 2, ..., D \tag{8.42}$$

it is necessary that whatever definition is chosen, the set of operators $\{\hat{U}^{(i)}_n\}$ must satisfy the constraint

$$\langle \Psi_n \hat{U}^{(i)}_n | \hat{U}^{(j)}_n \Psi_n \rangle = \delta_{ij} \quad , \quad i, j = 1, 2, ..., D \tag{8.43}$$

such that the product operation $(\hat{U}^{(i)*}_n \hat{U}^{(j)}_n)$ acting on the state Ψ_n maps it to an orthogonal vector[23] for all $i \neq j$.

[23]Note throughout that the 'starred' operator $\hat{U}^*$ denotes the complex conjugate transpose of the operator $\hat{U}$; this is equivalently represented in some textbooks as $\overline{\hat{U}^T}$, or $\hat{U}^\dagger$, or even $\hat{U}^+$. Clearly, for unitary operators $\hat{U}^* = \overline{\hat{U}^T} = \hat{U}^{-1}$, where the inverse operator $\hat{U}^{-1}$ gives $\hat{U}^{-1}\hat{U} = \hat{I}$, with $\hat{I}$ the identity.

In fact, from the above discussion it turns out that only $(D-1)$ of the operators $\{\hat{U}_n^{(i)}\}$ can be chosen arbitrarily at each time n, and not D as might be expected. Specifically, after the definition of $(D-1)$ operators $\{\hat{U}^{(i)} : i = 1, 2, ..., (D-1)\}$, the remaining operator is immediately defined by constraint.

To demonstrate this explicitly, observe that, without loss of generality, if $\{\hat{U}_n^{(1)}, \hat{U}_n^{(2)}, ..., \hat{U}_n^{(D-1)}\}$ are freely chosen according to the constraint that $\langle \Psi_n \hat{U}_n^{(i)} | \hat{U}_n^{(j)} \Psi_n \rangle = \delta_{ij}$ for $i, j = 1, 2, ..., (D-1)$, the "D^{th}" operator $\hat{U}_n^{(D)}$ is automatically defined by the condition that only one state is orthogonal to all of the vectors $\{\hat{U}_n^{(1)}\Psi_n, \hat{U}_n^{(2)}\Psi_n, ..., \hat{U}_n^{(D-1)}\Psi_n\}$. Assuming, then, that this 'last' vector is given by $\hat{U}_n^{(D)}\Psi_n$, it consequently follows that there can be no freedom in the definition of the D^{th} operator $\hat{U}_n^{(D)}$.

So, in D dimensional Hilbert spaces $\mathcal{H}^{(D)}$, an orthonormal basis set $\mathfrak{B}_{n+1} = \{\hat{U}_n^{(i)}\Psi_n : i = 1, ..., D)$ cannot be specified by using D unitary operators if each $\hat{U}_n^{(i)}$ is viewed as a free parameter. In reality, only $(D-1)$ of the operators $\{\hat{U}^{(i)}\}$ may actually be chosen freely.

Continuing, in fact, there is actually no need to define an operator $\hat{U}_n^{(D)}$ that maps Ψ_n to $\Phi_{n+1}^D = \hat{U}_n^{(D)}\Psi_n$ at all; the 'remaining' vector Φ_{n+1}^D is immediately determined by the operations $\{\hat{U}_n^{(j)}\Psi_n : i = 1, ..., (D-1)\}$ and by appealing to the mutual orthogonality of the elements of $\mathfrak{B}_{n+1}$.

Note that for clarity and to avoid confusion, from now on in this sub-section, Latin indices $i, j, ...$ will generally be used to run from $1, 2, ..., D$, whereas Greek indices $\mu, \nu, ...$ will be assumed to run from $1, 2, ..., (D-1)$.

With these comments in mind, it is possible to restate and clarify the Basis Method Rules. Specifically

- Given a state Ψ_n and a set of $(D-1)$ unitary operators $\{\hat{U}_n^{(\mu)} : \mu = 1, 2, ...(D-1)\}$ defined arbitrarily but obeying the rule

$$\langle \Psi_n \hat{U}_n^{(\mu)} | \hat{U}_n^{(\nu)} \Psi_n \rangle = \delta_{\mu\nu} \quad , \quad \mu, \nu = 1, 2, ..., (D-1) \tag{8.44}$$

  it is possible to construct a unique basis set of vectors, $\mathfrak{B}_{n+1}$, as

$$\mathfrak{B}_{n+1} = \{\hat{U}_n^{(1)}\Psi_n, \hat{U}_n^{(2)}\Psi_n, ..., \hat{U}_n^{(D-1)}\Psi_n, \Phi_{n+1}^D\} \tag{8.45}$$

  where the D^{th} vector is defined according to the constraint

$$\langle \Phi_{n+1}^D | \hat{U}_n^{(\mu)} \Psi_n \rangle = 0 \quad , \quad \mu = 1, 2, ..., (D-1). \tag{8.46}$$

- From this basis set $\mathfrak{B}_{n+1}$, an equivalence class of Hermitian operators $\hat{B}_{n+1}$ are implied, with eigenstates $\{\Phi_{n+1}^i\}$ equal to

$$\{\Phi_{n+1}^i\} = \{\{\hat{U}_n^{(\mu)}\Psi_n : \mu = 1, ..., (D-1)\}, \Phi_{n+1}^D\}. \tag{8.47}$$

The next test, $\hat{\Sigma}_{n+1}$, of the universe is then taken to be one of these operators $\hat{B}_{n+1}$, and the universe collapses to the state Ψ_{n+1}, which is an element of the set $\mathfrak{B}_{n+1}$.

Of course, since by definition $\Psi_{n+1} \neq \Psi_n$, none of the unitary operators $\hat{U}_n^{(i)}$ may be defined as the identity operator.

It will be useful in the following to consider the '*Reduced basis set*' $\mathfrak{B}_{n+1}^R$ at time $n+1$. Specifically, $\mathfrak{B}_{n+1}^R \subset \mathfrak{B}_{n+1}$ is defined as the set of vectors

$$\mathfrak{B}_{n+1}^R = \{\hat{U}_n^{(1)}\Psi_n, \hat{U}_n^{(2)}\Psi_n, ..., \hat{U}_n^{(D-1)}\Psi_n\}, \tag{8.48}$$

where the unitary operators $\{\hat{U}_n^{(\mu)} : \mu = 1, ..., (D-1)\}$ acting on the state Ψ_n obey the condition (8.44). The actual next basis set $\mathfrak{B}_{n+1}$ consequently comprises of this Reduced basis set $\mathfrak{B}_{n+1}^R$ and a vector Φ_{n+1}^D obeying (8.46) that is orthogonal to every element of $\mathfrak{B}_{n+1}^R$. Thus

$$\mathfrak{B}_{n+1} = \{\mathfrak{B}_{n+1}^R, \Phi_{n+1}^D\}. \tag{8.49}$$

Clearly, because the D^{th} vector Φ_{n+1}^D is provided by constraint, the central task for the Basis Method Rules lies in defining a set of $(D-1)$ unitary operators $\{\hat{U}_n^{(\mu)} : \mu = 1, ..., (D-1)\}$ that can be used to generate the Reduced basis set $\mathfrak{B}_{n+1}^R$.

Before discussing this further, however, note as an aside that in a single qubit universe governed by Basis Method Rules, there is only one free parameter: $\hat{U}_n^{(1)}$. For a single qubit universe in a two dimensional Hilbert space $\mathcal{H}^{(2)}$, only one unitary operator $\hat{U}_n^{(1)}$ needs be specified in order to generate a unique basis set of vectors, because if $\mathfrak{B}_{n+1} = \{\Phi_{n+1}^1, \Phi_{n+1}^2\}$ and Φ_{n+1}^1 is defined as $\Phi_{n+1}^1 = \hat{U}_n^{(1)}\Psi_n$, the remaining vector Φ_{n+1}^2 is given immediately from the orthogonality condition

$$\langle\Phi_{n+1}^2|\Phi_{n+1}^1\rangle = \langle\Phi_{n+1}^2|\hat{U}_n^{(1)}\Psi_n\rangle. \tag{8.50}$$

Equivalently, it is evident that the Reduced basis set for a single qubit universe contains only one member. This discussion is analogous to that presented later in Sub-section 8.5.2 regarding unitary rotation in single qubit spaces.

Of course, $\hat{U}_n^{(1)}$ is chosen freely, and could be any unitary operator in $\mathcal{H}^{(2)}$, obviously excluding the identity.

Similarly to every other mechanism used in this chapter to develop the universe, an important principle of the Basis Method dynamics is that the Rules should be repeatable. In the present case, such a principle implies that the Reduced basis set $\mathfrak{B}_{n+2}^R$ for the next step must be generated from the function f_{n+1} acting on Ψ_{n+1}, i.e. $\mathfrak{B}_{n+2}^R = \{\hat{U}_{n+1}^{(\mu)}\Psi_{n+1} : \mu = 1, ..., (D-1)\}$. The universe would then develop in an automatic, iterative way.

So, an issue that is immediately faced concerns how the $(D-1)$ unitary operators $\{\hat{U}_n^{(\mu)}\}$ used to determine $\mathfrak{B}_{n+1}^R$ from Ψ_n could relate to the set $\{\hat{U}_{n+1}^{(\mu)}\}$ that will be used to determine $\mathfrak{B}_{n+2}^R$ from Ψ_{n+1}. Specifically, a question of particular interest is whether the same set of operators $\{\hat{U}_n^{(\mu)}\}$ could be used in both cases, such that $\{\hat{U}_n^{(\mu)}\} \equiv \{\hat{U}_{n+1}^{(\mu)}\}$. In other words, this question is effectively asking whether it is possible to have Basis Method dynamics based upon functions $f_n^{(i)} = f^{(i)}$ that are constant for all n, or whether they have to change with n in order for the proposed mechanism to work.

In fact, if both possibilities are valid, it would consequently lead to two classes of Basis Method Rules:

Class 1: The set $\{\hat{U}_n^{(\mu)}\}$ is fixed for all n, such that

$$\Phi_{n+1}^{\mu} = \hat{U}_n^{(\mu)}\Psi_n \quad , \quad \mu = 1, ..., (D-1) \tag{8.51}$$

and

$$\Phi_{n+2}^{\mu} = \hat{U}_{n+1}^{(\mu)}\Psi_{n+1} = \hat{U}_n^{(\mu)}\Psi_{n+1} \quad , \quad \mu = 1, ..., (D-1) \tag{8.52}$$

and so on. By dropping the now redundant subscripts, such a fixed set of operators could be denoted by $U_F = \{\hat{U}^{(1)}, \hat{U}^{(2)}, ..., \hat{U}^{(D-1)}\}$, and, like the List in List-Sort dynamics, would be defined for all time at the outset. Moreover, the definition of U_F would be taken as a necessary pre-requisite without further justification, just as, for example, the existence of the underlying Hilbert space is assumed to be.

Class 2: The operators $\{\hat{U}_n^{(\mu)}\}$ do change with n, such that

$$\Phi_{n+1}^{\mu} = \hat{U}_n^{(\mu)}\Psi_n \quad , \quad \mu = 1, ..., (D-1) \tag{8.53}$$

but

$$\Phi_{n+2}^{\mu} = \hat{U}_{n+1}^{(\mu)}\Psi_{n+1} \quad , \quad \mu = 1, ..., (D-1) \tag{8.54}$$

where $\hat{U}_n^{(\mu)}$ is not (necessarily) equal to $\hat{U}_{n+1}^{(\mu)}$.

Evidently, Class 1 Rules are a special case of Class 2 Rules, in which $\hat{U}_{n+1}^{(\mu)} = \hat{U}_n^{(\mu)}$ for all n and μ.

These two possible cases are now discussed in turn.

Class 1 Basis Method

The short answer to the above question is that it does not seem likely that a universe is able to develop according to Basis Method Rules that incorporate a fixed set U_F of unitary

operators. In other words, whilst its non-existence has not yet been proved rigorously, no (Type III) Class 1 mechanism has been found that can be used to self-referentially develop the state of the universe from $\Psi_n \rightarrow \Psi_{n+1} \rightarrow \Psi_{n+2} \rightarrow ...$, continuing indefinitely, and no such mechanism is expected to be found.

This conclusion arises because in order for Class 1 Basis Method Rules to be valid, it is required that if $\mathfrak{B}^R_{n+1} = \{\hat{U}^{(\mu)}\Psi_n\}$ with $\langle \Psi_n \hat{U}^{(\mu)} | \hat{U}^{(\nu)} \Psi_n \rangle = \delta_{\mu\nu}$, then $\mathfrak{B}^R_{n+2} = \{\hat{U}^{(\mu)}\Psi_{n+1}\}$ with $\langle \Psi_{n+1} \hat{U}^{(\mu)} | \hat{U}^{(\nu)} \Psi_{n+1} \rangle = \delta_{\mu\nu}$, for all $\mu, \nu = 1, 2, ..., (D-1)$. This validity therefore rests on the assumption that the set $U_F = \{\hat{U}^{(\mu)}\}$ defined 'initially' to ensure the mutual orthogonality of the vectors $\{\hat{U}^{(\mu)}\Psi_n\}$ may also be used generate a set of orthogonal vectors $\{\hat{U}^{(\mu)}\Psi_{n+1}\}$ from Ψ_{n+1}.

However, a set of unitary operators $U_F = \{\hat{U}^{(\mu)} : \mu = 1, ..., (D-1)\}$ obeying the constraint $\langle \Psi_n \hat{U}^{(\mu)} | \hat{U}^{(\nu)} \Psi_n \rangle = \delta_{\mu\nu}$ for all μ, ν will not in general also satisfy the relationship $\langle \Theta \hat{U}^{(\mu)} | \hat{U}^{(\nu)} \Theta \rangle = \delta_{\mu\nu}$, where $\Theta \neq \Psi_n$ is an arbitrary vector in $\mathcal{H}^{(D)}$.

Specifically, in fact, given an 'initial' state Ψ_0, then if the operators $\{\hat{U}^{(\mu)}\}$ are in the first instance defined so that they obey $\langle \Psi_0 \hat{U}^{(\mu)} | \hat{U}^{(\nu)} \Psi_0 \rangle = \delta_{\mu\nu}$ to give the Reduced basis set $\mathfrak{B}^R_1$, the same set $\{\hat{U}^{(\mu)}\}$ will not then in general also satisfy the relationship $\langle \Psi_1 \hat{U}^{(\mu)} | \hat{U}^{(\nu)} \Psi_1 \rangle = \delta_{\mu\nu}$ required for the following step of the dynamics, where $\Psi_1 \in \mathfrak{B}_1$ and recalling that $\Psi_1 \neq \Psi_0$ by definition. So, the set of vectors $\{\hat{U}^{(1)}\Psi_1, \hat{U}^{(2)}\Psi_1, ..., \hat{U}^{(D-1)}\Psi_1\}$ will not in general be orthogonal, and so cannot be used to determine the next basis set $\mathfrak{B}_2$.

In fact, in order for such a set of vectors $\{\hat{U}^{(1)}\Psi_1, \hat{U}^{(2)}\Psi_1, ..., \hat{U}^{(D-1)}\Psi_1\}$ to be orthogonal, the set U_F must satisfy the following condition.

Condition. Assume an initial state Ψ_n and a set U_F of unitary operators $\{\hat{U}^{(\mu)}\}$ defined such that $\langle \Psi_n \hat{U}^{(\mu)} | \hat{U}^{(\nu)} \Psi_n \rangle = \delta_{\mu\nu}$ for $\mu, \nu = 1, 2, ..., (D-1)$. Each $\hat{U}^{(\mu)}$ generates a unique vector Φ^μ_{n+1}, given by $\Phi^\mu_{n+1} = \hat{U}^{(\mu)}\Psi_n$, such that the set $\{\Phi^\mu_{n+1}\}$ defines the Reduced basis set $\mathfrak{B}^R_{n+1}$.

Consider also the "D^{th}" operator $\hat{U}^{(D)}$, defined according to the constraint that it maps Ψ_n to the vector Φ^D_{n+1} that is orthogonal to every Φ^μ_{n+1}. Then, the set $\{U_F, \hat{U}^{(D)}\}$ acting on the state Ψ_n can be used to generate an orthonormal basis set $\mathfrak{B}_{n+1} = \{\Phi^i_{n+1} : i = 1, ..., D\}$.

Now, consider an additional set of D mutually orthogonal vectors $\{\Psi^k : k = 1, ..., D\}$, defined arbitrarily apart from the condition that the 'first' of these, Ψ^1, is identical to the state Ψ_n, i.e. $\Psi^1 = \Psi_n$. Then, the set $\{\Psi^k\}$ is effectively equivalent to some basis in $\mathcal{H}^{(D)}$, which may be labelled $\mathfrak{B}_K$.

Clearly, the subset $\{\Psi^j : j = 2, ..., D\}$ of $\mathfrak{B}_K$ contains an arbitrary set of vectors that are orthogonal to the current state Ψ_n and to each other.

It is possible to find a transformation that maps each of the vectors Ψ^k, $k = 1, ..., D$, to the state $\Psi_n = \Psi^1$. One such map involves an operator V defined as

$$V = |\Psi^1\rangle\langle\Psi^2| + |\Psi^2\rangle\langle\Psi^3| + ... + |\Psi^{D-1}\rangle\langle\Psi^D| + |\Psi^D\rangle\langle\Psi^1| \tag{8.55}$$

with the rule

$$(V)^{k-1}\Psi^k = \Psi^1 \tag{8.56}$$

where $(V)^{k-1}$ implies the operator V raised to the $(k-1)^{th}$ power.

So, as an example

$$\begin{aligned} (V)^2\Psi^3 &= VV|\Psi^3\rangle \\ &= \left(\begin{array}{c} (|\Psi^1\rangle\langle\Psi^2| + |\Psi^2\rangle\langle\Psi^3| + ... + |\Psi^D\rangle\langle\Psi^1|) \\ \times(|\Psi^1\rangle\langle\Psi^2| + |\Psi^2\rangle\langle\Psi^3| + ... + |\Psi^D\rangle\langle\Psi^1|) \end{array} \right) |\Psi^3\rangle \\ &= (|\Psi^1\rangle\langle\Psi^3| + |\Psi^2\rangle\langle\Psi^4| + ... + |\Psi^D\rangle\langle\Psi^2|)|\Psi^3\rangle = |\Psi^1\rangle = \Psi_n \end{aligned} \tag{8.57}$$

and note that $(V)^{1-1}\Psi^1 = \hat{I}\Psi^1$ as expected, where $\hat{I}$ is the identity.

Evidently, V is unitary

$$\begin{aligned} VV^* &= \left(\begin{array}{c} (|\Psi^1\rangle\langle\Psi^2| + |\Psi^2\rangle\langle\Psi^3| + ... + |\Psi^D\rangle\langle\Psi^1|) \\ \times(|\Psi^2\rangle\langle\Psi^1| + |\Psi^3\rangle\langle\Psi^2| + ... + |\Psi^1\rangle\langle\Psi^D|) \end{array} \right) \\ &= (|\Psi^1\rangle\langle\Psi^1| + |\Psi^2\rangle\langle\Psi^2| + ... + |\Psi^D\rangle\langle\Psi^D|) = \hat{I}. \end{aligned} \tag{8.58}$$

Moreover, $(V)^{k-1}$ is therefore also unitary for all k, as may be readily shown.

So as an aside, note that the vectors Ψ^k with the operators $(V)^{k-1}$ are defined very much in the spirit of the Basis Method from the state Ψ^1; that is, from the unitary transformation

$$\Psi^k = (V^*)^{k-1}\Psi^1. \tag{8.59}$$

Recall now that it is always possible to rotate any orthogonal basis set of vectors into a second orthogonal basis set of vectors, by using a suitably defined unitary operator. So, the basis set $\mathfrak{B}_K = \{\Psi^k\}$ may be rotated into the basis set $\mathfrak{B}_{n+1} = \{\Phi^i_{n+1}\}$ by a unitary operator, which may be labelled $\hat{U}^{(n+1,K)}$.

In fact, this possibility implies that each element $\Psi^k \in \mathfrak{B}_K$ may be uniquely mapped into an element $\Phi^i_{n+1} \in \mathfrak{B}_{n+1}$ by $\hat{U}^{(n+1,K)}$, such that

$$\Phi^i_{n+1} = \hat{U}^{(n+1,K)}\Psi^k \tag{8.60}$$

with the reverse also, of course, holding: $\Psi^k = (\hat{U}^{(n+1,K)})^*\Phi^i_{n+1}$.

Now, rewriting Ψ^k as $\Psi^k = (V^*)^{k-1}\Psi^1$ by definition, and associating the i^{th} vector Φ^i_{n+1} of $\mathfrak{B}_{n+1}$ with the k^{th} vector Ψ^k of $\mathfrak{B}_K$ without loss of generality, gives

$$\Phi^i_{n+1} = \hat{U}^{(n+1,K)}(V^*)^{i-1}\Psi^1. \tag{8.61}$$

Moreover, because the product operation $(\hat{U}^{(n+1,K)}(V^*)^{i-1})$ is unitary

$$\begin{aligned}(\hat{U}^{(n+1,K)}(V^*)^{i-1})(\hat{U}^{(n+1,K)}(V^*)^{i-1})^* &= (\hat{U}^{(n+1,K)}(V^*)^{i-1})((V)^{i-1}(\hat{U}^{(n+1,K)})^*) \\ &= \hat{U}^{(n+1,K)}(V^*V)^{i-1}(\hat{U}^{(n+1,K)})^* \\ &= \hat{U}^{(n+1,K)}\hat{I}(\hat{U}^{(n+1,K)})^* = \hat{I} \end{aligned} \tag{8.62}$$

it may be associated with a single unitary operator, and by denoting this single operator as $\hat{U}^{(i)}$, it is evident that

$$\Phi^i_{n+1} = \hat{U}^{(n+1,K)}(V^*)^{i-1}\Psi^1 = \hat{U}^{(i)}\Psi_n. \tag{8.63}$$

Clearly, then,

$$\begin{aligned}\langle\Phi^i_{n+1}|\Phi^j_{n+1}\rangle &= \langle\Psi_n\hat{U}^{(i)}|\hat{U}^{(j)}\Psi_n\rangle \\ &= \langle\Psi^1|(V)^{i-1}(\hat{U}^{(n+1,K)})^*\hat{U}^{(n+1,K)}(V^*)^{j-1}|\Psi^1\rangle \\ &= \langle\Psi^1|(V)^{i-1}(V^*)^{j-1}|\Psi^1\rangle = \langle\Psi^i|\Psi^j\rangle = \delta_{ij}\end{aligned} \tag{8.64}$$

as required, so that the product $(\hat{U}^{(n+1,K)}(V^*)^{i-1})$ can clearly be used to perform the unitary rotation $\hat{U}^{(i)}$ used in the 'standard' Basis Method, assuming of course that $\hat{U}^{(n+1,K)} \neq \hat{I}$.

Furthermore, because the vectors $\{\Psi^j : j = 2, ..., D\}$ were originally chosen arbitrarily, such an association of $\hat{U}^{(i)}$ with $\hat{U}^{(n+1,K)}(V^*)^{i-1}$ can always be found for any definition of the fixed set U_F of unitary operators $\{\hat{U}^{(\mu)} : \mu = 1, ..., (D-1)\}$, with the D^{th} operator $\hat{U}^{(D)}$ defined by constraint. In short, there is always a transformation $\hat{U}^{(n+1,K)}$ and a basis $\mathfrak{B}_K$ that defines an operator V, from which a set of unitary operators $\hat{U}^{(i)}$ may be defined that act on Ψ_n to generate an orthogonal basis $\mathfrak{B}_{n+1}$; similarly, any set of unitary operators $\hat{U}^{(i)}$ that relate Ψ_n to $\mathfrak{B}_{n+1}$ may be constructed in the above manner by defining a suitable transformation $\hat{U}^{(n+1,K)}$ and basis $\mathfrak{B}_K$.

Without loss of generality, assume that on testing Ψ_n by an operator $\hat{\Sigma}_{n+1}$ with eigenvectors $\{\Phi^i_{n+1} : i = 1, ..., D\}$, the universe collapses into the state $\Psi_{n+1} = \Phi^a_{n+1} = \hat{U}^{(a)}\Psi_n$, where $a \in \{1, ..., D\}$. The question faced by the Class 1 Basis Method Rules is: can the same set $U_F = \{\hat{U}^{(\mu)} : \mu = 1, ..., (D-1)\} \subset \{\hat{U}^{(i)} : i = 1, ..., D\}$ of unitary operators be used to construct a new *orthonormal* Reduced basis set of vectors from Ψ_{n+1}? In other words, given a set of vectors $\{\Theta^\mu : \mu = 1, ..., (D-1)\}$ defined as

$$\Theta^\mu = \hat{U}^{(\mu)}\Psi_{n+1} = \hat{U}^{(\mu)}\hat{U}^{(a)}\Psi_n \tag{8.65}$$

does it in general follow that

$$\langle\Theta^\mu|\Theta^\nu\rangle = \delta_{\mu\nu} \tag{8.66}$$

for all $\mu, \nu = 1, ..., (D-1)$?

Now, the condition is obviously satisfied for the case when $\mu = \nu$,

$$\langle \Theta^{\mu} | \Theta^{\mu} \rangle = \langle \Psi_{n+1} | \hat{U}^{(\mu)*} \hat{U}^{(\mu)} | \Psi_{n+1} \rangle = \langle \Psi_{n+1} | \hat{I} | \Psi_{n+1} \rangle = 1. \tag{8.67}$$

For $\mu \neq \nu$, however, it follows that

$$\begin{aligned} \langle \Theta^{\mu} | \Theta^{\nu} \rangle &= \langle \Psi_{n+1} | \hat{U}^{(\mu)*} \hat{U}^{(\nu)} | \Psi_{n+1} \rangle = \langle \Psi_{n+1} | (V)^{\mu-1} (\hat{U}^{(n+1,K)})^* \hat{U}^{(n+1,K)} (V^*)^{\nu-1} | \Psi_{n+1} \rangle \\ &= \langle \Psi_{n+1} | (V)^{\mu-1} (V^*)^{\nu-1} | \Psi_{n+1} \rangle = \langle \Psi_n | \hat{U}^{(a)*} (V)^{\mu-1} (V^*)^{\nu-1} \hat{U}^{(a)} | \Psi_n \rangle \\ &= \langle \Psi^1 | (V)^{a-1} (\hat{U}^{(n+1,K)})^* (V)^{\mu-1} (V^*)^{\nu-1} \hat{U}^{(n+1,K)} (V^*)^{a-1} | \Psi^1 \rangle \\ &= \langle \Psi^a | (\hat{U}^{(n+1,K)})^* (V)^{\mu-1} (V^*)^{\nu-1} \hat{U}^{(n+1,K)} | \Psi^a \rangle. \end{aligned} \tag{8.68}$$

Now, the product $(V)^{\mu-1}(V^*)^{\nu-1}$ gives

$$\begin{aligned} (V)^{\mu-1}(V^*)^{\nu-1} &= \left(\begin{array}{c} \left(|\Psi^1\rangle\langle\Psi^{\mu}| + |\Psi^2\rangle\langle\Psi^{\mu\oplus 1}| + ... + |\Psi^D\rangle\langle\Psi^{\mu\oplus(D-1)}| \right) \\ \times \left(|\Psi^{\nu}\rangle\langle\Psi^1| + |\Psi^{\nu\oplus 1}\rangle\langle\Psi^2| + ... + |\Psi^{\nu\oplus(D-1)}\rangle\langle\Psi^D| \right) \end{array} \right) \\ &= \sum_{x=1}^{D} |\Psi^x\rangle\langle\Psi^{\mu\oplus(x-1)}| \sum_{y=1}^{D} |\Psi^{\nu\oplus(y-1)}\rangle\langle\Psi^y| \\ &= \sum_{x=1}^{D}\sum_{y=1}^{D} |\Psi^x\rangle\langle\Psi^y| \delta_{[\mu\oplus(x-1)],[\nu\oplus(y-1)]} \end{aligned} \tag{8.69}$$

where $\oplus$ implies addition modulo D, i.e. $D \oplus z = z$ for $1 \leq z \leq D$.

So overall, the condition, $\langle \Theta^{\mu} | \Theta^{\nu} \rangle = 0$, for orthogonality is only satisfied iff

$$\sum_{x=1}^{D}\sum_{y=1}^{D} \delta_{[\mu\oplus x],[\nu\oplus y]} \langle \Psi^a | (\hat{U}^{(n+1,K)})^* | \Psi^x \rangle \langle \Psi^y | \hat{U}^{(n+1,K)} | \Psi^a \rangle = 0 \tag{8.70}$$

for $\mu \neq \nu$. ■

Now, noting that

$$\hat{U}^{(n+1,K)} | \Psi^a \rangle \notin \{ \Psi^1, \Psi^2, ..., \Psi^D \} \tag{8.71}$$

for a valid development from $\Psi_n \rightarrow \Psi_{n+1}$ to occur in this quantum universe, where $\Psi^1 = \Psi_n \neq \Psi_{n+1}$ by definition, such that the inner products $\langle \Psi^a | (\hat{U}^{(n+1,K)})^* | \Psi^x \rangle$ and $\langle \Psi^y | \hat{U}^{(n+1,K)} | \Psi^a \rangle$ are not equal to zero, the above condition is only fulfilled if the sum of products of amplitudes equals zero. Clearly, this will not in general be the case.

In fact, following on from this, it is observed that since the dimension of the Hilbert space of the real Universe is expected to be enormous, for the above result to be obtained in reality a truly remarkable level of cancellation must occur between the probability amplitudes in the relevant sum of terms.

It is expected, then, that in general $\langle \Theta^{\mu} | \Theta^{\nu} \rangle \neq \delta_{\mu\nu}$ for every μ, ν, so that a set of vectors given by $\{ \hat{U}^{(\mu)} \Psi_{n+1} : \mu = 1, ..., (D-1) \}$ are unlikely to be orthogonal, where the unitary operators $\{ \hat{U}^{(\mu)} \}$ are defined such that $\langle \Psi_n \hat{U}^{(\mu)} | \hat{U}^{(\nu)} \Psi_n \rangle = \delta_{\mu\nu}$ for $\mu, \nu = 1, 2, ..., (D-1)$.

An arbitrarily defined set U_F cannot therefore be expected to specify an orthonormal Reduced basis set $\mathfrak{B}^R_{n+2}$.

Furthermore, note that the above condition is only defined for the 'first' potential transition, that is from Ψ_{n+1} to Ψ_{n+2}. However, for the set U_F to provide valid Class 1 Basis Method dynamics, similar conditions must also hold for the indefinite series of transitions $\Psi_{n+2} \longrightarrow \Psi_{n+3} \rightarrow \Psi_{n+4} \rightarrow ...$ Thus, the set U_F must be defined such that it actually satisfies an 'infinite tower' of conditions, with a fortuitous level of cancellation required at each stage.

It is these observations that prompt the conclusion that such a set is unlikely to exist.

Summarising, given an arbitrary set U_F of unitary operators $\{\hat{U}^{(\mu)} : \mu = 1, ..., (D-1)\}$ defined such that they satisfy (8.44) for Ψ_n, and a vector Φ^D_{n+1} defined 'orthogonally' such that it satisfies (8.46), then whichever member Φ^i_{n+1} of the set $\mathfrak{B}_{n+1} = \{\{\Phi^\mu_{n+1} = \hat{U}^{(\mu)}\Psi_n\}, \Phi^D_{n+1}\}$ the universe collapses into when it becomes Ψ_{n+1}, the elements of a new set of vectors defined as $\{\hat{U}^{(1)}\Phi^i_{n+1}, \hat{U}^{(2)}\Phi^i_{n+1}, ..., \hat{U}^{(D-1)}\Phi^i_{n+1}\}$ are not expected to be orthogonal.

Thus, the set $\{\hat{U}^{(\mu)}\Psi_{n+1} : \mu = 1, ..., (D-1)\}$ is unlikely to form an orthogonal Reduced basis $\mathfrak{B}^R_{n+2}$ for $\mathcal{H}^{(D)}$, and so cannot be used to specify a unique basis set $\mathfrak{B}_{n+2}$, or, consequently, an equivalence class of operators $\hat{B}_{n+2}$. The next operator $\hat{\Sigma}_{n+2}$ cannot therefore be generated in this manner, from the state Ψ_{n+1} being rotated by the members of a fixed set U_F of unitary operators.

Concluding, the Type III Class 1 Basis Method is expected to be invalid.

As a caveat to this conclusion, note that it is in fact always possible for a single qubit universe in a two dimensional Hilbert space to be governed by Type III Class 1 Basis Method Rules. This result follows because any Reduced basis set $\mathfrak{B}^R_{n+1}$ for $\mathcal{H}^{(2)}$ has only one member; there is therefore no 'orthogonality problem' for its elements.

So, given any arbitrary vector $\theta \in \mathcal{H}^{(2)}$ and any unitary operator $\hat{U}^{(1)}$, it is always possible to specify an orthonormal basis for $\mathcal{H}^{(2)}$ that contains the vector $\hat{U}^{(1)}\theta$ and whichever vector in $\mathcal{H}^{(2)}$ is orthogonal to $\hat{U}^{(1)}\theta$. Effectively, the vector $\hat{U}^{(1)}\theta$ single-handedly implies a unique basis set for $\mathcal{H}^{(2)}$.

Thus, given $\hat{U}^{(1)}$ and a state $\Psi_n \in \mathcal{H}^{(2)}$, the basis set $\mathfrak{B}_{n+1}$ is readily generated, and either element of this can be used with $\hat{U}^{(1)}$ to generate a new basis set $\mathfrak{B}_{n+2}$. In fact, this process may be repeated indefinitely.

Such a possibility is unique to two dimensional Hilbert spaces.

Class 2 Basis Method

In the Class 2 Basis Method, the set of unitary operators $\{\hat{U}_n^{(\mu)} : \mu = 1, ..., (D-1)\}$, and hence the functions $f_n^{(i)}$, change as the universe jumps from one stage to the next. In other words, a new set of $(D-1)$ unitary operators is chosen at each time step, n. Clearly, this Class of Basis Method mechanism is immediately valid, because it is always possible to define a set of unitary operators that provides the next orthogonal Reduced basis set of vectors when acting upon a given state, for all n.

Summarising, then, a universe developing according to a Type III (state self-referential) Basis Method mechanism is governed by the general Rule

$$\mathfrak{B}_{n+1} = \{\mathfrak{B}_{n+1}^R, \Phi_{n+1}^D\} = \left\{\{\hat{U}_n^{(1)}\Psi_n, \hat{U}_n^{(2)}\Psi_n, ..., \hat{U}_n^{(D-1)}\Psi_n\}, \Phi_{n+1}^D\right\} \tag{8.72}$$

with the set of unitary operators $\{\hat{U}_n^{(\mu)}\}$ defined such that they satisfy the conditions

$$\langle \Psi_n \hat{U}_n^{(\mu)} | \hat{U}_n^{(\nu)} \Psi_n \rangle = \delta_{\mu\nu} \tag{8.73}$$

and the vector Φ_{n+1}^D defined as

$$\langle \Phi_{n+1}^D | \hat{U}_n^{(\mu)} \Psi_n \rangle = 0 \tag{8.74}$$

for all $\mu, \nu = 1, ..., (D-1)$ and n.

Of course, in order to ensure the above conditions, the actual choice of the operators $\{\hat{U}_n^{(\mu)}\}$ must rely to some extent on some sort of 'knowledge' of what the current state Ψ_n is. Certainly, it is difficult to imagine how $\hat{U}_n^{(\mu)}$ could be defined obeying $\langle \Psi_n \hat{U}_n^{(\mu)} | \hat{U}_n^{(\nu)} \Psi_n \rangle = \delta_{\mu\nu}$ if Ψ_n is unknown, especially considering the conclusion from the Class 1 case that no generalised set of such operators is expected to exist that can give orthogonal results when acting on arbitrary vectors. The suggestion, then, could be that the definition of these unitary operators might itself depend on self-referential rules, such that the members of the set $\{\hat{U}_n^{(\mu)}\}$ might themselves be some unknown function, F_n, of the state Ψ_n. Of course, F_n could perhaps be a $1 \rightarrow (D-1)$ function, such that $\{\hat{U}_n^{(\mu)}\} = F_n(\Psi_n)$, or could maybe 'contain' $(D-1)$ sub-functions, such that $\hat{U}_n^{(\mu)} = F_n^{(\mu)}(\Psi_n)$. Thus, the function F_n would be defined such that $F_n(\Psi_n)$ generates a set of unitary operators $\{\hat{U}_n^{(\mu)}\}$ that satisfy the conditions $\langle \Psi_n \hat{U}_n^{(\mu)} | \hat{U}_n^{(\nu)} \Psi_n \rangle = \delta_{\mu\nu}$ for $\mu, \nu = 1, ..., (D-1)$ and all n.

Overall, therefore, the Rules governing a universe that develops according to the Class 2 Basis Method could rely on a choice of Hermitian operator that is a result of unitary operators acting upon the state, that are themselves functions of the state.

Such a mechanism would overcome the non-orthogonality problem experienced by the Class 1 dynamics, but does lead to the question as to exactly how the unitary operators $\{\hat{U}_n^{(\mu)}\}$ are defined at each time step: what exactly is the form of the function F_n?

Indeed, at first glance it appears from such a question that very little progress is actually gained from analysing the presented method. After all, recall that the original aim of this chapter was to investigate how the universe chooses which operator it uses to develop itself with. So, if an answer to this question is that the actual mechanism relies on a particular choice of unitary operators that is itself unexplainable, or at least relies upon some higher-order self-referential process $F_n(\Psi_n)$, it is still unclear as to how the dynamics of a universe developing according to the Basis Method might actually proceed.

Nevertheless, the mathematical possibility of such a Class 2 Basis Method dynamics, and hence the possibility that the physical Universe itself develops according to such a mechanism, does imply that a full investigation into operators that are a result of operators that depend on the state is a necessary direction for future research.

Summary

Summarising, in order for a set of $(D-1)$ unitary operators $\{\hat{U}_n^{(\mu)}\}$ to generate a Reduced basis $\mathfrak{B}_{n+1}^R$ from a state Ψ_n in a universe governed by Type III Basis Method Rules, it is expected that the set $\{\hat{U}_n^{(\mu)}\}$ must be defined at each n in a way that depends upon this current state Ψ_n. In other words, Class 1 Basis Method Rules are unlikely to be valid for the development of a quantum universe (for $D > 2$). Class 2 Basis Method Rules, however, are.

Overall, then, in a universe governed by Type III Class 2 Basis Method Rules, the state develops as follows.

Given a state Ψ_n and a set of $(D-1)$ unitary operators $\{\hat{U}_n^{(\mu)}\}$ defined such that $\langle \Psi_n \hat{U}_n^{(\mu)} | \hat{U}_n^{(\nu)} \Psi_n \rangle = \delta_{\mu\nu}$ for $\mu, \nu = 1, ..., (D-1)$, a Reduced basis set of vectors $\mathfrak{B}_{n+1}^R$ is determined, where $\mathfrak{B}_{n+1}^R = \{\hat{U}_n^{(\mu)} \Psi_n\}$.

Moreover, given also a vector Φ_{n+1}^D defined such that $\langle \Phi_{n+1}^D | \hat{U}_n^{(\mu)} \Psi_n \rangle = 0$, a basis set $\mathfrak{B}_{n+1}$ of mutually orthogonal states can then be generated, where $\mathfrak{B}_{n+1} = \{\mathfrak{B}_{n+1}^R, \Phi_{n+1}^D\}$.

Equally, therefore, an equivalence class of operators $\{\hat{B}_{n+1}\}$ is also specified, and these may be associated with the operator $\hat{\Sigma}_{n+1}$ used to develop the state Ψ_n. The universe consequently jumps to the state Ψ_{n+1}, which is an eigenvector of $\hat{\Sigma}_{n+1}$, with probability given in the usual way.

The process may then be continued, so that a new set of unitary operators $\{\hat{U}_{n+1}^{(\mu)}\}$, defined such that $\langle \Psi_{n+1} \hat{U}_{n+1}^{(\mu)} | \hat{U}_{n+1}^{(\nu)} \Psi_{n+1} \rangle = \delta_{\mu\nu}$ for $\mu, \nu = 1, ..., (D-1)$, is used to generate the next Reduced basis set of vectors $\mathfrak{B}_{n+2}^R$ from the new state Ψ_{n+1} according to the Rule: $\mathfrak{B}_{n+2}^R = \{\hat{U}_{n+1}^{(\mu)} \Psi_{n+1}\}$.

Consequently, the next basis set $\mathfrak{B}_{n+2} = \{\mathfrak{B}^R_{n+2}, \Phi^D_{n+2}\}$ may immediately be determined from the conditions $\langle \Phi^D_{n+2} | \hat{U}^{(\mu)}_{n+1} \Psi_{n+1} \rangle = 0$, and this is turn implies an equivalence class of operators $\{\hat{B}_{n+2}\}$. The universe then jumps to the state Ψ_{n+2}, which is one of the members of $\mathfrak{B}_{n+2}$.

And so on; the procedure may be iterated indefinitely.

8.4.2 The One-to-Many Method

The development of a state $\Psi_n \in \mathcal{H}^{(D)}$ according to the Basis Method relies on D functions $f^{(i)}_n$, for $i = 1, ..., D$, each of which maps the state Ψ_n to a unique vector $f^{(i)}_n(\Psi_n)$. Assuming that these new vectors are orthogonal, a condition ensured by the actual definitions of $\{f^{(i)}_n\}$, they are then taken to comprise the next preferred basis set $\mathfrak{B}_{n+1}$. The universe subsequently jumps to one of these possible states.

In the One-to-Many Method, however, it is instead postulated that there exists a single function f_n that maps the state Ψ_n to D different, orthogonal vectors. In other words, f_n is defined to be a $1 \rightarrow D$ function which, when applied to Ψ_n, has D outcomes: $\Theta^1, \Theta^2, ..., \Theta^D$.

Now, because the function f_n is defined such that these D outcomes are all mutually orthogonal, then the application of f_n to Ψ_n effectively defines a basis set. Labelling this basis set $\mathfrak{B}_{n+1}$, it is possible to write

$$\mathfrak{B}_{n+1} \equiv \{\Theta^1_{n+1}, \Theta^2_{n+1}, ..., \Theta^D_{n+1}\} = f_n(\Psi_n) \tag{8.75}$$

where the 'temporal' subscript has now been added for completeness. Thus, the next basis set $\mathfrak{B}_{n+1}$ is generated from the current state by applying the $1 \rightarrow D$ function f_n to Ψ_n; such a Rule may be called a One-to-Many Method.

Evidently, the determination of the basis set $\mathfrak{B}_{n+1}$ implies the determination of an equivalence class of operators $\hat{B}_{n+1}$, and these may be associated in the usual way with the next test $\hat{\Sigma}_{n+1}$ used to develop the state.

