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# On the completability of incomplete orthogonal Latin rectangles

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**Abstract**

We address the problem of completability for 2-row orthogonal Latin rectangles ($OLR2$). Our approach is to identify all pairs of incomplete 2-row Latin rectangles that are not completable to an $OLR2$ and are minimal with respect to this property; i.e., we characterize all circuits of the independence system associated with $OLR2$. Since there can be no polytime algorithm generating the clutter of circuits of an arbitrary independence system, our work adds to the few independence systems for which that clutter is fully described. The result has a direct polyhedral implication; it gives rise to inequalities that are valid for the polytope associated with orthogonal Latin squares and thus planar multi-dimensional assignment. A complexity result is also at hand: completing a set of $(n-1)$ incomplete $MOLR2$ is $\mathcal{NP}$-complete.

## 1 Introduction

An $m$-row Latin rectangle $R$ of order $n$ is an $m \times n$ array where $m < n$, in which each value $1, ..., n$ appears exactly once in every row and at most once in every column [12]. For $m = n$, the above defines a *Latin square*, where each value $1, ..., n$ appears exactly once in every row and column. We call a Latin rectangle *normalized* if values $1, ..., n$ occur in the first row in natural order. Counting Latin rectangles is a topic broadly studied in combinatorics; some examples listed in chronological order are [16], [4], [7], [13] and [18].

**Definition 1** *Two $m$-row Latin rectangles of order $n$, with $m < n$, form an orthogonal pair (OLR) if and only if when superimposed each of the $n^2$ ordered pairs of values* $(1, 1), (1, 2), ..., (n, n)$ *appears at most once.*

An example of a normalized 2-row OLR ($OLR2$) of order 4 is shown in Table 1. Also note that for $m = n$ we have the case of *orthogonal Latin squares (OLS)* where each of the $n^2$ ordered pairs of values $(1, 1), (1, 2), ..., (n, n)$ appears exactly once when the two squares are superimposed.

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| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | | 1 | 2 | 3 | 4 |
| 2 | 3 | 4 | 1 | | 3 | 4 | 1 | 2 |

Table 1: An $OLR2$ of order 4

The definition for OLR naturally extends to a set $T$ of $m$-row Latin rectangles of order $n$, which are called *mutually orthogonal Latin rectangles (MOLR)*, if and only if all Latin rectangles are pairwise orthogonal. Note that for $m = n$ we have the case of of mutually orthogonal Latin squares (MOLS). Here, we are interested only in 2-row Latin rectangles (i.e., $MOLR2$). Hence, unless otherwise stated, whenever we refer to Latin rectangles we imply that they have two rows.

Latin rectangles and OLRs enjoy a close relationship to several areas of combinatorics like design theory and projective geometry (e.g., see [12] and references therein). Beyond that, they have recently received additional attention because of some quite important applications:

- [15] and [22] introduce the concept of physical layer network coding which has developed in to a sub-field of network coding with new results in the domains of wireless communication, wireless information theory and wireless networking. One branch of this new field works with de-noise-and-forward-protocol in the network coding maps that satisfy a requirement called the 'exclusive law', which reduces the impact of multiple access interference. In [21] it is established that the network coding maps that satisfy the 'exclusive law' are obtainable by the completion of incomplete Latin rectangles. Isotopic and transposed Latin squares are also used to create network coding maps with particular desirable characteristics.

- Fibre-optic signal processing techniques [17] deliver multi-access optical networks for fibre-optic communications. Relevant to that, an Optical Orthogonal Code (OOC) is a family of $(0, 1)$ sequences with good auto and cross-correlation properties, i.e., fast and low interference transmission properties. In [3] the authors propose two new coding schemes capable of cancelling the multi-user interference for certain systems based on MOLR and MOLS to accomplish large flexibility in choosing number of users, simplicity of construction and suitability to all important transmission technologies.

- LDPC codes are the lead technology used in hard disk drive read channels, wireless 10-GB, DVB-S2 and more recently in flash SSD as well as in communicating with space probes. Pseudo-random approaches and combinatorial approaches are the two main techniques for the construction of a specific LDPC code, based on finite geometries and first studied in [9]. In [20] and [10] a different construction is devised, based on balanced incomplete block designs constructed from $MOLR$ and $MOLS$.