In order to provide a consistent mechanism for the automatic development of a universe from $\Psi_n \rightarrow \Psi_{n+1} \rightarrow \Psi_{n+2} \rightarrow ...$, it is expected that the next basis set $\mathfrak{B}_{n+2}$ may be determined by applying a One-to-Many function f_{n+1} to the vector Ψ_{n+1}, where Ψ_{n+1} is defined as whichever member of the set $\{\Theta^1_{n+1}, \Theta^2_{n+1}, ..., \Theta^D_{n+1}\}$ the state of the universe collapsed into. As with the earlier Basis Method, an immediate question then arises regarding how the function f_{n+1} might be related to f_n. Also as before, two different classes of Rule consequently become apparent:

Class 1: The function f_n is constant for all n, so that $f_n = f$. Thus, the same $1 \rightarrow D$ function is used to generate $\mathfrak{B}_{n+2}$ from Ψ_{n+1} as was used to generate $\mathfrak{B}_{n+1}$ from Ψ_n, such that

$$\begin{aligned} \mathfrak{B}_{n+1} &= f(\Psi_n) \\ \mathfrak{B}_{n+2} &= f(\Psi_{n+1}) \\ &\vdots \end{aligned} \tag{8.76}$$

Class 2: The function f_n changes with n. Thus, a different $1 \rightarrow D$ function may be used to generate $\mathfrak{B}_{n+2}$ from Ψ_{n+1} than was used to generate $\mathfrak{B}_{n+1}$ from Ψ_n, such that

$$\begin{aligned} \mathfrak{B}_{n+1} &= f_n(\Psi_n) \\ \mathfrak{B}_{n+2} &= f_{n+1}(\Psi_{n+1}) \\ &\vdots \end{aligned} \tag{8.77}$$

Of course, and as with the Basis Method dynamics, the first Class is evidently a special example of the second Class in the case where $f_{n+1} = f_n$ for all n.

It is expected that there are many different functions f_n that could be used to generate a basis set of D orthogonal vectors from a given state Ψ_n; there might be many ways in which f_n might generally be constructed, and many forms it could then take. Indeed, this point may be reinforced by recalling that there are an infinite number of basis sets of orthogonal vectors spanning a Hilbert space $\mathcal{H}^{(D)}$, and hence, at least in principle, an infinite number of functions f_n relating them to a particular state Ψ_n.

However, it must be recalled that not every conceivable function acting in a vector space will provide D orthogonal outcomes when applied to a given state. In fact, the set of valid One-to-Many functions is a tiny subset of the set of all possible functions. Moreover, there are no obvious guidelines to suggest what a 'typical' such function should look like, and it is difficult to predict exactly how suitable One-to-Many relationships should in general be constructed. Clearly, then, it is a task for future research to attempt to discover what the various types and forms of valid One-to-Many functions might actually be.

For now, though, it is remarked that the validity of any proposed One-to-Many method depends entirely on the definition of the function in question, and hence relies on the underlying choice of Rules governing the system. This point is particularly important in regard to the question of when it is possible to construct a valid dynamics from a constant function $f_n = f$.

It is possible to provide a simple example of the Generated-Sort, One-to-Many Method using a single qubit universe represented by a state in a two dimensional Hilbert space. To

demonstrate this, however, observe first that the application of the function f_n to Ψ_n to give the basis set $\mathfrak{B}_{n+1} = \{\Theta^1_{n+1}, \Theta^2_{n+1}, ..., \Theta^D_{n+1}\}$ is, as expected, effectively the same as generating an equivalence class of operators $\hat{B}_{n+1}$ with eigenstates $\{\Theta^1_{n+1}, \Theta^2_{n+1}, ..., \Theta^D_{n+1}\}$. Moreover, the reverse of this is also clearly true: if an operator $\hat{B}_{n+1}$ could be constructed that is a function f'_n of Ψ_n, then this process also automatically defines the basis set $\mathfrak{B}_{n+1}$.

Thus, there are two equivalent ways of specifying One-to-Many Method Rules: either a $1 \rightarrow D$ function f_n should be defined that maps a state Ψ_n directly to D orthogonal vectors; or else a mechanism for obtaining an operator $\hat{B}_{n+1} = f'_n(\Psi_n)$ may be provided, where $\hat{B}_{n+1}$ is a Strong operator with D orthogonal eigenstates.

It is this latter possibility that will prove useful in the following examples.

Example A

Consider a state Ψ_n in the qubit Hilbert space $\mathcal{H}^{(2)}$ spanned by the 'usual' qubit basis

$$\mathcal{B} = \{|0\rangle, |1\rangle\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}. \tag{8.78}$$

Moreover, assume that the universe is governed by a One-to-Many Method mechanism, and develops according to the Class 1 Rule that the next operator $\hat{\Sigma}_{n+1}$ is defined simply as the projection of the n^{th} state. In other words, the dynamics of the universe proceeds by the general algorithm

- Ψ_n is tested by the Hermitian operator $\hat{\Sigma}_{n+1}$, to give the next state Ψ_{n+1} which is one of the eigenvectors of $\hat{\Sigma}_{n+1}$, where $\hat{\Sigma}_{n+1}$ is given by

$$\hat{\Sigma}_{n+1} = |\Psi_n\rangle\langle\Psi_n|. \tag{8.79}$$

Consider also an arbitrary state encountered in the universe's development. In fact, assume that at time n the universe may be described by the most general vector possible, that is, $\Psi_n = \frac{1}{\sqrt{\kappa}}(\alpha|0\rangle + \beta|1\rangle)$, where $\alpha, \beta \in \mathbb{C}$ and $\sqrt{\kappa} = \sqrt{\alpha\bar{\alpha} + \beta\bar{\beta}}$. Clearly, in the representation employed in this example, $\hat{\Sigma}_{n+1}$ is then given by the matrix

$$\hat{\Sigma}_{n+1} = \frac{1}{\kappa} \begin{pmatrix} \alpha\bar{\alpha} & \alpha\bar{\beta} \\ \beta\bar{\alpha} & \beta\bar{\beta} \end{pmatrix} \tag{8.80}$$

which has orthonormal eigenstates Θ^1_{n+1} and Θ^2_{n+1} equal to

$$\begin{aligned} \Theta^1_{n+1} &= \frac{1}{\sqrt{\kappa}}(\alpha|0\rangle + \beta|1\rangle) \\ \Theta^2_{n+1} &= \frac{1}{\sqrt{\kappa}}(\bar{\beta}|0\rangle - \bar{\alpha}|1\rangle) \end{aligned} \tag{8.81}$$

So, the eigenstates[24] of $\hat{\Sigma}_{n+1}$ define an orthogonal basis set $\mathfrak{B}_{n+1}$ given by $\mathfrak{B}_{n+1} = \{\Theta^1_{n+1}, \Theta^2_{n+1}\}$. Moreover, because $\hat{\Sigma}_{n+1}$ is a function of Ψ_n, i.e. $\hat{\Sigma}_{n+1} = \hat{\Sigma}_{n+1}(\Psi_n)$, a single qubit universe developing according to the Rule $\hat{\Sigma}_{n+1} = |\Psi_n\rangle\langle\Psi_n|$ provides a potential example of a Generated-Sort, One-to-Many Method mechanism.

Furthermore, because applying the same Rule (8.79) to whichever of the eigenstates Θ^1_{n+1} and Θ^2_{n+1} becomes Ψ_{n+1} gives rise to an operator $\hat{\Sigma}_{n+2} = |\Psi_{n+1}\rangle\langle\Psi_{n+1}|$ which also has two orthogonal eigenstates, and because this process may be continued indefinitely, where the next operator is always the same function of the current state for all n, a single qubit universe developing according to the Rule $\hat{\Sigma}_{n+1} = |\Psi_n\rangle\langle\Psi_n|$ actually provides an example of a Class 1 One-to-Many Method dynamics. Such a universe will develop with the next operator always dependent on the current state.

Of course, it must immediately be noted at this point that a Rule of the form $\hat{\Sigma}_{n+1} = |\Psi_n\rangle\langle\Psi_n|$ only really gives trivial dynamics. Clearly, Ψ_n is an eigenstate of $\hat{\Sigma}_{n+1}$, so the operator is ultimately equivalent to a null test, and $\Psi_n = \Psi_{n+1}$ for all n. Nevertheless, this example does, at least in principle, provide an illustration of the proposed mechanism.

Example B

A dynamics for a single qubit system that is perhaps more interesting than that of the above could instead be governed by the Class 1 One-to-Many Method Rule

- Ψ_n is tested by the Hermitian operator $\hat{\Sigma}_{n+1}$, to give the next state Ψ_{n+1} which is one of the eigenvectors of $\hat{\Sigma}_{n+1}$, where $\hat{\Sigma}_{n+1}$ is given by

$$\hat{\Sigma}_{n+1} = \hat{U}^*|\Psi_n\rangle\langle\Psi_n|\hat{U} \tag{8.82}$$

where $\hat{U}$ is a unitary operator of the form $\hat{U} = \exp(-i\varepsilon\hat{\sigma}_1)$, for $\varepsilon \in \mathbb{R}^+$ a real parameter and $\hat{\sigma}_1$ the usual Pauli operator. Clearly, such a Rule leads to a universe governed by an operator $\hat{\Sigma}_{n+1}$ with eigenvectors different from its current state.

As a visual interpretation of how this mechanism proceeds, it may be possible to imagine projecting the state Ψ_n onto a fixed vector $\hat{U}^*\Psi_n$ to get the next state Ψ_{n+1}, which is one of the eigenvectors of $\hat{U}^*|\Psi_n\rangle\langle\Psi_n|\hat{U}$. Continuing, a new 'fixed vector' may then be generated by slightly rotating Ψ_{n+1} to $\hat{U}^*\Psi_{n+1}$, and the state Ψ_{n+1} may be

[24] Strictly, $\hat{\Sigma}_{n+1}$ as given here is not a strong operator because it possesses a zero eigenvalue, so should technically not be allowed to develop the state according to the paradigm proposed in this thesis. However the reasons given in Chapter 5, for ruling out such operators in general, do not actually apply in the special case of two dimensional universes, and so the current example may still be validly discussed.

projected onto this new fixed vector $\hat{U}^*\Psi_{n+1}$ to get the subsequent state Ψ_{n+2}, which is one of the eigenvectors of $\hat{U}^*|\Psi_{n+1}\rangle\langle\Psi_{n+1}|\hat{U}$. And so on.

As a simple illustration, consider without loss of generality a universe initially in the state $\Psi_0 = |0\rangle$. According to the above Rule, the subsequent state Ψ_1 will be an eigenvector of the operator $\hat{\Sigma}_1$ given by

$$\begin{aligned}\hat{\Sigma}_1 &= \hat{U}^*|\Psi_0\rangle\langle\Psi_0|\hat{U} = e^{i\varepsilon\hat{\sigma}_1}|0\rangle\langle 0|e^{-i\varepsilon\hat{\sigma}_1} \\ &= \begin{pmatrix} \cos^2\varepsilon & -i\cos\varepsilon\sin\varepsilon \\ i\cos\varepsilon\sin\varepsilon & \sin^2\varepsilon \end{pmatrix}\end{aligned} \tag{8.83}$$

where the last line follows from the usual representation $|a\rangle = \binom{1-a}{a}$, for $a = 0, 1$, and from the identity $e^{-i\varepsilon\hat{\sigma}_1} = \left(\begin{smallmatrix}\cos\varepsilon & -i\sin\varepsilon \\ -i\sin\varepsilon & \cos\varepsilon\end{smallmatrix}\right) = \hat{\sigma}_0\cos\varepsilon - i\hat{\sigma}_1\sin\varepsilon$, which itself follows from the standard algebra (5.32) of the Pauli operators.

The operator $\hat{\Sigma}_1$ defined above has eigenvectors Θ_1^a and Θ_1^b given by

$$\begin{aligned}\Theta_1^a &= i\sin\varepsilon|0\rangle + \cos\varepsilon|1\rangle \\ \Theta_1^b &= -i\cos\varepsilon|0\rangle + \sin\varepsilon|1\rangle.\end{aligned} \tag{8.84}$$

So, the next state Ψ_1 of this universe will be either $\Psi_1 = \Psi_1^a = \Theta_1^a$ with probability $|\langle\Theta_1^a|\Psi_0\rangle|^2 = \sin^2\varepsilon$, or else $\Psi_1 = \Psi_1^b = \Theta_1^b$ with probability $|\langle\Theta_1^b|\Psi_0\rangle|^2 = \cos^2\varepsilon$, noting that $|\langle\Theta_1^a|\Psi_0\rangle|^2 + |\langle\Theta_1^b|\Psi_0\rangle|^2 = 1$ as expected.

Now, because Ψ_1 will be one of two possibilities, Ψ_1^a or Ψ_1^b, then according to the rule (8.82) it is evident that the subsequent test $\hat{\Sigma}_2$ will take one of two possible forms. Labelling these $\hat{\Sigma}_2^a$ and $\hat{\Sigma}_2^b$, it is clear that they are given by

$$\begin{aligned}\hat{\Sigma}_2^a &= \hat{U}^*|\Psi_1^a\rangle\langle\Psi_1^a|\hat{U} \\ \hat{\Sigma}_2^b &= \hat{U}^*|\Psi_1^b\rangle\langle\Psi_1^b|\hat{U}.\end{aligned} \tag{8.85}$$

Of course, which one of these is actually used to test the universe depends entirely upon which state, Ψ_1^a or Ψ_1^b, the system collapsed into when it became Ψ_1.

The operators $\hat{\Sigma}_2^a$ and $\hat{\Sigma}_2^b$ will themselves each possess two orthogonal eigenvectors. For $\hat{\Sigma}_2^a$ these may be labelled Ψ_2^{ac} and Ψ_2^{ad}, and are given by

$$\begin{aligned}\Psi_2^{ac} &= i\sin 2\varepsilon|0\rangle + \cos 2\varepsilon|1\rangle \\ \Psi_2^{ad} &= -i\cos 2\varepsilon|0\rangle + \sin 2\varepsilon|1\rangle\end{aligned} \tag{8.86}$$

whereas for $\hat{\Sigma}_2^b$ they may be labelled Ψ_2^{be} and Ψ_2^{bf}, with

$$\begin{aligned}\Psi_2^{be} &= -i\cos 2\varepsilon|0\rangle + \sin 2\varepsilon|1\rangle \\ \Psi_2^{bf} &= i\sin 2\varepsilon|0\rangle + \cos 2\varepsilon|1\rangle.\end{aligned} \tag{8.87}$$

So, given an initial state Ψ_0, then according to the rule (8.82) the wavefunction Ψ_2 after two steps will be one of these four possible states $\{\Psi_2^{ac}, \Psi_2^{ad}, \Psi_2^{be}, \Psi_2^{bf}\}$ with appropriate probabilities given by, for example,

$$\begin{aligned} P(\Psi_2^{ac}) &= P(\Psi_2^{ac}|\Psi_1^a|\Psi_0) = P(\Psi_2^{ac}|\Psi_1^a) \times P(\Psi_1^a|\Psi_0) \\ &= |\langle\Psi_2^{ac}|\Psi_1^a\rangle|^2 \times |\langle\Psi_1^a|\Psi_0\rangle|^2 \\ &= |(\sin\varepsilon \sin 2\varepsilon + \cos\varepsilon\cos 2\varepsilon)|^2 \sin^2\varepsilon = \cos^2\varepsilon \sin^2\varepsilon \end{aligned} \tag{8.88}$$

in obvious notation.

Further, the next test $\hat{\Sigma}_3$ will be one of four possibilities (one of which is $\hat{U}^*|\Psi_2^{ac}\rangle\langle\Psi_2^{ac}|\hat{U}$, etc.), each of which possesses two eigenstates. Clearly, the process continues such that the n^{th} state Ψ_n will be one of the eigenstates of one of 2^{n-1} possible operators $\hat{\Sigma}_n$.

The development of a universe described by a Rule such as $\hat{\Sigma}_{n+1} = \hat{U}^*|\Psi_n\rangle\langle\Psi_n|\hat{U}$ may be modelled by a simple computer simulation looping through a program a finite number of times. One successful method of achieving this has been to supply the program with an input containing an initial vector Ψ_0, an unitary matrix $\hat{U}$, and a number of iterations N to perform. The $i+1^{th}$ iteration, for $i = 0, 1, .., (N-1)$, has two parts: the first step of the program is to compute the matrix $\hat{U}^*|\Psi_i\rangle\langle\Psi_i|\hat{U}$, determine its two eigenvectors, and fill an array with the results of this evaluation. In the second step, a 'random number generator' is introduced that, when called, produces a number r_{i+1} that has either the value 0 or 1 with equal likelihood. If a "0" is found then the first element of the eigenvector array is recovered and set to Ψ_{i+1}; the second element is discarded from further discussion. If a "1" if found then the converse occurs and the second element is chosen instead to be Ψ_{i+1}.

The program loops through the process N times to yield a unique 'history' of states $\{\Psi_0, \Psi_1, \Psi_2, ..., \Psi_N\}$, where there are 2^N possible such histories, corresponding to the 2^N chains $r_1, r_2, ..., r_N$ of numbers produced by the random number generator, for $r_{i+1} = 0, 1$ for $i = 0, 1, .., (N-1)$. Of course, obtaining 2^N sets of results after time N is to be expected: any string of N characters $\{ r_{i+1} : i = 0, ..., (N-1)\}$, where each character may take one of two values, may be thought of as representing a binary number $< 2^N$. These binary numbers may thus be used to effectively label the quantum history of the system.

As with the model described in Example A, it is to be noted that the types of development resulting from the above Rule are extremely limited. In fact, this conclusion follows from two observations.

Firstly, and as is evident from (8.86) and (8.87), the eigenvectors of $\hat{\Sigma}_2^a$ are degenerate with those of $\hat{\Sigma}_2^b$, that is, $\Psi_2^{ac} \equiv \Psi_2^{bf}$ and $\Psi_2^{ad} \equiv \Psi_2^{be}$. It can be shown, moreover, that this is a trend that is continued throughout the universe's development, such that the state Ψ_n after n steps will not be one out of 2^n *different* possibilities, but will instead be

one of only 2 different vectors. Specifically, in fact, it may be shown that if Ψ_n^X and Ψ_n^Y are the two possible outcomes of a test $\hat{\Sigma}_n$, then the two subsequent potential operators $\hat{\Sigma}_{n+1}^X = \hat{U}^*|\Psi_n^X\rangle\langle\Psi_n^X|\hat{U}$ and $\hat{\Sigma}_{n+1}^Y = \hat{U}^*|\Psi_n^Y\rangle\langle\Psi_n^Y|\hat{U}$ share the same set of eigenstates.

This conclusion follows from the observations that Ψ_n^X and Ψ_n^Y are necessarily orthogonal, that the eigenstates of $\hat{\Sigma}_{n+1}^X$ must also be orthogonal, and that the eigenstates of $\hat{\Sigma}_{n+1}^Y$ must be orthogonal too. Now, labelling the eigenstates of $\hat{\Sigma}_{n+1}^X$ as $\{\psi, \varphi\}$, it is evident that the vector $\hat{U}^*|\Psi_n^X\rangle$ is an eigenstate of $\hat{\Sigma}_{n+1}^X$ with eigenvalue 1, because $\hat{\Sigma}_{n+1}^X(\hat{U}^*|\Psi_n^X\rangle) = \hat{U}^*|\Psi_n^X\rangle$; thus, ψ may be chosen as $\psi = \hat{U}^*|\Psi_n^X\rangle$.

Now, the other (unknown) eigenvector, φ, of $\hat{\Sigma}_{n+1}^X$ must be orthogonal to $\hat{U}^*|\Psi_n^X\rangle$, such that

$$\langle\varphi|\hat{U}^*\Psi_n^X\rangle = 0. \tag{8.89}$$

The question becomes: what is this vector φ? Clearly, one possible candidate for φ is the vector $\hat{U}^*\Psi_n^Y$, because $\hat{U}\hat{U}^* = \hat{I}$ and $\langle\Psi_n^Y|\Psi_n^X\rangle = 0$, where $\hat{I}$ is the identity operator. Furthermore, because the Hilbert space of the system is two dimensional, this candidate is the only choice. So, the eigenvectors of $\hat{\Sigma}_{n+1}^X$ must be $\hat{U}^*\Psi_n^X$ and $\hat{U}^*\Psi_n^Y$.

A similar analysis of $\hat{\Sigma}_{n+1}^Y$ can readily be used to demonstrate that the eigenstates of this operator are also $\hat{U}^*\Psi_n^X$ and $\hat{U}^*\Psi_n^Y$; the conclusion is shown.

Secondly, the above types of mechanism, where the tests $\hat{\Sigma}_{n+1}$ depend on projection operators $|\Psi_n\rangle\langle\Psi_n|$, can only provide a suitable dynamics for single qubit universes. This is because projection operators are not strong, as discussed in Section 5.5, on account of them possessing (degenerate) eigenvalues of zero. So, an operator of the form $\hat{U}^*|\Psi\rangle\langle\Psi|\hat{U}$, where Ψ is a vector in a $D > 2$ dimensional Hilbert space $\mathcal{H}^{(D)}$, does not specify a unique basis set of orthogonal eigenvectors, and hence cannot be used to provide valid dynamics in the scheme proposed here.

The problem is in fact symptomatic of the result that in situations with dimensions greater than 2, it is difficult to find a mechanism that uses a single vector Ψ to uniquely specify $D-1$ other vectors orthogonal to Ψ. This follows because there are very many sets of $D-1$ vectors in $\mathcal{H}^{(D)}$ that are mutually orthogonal, whilst also being orthogonal to Ψ, and so in general it is hard to find a Rule that effectively picks just one of these out.

For instance, consider the case when $D = 3$, and define an orthonormal basis $\mathcal{B}^{(3)}$ for $\mathcal{H}^{(3)}$ as $\mathcal{B}^{(3)} \equiv \{|0\rangle, |1\rangle, |2\rangle\}$. Now, without loss of generality, given a state $\Psi = |0\rangle$ it is possible to find many different pairs of vectors $[\phi, \theta]$ that are mutually orthogonal to one another, and also orthogonal to $|0\rangle$. One example is $[\phi, \theta] = [|1\rangle, |2\rangle]$, but of course

$$\begin{aligned} [\phi, \theta] &= \left[\frac{1}{\sqrt{2}}(|1\rangle + |2\rangle), \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)\right], \\ [\phi, \theta] &= \left[\frac{1}{5}(4|1\rangle - 3i|2\rangle), \frac{1}{5}(-3i|1\rangle + 4|2\rangle)\right], \end{aligned} \tag{8.90}$$

and so on, also satisfy this criterion. Thus, specifying just the vector $\Psi = |0\rangle$ does not imply an automatic choice for a unique orthonormal basis set $\{\Psi, \phi, \theta\}$ of states spanning $\mathcal{H}^{(3)}$, because ϕ and θ could take many different forms. In fact, more information needs to be provided in order to select a particular pair of orthonormal vectors from the infinite set of possibilities. In this case, only when, say, $\Psi = |0\rangle$ and ϕ are both given is it then possible to specify what θ must be.

In the context of this chapter, a unique basis set $\mathfrak{B}_{n+1}$ of states is hoped to be generated from Ψ_n in order for the universe to develop. So, in any One-to-Many mechanism based upon a Rule of the form $\mathfrak{B}_{n+1} = f_n(\Psi_n)$, it would clearly be unsatisfactory if the result of f_n acting on Ψ_n gave a number of orthonormal basis sets. Instead, f_n must be sufficiently well defined such that this process 'pins down' just one unique set.

A dynamics based on projection operators, however, cannot in general achieve this.

Fortunately, though, an exception to this conclusion occurs in two dimensional Hilbert spaces. In this circumstance, specifying just a single vector Ψ does imply a unique orthonormal basis set of states, because there is *only* one other vector that is orthogonal to Ψ in $\mathcal{H}^{(2)}$. So, when $D = 2$ it is possible to label an orthonormal basis set by using just one of its two states Ψ, and this principle may ultimately be exploited to formulate rules determining the system's development.

So, in the dynamics described in Examples A and B, a given $\Psi_n \in \mathcal{H}^{(2)}$ is able to generate a unique, preferred basis set $\mathfrak{B}_{n+1}$ for the next jump. Thus, a dynamics based upon projection operators may be justified in $\mathcal{H}^{(2)}$, and a unique basis set of states can still generated, even though an eigenvalue of zero is present.

Of course, despite potential objections questioning how physically 'interesting' a single qubit universe models might be, it does not detract from the overall point that the proposed examples show how a state in a Hilbert space can be developed according to Type III, Generated-Sort One-to-Many method Rules.

Example C

Examples A and B involve universes developing according to a deterministic (Type III) Class 1 One-to-Many Method mechanism. However, just as in previous sections of this chapter, it is also possible to consider probabilistic (Type $IIIa$) dynamics by extending the Rules in the obvious way.

For instance, instead of generating the next basis set $\mathfrak{B}_{n+1}$ from the current state Ψ_n according to the deterministic rule $\mathfrak{B}_{n+1} = f_n(\Psi_n)$, it is alternatively possible to consider Rules in which $\mathfrak{B}_{n+1} = f_n(\Psi_n)$ with probability P_{f_n}, but $\mathfrak{B}_{n+1} = g_n(\Psi_n)$ with probability

P_{g_n}, whilst $\mathfrak{B}_{n+1} = h_n(\Psi_n)$ with probability P_{h_n}, and so on, where f_n, g_n and h_n are different functions. As before, conservation of probability requires that $P_f + P_g + ... = 1$.

So, Example B may readily be augmented to a probabilistic Class 1 One-to-Many Method mechanism for a state Ψ_n in a two dimensional Hilbert space $\mathcal{H}^{(2)}$, by considering a Rule such as

- Ψ_n is tested by the Hermitian operator $\hat{\Sigma}_{n+1}$, to give the next state Ψ_{n+1} which is one of the eigenvectors of $\hat{\Sigma}_{n+1}$, where $\hat{\Sigma}_{n+1}$ is given by

$$\hat{\Sigma}_{n+1} = \left\{ \begin{array}{l} \hat{U}^{(1)*}|\Psi_n\rangle\langle\Psi_n|\hat{U}^{(1)} \text{ with Probability } P^{(1)} \\ \hat{U}^{(2)*}|\Psi_n\rangle\langle\Psi_n|\hat{U}^{(2)} \text{ with Probability } P^{(2)} \end{array} \right\} \tag{8.91}$$

where $P^{(1)} + P^{(2)} = 1$, and

$$\hat{U}^{(1)} = e^{-i\varepsilon\hat{\sigma}_1} \quad , \quad \hat{U}^{(2)} = e^{-i\mu\hat{\sigma}_2} \tag{8.92}$$

with $\hat{\sigma}_1$ and $\hat{\sigma}_2$ Pauli operators and $\varepsilon, \mu \in \mathbb{R}^+$.

As was the case in previous sections, extending a Type III Rule to a Type $IIIa$ one does not add anything significantly new to the discussion.

Example D

Up until now, attention has been focused on the Class 1 One-to-Many Method. It is, however, also possible to consider Class 2 models.

As an example of such a dynamics, consider as before a state Ψ_n in a two dimensional, single qubit Hilbert space $\mathcal{H}^{(2)}$, and assume that the universe is governed by the Rule

- Ψ_n is tested by the Hermitian operator $\hat{\Sigma}_{n+1}$, to give the next state Ψ_{n+1} which is one of the eigenvectors of $\hat{\Sigma}_{n+1}$, where $\hat{\Sigma}_{n+1}$ is given by

$$\hat{\Sigma}_{n+1} = (\hat{U}^*)^{n+1}|\Psi_n\rangle\langle\Psi_n|(\hat{U})^{n+1} \tag{8.93}$$

where $\hat{U}$ is an arbitrary unitary operator which, for the sake of illustration, could be defined again as $\hat{U} = \exp(-i\varepsilon\hat{\sigma}_1)$.

So, from a given state Ψ_0 at initial 'time' $n = 0$, it follows that

$$\begin{aligned} \hat{\Sigma}_1 &= \hat{U}^*|\Psi_0\rangle\langle\Psi_0|\hat{U} \\ \hat{\Sigma}_2 &= \hat{U}^*\hat{U}^*|\Psi_1\rangle\langle\Psi_1|\hat{U}\hat{U} \\ \hat{\Sigma}_3 &= \hat{U}^*\hat{U}^*\hat{U}^*|\Psi_2\rangle\langle\Psi_2|\hat{U}\hat{U}\hat{U} \end{aligned} \tag{8.94}$$

and so on.

A universe developing according to this type of Rule would proceed analogously to the models discussed in Example A and B; the universe always collapses to one of the eigenstates of $\hat{\Sigma}_n$, and the next operator $\hat{\Sigma}_{n+1}$ is then given as a function of this new state Ψ_n. The major difference, however, is that in the present case the function that generates the next operator $\hat{\Sigma}_{n+1}$ from the current state Ψ_n is not constant, but is instead a dynamic relationship that depends on the parameter n.

Thus, the above Rule provides an example of a universe that is developed according to operators that depend on both the current state and the current 'time': a Class 2 mechanism.

The difference between Class 1 and Class 2 dynamics might perhaps be likened to the differences encountered in laboratory quantum mechanics between systems that are evolved by a constant Hamiltonian, $\hat{H}$, and those that are alternatively evolved by a time dependent Hamiltonian $\hat{H}(t)$. After all, a jump dependent operator on the pregeometric level is directly analogous to a time dependent operator in conventional physics, because the parameter n is ultimately assumed to be the pregeometric origin of emergent time t.

However, this similarity should not of course be taken too literally: Hamiltonians are viewed in the proposed paradigm as emergent constructs, and as such are not defined on the pregeometric level discussed here.

8.4.3 A Type IV Extension

As an extension to this section, note that both List-Sort and Generated-Sort mechanisms are expected to have their analogies in universes developing according to Type IV, IVa, V and Va Rules. Moreover, in fact, such analogies may provide richer possibilities for dynamics than their Type III counterparts.

For example, note that although no Type III Class 1 Basis Method Rule has been found that can provide a valid dynamics for a universe developing self-referentially according to the current state (i.e. where the Reduced basis set $\mathfrak{B}^R_{n+1} = \{\hat{U}^{(1)}\Psi_n, \hat{U}^{(2)}\Psi_n, ..., \hat{U}^{(D-1)}\Psi_n\}$ is generated by a *fixed* set of $(D-1)$ unitary operators $\{\hat{U}^{(\mu)}\}$ acting on Ψ_n), such a Class 1 Basis Method mechanism may easily be implemented in universes developing according to Type IV rules.

To demonstrate this last point, recall from Section 8.1 that a Type IV Rule is defined as one in which the next basis set $\mathfrak{B}_{n+1}$ depends on the current basis set $\mathfrak{B}_n$ (unlike, of course, a Type III mechanism, where the next basis set $\mathfrak{B}_{n+1}$ depends on the current state Ψ_n, which is just one element of $\mathfrak{B}_n$).

So, by denoting the orthonormal elements of these two preferred bases as

$$\begin{aligned} \mathfrak{B}_n &\equiv \{\Phi_n^1, \Phi_n^2, ..., \Phi_n^D\} \\ \mathfrak{B}_{n+1} &\equiv \{\Phi_{n+1}^1, \Phi_{n+1}^2, ..., \Phi_{n+1}^D\} \end{aligned} \tag{8.95}$$

it is clear that the analogy of the Class 1 Basis Method mechanism for a Type IV universe requires a fixed set of unitary operators $\{\hat{U}^{(i)}\}$ to be found, where the operator $\hat{U}^{(i)}$ maps the element $\Phi_n^i \in \mathfrak{B}_n$ to the element $\Phi_{n+1}^i \in \mathfrak{B}_{n+1}$ in the manner

$$\Phi_{n+1}^i = \hat{U}^{(i)}\Phi_n^i \tag{8.96}$$

where the elements of the bases have been indexed in the simplest way, without loss of generality.

For completeness, note how this compares with the Type III Rule

$$\Phi_{n+1}^i = \hat{U}^{(i)}\Psi_n. \tag{8.97}$$

Of course, just as in the Type III situation, the constraint

$$\langle \Phi_{n+1}^i | \Phi_{n+1}^j \rangle = \langle \Phi_n^i \hat{U}^{(i)} | \hat{U}^{(j)} \Phi_n^j \rangle = \delta_{ij} \tag{8.98}$$

must be enforced on the definition of the unitary operators $\{\hat{U}^{(i)}\}$ to ensure that the vectors $\{\Phi_{n+1}^i\}$ are orthogonal.

Moreover, of course, in actuality only $(D-1)$ unitary operations $\{\hat{U}^{(\mu)}\}$ can be freely defined: as with the Type III case, a specification of the Reduced set $\{\Phi_{n+1}^1, \Phi_{n+1}^2, ..., \Phi_{n+1}^{D-1}\}$ automatically defines the "D^{th}" vector Φ_{n+1}^D because of the required orthogonality.

Now, in order for Type IV Class 1 Basis Method Rules to be accepted as providing a suitable mechanism for the universe's development from $\Psi_n \rightarrow \Psi_{n+1} \rightarrow \Psi_{n+2} \rightarrow ...$, the subsequent Reduced basis set $\mathfrak{B}_{n+2}^R$ defined as $\mathfrak{B}_{n+2}^R \equiv \{\Phi_{n+2}^1, \Phi_{n+2}^2, ..., \Phi_{n+2}^{D-1}\}$ must contain $(D-1)$ orthogonal elements Φ_{n+2}^μ that are generated from those of $\mathfrak{B}_{n+1}^R$ according to the map[25]: $\Phi_{n+2}^\mu = \hat{U}^{(\mu)}\Phi_{n+1}^\mu$, where $\mu = 1, ..., (D-1)$. Of course, these conditions must hold for all n.

However, unlike for the Type III case, in which no *fixed* set $\{\hat{U}^{(\mu)}\}$ has been found that can develop the system from $\Psi_n \rightarrow \Psi_{n+1} \rightarrow \Psi_{n+2} \rightarrow ...$ according to the Basis Method algorithm, in Type IV universes it is trivially easy to find a constant set of unitary operators that provide a valid mechanism for dynamics. In fact one such set occurs for the Rule

$$\hat{U}^{(\mu)} = \hat{U}_f \quad , \quad \forall \mu \tag{8.99}$$

[25]In principle, the $(D-1)$ 'free parameter' operators $\{\hat{U}^{(\mu)}\}$ could be defined such that they act on any of the D vectors in $\mathfrak{B}_n$. The permutations that result, however, do not seriously affect the situation discussed. In short, it does not matter which member of $\mathfrak{B}_n$ is 'left out' of the Reduced basis $\mathfrak{B}_{n+1}^R$.

where $\hat{U}_f$ is an arbitrary, fixed unitary operator, such that the set $\{\hat{U}^{(\mu)}\}$ becomes a set of $(D-1)$ equal members, $\{\hat{U}_f, \hat{U}_f, ..., \hat{U}_f\}$.

Under this circumstance, the constraint $\langle \Phi_n^\mu \hat{U}^{(\mu)} | \hat{U}^{(\nu)} \Phi_n^\nu \rangle = \delta_{\mu\nu}$ is clearly satisfied by definition, because

$$\hat{U}_f^* \hat{U}_f = \hat{I} \tag{8.100}$$

where $\hat{I}$ is the identity operator in $\mathcal{H}^{(D)}$, and the Reduced basis set $\mathfrak{B}_{n+1}^R$ is given by

$$\mathfrak{B}_{n+1}^R = \{\hat{U}_f \Phi_n^1, \hat{U}_f \Phi_n^2, ..., \hat{U}_f \Phi_n^{D-1}\}. \tag{8.101}$$

Moreover, for the 'remaining vector' Φ_{n+1}^D, it turns out in this case that

$$\Phi_{n+1}^D = \hat{U}_f \Phi_n^D \tag{8.102}$$

because in this instance

$$\langle \Phi_{n+1}^D | \hat{U}^{(\mu)} \Phi_n^\mu \rangle = \langle \Phi_n^D \hat{U}_f | \hat{U}_f \Phi_n^\mu \rangle = \langle \Phi_n^D | \Phi_n^\mu \rangle = 0 \tag{8.103}$$

for all $\mu = 1, ..., (D-1)$, as required.

So, assuming that $\hat{U}_f \neq \hat{I}$ it is evident that

$$\mathfrak{B}_{n+1} = \hat{U}_f \mathfrak{B}_n \neq \mathfrak{B}_n \tag{8.104}$$

such that $\mathfrak{B}_{n+1}$ does provide a suitable basis set of states for a jump from Ψ_n to Ψ_{n+1} to occur.

Furthermore, the same set of operators $\{\hat{U}^{(i)} = \hat{U}_f : i = 1, ..., D\}$ can then be applied to the elements of $\mathfrak{B}_{n+1}$ to give a new orthogonal basis, which may be labelled as $\mathfrak{B}_{n+2}$, and the process may be continued. Thus, the proposed mechanism may be described as valid, and the above conclusion is justified: Type *IV* Class 1 Basis Method Rules are indeed allowed.

Overall, then, the development of a universe according to this type of Type *IV* Basis Method Rule proceeds by the state Ψ_n collapsing to the state Ψ_{n+1}, which is one of the elements Φ_{n+1}^i of the basis set

$$\mathfrak{B}_{n+1} = \hat{U}_f \mathfrak{B}_n = \{\Phi_{n+1}^i = \hat{U}_f \Phi_n^i : i = 1, ..., D\} \tag{8.105}$$

with probability

$$P(\Psi_{n+1} = \Phi_{n+1}^i) = |\langle \hat{U}_f \Phi_n^i | \Psi_n \rangle|^2 \tag{8.106}$$

where Φ_{n+1}^i may equally be viewed as one of the eigenstates of $\hat{\Sigma}_{n+1}$, which is a member of the equivalence class of operators $\hat{B}_{n+1}$ implied by $\mathfrak{B}_{n+1}$. The universe then collapses to the state Ψ_{n+2}, which is one of the elements Φ_{n+2}^i of the basis set

$$\mathfrak{B}_{n+2} = \hat{U}_f \mathfrak{B}_{n+1} = \hat{U}_f \hat{U}_f \mathfrak{B}_n = \{\Phi_{n+2}^i = \hat{U}_f \hat{U}_f \Phi_n^i : i = 1, ..., D\} \tag{8.107}$$

with probability $P(\Psi_{n+2} = \Phi^i_{n+2}) = |\langle \hat{U}_f \hat{U}_f \Phi^i_n | \Psi_{n+1} \rangle|^2$. And so on.

Of course, the particular deterministic (Type IV) Rules presented above can be generalised to probabilistic (Type IVa) cases in the obvious way. For instance, the next basis $\mathfrak{B}_{n+1}$ could be given by

$$\mathfrak{B}_{n+1} = \hat{U}_f \mathfrak{B}_n \tag{8.108}$$

with probability P_f, or instead by

$$\mathfrak{B}_{n+1} = \hat{U}_g \mathfrak{B}_n \tag{8.109}$$

with probability P_g, and so on, where $\hat{U}_f$ and $\hat{U}_g$ are different unitary operators, and $P_f + P_g + ... = 1$.

In effect, in the above mechanisms the unitary operator $\hat{U}_f$ (or $\hat{U}_g$ etc.) may be thought of as 'rotating' the entire basis set $\mathfrak{B}_n$ into the set $\mathfrak{B}_{n+1}$, and then subsequently rotating this new basis set $\mathfrak{B}_{n+1}$ into the set $\mathfrak{B}_{n+2}$, and so on. In addition, because $\hat{U}_f$ is effectively behaving globally on the whole basis set, that is, because $\hat{U}_f$ is rotating each member of the basis set in the same way, its application automatically preserves the orthogonality between the individual elements, as required.

So, the proposed Rule clearly provides a simple, but valid, mechanism for dynamics, where the next operator $\hat{\Sigma}_{n+1} = \hat{B}_{n+1}$ chosen by the universe to test the state Ψ_n is strongly related to the previous operator $\hat{\Sigma}_n = \hat{B}_n$ of which Ψ_n is an eigenstate. Moreover, the procedure is valid for all n. The above mechanism therefore provides an example of a Basis Self-Referential, Class 1 Basis Method dynamics, a conclusion made particularly significant by the lack of any analogous State Self-Referential, Class 1 Basis Method dynamics.

Of course, analogies of the other sorts of mechanism discussed in this chapter are naturally expected to exist within the frameworks of Type IV and V Rules.

Note that the type of Rule proposed above could have an important physical consequence. Consider as before a universe developing according to the Rule $\mathfrak{B}_{n+1} = \hat{U}_f \mathfrak{B}_n$, but this time impose the additional condition that

$$\hat{U}_f = \hat{I} + \epsilon \hat{U}' \tag{8.110}$$

where $\hat{I}$ is the identity operator, ϵ a small parameter, and $\hat{U}'$ an operator chosen according to the constraint that $\hat{U}_f$ obeys the conditions required for the dynamics (i.e. $\hat{U}_f$ is unitary).

Now because ϵ is small, it follows that $\hat{U}_f$ approximates to $\hat{I}$, i.e. $\hat{U}_f \approx \hat{I}$. So, in this case

$$\mathfrak{B}_{n+1} = \hat{U}_f \mathfrak{B}_n \approx \hat{I}\mathfrak{B}_n = \mathfrak{B}_n \tag{8.111}$$

such that the preferred basis at time $n+1$ is 'roughly' the same as the preferred basis at time n. Moreover, this then implies that the next basis set $\mathfrak{B}_{n+1}$ will contain a member that is very 'similar', in some sense, to the current state Ψ_n. So, and due to the Born probability rule, the universe is highly likely to jump to this 'very similar' state, where the probability that the universe collapses to this vector is expected to approach unity in models with Hilbert spaces of high dimensionality. Overall, then, the outcome from such a Rule is that $\Psi_{n+1} \approx \Psi_n$ for all n.

In other words, in a universe developing according to this type of Rule, the state changes only very 'slightly' from one jump to the next. Importantly, then, such a mechanism might be useful to describe a possible origin of apparent persistence in the quantum universe. Moreover, it might also provide a dynamics in which the universe's development appears almost deterministic, just as seems to be the case in classical physics: given a state Ψ_n, it would be possible to predict what the next state Ψ_{n+1} will be like with near certainty, because $\Psi_{n+1} \approx \Psi_n$.

Of course, much work is required to fully justify these assertions, and to define exactly what the notion of 'similarity' might imply.

8.4.4 Summary

As a final remark to this section, it should be mentioned that it is also possible to envisage universes governed by Rules that are themselves subject to change. Indeed, the Type of Rule used to select the next operator $\hat{\Sigma}_{n+1}$ could actually depend on n, such that for example at 'time' m a particular Type III List-Sort mechanism could be used to select $\hat{\Sigma}_{m+1}$, whereas at time m' a different Type III List-Sort Rule might be employed instead to give $\hat{\Sigma}_{m'+1}$, whilst at time m'' the universe could adopt a Type III Generated-Sort dynamics, but at time m'''an entirely different Type IV Rule could be used. And so on.

Under such circumstances, it might be expected that there is some sort of 'Meta-Rule' governing the dynamics of how the Rules change with n, a concept that is analogous to Buccheri's idea of the "Rules of the rules" [27] which determines how the laws of physics may develop with time.

It is possible now to sum up the various Types of fixed Rule dynamics discussed so far in this chapter, and compare these with universes governed by Rules that change and develop over time.

Starting with classical physics, for example, if a scientist knows the current 'state' of the Universe (i.e. the position and momentum of every particle it contains) and the laws of physics, she is able to determine with certainty what its entire future will be. This is the deterministic physics of Newton's clockwork universe.

In Type 0 quantum universes, on the other hand, if the scientist knows the current state of the universe and decides upon a particular operator to test it with, she is able to determine with certainty what the next basis set of eigenstates will be, and hence estimate the next state within the bounds imposed by quantum probability.

Furthermore, the same comment is broadly true for the Type III List-sort dynamics discussed in Section 8.2: if the List $\mathfrak{L} \equiv \{\mathfrak{B}_i : i = 1, 2, ..., l\}$ of potential next operators is specified, and the Rules governing which of these is chosen to test the state are understood, then given a state Ψ_n it is always possible to determine what the next set of eigenstates will be. Moreover, under such circumstances the state of the universe will always be an element of one of the basis sets from the list $\mathfrak{L}$, and this state will always be tested by one of the operators $\{\hat{B}_i\}$.