In this paper, after establishing that $MOLR$ completion is $\mathcal{NP}$-complete, we address the problem of completability for $OLR2$. To achieve this, it suffices to characterize all pairs of incomplete 2-row Latin rectangles that are not completable to an $OLR2$. Minimal such pairs define *circuits* of the *independence system* ($IS$) associated with $OLR2$ of order $n$ (formal definitions

| | $j_1$ | $j_2$ | $j_3$ | $j_4$ | | $j_1$ | $j_2$ | $j_3$ | $j_4$ |
|---|---|---|---|---|---|---|---|---|---|
| $i_1$ | $k_1^1$ | $k_2^1$ | $k_3^1$ | $k_4^1$ | | $k_1^2$ | $k_2^2$ | $k_3^2$ | $k_4^2$ |
| $i_2$ | $k_2^1$ | $k_3^1$ | $k_4^1$ | $k_1^1$ | | $k_3^2$ | $k_4^2$ | $k_1^2$ | $k_2^2$ |
| | $R_1$ | | | | | $R_2$ | | | |

Table 2: The $OLR2$ of Table 1

appear in the next section). In this system, a pair of incomplete 2-row Latin rectangles is independent if and only if it is completable to an $OLR2$ or, equivalently, it contains no circuits. Notably, the circuits for the IS associated with 2-row Latin rectangles have been described in [6]; this description has been based on the notion of an availability matrix, which is also employed here to provide a concise proof, despite the enumerative nature of our exposition.

Since there can be no polytime algorithm (unless $\mathcal{P} = \mathcal{NP}$) generating the clutter of either bases or circuits of an arbitrary IS [19], our work adds to the (few) independence systems in the literature for which the clutter of circuits is fully characterized (see [5], [19] and [11]). The results presented here have some polyhedral implications that are also discussed, namely they directly give rise to *lifted circuit inequalities* for the polytope associated with both $OLR$ and $OLS$. Finally, our approach could be useful for the characterization of both circuits and associated inequalities for $MOLR, MOLS$ and possibly other highly symmetric combinatorial problems.

The remainder of this paper is organized as follows. In Section 2, we introduce our notation and present some initial results, including the complexity of $MOLR$ completion. After reviewing the results of [6], along with counting the circuits associated with the completion of 2-row (single) Latin rectangles in Section 3, we present our main contribution in Section 4. In Section 5, we discuss the implications of our work regarding the polytope associated with orthogonal Latin squares and thus planar multi-dimensional assignment [1]. We conclude in Section 6 with ideas for future work. The proofs of some intermediate results appear in the Appendix.

## 2 Notation and basic results

Let us introduce our notation. For a given order $n$, let $T = \{1, ..., |T|\}$, where $|T| \leq (n-1)$ is the number of $MOLR2$. The two sets $I = \{i_1, i_2\}$ and $J = \{j_1, \ldots, j_n\}$ correspond to the rows and columns of each Latin rectangle, while the $|T|$ disjoint sets $K_t = \{k_1^t, \ldots, k_n^t\}$ $(t \in T)$ define the $n$ elements appearing in the $t^{th}$ $MOLR2$. Define $G_t = I \times J \times K_t$, $t \in T$, i.e., each $G_t$ contains $2n^2$ triples and $\bigcup_{t \in T} G_t$ contains $2|T|\,n^2$ triples. Based on this notation, Table 1 is revised in Table 2, while a Latin rectangle can be represented as an $R_t \subseteq G_t$, e.g., $R_1$ in Table 2 can be written as

$$R_1 = \{(i_1, j_1, k_1^1), (i_1, j_2, k_2^1), (i_1, j_3, k_3^1), (i_1, j_4, k_4^1), (i_2, j_1, k_2^1), (i_2, j_2, k_3^1), (i_2, j_3, k_4^1), (i_2, j_4, k_1^1)\}$$

Similarly, an $OLR2$ is represented as $R_t \cup R_{t'}$, where $R_t \subseteq G_t, R_{t'} \subseteq G_{t'}$ and $\{t, t'\} \subseteq T$; for example the $OLR2$ of Table 2 is represented as $R_1 \cup R_2$.

We call two $MOLR2$ *equivalent*, if one is obtainable from the other by permuting the sets

| | $j_1$ | $j_2$ | $j_3$ | $j_4$ |
|---|---|---|---|---|
| $i_1$ | | $k_1^1$ | $k_3^1$ | $k_4^1$ |
| $i_2$ | $k_2^1$ | $k_4^1$ | $k_1^1$ | $k_3^1$ |

$R_{1-}$

| | $j_1$ | $j_2$ | $j_3$ | $j_4$ |
|---|---|---|---|---|
| $i_1$ | | $k_1^1$ | $k_3^1$ | $k_2^1$ |
| $i_2$ | $k_1^1$ | $k_3^1$ | | $k_4^1$ |

$R_{2-}$

Table 3: A pair of incomplete Latin rectangles

$I, J, K_1, ..., K_{|T|}$ and $T$. Note that directly interchanging the roles of sets $I$ and $J$ is not allowed since we stipulate the existence of only 2 rows but $n$ columns; the same applies to the interchange of $I$ or $J$ with a set $K_t, t \in T$. In contrast, interchanging two elements $t, t' \in T$ swaps the roles of sets $K_t$ and $K_{t'}$.