So, it is consequently possible not only to predict the probability of obtaining a particular next state, but also to ask questions of the form: "if the next state turns out to be X, what is the probability that the subsequent state will be Y?"; or, "what is the probability that the test after n steps will be a particular operator $\hat{B}_a$?".

In Class 1 Type III Generated-Sort dynamics, if the current state of the universe is known, it is only ever possible to determine the next operator that acts. Unlike the List-sort dynamics, there is now no pre-existing List of basis sets specifying every operator that will ever be used in the universe's development, because the universe is 'making-up' its tests as it goes along. Additionally, unlike for List-Sort dynamics, in which the number of different states the universe could ever potentially exist in (its phase space) is fixed by the length of the List and the dimension of the Hilbert space, in Generated-Sort dynamics this number of different states could be unbounded. However, whilst it may not be known in advance what every future operator will be, because the Rules are known and the function f relating $\mathfrak{B}_{n+1}$ to $\Psi_n \in \mathcal{H}^{(D)}$ is fixed, it is possible to say that if the universe were ever in a particular state Z, the subsequent state would be one of the D eigenvectors of an operator $\hat{\Sigma} = f(Z)$ determined by Z.

Conversely, in Class 2 Type III Generated-Sort dynamics it is not possible to determine what the next operator will be if only the current state is known. In this case, if the universe were ever in a state Z, it could not immediately be inferred what the next set of eigenvectors must be, because the relationship f_n between states and operators is always changing. Specifically, if a universe is governed by a Rule of the form $\hat{\Sigma}_{n+1} = f_n(\Psi_n)$, it is necessary to know both the state Ψ_n and the 'time' n in order to determine the next

operator $\hat{\Sigma}_{n+1}$. In this instance, information additional to the knowledge of the current state is required (for example how many jumps have taken place since a particular 'reference state' Ψ_0, or at least what the previous operator was), because even if it is known how the operator changes with 'time', it is still necessary to specify what the time is in order to say how much it has changed. However, given Ψ_n and n it is then possible to determine $\hat{\Sigma}_{n+1}$, because the Rule $\hat{\Sigma}_{n+1} = f_n(\Psi_n)$ relating $\hat{\Sigma}_{n+1}$ to Ψ_n is defined for all time at the outset.

Finally, in a universe governed by Rules which also change, it is not sufficient just to know the current Rule, state and time in order to determine $\hat{\Sigma}_{n+1}$, but it is also necessary to know the Rules of the Rules. Such 'Meta-Rules' could then be used to select a particular Type of Rule, which could then choose a particular operator $\hat{\Sigma}_{n+1}$ based somehow, perhaps, on the current state Ψ_n and/or the last basis $\mathfrak{B}_n$ and/or the current time n. Of course, these choices could also even depend upon some sort of additional variable previously indiscussed.

In reality, it would be very difficult for endo-physical observers to ever ascertain what the Rules of the Rules governing their universe actually are. After all, a physicist could (at best) only ever really be sure of what the current Rule is, and the Rules are assumed to be constantly changing. So, although such universes will not be discussed further in this work, note that this point has analogies with some of the recent speculations in fundamental physics regarding whether the speed of light or the electron charge have remained constant throughout the history of the Universe (e.g. [86] and [87], respectively). In both of these cases, it is difficult to reconstruct what the laws of physics were like in the distant past when only the current state of the Universe is available for study.

It is also far beyond the scope of this thesis to take the logically greater step and consider a completely 'free' universe, that is, one in which neither the Rules, nor the Rules governing the Rules, are fixed and specified in advance. Indeed, it is difficult to imagine how such a model could even be envisaged that required no order or direction, at least at the outset. After all, even a universe incorporating Meta-Rules relies on a definition of what this Meta-Rule is, and additionally on what the boundary or initial condition Ψ_0 was. In fact, in any such 'free universe', order, Rules, and even Rules of Rules would have to be defined or 'created' somehow on their own account as the universe develops, and it is almost impossible to comprehend how this process could occur. A mechanism governed by Rules of the Rules might therefore represent the 'final level' that can be used to describe a fully quantum universe.

Ultimately, then, the actual definition of the Rules governing a universe (or at least the choice of the Rules of the Rules) could have no origin that is explainable in terms of any

sort of higher order mechanism. Consequently, their presence and form in any given model may have to be accepted merely as a fundamental pre-requisite, just as the existence of the underlying Hilbert space or the List $\mathfrak{L}$ is taken to be.

Of course, this is similar to the philosophical problem faced in the real Universe: physicists might one day be able to determine *what* the Theory of Everything is, but to say *why* it is like this without appealing to blind chance, the Anthropic Principle, or a Higher Being may be beyond the scope of empirical physics. Scientists may never be able to say why the constants of nature have the values that they do, but just that they are predicted by a theory that happens to describe the reality they exist in.

8.5 Examine-Decision Mechanisms

Each of List-Sort and Generated-Sort dynamics attracts an obvious comment.

The List-Sort Rules rely on the decision of operator $\hat{\Sigma}_{n+1}$ being made based upon a particular 'property' of the state Ψ_n, for example its separability. No explanation is given, however, as to how the Rules actually get to 'know' what this property is, such that they can then make the selection. Paraphrasing, there is no 'self-examining' part of the List-Sort algorithm that allows the universe to explicitly investigate its state for a particular property (e.g. count how many factors Ψ_n has), such that the next operator may then be chosen appropriately from the List.

The same remark is not necessary in Generated-Sort mechanisms, because in these the next operator is a direct function of the current state. However, the types of Rules discussed in Section 8.4 suffer instead from the fact that the dynamics which results is rather inflexible: once a state Ψ_n is specified, there is no choice about how the next operator will be defined. In other words, there is no 'decision making' part of the Generated-Sort algorithm applying logic of the form: *if* the universe finds itself in state $\Psi_n = x$ *then* use an operator $\hat{\Sigma}_{n+1} = f(x)$; but *if instead* the universe is in state y *then instead* use an operator $\hat{\Sigma}_{n+1} = g(y)$ generated in an alternative way, where $f \neq g$; and so on. For example, there is no freedom in the mechanism to allow the state to be tested by, say, $\hat{\Sigma}_{n+1} = h(\Psi_n)$ if Ψ_n has F_H factors, but instead by $\hat{\Sigma}_{n+1} = k(\Psi_n)$ if it has F_K factors; according to Generated-Sort Rules, the actual 'properties' of Ψ_n are not used to decide how the state develops.

Whilst these two comments are not serious problems, such that both List-Sort and Generated-Sort Rules can still be taken to provide valid dynamics for the quantum universe, it is natural to speculate on whether mechanisms could exist that appear to develop the state without these limitations. After all, on the small scale this type of development is

what real physicists tend to experience in the laboratory: in general, scientists do believe themselves to be able to examine and investigate quantum sub-systems, and then choose how to develop them, from a huge number of different ways, based upon what they have learnt.

In this section, therefore, it is hoped to explore the possibility of universes that are somehow able to 'examine' their state themselves for a particular property, and then develop it in a way that depends on what this property is. The desire, then, is to investigate sets of Rules that could provide a fully automatic and self-referential mechanism that leads to a dynamics equivalent to a process of examination, decision, development, examination, decision, development,..., continuing indefinitely. Universes governed by such mechanisms could be described as obeying *Examine-Decision* (ED) Rules.

8.5.1 Preliminary Considerations

In order to be valid, the Rules governing ED dynamics must define a single quantum computation that, in one time step, 'examines' the universe's state Ψ_n, 'decides' upon a course of action according to the result of the initial examination, and then consequently develops it to the next state Ψ_{n+1}. So, the question becomes: how might it be possible to construct mechanisms that examine the state, and then develop it in a way that depends on the outcome of this investigation?

The overall goal of this section is to investigate how such computations might be achieved.

The first point to note, however, is that any *measurement* of any property of the state Ψ_n of the universe necessarily involves a process of information extraction. This fact would not cause any problems in a classical universe, because in classical physics it is possible to observe an object and expect it to remain unchanged. In a fully quantum universe, on the other hand, such non-invasive techniques are forbidden: it is not possible to measure a quantum state for one property, and then test the same state for something else. The first measurement destroys the original state and creates a new one that is an eigenstate of whichever operator was used, and it is this new state that then has to be tested in subsequent measurements.

Now, *if* a hypothetical ED mechanism was proposed that followed the general algorithm

1. test the state Ψ_n for a particular 'property' $p^{(i)}$ from the set $\{p^{(i)}, i = 1, 2, ...\}$, by applying an operator $\hat{P}$;

2. then collapse the state Ψ_n into one of the eigenvectors of a particular Hermitian operator $\hat{\Sigma}_{n+1}$, where, if the result of "1." is $p^{(1)}$ the operator $\hat{\Sigma}_{n+1} = \hat{O}^{(1)}$ is used; but, if instead the result of "1." is $p^{(2)}$ then the operator $\hat{\Sigma}_{n+1} = \hat{O}^{(2)}$ is used instead; and so on, accounting for every possible outcome $p^{(i)}$ of $\hat{P}$;

the above discussion would consequently cause a problem. Specifically, if the examination by $\hat{P}$ of the property $p^{(i)}$ of the state Ψ_n is taken to be a measurement process, then it would lead to a wavefunction collapse, and so after this examination the universe will be in a new state Ψ_{n+1}, which is one of the eigenvectors of $\hat{P}$. Clearly, it is now too late to test the 'old' state Ψ_n by whichever operator $\hat{\Sigma}_{n+1} \in \{\hat{O}^{(1)}, \hat{O}^{(2)}, ...\}$ is implied by Rule "2." from the information gained as a result of this measurement.

In fact, the universe would next have to be developed by some operator $\hat{\Sigma}_{n+2}$, and would accordingly jump to a new state Ψ_{n+2}.

This conclusion highlights the fact that any method used to measure the state Ψ for a particular property must be seen as equivalent to the operators $\hat{\Sigma}$ used in the universe's development. Indeed, this is not surprising: recall that the Hermitian operators $\hat{\Sigma}$ have been regarded throughout this thesis as being synonymous with physical tests anyway. Thus, in this case the test $\hat{P}$ was effectively used as the test $\hat{\Sigma}_{n+1}$, so in this universe $\hat{\Sigma}_{n+1} = \hat{P}$.

So, any examination procedure that involves a measurement does not fulfil the intention of finding an ED computation that proceeds in a single time step. Consequently, in any suggested ED mechanism the examination part of the algorithm cannot rely on any sort of physical measurement or information extraction process: two tests per time step are not allowed in the proposed paradigm.

It is therefore necessary to choose the ED Rules very carefully, such that whatever 'examination' procedure is employed avoids an actual physical measurement of the state.

8.5.2 Selective Global Evolution

In an attempt to find a way around the above difficulty, recall that operators used in standard quantum mechanics are generally one of two types, namely, either Hermitian or unitary. Hermitian operators represent physical measurements, and are used to test the state, thereby resulting in its collapse into one of the operator's eigenvectors. Unitary operators, conversely, are used to evolve the state, or, in the sense discussed in this thesis, 'rotate' it into a new vector in its Hilbert space. Unlike Hermitian operators, unitary operators do not extract any physical information from the state, and are hence traditionally

used in quantum theory to describe the (Schrödinger) development of the system in the absence of observation.

Now, since it has been shown that the examination part of an ED mechanism cannot be based upon a physical extraction of information from the state, if Rules are to be proposed that do provide suitable Examine-Decision dynamics for the development of a quantum universe, they must rely on non-invasive techniques.

Particularly, if the examination part cannot rely on Hermitian tests, the suggestion might be that it should instead be based upon unitary operators.

So, as an alternative to a dynamics based upon a universe that is *tested* by an operator chosen according to some property of its state (as hypothesised in Section 8.5.1), it might instead be possible to conject a dynamics in which the universe is *evolved* in a way that depends on some property of its state.

The suggestion, then, is that the examination and decision parts of an ED mechanism could be governed by a unitary operator. Consequently, the existence of a unitary operator $\hat{U}_n$ is hypothesised that appears to 'examine' the state Ψ_n for a particular property, 'decides' how it is to be developed, and then accordingly develops it into $\Psi'_n = \hat{U}_n \Psi_n$ in a way that depends upon this 'property'.

So, Examine-Decision Rules are considered that involve both unitary operators $\hat{U}_n$ and Hermitian tests $\hat{\Sigma}_{n+1}$. Specifically, when a certain, carefully defined unitary operator $\hat{U}_n$ is applied to the state Ψ_n, if Ψ_n has one particular property the universe will be evolved in one way, whilst if instead Ψ_n has a second property it is rotated in a different way, and so on. The evolved state Ψ'_n may then be tested in the usual manner by some Hermitian operator $\hat{\Sigma}_{n+1}$ that is chosen, perhaps, by one of the various Types of Rule discussed in Section 8.1, and the universe subsequently collapses to Ψ_{n+1}.

Clearly, because no physical information gets extracted from the state Ψ_n during the application of $\hat{U}_n$ (i.e. regarding what its properties actually are; the choice of $\hat{\Sigma}_{n+1}$ does not depend on this initial examination), no quantum collapse occurs, and the overall development of the state using the operators $\hat{U}_n$ and $\hat{\Sigma}_{n+1}$ proceeds in one time step, as required. Of course, this condition is ensured because the 'examination' process is part of $\hat{U}_n$, and is hence not a Hermitian test.

Such a mechanism may be described as *Selective Global Evolution* (SGE), because properties of the state are being used to select the way in which Ψ_n is globally evolved.

The development of a universe according to a SGE mechanism would therefore proceed in two distinct parts: an evolution part and a test part. Thus, the general Rules governing such a universe may be of the form

- Apply $\hat{U}_n$ to Ψ_n to give the 'rotated' state $\Psi'_n = \hat{U}_n \Psi_n$;

- Test Ψ'_n with $\hat{\Sigma}_{n+1}$ to give the new state Ψ_{n+1}, which is one of the eigenvectors of $\hat{\Sigma}_{n+1}$;

where $\hat{U}_n$ is defined such that it can Selectively Globally Evolve the state Ψ_n. The process may then be repeated, such that Ψ_{n+1} is selectively rotated by $\hat{U}_{n+1}$.

The actual development of a SGE governed universe will become clearer in the following.

The overall goal of SGE dynamics is to use an operator $\hat{U}_n$ that evolves the universe in a manner that depends upon some property of its state. So, the obvious first question is: what sort of unitary operators could give rise to such a selective development? How could $\hat{U}_n$ be constructed such that it evolves different vectors in different ways, depending on what this vector is?

Perhaps the most obvious answer to this question would be to suggest that $\hat{U}_n$ is an operator that somehow 'contains' (in a sense to be defined) many other unitary operators $\{\hat{u}_n^{(i)} : i = 1, 2, ...\}$, each of which can be somehow 'turned on or off' depending on the particular properties of the state Ψ_n on which $\hat{U}_n$ acts. In other words, a mechanism might schematically be suggested in which under some circumstances the operator $\hat{U}_n$ 'looks' like the unitary operator $\hat{u}_n^{(1)}$, whereas under different circumstances it instead effectively behaves like the unitary operator $\hat{u}_n^{(2)}$, etc., where $\hat{u}_n^{(1)}, \hat{u}_n^{(2)}, ...$ are different unitary operators. Clearly, the "actual circumstances" would be dictated by the Rules governing the universe's development, and it would be a 'property' of the state Ψ_n that actually causes $\hat{U}_n$ to resemble one particular operator, $\hat{u}_n^{(i)}$, over another, $\hat{u}_n^{(j)}$.

Continuing this schematic viewpoint, it is possible to imagine a mechanism in which $\hat{U}_n$ is defined such that if Ψ_n has one particular form, then the application of $\hat{U}_n$ to Ψ_n gives the same result as an application of $\hat{u}_n^{(1)}$ to Ψ_n, whereas if Ψ_n has a different property, then $\hat{U}_n\Psi_n$ is instead effectively equivalent to $\hat{u}_n^{(2)}\Psi_n$, and so on. For example, $\hat{U}_n$ could be such that if Ψ_n is the state $\Psi_n = \Theta$, then $\hat{U}_n\Psi_n = \hat{U}_n\Theta = \hat{u}_n^{(i)}\Theta$, whereas if Ψ_n is instead the state $\Psi_n = \Phi$, then $\hat{U}_n\Psi_n = \hat{U}_n\Phi = \hat{u}_n^{(j)}\Phi$, where $\hat{u}_n^{(i)} \neq \hat{u}_n^{(j)}$.

Overall, then, different states are taken to cause the same operator $\hat{U}_n$ to behave differently.

Of course, even in such a schematic model the issue of how the various operators $\{\hat{u}_n^{(i)}\}$ are actually "turned on or off" by the properties of the state remains to be addressed.

So, one potential suggestion might be to propose that $\hat{U}_n$ could in fact 'contain' (again, in a sense to be defined) many pairs of operators $\{\hat{u}_n^{(i)}\hat{S}_n^{(i)}\}$, where the $\{\hat{S}_n^{(i)}\}$ are defined as 'asking' operators. Thus, the idea, again schematic, is that when $\hat{U}_n$ acts on Ψ_n, the operator $\hat{S}_n^{(i)}$ in each pair 'asks' a question of the state, and the remaining operator $\hat{u}_n^{(i)}$

either rotates Ψ_n if the answer to this question is "Yes", but is not applied to Ψ_n if the answer is instead "No". It is this potential mechanism that is explored now.

Each 'asking' operator $\hat{S}_n^{(i)}$ must obey the condition that it provides a definite answer, either "Yes" or "No", for a given state Ψ_n, such that there is no ambiguity in whether the corresponding unitary operator $\hat{u}_n^{(i)}$ is applied or not. Furthermore, note that if the set $\{\hat{S}_n^{(i)}\}$ is constrained such that the individual 'questions' are mutually exclusive of the others, that is

$$\hat{S}_n^{(i)}\Psi_n \rightarrow \left\{ \begin{array}{c} \text{"Yes" for } i = a \\ \text{"No" for all } i \neq a \end{array} \right\} \tag{8.112}$$

it follows in this case that $\{\hat{u}_n^{(i)}\hat{S}_n^{(i)}\}$ acting on Ψ_n is equivalent to just $\hat{u}_n^{(a)}$ acting on Ψ_n, as desired; no other operator $\hat{u}_n^{(j)}$, $j \neq a$, is applied.

Continuing, it is also observed that the Yes/No answers to the 'questions' $\{\hat{S}_n^{(i)}\}$ could be associated with binary logic of the form "Yes"$\Rightarrow 1$ and "No"$\Rightarrow 0$.

So, one choice for the operators $\{\hat{S}_n^{(i)}\}$ could be to define them according to the rule

$$\hat{S}_n^{(i)}\Psi_n = \left\{ \begin{array}{rcl} 1 \times \Psi_n \text{ if } \hat{S}_n^{(i)}\Psi_n & \rightarrow & \text{"Yes"} \\ 0 \times \Psi_n \text{ if } \hat{S}_n^{(i)}\Psi_n & \rightarrow & \text{"No"} \end{array} \right\} \tag{8.113}$$

In this case, $\hat{U}_n$ could be written as a linear sum of the pairs $\{\hat{u}_n^{(i)}\hat{S}_n^{(i)}\}$, that is, in the form

$$\hat{U}_n = \hat{u}_n^{(1)}\hat{S}_n^{(1)} + \hat{u}_n^{(2)}\hat{S}_n^{(2)} + \hat{u}_n^{(3)}\hat{S}_n^{(3)} + ... \tag{8.114}$$

and this imposes an additional, obvious constraint on the operators $\{\hat{u}_n^{(i)}\hat{S}_n^{(i)}\}$: the pairs $\hat{u}_n^{(i)}\hat{S}_n^{(i)}$ must be chosen such that the unitarity of the overall operator $\hat{U}_n$ is preserved.

Under the above circumstances, for each pair $\hat{u}_n^{(i)}\hat{S}_n^{(i)}$ acting on Ψ_n, if the operator $\hat{S}_n^{(i)}$ applied to Ψ_n gives the result 1, the product $\hat{u}_n^{(i)}\hat{S}_n^{(i)}\Psi_n$ equals $\hat{u}_n^{(i)}\Psi_n$, and so $\hat{u}_n^{(i)}$ is used to evolve the state Ψ_n. However, if instead $\hat{S}_n^{(i)}\Psi_n$ gives the result 0, the combination $\hat{u}_n^{(i)}\hat{S}_n^{(i)}\Psi_n$ also becomes 0, and the unitary operator $\hat{u}_n^{(i)}$ is effectively 'removed' from the equation. In essence, the state Ψ_n only 'sees' the unitary operator $\hat{u}_n^{(i)}$ if $\hat{S}_n^{(i)}\Psi_n = 1 \times \Psi_n$; the operator $\hat{S}_n^{(i)}$ is effectively being used to turn $\hat{u}_n^{(i)}$ 'on or off' depending entirely on the properties of the state Ψ_n. As throughout this chapter, the parallels between the above type of logic and that exhibited in (quantum) computational gates are evident.

So, if $i = a$ is the only value for which $\hat{S}_n^{(i)}\Psi_n \rightarrow "Yes"$, it consequently follows that

$$\hat{U}_n\Psi_n = \hat{u}_n^{(a)}\Psi_n \tag{8.115}$$

as desired.

Note that $\{\hat{S}_n^{(i)}\}$ could be associated with a suitable set of projection operators, as suggested below, because these can conventionally be interpreted as Yes/No operators in quantum mechanics.

As an illustration of how a possible such SGE mechanism might be constructed, consider a universe represented by a state Ψ_n in a D dimensional Hilbert space $\mathcal{H}^{(D)}$ spanned by the orthonormal basis $\mathcal{B}^{(D)} = \{|i\rangle : i = 0, ..., (D-1)\}$. Further, assume that the dynamics of the universe are governed by the SGE two part Rules: Ψ_n is evolved to Ψ'_n by applying the unitary operator $\hat{U}_n$, i.e. $\Psi_n \rightarrow \Psi'_n = \hat{U}_n \Psi_n$, where $\hat{U}_n$ acts selectively; and then Ψ'_n is collapsed to Ψ_{n+1} by a test with the Hermitian operator $\hat{\Sigma}_{n+1}$, where Ψ_{n+1} is an eigenstate of $\hat{\Sigma}_{n+1}$. Finally, consider for all n defining $\hat{\Sigma}_{n+1}$ as an operator with a basis set of eigenstates $\mathfrak{B}_{n+1} = \{|i\rangle : i = 0, ..., (D-1)\}$, such that the collapsed state is always a member of the set $\{|i\rangle\}$.

Now, assume that $\hat{U}_n$ is of the form given in (8.114), but that each $\hat{S}_n^{(i)}$ is defined as the projection operator $\hat{S}_n^{(i)} = |i\rangle\langle i|$, for $i = 0, ..., (D-1)$. That is

- $\hat{U}_n = \hat{u}_n^{(0)}|0\rangle\langle 0| + \hat{u}_n^{(1)}|1\rangle\langle 1| + ... + \hat{u}_n^{(D-1)}|D-1\rangle\langle D-1|$

  where the $\{\hat{u}_n^{(i)}\}$ are particular unitary operators to be defined in due course.

Clearly, each pair of operators $\hat{u}_n^{(i)}|i\rangle\langle i|$ acts sequentially on Ψ_n; first the operator $|i\rangle\langle i|$ is applied to the state Ψ_n, then the resulting vector $(|i\rangle\langle i|\Psi_n)$ is rotated by the unitary operator $\hat{u}_n^{(i)}$. However, since $\Psi_n \in \mathfrak{B}_n = \{|i\rangle\}$ for all n, the expression

$$|i\rangle\langle i|\Psi_n = 1 \times \Psi_n \tag{8.116}$$

is true for only one value of i; for all other values, the application of $|i\rangle\langle i|$ to Ψ_n gives $|i\rangle\langle i|\Psi_n = 0 \times \Psi_n$. Labelling this one value $i = a$, it implies that $\Psi_n = |a\rangle$, and so it consequently follows that

$$\hat{U}_n \Psi_n = \left[\sum_{i=0}^{D-1} \hat{u}_n^{(i)} |i\rangle\langle i| \right] |a\rangle = 0 + 0 + ... + \hat{u}_n^{(a)}|a\rangle + 0 + ... + 0. \tag{8.117}$$

Thus, the state is evolved globally by an operator $\hat{u}_n^{(a)}$, selected from the set $\{\hat{u}_n^{(i)}\}$ 'contained' in $\hat{U}_n$, because the universe is initially in the state $\Psi_n = |a\rangle$.

Overall, the projection operator $|i\rangle\langle i|$ is effectively 'asking' whether or not Ψ_n is in the state $\Psi_n = |i\rangle$. The interpretation of the above type of Selection mechanism is that the projection operators cause $\hat{U}_n$ to act like a set of "*If*" statements: if the state Ψ_n is $|0\rangle$, then it is rotated by $\hat{u}_n^{(0)}$, but if the state Ψ_n is instead $|1\rangle$, then it is instead rotated by $\hat{u}_n^{(1)}$, and so on. In other words, given a universe prepared as $\Psi_n = |i\rangle$, the $|i\rangle\langle i|$ part of the operator $\hat{U}_n$ effectively selects the operator $\hat{u}_n^{(i)}$ to evolve the state.

The overall SGE mechanism is then concluded by the second part of the Rules. So, the evolved state $\Psi'_n = \hat{U}_n \Psi_n = \hat{u}_n^{(a)}|a\rangle$ is collapsed back into one of the vectors $\{|i\rangle\}$ by the operator $\hat{\Sigma}_{n+1}$, with the usual quantum probabilities.

The two part process may then be repeated, noting that because $\Psi_{n+1} \in \mathfrak{B}_n = \{|i\rangle\}$, the projection operators $|i\rangle\langle i|$ in the next SGE operator $\hat{U}_{n+1} = \left[\sum_{i=0}^{D-1} \hat{u}_{n+1}^{(i)} |i\rangle\langle i|\right]$ are still able to provide a mutually exclusive and exhaustive set of 'questions' for the new state Ψ_{n+1}. Clearly, this point will be true for all n.

Whilst the methodology behind the mechanism suggested above is sound, it turns out that it cannot actually accomplish non-trivial SGE development. In particular, in order for the proposed mechanism to provide valid dynamics, the set of operators $\{\hat{u}_n^{(i)}\}$ must be so restrictively defined that any non-trivial selection is effectively removed. Specifically, the necessary constraint of choosing the set $\{\hat{u}_n^{(i)}\}$ such that the overall unitarity of the operator $\hat{U}_n$ is preserved prevents the suggested SGE mechanism from developing the universe in a non-trivial way.

Phrasing this more mathematically, it can be shown that the overall operator $\hat{U}_n$ is only unitary if

$$\langle i|\hat{u}_n^{(i)*}\hat{u}_n^{(j)}|j\rangle = \delta_{ij} \tag{8.118}$$

for all $i, j = 0, 1, ..., (D-1)$, and this is only achieved if $\hat{u}_n^{(i)} = \hat{u}_n^{(j)}$.

This result is derived now. Note how the present ideas may be related to the discussions given in Section 8.4.1 regarding Class 1 and 2, Type III Basis Method dynamics.

Proof. Consider an operator $\hat{U}_n$ defined, as above, as

$$\hat{U}_n = \hat{u}_n^{(0)}|0\rangle\langle 0| + \hat{u}_n^{(1)}|1\rangle\langle 1| + ... + \hat{u}_n^{(D-1)}|D-1\rangle\langle D-1| \tag{8.119}$$

where the $\hat{u}_n^{(i)}$ are unitary, for $i = 0, 1, ..., (D-1)$.

In order for $\hat{U}_n$ to be unitary it must be the case that $\hat{U}_n^*\hat{U}_n = \hat{I}$, where $\hat{I}$ is the identity operator, and $\hat{U}_n^*$ is the transpose conjugate of $\hat{U}_n$ given by

$$\hat{U}_n^* = |0\rangle\langle 0|\hat{u}_n^{(0)*} + |1\rangle\langle 1|\hat{u}_n^{(1)*} + ... + |D-1\rangle\langle D-1|\hat{u}_n^{(D-1)*} \tag{8.120}$$

with $\hat{u}_n^{(i)*}\hat{u}_n^{(i)} = \hat{I}$. So,

$$\begin{aligned} \hat{I} &= \left(|0\rangle\langle 0|\hat{u}_n^{(0)*} + |1\rangle\langle 1|\hat{u}_n^{(1)*} + ... + |D-1\rangle\langle D-1|\hat{u}_n^{(D-1)*}\right) \\ &\quad \times \left(\hat{u}_n^{(0)}|0\rangle\langle 0| + \hat{u}_n^{(1)}|1\rangle\langle 1| + ... + \hat{u}_n^{(D-1)}|D-1\rangle\langle D-1|\right) \\ &= \sum_{i=0}^{D-1}\sum_{j=0}^{D-1} |i\rangle\langle i|\hat{u}_n^{(i)*}\hat{u}_n^{(j)}|j\rangle\langle j|. \end{aligned} \tag{8.121}$$

The sum of terms $\sum_{i=0}^{D-1}\sum_{j=0}^{D-1} |i\rangle\langle i|\hat{u}_n^{(i)*}\hat{u}_n^{(j)}|j\rangle\langle j|$ may be used to generate a $D \times D$ matrix. Moreover, the specific term $|i\rangle\langle i|\hat{u}_n^{(i)*}\hat{u}_n^{(j)}|j\rangle\langle j|$ gives the value $\langle i|\hat{u}_n^{(i)*}\hat{u}_n^{(j)}|j\rangle \in \mathbb{C}$ of the component in the $[(i+1)^{th}$ row, $(j+1)^{th}$ column] of this matrix.

Now, recall that the identity operator $\hat{I}$ may also be represented by a $D \times D$ matrix, which contains components of zero everywhere apart from the leading diagonal, where the values are one. So, for the above equality (8.121) to hold, it must be the case that

$$\langle i|\hat{u}_n^{(i)*}\hat{u}_n^{(j)}|j\rangle = \delta_{ij} \tag{8.122}$$

for all i and j.

Clearly, the equation $\langle i|\hat{u}_n^{(i)*}\hat{u}_n^{(j)}|j\rangle = \delta_{ij}$ is equivalent to the matrix elements of the identity $\hat{I}$ operator: $\langle i|\hat{I}|j\rangle = \delta_{ij}$. It is therefore the case that $\hat{u}_n^{(i)*}\hat{u}_n^{(j)} = \hat{I}$. Moreover, since $\hat{u}_n^{(i)}$ and $\hat{u}_n^{(j)}$ are unitary by definition, the relations $\hat{u}_n^{(i)*}\hat{u}_n^{(i)} = \hat{I}$ and $\hat{u}_n^{(j)*}\hat{u}_n^{(j)} = \hat{I}$ must also be true. Thus, because each unitary operator has one, and only one, inverse, it must follow that $\hat{u}_n^{(j)*} = \hat{u}_n^{(i)*}$, such that $\hat{u}_n^{(i)} = \hat{u}_n^{(j)}$ for all i and j.

Summarising, $\hat{U}_n$ is only unitary if $\hat{u}_n^{(i)} = \hat{u}_n^{(j)}$ for all $i, j = 0, 1, ..., (D-1)$. ■

Thus, $\hat{U}_n$ is only unitary if $\hat{u}_n^{(i)} = \hat{u}_n^{(j)}$. In this case,

$$\langle i|\hat{u}_n^{(i)*}\hat{u}_n^{(j)}|j\rangle = \langle i|\hat{u}_n^{(i)*}\hat{u}_n^{(i)}|j\rangle = \langle i|\hat{I}|j\rangle = \langle i|j\rangle = \delta_{ij} \tag{8.123}$$

and it then follows that $\hat{U}_n = \hat{u}_n^{(i)}$, because

$$\begin{aligned} \hat{U}_n &= \hat{u}_n^{(0)}|0\rangle\langle 0| + \hat{u}_n^{(1)}|1\rangle\langle 1| + ... + \hat{u}_n^{(D-1)}|D-1\rangle\langle D-1| \\ &= \hat{u}_n^{(i)}(|0\rangle\langle 0| + |1\rangle\langle 1| + ... + |D-1\rangle\langle D-1|) = \hat{u}_n^{(i)}\hat{I} = \hat{u}_n^{(i)}. \end{aligned} \tag{8.124}$$

Evidently, such a choice of $\{\hat{u}_n^{(i)}\}$ or $\hat{U}_n$ does not lead to any Selective Global Evolution of the sort aimed at in this section, because the universe would be evolved by the same operator $\hat{u}_n^{(i)}$ regardless of which state it is in.

So, the conclusion is that it is not possible to use the mechanism proposed above to define a SGE dynamics for a universe based upon the selection of one of a number of *different* unitary operators $\{\hat{u}_n^{(i)} : i = 0, ..., (D-1)\}$; the procedure can only work if $\hat{u}_n^{(i)} = \hat{u}_n^{(j)}$ for all i, j, because only under this circumstance is the overall operator $\hat{U}_n$ unitary. Clearly, then, such a $\hat{U}_n$ is not selectively evolving the universe in a way that non-trivially depends on what Ψ_n is: whichever of the set $\{|i\rangle : i = 0, ..., (D-1)\}$ the state is in, it will always be rotated in the same way.

It appears that Selective Global Evolution mechanisms of the sort described above cannot be used to self-referentially develop the state of the universe.

To demonstrate this point explicitly, consider a single qubit universe represented by a state Ψ_n in a two dimensional Hilbert space spanned by the basis $\mathcal{B}^{(2)} = \{|0\rangle, |1\rangle\}$, where $\mathcal{B}^{(2)}$ may equally be given in the usual representation as $\left\{\binom{1}{0}, \binom{0}{1}\right\}$.

If this universe was to be governed by the above type of Selective Global Evolution mechanism, two unitary operators, $\hat{u}_n^{(0)}$ and $\hat{u}_n^{(1)}$, would be required for its development.

Moreover, these operators would in turn define a third operator $\hat{U}_n$ according to the Rule $\hat{U}_n = \hat{u}_n^{(0)}|0\rangle\langle 0| + \hat{u}_n^{(1)}|1\rangle\langle 1|$, where $\hat{U}_n$ is also required to be unitary, which would be used to evolve the state Ψ_n.

The three unitary operators $\hat{u}_n^{(0)}$, $\hat{u}_n^{(1)}$ and $\hat{U}_n$ may be given by the general matrices

$$\hat{U}_n = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad , \quad \hat{u}_n^{(0)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad , \quad \hat{u}_n^{(1)} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \tag{8.125}$$

with the values of $A, ..., D, a, ..., h \in \mathbb{C}$ to be investigated.

Now, in order for $\hat{U}_n$, $\hat{u}_n^{(0)}$ and $\hat{u}_n^{(1)}$ to fulfil the unitarity conditions $\hat{U}_n^* \hat{U}_n = \hat{u}_n^{(0)*}\hat{u}_n^{(0)} = \hat{u}_n^{(1)*}\hat{u}_n^{(1)} = \hat{I}$, it must be the case that $B = -\bar{C}$, $D = \bar{A}$, $b = -\bar{c}$, $d = \bar{a}$, $f = -\bar{g}$ and $h = \bar{e}$ (ignoring row exchange permutations), where the bar denotes complex conjugation, with $AD - BC = ad - bc = eh - gh = 1$.

So by substituting these, the relation $\hat{U}_n = \hat{u}_n^{(0)}|0\rangle\langle 0| + \hat{u}_n^{(1)}|1\rangle\langle 1|$ becomes

$$\begin{aligned} \begin{pmatrix} A & -\bar{C} \\ C & \bar{A} \end{pmatrix} &= \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \bar{h} & f \\ -\bar{f} & h \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & f \\ 0 & h \end{pmatrix} = \begin{pmatrix} a & f \\ c & h \end{pmatrix} \end{aligned} \tag{8.126}$$

such that clearly $a = A$, $c = C$, $f = -\bar{C}$ and $h = \bar{A}$. Thus, by inspection of their components' complex conjugates, it immediately follows that $\hat{U}_n = \hat{u}_n^{(0)} = \hat{u}_n^{(1)}$.

Evidently, there can be no choice of operator, $\hat{u}_n^{(0)}$ or $\hat{u}_n^{(1)}$, in this single qubit universe; it does not matter what the state Ψ_n actually is, it will always be rotated in the same way.

8.5.3 Selective Local Evolution and Endophysics

In universes developing according to the Rules described in Section 8.5.2, every part of the state would be evolved at the same time and in the same way.

This, however, would lead to a physical limitation. An outcome of Chapter 6 was that the result of a global application of a unitary operator to a state is effectively unobservable, at least from the point of view of an endo-observer who is only able to witness the universe developing if it changes relative to herself. But, real endo-physical observers do appear to be able to witness relative changes occurring in the real physical Universe.

As an illustration of this issue, consider again the notion of an idealised physics experiment. Traditionally, in such an experiment a physicist prepares an apparatus and some sort of sample to be investigated. She then decides what she wants to measure, tests the sample, and records the result. She may then go on to do any number of further investigations on the sample based upon what she has learnt.

In the context of a fully quantum reality, however, it should be recalled that the physicist, the sample, and the apparatus are just sub-systems within the state Ψ_n of the universe. Each may be represented by a factor of the universe's state, such that Ψ may be written in the form

$$|\Psi\rangle = |Physicist\rangle \otimes |Sample\rangle \otimes |Apparatus\rangle \otimes |Rest\ of\ Universe\rangle. \tag{8.127}$$

Of course, these sub-states of Ψ are likely to change as the experiment proceeds; they must also necessarily entangle with one another as new information is exchanged. Thus, the separability of the state changes as the universe develops, and this gives rise to apparent measurements of the sample's sub-state, movement of the apparatus' pointer, changes in the scientist's brain as she learns the result, etc.

So, as the universe develops from state Ψ_n to Ψ_{n+1} to Ψ_{n+2} etc., the factors representing the sub-systems also develop. However, if this sequence is to be consistent with the reality experienced by scientists, the development must be such that, overall and from the endo-physical perspective of the physicist, the factor representing the physicist appears to use the factor representing the apparatus to independently prepare and test the factor representing the sample.

Thus in the context of a quantum universe, the above experiment may be viewed as one part of the universe's wavefunction apparently developing another part of the universe's wavefunction. Further, in such a universe the decision about which operator is used to develop one part of the universe's state may be made by considering changes in a different part of the universe's state. In other words, what the scientist decides to do next to the sample may be based upon the result contained in the position of the apparatus' pointer.

This then leaves a question. If physicists are part of the Universe, yet appear to be able to develop their surroundings with an apparent freedom that depends (literally) on their current 'state of mind', how does this procedure actually work? How are physicists able to get the impression that the evolution of the Universe around them actually depends on their sub-state and what they are doing? In other words, what mechanisms could be used to self-referentially develop the universe in a manner that appears to depend on one or some of its sub-states, but not on others? Thus, how might it be possible for the universe to develop in a way such that some parts of it are evolved, whilst other parts appear to remain unaffected, so that changes and relative differences may be observed and catalogued?

So, to begin to answer to these questions, and as an extension to the previous Selective Global Evolution Rules, it may be possible to consider Examine-Decision mechanisms in which one factor of the state is examined, and where the outcome of this is then used to

determine how a different factor is evolved. Such a mechanism may be called a *Selective Local Evolution* (SLE).

Care must be taken when interpreting this type of evolution. Emphatically, the suggestion is not to introduce a dynamics in which only a portion of the state Ψ_n is evolved, measured or collapsed at any one time. After all, recall that quantum mechanics is a holistic theory, and so the entire state of a universe based upon quantum principles must be evolved and tested at the same time, and not just sub-states or factors of it. Indeed, even in the simple toy-universes described in this chapter as tensor products of qubits, it has not been possible to evolve or test just one of these qubits on its own, but has instead relied on the projection of the entire wavefunction Ψ_n into one of members of the basis sets of eigenvectors of $\hat{\Sigma}_{n+1}$ that span the overall Hilbert space of the universe's state.

So in the hope of proposing Rules for a quantum universe that appears to evolve according to examinations of parts of its state, the 'trick' is therefore to devise a mechanism in which although every part of the universe is evolved or tested at the same time, it appears as if, after each evolution or state reduction, only certain factors have changed whereas others have been unaffected. Moreover, the way in which a changed factor is rotated appears to depend on the sub-state of an unaffected factor.

Putting this in context, the aim is to investigate mechanisms in which the 'Physicist' factor remains unchanged during the evolution of the 'Sample' factor under investigation, and where the sub-state of the Physicist somehow determines how the Sample sub-state is actually rotated.

Universes incorporating SLE Rules necessarily require a separable state, and hence a factorisable Hilbert space. Without loss of generality, and for simplicity, consider a bi-partite factorisation of the Hilbert space $\mathcal{H}$ of the universe, such that $\mathcal{H} \equiv \mathcal{H}_{[AB]} = \mathcal{H}_A \otimes \mathcal{H}_B$. Consider also an arbitrary separable state $\Psi_n \in \mathcal{H}_{AB}$ of the form

$$\Psi_n = |a\rangle_A \otimes |b\rangle_B. \tag{8.128}$$

In order to generate the SLE dynamics desired, a unitary operator $\hat{U}_n$ is sought that, when acting upon the state Ψ_n, examines the 'properties' of one of the factors (i.e. the "Physicist"), and then evolves the other factor (i.e. the "Sample") in a way that depends upon the result of this examination. Moreover, the sub-state of the physicist must be left unchanged by the application of $\hat{U}_n$. The resulting vector $\Psi'_n = \hat{U}_n \Psi_n$ (consisting of the unchanged Physicist and the evolved Sample) may then be tested and collapsed by some Hermitian operator $\hat{\Sigma}_{n+1}$, to give the next state Ψ_{n+1}. For obvious reasons, the particular SLE mechanism described here may hence be labelled a Physicist-Sample (PS) mechanism.

Overall, then, this type of Physicist-Sample, Selective Local Evolution, Examine-Decision mechanism is based upon general Rules of the form

1. Apply a particular unitary operator $\hat{U}_n$ to Ψ_n to give the 'rotated' state $\Psi'_n = \hat{U}_n \Psi_n$;

2. Test Ψ'_n with $\hat{\Sigma}_{n+1}$ to give the new state Ψ_{n+1}, which is one of the eigenvectors of $\hat{\Sigma}_{n+1}$;

Of course, the actual mechanism employed to choose the Hermitian operator $\hat{\Sigma}_{n+1}$ is left unspecified in this discussion, and may, perhaps, involve any of the Types of Rule described in Section 8.1. In fact, to ensure that the state Ψ_{n+1} is also separable relative to $\mathcal{H}_A \otimes \mathcal{H}_B$, as would be necessary for the procedure to repeat, the choice of operator $\hat{\Sigma}_{n+1}$ could actually be constrained such that it is itself factorisable relative to this split, as discussed in Chapter 5.

In order to provide the type of Physicist-Sample dynamics hoped for, the unitary operator $\hat{U}_n$ must be defined such that

$$\Psi'_n = \hat{U}_n \Psi_n = \hat{U}_n(|a\rangle_A \otimes |b\rangle_B) = |a\rangle_A \otimes |c^{(a)}\rangle_B \tag{8.129}$$

where $|c^{(a)}\rangle \in \mathcal{H}_B$ is a vector in $\mathcal{H}_B$ whose form depends somehow on $|a\rangle$. Thus $|c^{(a)}\rangle = |b'\rangle = \hat{u}_B^{(a)}|b\rangle$, where $\hat{u}_B^{(a)}$ is a unitary operator acting locally in the sub-space $\mathcal{H}_B$ that is chosen according to some property of $|a\rangle$.