Let us now define the independence systems (IS) associated with $MOLR2$. For $|T| = 1$, every $B \subseteq G_1$ that forms a 2-row Latin rectangle is called a *basis*. Hence the clutter of bases $\mathcal{B}_1 = \{B \subseteq G_1 : B \text{ is a 2-row Latin rectangle}\}$; i.e., $|B| = 2n$ for all $B \in \mathcal{B}_1$. Accordingly, the 2-row Latin rectangle IS is $\mathcal{S}_1 = (G_1, \mathcal{J}_1)$ where $\mathcal{J}_1 = \{X \subseteq G_1 : X \subseteq B, B \in \mathcal{B}_1\}$. As usual, $\mathcal{S}_1$ induces the unique clutter of circuits defined as

$$\mathcal{C}_1 = \{C \subseteq G_1 : C \notin \mathcal{J}_1, C \backslash \{c\} \in \mathcal{J}_1 \text{ for all } c \in C\}.$$

The IS associated with $MOLR2$, denoted as $\mathcal{S}_{|T|}$, is again defined in terms of the clutter of bases $\mathcal{B}_{|T|}$ (each basis corresponding to an $MOLR2$) and induces the clutter of circuits $\mathcal{C}_{|T|}$, i.e., the set of all $C$ that are not contained in any $B \in \mathcal{B}_{|T|}$ (exclusion property) but all their subsets are (minimality property); $\mathcal{C}_{|T|}$ is also known as the set of minimal dependent subsets of the *ground set* $\bigcup_{t \in T} G_t$.

An *incomplete* Latin rectangle (also called *partial* Latin rectangle) is an $m \times n$ array (with $m < n$) whose cells receive values 1 to $n$ but may also be empty; in our notation, such a rectangle is represented by an $R_- \subset G_1$. An incomplete Latin rectangle is called *completable* if there exists $R'_- \subset G_1$ such that $(R_- \cup R'_-) \in \mathcal{B}_1$ and *incompletable* otherwise. The incomplete Latin rectangle $R_{1-}$ in Table 3 is incompletable since the only value allowed for the empty cell $(i_1, j_1)$ is $k_2^1$, which violates the Latin rectangle structure as it appears twice in column $j_1$. In contrast, the incomplete Latin rectangle $R_{2-}$ is completable since for $R'_{2-} = \{(i_1, j_1, k_4^1), (i_2, j_3, k_2^1)\}$ it holds that $R_{2-} \cup R'_{2-} \in \mathcal{B}_1$.

In a similar fashion, a set of $|T|$ *incomplete* Latin rectangles $\{R_{t-}, t \in T\}$ is completable if and only if there are $\{R'_{t-}, t \in T\}$ such that $\bigcup_{t \in T} \left(R_{t-} \cup R'_{t-}\right) \in \mathcal{B}_{|T|}$, i.e., if these rectangles can be completed to an $MOLR2$. Equivalence applies to (sets of) incomplete rectangles exactly as for $MOLR2$.

Clearly, $\{R_{t-}, t \in T\}$ being completable implies that it is a subset of some $B \in \mathcal{B}_{|T|}$ thus containing no circuits. Equivalently, if incompletable and therefore contained in no $B \in \mathcal{B}_{|T|}$, it contains some circuit $C \in \mathcal{C}_{|T|}$. Hence the following.

**Proposition 2** *An set of incomplete Latin rectangles $\{R_{t-}, t \in T\}$ is completable to an $MOLR2$ if and only if $C \backslash \left(\bigcup_{t \in T} R_{t-}\right) \neq \emptyset$ for all $C \in \mathcal{C}_{|T|}$.*

| | $j_1$ | $j_2$ | $j_3$ | $j_4$ |
|---|---|---|---|---|
| $i_1$ | | $k_2^t$ | | $k_4^t$ |
| $i_2$ | | $k_4^t$ | $k_1^t$ | $k_2^t$ |

$R_-^t$

Table 4: An incomplete Latin rectangle

Notice, that any $C \in \mathcal{C}_{|T|}$ is itself a set of incomplete rectangles that is also not completable. In fact, sets of incomplete Latin rectangles that are incompletable are exactly the dependent subsets of $\bigcup_{t \in T} G_t$.

Let us introduce ways of presenting *available* values. The values available for filling the empty cell $(i_1, j_1)$ in the rectangle of Table 4 are $k_1^t$ and $k_3^t$; any of them can be used without violating the Latin rectangle structure. In general, the set of all available values for a row $i$ are

$$V_i(R_{t-}) = \{a \in G_t \,:\, a \cup R_{t-} \text{ does not violate the Latin rectangle structure}\}$$

We illustrate $V_i(R_{t-})$ by the *availability matrix* of row $i$ as first described in [6].