A schematic description of the SLE dynamics discussed here is analogous to the SGE case introduced in Sub-section 8.5.2. In particular, a mechanism is similarly imagined in which the operator $\hat{U}_n$ is taken to 'contain' within it a further set of unitary operators, each of which may be appropriately 'turned on or off'. This time, however, which one of these unitary operators is actually 'activated' depends on the properties of one particular factor of Ψ_n, and not on overall properties of the whole state. Thus, the factor $|a\rangle$ in $\mathcal{H}_A$ is used to select how the factor $|b\rangle$ in $\mathcal{H}_B$ is evolved, such that the selection of the particular operator $\hat{u}_B^{(a)}$ depends somehow on $|a\rangle$.

Also similarly to the SGE mechanism suggested previously, the proposal here is to construct $\hat{U}_n$ from a linear sum of pairs of operators. This time, however, one member of the pair is used to 'ask' about the factor of Ψ_n in $\mathcal{H}_A$, whilst the other operator is used to evolve the factor of Ψ_n in $\mathcal{H}_B$ in a way that depends on the result of this 'question'. Consequently, in the present circumstance the operator $\hat{U}_n$ is given by a linear sum of tensor products of such pairs of operators, and may be given in the form

$$\hat{U}_n = \hat{A}_A^{(1)} \otimes \hat{u}_B^{(1)} + \hat{A}_A^{(2)} \otimes \hat{u}_B^{(2)} + ... \tag{8.130}$$

where $\hat{A}_A^{(i)}$ is an operator that 'asks' about the factor in $\mathcal{H}_A$, and $\hat{u}_B^{(i)}$ evolves the factor in $\mathcal{H}_B$ according to the answer. As before, the 'questions' $\{\hat{A}_A^{(i)}\}$ are assumed to be mutually exclusive and exhaustive, and are taken to provide a definite 'Yes' or 'No' (i.e. 1 or 0) for each Ψ_n. So, if the answer to $\hat{A}_A^{(i)}$ 'asking' about the factor in $\mathcal{H}_A$ is 'Yes' (or 1), then the factor in $\mathcal{H}_B$ is rotated by $\hat{u}_B^{(i)}$; otherwise, if the answer is 'No' (or 0), $\hat{u}_B^{(i)}$ is not applied.

The actual details of the Selective Local Evolution mechanism are perhaps best illustrated by example, as given in the following sub-sections. As will be seen, the asking operators may again be associated with projection operators.

First, however, note that in any SLE governed universe, the apparent dynamics depends very much upon point of view. In fact, this comment itself reflects the opposing viewpoints of exo- and endo-physics.

From the exo-physical point of view of an observer standing outside the quantum universe and examining the system as a whole, the dynamics describes a single state Ψ_n evolving as $\Psi_n \rightarrow \Psi'_n = \hat{U}_n \Psi_n$ according to a global operator $\hat{U}_n$, before undergoing collapse to one of the non-degenerate eigenvectors of a particular operator $\hat{\Sigma}_{n+1}$. The new state Ψ_{n+1} is then evolved and collapsed, and the process continued.

This 'external' point of view is generally the most convenient way to discuss the development of a universe, and is the one that has been used almost exclusively throughout this thesis. Of course, such a perspective is also inherently unphysical, because by definition nothing can stand outside of the universe. However whilst this may be the case, it is still valid to discuss this hypothetical point of view if it is specified that such an external 'observer' does not interact with the universe in any way; it is merely a privileged vantage point illustrative when discussing the development of the state as a whole. Thus, such observers do not actually observe anything, in the true quantum sense of the word.

Now, a feature of the Physicist-Sample mechanism suggested above is that some substates that were present as factors of Ψ_n may still exist as factors of Ψ_{n+1}. Say, for example, that the operator $\hat{\Sigma}_{n+1}$ is such that the next state Ψ_{n+1} is of the form

$$\Psi_{n+1} = |a\rangle_A \otimes |d\rangle_B \tag{8.131}$$

where $|d\rangle_B \in \mathcal{H}_B$. Clearly, then, $|a\rangle_A$ is a factor of each of Ψ_n, Ψ'_n and Ψ_{n+1}.

Now, from the endo-physical point of view of this unchanged factor $|a\rangle_A$, it would look like nothing had been done to it during the transition from Ψ_n to Ψ_{n+1} whilst other, apparently isolated, parts of the universe have changed. From such a factor's perspective, it would appear as if the rest of the universe had evolved 'around' it, whilst it had been unaffected by either the evolution from Ψ_n to Ψ'_n or the state reduction from Ψ'_n to Ψ_{n+1}.

Alternatively, from the point of view of the rest of the universe, it appears that the unchanged factor has been 'frozen in time'. The frozen factor is a part of the universe that seems to have been created at some time in the past, but has since appeared to have been left alone in the subsequent development.

So, if the way in which the universe evolves depends somehow on properties of one of these unchanged factors, it could appear, again from the perspective of such a factor, that

it was these properties that caused the change in the rest of the universe. From this point of view, it is as if the way in which the universe is developed depends on one of the factors of its state.

Summarising, from the point of view of an endo-physical observer, it is possible to devise a mechanism in which the dynamical evolution of the universe appears to depend on parts of its state, as will be shown in the following. From such a perspective, it would seem that the unchanged factor (i.e. the Physicist) 'chooses' how the Sample evolves, according to which sub-state this 'endo-observer' is in.

From the external point of view of the entire universe, however, the state Ψ_n will be seen to evolve to Ψ'_n in a deterministic, global fashion, as expected from the unitary relationship $\Psi_n \rightarrow \Psi'_n = \hat{U}_n \Psi_n$.

SLE rules are therefore a variant of SGE dynamics in which local examinations and relative evolutions seem to become apparent from an endo-physical perspective.

The exact details behind such SLE mechanisms will be introduced, elaborated upon, and demonstrated in the following examples.

8.5.4 A Two Qubit 'Physicist-Sample' Universe

To illustrate the type of Selective Local Evolution, Physicist-Sample dynamics suggested in the previous subsection, consider a two qubit universe represented by a state Ψ_n in the factorisable Hilbert space $\mathcal{H}_{[12]} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Further, assume that Ψ_n is separable relative to $\mathcal{H}_{[12]}$, such that $\Psi_n \in \mathcal{H}_{12} \subset \mathcal{H}_{[12]}$, and label the factor in $\mathcal{H}_1$ as 'qubit 1', or q_1, and the factor in $\mathcal{H}_2$ as 'qubit 2', or q_2. Additionally, consider the usual orthonormal basis sets $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_{12}$ for the Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_{[12]}$ respectively, defined as

$$\begin{aligned} \mathcal{B}_1 &= \{|0\rangle_1, |1\rangle_1\} \quad , \quad \mathcal{B}_2 = \{|0\rangle_2, |1\rangle_2\} \\ \mathcal{B}_{12} &= \{|00\rangle_{12}, |01\rangle_{12}, |10\rangle_{12}, |11\rangle_{12}\}. \end{aligned} \tag{8.132}$$

As before, the matrix representation $\{|0\rangle_a, |1\rangle_a\} \equiv \left\{ \binom{1}{0}_a, \binom{0}{1}_a \right\}$ may be adopted for $a = 1, 2$.

The intention of this sub-section is to introduce a model in which the development of qubit 1 is controlled somehow by the state of qubit 2. In particular, and to illustrate the general principle, the aim will be to analyse a system in which if $q_2 = |0\rangle$ then q_1 is evolved using an unitary operator $\hat{u}_0$, whereas if $q_2 = |1\rangle$ then q_1 is evolved using a different unitary operator $\hat{u}_1$.

As should be evident from before, such a mechanism is analogous to introducing a 'physicist' into the universe. The physical interpretation is that qubit 2 acts like the decision making scientist: if the 'physicist' is in one particular state then a certain experiment

is performed on qubit 1, but if 'she' is in another state then something completely different is done to qubit 1.

An important comment, however, must first be made at this point. Note that in the following, a sub-script on an *operator* is used to distinguish it, whereas its super-script denotes which qubit(s) Hilbert space(s) it is acting upon. Thus, for example, the operator written $\hat{u}_0^1$ indicates the operator $\hat{u}_0$ acting in $\mathcal{H}_1$, whereas $\hat{u}_0^2$ implies the same operator $\hat{u}_0$ acting instead in $\mathcal{H}_2$. Similarly, the operator $\hat{U}_n \equiv \hat{U}_n^{[12]}$ acts across the entire Hilbert space $\mathcal{H}_{[12]}$. This notation is converse to both the usual convention adopted generally throughout this thesis to label operators, and to the usual reservation of sub-scripts for labelling Hilbert spaces, and results from a desire to keep the sub-script n on $\hat{U}_n$ as a 'temporal' parameter.

Sub-scripts on *states* are still used to denote Hilbert space affiliation, such that $|0\rangle_1 \in \mathcal{H}_1$ etc., apart from on Ψ_n where it indicates the state of the universe at time n in the usual way.

Note also that labels may be omitted for clarity when no confusion is likely to occur, such that for example $\hat{U}_n \equiv \hat{U}_n^{[12]}$, and $|0\rangle_1 \otimes |1\rangle_2 \equiv |0\rangle \otimes |1\rangle \equiv |01\rangle$. Further, note that the 'lower case' unitary operators $\hat{u}_0$ and $\hat{u}_1$ will be defined constantly for all time: $\hat{u}_0$ is the 'zeroth' operator, and not an operator $\hat{u}$ at 'time' $n = 0$. In general, of course, the purpose of a particular sub- or super-script in any individual case in the following should be fairly obvious from context.

As expected from (8.130), the evolution of a two qubit PS universe is taken to be governed by a unitary operator $\hat{U}_n$ of the form[26] $\hat{U}_n = \hat{u}_0^1 \otimes \hat{A}_0^2 + \hat{u}_1^1 \otimes \hat{A}_1^2 + ...$, where $\hat{A}_j^i$ is an operator that 'asks' the j^{th} 'question' of the i^{th} qubit. In fact for simplicity in the current two qubit universe, attention may be restricted to operators $\hat{U}_n$ of the form

$$\hat{U}_n = \hat{u}_0^1 \otimes \hat{A}_0^2 + \hat{u}_1^1 \otimes \hat{A}_1^2 \tag{8.133}$$

where $\hat{A}_0^2$ and $\hat{A}_1^2$ provide mutually exclusive and exhaustive 'questions': if $\hat{A}_0^2$ acting on q_2 is 'Yes', then $\hat{A}_1^2$ acting on q_2 must be 'No', and vice versa.

In order to define a valid Physicist-Sample mechanism, unitary operators $\hat{U}_n$, $\hat{u}_0$ and $\hat{u}_1$ are sought such that, from the exo-physical point of view, the entire state of the universe is evolved globally by a single unitary operator $\hat{U}_n$, i.e. $\Psi_n \rightarrow \Psi'_n = \hat{U}_n \Psi_n$, but from the endo-physical viewpoint of qubit 2 this operation $\hat{U}_n$ appears to be equivalent to a selection of either $\hat{u}_0$ or $\hat{u}_1$ to act locally upon q_1, by a decision made in reference to the state of q_2. Clearly, a definition for the operators $\hat{A}_j^2$, that 'ask' whether qubit 2 is in the state $|0\rangle$ or $|1\rangle$, will therefore also be required.

[26] Noting the now changed sub- and super-script convention.

This observation provokes a second important comment; namely, observe that there is immediately an inherent difference between the operators $\hat{U}_n$, $\hat{u}_0$ and $\hat{u}_1$. The operator $\hat{u}_0^1$ may be represented by a 2×2 matrix, because it is to act locally on qubit 1 in the two dimensional Hilbert space $\mathcal{H}_1$. Similarly, $\hat{u}_1^1$ may also be represented by a 2×2 matrix because it also acts just on q_1 in $\mathcal{H}_1$. However, $\hat{U}_n$ acts globally on the state of the entire universe, i.e. on the state Ψ_n of both qubits in the four dimensional Hilbert space $\mathcal{H}_{[12]}$, and so must be represented by a 4×4 matrix. Clearly, if $\hat{U}_n$ is given in the form (8.133), its dimension must equal the product of the dimensions of the operator acting in $\mathcal{H}_1$ (i.e. $\hat{u}_0^1$ or $\hat{u}_1^1$) and the operator $\hat{A}_j^2$ acting in $\mathcal{H}_2$ that 'asks' which state qubit 2 is in. Thus, the 'asking' operator must also be represented by a 2×2 matrix, as expected from the observation that it is to 'ask' about the state of a single qubit.

As before, suitably defined projection operators are obvious candidates for the $\hat{A}_j^i$.

The following example illustrates how a two qubit Physicist-Sample mechanism might be constructed. The development of the presented model will proceed through two different steps, each incorporating an evolution part and a state reduction. Thus, one 'cycle' of the dynamics takes place in two time steps, as will become evident. Moreover, under this circumstance it is necessary to define an initial stage as a type of 'reference', so that the Rules 'know' which of the first or second steps should be applied to the current state.

Defining the initial state as Ψ_N at initial 'time' $n = N$, the development of the proposed two qubit PS universe is governed by the Rules

1. Evolve the initial state Ψ_N with the particular unitary operator $\hat{U}_N = \hat{U}_X$, such that $\Psi_N \rightarrow \Psi'_N = \hat{U}_X \Psi_N$;

2. This evolved state Ψ'_N is then collapsed into one of the eigenstates of a particular Hermitian operator $\hat{\Sigma}_{N+1} = \hat{B}$, and this vector may now be associated with the next state Ψ_{N+1}, with the usual probability amplitudes $\langle \Psi_{N+1} | \Psi'_N \rangle$;

3. The new state Ψ_{N+1} is then evolved with a different unitary operator, $\hat{U}_{N+1} = \hat{U}_Y$, into the state Ψ'_{N+1}, such that $\Psi_{N+1} \rightarrow \Psi'_{N+1} = \hat{U}_Y \Psi_{N+1}$;

4. Finally, this new evolved state is collapsed back into one of the eigenstates of $\hat{\Sigma}_{N+2} = \hat{B}$ to give the next state Ψ_{N+2}, with probabilities $|\langle \Psi_{N+2} | \Psi'_{N+1} \rangle|^2$.

The 'first step' hence contains procedures "1." and "2.", whilst the second step is parts "3." and "4.". The two step mechanism then repeats, such that Ψ_{N+2} is next rotated by $\hat{U}_{N+2} = \hat{U}_X = \hat{U}_N$, and so on. Of course, the operators $\hat{U}_X$, $\hat{U}_Y$ and $\hat{B}$ must be carefully defined in order for the universe to follow SLE dynamics.

In general, then, the Rules of the model are such that

$$\begin{aligned} \hat{\Sigma}_{N+m} &= \hat{\Sigma}_{N+m+1} = \hat{B} \quad , \quad m = 0, 1, 2, ... \\ \hat{U}_{N+m} &= \left\{ \begin{array}{c} \hat{U}_N = \hat{U}_X \text{ if } m = 0, 2, 4, ... \\ \hat{U}_{N+1} = \hat{U}_Y \text{ if } m = 1, 3, 5, ... \end{array} \right\} \end{aligned} \tag{8.134}$$

noting the unavoidable clash of notation: $\hat{U}_N$ is taken to indicate the relevant unitary operator at time N, whereas $\hat{U}_X$ denotes a fixed unitary operator. Similarly, $\hat{U}_Y$ is also a fixed operator, and is not meant to imply $\hat{U}$ at time Y.

In order to provide a suitable mechanism for a universe developing according to Physicist-Sample, SLE Rules, the unitary operators $\hat{U}_X$ and $\hat{U}_Y$ are defined to be

$$\hat{U}_X = \hat{u}_0^1 \otimes \hat{P}_0^2 + \hat{u}_1^1 \otimes \hat{P}_1^2 \tag{8.135}$$

where $\hat{P}_s^r \equiv |s\rangle_{rr}\langle s|$ for $s = 0, 1$ is the s^{th} projection operator acting in $\mathcal{H}_r$, and

$$\hat{U}_Y = \hat{I}^1 \otimes \hat{u}_2^2 \tag{8.136}$$

where $\hat{I}^t$ is the identity operator in $\mathcal{H}_t$. The exact reasons for these choices will become apparent, noting immediately, however, that $\hat{U}_X$ is in the form expected from (8.133).

Similarly, the Hermitian operator $\hat{B}$ is chosen to be one that possesses a completely separable basis set $\mathfrak{B}_{(0,4)}$ of orthonormal eigenvectors,

$$\mathfrak{B}_{(0,4)} = \{|00\rangle_{12}, |01\rangle_{12}, |10\rangle_{12}, |11\rangle_{12}\} \tag{8.137}$$

such that in fact $\hat{B} \equiv \hat{B}_{(0,4)}$ and $\mathfrak{B}_{(0,4)} = \mathcal{B}_{12}$, again for reasons given below.

The significance of this model, and in particular the appearance of Selective Local Evolution dynamics, can be demonstrated from a comparison of the exo- and endo-physical interpretations of the universe's development.

From the point of view of an observer external to the system, the first step involves the global rotation of the entire initial state Ψ_N by $\hat{U}_X$, followed by its subsequent measurement with $\hat{B}$, whilst the second step involves the new state Ψ_{N+1} being globally rotated in a different way by $\hat{U}_Y$, before the whole universe is again measured with $\hat{B}$. So, from the exo-physical point of view, the development of the universe proceeds in a semi-deterministic globalised fashion according to the operators $\hat{U}_X, \hat{B}, \hat{U}_Y$ and $\hat{B}$ being applied in turn, and the only randomness occurs as a result of the stochastic nature of the wavefunction collapse process, when a particular state Ψ_{N+m} is obtained from the set of eigenvectors $\mathfrak{B}_{(0,4)}$ of $\hat{\Sigma}_{N+m} = \hat{B}$.

However, to see the apparent Selective Local Evolution present in the model, it is necessary to examine the endo-physical point of view of one of the qubits. From such a

perspective, it appears that the unitary operators $\hat{U}_X$ and $\hat{U}_Y$ are evolving the universe in a manner that depends on one of its sub-states; this conclusion is demonstrated now.

Firstly, and for simplicity, assume that the universe has been prepared such that its initial state Ψ_N is separable (i.e. the qubits are not entangled with one another), that the factor of Ψ_N representing q_1 in $\mathcal{H}_1$ is either in the state $|0\rangle_1$ or the state $|1\rangle_1$, and that the factor of Ψ_N representing q_2 in $\mathcal{H}_2$ is either in the state $|0\rangle_2$ or the state $|1\rangle_2$ (i.e. neither qubit is in an arbitrary superposition of its basis vectors). These assumptions will be justified later, noting that the former has already been taken to be essential if a classically *distinct* 'Physicist' and 'Sample' are to be discussed.

Consider now the application of the operator $\hat{U}_X$ to this state Ψ_N. A projection operator $\hat{P}_s^2 \equiv |s\rangle_{22}\langle s|$, where $s = 0, 1$, may be interpreted as an operator that 'asks' which state qubit 2 is in: if q_2 is in the state $|q_2\rangle_2 = |s\rangle_2$, then $\hat{P}_s^2|q_2\rangle_2$ gives the 'answer'

$$\hat{P}_s^2|q_2\rangle_2 = |s\rangle_{22}\langle s||s\rangle_2 = 1 \times |s\rangle_2 \quad , \quad s = 0, 1 \tag{8.138}$$

whereas if q_2 is in a state orthogonal to $|s\rangle_2$ then $\hat{P}_s^2|q_2\rangle_2$ gives the 'answer' $0 \times |s\rangle_2$.

Thus, the combined operator $\hat{u}_t^1 \otimes \hat{P}_s^2$ acting on the general qubit product state $|q_1\rangle_1 \otimes |q_2\rangle_2$, where $|q_2\rangle_2$ is $|0\rangle_2$ or $|1\rangle_2$ by design and $t = 0, 1$, gives the result

$$[\hat{u}_t^1 \otimes \hat{P}_s^2]|q_1\rangle_1 \otimes |q_2\rangle_2 = \left\{ \begin{array}{l} [\hat{u}_t^1|q_1\rangle_1] \otimes [1 \times |q_2\rangle_2] \text{ if } |q_2\rangle_2 = |s\rangle_2 \\ [\hat{u}_t^1|q_1\rangle_1] \otimes [0 \times |q_2\rangle_2] \text{ if } |q_2\rangle_2 \neq |s\rangle_2 \end{array} \right\} \quad , \quad s, t = 0, 1. \tag{8.139}$$

Overall, then, the evolution $\Psi_N \rightarrow \Psi'_N = \hat{U}_X \Psi_N$ is consequently equivalent to an operation that leaves q_2 in its initial state (either $|0\rangle$ or $|1\rangle$) whilst rotating the state of q_1. Further, if qubit 2 is initially in the state $q_2 = |0\rangle$ then qubit 1 is evolved by $\hat{u}_0^1$, whereas if qubit 2 is in the state $q_2 = |1\rangle$ then qubit 1 is evolved by $\hat{u}_1^1$.

So, from the endo-physical point of view of the 'Physicist' qubit, q_2, it appears that the rest of the universe, q_1, has developed in a way that depends on 'her' state, whilst 'she' has remained unchanged. Qubit 2 could conclude that it was she who determined how the universe evolved, by 'choosing' to apply an operator $\hat{u}_0^1$ or $\hat{u}_2^1$ to q_1; she would not automatically assume that it was actually the whole universe that was globally evolved by an operator $\hat{U}_X$.

The next part of the first step involves the collapse of the wavefunction Ψ'_N.

In many ways, it is possible to consider qubit 2 as being a classical 'object'; qubit 2 is still in a 'classical' looking state, $|0\rangle$ or $|1\rangle$, because the unitary operators $\hat{u}_0^1$ and $\hat{u}_1^1$ act solely on q_1, such that only this first qubit may be in a superposition. It might therefore be tempting to disregard q_2 entirely when discussing the collapse of the statefunction Ψ'_N. This follows from the general logic that if the initial state of qubit 2 is known, no

new information may be extracted from the system by the application of the projection operators $\hat{P}_0^2$ or $\hat{P}_1^2$.

From the point of view of the physical interpretation of the current model, this disregarding seems a natural conclusion, and represents the normal, 'everyday' exo-physical approach to quantum theory in which it appears possible to perform some sort of quantum experiment on one particular sub-system of the universe whilst leaving other sub-systems alone.

So, following on from this (erroneous) perspective, a suitable next operator $\hat{\Sigma}_{N+1} = \hat{B}_{suit?}$ used to test the universe's state Ψ'_N might therefore be expected to be of the form

$$\hat{B}_{suit?} = \hat{\sigma}_3^1 \otimes \hat{I}^2 \tag{8.140}$$

which could be interpreted as the product of a Pauli operator $\hat{\sigma}_3^1$ that collapses qubit 1 into either $|0\rangle$ or $|1\rangle$, with the identity operator $\hat{I}^2$ doing nothing to qubit 2.

Such an analysis, however, is incorrect. The operator $\hat{B}_{suit?}$ can be shown to have degenerate eigenvalues, and as such does not possess a unique basis set of orthogonal eigenvectors; consequently it is not a valid operator to use when discussing tests in the proposed fully quantum paradigm.

The general problem associated with operators such as $\hat{B}_{suit?}$ results from the fact that the universe cannot be tested simply as a product of classical objects: in quantum theory it is *not* possible to isolate a sub-state from everything else. Phrasing this more mathematically, it is not always possible to test the universe by independently testing its parts: an operator $\hat{\Lambda}^1$ acting on qubit 1 in $\mathcal{H}_1$ may have two, unique orthogonal eigenvectors, as might an operator $\hat{\Delta}^2$ acting on qubit 2 in $\mathcal{H}_2$, but this does not imply that the combined operator $\hat{\Lambda}^1 \otimes \hat{\Delta}^2$ must necessarily have four, unique orthogonal eigenvectors. Such an argument was presented in the discussion of 'Strong' and 'Weak' operators given in Chapter 5, and is also related to the conclusion of Chapter 4 that separable states are just a tiny subset of the set of all vectors in a Hilbert space.

In short, the state of the universe may be described as greater than the sum of its parts, and so care must therefore always be taken to choose an operator $\hat{\Sigma}_{N+1}$ that acts on the entire quantum state, yet also possesses a basis set of four orthogonal eigenvectors.

The operator $\hat{B}_{(0,4)}$, however, defined by

$$\hat{B}_{(0,4)} = w|00\rangle\langle 00| + x|01\rangle\langle 01| + y|10\rangle\langle 10| + z|11\rangle\langle 11| \tag{8.141}$$

in accordance with the basis set $\mathfrak{B}_{(0,4)}$ in (8.137), does satisfy this condition, where $w, x, y, z \in \mathbb{R}^+$ are non-degenerate eigenvalues of no further consequence to the discussion. Thus, $\hat{B}_{(0,4)}$ is a valid Hermitian operator, and so may be used to test the overall state Ψ'_N of the universe.

Moreover, an operator of the form $\hat{\Sigma}_{N+1} = \hat{B}_{(0,4)}$, which defines the basis set $\mathfrak{B}_{N+1} = \mathfrak{B}_{(0,4)}$ for the next state Ψ_{N+1}, also has the important consequence that there is no eigenstate of $\hat{\Sigma}_{N+1}$ for which q_1 and q_2 are entangled with one another, nor is there an eigenstate containing a linear superposition of the form $(a|0\rangle_1+b|1\rangle_1)\otimes(c|0\rangle_2+d|1\rangle_2)$, for $a, b, c, d \neq 0$ and $a, b, c, d \in \mathbb{C}$. Thus after testing Ψ'_N with $\hat{\Sigma}_{N+1} = \hat{B}_{(0,4)}$, both qubits have the 'classical' form $q_1 = |0\rangle$ or $q_1 = |1\rangle$ and $q_2 = |0\rangle$ or $q_2 = |1\rangle$.

Furthermore, since the Rules are such that $\hat{\Sigma}_{N+m} = \hat{B}_{(0,4)}$ for all m, it implies that $\Psi_{N+m} \in \mathfrak{B}_{(0,4)}$ for all m, and this justifies the assumption made earlier that the 'initial' wavefunction is always a separable state with neither qubit superposed.

So, after the test of Ψ'_N by $\hat{\Sigma}_{N+1} = \hat{B}_{(0,4)}$, the subsequent state Ψ_{N+1} will be one of the eigenstates of $\hat{B}_{(0,4)}$, with appropriate probability amplitudes given in the usual way.

The above discussion highlights the fact that from the holistic point of view of a quantum universe, it is not possible to naively segregate the state into factors under investigation and everything else. In fact, it is the entire state of the universe that must instead be measured.

The endo-physical observation that qubit 2 *appears* to be unaffected by the application of $\hat{U}_X$ or $\hat{\Sigma}_{N+1} = \hat{B}_{(0,4)}$, however, is because the jump from the state

$$\Psi'_N = (a|0\rangle_1 + b|1\rangle_1) \otimes (c|0\rangle_2 + d|1\rangle_2) \tag{8.142}$$

where $a, b \in \mathbb{C}$ and *either* $c = 0$ and $d = 1$ *or* $c = 1$ and $d = 0$ (noting that Ψ'_N is still separable, as $\hat{u}^1_0$, $\hat{u}^1_1$, $\hat{P}^2_0$ and $\hat{P}^2_1$ act locally), to a subsequent state

$$\Psi_{N+1} = (A|0\rangle_1 + B|1\rangle_1) \otimes (C|0\rangle_2 + D|1\rangle_2) \tag{8.143}$$

which is an eigenstate of $\hat{B}_{(0,4)}$ (one of which, for example, is $|00\rangle$, where $A = C = 1$, $B = D = 0$), has a non-zero probability, $|\langle\Psi_{N+1}|\Psi'_N\rangle|^2$, of occurring if and only if $C = c$ and $D = d$.

So, from the point of view of qubit 2, the test $\hat{B}_{(0,4)}$ therefore *appears* equivalent to the product of a projection of the evolved sub-state of qubit1 onto the basis set $\{|0\rangle_1, |1\rangle_1\}$, with a 'null' operator acting upon itself. In reality, both qubits are actually tested, but no new information is acquired about q_2.

It is these apparent null tests, i.e. the observation that every factor of a state is involved in a quantum test but that some outcomes of this measurement have a zero probability of being realised, that may be a root cause of apparent permanence in the Universe. Specifically, this mechanism gives rise to the "frozen factors" described previously, and explains why it is possible to have sub-states that appear unchanged as the Universe evolves from Ψ_n to Ψ_{n+1}.

Note that, of course, this 'null process' may in principle continue over several jumps of the real Universe: a sub-state that was present as a factor of both Ψ_n and Ψ_{n+1} may remain as a factor of Ψ_{n+2}, $\Psi_{n+3}, ...$ until some later state Ψ_{n+m}, which may have completely different factors. Again, this could contribute to the phenomena of persistence and longevity.

It should be reiterated how important it is that the null tests are included in the operators. In the present case, qubit 2 must still be involved in the measurement of Ψ'_N, because, apart from the degenerate eigenvalue problem, if this were not the case the question would remain as to why only parts of the state are evolved or tested when others are left alone, and this is contrary to the notion of a single set of rules of physics applying to the whole universe at the same time. Further, if a test did include eigenstates in which qubit 2 could be neglected, that is if, say, $\Psi_{N+1} \equiv \chi = (A|0\rangle_1 + B|1\rangle_1)$, the question would remain as to what the amplitude $\langle\chi|\Psi'_N\rangle$ may mean mathematically, given that the dimensions of the Hilbert spaces of χ and Ψ'_N differ.

Of course, it is possible to restrict attention to individual factors of the initial and final states, and correctly evaluate amplitudes such as $\langle\chi|\varphi\rangle$, where $\varphi = (a|0\rangle_1 + b|1\rangle_1)$ is a factor of $\Psi'_N = \varphi \otimes (c|0\rangle_2 + d|1\rangle_2)$, for $a, b \in \mathbb{C}$ and *either* $c = 0$ and $d = 1$ *or* $c = 1$ and $d = 0$. However whilst this is mathematically sound, the interpretation is really only valid from the exo-physical point of view of one sub-system (in this case q_2) describing changes in another isolated sub-system (in this case q_1). It is therefore a bit misleading when attempting to consider a universe that is a complete quantum system, in which the endo-physical 'observer' (here q_2) is itself part of the state it is trying to measure.

In a fully quantum universe, everything has to be evolved and tested at the same time, though some factors of the universe's state may be unchanged by the evolution, and may appear unchanged by the test. Despite appearances, it is not possible to just evolve or measure part of a fully quantum universe, though it is possible to discuss effects that appear local by ignoring null tests and identity evolutions and considering the physical interpretation of the model.

If the development of the two-qubit universe was governed by just repeating the first step of the Rules (i.e. just parts "1." and "2."), its dynamics would be rather restricted. Specifically, if the development from $...\Psi_N \rightarrow \Psi_{N+1} \rightarrow \Psi_{N+2} \rightarrow ...$ was due solely to an application of the operators $\hat{U}_X$, $\hat{\Sigma}_{N+1}$, $\hat{U}_X$, $\hat{\Sigma}_{N+2}$, $\hat{U}_X, ...$, its Evolution would actually contain very little Selection. This is because if qubit 2 is initially in the state $|0\rangle$, it implies that qubit 1 will always be evolved by $\hat{u}_0^1$ from then on, whereas if at initial 'time' N qubit 2 is instead in the state $|1\rangle$, then qubit 1 would instead be evolved by $\hat{u}_1^1$ for all $N + m$, where $m = 0, 1, 2, ...$

The above conclusion follows because there is currently no mechanism for changing the state of qubit 2, and therefore obtaining a more interesting dynamics based upon selection.

This problem, however, may be remedied by introducing the second step. The second step is defined such that it begins by evolving q_2 whilst appearing to leave q_1 unaffected. A unitary operator $\hat{U}_{N+1}$ is hence used that rotates the factor of Ψ_{N+1} in $\mathcal{H}_2$ whilst doing nothing to the factor in $\mathcal{H}_1$.

Specifically, this "doing nothing" operation may be achieved by the identity operator $\hat{I}^1$ acting locally on q_1, such that a suitable unitary operator $\hat{U}_{N+1} = \hat{U}_Y$ is given by

$$\hat{U}_{N+1} = \hat{U}_Y = \hat{I}^1 \otimes \hat{u}_2^2 \tag{8.144}$$

as suggested earlier. Clearly, both qubits are involved in the evolution, but only the state of q_2 is actually changed.

The final procedure of the second step is then to collapse the wavefunction in order to obtain the next state Ψ_{N+2}. The operator $\hat{B}_{(0,4)}$ may again be used, such that the state after reduction is 'reset' back to one of the members of the basis $\mathfrak{B}_{(0,4)}$.

Evidently, from the endo-physical point of view of the individual qubits, it appears that during the second step qubit 1 is not taking any part in the evolution or collapse process. From the exo-physical point of view of the entire universe, of course, both qubits are involved.

The two step process may then be repeated, starting with the application of $\hat{U}_X$ to Ψ_{N+2}.

The proposed mechanism may now be summarised from the endo-physical perspective. Given an 'initial' state $\Psi_N \in \mathfrak{B}_{(0,4)}$, the application of the operator $\hat{U}_X$ evolves this vector in a manner that appears to depend on whether qubit 2 is $|0\rangle$ or $|1\rangle$. The rotated state Ψ'_N is then tested by an operator $\hat{\Sigma}_{N+1} = \hat{B}_{(0,4)}$, which collapses it back into one of the members $\mathcal{B}_{(0,4)}$, with probabilities given in the usual way.

Then, during the second step, the state Ψ_{N+1} of the universe is evolved by $\hat{U}_Y$, which is equivalent to just rotating the sub-state of qubit 2 with $\hat{u}_2^2$. Finally, the state Ψ'_{N+1} is tested with $\hat{B}_{(0,4)}$, and the universe collapses back into one of the members of $\mathfrak{B}_{(0,4)}$, noting that whichever member this may be, Ψ_{N+2} is separable and q_2 is definitely in either $|0\rangle$ or $|1\rangle$, as required for the reapplication of $\hat{U}_X$ when the first step is repeated.

PS Example

It is now shown implicitly how a two qubit universe may develop when governed by the above Selective Local Evolution, PS Rules incorporating the operators $\hat{U}_X$, $\hat{U}_Y$ and $\hat{B}_{(0,4)}$.

Consider as an example a two qubit system initially in a state Ψ_0 given by

$$\Psi_0 = |00\rangle_{12} = |0\rangle_1 \otimes |0\rangle_2 = \begin{pmatrix}1\\0\end{pmatrix}_1 \otimes \begin{pmatrix}1\\0\end{pmatrix}_2 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}. \tag{8.145}$$

Consider also unitary operators $\hat{u}_0$, $\hat{u}_1$ and $\hat{u}_2$ defined as

$$\hat{u}_0 = e^{-i\varepsilon\hat{\sigma}_1} \quad , \quad \hat{u}_1 = e^{-i\mu\hat{\sigma}_2} \quad , \quad \hat{u}_2 = e^{-i\nu\hat{\sigma}_1} \tag{8.146}$$

where $\varepsilon, \mu, \nu \in \mathbb{R}^+$ are real parameters, and $\hat{\sigma}_i$ is the i^{th} Pauli operator. Thus the operator $\hat{U}_X$ becomes

$$\begin{aligned} \hat{U}_X &= \hat{u}_0^1 \otimes \hat{P}_0^2 + \hat{u}_1^1 \otimes \hat{P}_1^2 \\ &= \begin{pmatrix}\cos\varepsilon & -i\sin\varepsilon\\ -i\sin\varepsilon & \cos\varepsilon\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + \begin{pmatrix}\cos\mu & -\sin\mu\\ \sin\mu & \cos\mu\end{pmatrix} \otimes \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \end{aligned} \tag{8.147}$$

whilst $\hat{U}_Y$ is given by

$$\hat{U}_Y = \hat{I}^1 \otimes \hat{u}_2^2 = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \otimes \begin{pmatrix}\cos\nu & -i\sin\nu\\ -i\sin\nu & \cos\nu\end{pmatrix}. \tag{8.148}$$

The first step of the mechanism evolves the state of the universe to Ψ_0' by an application of $\hat{U}_0 = \hat{U}_X$ to Ψ_0. So,

$$\begin{aligned} \Psi_0' &= \hat{U}_X\Psi_0 = \hat{U}_X(|0\rangle_1 \otimes |0\rangle_2) \\ &= [\hat{u}_0^1|0\rangle_1] \otimes [|0\rangle_{22}\langle 0|0\rangle_2] + [\hat{u}_1^1|0\rangle_1] \otimes [|1\rangle_{22}\langle 1|0\rangle_2] \\ &= [\hat{u}_0^1|0\rangle_1] \otimes [|0\rangle_2 \times 1] + 0 = (\cos\varepsilon|0\rangle_1 - i\sin\varepsilon|1\rangle_1) \otimes |0\rangle_2. \end{aligned} \tag{8.149}$$

The universe is then tested with the Hermitian operator $\hat{\Sigma}_1 = \hat{B}_{(0,4)}$, such that its next state Ψ_1 will be a member of the basis set $\mathfrak{B}_{(0,4)} \equiv \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, with relative probabilities given by

Ψ_1	$\|00\rangle$	$\|10\rangle$	$\|01\rangle$	$\|11\rangle$
Prob.$= \|\langle\Psi_1\|\Psi_0'\rangle\|^2$	$\cos^2\varepsilon$	$\sin^2\varepsilon$	0	0

Table 8.4

As an aside, note that if $\varepsilon \sim 0$, it is highly probable that Ψ_1 is the same state as Ψ_0; this type of argument could play an important role in discussions regarding the origins of apparent persistence.

In accordance with the second step of the proposed Rules, the new state Ψ_1 is now evolved by the unitary operator $\hat{U}_1 = \hat{U}_Y$.

From Table 8.4 it is clear that Ψ_1 will be one of two possible states, which may be labelled $\Psi_1^a = |00\rangle$ and $\Psi_1^b = |10\rangle$. The next evolved state $\Psi_1' = \hat{U}_Y \Psi_1$ will consequently also be one of two possible states, viz. $\Psi_1^{a\prime} = \hat{U}_Y \Psi_1^a$ or $\Psi_1^{b\prime} = \hat{U}_Y \Psi_1^b$.

Specifically, it can be shown that $\Psi_1^{a\prime}$ is given by

$$\Psi_1^{a\prime} = \hat{U}_Y \Psi_1^a = |0\rangle_1 \otimes (\cos\nu|0\rangle_2 - i\sin\nu|1\rangle_2) \tag{8.150}$$

whereas $\Psi_1^{b\prime}$ is given by

$$\Psi_1^{b\prime} = \hat{U}_Y \Psi_1^b = |1\rangle_1 \otimes (\cos\nu|0\rangle_2 - i\sin\nu|1\rangle_2). \tag{8.151}$$

The second part of the second step involves the collapse of the wavefunction Ψ_1' back into one of the eigenstates of the operator $\hat{\Sigma}_2 = \hat{B}_{(0,4)}$.

For the case where the state Ψ_1' turned out to be $\Psi_1' = \Psi_1^{a\prime}$, the probabilities that the subsequent state Ψ_2 will be a particular member of $\mathfrak{B}_{(0,4)}$ are given by

Ψ_2	$\|00\rangle$	$\|10\rangle$	$\|01\rangle$	$\|11\rangle$
Prob.$= \|\langle\Psi_2\|\Psi_1^{a\prime}\rangle\|^2$	$\cos^2\nu$	0	$\sin^2\nu$	0

Table 8.5

So, if the state Ψ_1 at 'time' $n = 1$ is $\Psi_1 = \Psi_1^a$, the next state Ψ_2 is clearly going to be one of two possibilities, which may be labelled $\Psi_2^{ac} = |00\rangle$ or $\Psi_2^{ad} = |01\rangle$.

Conversely, if the 'first' state Ψ_1 is instead found to be $\Psi_1 = \Psi_1^b = |10\rangle$, and not Ψ_1^a, then the corresponding probabilities of obtaining a particular eigenstate of $\hat{B}_{(0,4)}$ for Ψ_2 would alternatively be given by

Ψ_2	$\|00\rangle$	$\|10\rangle$	$\|01\rangle$	$\|11\rangle$
Prob.$= \|\langle\Psi_2\|\Psi'_{1,b}\rangle\|^2$	0	$\cos^2\nu$	0	$\sin^2\nu$

Table 8.6

As before, Ψ_2 would again clearly be one of two possibilities in this case, which may now be labelled $\Psi_2^{be} = |10\rangle$ or $\Psi_2^{bf} = |11\rangle$.

The Rules next dictate that the first step is repeated again, such that $\hat{U}_X$ is used to evolve whichever of Ψ_2^{ac} or Ψ_2^{ad} or Ψ_2^{be} or Ψ_2^{bf} is actually realised. Now, if Ψ_2 turns out to be either $\Psi_2^{ac} = |00\rangle$ or $\Psi_2^{be} = |10\rangle$, then $\hat{U}_X$ will effectively be equivalent to $\hat{u}_0^1 \otimes \hat{P}_0^2$, just as it was for Ψ_0. However, if it is instead the case that Ψ_2 is either $\Psi_2^{ad} = |01\rangle$ or $\Psi_2^{bf} = |11\rangle$, then $\hat{U}_X$ instead effectively becomes equivalent to $\hat{u}_1^1 \otimes \hat{P}_1^2$. In this latter circumstance, $\cos^2\mu$ or $\sin^2\mu$ terms are now consequently introduced into the probability amplitudes, in the obvious way.

And so on; the two step algorithm may be continued indefinitely.

As with the examples of previous sections of this chapter, it is possible to write elementary computer programs that iterate the above procedure through a number of cycles. Also as previously it is possible to 'interrogate' the results in a number of ways, as desired.

For example, even after just one iteration it is possible to examine the probability of proceeding from an initial state Ψ_0 to a particular state Ψ_2. Specifically, defining

$$P_{ac} = P(\Psi_2 = \Psi_2^{ac} = |00\rangle | \Psi_0 = |00\rangle) \tag{8.152}$$

as the probability that the state Ψ_2 at 'time' $n = 2$ will be Ψ_2^{ac} given that the initial state is $\Psi_0 = |00\rangle$ (which is equivalent to the product of the probability of jumping from state Ψ_0 to Ψ_1^a and the probability of then jumping from state Ψ_1^a to Ψ_2^{ac}), the result (8.153) is readily obtained.

$$\begin{aligned} P_{ac} &= P(\Psi_2^{ac} \mid \Psi_1^a) \cdot P(\Psi_1^a \mid \Psi_0) \\ &= \cos^2 \nu \cos^2 \varepsilon. \end{aligned} \tag{8.153}$$

Of course, other 'histories' of $\Psi_0 \rightarrow \Psi_1 \rightarrow \Psi_2$ may alternatively be chosen. In obvious notation, it may similarly be found that $P_{ad} = \sin^2 \nu \cos^2 \varepsilon$, $P_{be} = \cos^2 \nu \sin^2 \varepsilon$ and $P_{bf} = \sin^2 \nu \sin^2 \varepsilon$. Clearly, $P_{ac} + P_{ad} + P_{be} + P_{bf} = 1$ as expected.

Continuing, the probability that the universe will develop from the initial state $|00\rangle$ through the sequence $|00\rangle \rightarrow |10\rangle \rightarrow |11\rangle \rightarrow |11\rangle$ (i.e. $|00\rangle \rightarrow \Psi_1^b \rightarrow \Psi_2^{bf} \rightarrow |11\rangle$) is given by $\cos^2 \mu \sin^2 \nu \sin^2 \varepsilon$, as may be readily verified. And so on.

The physical interpretation of the above model should be emphasised from the endophysical point of view of qubit 2. Initially q_2 is in the state $|0\rangle$, so the projection operator part of $\hat{U}_X$ 'picks out' $\hat{u}_0^1$, and qubit 1 is evolved accordingly. In other words the projection operator ensures that the global transformation $\Psi_0 \rightarrow \Psi_0' = \hat{U}_X \Psi_0$ is effectively equivalent to the local transformations $|q_1\rangle_1 \rightarrow |q_1'\rangle_1 = \hat{u}_0^1 |q_1\rangle_1$ and $|q_2\rangle_2 \rightarrow |q_2'\rangle_2 = |q_2\rangle_2$.