**Definition 3** *Let $R_{t-}$ $(t \in T)$ denote an incomplete Latin rectangle with $n$ columns, $K(i)$ denote the set of symbols appearing in row $i$ and $J(i)$ the set of column indices of the $v$ empty cells in that row, where $v < n$. The availability matrix $A(R_{t-}, i)$ is the $v \times v$ matrix obtained from the $n \times n$-matrix*

$$A = \begin{pmatrix} k_1^t & \dots & k_1^t \\ k_2^t & \dots & k_2^t \\ & \ddots & \\ k_n^t & \dots & k_n^t \end{pmatrix}$$

*after deleting from $A$ all rows of elements of $K(i)$ and all columns that are not members of $J(i)$. We mark an element of $A(R_{t-}, i)$ in column $j$ with the symbol '$*$' to indicate that the value is not available if and only if that element appears in column $j$ of $R_{t-}$.*

We use curved () and square [] brackets for the availability matrix of the first and the second row respectively, i.e., we write $A(R_{t-}, i_1)$ and $A[R_{t-}, i_2]$. Notice that there is a one-to-one correspondence between $V_i(R_t)$ and $A(R_{t-}, i)$. Therefore, in terms of set notation for a particular row $i$, every combination of column $j$ and value $k^t$ of the availability matrix represents a member $(i, j, k^t)$ of $V_i(R_{t-})$. Therefore, both $A(R_{t-}, i_1)$ and $A[R_{t-}, i_2]$, denoted simply by $A_t$, are hereafter considered as a subset of $G_t$; that is (with a slight abuse of notation) $A_t$ is used to refer both to the set $V_{i_1}(R_{t-}) \cup V_{i_2}(R_{t-})$ and to its matrix representation.

To complete any $R_{t-}$ to a 2-row Latin rectangle (having the same rows), a single value must be selected from every row and column in $A(R_{t-}, i_1)$ and $A[R_{t-}, i_2]$ such that the value selected in column $j$ of $A(R_{t-}, i_1)$ is different from the one selected in column $j$ of $A[R_{t-}, i_2]$; values with an '$*$' cannot be selected. To ease our illustrations, we merge the availability matrices of the two rows into one figure (see Table 5 for an example).

$$\begin{array}{cc} j_1 & j_3 \\ \end{array}$$
$$\begin{pmatrix} k_1^t & k_1^{t*} \\ [k_3^t] & k_3^t \end{pmatrix}$$

Table 5: The availability matrix for the rectangle of Table 4

| | $j_1$ | $j_2$ | $j_3$ | $j_4$ |
|---|---|---|---|---|
| $i_1$ | $k_1^1$ | $k_2^1$ | $k_3^1$ | $k_4^1$ |
| $i_2$ | $k_2^1$ | $k_3^1$ | $k_4^1$ | $k_1^1$ |
| $i_3$ | $k_3^1$ | $k_4^1$ | $k_1^1$ | $k_2^1$ |

$R$

Table 6: The rectangle $R$ obtained from the $OLR2$ of Table 2

**Example 1** *For the first row of Table 4, $K(i_1) = \{k_2^t, k_4^t\}$ and $J(i_1) = \{j_1, j_3\}$, therefore*

$$A(R_{t-}, i_1) = \begin{array}{c} \begin{array}{cc} j_1 & j_3 \end{array} \\ \begin{pmatrix} k_1^t & k_1^{t*} \\ k_3^t & k_3^t \end{pmatrix} \end{array} .$$

*For the second row, $K(i_2) = \{k_1^t, k_2^t, k_4^t\}$ and $J(i_2) = \{j_1\}$, thus*

$$A[R_{t-}, i_2] = \begin{array}{c} j_1 \\ \begin{bmatrix} k_3^t \end{bmatrix} \end{array}$$

*Also, $A_t = \{(i_1, j_1, k_1^t), (i_1, j_1, k_3^t), (i_1, j_3, k_3^t), (i_2, j_1, k_3^t)\}$. Element $(i_1, j_3, k_1^t)$ is not included in $A_t$ as it has an '$*$' indicating it is not available. It is easy to complete $R_{t-}$, since $k_3^t$ is the single value available for cells $(i_2, j_1), (i_1, j_3)$ and then $k_1^t$ becomes the single value available for $(i_1, j_1)$.*

Let us now present some initial results, which motivate on their own the study of the aforementioned independence systems.

**Proposition 4** *Any set $T$ of normalized $MOLR2$ of order $n$ is representable as a normalized $(|T| + 1)$-row Latin rectangle of order $n$ and vice versa.*

**Proof.** The $(|T|+1)$-row Latin rectangle of order $n$, denoted as $R$, is obtained from the $|T|$ normalized rectangles $MOLR2$ by placing the entries of the second row of $R_t$ ($t \in T$) at the $(t + 1)$ row of $R$; that is, if the value of cell $(2, j)$ of $R_t$ is $k_l^t$ then cell $(t + 1, j)$ receives value $k_l^1$ (Table 6 shows an example). It follows that every value appears once in each row of $R$, since appearing once in every row of $R_t$ ($t \in T$). It remains to show that every value occurs at most once in each column of $R$.

Since all rectangles in $T$ are normalized, for any two of them, say $t$ and $t'$, the $n$ pairs of values $(k_1^t, k_1^{t'}), \ldots, (k_n^t, k_n^{t'})$ appear at the first row. Then, to avoid repeating a pair, value $k_l^t$ ($l = 1, \ldots, n$) occurrence at the second row of each $R_t$ ($t \in T$) is bound to be at a different column per $R_t$. But then, value $k_l^1$ ($l = 1, \ldots, n$) appears at most once per column of $R$.

It is easy to see that the construction is applicable in the reverse direction, i.e., given a normalized $(|T|+1)$-row Latin rectangle of order $n$, one can obtain $|T|$ normalized $MOLR2$ of the same order. $\blacksquare$

Proposition 4 has some interesting implications. The first one follows from Hall's theorem [8], i.e., from the fact that every $m$-row Latin rectangle of order $n$ is completable to a Latin square of order $n$.

**Corollary 5** *Any set $T$ of $MOLR2$ ($1 \leq |T| \leq n-1$) of order $n$ can be completed to a set of $n-1$ $MOLR2$ of order $n$.*

Moreover, a Latin rectangle of order $n$ can have at most $n$ rows, in which case it would be a Latin square. Hence, Proposition 4 implies that there exist $(n-1)$ $MOLR2$ of order $n$; the latter directly yields that there can be at most $(n-1)$ $MOLS$ [12, Theorem 2.1].

Now consider the following decision problem: Is a given set of $n-1$ incomplete Latin rectangles of order $n$ completable to a set of $|T|$ $MOLR2$? Clearly this problems is in $\mathcal{NP}$, since given a solution we can easily verify its correctness by simply listing all pairs of values obtained form the superimposed rectangles, and checking whether there appears a repetition of a pair. Now the problem of completing an incomplete Latin square of order $n$ to a Latin square, known to be $\mathcal{NP}$-complete [2], reduces to the problem of completing a set of incomplete Latin rectangles (as in the proof of Proposition 4). Hence the following.

**Corollary 6** *Deciding whether a set of $n-1$ incomplete 2-row Latin rectangles of order $n$ is completable to an $MOLR2$ is also $\mathcal{NP}$-complete.*

It becomes apparent by Proposition 4 that any set of $|T| < n-2$ $MOLR2$ is included into a set of $(|T|+1)$ $MOLR2$ of the same order; it easily follows that $\mathcal{J}_t \subseteq \mathcal{J}_{t+1}$, $t = 1, ..., |T|-1$. But then, any set of $|T|$ incomplete Latin rectangles that is not completable to a set of $|T|$ $MOLR2$, is not completable to a set of $(|T|+1)$ $MOLR2$ either. Hence our last implication.

**Corollary 7** $\mathcal{C}_t \subseteq \mathcal{C}_{t+1}$, $t = 1, ..., n-1$.

Since the clutter $\mathcal{C}_1$, presented next, has been described in [6], it remains to characterize $\mathcal{C}_2 \backslash \mathcal{C}_1$.

## 3 Circuits in $\mathcal{C}_1$

There are five equivalence classes in $\mathcal{C}_1$, denoted as $\mathcal{C}_{1,d}$ with $d = 1, ..., 5$. Let us present a representative from each equivalence class and count $|\mathcal{C}_1|$, assuming without loss of generality that $K_1 = \{1, ..., n\}$. The representative of $\mathcal{C}_{1,1}$ is shown in Table 7, where $C = \{(1, j_1, 1), (2, j_1, 1)\}$. To see that $C$ is a circuit, notice first that $C$ is a dependent set of $\mathcal{S}_1$, since 1 appears twice in column $j_1$ thus violating the Latin rectangle structure (i.e., $C$ is not contained in any $B \in \mathcal{B}_1$); it becomes easy to see that the removal of any $c \in C$ makes $C \backslash \{c\}$ completable to a 2-row Latin rectangle hence $C$ is minimal.

| $j_1$ | $j_2$ | $\cdots$ | $j_{n-1}$ | $j_n$ |
|---|---|---|---|---|
| 1 | | | | |
| 1 | | | | |