After the subsequent collapse of the state into either $|00\rangle$ or $|10\rangle$, qubit 2 is evolved into a superposition by the global operator $\hat{U}_Y$. This time, the global transformation $\Psi_1 \rightarrow \Psi_1' = \hat{U}_Y \Psi_1$ is clearly equivalent to the local transformations $|q_1\rangle_1 \rightarrow |q_1'\rangle_1 = |q_1\rangle_1$ and $|q_2\rangle_2 \rightarrow |q_2'\rangle_2 = \hat{u}_2^2 |q_2\rangle_2$. So, although the system was initially in the state $\Psi_0 = |00\rangle$, after a second state reduction the wavefunction Ψ_2 of the universe could be any member of the set $\mathfrak{B}_{(0,4)} \equiv \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, with appropriate probabilities.

The dynamics become particularly interesting during the next application of $\hat{U}_X$. If Ψ_2 is either $|00\rangle$ or $|10\rangle$, then the projection operator will again pick out the $\hat{u}_0^1$ part of $\hat{U}_X$, and q_1 will be evolved with this. However, if Ψ_2 is instead either $|01\rangle$ or $|11\rangle$, then the projection operator will alternatively pick out the $\hat{u}_1^1$ part of $\hat{U}_X$, and q_1 will be evolved in a completely different way.

Thus from the endo-physical point of view, the way in which qubit 1 develops depends on the sub-state of qubit 2. For an observer inside the universe it appears as if the way in which the universe develops depends upon a 'property' of part of it.

As a final comment to this discussion it should be remarked that, despite the name, the above two qubit Physicist-Sample mechanism is in no way imagined to be completely descriptive of a real, physical quantum experiment. After all, from a practical point of view, real experiments in real laboratories generally occur over very many jumps, and between apparatus, equipment, scientists and samples that may each be represented by enormous groups of factors. Furthermore, real experiments generally involve extended spatial objects, and so could perhaps only be truly discussed in the large scale limit of very many subregisters, when a quantum causal set description of emergent space may be incorporated[27]. Additionally, real physical objects are generally made from enormous collections of fundamental physical particles, and this perhaps implies that a quantum field theoretic description should also ultimately be employed in any discussion of real measurements in physics[28].

More importantly, though, an outcome of Chapter 6 was that a real physical measurement between a physicist and a sample necessarily requires a degree of entangling to occur between their sub-states if any physical information is to be exchanged. Specifically, a conclusion was that endophysical interactions cannot just be the result of local unitary transformations. Thus, in the two qubit universes investigated above, the Physicist qubit would not actually witness the selective evolution of the Sample qubit, because no information is physically exchanged between them during the system's development.

Having noted these points, however, they are subsequently ignored in the present chapter, because it is the principles behind the Selective Evolution mechanisms that is of interest. In particular, the success of this section is that even in the absence of physical information extraction or exchange, it is still possible for the universe to develop in a way in which parts of it appear to evolve relatively to others, and where the development of one factor appears to determine the development of another. The fact that neither part is actually 'aware' of how the other qubit is developing is not currently important; what is important is that the overall universe is able to develop in this apparently self-referential manner. In short, as long as the overall universe 'knows' what it is doing, it does not matter that its constituent parts do not.

Indeed, an additional entangling step could be added to the presented SLE Rules without great conceptual difficulty, and this could effectively enable the Physicist qubit to investigate, in some sense, how the Sample qubit has been evolved. 'She' could then

[27] See Chapter 5.
[28] See Chapter 7.

potentially ascertain how her state influenced the development of the Sample qubit.

Of course, this type of proposed extension is still highly schematic at this stage; presumably for a Physicist to really make conscious measurements and deductions would actually require her to possess enormous numbers of degrees of freedom and be highly and complexly organised. The conclusion, then, is that the issue of exactly how the suggested SLE mechanism could be extended and incorporated into the discussions of the previous chapters, so that real Physicists perform real measurements on real Samples, remains an important question for the future.

8.5.5 A Two Qubit 'Double Experiment' Universe

The Physicist-Sample, Selective Local Evolution Rules of sub-section 8.5.4 govern a universe in which, from an endo-physical point of view, the development of qubit 1 (the 'Sample') appears to be determined by the state of qubit 2 (the 'Physicist'), whereas the state of qubit 2 is evolved independently: during the second step, q_2 is rotated by $\hat{u}_2^2$ regardless of the state of q_1.

A natural extension to this mechanism therefore involves a two-factor system in which *each* factor appears to be evolved in a way that depends upon the sub-state of the other. Thus, the development of one individual sub-state of such a universe seems, from the endo-physical perspective, to be determined by the factor that comprises the remainder of the universe; during the second step, q_2 is now rotated by an unitary operator that is selected according to the state of q_1. This is a truly self-referential system: at each stage, the universe appears to develop by examining one part of itself and evolving the other part accordingly.

The suggested Rules may thus be described as giving rise to '*Double Experiment*' (DE) dynamics; they are still a type of Selective Local Evolution.

The physical interpretation of such a universe is of a 'Sample' whose state is evolved according to the state of a 'Physicist', and where the state of the 'Physicist' is then influenced by the outcome of this experiment. Furthermore, on repetition of the procedure, the analogy is of a Physicist who subsequently 'decides' to develop the Sample in a way that is based upon how 'she' has been affected. These ideas are consistent with the notion that when an experiment is performed in reality on a subject, its result is often registered as a changed 'pointer state' of the apparatus and, ultimately, as a change in the observing scientist's brain. Moreover, the result of an initial test often dictates how a scientist may decide to perform further experiments.

In addition, such a dynamics is also fully compatible with one of the central tenets of quantum theory: in any quantum measurement, there should be an element of symmetry

between the 'observer' and the 'observed', because there is no real criterion for deciding exactly which is which anyway.

The Rules that govern such a Double Experiment universe could be similar to that of the 'single experiment', Physicist-Sample mechanism discussed in the previous sub-section, but modified in the obvious way. As before, a separable state is required such that a classical distinction may be made between the Physicist and the Sample.

Defining again a 'reference' time N in order to keep track of which step is currently applicable, for an 'initial' separable state $\Psi_N = |a\rangle_A \otimes |b\rangle_B$ in a Hilbert space $\mathcal{H}$ factorised in the bi-partite form $\mathcal{H} \equiv \mathcal{H}_{[AB]} = \mathcal{H}_A \otimes \mathcal{H}_B$, the Rules could dictate a two step sequence of the form

1. Evolve Ψ_N to $\Psi'_N = \hat{U}_N \Psi_N$ by applying the unitary operator $\hat{U}_N$, where $\hat{U}_N$ is defined such that it selectively evolves the factor of Ψ_N in $\mathcal{H}_A$ according to the factor of Ψ_N in $\mathcal{H}_B$. Thus, $\hat{U}_N$ obeys the relationship

$$\hat{U}_N \Psi_N = \hat{U}_N(|a\rangle_A \otimes |b\rangle_B) = |a'\rangle_A \otimes |b\rangle_B \tag{8.154}$$

   with $|a'\rangle_A \in \mathcal{H}_A$ defined as $|a'\rangle_A = \hat{u}_b^A |a\rangle_A$, and where the actual choice of the unitary operator $\hat{u}_b^A$ depends somehow on the factor $|b\rangle_B$. Note that the sub-script and super-script convention adopted here is the same as in Sub-section 8.5.4;

2. Collapse Ψ'_N to Ψ_{N+1} with an operator $\hat{\Sigma}_{N+1}$ that has a separable eigenstate of the form $|c\rangle_A \otimes |b\rangle_B$. Thus, $\Psi_{N+1} \in \mathcal{H}_{AB}$, as would be ensured if $\hat{\Sigma}_{N+1}$ is chosen such that it is factorisable relative to $\mathcal{H}_A \otimes \mathcal{H}_B$;

3. Evolve Ψ_{N+1} to $\Psi'_{N+1} = \hat{U}_{N+1} \Psi_{N+1}$, where $\hat{U}_{N+1}$ is defined such that it selectively evolves the factor of Ψ_{N+1} in $\mathcal{H}_B$ according to the factor of Ψ_{N+1} in $\mathcal{H}_A$. Thus, $\hat{U}_{N+1}$ obeys the relationship

$$\hat{U}_{N+1} \Psi_{N+1} = \hat{U}_{N+1}(|c\rangle_A \otimes |b\rangle_B) = |c\rangle_A \otimes |b'\rangle_B \tag{8.155}$$

   with $|b'\rangle_B \in \mathcal{H}_B$ given by $|b'\rangle_B = \hat{u}_c^B |b\rangle_B$, where the actual choice of the unitary operator $\hat{u}_c^B$ depends somehow on the sub-state $|c\rangle_A$;

4. Collapse Ψ'_{N+1} to Ψ_{N+2} with an operator $\hat{\Sigma}_{N+2}$ that has a separable eigenstate of the form $|c\rangle_A \otimes |d\rangle_B$. Thus, $\Psi_{N+2} \in \mathcal{H}_{AB}$.

Procedures "1." and "2." are taken to comprise the first step, whilst "3." and "4." define the second step, as analogous to the PS Rules.

Clearly, because $\Psi_{N+2} \in \mathcal{H}_{AB}$, the first step may now be repeated, and the overall process continued indefinitely. Also clearly, the universe is developing according to Selective Local Evolution, where in any given step of the mechanism the sub-state of one

factor determines the evolution of the other factor, before these roles are reversed in the subsequent step.

To illustrate how a typical DE mechanism might proceed, consider as before a toy-universe represented by a state in a four dimensional, two qubit Hilbert space $\mathcal{H}^{(4)} = \mathcal{H}_1 \otimes \mathcal{H}_2$ spanned by the orthonormal basis $\mathcal{B}_{12} = \{|00\rangle_{12}, |01\rangle_{12}, |10\rangle_{12}, |11\rangle_{12}\}$, where $\mathcal{B}_1 = \{|0\rangle_1, |1\rangle_1\}$ and $\mathcal{B}_2 = \{|0\rangle_2, |1\rangle_2\}$ are bases for $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Consider also the above Rules, but specified by operators of the form

- $\hat{\Sigma}_{N+m} = \hat{\Sigma}_{N+m+1} = \hat{B}$, for all $m = 0, 1, 2, ...$
- $\hat{U}_{N+m}$ given by

$$\hat{U}_{N+m} = \left\{ \begin{array}{c} \hat{U}_N = \hat{U}_S \text{ if } m = 0, 2, 4, ... \\ \hat{U}_{N+1} = \hat{U}_T \text{ if } m = 1, 3, 5, ... \end{array} \right\} \tag{8.156}$$

where $\hat{B}$ has four separable eigenstates defining the basis set

$$\mathfrak{B} = \mathfrak{B}_{(0,4)} = \{|00\rangle_{12}, |01\rangle_{12}, |10\rangle_{12}, |11\rangle_{12}\} = \mathcal{B}_{12} \tag{8.157}$$

and

$$\begin{aligned} \hat{U}_S &= \hat{u}_0^1 \otimes \hat{P}_0^2 + \hat{u}_1^1 \otimes \hat{P}_1^2 \\ \hat{U}_T &= \hat{P}_0^1 \otimes \hat{u}_2^2 + \hat{P}_1^1 \otimes \hat{u}_3^2. \end{aligned} \tag{8.158}$$

and a suitable 'initial' time, N, has been chosen for reference, noting that the subscripts S and T on the fixed operators $\hat{U}_S$ and $\hat{U}_T$ are obviously labels, and not temporal parameters. Moreover, $\hat{P}_j^i$ is the projection operator $\hat{P}_j^i = |j\rangle_{ii}\langle j|$ for $i = 1, 2$ and $j = 0, 1$, whilst $\hat{u}_0^1$ and $\hat{u}_1^1$ are different unitary operators acting in $\mathcal{H}_1$, but $\hat{u}_2^2$ and $\hat{u}_3^2$ are different unitary operators acting in $\mathcal{H}_2$.

As with the Physicist-Sample mechanism of the previous sub-section, the interpretation of the dynamics of a universe developing according to the above types of Rule depends very much upon whether an exo-physical or an endo-physical perspective is being discussed.

From a viewpoint external to the system, an 'observer' would witness the state of the universe changing as follows[29]. The initial wavefunction Ψ_N is globally evolved into the state Ψ'_N by an application of the unitary operator $\hat{U}_N = \hat{U}_S$, i.e. $\Psi_N \rightarrow \Psi'_N = \hat{U}_S \Psi_N$. This evolved state Ψ'_N is then tested by the Hermitian operator $\hat{\Sigma}_{N+1} = \hat{B}$, and consequently collapses into one of the members of the basis set $\mathfrak{B}_{(0,4)}$ with probability given by the usual Born rule, thereby becoming the new state, Ψ_{N+1}.

[29]Note that as before, the 'observer' merely possesses a priviliged vantage point useful for the discussion, and does not interact with the universe in any way.

The second step in the universe's development begins by the global evolution of the state Ψ_{N+1} by the operator $\hat{U}_{N+1} = \hat{U}_T$, that is, $\Psi_{N+1} \rightarrow \Psi'_{N+1} = \hat{U}_T \Psi_{N+1}$. Finally, the operator $\hat{B}$ is again used to test the universe, and the state again collapses into one of the members of $\mathfrak{B}_{(0,4)}$ with a new set of appropriate probabilities. The resulting eigenvector now becomes the subsequent state Ψ_{N+2}, and the process begins again with a repetition of the first step and an application of the operator $\hat{U}_S$ to Ψ_{N+2}, such that $\Psi_{N+2} \rightarrow \Psi'_{N+2} = \hat{U}_S \Psi_{N+2}$. The two-step procedure may be iterated indefinitely.

Thus, from the external point of view the universe develops in a semi-deterministic, globalised fashion, with the choice of unitary operator $\hat{U}_S$ or $\hat{U}_T$ used to globally evolve the state depending only on whether the procedure is in its first or second step. Of course, randomness does occur in the model, but only due to the stochastic nature of the collapse mechanism.

Ultimately, then, an exo-physical observer would conclude that the universe is not developing according to operators chosen as a result of any of the 'properties' of the current state.

As in the case of the Physicist-Sample universe, the interesting physics in the current system's development arises when considering the endo-physical perspective of the individual qubits. From the point of view of one of these factors, the unitary operator $\hat{U}_S$ described by (8.158) is an object that appears to 'ask' whether qubit 2 is in the state $|0\rangle$ or $|1\rangle$, whilst locally evolving qubit 1 with either $\hat{u}_0^1$ or $\hat{u}_1^1$ according to the 'answer' to this question. Specifically, if q_2 is in the state $|0\rangle$, then q_1 is evolved by $\hat{u}_0^1$, but if q_2 is instead in the state $|1\rangle$, then q_1 is alternatively evolved by $\hat{u}_1^1$.

Similarly, from this endo-physical point of view, the unitary operator $\hat{U}_T$ appears to 'ask' about the state of qubit 1 before locally evolving qubit 2 appropriately with either $\hat{u}_2^2$ or $\hat{u}_3^2$; if q_1 is $|0\rangle$, then q_2 is evolved by $\hat{u}_2^2$, whereas if q_1 is $|1\rangle$, then q_2 is evolved by $\hat{u}_3^2$.

Also congruent to the earlier PS example, the repeated use of the operator $\hat{\Sigma}_{N+m} = \hat{B}$ constrains, for all m, every collapsed state Ψ_{N+m} to be one of the four separable and non-superposed eigenvectors defined by the basis set $\mathfrak{B}_{(0,4)}$.

Moreover, as before an important fact is that only those eigenstates with non-zero probability amplitudes with Ψ'_{N+m-1} can actually be realised physically. Consequently, because during the application of either $\hat{U}_S$ or $\hat{U}_T$ only one of the qubits actually changes, only two of the possible eigenvectors of $\hat{\Sigma}_{N+m} = \hat{B}$ will give rise to non-vanishing inner products with Ψ'_{N+m-1}. In practice, therefore, the randomly selected Ψ_{N+m} can only ever be one of these two eigenstates. So, from an internal point of view it appears that only one of the qubits was actually involved in the development of the state from Ψ_{N+m-1} to Ψ'_{N+m-1} to Ψ_{N+m}, because nothing appears to have been done to the other qubit during

this transition.

Thus, from the endo-physical perspective, the development of the universe proceeds in a manner that appears to depend on parts of its state. Paraphrasing, during one of the steps an observer associated with a particular qubit would believe that 'she' was 'deciding' how the other qubit is being evolved, whilst she would then conclude during the remaining step that she was herself being evolved by an operator chosen according to the sub-state of the other qubit.

The development of this universe may now be summarised. From the point of view of an external observer looking at the entire state, the overall two qubit system is globally developed according to a deterministic Rule. Broadly speaking, this Rule implies the successive application of the operators $\hat{U}_S$, $\hat{B}$, $\hat{U}_T$, $\hat{B}$, $\hat{U}_S$, ... to the changing state Ψ.

However, the specific construction of the operators $\hat{U}_S$ and $\hat{U}_T$, and the fact that the state prior to evolution is always one of the members of $\mathfrak{B}_{(0,4)}$, ensures that the individual qubits only ever 'see' half of each of these operators at any one time, that is, either $\hat{u}_0^1 \otimes \hat{P}_0^2$ *or* $\hat{u}_1^1 \otimes \hat{P}_1^2$ for $\hat{U}_S$, and either $\hat{P}_0^1 \otimes \hat{u}_2^2$ *or* $\hat{P}_1^1 \otimes \hat{u}_3^2$ for $\hat{U}_T$. For example, if at the beginning of the second step the state has the form $\Psi = |01\rangle$, then the application of $\hat{U}_T$ is effectively equivalent to an application of just the operator $\hat{P}_0^1 \otimes \hat{u}_2^2$, and it would appear that the unitary operator $\hat{u}_2^2$ has been 'selected' to evolve qubit 2 according to the state of qubit 1.

Thus, from the point of view of an individual qubit, it is the state of the other factor that appears to determine its evolution. From this perspective, an internal observer associated with an individual qubit would believe herself to exist in a fully self-referential universe.

DE Example

As with the earlier Physicist-Sample model, it is beneficial to illustrate the Double-Experiment mechanism by example.

Consider a separable state Ψ_n, in the factorisable two qubit Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, whose development is governed by the operators $\hat{U}_S$, $\hat{U}_T$ and $\hat{B}$ according to the above DE Rules, and where $\hat{u}_0$, $\hat{u}_1$, $\hat{u}_2$, and $\hat{u}_3$ are given by

$$\begin{aligned} \hat{u}_0 &= e^{-i\varepsilon\hat{\sigma}_1} \quad , \quad \hat{u}_1 = e^{-i\mu\hat{\sigma}_2} \\ \hat{u}_2 &= e^{-i\nu\hat{\sigma}_1} \quad , \quad \hat{u}_3 = e^{-i\tau\hat{\sigma}_2} \end{aligned} \tag{8.159}$$

for $\varepsilon, \mu, \nu, \tau \in \mathbb{R}^+$ with $\hat{\sigma}_j$ the j^{th} Pauli operator.

The unitary operators $\hat{U}_S = \hat{u}_0^1 \otimes \hat{P}_0^2 + \hat{u}_1^1 \otimes \hat{P}_1^2$ and $\hat{U}_T = \hat{P}_0^1 \otimes \hat{u}_2^2 + \hat{P}_1^1 \otimes \hat{u}_3^2$ are hence given by

$$\hat{U}_S = \begin{pmatrix} \cos\varepsilon & 0 & -i\sin\varepsilon & 0 \\ 0 & \cos\mu & 0 & -\sin\mu \\ -i\sin\varepsilon & 0 & \cos\varepsilon & 0 \\ 0 & \sin\mu & 0 & \cos\mu \end{pmatrix}, \; \hat{U}_T = \begin{pmatrix} \cos\nu & -i\sin\nu & 0 & 0 \\ -i\sin\nu & \cos\nu & 0 & 0 \\ 0 & 0 & \cos\tau & -\sin\tau \\ 0 & 0 & \sin\tau & \cos\tau \end{pmatrix} \tag{8.160}$$

with the matrices constructed from the usual representations of the bases. As expected, $\hat{U}_S^* \hat{U}_S = \hat{U}_T^* \hat{U}_T = \hat{I}$, as may be readily shown.

The development of this universe proceeds as follows. Without loss of generality, let the initial state Ψ_0 of the system be $\Psi_0 = |00\rangle$. Then, the evolved state Ψ_0' is given by

$$\begin{aligned} \Psi_0' &= \hat{U}_S \Psi_0 = \hat{U}_S |00\rangle \\ &= (\hat{u}_0^1 |0\rangle_1) \otimes (1 \times |0\rangle_2) + 0 = (\cos\varepsilon |0\rangle_1 - i\sin\varepsilon |1\rangle_1) \otimes |0\rangle_2. \end{aligned} \tag{8.161}$$

The subsequent state will be one of the eigenvectors of $\hat{\Sigma}_1 = \hat{B}$, with appropriate probabilities given by

$$\begin{aligned} P(\Psi_1 &= |00\rangle, \Psi_0) = |\langle 00|((\hat{u}_0^1|0\rangle_1) \otimes |0\rangle_2)|^2 = |{}_1\langle 0|\hat{u}_0^1|0\rangle_1|^2 \times 1 = \cos^2\varepsilon \\ P(\Psi_1 &= |10\rangle, \Psi_0) = |\langle 10|((\hat{u}_0^1|0\rangle_1) \otimes |0\rangle_2)|^2 = |{}_1\langle 1|\hat{u}_0^1|0\rangle_1|^2 \times 1 = \sin^2\varepsilon \\ P(\Psi_1 &= |01\rangle, \Psi_0) = P(\Psi_1 = |11\rangle, \Psi_0) = 0. \end{aligned} \tag{8.162}$$

Whichever eigenvector becomes the new state Ψ_1 is then evolved by the unitary operator $\hat{U}_T$. This time, however, it is qubit 1 that is used to 'select' how $\hat{U}_T$ 'works'. Specifically, if $\Psi_1 = |00\rangle$ then

$$\begin{aligned} \Psi_1' &= \hat{U}_T \Psi_1 = \hat{U}_T |00\rangle \\ &= (1 \times |0\rangle_1) \otimes (\hat{u}_2^2 |0\rangle_2) + 0 = |0\rangle_1 \otimes (\cos\nu |0\rangle_2 - i\sin\nu |1\rangle_2) \end{aligned} \tag{8.163}$$

whereas if alternatively $\Psi_1 = |10\rangle$ then

$$\begin{aligned} \Psi_1' &= \hat{U}_T \Psi_1 = \hat{U}_T |10\rangle \\ &= 0 + (1 \times |1\rangle_1) \otimes (\hat{u}_3^2 |0\rangle_2) = |1\rangle_1 \otimes (\cos\tau |0\rangle_2 + \sin\tau |1\rangle_2). \end{aligned} \tag{8.164}$$

The evolved state Ψ_1' is then tested again by the Hermitian operator $\hat{\Sigma}_2 = \hat{B}$ to give the next state $\Psi_2 \in \mathfrak{B}_{(0,4)}$ with a new set of appropriate probabilities, and the process repeated. From the endo-physical perspective, qubit 1 could then be evolved by either $\hat{u}_0^1$ or $\hat{u}_1^1$ during this second application of $\hat{U}_S$, depending of course on which particular element of $\mathfrak{B}_{(0,4)}$ the universe collapses into when it becomes Ψ_2. And so on.

The Double Experiment Rules featured in this sub-section may be extended to 'higher order' mechanisms in obvious, though non-trivial, ways. Indeed, for a universe in a Hilbert space factorisable into a large number F of sub-registers, with a state Ψ_n that is constrained to be separable into F factors for all n, it is possible to imagine Selective Local Evolution dynamics of many types. Continuing, obvious such mechanisms include

- *Many-Physicist* Rules, where the evolution of a Sample sub-state appears to be determined by the sub-states of a number of different Physicist factors;
- *Many-Sample* Rules, where a single Physicist sub-state appears to determine how a number of Sample factors evolve;
- *Many-Physicist/Sample* Rules, where sets of Physicist sub-states appear to determine how groups of Sample factors evolve;
- *Chain-Experiment* Rules, which are effectively F step models: in the first step factor 1 determines how factor 2 evolves, whilst in the second step factor 2 determines how factor 3 evolves, and in the third step factor 3 determines how factor 4 evolves, and so on. In each step, only one of the state's factors actually changes, with the remaining $F - 1$ sub-states apparently unaffected;
- *Simultaneous-Experiment* Rules, in which different 'experiments' occur simultaneously within separate groups of factors of the universe's state, and where the evolution of a member of one group is independent of any member in another group. The simplest Simultaneous-Experiment mechanism would require a four qubit universe, where perhaps in each two step cycle, qubit 1 and qubit 2 are used to determine the evolution of each other, whilst qubit 3 and qubit 4 are also used to evolve each other. Each set of two qubits (say qubits 1 and 2), however, neither influences, nor is influenced by, the other set of qubits (that is, qubits 3 and 4). Thus, the two groups appear to evolve self-referentially within themselves, but independently of the other;

and many others, including potential hybrid 'cross'-variants of those Rules mentioned above.

Clearly, the level of complexity of the different types of Selective Local Evolution mechanisms increases rapidly as the factorisability of the Hilbert space increases. Of course, and as with previous discussions throughout this thesis, such levels are expected to be echoed in the real, physical Universe, where many different strata of systems, sub-systems, sub-sub-systems etc., exist and are known to evolve and interact within themselves and with each other.

8.6 Concluding Remarks

The intention of this chapter has been to investigate a number of closed, quantum systems whose developments are not reliant on the scrutiny, whim or decisions of any sort of external agent. The focus has been to investigate the various types of Rules that could govern such a universe, and could lead it to develop in a wholly isolated and self-consistent manner.

A number of issues and outcomes related to this, however, still remain to be discussed.

8.6.1 Self-referential Quantum Computation

At the beginning of this chapter, an analogy was drawn between self-referential quantum universes and the self-diagnostic software of a hypothetical quantum computer. It is now possible to reinforce this comparison by observing that the mechanisms described in the above sections are themselves nothing more than quantum computations. From this standpoint, the developing quantum universe is therefore viewed as equivalent to a gigantic, self-referential quantum computation, with the universe's wavefunction Ψ_n at time n analogous to the internal state of the quantum computer after n steps of its algorithm, and with the various operators acting upon Ψ_n interpreted in the form of elementary quantum logic 'gates'.

Continuing on from this, given that it can be shown that all qubit evolutions may be re-written in terms of local unitary matrices and the CNOT operator [67], it should consequently be possible to break down some of the mechanisms discussed in this chapter into combinations of these basic quantum computational gates, and this would be taken to additionally illustrate the equivalence.

Ultimately, the above comments complete a central discussion of this thesis. Recall that a developing quantum universe is able to give rise to spatial and quantum field theoretic degrees of freedom, as explored in Chapters 5, 6 and 7. So, it has effectively been shown in this body of work that space and particle physics may be generated by considering the quantum universe as an enormous self-referential quantum computer.

Furthermore, this particular conclusion is granted additional physical weight by noting that in the Standard Model of modern physics, the real Universe is often viewed as containing a vast collection of interacting quantum fields, and by recalling the idea of Feynman [20] that all quantum field theories should be reinterpreted in terms of quantum computation.

Summarising, then, the suggestion is that real physics *is* quantum computation, and that a developing quantum universe behaves like a quantum computer. The quantum

universe paradigm proposed in this thesis could therefore be hoped to simulate any physical phenomenon observed in the real Universe.

When it is remembered that the dimension of the Hilbert space containing the state Ψ_n representing the real physical Universe is at least of the order of $2^{10^{184}}$, the enormity of the quantum computations required to model the actual Universe highlights just how limited the qubit examples described in the chapter actually are.

However, even the modelling of these relatively small dimensional quantum systems can lead to computational problems, and this would be expected to become far worse as the number of qubits is increased. Indeed, as a simple illustration of how even low numbers of qubits can cause severe computational demands, recall that because a system of just fifty qubits is represented by a vector of dimension 2^{50} (i.e. $\sim 10^{15}$), even a modern processor of 1 GHz could require something like 10^6 seconds to compute just one simple step of its evolution.

Going further, if the physical Universe really does behave like a vast quantum computation, it could be argued that any device on which a scientist tried to model it would itself have to be a quantum computer with a dimension even greater than that of the Universe it is modelling. This would lead to the amusing consequence that such a computer would presumably have to be larger than the Universe containing it.

The point is that the low-dimensional qubit models discussed in this chapter provide an illustration of how quantum computational methods can be applied to a universe as a whole. More importantly, the thrust is that it is possible to develop these toy-universes in ways that do not require external guidance, as required for a self-contained and all encompassing view of physics. The hope, then, is that if the principles applied are valid in the low dimensional region, it might be possible to extend them to cope with situations where the complexity of the quantum state is increased. Such extensions will form a basis for future work.

In fact, it is noted that vectors of increased dimensionality would provide incredibly rich scopes for potential dynamics, involving, for example, whole sub-systems of factors evolving and appearing to test one another independently of other sub-systems that are apparently evolving and testing yet more sub-systems that are apparently evolving and... just like the interactions and hierarchies present in the real, classical Universe that humans seem to perceive.

8.6.2 The Real Universe

Nevertheless, despite the fact that many different types of Rules and mechanisms have been successfully proposed and illustrated, a number of issues still remain when faced with the question of the dynamics of the real Universe.

Firstly, and perhaps most obviously, is: what Rules govern the development of the real quantum Universe? Is real physics best described, for example, by Type $IIIa$ Probabilistic List-Sort Rules, or is a Type IV Generated-Sort Class 1 Basis Method mechanism more appropriate? In fact, would some Rules be favoured over others when constrained by attempts to recreate the physical phenomena known to empirical science? Is one particular mechanism, for instance, more able or more likely to provide the sorts of 'persistence' of groups of factors that is generally observed in physics, or provide the sort of causal set structure required to generate apparently continuous Minkowski spacetime?

Indeed, could the Rules governing the development of the Universe ever change, perhaps according to some Rules of the Rules [27], as discussed previously? Such a possibility could provide, potentially, abrupt changes in values of the constants of Nature over time, an idea favoured by some theorists when attempting to answer some of the problems of cosmology; the proposed variable speed of light [86] in the early Universe is an obvious example here.

Of course, the answers to these questions rely on a better understanding of the principles that govern emergence in physics, and will only really become more apparent in the future when states in many sub-register Hilbert spaces have been properly investigated and modelled. In short, exactly how the real Universe develops, and hence gives rise to the spatial and particle degrees of freedom observed in physics, remains an enormous question for the future.

A second issue regarding the choice of operators in the real Universe is: how do they translate to the types of test familiar to empirical physics? How exactly does it arise that physicists appear able to develop their surroundings in an almost unimaginable number of ways, and with complete apparent freedom?

This problem is made more pertinent by noting that not every question that a physicist can ask of a quantum state in the laboratory may be asked of the state of the Universe. In a laboratory, for example, physicists are often able to ascertain whether a given state is entangled or not; given a large number M of identically prepared states $\psi = \{\psi^x : x = 1, 2, ..., M\}$ of the form

$$\psi^x = a|00\rangle_{x_1x_2} + b|01\rangle_{x_1x_2} + c|10\rangle_{x_1x_2} + d|11\rangle_{x_1x_2} \tag{8.165}$$

in factorisable Hilbert sub-spaces $\mathcal{H}_{[x_1x_2]} = \mathcal{H}_{x_1} \otimes \mathcal{H}_{x_2}$ spanned by bases $\mathcal{B}_{x_1x_2} = \{|ij\rangle_{x_1x_2} :$

$i, j = 0, 1\}$ (with $\mathcal{B}_{x_1} = \{|i\rangle_{x_1} : i = 0, 1\}$ and $\mathcal{B}_{x_2} = \{|j\rangle_{x_2} : j = 0, 1\}$), the coefficients $a, b, c, d \in \mathbb{C}$ may be statistically determined, and the separability of ψ in turn discovered. In particular, each laboratory state ψ^x may be tested by an operator $\hat{B}^{x_1 x_2}_{(0,4)}$ with eigenstates $\{|ij\rangle_{x_1 x_2} : i, j = 0, 1\}$, and the frequencies of particular results used to determine the values of the coefficients a, b, c and d.

There is, however, no known general 'entanglement operator' $\hat{E}$ of the form $\hat{E}\Psi = \lambda\Psi$ in physics, where if an arbitrary state Ψ is entangled then $\lambda = \lambda_1$ (which might be interpreted as 'yes'), but if Ψ is instead separable then $\lambda = \lambda_2$ (or 'no'). So, if $\Psi_n = \psi^x$ were taken to represent the state of a two qubit universe, there is no operator $\hat{\Sigma}_{n+1} = \hat{E}$ that could be directly applied to test its separability.

Indeed, it is difficult to imagine how any such operator $\hat{E}$ could actually be constructed anyway: aside from the fact that a yes/no response is 'binary', whereas any test of ψ^x should give four possible results; and the point that there are an infinite number of entangled states in $\mathcal{H}_{[x_1 x_2]}$, whereas $\hat{E}$ must have a finite number of eigenvectors; it is difficult to see how the measurements of a, b, c and d could ever be achieved in a single jump when a statistical approach is instead generally required. Apart from anything else, it is not possible to prepare an ensemble of M identical Universes.

So, a situation appears to arise in which endophysical tests of one factor of the universe by another seem, over a number of jumps, to be able to gain more information than exophysical tests of its entire state. The obvious parallels with Gödel type incompleteness [24] may be drawn here.

The question is, then: how can the self-referential nature of the Universe organise itself in such a way so that at one stage a physicist group of factors is prepared along with M separate entangled factors $\{\psi^x\}$, and then over a series of jumps the state develops such that the physicist believes she is applying the local operators $\hat{B}^{x_1 x_2}_{(0,4)}$, and is hence determining the values of a, b, c and d? In a many subregister universe developing over a number of jumps, how might it be possible for one persistent set of factors to determine whether another factor is entangled or not?

8.6.3 Memory and Information

In Section 8.5 it was shown that it is possible for a universe to self-referentially develop if it is governed by certain carefully defined, two-step, Selective Local Evolution Rules. However, in accordance with the notion of Process time and the Kochen-Specker theorem [21][22], it is noted that once the state Ψ_n has been realised, then the previous state Ψ_{n-1} can no longer be said to exist in any sense. Given this, a natural question faced by such

a two-step mechanism is: how does the universe 'know' which step it should be in? If the universe can only refer to its current state Ψ_n, how does it 'keep track' of which part of the two step Rules are applicable at that time? Specifically, and from the example given at the end of Sub-section 8.5.4, after the state has collapsed to Ψ_n via a test with an operator $\hat{\Sigma}_n = \hat{B}_{(0,4)}$, how does the universe 'know' that the state Ψ_{n-1} was evolved by the operator $\hat{U}_Y$, say, and not $\hat{U}_X$, such that the current state Ψ_n must now be rotated by the operator $\hat{U}_X$?

One suggestion could therefore be that some sort of 'memory' is required in a completely self-referential universe. This memory could perhaps be used to record what the previous state was, what it did, and how it was developed (or even, by extension, what the previous few states were). Alternatively, the memory could maybe equivalently take the form of a type of 'clock' that counts the number of steps taken since a particular 'reference' time, as alluded to in Sub-section 8.5.4. Either way, the Rules would 'consult' this memory store in order to keep track of which step it is in, and to determine what the universe should do next.

The question now becomes: what exactly is this memory and where could it be stored?

Up until now, the current state Ψ_n has been taken to completely specify everything about the universe. It might seem natural, therefore, to somehow try to incorporate the proposed memory store into this vector. In fact, there are two obvious ways to attempt this, each with associated problems.

Firstly, if Ψ_n is assumed to completely represent every conceivable current property of the universe *and* everything about the previous state (i.e. if a mechanism is suggested in which Ψ_{n-2} somehow gets 'absorbed' into Ψ_{n-1}, which itself then somehow gets 'absorbed' into Ψ_n, and so on), then the dimension of the Hilbert space of Ψ_n must be at least twice that of Ψ_{n-1}. This leads to a situation in which as the universe proceeds from Ψ_{n-2} to Ψ_{n-1} to Ψ_n etc., the associated vector space is growing at an exponential rate, and it is unclear what this may mean mathematically.

The second way may be to redefine the actual n^{th} state of the universe as the larger vector Φ_n, where $\Phi_n = \Psi_n \otimes M_n$, which contains the usual 'physical' current state Ψ_n responsible for the physically observable universe, and a 'memory' factor M_n. Particularly, M_n might contain some sort of 'information' regarding what the previous states $\Psi_{n-1}, \Psi_{n-2}, ...$, previous tests $\hat{\Sigma}_n, \hat{\Sigma}_{n-1}, ...$, and previous operators $\hat{U}_{n-1}, \hat{U}_{n-2}, ...$, were, or even just be a 'clock' that somehow registers the current 'time' n. This memory factor, M_n, could then be 'examined' somehow to determine how the 'physical state' factor Ψ_n is developed. Then, and overall, during the development of the universe from Φ_n to Φ_{n+1}, where $\Phi_{n+1} = \Psi_{n+1} \otimes M_{n+1}$, the 'memory' part M_n in Φ_n might somehow be erased and

replaced by M_{n+1} in Φ_{n+1} (now containing information about Ψ_n and/or $\hat{\Sigma}_{n+1}$ and/or $\hat{U}_n$ and/or $n+1$), whilst the 'physical state' part Ψ_n of Φ_n would be replaced by Ψ_{n+1} in Φ_{n+1}.

However, segregation of the overall vector Φ_n into a 'physical state' and a 'memory' could also lead to difficulties. For example, if the memory factor M_{n+1} is to be a direct copy of Ψ_n, i.e. $M_{n+1} = \Psi_n$, then it is difficult to see how the transition $M_n \rightarrow M_{n+1}$ could occur. In particular, the 'erase and replacement' procedure may not be governed by unitary evolution, because it is manifestly irreversible. Whilst this might appear good from the point of view of a universe developing according to the second law of thermodynamics, it is forbidden by an argument similar to the No-Cloning theorem which prevents general unitary evolutions $\hat{U}$ of the form $\hat{U}(\psi \otimes \phi) \rightarrow \psi \otimes \psi$.

Alternatively, if the 'erase and replacement' procedure is to result from a state reduction, then an operator $\hat{\Sigma}_{n+1}$ with an eigenstate $\Psi_{n+1} \otimes \Psi_n$ needs to be used to test $\Phi_n = \Psi_n \otimes \Psi_{n-1}$, and it is unclear how this should in general be constructed; the Memory factor of Φ_{n+1} might be expected to result from the factor Ψ_n of Φ_n, whilst the 'physical state' factor of Φ_{n+1} might equally be expected to result somehow from details of the memory factor of Φ_n. In short, it is difficult to see how the memory could be both referred to and changed at the same time. It would also be required that the probabilistic nature of the collapse from Φ_n to Φ_{n+1} is taken into account by the Rules, and additionally that $\hat{\Sigma}_{n+1}$ is defined such that every physically realisable outcome of it is separable into a 'physical state' factor and a 'memory' factor.

Moreover, observe that the form of Φ_n is similar to the partitioning of the Double-Experiment universe described in Sub-section 8.5.5. However, given that it was this separation that led to the need for a two-step dynamics in the first place, it is difficult to envisage how such a form may then be able to solve the problem of specifying which step of the mechanism the Rules should follow.

Of course, the memory M_n does not have to contain the entire state Ψ_{n-1}. Indeed, the above problems may not occur if the memory factor instead takes the form of a type of 'clock'. However, one difficulty that would now arise is that if each possible 'time' of this clock is assumed to be represented by a different basis state in the memory's sub-space, the dimension of this Hilbert space might be expected to be very large. Specifically, in an eternally enduring universe, the dimension of the clock's Hilbert sub-space may be required to be infinite, and this is clearly undesirable.

Evidently, the above suggestions are just embryonic ideas at this stage. However, if the difficulties encountered are indeed insurmountable problems, the conclusion might be that in order to account for two (or more) step Rules, it may be necessary to parameterise

the universe with two vectors: a state vector and a memory vector. Exactly how a two vector mechanism might be defined, what form the memory vector could take, what its implications might be for the dynamics, and how it might influence the development of the state Ψ_n, are left as questions for the future. However, as remarked in Chapter 3, it could appear that the state Ψ_n and the rules R_n in the n^{th} stage Ω_n of the Universe's development might only be parts of the story; in a complete and consistent quantum universe some sort of information store I_n might also be vital.

Furthermore, in fact, an information content I_n may not just be necessary in universes governed by Selective Evolution dynamics. In List-Sort dynamics, for instance, an Information I_n may be needed to 'contain' the List of possible operators $\mathfrak{L}$, whilst in Type IV Class 1 Basis Method mechanisms an Information I_n may be required to specify the set of constant unitary transformations $\{\hat{U}^{(i)}\}$.

Consequently, and continuing the computational analogy central to this thesis, if the Universe may be described as an enormous quantum computer whose wavefunction Ψ_n describes the state during the n^{th} step of an algorithm specified by the Rules R_n, the Information I_n is like a 'cosmic hard drive' that keeps track of the time n and stores the set of possible instructions, $\hat{\Sigma}_n$, $\mathfrak{L}$ and $\hat{U}_n$ etc.

8.6.4 Reduction without Observers

One point that has not been addressed so far is the actual cause of the state reduction process ubiquitously present in all of the mechanisms discussed in this chapter, and, indeed, throughout this thesis.

Every mechanism has assumed that the n^{th} state Ψ_n of the universe is tested by some Hermitian operator $\hat{\Sigma}_{n+1}$, and consequently collapses into the next state Ψ_{n+1} which is one of the eigenvectors of $\hat{\Sigma}_{n+1}$; the dynamics are therefore analogous to those empirically known to govern quantum systems in the laboratory. With this analogy in mind, however, the issue remains as to why the application of an operator $\hat{\Sigma}_{n+1}$ actually causes the state of the universe to jump from Ψ_n to Ψ_{n+1}. In other words, how exactly does the proposed paradigm view the infamous 'Measurement Problem' of laboratory physics?

The Measurement Problem of laboratory quantum mechanics traditionally raises two main questions: firstly, how is a test actually chosen, and secondly, why does this lead to a collapse of the wavefunction?

In the conventional Copenhagen interpretation of quantum theory, the first question is often swept under the carpet by assuming the role of the observer. In this exo-physical "solution", the state reduction process is initiated by an external physicist deciding to

measure the quantum state with a particular operator, and the collapse then manifests itself as the quantum state reducing to a classically observed object. In short, in laboratory quantum theory an external agent or environment is assumed to 'do something' to the quantum system, and the quantum state then reacts by jumping to the observed (semi-) classical form.

Of course, this 'explanation' has a number of problems associated with it. For example, which observer or observers actually get to choose the test? At what level of observation does the collapse ultimately occur (i.e. the Schrödinger's Cat paradox [32])? How does the physicist decide to test the state in a particular way in the first place, given that she is presumably a complex of quantum particles herself?

The second question also remains unanswered in conventional quantum theory. However, whilst there has never been a satisfactory explanation for the existence of this discontinuous and irreversible process in a Universe that otherwise seems to run on continuous and reversible laws, in many elementary texts on quantum mechanics a number of different interpretations are given that attempt to account for the apparent collapse of the wavefunction in laboratory physics. For example, some explanations involve particular deterministic evolutions, suggestions being due to the Decoherence paradigm discussed in Chapter 3, or because of the Many-Worlds interpretation [19] of Multiverse splitting. Other attempts assume that the state collapses spontaneously, perhaps because of the non-trivial dynamics of the Hamiltonian in GRW theory [88], or due to quantum gravitational effects induced by superposed spacetimes [26].

Each of these interpretations, however, is associated with its own set of problems, inconsistencies and difficulties, and it is by no means clear as to whether any of them is able to provide a coherent, complete and verifiable explanation for the phenomenon of state reduction.

Now, it is not the intention of this thesis to bias any one of the above exo-physical 'explanations' over the others, nor is there a desire to provide fresh insight into possible solutions for the exo-physical Measurement Problem. Instead, in the presented fully quantum universe paradigm it is implicitly assumed that state reduction is a necessary part of quantum theory, and must hence also be a necessary part of any universe running according to quantum principles. Generally speaking, the way in which a state appears to collapse in the laboratory is less important here than the fact that it does indeed collapse.

Summarising, in fact, in the paradigm proposed in this thesis involving the measurement of a state Ψ_n by a test $\hat{\Sigma}_{n+1}$, the existence of the operator $\hat{\Sigma}_{n+1}$ is considered to be just as fundamental as the existence of the vector Ψ_n or the Hilbert space $\mathcal{H}$. In other words, the existence of the operator $\hat{\Sigma}_{n+1}$ is taken to be an integral feature of the dynam-

ics of the universe, and it is assumed that it is an automatic application of this test that results in an automatic collapse of the state. Moreover, the choice of the operator $\hat{\Sigma}_{n+1}$ is governed by a quantum algorithm, as suggested by the various Rules discussed in this chapter.

There is consequently no real "Measurement Problem" in the quantum universe paradigm, at least not in the sense generally understood. The traditional exo-physical difficulties of laboratory physics, which should now be associated with one group of factors appearing to measure another group of factors, consequently find a natural solution emerging from the discussion of information exchange (Chapter 6), and from the operator selection mechanisms proposed in the present chapter.

In conventional laboratory physics, with the scientist and quantum experiment standing isolated in a much larger universe, a question naturally arises concerning the associated timescale between the preparation of the quantum state by the physicist and its subsequent collapse. This question is especially important in, for example, the GRW and superposed spacetime interpretations mentioned above, because in these the quantum wavefunction is assumed to collapse spontaneously after a period of time that depends on the model in question.

However, in the case presented here, where the state represents the entire Universe and not just a tiny sub-system within it, such a concern could not be an issue. Under this circumstance, and because there is no absolute, external time in which the universe is developing, the question as to 'how long' it takes for the universe to develop from the state Ψ_n to the state $\Psi'_n \equiv \hat{U}\Psi_n$ to the state Ψ_{n+1} is meaningless.

Conversely, in fact, in the universe described in this thesis, time is nothing but a concept that emerges from the observation that the state Ψ_n is not the same as the state Ψ_{n-1}, which was itself not the same as Ψ_{n-2}. Thus, in this sense time is viewed as synonymous with *change*. It is consequently no more pertinent to ask about the timescale involved between the preparation of Ψ_n and the collapse of Ψ'_n into Ψ_{n+1}, than it is to ask about the spatial distance between different factors of Ψ_n. The actual process is seen as nothing but the mathematical transformation of a vector in a Hilbert space; time, space and physics are concepts that somehow emerge internally from the dynamics of this state.

8.6.5 Time without Time

The actual origin of continuous-looking physical time in the quantum Universe is naturally a complicated process due to the dimensionality of the Hilbert space involved, but the logic behind it is fairly straightforward.

Consider first an exo-physical perspective, and consider two arbitrary vectors Ψ_a and Ψ_b in a Hilbert space $\mathcal{H}$. If the vector Ψ_b results from an operation on Ψ_a, such that Ψ_b does not exist without the prior existence of Ψ_a, it may be concluded that Ψ_a is a 'cause' of Ψ_b, in some sense.

Moreover, if Ψ_a and Ψ_b cannot both exist simultaneously, and if Ψ_b is known to exist 'now', it follows that Ψ_a existed at one point, but no longer does. It may hence be said that Ψ_a existed 'before' Ψ_b.

Furthermore, if Ψ_b is taken to be an eigenstate of a physical test $\hat{\Sigma}_b$ on Ψ_a, such that no intermediate state exists[30] between the existence of Ψ_a and the existence of Ψ_b, it may be argued that Ψ_a and Ψ_b are separated by one 'step'. It may therefore be justifiable to relabel the sub-scripts as $\Psi_a = \Psi_{n-1}$ and $\Psi_b = \Psi_n$.

In addition, if Ψ_a and Ψ_b are not orthogonal (such that the inner product $|\langle\Psi_b|\Psi_a\rangle| = |\langle\Psi_a|\Psi_b\rangle| > 0$), and if $\Psi_a \neq \Psi_b$ (such that Ψ_b and Ψ_a may be distinguished and $|\langle\Psi_b|\Psi_a\rangle| < 1$), the vectors $\Psi_a = \Psi_{n-1}$ and $\Psi_b = \Psi_n$ may be used, from this perspective, to represent successive states of a quantum universe.

Continuing the logic, if the state Ψ_{n-1} resulted from a test $\hat{\Sigma}_{n-1}$ on a different, but not orthogonal, state Ψ_{n-2} (i.e. $0 < |\langle\Psi_{n-1}|\Psi_{n-2}\rangle| < 1$), and if this chain may be repeatedly extended to the observation that the state Ψ_{n-N+1} resulted from a test $\hat{\Sigma}_{n-N+1}$ on a different, but not orthogonal, state Ψ_{n-N} (i.e. $0 < |\langle\Psi_{n-N+1}|\Psi_{n-N}\rangle| < 1$), it could be interpolated that the state Ψ_{n-N} appeared to develop into the state Ψ_n through a sequence of intermediate states Ψ_{n-N+1}, Ψ_{n-N+2}, ..., Ψ_{n-1}, and hence through a series of discrete jumps. It might consequently be argued that the universe developed from Ψ_{n-N} to Ψ_n in N distinct changes, or steps.

So, it is now possible to define 'exo-time' as a measure of the number of steps taken to get from one state to another in the chain $\Psi_{n-N}, \Psi_{n-N+1}, \Psi_{n-N+2}, ...$ Thus, the exo-time taken for the universe to develop from Ψ_{n-N} to Ψ_n is N. Paraphrasing, according to this definition, time is at root a counting process.

Once a definition of exo-time has been established, it is possible to consider notions of endo-time. In particular, endo-time is defined in terms of the number of changes experienced by a particular factor (representing, for example, a human endo-observer) as the state of the universe develops through a series of steps. This definition will be clarified in the following.

Note first of all, however, that since such an endo-observer can at most only ever be sure of parts of the current state Ψ_n, can possibly recollect factors of a state Ψ_{n-1} that

[30] Recall that in the context of quantum mechanics, nothing can be said to physically exist unless it is observed or measured.

appeared different from the current state, and can predict other possible states Ψ_{n+1} that are eigenvectors of subsequent potential operators, they could immediately infer that Ψ_n is later than Ψ_{n-1}, but Ψ_{n+1} does not yet exist. Moreover from this logic, such complex, macroscopic and allegedly intelligent parts of the universe called human beings, who are aware of parts of the state Ψ_n, can remember parts of sets of states Ψ_{n-1}, Ψ_{n-2}, ... but cannot recall anything about sets such as Ψ_{n+1}, Ψ_{n+2}, ..., are moved to construct concepts such as past, present and future in order to describe things that they think have happened, are currently happening, and may well happen.

Humans could then go on to quantify measures of endo-time by a process that involves counting the changes of the universe around them. They might, for instance, define the 'second' in terms of how many times a particular part of the universe changed as the universe developed from the remembered state Ψ_{n-X} to the current state Ψ_n, where X implies a huge number of jumps. If, for example, they notice that over Y jumps particular factors of the universe representing photons emitted by Caesium-133 atoms are able to change $9,192,631,770$ times [89] under certain circumstances, then they might define Y to constitute one second.

Going further, physicists could even attempt to define time by using laws of physics that were themselves discovered by observing changes in the universe. If, for example, constraints on laboratory quantum mechanics and general relativity seem to indicate that any time scale less than about $\tau_p = (\frac{Gh}{c})^{1/2} \sim 1.35 \times 10^{-43}$ seconds is undefined, they might conclude that in one second the overall state of the universe is able to change $Y = 1/\tau_p$ times. However, this is not the clearest way to view the process: it is not that in one second the universe may change Y times, but that by changing Y times, parts of the universe may go on to provide a definition of one second. In essence, one second *is* the fact that the universe changes Y times.

Of course, this then leads to an apparently paradoxical situation in which although Y jumps might give rise to the definition of one second, it does not imply that the change from Ψ_i to[31] Ψ_{i+1} has a duration of $1/Y^{\,th}$ of a second. After all, recall that this development is just a mathematical procedure, and is therefore without duration. Paraphrasing, because, for example, the evolution of Ψ_n to $\Psi'_n = \hat{U}_n\Psi_n$ by $\hat{U}_n$ is nothing but a mathematical relation, to question its timescale is effectively equivalent to asking how long it takes for one plus one to equal two.

The resolution of this paradox is to note that it only arises from an exo-physical perspective, that is, when an observer believes she can stand isolated from the universe

[31]Or even from Ψ_i to $\Psi'_i = \hat{U}_i\Psi_i$ to Ψ_{i+1}, if the Rules governing the dynamics dictate that the state is rotated during its development.

and witness it evolving in her own external time. From the endo-physical point of view of a scientist who is part of the universe she is trying to observe, the paradox does not arise because only relative differences can be granted any real, physical significance. In fact, even if an external time did exist in which the universe evolves, an observer who is part of the universe would be unable to say whether the development of the state from Ψ_{n-1} to Ψ'_{n-1} to Ψ_n took the tiniest fraction of an "exo-second" or many billions of "exo-years", because all that she can ever be aware of is that Ψ_n is different from Ψ_{n-1} (c.f. the discussion of passive and active transformations given in Chapter 6).

Moreover, from a strict quantum mechanical point of view, the actual evolution of the state from Ψ_{n-1} to Ψ'_{n-1} can have no duration, external or internal, because according to the interpretation of Wheeler ([15]), and as discussed in Chapter 3, no attributes of the state can even be said to exist until it is measured. Only the measurements, that is the changes from Ψ_{n-1} to Ψ_n, are physically relevant, so it is only relative to these changes that physical phenomena such as time may be discussed.

Like the apparent existence of Euclidean space, and as proposed in Chapter 5, the emergence of a linear, temporal dimension is something that appears under specific circumstances, according to the unique point of view of a particular endo-observer, as the universe jumps from Ψ_{n-X} to Ψ_{n-X+1} to... to Ψ_n, where $X \gg 1$. Schematically, if X is very large and if Ψ_{m-1} is sufficiently 'similar'[32] to Ψ_m for all $(n - X + 1) \leq m \leq n$, a particular causal set description might begin to generate flat Minkowski spacetime.

The point is that under these special circumstances (which certainly appear to be the case from the perspective of physicists in the real Universe), endo-physical observers might falsely conclude that they live in a universe that is evolving in an external, continuous time that exists independently of the state.

Moreover, they might therefore make the mistake in this case of asking how long it takes for the state of such a universe to develop from Ψ_m to Ψ'_m to Ψ_{m+1}. They might also be surprised when, under specially controlled laboratory conditions, they witness discontinuous processes such as quantum state collapse occurring, because their 'everyday', large scale, emergent, classical time and laws of physics appear continuous. And when they extrapolate this continuous time dimension to the smallest scales, they might find problems with their classical theories of general relativity and spacetime.

In short, the problem lies in assuming that the state of the universe is developing in an external, continuous, background (space)time; the mistake is to apply the Block universe approach to a system that is running according to quantum principles, and hence according to Process time. Instead, physical time might really only be defined in terms

[32] In some sense. Certainly, for example, Ψ_{m-1} could not be in a vastly different partition to Ψ_m if apparent continuity is to result.

of counting the number of changes of this quantum state, and should hence be considered an emergent feature that is 'created' as the universe develops.

It is at this stage that a simple and schematic discussion of the relative durations of different systems within the universe is permissible. Suppose that the Hilbert space of the universe may be split in the form $\mathcal{H}_{[OABR]} = \mathcal{H}_O \otimes \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_R$, where each factor sub-space needs not be of prime dimension. Suppose also that $\mathcal{H}_O$ may be associated with the Hilbert sub-space of an 'Observer' (i.e. a part of the universe representing a physicist); $\mathcal{H}_A$ may be associated with the Hilbert sub-space of a particular 'sample' (an electron, say, in order to draw parallels with the EPR system discussed in Chapter 3); $\mathcal{H}_B$ may be associated with the Hilbert sub-space of a different 'sample' (say, a positron, for the same reasoning); and $\mathcal{H}_R$ may be associated with the Hilbert sub-space comprising the rest of the Universe.

Suppose further that four successive states, Ψ_{n-3}, Ψ_{n-2}, Ψ_{n-1}, and Ψ_n, in the universe's history are separable in the forms

$$\begin{aligned} \Psi_{n-3} &\in \mathcal{H}_R^{OAB} \quad , \quad \Psi_{n-2} \in \mathcal{H}_{OABR} \\ \Psi_{n-1} &\in \mathcal{H}_{BR}^{OA} \quad , \quad \Psi_n \in \mathcal{H}_R^{OAB}. \end{aligned} \tag{8.166}$$

Then, the interpretation of the development of this universe is as follows.

First, the jump from Ψ_{n-3} to Ψ_{n-2} may be schematically imagined to imply the creation at 'time' $n-2$ of an 'Observer' sub-state in $\mathcal{H}_O$, an isolated electron sub-state in $\mathcal{H}_A$, and an isolated positron sub-state in $\mathcal{H}_B$.

Continuing, from the endo-physical point of view of the sub-state representing the Observer, the sequence $\Psi_{n-2} \rightarrow \Psi_{n-1} \rightarrow \Psi_n$ schematically appears to represent a progression of a quantum universe from an initial state Ψ_{n-2} prepared as four separate sub-systems, to the state Ψ_{n-1} in which the Observer has 'measured' the electron, and then to the state Ψ_n in which both the electron and positron sub-states have been 'measured' by the Observer[33]. Of course, and as in Section 8.5, assuming that the factor of Ψ_{n-2} in $\mathcal{H}_B$ is identical to the factor of Ψ_{n-1} in $\mathcal{H}_B$, the operator $\hat{\Sigma}_{n-1}$ of which Ψ_{n-1} is an eigenstate must be carefully chosen to ensure that, from an endo-physical perspective, the positron sub-state appears to be unaffected by the transition from Ψ_{n-2} to Ψ_{n-1}.

From the endo-Observer's point of view, a physicist would argue that the time-scale involved between the preparation of the positron and its measurement was twice that involved between the preparation of the electron and its measurement. The reasoning is that although both the electron and positron are apparently prepared at the same time as factors of Ψ_{n-2}, the factor in $\mathcal{H}_A$ is 'measured' by the Observer during the transition from

[33]The actual meaning of the word 'measurement' is left deliberately vague here, with reference made to Chapter 6.

Ψ_{n-2} to Ψ_{n-1}, whereas the 'measurement' of the positron by the Observer does not occur until the transition from Ψ_{n-1} to Ψ_n. In other words, the Observer witnesses two changes of the universe's state between the preparation of the positron and its measurement, whilst only one step appears to occur between the preparation of the electron factor and its subsequent re-entangling. The Observer could therefore conclude that the positron factor exists for twice as long as the electron, from her endo-physical perspective.

On the other hand, from the perspective of the positron, only one time step appears to occur between its creation as a factor of Ψ_{n-2} and its measurement by (or, indeed, of) the Observer, because it is unchanged during the transition from Ψ_{n-2} to Ψ_{n-1}. Paraphrasing, because the factor in $\mathcal{H}_B$ does not witness any changes occurring between exo-times $n-2$ and $n-1$, time appears not to pass for it; the factor representing the positron effectively behaves as if it is 'frozen' in time during this period.

So overall, the apparent null test on the factor in $\mathcal{H}_B$ during the jump from Ψ_{n-2} to Ψ_{n-1} leads to the type of 'route dependent' endo-time discussed in Chapter 5, and results in concepts analogous to the notion of proper time in relativity. In short, the Observer would believe that two time steps occurred between the preparation of the positron and its measurement, whereas the positron would contend that only one step occurred between these two events.

Of course, the Observer cannot say *absolutely* how long it took for the universe to develop from the state Ψ_{n-2} to the state Ψ_n, because such an absolute measure is meaningless when time is only defined relative to the changes themselves. Time is defined in the proposed paradigm as nothing but a counting process, and so should not be confused with the mathematical developments of the individual states themselves, such as would occur by falsely associating durations to mathematical procedures. Only relative endo-times have any physical significance in a fully quantum universe.

Moreover, once a concept of relative time-scales has been established, it is possible to ignore the definition of time as a counting procedure simply by appealing to the 'duration' of a specific process compared to that of an accepted standard. From this, scientists are consequently able to say that a certain factor $|F\rangle$ of the universe's state exists for Z seconds if it persists for, say, $Z \times (9,192,631,770)$ cycles of the radiation emitted by a particular atom, where, ultimately, this value is itself only defined relative to the universe developing through a series of Y states. The 'standard' definition of time used in conventional physics is therefore recovered, without the need to count actual numbers of jumps.

It is hence possible to reinterpret the question faced by proponents of spontaneous collapse models of quantum mechanics (e.g. the GRW process) for conventional quantum sub-systems of the universe. Any time-scale involved between the preparation of a

quantum sub-state and its apparent collapse is only relevant either relative to a number of changes of the universe, or equivalently relative to the duration of another sub-system that itself only endures relative to a number of changes of the universe.

Clearly, however, similar such questions are not relevant for the Universe itself. There is no time in which the state of the Universe is evolving; rather the Universe is generating time as it develops through a series of states. This is truly a self-contained perspective.

9 Summary, Conclusions and Future Directions

The purpose of this thesis has been to propose a perspective on the overall structure of the Universe that is fully compatible and consistent with the empirically verified principles of quantum mechanics. In effect, the proposal resulted in extending the standard principles of quantum mechanics to the case where the state in question represents the Universe itself, and not just some microscopic sub-system within it.

Despite such an inevitable conclusion, however, it is noticed that the Universe observed by physicists does not generally appear to resemble a quantum wavefunction. So from this viewpoint, and by considering the various properties of a state developing in a Hilbert space factorisable into an enormous number of subregisters, attempts have been made to suggest how the aspects of physics familiar to laboratory science could begin to emerge from this fundamental, mathematical picture. Some success may therefore be claimed for the endeavour of investigating the potential bridges between the quantum computational, pregeometric vision of reality unavoidably proposed, and the semi-classical world experienced by humans.

These attempts are now summarised in this final chapter, with the conclusions that may be drawn from such work given, and some of the remaining questions and future directions for research highlighted.

In Chapter 3 it was shown that quantum mechanics is a 'valid' theory, in the sense that empirical results confirm the predictions of quantum physics, but do not support the conclusions of theories based upon classical Hidden Variables. Furthermore, from a basic set of observations regarding experimentally known features of physics, it was then argued that the entire Universe should in fact be treated according to quantum principles. Specifically, it was suggested that the Universe may be represented by a pure state Ψ_n in a Hilbert space $\mathcal{H}^{(D)}$ of enormous dimension $D > 2^{10^{184}}$, and that this state is subject to 'rotations' by unitary operators $\hat{U}_n$ and 'testing' by Hermitian operators $\hat{\Sigma}_{n+1}$; moreover, it is this discontinuous process of information extraction by $\hat{\Sigma}_{n+1}$ that justifies the use of the discrete, 'temporal-like' label, n.

From this line of thinking, the concept of a Stage was conjectured. Thus, it was suggested that the operators $\hat{U}_n$ and $\hat{\Sigma}_{n+1}$ used to develop the state Ψ_n are chosen by a quantum algorithm according to a set of Rules R_n, possibly making reference to some sort of Information store, I_n. In the paradigm proposed in this thesis, the development of the Universe is therefore envisaged to be analogous to a gigantic quantum computation, with its state proceeding eternally through a sequence of collapse, evolution, testing, collapse, evolution, testing,... in an automatic and self-referential way.

Based on the observation that the Universe that humans perceive generally appears to be classical, and consequently not indicative of the types of phenomena typically exhibited by conventional quantum states, Chapter 4 attempted to discuss the necessary requirements for arguing that two physical objects may be described as classically distinct and distinguishable. To this end, it was shown that if a Hilbert space $\mathcal{H}$ may be factorised into two factor sub-spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, such that $\mathcal{H} = \mathcal{H}_{[AB]} = \mathcal{H}_A \otimes \mathcal{H}_B$, then the components of a state $\Phi \in \mathcal{H}_{[AB]}$ in $\mathcal{H}_A$ are distinguishable from the components of Φ in $\mathcal{H}_B$ if Φ is separable relative to the split $\mathcal{H}_A \otimes \mathcal{H}_B$. That is, if Φ may be written in the form $\Phi = \phi \otimes \varphi$, where $\phi \in \mathcal{H}_A$ and $\varphi \in \mathcal{H}_B$, then the factor ϕ is classically distinct from the factor φ.

Continuing, a test to determine the separability of an arbitrary state was then given, and the conclusion thereby drawn that separability should be a surprisingly uncommon feature in a fully quantum Universe. The fact that this does not seem to be the case in Nature, however, because the Universe does seem to possess enormous numbers of classically distinct objects, therefore strongly suggested that very tight constraints must be placed upon the operators used to produce the states, such that the occurrence of this result is ensured.

It was then shown that states that are separable relative to one particular split of the overall Hilbert space may be entangled relative to an alternative split. This result in turn suggested that a preferred factorisation of the Hilbert space may be appropriate for the case of the Universe.

Chapter 4 also raised the issue of basis sets of vectors, and showed that even in a four dimensional Hilbert space not every combination of entangled and separable elements exists; specifically, no type $(1, 3)$ basis set $\mathfrak{B}_{(1,3)}$ is permitted. This then immediately raised the question of preferred bases for the Universe, and it is consequently a task for future research to discover which types of basis are allowed in Hilbert spaces of higher dimensions, and which are forbidden by vector space mathematics. Is it possible, for example, to find a $(1, [D-1])$ type basis set for a $D > 4$ dimensional Hilbert space split into two factors? What limitations exist in tri-, quad-,... or N-partite splits of a Hilbert space?

The last part of Chapter 4 addressed the question of real classicity. The suggestion was that classical objects on the macroscopic scale may be somehow associated with groups of factors of the state Ψ_n of the Universe, not least because if two objects may be described as classically distinct and distinguishable from one another, they cannot by definition be entangled. Exactly how these groups of factors translate to the classical looking objects of the laboratory is however left as an important question for the future, but it is interesting to speculate on whether paradigms related to emergent theories of decoherence may play

a central role in this discussion.

So overall, the conclusion was that the semi-classical degrees of freedom of the observed Universe somehow ultimately originate from the factors of the state Ψ_n, because the factors of a separable state may be considered to be distinct and distinguishable, as required for classicity.

One necessary requirement for a system of objects to be described as classical is that it is possible to argue that "*this* object with *these* qualities is *here,* whereas *that* object with *those* properties is *there*." The issues of 'here' and 'there' were therefore addressed in Chapter 5. In response, it was consequently shown that during the development of a universe represented by a state contained in a highly factorisable Hilbert space, causal set relationships may begin to arise between the factors of successive states, and these may in turn give rise to spatial degrees of freedom in the emergent limit. Specifically, embryonic lightcone structures were introduced by considering how counterfactual changes in the factors of Ψ_{n-1} could affect the factors of Ψ_n, and it was conjectured that, over a large number of jumps, these could ultimately be used to generate manifolds, metrics and geodesics. Spatial relationships are therefore introduced between the physical objects that these factors represent on the emergent scale.

Furthermore, because the state Ψ_n is itself the result of a test $\hat{\Sigma}_n$, the factorisability of the operators was also investigated; an important conclusion was that factorisable operators can only have separable eigenstates. The changing sequence of operators $\hat{\Sigma}_n, \hat{\Sigma}_{n+1}, \hat{\Sigma}_{n+2}, ...$ was therefore also demonstrated to exhibit causal set type patterns, and this fact was asserted to be responsible for driving the conditions necessary for continuous space and time to arise. Moreover, and unlike the states, the operators were also conjectured to be constrained to obey Einstein locality, and it was suggested that these assertions may play important parts in discussions explaining why states in quantum physics may exhibit apparently superluminal correlations, whilst observables are restricted to follow classical causality.

A number of physical examples were finally given to illustrate these general points.

Overall, the outcome of Chapter 5 was to show that the changing factorisability of the operators, as the Universe proceeds through a series of Stages, can give rise to a changing separability of the state, and this in turn might consequently begin to exhibit causal set-like relationships between its factors. In addition, the presence of local null tests as factors of the operators generates the appearance of 'route dependent' endo-times for the various factors of the changing state, and this may be interpreted as analogous to the existence of proper time in relativity. So, the 'fictitious' exo-time parameter n gives way to physical, local endo-times in terms of counting the changes experienced by particular groups of factors, and this could in turn provide a possible pregeometric origin

for discussions involving the possession of unique inertial frames of reference by individual endo-observers.

Despite the successes mentioned above, a number of questions still remain unanswered concerning the origin of space from the proposed fully quantum paradigm. Perhaps the greatest of these is: how exactly does conventional general relativity emerge from the underlying statevector description of the Universe?

Now, this question is not just about how the separations of a state in a $D > 2^{10^{184}}$ dimensional Hilbert space could give rise to the continuous looking spacetime experienced in physics, though this task is, of course, itself an enormous issue to be addressed. Neither is it directly concerned with the mechanics of exactly how a spacetime existing as an apparently linear, $3+1$ dimensional Block Universe arena could emerge from the prege-ometric, causal set relationships between factors, though this too is an important point. More importantly, the question of particular concern is: what features of the separable state picture could be used to contain information regarding an emergent object's mass, and how could this be used to affect and distort the lightcone structure so that it appears to result in gravitationally curved spacetime? In other words, how can a sort of 'mass parameter' be introduced into the statevector description, such that the self-referential interplay between the factorisable operators and separable states results in an apparently curved spacetime in the emergent limit?

In short, the emergence of mass curved, four dimensional spacetime from the quantum universe is an enormous question for the future.

Chapter 6 explored the links between quantum computation, information, and the quantum universe paradigm.

Section 6.1 was used to set up the necessary framework for the work of the following chapters; in particular the issues of logic gates and the CNOT operator were discussed, and the concept of 'Transformation' operators that act between the basis vectors of the individual subregisters introduced. It was then shown how elementary computations may be performed, with the accompanying Bell correlations used to provide an example of how care must be taken when interpreting the results of quantum questions.

The remaining section of Chapter 6 discussed the definition and role of information in closed quantum systems. In particular, the notions of active and passive transformations were discussed, and this was followed by definitions for information change and exchange. Summarising, the conclusion was that information changing processes necessarily rely on an active transformation, and as such cannot be achieved simply by a convenient transformation (e.g. relabelling) of the basis. Moreover, it was argued that if Ψ_{n-1} and Ψ_n are in different partitions, the change from Ψ_{n-1} to Ψ_n is synonymous with an

information changing process. Such physically significant processes cannot be achieved by passive transformations, and cannot be removed by unitary rotations of the bases of the individual subregisters.

Related to information change was information exchange, where a component of the state in a particular subregister $\mathcal{H}_i$ may be said to have exchanged information with a component of the state in a different subregister $\mathcal{H}_j$ during a jump from Ψ_{n-1} to Ψ_n if the 'relationship' between them changes. Specifically, if the component of Ψ_{n-1} in $\mathcal{H}_i$ is in a different block of the partition containing Ψ_{n-1} from the component of Ψ_{n-1} in $\mathcal{H}_j$, but the component of Ψ_n in $\mathcal{H}_i$ is in the same block of the partition containing Ψ_n as the component of Ψ_n in $\mathcal{H}_j$, then these two components may be said to have exchanged information during the transition from Ψ_{n-1} to Ψ_n.

Ultimately, the conclusion was that if the number of factors of the probability amplitude $\langle\Psi_n|\Psi_{n-1}\rangle$ is less than the number of factors of either the initial or final states, then the transition from Ψ_{n-1} to Ψ_n is an information exchanging process.

Following on from these definitions, the question of endo-physical measurements was addressed. Specifically, if $\Psi_{n-1} \in \mathcal{H}_{ABR}$ but $\Psi_n \in \mathcal{H}_R^{AB}$, where $\mathcal{H}_{[ABR]} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_R$, with $\mathcal{H}_R$ interpreted as a 'rest of the universe' factor space, and $\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_R$ need not be of prime dimension, then the factor of Ψ_{n-1} in $\mathcal{H}_A$ may be said to have 'measured' the factor of Ψ_{n-1} in $\mathcal{H}_B$ (and vice versa) as the universe jumped to Ψ_n.

The concepts of 'split partition' and 'partition overlap' were in turn usefully introduced.

As with the work of Chapter 5, the results of Chapter 6 also raise a number of questions when attempting to apply them to the real world of laboratory physics; again, it is the issue of how they relate in the emergent limit that is of issue. For example, for a semi-classical object consisting of large numbers of factors, exactly 'how much' information needs to be exchanged to constitute the sort of experiment familiar to physics? Paraphrasing, if a laboratory apparatus is represented by, say, 10^6 components of the state of the universe, whilst a laboratory subject by, say, 10^3, how many of these must become entangled during a jump from Ψ_{n-1} to Ψ_n in order to say that the apparatus has measured the subject? 1 of each? 46 of one, but 23 of the other? All 10^3 and 10^6?

Additionally, how do actual laboratory measurements, that may in reality take place over very many jumps of the universe, translate to the pregeometric, single-jump endo-measurements discussed in this thesis?

In Chapter 7 the remaining part of the statement concerning when "*this* object with *these* qualities is *here*" was addressed, by considering how particular groups of factors might begin to exhibit the types of physical property possessed by particle fields in Nature. From the starting point of the Transformation operators introduced in the previous

chapter, pregeometric ladder operators were defined that caused, under certain circumstances, qubit states to be 'raised' or 'lowered' within their individual Hilbert sub-spaces. Moreover, these pregeometric ladder operators were shown to obey the same statistics and anti-commutation relations as fermionic annihilation and creation operators, and also exhibited the characteristics of $U(2)$ symmetry.

From this definition, it was then shown how the Dirac field may be accounted for from the suggested pregeometric structure, by considering a Hilbert space factorisable into $4(2M+1)$ subregisters, where M is large. Specifically, momentum space creation and annihilation operators were defined in terms of discrete Fourier transforms of the pregeometric ladder operators over this large number of subregisters, and these were shown to obey the anti-commutation relationships necessary for physical spin-half particles. Moreover, Hamiltonian, momentum and charge operators were then derived for these particles in terms of the pregeometric ladder operators, by substituting the usual momentum space operators for their subregister defined counterparts.

To complete the discussion, it was finally shown how these operators may be re-written as sums of products of two-qubit CNOT gates and unitary operators acting locally in the individual subregisters. Feynman's vision of reinterpreting quantum field theory as a form of quantum computation was therefore demonstrated from the perspective of the proposed paradigm.

In conclusion, then, it was shown that by considering particular combinations of pregeometric transformation operators defined in a highly factorisable Hilbert space, the types of operator familiar to experimental particle physics may be constructed. From such considerations, the physical properties exhibited by particular physical objects may therefore begin to emerge from the quantum universe picture envisaged in this thesis.

Of course, real objects in the real Universe tend to exhibit enormous varieties of physical properties, and these are still generally unaccounted for in the proposed model. It is therefore a task for future work to attempt to discover how alternative fields may be encoded into the suggested paradigm. How, for example, could colour degrees of freedom emerge from the suggested quantum vision? Or flavour? What about more 'exotic' fields and particle species? Indeed, would the existence of, for example, (the so far unexplained) Dark Energy matter emerge as an inevitable consequence of the types of pregeometric treatment employed to obtain quantum fields from the state of the quantum universe? What about string theory: does this fit into the proposed paradigm, and if so, how?

On a related note, how could the Higgs field be accounted for, or even the inflation fields conjectured in the early universe, and how would such mass-involving processes influence the quantum causal sets produced, and consequently the emergence of spacetime from the statevector description?

Indeed, how could the suggested approach actually be applied in real space anyway? After all, Chapter 7 discussed the emergence of the Dirac field from a set of qubits that map to an emergent, one dimensional 'lattice'; how exactly should this approach best be extended to cope with the three dimensional volumes present in the real Universe, as accounted for by conventional quantum field theory?

A further extension to the work presented in Chapter 7 asks the question: is the emergence of quantum field theoretic descriptions limited to qubit subregisters, or could higher dimensional factor spaces be considered? Certainly, non-qubit subregisters would intuitively seem to be required for $SU(3)$ colour gauge symmetry, but is this actually the case; is it necessary? Going further, could such a proposal account for the appearance of bosonic particles, with subregisters of enormous, 'near infinite' dimension playing an important part? If so, would the emergence of bosonic particles from such a para-fermionic treatment of physics make important comments regarding the theories of supersymmetry currently hypothesised?

Finally, would the suggested approaches to quantum field theory eventually be able to explain why some of the parameters of the Standard Model have the values that they do? In fact, could the proposed links between quantum field theory and the origin of spacetime in the united paradigm be used to explain some of the other puzzles of fundamental physics, such as why, for example, the constants of nature have the values they do, or why the curvature of the Universe is so close to unity?

All of these are necessary questions for the future development of the quantum universe vision. Many, it is hoped, may be answered from the type of analysis discussed in Chapter 7.

Chapter 8 attempted to classify and explore some of the different ways that a fully quantum universe free from external observers might be able to develop. First, the various Types of way in which Ψ_n could develop into the next state Ψ_{n+1} were classified. Then it was argued that a self-referential mechanism is required in order to provide some of the empirical properties of the physical Universe, and attention was therefore turned onto situations in which the next operator $\hat{\Sigma}_{n+1}$ depends somehow on the present state Ψ_n. Two particular Sorts of mechanism consequently became apparent: those in which the state Ψ_n is used to select a particular operator from a pre-existing List, and those in which the next operator is instead Generated from the current state at time n; these two possibilities were explored in turn. Moreover, it was also concluded that not every method of determining $\hat{\Sigma}_{n+1}$ from Ψ_n is necessarily permitted, even though it might be possible to determine $\hat{\Sigma}_{n+1}$ from $\hat{\Sigma}_n$ in an analogous way.

In all cases it was shown that particular types of Rule could give rise to various phys-

ical features for the developing universe; certain List-Sort Rules, for example, may be particularly suitable to produce the sorts of embryonic lightcone structure required for a causal set description of space to begin to emerge.

Chapter 8 finished by discussing the possibility of allowing the state to be evolved self-referentially, and unitary operators $\hat{U}_n$ were proposed that could rotate Ψ_n in ways that appeared to depend on what it is. However, the crucial point concluded in this work was that an interpretation of how $\hat{U}_n$ acting on Ψ_n behaves depends very much on whether an exo- or an endo-physical perspective is adopted. Specifically, although from an external point of view the outcome of $\hat{U}_n\Psi_n$ may not seem to be determined self-referentially, from an internal viewpoint it could appear that $\hat{U}_n$ is examining Ψ_n, and then developing it in a manner that depends on the result of this investigation.

Moreover, it was also shown that by defining the operators such that $\hat{U}_n$ appears to just examine part of Ψ_n, endo-physical experiment type effects could consequently arise, again from an endo-physical perspective. As before, a conclusion drawn was that local null tests play an important part in the emergence of real physics from the quantum universe.

As discussed in the final part of Chapter 8, a huge number of questions still remain regarding the development of the state in the proposed paradigm. What Type of Rule, for example, is the physical Universe actually governed by, and what exactly is this Rule? Which sorts of mechanism are most suitable for the generation of particular physical phenomena, highly separable states persisting over large numbers of jumps being an obvious example? Could the Rules governing the development of the state change over 'time', n, and how would this affect the resulting physics? Indeed, if this is the case, how does the universe keep track of what the current time is, noting that this is also a question faced in the two-step Selective Local Evolution mechanisms? Is the suggestion therefore to be accepted that some sort of information store is required for the actual development of the Universe, and if so, what form could this 'memory' take?

In short, although the conclusion may be made that it is possible to successfully specify consistent quantum algorithms that automatically develop the state of a universe through an endless series of stages, without the need or intervention of any sort of external guidance, a great deal of future research will be required before it is known what the Rules governing the development of the real Universe actually are, and how these might physically operate.

By far the greatest question for future research concerns the issue of emergence: how exactly does the semi-classical world familiar to physicists arise from the pregeometric, statevector description proposed in this thesis?

Throughout this work, a number of points that begin to answer this issue have been addressed. The general conclusion is that each successive state Ψ_n must be highly separa-

ble, such that the vision of reality perceived by physicists, involving countless numbers of quantum 'micro-systems', may be readily generated. Moreover, it additionally follows that the operators must also be highly factorisable, such that they can appear to control the development of these quantum micro-systems in apparently local and microscopic ways.

However, an enormous number of details still remain to be investigated in this picture, and it is by no means clear as to exactly how the proposed vision can give rise to every emergent property known to classical physics. How, for example, is the continuous time Schrödinger equation able to emerge as an accurate tool useful in describing the apparent evolution of these quantum microsystems in the absence of measurement by emergent semi-classical observers? Similarly, how do the emergent operators of laboratory physics arise from the properties of the 'universe operators' $\hat{\Sigma}_{n+1}, \hat{\Sigma}_{n+2}, ...$?

Of course, the relationships between the 'laboratory tests' and the 'universe tests', $\hat{\Sigma}$, and operators $\hat{U}$ are expected to be highly complex and non-trivial. For instance, field theoretic operators such as the Hamiltonian may initially be constructed from incredibly complicated relations between pregeometric transformations, as was discussed in Chapter 7. But, it both interesting and necessary to speculate on how such a vision could be incorporated into the picture of the developing universe discussed in Chapter 8. After all, the interpretations of laboratory experiments are ultimately expected to rely somehow on subjects and apparatus both being represented by large groups of factors of a state Ψ_n, and the universe then self-referentially choosing operators $\hat{U}_n$ and $\hat{\Sigma}_{n+1}$ according to these sets of factors; the resulting state Ψ_{n+1} is then taken to represent the outcome of this experiment. It is, however, unclear at this stage as to exactly how this mechanism might work in practice, and so a consistent, self-referential version of field theory is even further away.

In fact, the emergent operators familiar to emergent physicists may be expected to bear no resemblance at all to the 'universe operators' $\hat{\Sigma}$, and the operators that represent real laboratory measurements may, perhaps, really only emerge from considering average properties of the operators $\hat{\Sigma}_{n+1}, \hat{\Sigma}_{n+2}, ..., \hat{\Sigma}_{n+m}$ as the universe develops over an enormous number m of jumps. This type of general point was again exemplified in Chapter 7, where it was demonstrated how conventional field theory annihilation and creation operators may emerge from Fourier transforms of enormous numbers of pregeometric ladder operators. It may also go some way towards explaining the "how much information needs to be exchanged" question of Chapter 6: groups of factors representing subjects and apparatus could become slowly entangled, a few components at a time, over a large number of jumps, and it could only be over this set of transitions $\Psi_n \rightarrow \Psi_{n+1} \rightarrow ... \rightarrow \Psi_{n+m}$ that a large scale measurement may be said to have occurred between them. Overall properties of the corresponding set of operators $\hat{\Sigma}_{n+1}, \hat{\Sigma}_{n+2}, ..., \hat{\Sigma}_{n+m}$ may then be used, somehow, to

describe this single laboratory test.

The real operator structure of the developing universe may therefore be vastly different from the types of laboratory Hermitian operators that physicists are familiar with. Moreover, the ultimate, self-referential interplay between the states Ψ_n and the operators $\hat{\Sigma}_{n+1}$ could provide a vision of reality completely different to that perceived by scientists on the classical, emergent scale. Indeed, to quote Jaroszkiewicz [90]: "*...almost everything that we humans believe in is a sort of illusion, a convenient fabrication of the brain, designed to rationalize the massive amounts of stimuli that we constantly receive from our immediate environment. This includes space. This illusion gives us a fighting chance of survival. From this point of view, nothing is really what it seems. If you have seen the film "The Matrix", you may have some idea of what I mean.*".

Exactly how the patterns and constructs recognisable to the human brain emerge from the developing quantum state description therefore remains an enormous question. Indeed, exploring how the human brain creates this illusion from the constantly changing state is a task potentially beyond the scope of neuroscientists and psychologists, despite that fact that some scientists are already beginning to explore the possible quantum origins of consciousness (e.g. [91]).

The point is that the reality humans perceive, consisting of macroscopic semi-classical objects representing subjects, laboratories and apparatus, and with laboratory tests represented by single Hermitian operators, could really bear no apparent similarity to the underlying subregister structure of states and jumps from which they emerge. The sequence of evolutions, tests, and partition changes occurring on the pregeometric level as the Universe's state develops could bear very little resemblance to the large scale reality perceived by human observers.

Humans may therefore never truly be able to understand the apparently bizarre properties of the underlying quantum structure, because the notions of pregeometric states and operators are far beyond their sphere of rational experience. Indeed, what exactly *is* a quantum state?

The conclusion of this thesis is not that the conjectured quantum universe paradigm provides a 'Theory of Everything'. It is hoped, however, that it could provide a valid and correct framework for such a theory to begin to be discussed. Thus, the overall desire is that from this work, it will eventually be possible to describe an all-encompassing and consistent view of physics, in which the properties of a quantum state undergoing an endless series of evolutions and tests in a factorisable Hilbert space of enormous dimension is able to generate, in the emergent limit, every phenomenon associated with the observed physical Universe.

A real 'Theory of Everything' will therefore take the form of a complete set of self-

referential Rules, which may be used to select particular operators, which in turn cause the development of this state to give rise to these particular properties. From the underlying pregeometric structure, such a Theory will therefore consequently govern the emergent scale presence of classicity, continuous space, an expanding Universe, particle field theories, interactions, and even human physicists performing tests in laboratories. And, of course, the phenomenon of time.

A Classical and Quantum Computation

Culturally, technologically and epistemologically, the Theory of Computation was one of the revolutionary successes of Twentieth century science. Now in the Twenty-first century, a great deal of research time is being spent on extending the idea of a computer acting classically to one obeying the rules of quantum mechanics. There are a number of reasons for this, both practical and theoretical.

Practically, modern computer chip technology is reaching its limits. One reason is that the overall size of the device is bounded by thermodynamic concerns over the dissipation of heat, and so there is an increasing need to manufacture silicon chips of highly compact structure. This, however, can only go so far, and current technology is rapidly approaching scales where quantum effects become significant.

On the other hand, as suggested by Manoharan [92], it might be expected that the theory of quantum computation is the natural extension to the classical case. An analogy is drawn here to mechanics: Newtonian mechanics is the limiting case of relativity ($c \to \infty$) and of quantum theory ($\hbar \to 0$), which are in turn both special cases of relativistic quantum field theory. In a similar vein, Classical Computation (CC) might be a limit to Quantum Computation (QC), itself just a subset of Quantum Field Computation (QFC). It is even conjectured that quantum field theory itself is only an approximation of higher order theories, for example supersymmetry or quantum gravity, and so this area of computation might also eventually need expanding as subsequent models of reality become better understood.

Furthermore, it has been conjectured throughout this thesis that the entire Universe may be running as a giant quantum computer. If this viewpoint is correct, a better understanding of the principles of quantum computation is essential if any type of 'Theory of Everything' is to be achieved.