Table 7: A circuit of $\mathcal{C}_{1,1}$

| $j_1$ | $j_2$ | $\cdots$ $j_n$ |
|---|---|---|
| 1 | 1 | |
| | | |

A circuit of $\mathcal{C}_{1,2}$

| $j_1$ | $j_2$ $\cdots$ $j_n$ |
|---|---|
| $1,2$ | |
| | |

A circuit of $\mathcal{C}_{1,3}$

| $j_1$ | $\cdots$ $j_n$ |
|---|---|
| | $K_1\backslash\{1\}$ |
| | $K_1\backslash\{1\}$ |

A circuit of $\mathcal{C}_{1,4}$

| $j_1$ | $\cdots$ $j_n$ |
|---|---|
| | $K_1\backslash\{1\}$ |
| 1 | |

A circuit of $\mathcal{C}_{1,5}$

Table 8: Circuits of $\mathcal{C}_{1,2} - \mathcal{C}_{1,5}$

Table 8 illustrates a representative from each of the remaining four classes. For the last two circuits presented, the notation $K_1\backslash\{1\}$ means that all values of set $K_1$ except for 1, appear in columns $j_2$ to $j_n$.

**Lemma 8**

$$|\mathcal{C}_1| = n^2\left(1 + 2(n-1) + ((n-1)!)^2\sum_{q=0}^{n-1}(-1)^q\frac{1}{q!} + 2(n-1)!\right)$$

**Proof.** Family $\mathcal{C}_{1,1}$ includes $n^2$ circuits, since there is one such circuit per column and value, i.e., per member of $J$ and $K_1$. To obtain a circuit in $\mathcal{C}_{1,2}$ (notice its representative in Table 8), there are 2 options for the row, $n$ options for the value in $K_1$ and $\binom{n}{2}$ options for the two columns in which the value appears, i.e., a total of $n^2(n-1)$ options. An analogous reasoning yields the same size for the class $\mathcal{C}_{1,3}$.

Regarding $\mathcal{C}_{1,4}$, notice that there are $n$ options for the value and $n$ options for the column left empty. Notice also that, for columns in $J\backslash\{j_1\}$, the second row must be a *derangement* of the first (i.e., a permutation without a fixed point) in order to comply with the Latin rectangle structure; since (as shown in [16]) the number of derangements of $n$ symbols is

$$r_2[n] = n!\sum_{q=0}^{n}(-1)^q\frac{1}{q!},$$

there are $r_2[n-1]$ options for filling the second row per each of the $(n-1)!$ options of filling the first one. Overall, $\mathcal{C}_{1,4}$ contains $n^2 \cdot r_2[n-1] \cdot (n-1)!$ circuits. Last, $|\mathcal{C}_{1,5}| = 2n^2(n-1)!$ since there are two options for the row where a single value appears, $n$ options for the value, $n$ for the column and $(n-1)!$ options for filling the remaining row. The result follows from the fact that the five classes in $\mathcal{C}_1$ are disjoint. ■

| $j_1$ | $j_2$ | $j_3$ | $\cdots$ | $j_n$ |
|---|---|---|---|---|
| 1 | | | | |
| | 1 | | | |

| $j_1$ | $j_2$ | $j_3$ | $\cdots$ | $j_n$ |
|---|---|---|---|---|
| 1 | | | | |
| | 1 | | | |

Table 9: A circuit of $\mathcal{C}_2 \setminus \mathcal{C}_1$ (set $E$)

| $j_1$ | $j_2$ | $j_3$ | $\cdots$ | $j_{n-1}$ | $j_n$ |
|---|---|---|---|---|---|
| 1 | 2 | | $\cdots$ | | $n$ |
| 2 | 3 | | $\cdots$ | $n$ | 1 |

| $j_1$ | $j_2$ | $j_3$ | $\cdots$ | $j_n$ |
|---|---|---|---|---|
| 1 | 2 | | $\cdots$ | $n$ |
| $n$ | 1 | 2 | $\cdots$ | $n-1$ |

Table 10: Completion of $E \setminus c$ to an $OLR2$

Lemma 8 yields that $|\mathcal{C}_1|$ is of $O\left(n^2 \cdot ((n-1)!)^2 \cdot r_2[n-1]\right)$.