The purpose of this appendix is to introduce and elaborate upon the conventional ideas of classical and quantum computation. It should be noted, however, that the type of quantum computation discussed here is not strictly identical to the mechanism adopted by the quantum Universe, as proposed in the body of this work. In fact, and as has been the case before, the difference arises from a conflict between endo- and exo-physics. From the endo-physical viewpoint of a system free from guiding observers, the Universe prepares, evolves and tests itself according to the Rules governing its dynamics. Conversely, in the conventional quantum computations examined below, the concern is for how semi-classical human observers are able to manipulate an isolated quantum state in order to perform a specific calculation, and how they can then test this state to obtain a specific answer.

Thus, the discussion below provides a summary of the principles of laboratory computation, and therefore forms a useful completion and comparison to the ideas of Chapter 6.

A.1 Classical Computation

In the 1930s Alan Turing wrote his seminal paper on computation [93] and proposed the Turing Machine (TM), the archetypal Classical Computer and forerunner to all modern electronic computers. Contained in this work is the definition of computability: "*A number is computable if its decimal can be written down (by a machine)*". Obvious examples are integer quantities and rational fractions such as 1/2, 3/8 etc.

In fact, it is possible to extend the definition to numbers that can be written down as decimals to a given degree of accuracy. This extends computability to quantities that are the result of Cauchy convergent sequences if a level of approximation is specified. For example, the number 2.7183, which is the estimate to five significant figures of the value of e^1, provides an approximation to the sequence

$$e^1 = 1 + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{r!} + ... \tag{A-1}$$

for very large values of r. Hence, 2.7183 is a good approximation to the infinite sum generating the exponential, so e^1 can be considered computable.

Mathematics is generally concerned with the processing of numbers via specific operations. Calculations, for example, often follow the general logic: *"what is the output number O resulting from the operation A acting on the input number I ?"*. From this, and the definition of computability, it is possible to conject an Automated Computing Machine (or TM) which, given an input and a set of rules for computation, is able to solve (or 'compute') a specific problem and return an output.

Turing provided his hypothetical machine with a certain set of characteristics and components necessary for it to work. The actual physical design of the machine is taken to be immaterial, and any particular hypothesised TM is not automatically assumed to be the only (or even the best) way of encoding and processing information. It is the hierarchy of how the characteristics and components interrelate that is important, and how the algorithm proceeds. This principle is reinforced by Church's Thesis, which argues that all reasonable models of computation are equivalent [94].

Turing's necessary conditions are as follows:

1. A TM has a finite number of internal states (called m-configurations). These are analogous to a set of rules to be followed during the computation, and consequently define it.

2. The machine is supplied with a 'tape', that is, a medium of infinite capacity on which the Input is recorded, the Output is displayed, and the result of any intermediate 'rough workings' can be temporarily recorded.

3. The tape itself is divided into a series of sites or 'squares'. Each square can bear one, and only one, 'symbol' from a set of possibilities.

4. The tape is moved along and is 'scanned' by the TM one square at a time. The TM is 'aware' of only one square at any one time.

5. Scanning the symbol in the square may cause the internal state of the TM to change.

6. Define the Configuration of the TM as (S_r, i), where S_r is the symbol in the r^{th} square, and i is the current internal state of the machine. The Configuration determines the behaviour of the machine.

7. The machine may erase, amend or do nothing to the symbol, and may move the tape one square to the left or right, according to the rules specified by its current configuration.

A simple TM can perform all possible computations using just one of two possible symbols in each 'square'; this includes modern computers which run on binary logic based on microelectronic components that are either switched 'on' or 'off'. Thus, each square has a value corresponding to a binary digit, and so may be called an individual *bit*. Labelling these two possible values 0 and 1, it is easy to show that any input number can be represented by a string of these bits according to the rules of binary mathematics, as demonstrated later.

Additionally, special sequences of 0's and 1's can also be implemented to incorporate necessary functions or instructions, for example a code to inform the TM that the input string has ended, and computation can begin.

The m-configurations contain all of the possible instructions required during a computation. Of course, *which* instructions are used depends on the actual calculation to be performed. Four of the eight simplest instructions that might be used are:

$$a \rightarrow b : L \tag{A-2}$$

where $a, b = 0$ or 1. In words, this command implies: "If the current symbol is a then amend it to b and move the tape one square to the left". The complement four instructions, '$a \rightarrow b : R$', would move the tape to the right.

The next simplest set of instructions are of the form

$$(c) \qquad a \rightarrow b : L \tag{A-3}$$

with $a, b, c = 0, 1$ and implying: "If the current symbol is a and the last encountered symbol was c then amend a to b and move the tape one square to the left". The complexity of the instructions can of course be extended in an obvious way by considering the last $2, 3, ..., n$ encountered symbols.

A computation is conventionally taken to begin with the Input string (input number + any instructions) on the far left of the tape. The schematics of this basic computation can now be written as a generic algorithm:

1. The TM has a finite number of m-configurations which determine the nature of the computation. It is initially in a particular state.

2. The Input is read (from the left) until a certain sequence of symbols is encountered. This signals the end of the Input, and the actual calculating can begin.

3. The TM scans a particular square with the algorithm: If the current symbol is a and the last N symbols were $ABC...$, then:

   - Amend the symbol to b,
   - Move the tape one step to the left or right,
   - Change the internal state of the TM from m-configuration X to m-configuration Y.

Exactly what action the symbols a, A, B, C, ... produce is determined by the particular m-configuration at that time.

4. Step 3 is looped until symbol z is encountered when the last N' symbols were $A'B'C'...$ and the TM is in m-configuration Z. At this point the computation is halted.

5. The 'answer' to the problem is encoded as the remaining symbols on the tape.

It is possible that step 4 may never be encountered, for example if the internal state Z never arises; in this case, the computation effectively loops forever without producing an answer. In fact, it is a central problem in computer science to determine whether a given computation will ever yield an output or will run on indefinitely. There are many examples of this 'Halting problem', Gödel's Incompleteness Theorem [24] being a famous case.

Note that it is additionally possible to encode into the Input (as a series of 0's and 1's) the rules telling the TM which m-configurations to use. This gives a binary representation of all the m-configurations used in a particular TM calculation, and by transforming this binary number into the decimal number n it is possible to label the TM as the n^{th}-Turing Machine.

This idea can be extended to the concept of a Universal Turing Machine (UTM), defined as a TM which has all possible m-configurations inbuilt. Any particular TM, i.e. any specific computation, can be simulated on this Universal machine simply by supplying the number n, because this consequently 'informs' the UTM which m-configurations are relevant. A modern PC is effectively a Universal Turing Machine.

Generally speaking, computations involve mathematics, and mathematics involves numbers. It is therefore necessary to be able to explicitly encode numerical concepts into the Turing machines if they are to be useful. This can be achieved by recalling that any non-negative integer $z < 2^{r+1}$ may be represented in binary notation by the $(r+1)$ bit string

$$z = z_r z_{r-1} ... z_2 z_1 z_0 \tag{A-4}$$

which is shorthand for

$$z = (z_r)2^r + (z_{r-1})2^{r-1} + ... + (z_2)2^2 + (z_1)2^1 + (z_0)2^0 \tag{A-5}$$

where $z_i = 0, 1$ for $i = 0, 1, ..., r$. This obviously extends in a natural way to incorporate, for example, negative integers where $(z_t) < 0$ for all t, but this is not important here.

As an illustration, by using this binary notation the results: $2 = 10$, $3 = 11$, $4 = 100$, $9 = 1001$, $23 = 10111$ etc. are readily obtained. The important point for the present discussion on computation is that by employing this method every number z may be uniquely defined by a string of 0's and 1's, exactly as required for the classical Turing machine. Any number $z < 2^{r+1}$ can consequently be written as a sequence of $(r+1)$ bits, each of which has a definite value.

In fact, it is also possible to cleverly incorporate instructions for mathematical operations as particular combinations of bit values [25]. A full discussion of how this may be achieved, however, is beyond the scope of this short introduction.

It is possible now to provide a simple example of how an idealised Turing machine might actually perform a calculation. For instance, consider the sum $1 + 3 = 4$; in binary notation, this equation is equivalent to the relation

$$...00001 + ...00011 = ...00100 \tag{A-6}$$

By noting that the right-most position in a string may be labelled the 'first' bit (that is, perhaps even more confusingly, $r = 0$), a computer is able to generate the output string according to the following algorithm:

1. The computer is initially provided with the input. In this case, the input takes the form of information regarding the number '1' (i.e. the string ...00001), some sort of code telling the machine that an addition is required, and a number '3' (indicated by another string ...00011).

2. The value of the first bit of the output is given by the modulo two addition of the value of the first bit, 1, of the first string (...00001) of the input to the value of the first bit, 1, of the second string (...00011) of the input. So, in this case the value of the first bit of the output is 0, because $1 \oplus 1 = 0$.

3. This first addition leaves a 'carry' of 1. The value of the second bit of the output is then given by the modulo two addition of the value of the second bit, 0, of the first string of the input, to the value of the second bit, 1, of the second string of the input, followed by the modulo two addition of this result to the remainder (i.e. the 'carry'), 1, of the previous addition used to generate the value of the first bit (i.e. "Step 2".). So, in this case the value of the second bit of the output is 0, because $(0 \oplus 1) \oplus 1 = 0$.

4. The procedure is repeated for all of the bits in the string. In general, the value of the j^{th} bit of the output is given by the modulo two addition of the values of the j^{th} bits of the two strings (...00001 and ...00011) of the input, added to the remainder of the result of the modulo two addition of the values of the $(j-1)^{th}$ bits of these two strings and the 'carry' from the determination of the output's $(j-2)^{th}$ bit.

Thus, given an input incorporating the number '1' ($\equiv$ ...00001), a code telling the machine to do addition, and another number '3' ($\equiv$ ...00011), the computer outputs the number 4 (represented by the string ...00100).

The actual 'mechanics' of the above type of computation may be performed using particular logic gates to manipulate the values of the bits of a string in order to produce an output [95]. Consider the AND, the OR and the Exclusive-OR (XOR) operations that obey the truth table

x	**0**	**0**	**1**	**1**
y	**0**	**1**	**0**	**1**
x AND y	0	0	0	1
x OR y	0	1	1	1
x XOR y	0	1	1	0

Table A-1

Given an input that incorporates the two $(r+1)$-bit strings of value $x_r x_{r-1}...x_2 x_1 x_0$ and $y_r y_{r-1}...y_2 y_1 y_0$, the result of the sum $(x_r x_{r-1}...x_2 x_1 x_0 + y_r y_{r-1}...y_2 y_1 y_0)$ is expected to be a string of the form $z_{r+1} z_r z_{r-1}...z_2 z_1 z_0$, where $x, y, z = 0, 1$, noting that the additional bit z_{r+1} may be required in the output to cope with a potential 'carry' from the addition of x_r and y_r.

Now, it is evident that the calculation of the value z_0 of the 1^{st} bit of the output is given by

$$z_0 = (x_0 \; XOR \; y_0). \tag{A-7}$$

Then, the calculation of the value z_1 of the 2^{nd} bit of the output may be given by

$$z_1 = (x_1 \; XOR \; y_1) \; XOR \; (c_0), \tag{A-8}$$

where c_0 is the 'carry' from the sum of the values of the first bits of the input, such that c_0 is clearly $c_0 = x_0 \; AND \; y_0$.

In general, the value z_j of the j^{th} bit of the output is

$$z_j = (x_j \; XOR \; y_j) \; XOR \; (c_{j-1}) \tag{A-9}$$

where the carry c_{j-1} from the earlier calculation of z_{j-1} is given by the recursive formula

$$c_{j-1} = (x_{j-1} \; AND \; y_{j-1}) \; OR \; [(x_{j-1} \; OR \; y_{j-1}) \; AND \; c_{j-2}]. \tag{A-10}$$

Of course, computers are not simply limited to addition, and alternative calculations can be achieved by using different combinations of logic gates.

Another function useful in computation is the NOT-AND gate (NAND), which produces the truth table

x	**0**	**0**	**1**	**1**
y	**0**	**1**	**0**	**1**
x NAND y	1	1	1	0

Table A-2

This operation is particularly important because it is a standard result [95] that all classical computations can be performed just by using combinations of NAND gates. Thus, the NAND gate is said to be 'complete', and can consequently form the basis for a truly universal Turing machine.

Moreover it is easy to physically build a NAND gate, for example by connecting a set of transistors in a certain way. Thus, by incorporating the principle that low and high voltages may be used to represent the off/on states associated with the bit values 0 and 1, it is possible to construct an electronic device whose output is related to its input potentials according to the logic of the NAND truth table. In fact, a modern personal computer is effectively just a "black-box" containing many such devices. Thus by representing the

input string of bits of value 0 or 1 as a set of low and high voltages, and by instructing the computer as to which groups of NAND gates are to be used in which order and on which bits for a particular desired calculation, the values of an input sequence of electrical pulses may be used to generate a digital output sequence of 0's and 1's. From this starting point any classical computation is theoretically possible.

As noted previously, a Turing Machine can act on any computable number. Computable numbers, however, are only a subset of the field of real numbers, because Real analysis contains non-computable quantities, that is, those which do not possess a sequence that is Cauchy convergent and hence those for which no level of approximation can be used to specify them to an acceptable degree. As an example, non-computability is exhibited in non-recursive sets, for instance the Mandlebrot fractal[34] (described in [25] and others).

For both mathematical and computational completeness, and maybe even for physics as well, it is necessary to extend the encompass of computations operating over the set of computable numbers to that of 'Real Computations' that also operate over the set of non-computable numbers. This extension has recently been analysed [96] with an algorithm found in polynomial time, but is beyond the scope of this short introduction. It is, however, encouraging to note the completeness of mathematics in classical computation.

A.2 Quantum Computation

A number of authors have given accounts of how quantum computation may work, though much of the original idea is accredited to Feynman [72]. Gramß *et al* have written a good introductory text [97], from which much of this section of Appendix A is based.

A quantum computer has the same general structure as a classical computer: there is an Output which is the result of some computation on an initial Input. The Input of a quantum computer, however, is not a classical series of bits but a wavefunction represented at time $t = 0$ by $\psi(0)$. This wavefunction is dynamically evolved during the computation into an Output wavefunction $\psi(T)$, which represents the state of the system at some later time $t = T$. The actual evolution is governed by an operator $\hat{U}$, and this determines the type of computation to be performed.

The information contained in the state $\psi(t)$ may be encoded in a way analogous to a classical computer that incorporates bits of value 0 and 1. Each quantum bit, or 'qubit',

[34] Strictly, the Mandlebrot set is based on complex numbers. This is unimportant for the present discussion since moduli may be taken, and it is the fact that it is irrational and non-Cauchy convergent that is of issue.

q is a component of $\psi(t)$ and is contained in a two dimensional Hilbert space spanned by an orthonormal basis set of vectors conventionally represented by $|0\rangle$ and $|1\rangle$. However, unlike the bits of a classical computer which can only take the values 0 or 1, the state of a qubit can exist as a linear superposition of the form $\alpha|0\rangle + \beta|1\rangle$, for $\alpha, \beta \in \mathbb{C}$.

In practical terms, qubits could be physically associated with the two orthogonal eigenstates of a 'binary' quantum system. Traditionally, the qubit is identified with the eigenstates of a spin-1/2 particle, where perhaps $|down\rangle$ represents $|0\rangle$ and $|up\rangle$ represents $|1\rangle$, but of course the ground and first excited states of any other two-level system could equally well be used, as could, for example, left and right handed photonic polarisation states. The two states can also be identified with the column matrices

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{A-11}$$

The most general state $\psi(t)$ of the quantum computer may be represented in the usual way by a vector in a Hilbert space $\mathcal{H}$. However, since a desire will be to retain the binary logic common to both classical and quantum bits, attention is restricted to Hilbert spaces of dimension 2^N, where N is the number of qubits chosen to comprise the system. Thus, $\mathcal{H}$ may be written $\mathcal{H} \equiv \mathcal{H}_{[1...N]}$, where $\mathcal{H}_m$ is the two-dimensional subregister containing the m^{th} qubit, for $m = 1, 2, ..., N$. The state $\psi(t)$ is now taken to be an arbitrary vector in $\mathcal{H}_{[1...N]}$ with all the separability and entanglement properties familiar to quantum theory and discussed elsewhere in this work.

This general vector approach may be usefully simplified and constrained in order to draw further parallels with the classical computers described previously. For example, just as the state of a classical Turing Machine is given by a string of classical bits, the wavefunction of the quantum computer could be the tensor product of N qubit factor substates; such a product may also be called a 'string'. Continuing the analogy, the classical symbol 0 or 1 in the m^{th} square of the Input section of the TM's tape may be seen as related to the particular spin eigenstate of the m^{th} qubit.

Thus a classical N bit string $a_N a_{N-1} ... a_1$, where $a_m = 0$ or 1 for $m = 1, 2, ..., N$, is *analogous* (in some sense) to an N qubit product represented by a wavefunction ψ defined as $|\psi\rangle = |a_1\rangle \otimes |a_2\rangle \otimes ... \otimes |a_N\rangle$, with the important difference being that in the quantum case each of these sub-states $|a_m\rangle$ may exist as a linear superposition of their bases.

So, $|a_m\rangle = \alpha_m|0\rangle_m + \beta_m|1\rangle_m$ for $m = 1, 2, ..., N$ and $\alpha_m, \beta_m \in \mathbb{C}$, which clearly gives

$$|\psi\rangle = \left[\alpha_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 + \beta_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1\right] \otimes \left[\alpha_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 + \beta_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2\right] \otimes ... \otimes \left[\alpha_N \begin{pmatrix} 1 \\ 0 \end{pmatrix}_N + \beta_N \begin{pmatrix} 0 \\ 1 \end{pmatrix}_N\right] \tag{A-12}$$

noting how the left-right ordering of the state has been reversed between the classical ($a_N \rightarrow a_1$) and quantum ($a_1 \rightarrow a_N$) cases. As throughout this thesis, tensor product

symbols may be emitted for brevity, with the position being used instead as the marker of distinction. For example, the state $|110...1\rangle$ will be taken to imply $|1\rangle_1 \otimes |1\rangle_2 \otimes |0\rangle_3 \otimes ... \otimes |1\rangle_N$ etc.

If a string of qubit sub-states may be written as a product that involves no quantum superposition, it may be seen to directly represent a classical input string. Of course, one way of achieving this would be if either α_m or β_m is zero for each m. In such cases, products of qubits may also be associated with unique numbers according to the rules of binary mathematics, just as with the strings of classical bits. Thus, in the instance where either α_m or β_m is zero for each m, the state $|a_1 a_2 ... a_N\rangle$ would be classically equivalent to the string $a_N a_{N-1} ... a_1$, and may hence be labelled by the number $a_N 2^{N-1} + a_{N-1} 2^{N-2} + ... + a_1 2^0$. For instance, the product quantum state $|101\rangle$ is equivalent to the classical string 101, and hence represents the binary number 5.

This idea may be extended in a way that will become important later. If the individual products of qubit sub-states may be superposed, the overall wavefunction may then be interpreted as representing a superposition of numbers. As an illustrative example, the superposed state

$$|101 + 111\rangle = |101\rangle + |111\rangle \tag{A-13}$$

which may be thought of as a superposition of the quantum strings $|101\rangle$ and $|111\rangle$ and is equivalent to the single qubit superposition

$$|1\rangle_1 \otimes [|0\rangle_2 + |1\rangle_2] \otimes |1\rangle_3 \tag{A-14}$$

is analogous to a quantum superposition of the classical strings 101 and 111, and consequently represents a superposition of the numbers 5 and 7. Note, however, that the actual superposed state $|1(0 + 1)1\rangle$ has no classical equivalent itself, because classical physics does not support superpositions. Reversing this statement: there is no *single* string of classical bits $a_N a_{N-1} ... a_1$ that has the quantum equivalent $|101 + 111\rangle$.

The evolution of the state is governed by an operator $\hat{U}$. For a useful quantum computation, this operator must be: $^{a)}$ Reversible; $^{b)}$ Universal, so that all computations can be performed (c.f. NAND in classical logic). The first condition is important because it implies the existence of the inverse operator $\hat{U}^{-1}$, ensuring that the operator is unitary as required for the Schrödinger evolution of a state. Furthermore, the one-to-one mapping that then arises from the reversibility of $\hat{U}$ implies that a given Output state is the result of a unique Input state.

The operator $\hat{U}$ could be seen as a type of logic gate, for example the Fredkin-Toffoli gate [98] $\hat{U}_{FT}$. An Input wavefunction $|\psi(0)\rangle$ would be evolved by such a gate into an output wavefunction, such that, for example, the state $|\psi(1)\rangle$ after one 'application' is

given by:

$$\hat{U}_{FT}|\psi(0)\rangle = |\psi(1)\rangle. \tag{A-15}$$

In reality, the desired form of the unitary operator $\hat{U}$ is achieved by carefully modifying the Hamiltonian used to determine the dynamical evolution of the system. Exactly how this is accomplished is, therefore, an important technical question. However whilst this might be the case, the issue should really just be seen as a physical practicality that does not alter the following theoretical discussion.

A Universal Quantum Turing Machine (i.e. Quantum Computer (QC)) is the quantum version of the reversible classical Turing Machine. There are, however, important differences between how the two 'devices' work. In general, for example, classical UTM's operate by performing a series of computations ('Serial Computation'), i.e. by performing one step after another, where the TM only 'reads' and acts upon one particular bit at any one time. Conversely, the power of quantum computation lies within 'Quantum Parallelism' (QP), as shown below. The Input wavefunction can exist as a linear superposition of its qubit sub-states, so the quantum computer can in principle act on more than one 'string' of qubits at a time, where each string has a different classical equivalent. For example, if the input state was of the form $\psi(0) = |00\rangle + |11\rangle$, the computation could act simultaneously on the strings $|00\rangle$ and $|11\rangle$ (with classical equivalents 00 and 11). The general idea is that the computations of the strings (e.g. $|00\rangle$ and $|11\rangle$) are performed in parallel (i.e. as $|00\rangle + |11\rangle$), and then brought together at the end to give a result in less time than would be the case if the computations had been performed on each string serially (e.g. $|00\rangle$ followed by $|11\rangle$).

This power of QC can be illustrated by example. Consider some rule or function $f(i)$ that, given an input integer i, computes an output integer i' (i.e. $f(i) = i' : i, i' \in \mathbb{Z}^*$). Because any computable function can be constructed from reversible logic gates, the function $f(i)$ is described by a particular unitary operator $\hat{U}_f$.

Assume that the state of the system may be labelled by Ψ. This state must necessarily represent everything that is involved in the computation; if the quantum computer is viewed as a 'black box', the state Ψ must incorporate the part of the computer's memory that stores the input state, the part used to perform the calculation, the part used to store the outcome, etc.

So, in order to abbreviate the notation, consider the sub-state ψ of Ψ defined as $|\psi\rangle = |i, j\rangle$, where i is the state of the input and j the current state of the output. Moreover, if i and j are both assumed to be integers, they may naturally be written in binary notation as a string of 0's and 1's, and so may readily be encoded as a product of non-superposed qubits of given spin.

Consider now an initial state $\psi_i(0)$ defined as $|i, \underline{0}\rangle$, where $\underline{0}$ is a 'ground state' string of 0's, i.e. $\underline{0} = |000...0\rangle$, representing the obvious observation that there is no output j yet. If the function $f(i)$ is associated with the computation $\hat{U}_f$, the evolution of the initial state $\psi_i(0)$ to the final state $\psi_i(f)$ is described by

$$\psi_i(f) = \hat{U}_f \psi_i(0) = \hat{U}_f |i, \underline{0}\ \rangle = |i, f(i)\rangle. \tag{A-16}$$

Note that the number i features in both the Input $\psi_i(0)$ and Output $\psi_i(f)$ wavefunctions. This feature is a result of the constraint that the evolution is unitary; if the input information had been overwritten or 'forgotten', reversibility would be violated.

As required for quantum parallelism, it is desirable to write the input state as a linear superposition of many alternative classical strings of qubits. Supposing there are n such possible strings, i.e. $i = 1, 2, ..., n$ corresponding ultimately to the 'binary' product states $\psi_1(0) = |10...0, \underline{0}\rangle$, $\psi_2(0) = |01...0, \underline{0}\rangle, ...,$ $\psi_n(0) = |11...1, \underline{0}\rangle$, the superposed Input state $|\psi\ \rangle_I$ may be given by the sum

$$|\psi\rangle_I = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi_i(0) = \ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} |i, \underline{0}\rangle. \tag{A-17}$$

The final state $|\psi\rangle_F$ is generated by evolving the input state $|\psi\rangle_I$ with $\hat{U}_f$, i.e.

$$|\psi\rangle_F = \hat{U}_f |\psi\rangle_I = \hat{U}_f \ \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} |i, \underline{0}\rangle \right) = \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} |i, f(i)\rangle \right). \tag{A-18}$$

Clearly, this final state $|\psi\rangle_F$ contains n 'solutions' corresponding to the n many $f(i)$ for $i = 1, 2, ..., n$. However, the generation of $|\psi\rangle_F$ from $|\psi\rangle_I$ has been achieved during one time step (evolution) of the calculation on only one quantum computer, i.e. by one application of the gate $\hat{U}_f$ to the Input state $|\psi\rangle_I$. Conversely, if performed serially on each of the n states $\psi_i(0)$, it would take n time steps to produce n results for $f(i)$.

There is, however, unfortunately an inherent problem here: it is not possible to access more than one of these solutions. As soon as the superposed Output $|\psi\rangle_F$ is observed its state vector collapses to one of the eigenfunctions of whichever Hermitian operator was used to measure it. From this perspective all that can be known about $|\psi\rangle_F$ is that it collapses to, say, the eigenstate $|e\rangle$ with relative probability $|\langle e|\psi\rangle_F|^2$. Moreover, assuming that the Hermitian operator is chosen such that its n eigenvectors are the 'answer' states $|i, f(i)\rangle$, then the probability $|\langle e|\psi\rangle_F|^2 = |\langle e, f(e)|\psi\rangle_F|^2$ of obtaining the e^{th} one of these is given by $1/n$, with each outcome taken to be equally likely.

Furthermore, once the state $|\psi\rangle_F$ has collapsed, any additional measurements of the system with the same Hermitian operator produce the same result. Thus, there is no way

to retrieve any information about any of the other $n-1$ parts of the superposition $|i, f(i)\rangle$ of $|\psi\rangle_F$ for $i \neq e$, and so the fact that all of this other information is lost renders the QC described above as no more efficient than a classical computer.

The problem may be rephrased by emphasising that the quantum state $|\psi\rangle_F$ has been 'asked a direct question', thereby forcing it into a single eigenstate. To avoid this, a more stochastic approach needs to be employed, where sets of questions are simultaneously posed and the results are given in terms of the probabilities of ensembles of answers.

As it turns out, this procedure is incredibly difficult, and only a few such possible solutions to this type of problem have been found where quantum parallel computation can better classical serial computation. Examples are Shor's algorithm for the fast (polynomial time) factorisation of a large number into two primes [99], and the work by Deutsch and Jozsa [100] described below.

Deutsch and Jozsa's model (henceforth referred to as DJ) begins by considering a function f that maps a positive integer z randomly to either 0 or 1, that is

$$f(z) = 0 \text{ or } 1 \quad , \quad \forall\ z \in \mathbb{Z}^+. \tag{A-19}$$

Consider now a string of n numbers $\{n\} = \left\{\sum_{i=1}^{2N} i\right\} = \{1, 2, ..., 2N\}$, where $n = 2N$ is clearly even. In DJ's model the computation f acts on each of these numbers to yield a bit string x defined as

$$x = f(1)f(2)...f(2N) \tag{A-20}$$

which is evidently a sequence $2N$ characters long of 0's and 1's that will randomly take one of the 2^{2N} forms:

$$x = \{(000...0),\ (100...0),\ (010...0),\ ...\ ,\ (000...1),\ (110...0),\ (101...0),\ .....\ ,\ (111...1)\}. \tag{A-21}$$

Given an initial sequence n, the thrust of DJ's task is then to find at least one true statement about the resulting string x from the following two assertions:

1. The string x is neither just a string of 0's nor just a string of 1's (i.e. x is neither 000...0 or 111...1). This is equivalent to the statement that f is not a constant function.

2. The number of 0's in x is not equal to the number of 1's in x. In other words, the function f acting on the $2N$ numbers $1, 2, ..., 2N$ will not give exactly N many 0's and N many 1's.

Clearly, for a string x picked at random both statements are likely to be true.

A schematic algorithm for a classical computation of this sort could be to compute $f(1)$, then compute $f(2)$, then compare the values of $f(1)$ and $f(2)$, then compute $f(3)$ before comparing its value to $f(1)$ and $f(2)$, then compute $f(4)$, and so on. Assuming that each computation takes one time step to complete, and that the comparison procedure is effectively instant, a worst case scenario for the efficiency of such a serial method of testing the validity of statements "1." and "2." consequently takes $N+1$ steps: if the first N bits all turn out to be 0's, and if the $N+1^{th}$ bit is another 0, it implies that Assertion "2." is true, whereas if the $N+1^{th}$ bit is alternatively a 1 it follows that Assertion "1." must be true (and similarly, of course, if the first N bits are all 1's). In other words, for a serial classical computer the computation f may need to be called $N+1$ times before an answer can be obtained to statements "1." and "2." for an initial sequence of $2N$ numbers.

The question is: "Can a quantum computer improve on this efficiency?". Is it possible to find a quantum method that appears to compute every number simultaneously?

The quantum computation in DJ's proposed method makes use of three distinct stages: preparation of the Input state; computation via dynamical evolution; and measurement of the Output state.

In a classical computation, the string n comprises of a 'chain' of (binary) numbers $1, 2, 3, ..., 2N$, and the function $f(i)$ acts on each of them in turn, i.e. serially. In quantum computation, however, the quantum strings equivalently representing these $2N$ numbers may be linearly superposed into a single state. Thus, the Input state $|\psi\rangle_I$ for the present calculation is taken to be a linear superposition of the $2N$ numbers $\left\{\sum_{i=1}^{2N} i\right\}$ and may be written

$$|\psi\rangle_I = \frac{1}{\sqrt{2N}} \sum_{i=1}^{2N} \psi_i(0) = \frac{1}{\sqrt{2N}} \sum_{i=1}^{2N} |i, \underline{0}\rangle \tag{A-22}$$

with the $|i = input, j = output\rangle$ defined as before and the $\underline{0}$ implying a string of 0's. Of course, each i is taken to represent a binary number between 1 and $2N$, and is hence a string of non-superposed qubits of definite value; this therefore requires at least R qubits, where R is the smallest integer for which $2^R > 2N$.

As an aside, note that in order to actually prepare the initial state $|\psi\rangle_I$ it is necessary to consider the 'pre-Input' state $|\psi\rangle_p$. This is taken to be $|\psi\rangle_p = |\underline{0}, \underline{0}\rangle$, such that every qubit of that part of the quantum computer allocated to store the input state i is assumed to be in the ground state $|0\rangle$. Now, because any number i can be represented in binary notation by a unique string of 0's and 1's, every quantum 'number state' $|i\rangle$ is represented by a product of qubits, each of which is definitely in the state $|0\rangle$ or $|1\rangle$. To represent a particular number it is therefore necessary to transform some of the state $|0\rangle$ qubits contained in the input product $\underline{0}$ into state $|1\rangle$ qubits. Moreover, because the eventual Input state $|\psi\rangle_I$ is defined to be a superposition of all of these different combinations of

product qubit states, this must also be taken into account.

One way of achieving this is therefore to use a suitable unitary operator $\hat{A}$ acting on $|\psi\rangle_p$ that evolves it into the superposition state $|\psi\rangle_I$. Thus,

$$|\psi\rangle_I = \hat{A}|\psi\rangle_p = \hat{A}|\underline{0},\underline{0}\rangle \tag{A-23}$$

where $\hat{A}$ may act locally on the individual qubit spaces.

The actual quantum computation makes use of two operators: an evolution operator $\hat{U}_f$ that evolves a state in the manner

$$\hat{U}_f\psi_i(0) = \hat{U}_f|i,\underline{0}\ \rangle = |i,f(i)\rangle \tag{A-24}$$

where $f(i) = 0$ or 1, and a 'parity' operator $\hat{S}$ defined as

$$\hat{S}|i,j\rangle = (-1)^j|i,j\rangle. \tag{A-25}$$

The computation is achieved by evolving the state, performing a parity operation, and then applying the inverse operator $\hat{U}_f^{-1}$ to obtain an Output wavefunction $|\psi\rangle_F$. Specifically,

$$\begin{aligned} |\psi\rangle_F &= \hat{U}_f^{-1}\hat{S}\hat{U}_f|\psi\rangle_I = \hat{U}_f^{-1}\hat{S}\hat{U}_f\left(\frac{1}{\sqrt{2N}}\sum_{i=1}^{2N}|i,\underline{0}\rangle\right) \\ &= \hat{U}_f^{-1}\hat{S}\left(\frac{1}{\sqrt{2N}}\sum_{i=1}^{2N}|i,f(i)\rangle\right) = \hat{U}_f^{-1}\left(\frac{1}{\sqrt{2N}}\sum_{i=1}^{2N}(-1)^{f(i)}|i,f(i)\rangle\right) \\ &= \left(\frac{1}{\sqrt{2N}}\sum_{i=1}^{2N}(-1)^{f(i)}|i,\underline{0}\rangle\right). \end{aligned} \tag{A-26}$$

Consequently, the $2N$ results of the computation $f(i)$ are stored as phase information in the $(-1)^{f(i)}$ part of the Output state $|\psi\rangle_F$.

Measurement of the Output wavefunction $|\psi\rangle_F$ is achieved by a Hermitian operator that possesses ϕ as an eigenvector, where $\phi = |\psi\rangle_I$ is the initial state. The probability of recording the eigenvalue associated with this is given in the usual way by P, where $P = |\langle\phi|\psi\rangle_F|^2$. So

$$P = \left|\frac{1}{2N}\sum_{i=1}^{2N}\sum_{j=1}^{2N}(-1)^{f(i)}\langle j,\underline{0}|i,\underline{0}\rangle\right|^2, \tag{A-27}$$

and assuming orthogonality of the states, $\langle j,\underline{0}|i,\underline{0}\rangle = \delta_{ij}$, gives

$$P = \left|\frac{1}{2N}\sum_{i=1}^{2N}(-1)^{f(i)}\right|^2. \tag{A-28}$$

The statements "1." and "2." can be answered by examining P. Three distinct cases are present:

i) If $P = 0$ then the sum must have vanished. This implies that $f(i)$ has produced as many 0's as it has 1's and consequently Assertion "1." must be true (and "2." must be false).

ii) If $P = 1$ then all of the $f(i)$'s must be either 0 or 1 and hence Assertion "2." is true (and "1." is false).

iii) If $0 < P < 1$ then $f(i)$ has produced an unequal number of 0's and 1's, but has produced at least one of each. Both Assertions "1." and "2." must be true.

The important point is that the time taken for $\hat{U}_f$ to act is assumed to be the same as the time needed to perform just one computation $f(i)$ on a single number i in the classical (serial) case. If $\hat{U}_f^{-1}$ is assumed to take one time step also, and if $\hat{S}$ is taken to act comparatively 'instantly', the entire quantum computation has proceeded in just two time steps. Thus the quantum computation is performing $2N$ computations in parallel in only two time steps. Moreover, the testing of the validity of statements "1." and "2." has also been achieved in just two time steps, which compares with a serial, classical computer taking (at worst) $N + 1$ time steps to arrive at the same conclusion.

So, quantum computers clearly have an enormous advantage over their classical counterparts in certain specifically defined computations. Unfortunately, of course, they also have the even greater disadvantage that they cannot (currently?) actually be built: the effects of their external surroundings destroy the superposition of the evolving state before any significant computation can take place.

Whether or not this technological difficulty will ever be overcome is a question for the future. However, even if the 'decohering' presence of an environment fundamentally prohibits the construction of a working quantum computer *inside* the Universe, it does not prevent the principles of quantum computation being applied to the Universe as a whole, as proposed in this work. After all, the physical Universe has no external environment to interfere with it.

As relativity is an extension of Newtonian mechanics, and quantum field theory an extension of quantum mechanics, we might expect the extension of quantum computation into relativistic quantum field computation. After all, quantum computation proceeds as the time evolution of an initial state (generally comprising of products of qubits located at definite sites), where time is treated as a continuous variable. Lorentz invariance, however, expects space and time to be interchangeable, so a covariant case of quantum computation might ultimately be sought: quantum field computation.

Quantum field computation would be a new branch of computation drawing from both Real and Quantum analogues. At this stage, very little is clear about exactly how a QFC could work, and authors mainly describe it as a necessary new direction instead of as a well understood procedure with defined mathematical structure [92][101][102]. The actual preparation of input states, and the eventual defining of the system via information encoded into Lagrangian formulations, are both interesting considerations for the future.

The primary difference between quantum computation and quantum field computation is that whilst QC permits superpositions of qubits, QFC allows superpositions of entire fields. So, whereas in quantum computations an Input exists as linear superpositions of 'classical' n qubit strings in a Z^n dimensional Hilbert space (where Z is the number of states per qubit[35]), each field in a quantum field computation would possess an infinite number of degrees of freedom, so the computation would take place in an infinite dimensional Hilbert space. Additionally, whereas the results of a quantum computation may be exhibited as single eigenstates, the output of a quantum field computation might be given in terms of expectation values of field operators.

Quantum field computation is an extension from real computation in that it includes computation over the continuum. This extension naturally increases the computational power of the system at the cost of an increasingly complex mathematical formalism.

Exactly what this might imply for a Universe running as a quantum computer is an intriguing question for the future.

[35]Strictly speaking, "qubit" is an acronym for QUantum Binary digIT, so Z can only ever equal 2, but it is in principle possible to build quantum computers out of suitably named qutrits, ququads, ..., quzits (?), represented by $3, 4, ..., Z$ level systems.

B The Dirac Field

In this appendix, the standard Hamiltonian, momentum and charge operator representations are derived for spin-$\frac{1}{2}$ fermions. The presented approach follows closely the treatment given in the text of Mandl and Shaw [78].

B.1 Lagrangian Dynamics

The Dirac equation of motion for free particles of rest mass m is conventionally given by

$$i\hbar\gamma^\mu\partial_\mu\psi(x) - mc\psi(x) = 0 \tag{B-1}$$

where c is the speed of light, $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ for $\mu = 0, 1, 2, 3$ and $x^\mu \equiv (ct, \mathbf{x}) \equiv (ct, x^j)$ for $j = 1, 2, 3$, and γ^μ are 4×4 matrices satisfying the anti-commutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{B-2}$$

and the Hermiticity conditions $\gamma^{0\dagger} = \gamma^0$ and $\gamma^{j\dagger} = -\gamma^j$, so that

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \tag{B-3}$$

with $g^{\mu\nu}$ the metric tensor of signature $(+, -, -, -)$ such that $x_\mu = (ct, -\mathbf{x})$; the gamma matrices form a Clifford algebra [103].

The 'adjoint' field $\bar{\psi}(x)$ is defined as $\bar{\psi}(x) \equiv \psi^\dagger(x)\gamma^0$, and this satisfies the 'adjoint' Dirac equation $i\hbar\partial_\mu\bar{\psi}(x)\gamma^\mu + mc\bar{\psi}(x) = 0$. Note that there is no standardised notation in the texts on quantum field theory, and is often incongruent with 'conventional' mathematics: the field $\psi^\dagger(x)$ is taken here to represent the Hermitian (or transpose) conjugate of the field $\psi(x)$, whereas in linear algebra such an operator $\psi^\dagger(x)$ would often be called the adjoint of $\psi(x)$ and may instead be denoted by $\psi^*(x)$.

Both the Dirac equation and the adjoint Dirac equation can be derived from the Euler-Lagrange condition, given the Dirac Lagrangian density L defined[36] as

$$\begin{aligned} L &= c\bar{\psi}(x)\left[i\hbar\gamma^\mu\partial_\mu - mc\right]\psi(x) \\ &= c\bar{\psi}(x)\left[i\hbar\gamma^0\frac{\partial}{\partial(ct)} + i\hbar\gamma^j\frac{\partial}{\partial x^j} - mc\right]\psi(x) \\ &= \bar{\psi}(x)\left[i\hbar\gamma^0\dot{\psi}(x) + i\hbar c\gamma^j\partial_j\psi(x) - mc^2\psi(x)\right] \end{aligned} \tag{B-4}$$

[36] Note that for convenience in this appendix an asymmetric Lagrangian has been used. As can be readily verified, however, a symmetrised version would lead to the same results.

where the dot denotes differentiation with respect to time, t. Consider now the conjugate momenta to $\psi(x)$ and $\bar{\psi}(x)$, written as $\pi(x)$ and $\bar{\pi}(x)$ respectively. These are given by

$$\begin{aligned} \pi(x) &\equiv \frac{\partial L}{\partial \dot{\psi}(x)} = \bar{\psi}(x) i\hbar\gamma^0 = i\hbar\psi^\dagger(x) \\ \bar{\pi}(x) &\equiv \frac{\partial L}{\partial \dot{\bar{\psi}}(x)} = 0. \end{aligned} \tag{B-5}$$

Together with the fields, the conjugate momenta satisfy the canonical anti-commutation algebra

$$\begin{aligned} \{\psi(x), \pi(x')\} &= i\hbar\delta(x - x') \\ \{\psi(x), \psi(x')\} &= \{\pi(x), \pi(x')\} = 0. \end{aligned} \tag{B-6}$$

So, the Hamiltonian density $\tilde{H}(x)$ defined as

$$\tilde{H}(x) \equiv \pi(x)\dot{\psi}(x) + \bar{\pi}(x)\dot{\bar{\psi}}(x) - L \tag{B-7}$$

becomes

$$\begin{aligned} \tilde{H}(x) &= i\hbar\psi^\dagger(x)\dot{\psi}(x) + 0 - \left[i\hbar\psi^\dagger(x)\dot{\psi}(x) + i\hbar c\bar{\psi}(x)\gamma^j\partial_j\psi(x) - mc^2\bar{\psi}(x)\psi(x)\right] \\ &= mc^2\bar{\psi}(x)\psi(x) - i\hbar c\bar{\psi}(x)\gamma^j\partial_j\psi(x) \end{aligned} \tag{B-8}$$

producing the Hamiltonian H

$$H = \int \bar{\psi}(x)[mc^2 - i\hbar c\gamma^j\partial_j]\psi(x)\ d^3x. \tag{B-9}$$

Similarly, the 3-momentum $\mathbf{P}$ defined as

$$\mathbf{P} \equiv -\int [\pi(x)\partial_j\psi(x) + \bar{\pi}(x)\partial_j\bar{\psi}(x)]\ d^3x \tag{B-10}$$

becomes

$$\mathbf{P} = -\int [i\hbar\psi^\dagger(x)\partial_j\psi(x) + 0]\ d^3x. \tag{B-11}$$

such that the relativistic energy-momentum vector $P^\mu = (H/c, \mathbf{P})$ may be evaluated.