# 4 Circuits in $\mathcal{C}_2 \backslash \mathcal{C}_1$

Recall that two Latin rectangles $R_1, R_2$ are orthogonal if and only if, once superimposed, no pair of values is repeated, i.e., if $R_1 \cup R_2 \in \mathcal{B}_2$. Two incomplete Latin rectangles $R_{1-}, R_{2-}$ are completable if and only if $R_{1-} \cup R_{2-} \subseteq B \in \mathcal{B}_2$ (i.e., if $R_{1-} \cup R_{2-}$ is in $\mathcal{J}_2$) or, equivalently, $R_{1-} \cup R_{2-}$ does not contain a member of $\mathcal{C}_2$. Specifically, $R_{1-} \cup R_{2-}$ does not contain a member of $\mathcal{C}_2 \backslash \mathcal{C}_1$ if and only if each of $R_{1-}, R_{2-}$ is individually completable i.e., if $R_{t-} \subseteq B \in \mathcal{B}_1$, $t = 1, 2$. Therefore, notice that each of $R_{1-}$, $R_{2-}$ neither violates the Latin rectangle structure (i.e., does not contain a circuit of $\mathcal{C}_{1,1} - \mathcal{C}_{1,3}$) nor its completion can be done in a manner that violates this structure (i.e., does not contain a circuit in $\mathcal{C}_{1,4}$ or $\mathcal{C}_{1,5}$).

It follows that $R_{1-} \cup R_{2-}$ containing a member of $\mathcal{C}_2 \backslash \mathcal{C}_1$ implies that the completion of $R_{1-}$ and $R_{2-}$ forces a pair of values to be repeated. Also, equivalence yields that we may assume that pair $(k_1^1, k_1^2)$ appears twice in columns $j_1$ and $j_2$. That is, we assume that any completion of $R_{1-} \cup R_{2-}$ is bound to include the set $E = \{(1, j_1, k_1^1), (2, j_2, k_1^1), (1, j_1, k_1^2), (2, j_2, k_1^2)\}$. Table 9 shows this by assuming $k_1^1 = k_1^2$; in fact, to simplify our exposition, let us hereafter assume that $K_1 = K_2 = \{1, \ldots, n\}$.

**Lemma 9** *$E$ belongs to $\mathcal{C}_2 \backslash \mathcal{C}_1$.*

**Proof.** It becomes easy to see that each incomplete Latin rectangle in Table 9 is completable to a Latin rectangle, i.e., $E$ does not contain a member of $\mathcal{C}_1$, while the pair is not completable to an $OLR2$. Also, it holds that $E$ is minimal, since $E \backslash \{c\} \subset B$ for some $B \in \mathcal{B}_2$, for any $c \in E$. By symmetry, it suffices to show that for element $(2, j_2, 1)$ of the left rectangle and this is exactly illustrated in Table 10. ■

Most importantly, up to equivalence, any member of $\mathcal{C}_2 \backslash \mathcal{C}_1$, may contain elements of $E$. Hence, define $\mathcal{C}_{2,d} \subset \mathcal{C}_2 \backslash \mathcal{C}_1$, $d = 0, \ldots, 4$ as $\mathcal{C}_{2,d} = \{C \in \mathcal{C}_2 \backslash \mathcal{C}_1 : C$ is equivalent to some $C'$ such that $|C' \cap E| = d\}$; evidently, classes $\{\mathcal{C}_{2,d},\ d = 0, \ldots, 4\}$ form a partition of $\mathcal{C}_2 \backslash \mathcal{C}_1$. Also, it is direct that $E$ is, up to equivalence, the single circuit in class $\mathcal{C}_{2,4}$. For any other class we provide a list of all non-equivalent circuits (i.e., we consider that it splits into more than one sub-classes and list one representative per sub-class).

**Theorem 10** *The pairs of incomplete Latin rectangles of Table 11 comprise, up to equivalence, the complete list of circuit members of $\mathcal{C}_{2,0}$.*

| | $j_1$ | $j_2$ | $j_3$ | $j_4$ | $\cdots$ | $j_n$ |
|---|---|---|---|---|---|---|
| $R^*_{6-}$ | | 3 | | $K_1\backslash\{1,2,3\}$ | | |
| | 2 | | $n$ | | $K_1\backslash\{1,2,3,n\}$ | |

| | $j_1$ | $j_2$ | $j_3$ | $j_4$ | $\cdots$ | $j_n$ |
|---|---|---|---|---|---|---|
| $R^*_{6-}$ | | 3 | | $K_2\backslash\{1,2,3\}$ | | |
| | 2 | | $n$ | | $K_2\backslash\{1,2,3,n\}$ | |

| | $j_1$ | $j_2$ | $j_3$ | $j_4$ | $\cdots$ | $j_n$ |
|---|---|---|---|---|---|---|
| $R^*_{6-}$ | | 3 | | $K_1\backslash\{1,2,3\}$ | | |
| | 2 | | $n$ | | $K_1\backslash\{1,2,3,n\}$ | |