Lastly, from the quantity Q defined as

$$Q \equiv -\frac{iq}{\hbar}\int [\pi(x)\psi(x) - \bar{\pi}(x)\bar{\psi}(x)]\ d^3x \tag{B-12}$$

for particles possessing 'charge' of magnitude q, it follows that

$$\begin{aligned} Q &\equiv -\frac{iq}{\hbar}\int [i\hbar\psi^\dagger(x)\psi(x) - 0]\ d^3x \\ &= q\int \psi^\dagger(x)\psi(x)\ d^3x. \end{aligned} \tag{B-13}$$

The result (B-13) for Q is associated with the conserved electric charge, and leads to an invariance of the Lagrangian density L under a global phase transformation of the fields $\psi(x) \to \psi(x)'$ and $\psi^\dagger(x) \to \psi^\dagger(x)'$ by $e^{-i\alpha q}$, where

$$\begin{aligned} \psi(x) &\to \psi(x)' \equiv e^{-i\alpha q}\psi(x) \sim (1 - i\alpha q + ...)\psi(x) \\ \psi^\dagger(x) &\to \psi^\dagger(x)' \equiv e^{+i\alpha q}\psi^\dagger(x) \sim (1 + i\alpha q + ...)\psi^\dagger(x) \end{aligned} \tag{B-14}$$

for small $\alpha \in \mathbb{R}$, such that

$$L = L(\psi, \bar{\psi}) = L(\psi', \bar{\psi}') \equiv L([e^{-i\alpha q}\psi], [e^{+i\alpha q}\bar{\psi}]) \tag{B-15}$$

as may be readily shown from the condition of invariance, $\delta L = 0$, for a Lagrangian density affected as $L \to L' \equiv L + \delta L$ by a change of the fields (B-14), where

$$\delta L \equiv \frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial \psi_{,\mu}}\delta\psi(x)\right) \tag{B-16}$$

and by using the Dirac equation.

Specifically, the unitary transformation U associated with this unobservable phase change in the fields is given by

$$U = \exp(i\alpha Q) \tag{B-17}$$

such that from Schrödinger's equation

$$\begin{aligned} \psi(x)' &= e^{i\alpha Q}\psi(x)e^{-i\alpha Q} \\ &= \psi(x) + i\alpha[Q, \psi(x)] \end{aligned} \tag{B-18}$$

which is seen to result in (B-14) by using the equations

$$\begin{aligned} [Q, \psi(x)] &= -q\psi(x) \\ [Q, \psi^\dagger(x)] &= q\psi^\dagger(x) \end{aligned} \tag{B-19}$$

which themselves follow from the anti-commutation relations of on the field [104].

The conserved nature of the charge Q is thus shown from Noether's theorem regarding the invariance of the Lagrangian density under a given continuous transformation. Furthermore, the invariance of the dynamics under this unitary transformation may be incorporated into Heisenberg's equation of motion.

B.2 Quantisation

The system may now be quantised. Firstly, consider a large, cubic region of space of volume V containing the fields. For every periodic mode of momentum $\mathbf{p}$ and positive energy E_p given by

$$E_p = \sqrt{(\mathbf{p} \cdot \mathbf{p})c^2 + m^2c^4} \quad , \quad E_p \geq 0 \tag{B-20}$$

that is inside this 'box' and is constrained to tend to zero at the boundary, the Dirac equation has four independent, plane wave solutions represented by

$$\psi_r^+(x) = K_{\mathbf{p}} u_r(\mathbf{p}) \frac{e^{-ip\cdot x/\hbar}}{\sqrt{V}} \quad \text{and} \quad \psi_r^-(x) = K_{\mathbf{p}} v_r(\mathbf{p}) \frac{e^{+ip\cdot x/\hbar}}{\sqrt{V}} \quad , \quad r = 1, 2 \tag{B-21}$$

where $r = 1, 2$ and $K_{\mathbf{p}}$ is a constant, with $p = p_\mu = (E_p/c, -\mathbf{p})$. The $u_r(\mathbf{p})$ and $v_r(\mathbf{p})$ are constant, four-component spinors, which, together with their adjoints $\bar{u}_r(\mathbf{p}) \equiv u_r^\dagger(\mathbf{p})\gamma^0$ and $\bar{v}_r(\mathbf{p}) \equiv v_r^\dagger(\mathbf{p})\gamma^0$, satisfy

$$\begin{aligned} (\gamma^\mu p_\mu - mc)u_r(\mathbf{p}) &= 0 \quad , \quad (\gamma^\mu p_\mu + mc)v_r(\mathbf{p}) = 0 \\ \bar{u}_r(\mathbf{p})(\gamma^\mu p_\mu - mc) &= 0 \quad , \quad \bar{v}_r(\mathbf{p})(\gamma^\mu p_\mu + mc) = 0 \end{aligned} \tag{B-22}$$

as may be verified by substituting the solutions (B-21) into the original Dirac equation.

The index $r = 1, 2$ labels two distinct solutions for each momentum $\mathbf{p}$; choosing these to be orthogonal, the solutions are ultimately taken to represent the two spin components required for a spin-$\frac{1}{2}$ theory. Thus, the states containing $u_r(\mathbf{p})$ are interpreted as corresponding to positive energy particles of momentum $\mathbf{p}$ (e.g. spin-up and spin-down electrons), whereas the states containing $v_r(\mathbf{p})$ are interpreted as corresponding to negative energy particles (e.g. spin-up and spin-down positrons). Note, however, that the negative energy solutions are also traditionally taken to represent positive energy anti-particles travelling backwards in time, but from the point of view advocated in this thesis it is debateable as to whether this interpretation really makes consistent sense.

The normalisation of the spinors is defined [105] as

$$u_r^\dagger(\mathbf{p})u_r(\mathbf{p}) = v_r^\dagger(\mathbf{p})v_r(\mathbf{p}) = \frac{E_p}{mc^2} \tag{B-23}$$

such that

$$\begin{aligned} u_r^\dagger(\mathbf{p})u_s(\mathbf{p}) &= v_r^\dagger(\mathbf{p})v_s(\mathbf{p}) = \frac{E_p}{mc^2}\delta_{rs} \\ u_r^\dagger(\mathbf{p})v_s(-\mathbf{p}) &= 0 \end{aligned} \tag{B-24}$$

and

$$\begin{aligned} \bar{u}_r(\mathbf{p})u_s(\mathbf{p}) &= -\bar{v}_r(\mathbf{p})v_s(\mathbf{p}) = \delta_{rs} \\ \bar{u}_r(\mathbf{p})v_s(\mathbf{p}) &= \bar{v}_r(\mathbf{p})u_s(\mathbf{p}) = 0. \end{aligned} \tag{B-25}$$

Consider now the expansion of the field $\psi(x)$ into a complete set of plane wave states, that is $\psi(x) = \psi^+(x) + \psi^-(x)$, where $\psi^+(x)$ represents the solutions propagating forwards (i.e. the 'electrons') and $\psi^-(x)$ the solutions propagating backwards (i.e. the 'positrons'). This expansion is given by

$$\psi(x) = \sum_{r=1}^{2} \sum_{\mathbf{p}=0}^{\infty} \left(\frac{mc^2}{E_p}\right)^{1/2} \left[c_r(\mathbf{p}) u_r(\mathbf{p}) \frac{e^{-ip\cdot x/\hbar}}{\sqrt{V}} + d_r^\dagger(\mathbf{p}) v_r(\mathbf{p}) \frac{e^{+ip\cdot x/\hbar}}{\sqrt{V}} \right] \tag{B-26}$$

where the sum is over all possible spin states, r, and all possible momenta, $\mathbf{p}$, noting that this last point leads to some of the divergence problems associated with quantum field theory. The variable[37] $c_r(\mathbf{p}) = c(\mathbf{p}, r)$ provides the amplitude of the $\mathbf{p}^{th}$ contribution of $u_r(\mathbf{p})$ of spin r to $\psi^+(x)$, whilst $d_r^\dagger(\mathbf{p}) = d^\dagger(\mathbf{p}, s)$ similarly provides the amplitude of the $\mathbf{p}^{th}$ contribution of $v_r(\mathbf{p})$ of spin r to $\psi^-(x)$, and both are scaled such that the multiplying constant $\left(mc^2/E_p\right)^{\frac{1}{2}}$ is chosen for later convenience.

In a similar manner, the adjoint field $\bar{\psi}(x)$ may be expanded as

$$\bar{\psi}(x) = \sum_{r,\mathbf{p}} \left(\frac{mc^2}{VE_p}\right)^{1/2} \left[d_r(\mathbf{p})\bar{v}_r(\mathbf{p})e^{-ip\cdot x/\hbar} + c_r^\dagger(\mathbf{p})\bar{u}_r(\mathbf{p})e^{+ip\cdot x/\hbar}\right]. \tag{B-27}$$

Taking now the continuous limit [78], the discrete sum over all momenta in $V \to \infty$ may be replaced by an integral, such that

$$\psi(x) = \int_{-\infty}^{\infty} \frac{d^3p}{(2\pi\hbar)^{3/2}} \sum_{r=1}^{2} \left(\frac{mc^2}{E_p}\right)^{1/2} \left[c_r(\mathbf{p})u_r(\mathbf{p})e^{-ip\cdot x/\hbar} + d_r^\dagger(\mathbf{p})v_r(\mathbf{p})e^{+ip\cdot x/\hbar}\right] \tag{B-28}$$

and similarly for $\bar{\psi}(x)$.

In the standard procedure of quantum field theory, quantisation is achieved by directly quantising the individual harmonic oscillator modes of the field. Thus the field amplitudes $c_r(\mathbf{p})$ and $d_r(\mathbf{p})$ are associated with operators, such that for example the operation $\hat{c}_r^\dagger(\mathbf{p})|0\rangle = |\mathbf{p}_r^u\rangle$ is taken to imply the creation of a positive energy particle $|\mathbf{p}_r^u\rangle$ of 'type' $u_r(\mathbf{p})$ with spin r and momentum $\mathbf{p}$ from the vacuum $|0\rangle$, whilst $\hat{c}_r(\mathbf{p})$ is taken to result in the destruction of this state.

Moreover, different particles can be created by applying different operators, such that for example $\hat{c}_r^\dagger(\mathbf{p})\hat{c}_s^\dagger(\mathbf{p}')|0\rangle = |\mathbf{p}_r^u, \mathbf{p}_s'^u\rangle$ produces the two particle state $|\mathbf{p}_r^u, \mathbf{p}_s'^u\rangle$. Note that the vacuum is defined as the lowest possible 'occupation' of particles, such that $\hat{c}_r(\mathbf{p})|0\rangle = 0$.

Care is needed, however, when applying such an interpretation, because the spin-$\frac{1}{2}$ particles of Dirac theory are physically observed to obey the statistics of Fermi. Paraphrasing, this condition ensures that no two identical particles can exist in the same state, so that $\hat{c}_r^\dagger(\mathbf{p})\hat{c}_r^\dagger(\mathbf{p})|0\rangle = 0$. This constraint leads to the result that $[\hat{c}_r^\dagger(\mathbf{p})]^2 = [\hat{d}_r^\dagger(\mathbf{p})]^2 = 0$, which is ensured by assuming the anticommutation relations for the operators:

$$\begin{aligned}
\{\hat{c}_r^\dagger(\mathbf{p}), \hat{c}_s^\dagger(\mathbf{p}')\} &= \{\hat{c}_r(\mathbf{p}), \hat{c}_s(\mathbf{p}')\} = 0 \\
\{\hat{d}_r^\dagger(\mathbf{p}), \hat{d}_s^\dagger(\mathbf{p}')\} &= \{\hat{d}_r(\mathbf{p}), \hat{d}_s(\mathbf{p}')\} = 0 \\
\{\hat{c}_r(\mathbf{p}), \hat{d}_s(\mathbf{p}')\} &= \{\hat{c}_r(\mathbf{p}), \hat{d}_s^\dagger(\mathbf{p}')\} = 0 \\
\{\hat{c}_r^\dagger(\mathbf{p}), \hat{d}_s(\mathbf{p}')\} &= \{\hat{c}_r^\dagger(\mathbf{p}), \hat{d}_s^\dagger(\mathbf{p}')\} = 0
\end{aligned} \tag{B-29}$$

[37] Noting the interchangeability of the notation between this appendix and Chapter 7, in which subscripts were replaced by bracketed parameters for clarity.

and

$$\{\hat{c}_r(\mathbf{p}), \hat{c}_s^\dagger(\mathbf{p}')\} = \{\hat{d}_r(\mathbf{p}), \hat{d}_s^\dagger(\mathbf{p}')\} = \delta_{rs}\delta_{\mathbf{pp}'}. \tag{B-30}$$

It is beneficial to define two further operators $\hat{N}_r(\mathbf{p})$ and $\overline{\hat{N}}_r(\mathbf{p})$ as

$$\hat{N}_r(\mathbf{p}) = \hat{c}_r^\dagger(\mathbf{p})\hat{c}_r(\mathbf{p}) \quad , \quad \overline{\hat{N}}_r(\mathbf{p}) = \hat{d}_r^\dagger(\mathbf{p})\hat{d}_r(\mathbf{p}). \tag{B-31}$$

$\hat{N}_r(\mathbf{p})$ and $\overline{\hat{N}}_r(\mathbf{p})$ are then seen as analogous to the number operators of the conventional harmonic oscillator, and have product rules evidently given by

$$\begin{aligned} [\hat{N}_r(\mathbf{p})]^2 &= \hat{c}_r^\dagger(\mathbf{p})\hat{c}_r(\mathbf{p})\hat{c}_r^\dagger(\mathbf{p})\hat{c}_r(\mathbf{p}) \\ &= \hat{c}_r^\dagger(\mathbf{p})[1 - \hat{c}_r^\dagger(\mathbf{p})\hat{c}_r(\mathbf{p})]\hat{c}_r(\mathbf{p}) = \hat{N}_r(\mathbf{p}) - 0 \end{aligned} \tag{B-32}$$

and so on.

As noted above, the vacuum state $|0\rangle$ is defined as

$$\hat{c}_r(\mathbf{p})|0\rangle = \hat{d}_r(\mathbf{p})|0\rangle = 0 \tag{B-33}$$

which is equivalent to

$$\psi^+(x)|0\rangle = \bar{\psi}^+(x)|0\rangle = 0. \tag{B-34}$$

B.3 The Hamiltonian Operator

It is now possible to rewrite the Hamiltonian (B-9) in terms of the annihilation and creation operators.

Substituting gives

$$\begin{aligned} \hat{H} &= \int d^3x \left\{ \int_{-\infty}^{\infty} \frac{d^3p}{(2\pi\hbar)^{3/2}} \sum_r \left[\hat{d}_r(\mathbf{p})\bar{v}_r(\mathbf{p})e^{-ip\cdot x/\hbar} + \hat{c}_r^\dagger(\mathbf{p})\bar{u}_r(\mathbf{p})e^{+ip\cdot x/\hbar} \right] \right\} \\ &\quad \times \left(\frac{mc^2}{E_p} \right)^{1/2} (mc^2 - i\hbar c\gamma^j\partial_j) \\ &\quad \left\{ \int_{-\infty}^{\infty} \frac{d^3p'}{(2\pi\hbar)^{3/2}} \sum_s \left(\frac{mc^2}{E_{p'}} \right)^{1/2} \left[\hat{c}_s(\mathbf{p}')u_s(\mathbf{p}')e^{-ip'\cdot x/\hbar} + \hat{d}_s^\dagger(\mathbf{p}')v_s(\mathbf{p}')e^{+ip'\cdot x/\hbar} \right] \right\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d^3p\, d^3p'}{(2\pi\hbar)^3} \sum_r \sum_s \int d^3x \left[\hat{d}_r(\mathbf{p})\bar{v}_r(\mathbf{p})e^{-ip\cdot x/\hbar} + \hat{c}_r^\dagger(\mathbf{p})\bar{u}_r(\mathbf{p})e^{+ip\cdot x/\hbar} \right] \\ &\quad \times \left(\frac{mc^2}{\sqrt{E_pE_{p'}}} \right) (mc^2 - i\hbar c\gamma^j\partial_j) \left[\hat{c}_s(\mathbf{p}')u_s(\mathbf{p}')e^{-ip'\cdot x/\hbar} + \hat{d}_s^\dagger(\mathbf{p}')v_s(\mathbf{p}')e^{+ip'\cdot x/\hbar} \right]. \end{aligned} \tag{B-35}$$

By rearranging the Dirac equation as

$$i\hbar\gamma^i\partial_j\psi(x) = mc\psi(x) - i\hbar\gamma^0\partial_0\psi(x) \tag{B-36}$$

the Laplacian derivatives $c(i\hbar\gamma^j\partial_j)$ may be removed, and (B-35) may be rewritten as

$$\begin{aligned}
\hat{H} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d^3p\ d^3p'}{(2\pi\hbar)^3}\sum_r\sum_s\int d^3x\left(\frac{mc^2}{\sqrt{E_pE_{p'}}}\right)\left[\hat{d}_r(\mathbf{p})\bar{v}_r(\mathbf{p})e^{-ip\cdot x/\hbar}+\hat{c}_r^\dagger(\mathbf{p})\bar{u}_r(\mathbf{p})e^{+ip\cdot x/\hbar}\right]\\
&\quad\times\left(mc^2-mc^2+i\hbar c\gamma^0\frac{\partial}{\partial x^0}\right)\left[\hat{c}_s(\mathbf{p}')u_s(\mathbf{p}')e^{-ip'\cdot x/\hbar}+\hat{d}_s^\dagger(\mathbf{p}')v_s(\mathbf{p}')e^{+ip'\cdot x/\hbar}\right]\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d^3p\ d^3p'}{(2\pi\hbar)^3}\sum_r\sum_s\int d^3x\left(\frac{mc^2}{\sqrt{E_pE_{p'}}}\right)\\
&\quad\times\left[\hat{d}_r(\mathbf{p})\bar{v}_r(\mathbf{p})e^{-ip\cdot x/\hbar}+\hat{c}_r^\dagger(\mathbf{p})\bar{u}_r(\mathbf{p})e^{+ip\cdot x/\hbar}\right]\\
&\quad\times(i\hbar c\gamma^0)\left[(-iE_{p'}/\hbar c)\hat{c}_s(\mathbf{p}')u_s(\mathbf{p}')e^{-ip'\cdot x/\hbar}+(iE_{p'}/\hbar c)\hat{d}_s^\dagger(\mathbf{p}')v_s(\mathbf{p}')e^{+ip'\cdot x/\hbar}\right]\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d^3p\ d^3p'}{(2\pi\hbar)^3}\sum_r\sum_s\int d^3x\left(mc^2\sqrt{\frac{E_{p'}}{E_p}}\right)\\
&\quad\times\left[\hat{d}_r(\mathbf{p})\bar{v}_r(\mathbf{p})e^{-ip\cdot x/\hbar}+\hat{c}_r^\dagger(\mathbf{p})\bar{u}_r(\mathbf{p})e^{+ip\cdot x/\hbar}\right]\\
&\quad\times\left[\gamma^0\hat{c}_s(\mathbf{p}')u_s(\mathbf{p}')e^{-ip'\cdot x/\hbar}-\gamma^0\hat{d}_s^\dagger(\mathbf{p}')v_s(\mathbf{p}')e^{+ip'\cdot x/\hbar}\right].
\end{aligned}\tag{B-37}$$

Clearly, multiplying out the square brackets gives four terms, which in turn become

1.

$$\begin{aligned}
\hat{d}_r(\mathbf{p})\bar{v}_r(\mathbf{p})e^{-ip\cdot x/\hbar}\gamma^0\hat{c}_s(\mathbf{p}')u_s(\mathbf{p}')e^{-ip'\cdot x/\hbar} &= e^{-i(p+p')\cdot x/\hbar}\hat{d}_r(\mathbf{p})\hat{c}_s(\mathbf{p}')\bar{v}_r(\mathbf{p})\gamma^0u_s(\mathbf{p}')\\
&= e^{-i(p+p')\cdot x/\hbar}\hat{d}_r(\mathbf{p})\hat{c}_s(\mathbf{p}')\times 0=0
\end{aligned}\tag{B-38}$$

2.

$$\begin{aligned}
\hat{d}_r(\mathbf{p})\bar{v}_r(\mathbf{p})e^{-ip\cdot x/\hbar}(-\gamma^0\hat{d}_s^\dagger(\mathbf{p}')v_s(\mathbf{p}')e^{+ip'\cdot x/\hbar}) &= -e^{-i(p-p')\cdot x/\hbar}\hat{d}_r(\mathbf{p})\hat{d}_s^\dagger(\mathbf{p}')\bar{v}_r(\mathbf{p})\gamma^0v_s(\mathbf{p}')\\
&= -e^{-i(p-p')\cdot x/\hbar}\hat{d}_r(\mathbf{p})\hat{d}_s^\dagger(\mathbf{p}')v_r^\dagger(\mathbf{p})v_s(\mathbf{p}')
\end{aligned}\tag{B-39}$$

3.

$$\begin{aligned}
\hat{c}_r^\dagger(\mathbf{p})\bar{u}_r(\mathbf{p})e^{+ip\cdot x/\hbar}\gamma^0\hat{c}_s(\mathbf{p}')u_s(\mathbf{p}')e^{-ip'\cdot x/\hbar} &= e_r^{+i(p-p')\cdot x/\hbar}\hat{c}_r^\dagger(\mathbf{p})\hat{c}_s(\mathbf{p}')\bar{u}_r(\mathbf{p})\gamma^0u_s(\mathbf{p}')\\
&= e^{+i(p-p')\cdot x/\hbar}\hat{c}_r^\dagger(\mathbf{p})\hat{c}_s(\mathbf{p}')u_r^\dagger(\mathbf{p})u_s(\mathbf{p}')
\end{aligned}\tag{B-40}$$

4.

$$\begin{aligned}
\hat{c}_r^\dagger(\mathbf{p})\bar{u}_r(\mathbf{p})e^{+ip\cdot x/\hbar}(-\gamma^0\hat{d}_s^\dagger(\mathbf{p}')v_s(\mathbf{p}')e^{+ip'\cdot x/\hbar}) &= -e^{+i(p+p')\cdot x/\hbar}\hat{c}_r^\dagger(\mathbf{p})\hat{d}_s^\dagger(\mathbf{p}')\bar{u}_r(\mathbf{p})\gamma^0v_s(\mathbf{p}')\\
&= -e^{+i(p+p')\cdot x/\hbar}\hat{c}_r^\dagger(\mathbf{p})\hat{d}_s^\dagger(\mathbf{p}')\times 0=0
\end{aligned}\tag{B-41}$$

where the simplification $p' = p$ has been made in "1." and "4." without affecting the outcome, in anticipation of the assumed orthogonality of the terms in the Fourier expansion as given below and the appearance of the Dirac delta $\delta_{\mathbf{pp}'}$ in (B-30), and using [106]

$$\begin{aligned} u_r^\dagger(\mathbf{p})v_s(\mathbf{p}) &= 0 \\ v_r^\dagger(\mathbf{p})u_s(\mathbf{p}) &= 0. \end{aligned} \tag{B-42}$$

Substituting, (B-37) evidently becomes

$$\begin{aligned} \hat{H} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d^3p\ d^3p'}{(2\pi\hbar)^3} \sum_r \sum_s \int d^3x \left(mc^2 \sqrt{\frac{E_{p'}}{E_p}} \right) \\ &\quad \times \left[\begin{array}{c} 0 - e^{-i(p-p')\cdot x/\hbar} \hat{d}_r(\mathbf{p}) \hat{d}_s^\dagger(\mathbf{p}') v_r^\dagger(\mathbf{p}) v_s(\mathbf{p}') \\ + e^{+i(p-p')\cdot x/\hbar} \hat{c}_r^\dagger(\mathbf{p}) \hat{c}_s(\mathbf{p}') u_r^\dagger(\mathbf{p}) u_s(\mathbf{p}') + 0 \end{array} \right] \end{aligned} \tag{B-43}$$

and taking the Fourier transform of the exponential over d^3x to give a Dirac delta function, it follows that

$$\begin{aligned} \hat{H} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^3p\ d^3p' \sum_r \sum_s \delta_{pp'} \left(mc^2 \sqrt{\frac{E_{p'}}{E_p}} \right) \\ &\quad \times \left[\hat{c}_r^\dagger(\mathbf{p}) \hat{c}_s(\mathbf{p}') u_r^\dagger(\mathbf{p}) u_s(\mathbf{p}') - \hat{d}_r(\mathbf{p}) \hat{d}_s^\dagger(\mathbf{p}') v_r^\dagger(\mathbf{p}) v_s(\mathbf{p}') \right]. \end{aligned} \tag{B-44}$$

Furthermore, from the normalisation of the spinors defined in (B-24) the above relationship becomes

$$\begin{aligned} \hat{H} &= \int_{-\infty}^{\infty} d^3p \ \sum_r \sum_s \left(mc^2 \sqrt{\frac{E_p}{E_p}} \right) \left[\hat{c}_r^\dagger(\mathbf{p}) \hat{c}_s(\mathbf{p}) \left(\frac{E_p}{mc^2} \right) \delta_{rs} - \hat{d}_r(\mathbf{p}) \hat{d}_s^\dagger(\mathbf{p}) \left(\frac{E_p}{mc^2} \right) \delta_{rs} \right] \\ &= \int_{-\infty}^{\infty} d^3p \ \sum_r E_p \left[\hat{c}_r^\dagger(\mathbf{p}) \hat{c}_s(\mathbf{p}) - \hat{d}_r(\mathbf{p}) \hat{d}_s^\dagger(\mathbf{p}) \right]. \end{aligned} \tag{B-45}$$

This last equation can be rearranged, by making use of the anti-commutation relations $\{\hat{d}_r(\mathbf{p}), \hat{d}_r^\dagger(\mathbf{p}')\} = \delta_{rs}\delta_{\mathbf{pp}'}$ of the ladder operators, to give

$$\hat{H} = \int_{-\infty}^{\infty} d^3p \sum_r E_p \left[\hat{c}_r^\dagger(\mathbf{p}) \hat{c}_r(\mathbf{p}) + \hat{d}_r^\dagger(\mathbf{p}) \hat{d}_r(\mathbf{p}) - 1 \right]. \tag{B-46}$$

Equation (B-46) evidently contains two terms involving operators and a constant term of the form $\left(\int_{-\infty}^{\infty} d^3p \sum_r E_p \right)$. Moreover, because of the integral this constant provides an infinite contribution to the Hamiltonian. To 'overcome' this problem the constant is, perhaps rather dubiously, ignored in conventional quantum field theory by arguing that only relative differences in energy are observable, such that the vacuum is consequently instilled with a non-zero expectation value. Whilst it is not intended to fully discuss the issue here, it is noted that this property leads to all sorts of interpretational difficulties when considering general relativity.

The above procedure is implied by adopting the 'Normal Order' convention, denoted by $\mathcal{N}$, in which it is assumed that all anti-commutators vanish, such that in a product every creation operator is placed to the left of the absorption operators. Thus, the normal ordered Hamiltonian may be rewritten as

$$\begin{aligned}\hat{H} &= \int_{-\infty}^{\infty} d^3p \sum_r E_p \mathcal{N}[\hat{c}_r^\dagger(\mathbf{p})\hat{c}_r(\mathbf{p}) - \hat{d}_r(\mathbf{p})\hat{d}_r^\dagger(\mathbf{p})] \\ &= \int_{-\infty}^{\infty} d^3p \sum_r E_p \left[\hat{c}_r^\dagger(\mathbf{p})\hat{c}_r(\mathbf{p}) + \hat{d}_r^\dagger(\mathbf{p})\hat{d}_r(\mathbf{p})\right].\end{aligned} \tag{B-47}$$

Finally, by making use of the number operators defined in (B-31), the Hamiltonian becomes

$$\hat{H} = \int_{-\infty}^{\infty} d^3p \sum_r E_p[\hat{N}_r(\mathbf{p}) + \overline{\hat{N}}_r(\mathbf{p})]. \tag{B-48}$$

B.4 The Momentum Operator

The momentum operator $\mathbf{P}$ of (B-11) can also be investigated. Substituting into this expression the plane wave expansions for $\psi(x)$ and $\bar{\psi}(x)$ gives

$$\begin{aligned}\hat{\mathbf{P}} &= -\int d^3x[i\hbar\psi^\dagger(x)\partial_j\psi(x)] = -\int d^3x[i\hbar\bar{\psi}(x)\gamma^0\partial_j\psi(x)] \\ &= -\int d^3x \int_{-\infty}^{\infty} \frac{d^3p}{(2\pi\hbar)^{3/2}} \sum_r i\hbar \left[\left(\frac{mc^2}{E_p}\right)^{1/2} \left[\hat{d}_r(\mathbf{p})\bar{v}_r(\mathbf{p})\gamma^0 e^{-ip\cdot x/\hbar} + \hat{c}_r^\dagger(\mathbf{p})\bar{u}_r(\mathbf{p})\gamma^0 e^{+ip\cdot x/\hbar}\right]\right] \\ &\quad \times\partial_j \left[\int_{-\infty}^{\infty} \frac{d^3p'}{(2\pi\hbar)^{3/2}} \sum_s \left(\frac{mc^2}{E_{p'}}\right)^{1/2} \left[\hat{c}_s(\mathbf{p}')u_s(\mathbf{p}')e^{-ip'\cdot x/\hbar} + \hat{d}_s^\dagger(\mathbf{p}')v_s(\mathbf{p}')e^{+ip'\cdot x/\hbar}\right]\right].\end{aligned} \tag{B-49}$$

So,

$$\begin{aligned}\hat{\mathbf{P}} &= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d^3p\ d^3p'}{(2\pi\hbar)^3} \sum_r \sum_s \int d^3x(i\hbar) \left(\frac{mc^2}{\sqrt{E_pE_{p'}}}\right) \\ &\quad \times \left[\hat{d}_r(\mathbf{p})v_r^\dagger(\mathbf{p})\gamma^0\gamma^0 e^{-ip\cdot x/\hbar} + \hat{c}_r^\dagger(\mathbf{p})u_r^\dagger(\mathbf{p})\gamma^0\gamma^0 e^{+ip\cdot x/\hbar}\right] \\ &\quad \times \left[(i\mathbf{p}'/\hbar)\hat{c}_s(\mathbf{p}')u_s(\mathbf{p}')e^{-ip'\cdot x/\hbar} + (-i\mathbf{p}'/\hbar)\hat{d}_s^\dagger(\mathbf{p}')v_s(\mathbf{p}')e^{+ip'\cdot x/\hbar}\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d^3p\ d^3p'}{(2\pi\hbar)^3} \sum_r \sum_s \int d^3x \left(\frac{mc^2}{\sqrt{E_pE_{p'}}}\right)\mathbf{p}' \\ &\quad \times \left[\hat{d}_r(\mathbf{p})v_r^\dagger(\mathbf{p})e^{-ip\cdot x/\hbar} + \hat{c}_r^\dagger(\mathbf{p})u_r^\dagger(\mathbf{p})e^{+ip\cdot x/\hbar}\right] \\ &\quad \times \left[\hat{c}_s(\mathbf{p}')u_s(\mathbf{p}')e^{-ip'\cdot x/\hbar} - \hat{d}_s^\dagger(\mathbf{p}')v_s(\mathbf{p}')e^{+ip'\cdot x/\hbar}\right].\end{aligned} \tag{B-50}$$

As before, the product may be expanded into four terms:

1.

$$\begin{aligned}\hat{d}_r(\mathbf{p})v_r^\dagger(\mathbf{p})e^{-ip\cdot x/\hbar}\hat{c}_s(\mathbf{p}')u_s(\mathbf{p}')e^{-ip'\cdot x/\hbar} &= e^{-i(p+p')\cdot x/\hbar}\hat{d}_r(\mathbf{p})\hat{c}_s(\mathbf{p}')v_r^\dagger(\mathbf{p})u_s(\mathbf{p}') \\ &= e^{-i(p+p')\cdot x/\hbar}\hat{d}_r(\mathbf{p})\hat{c}_s(\mathbf{p}')\times 0 = 0\end{aligned} \tag{B-51}$$

2.

$$\hat{d}_r(\mathbf{p})v_r^\dagger(\mathbf{p})e^{-ip\cdot x/\hbar}(-d_s^\dagger(\mathbf{p}')v_s(\mathbf{p}')e^{+ip'\cdot x/\hbar}) = -e^{-i(p-p')\cdot x/\hbar}\hat{d}_r(\mathbf{p})\hat{d}_s^\dagger(\mathbf{p}')v_r^\dagger(\mathbf{p})v_s(\mathbf{p}') \tag{B-52}$$

3.

$$\hat{c}_r^\dagger(\mathbf{p})u_r^\dagger(\mathbf{p})e^{+ip\cdot x/\hbar}\hat{c}_s(\mathbf{p}')u_s(\mathbf{p}')e^{-ip'\cdot x/\hbar} = e^{+i(p-p')\cdot x/\hbar}\hat{c}_r^\dagger(\mathbf{p})\hat{c}_s(\mathbf{p}')u_r^\dagger(\mathbf{p})u_s(\mathbf{p}') \tag{B-53}$$

4.

$$\begin{aligned}\hat{c}_r^\dagger(\mathbf{p})u_r^\dagger(\mathbf{p})e^{+ip\cdot x/\hbar}(-\hat{d}_s^\dagger(\mathbf{p}')v_s(\mathbf{p}')e^{+ip'\cdot x/\hbar}) &= -e^{+i(p+p')\cdot x/\hbar}\hat{c}_r^\dagger(\mathbf{p})\hat{d}_s^\dagger(\mathbf{p}')u_r^\dagger(\mathbf{p})v_s(\mathbf{p}') \\ &= -e^{+i(p+p')\cdot x/\hbar}\hat{c}_r^\dagger(\mathbf{p})\hat{d}_s^\dagger(\mathbf{p}')\times 0 = 0\end{aligned} \tag{B-54}$$

So, $\mathbf{P}$ becomes

$$\begin{aligned}\hat{\mathbf{P}} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d^3p\;d^3p'}{(2\pi\hbar)^3}\sum_r\sum_s\int d^3x\left(\frac{mc^2}{\sqrt{E_pE_{p'}}}\right)\mathbf{p}' \\ &\quad\times\begin{bmatrix} 0 - e^{-i(p-p')\cdot x/\hbar}\hat{d}_r(\mathbf{p})\hat{d}_s^\dagger(\mathbf{p}')v_r^\dagger(\mathbf{p})v_s(\mathbf{p}') \\ +e^{+i(p-p')\cdot x/\hbar}\hat{c}_r^\dagger(\mathbf{p})\hat{c}_s(\mathbf{p}')u_r^\dagger(\mathbf{p})u_s(\mathbf{p}') + 0\end{bmatrix}\end{aligned} \tag{B-55}$$

which may be Fourier transformed to give

$$\begin{aligned}\hat{\mathbf{P}} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d^3p\;d^3p'\sum_r\sum_s\left(\frac{mc^2}{\sqrt{E_pE_{p'}}}\right)\mathbf{p}'\delta_{pp'} \\ &\quad\times\left[\hat{c}_r^\dagger(\mathbf{p})\hat{c}_s(\mathbf{p}')u_r^\dagger(\mathbf{p})u_s(\mathbf{p}') - \hat{d}_r(\mathbf{p})\hat{d}_s^\dagger(\mathbf{p}')v_r^\dagger(\mathbf{p})v_s(\mathbf{p}')\right].\end{aligned} \tag{B-56}$$

Using again the normalisation of the spinors

$$\hat{\mathbf{P}} = \int_{-\infty}^{\infty}d^3p\;\sum_r\sum_s\left(\frac{mc^2}{E_p}\right)\mathbf{p}\left[\hat{c}_r^\dagger(\mathbf{p})\hat{c}_s(\mathbf{p})\left(\frac{E_p}{mc^2}\right) - \hat{d}_r(\mathbf{p})\hat{d}_s^\dagger(\mathbf{p})\left(\frac{E_p}{mc^2}\right)\right]\delta_{rs}$$

and with the anti-commutation relations of the ladder operators, this may be written in the normal order convention as

$$\begin{aligned}\hat{\mathbf{P}} &= \int_{-\infty}^{\infty}d^3p\;\sum_r\mathbf{p}\mathcal{N}[\hat{c}_r^\dagger(\mathbf{p})\hat{c}_r(\mathbf{p}) - \hat{d}_r(\mathbf{p})\hat{d}_r^\dagger(\mathbf{p})] \\ &= \int_{-\infty}^{\infty}d^3p\sum_r\mathbf{p}[\hat{c}_r^\dagger(\mathbf{p})\hat{c}_r(\mathbf{p}) + \hat{d}_r^\dagger(\mathbf{p})\hat{d}_r(\mathbf{p})].\end{aligned} \tag{B-57}$$

Lastly, expressing this as a sum of number operators gives

$$\hat{\mathbf{P}} = \int_{-\infty}^{\infty} d^3p \sum_r \mathbf{p} \left[\hat{N}_r(\mathbf{p}) + \overline{\hat{N}}_r(\mathbf{p}) \right]. \tag{B-58}$$

B.5 The Charge Operator

Finally, the charge operator $\hat{Q}$ may also be evaluated. From equation (B-13), $\hat{Q}$ becomes

$$\begin{aligned} \hat{Q} &= q \int \psi^\dagger(x)\psi(x)\ d^3x \qquad \text{(B-59)} \\ &= q \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d^3p\ d^3p'}{(2\pi\hbar)^3} \sum_{r=1}^{2} \sum_{s=1}^{2} \int d^3x \left(\frac{mc^2}{\sqrt{E_p E_{p'}}} \right) \\ &\quad \times \left[d_r(\mathbf{p}) \bar{v}_r(\mathbf{p}) e^{-ip\cdot x/\hbar} + c_r^\dagger(\mathbf{p}) \bar{u}_r(\mathbf{p}) e^{+ip\cdot x/\hbar} \right] \\ &\quad \times \gamma^0 \left[c_s(\mathbf{p}') u_s(\mathbf{p}') e^{-ip'\cdot x/\hbar} + d_s^\dagger(\mathbf{p}') v_s(\mathbf{p}') e^{+ip'\cdot x/\hbar} \right] \end{aligned}$$

which again gives four terms upon multiplication:

1.

$$\begin{aligned} \hat{d}_r(\mathbf{p}) \bar{v}_r(\mathbf{p}) e^{-ip\cdot x/\hbar} \gamma^0 \hat{c}_s(\mathbf{p}') u_s(\mathbf{p}') e^{-ip'\cdot x/\hbar} &= e^{-i(p+p')\cdot x/\hbar} \hat{d}_r(\mathbf{p}) \hat{c}_s(\mathbf{p}') v_r^\dagger(\mathbf{p}) u_s(\mathbf{p}') \\ &= e^{-i(p+p')\cdot x/\hbar} \hat{d}_r(\mathbf{p}) \hat{c}_s(\mathbf{p}') \times 0 = 0 \end{aligned} \tag{B-60}$$

2.

$$\hat{d}_r(\mathbf{p}) \bar{v}_r(\mathbf{p}) e^{-ip\cdot x/\hbar} \gamma^0 d_s^\dagger(\mathbf{p}') v_s(\mathbf{p}') e^{+ip'\cdot x/\hbar} = e^{-i(p-p')\cdot x/\hbar} \hat{d}_r(\mathbf{p}) \hat{d}_s^\dagger(\mathbf{p}') v_r^\dagger(\mathbf{p}) v_s(\mathbf{p}') \tag{B-61}$$

3.

$$\hat{c}_r^\dagger(\mathbf{p}) \bar{u}_r(\mathbf{p}) e^{+ip\cdot x/\hbar} \gamma^0 \hat{c}_s(\mathbf{p}') u_s(\mathbf{p}') e^{-ip'\cdot x/\hbar} = e^{+i(p-p')\cdot x/\hbar} \hat{c}_r^\dagger(\mathbf{p}) \hat{c}_s(\mathbf{p}') u_r^\dagger(\mathbf{p}) u_s(\mathbf{p}') \tag{B-62}$$

4.

$$\begin{aligned} \hat{c}_r^\dagger(\mathbf{p}) \bar{u}_r(\mathbf{p}) e^{+ip\cdot x/\hbar} \gamma^0 \hat{d}_s^\dagger(\mathbf{p}') v_s(\mathbf{p}') e^{+ip'\cdot x/\hbar} &= e^{+i(p+p')\cdot x/\hbar} \hat{c}_r^\dagger(\mathbf{p}) \hat{d}_s^\dagger(\mathbf{p}') u_r^\dagger(\mathbf{p}) v_s(\mathbf{p}') \\ &= e^{+i(p+p')\cdot x/\hbar} \hat{c}_r^\dagger(\mathbf{p}) \hat{d}_s^\dagger(\mathbf{p}') \times 0 = 0 \end{aligned} \tag{B-63}$$

So, $\hat{Q}$ becomes

$$\begin{aligned}\hat{Q} &= q\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d^3p\;d^3p'}{(2\pi\hbar)^3}\sum_{r=1}^{2}\sum_{s=1}^{2}\int d^3x\left(\frac{mc^2}{\sqrt{E_pE_{p'}}}\right) \\ &\quad\times\left[\begin{array}{c} 0+e^{-i(p-p')\cdot x/\hbar}\hat{d}_r(\mathbf{p})\hat{d}_s^{\dagger}(\mathbf{p}')v_r^{\dagger}(\mathbf{p})v_s(\mathbf{p}') \\ +e^{+i(p-p')\cdot x/\hbar}\hat{c}_r^{\dagger}(\mathbf{p})\hat{c}_s(\mathbf{p}')u_r^{\dagger}(\mathbf{p})u_s(\mathbf{p}')+0\end{array}\right]\end{aligned} \tag{B-64}$$

which is Fourier transformed as

$$\begin{aligned}\hat{Q} &= q\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d^3p\;d^3p'\sum_{r=1}^{2}\sum_{s=1}^{2}\left(\frac{mc^2}{\sqrt{E_pE_{p'}}}\right)\delta_{pp'} \\ &\quad\times\left[\hat{d}_r(\mathbf{p})\hat{d}_s^{\dagger}(\mathbf{p}')v_r^{\dagger}(\mathbf{p})v_s(\mathbf{p}')+\hat{c}_r^{\dagger}(\mathbf{p})\hat{c}_s(\mathbf{p}')u_r^{\dagger}(\mathbf{p})u_s(\mathbf{p}')\right]\end{aligned} \tag{B-65}$$

and, upon substituting the spinor normalisation, gives

$$\hat{Q}=q\int_{-\infty}^{\infty}d^3p\sum_{r=1}^{2}\sum_{s=1}^{2}\left(\frac{mc^2}{E_p}\right)\left[\hat{d}_r(\mathbf{p})\hat{d}_s^{\dagger}(\mathbf{p})\left(\frac{E_p}{mc^2}\right)+\hat{c}_r^{\dagger}(\mathbf{p})\hat{c}_s(\mathbf{p}')\left(\frac{E_p}{mc^2}\right)\right]\delta_{rs}. \tag{B-66}$$

Thus, applying the anti-commutation relations of the ladder operators, $\hat{Q}$ may be written in normal order as

$$\begin{aligned}\hat{Q} &= q\int_{-\infty}^{\infty}d^3p\sum_{r=1}^{2}\mathcal{N}\left[\hat{d}_r(\mathbf{p})\hat{d}_r^{\dagger}(\mathbf{p})+\hat{c}_r^{\dagger}(\mathbf{p})\hat{c}_r(\mathbf{p}')\right] \\ &= q\int_{-\infty}^{\infty}d^3p\sum_{r=1}^{2}\left[-\hat{d}_r^{\dagger}(\mathbf{p})\hat{d}_r(\mathbf{p})+\hat{c}_r^{\dagger}(\mathbf{p})\hat{c}_r(\mathbf{p}')\right].\end{aligned} \tag{B-67}$$

So, charge is defined as

$$\hat{Q}=q\int_{-\infty}^{\infty}d^3p\sum_{r=1}^{2}\left[\hat{N}_r(\mathbf{p})-\overline{\hat{N}}_r(\mathbf{p})\right]. \tag{B-68}$$

Clearly, the commutation relation

$$[\hat{Q},\hat{H}]=0 \tag{B-69}$$

vanishes due to the relations (B-29), such that $\frac{\partial\hat{Q}}{\partial t}=0$ as expected from Heisenberg's equation.

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