| | $j_1$ | $j_2$ | $j_3$ | $\cdots$ | $j_n$ |
|---|---|---|---|---|---|
| $R_{10-}$ | | | $K_2\backslash\{1,2\}$ | | |
| | $n$ | | | $K_2\backslash\{1,2,n\}$ | |

| | $j_1$ | $j_2$ | $j_3$ | $\cdots$ | $j_n$ |
|---|---|---|---|---|---|
| $R_{10-}$ | | | $K_1\backslash\{1,2\}$ | | |
| | $n$ | | | $K_1\backslash\{1,2,n\}$ | |

| | $j_1$ | $j_2$ | $j_3$ | $\cdots$ | $j_n$ |
|---|---|---|---|---|---|
| $R_{10-}$ | | | $K_2\backslash\{1,2\}$ | | |
| | $n$ | | | $K_2\backslash\{1,2,n\}$ | |

| | $j_1$ | $j_2$ | $j_3$ | $j_4$ | $\cdots$ | $j_n$ |
|---|---|---|---|---|---|---|
| $R_{23-}$ | | | | $K_1\backslash\{1,2,3\}$ | | |
| | | | $n$ | | $K_1\backslash\{1,2,3,n\}$ | |

| | $j_1$ | $j_2$ | $\cdots$ | $j_n$ |
|---|---|---|---|---|
| $R_{11-}$ | | | $K_2\backslash\{1,2\}$ | |
| | | | $K_2\backslash\{1,2\}$ | |

| | $j_1$ | $j_2$ | $j_3$ | $\cdots$ | $j_n$ |
|---|---|---|---|---|---|
| $R_{24-}$ | | | | $K_1\backslash\{1,2,3\}$ | |
| | | | | $K_1\backslash\{1,2,3\}$ | |

| | $j_1$ | $j_2$ | $\cdots$ | $j_n$ |
|---|---|---|---|---|
| $R_{11-}$ | | | $K_2\backslash\{1,2\}$ | |
| | | | $K_2\backslash\{1,2\}$ | |

Table 11: Circuits in $\mathcal{C}_{2,0}$

To prove Theorem 10, we need some intermediate results and definitions, which are to be employed for revealing also the non-equivalent circuits per class $\mathcal{C}_{2,d}, d = 0, \ldots, 4$.

To obtain a circuit of class $\mathcal{C}_{2,0}$, we start with an arbitrary pair of 2-row Latin rectangles, say $R_1$ and $R_2$, that do not form an $OLR2$ due to the repetition of exactly one pair of values in the first two columns. That is, $R_1 \cup R_2$ is dependent containing a circuit of class $\mathcal{C}_{2,4}$, i.e., containing $E$. Notice that the set $(R_1 \cup R_2)\backslash E$ remains dependent, since completing each of the rectangles corresponding to $R_1\backslash E$ and $R_2\backslash E$ forces the repetition of pair $(1,1)$ in cells $(1, j_1)$ and $(2, j_2)$ (see Table 12). But then, $(R_1 \cup R_2)\backslash E$ still contains a circuit (since dependent) but no element of $E$. This yields that $(R_1 \cup R_2)\backslash E$ contains a circuit of class $\mathcal{C}_{2,0}$. It follows that, starting with an arbitrary $(R_1 \cup R_2)\backslash E$ and after removing elements we are bound to end up with such a circuit.

For our search to be more concise, we observe first that any of $R_1\backslash E$ or $R_2\backslash E$ (after removing some of its elements) must always be completable in a way that forces some value to appear twice in the first two columns, i.e., in cells $(1, j_1), (1, j_2), (2, j_1)$ and $(2, j_2)$. For convenience, let us introduce the following.

| $j_1$ | $j_2$ | $\cdots$ | $j_n$ |
|---|---|---|---|
| | $K_1\backslash\{1\}$ | | |
| $k_1$ | | $K_1\backslash\{1,k_1\}$ | |

$R_1\backslash E$

| $j_1$ | $j_2$ | $\cdots$ | $j_n$ |
|---|---|---|---|
| | $K_2\backslash\{1\}$ | | |
| $k_2$ | | $K_2\backslash\{1,k_2\}$ | |

$R_2\backslash E$

Table 12: A pair of incomplete Latin rectangles containing a circuit of class $\mathcal{C}_{2,0}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ | $\cdots$ | $j_n$ |
|---|---|---|---|---|---|
| 1 | 3 | 2 | $K_1\backslash\{1,2,3\}$ | | |
| 2 | 1 | $n$ | 3 | $K_1\backslash\{1,2,3,n\}$ | |

$R_6^*$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ | $\cdots$ | $j_n$ |
|---|---|---|---|---|---|
| 1 | 3 | 2 | $K_2\backslash\{1,2,3\}$ | | |
| 3 | 2 | $n$ | 1 | $K_2\backslash\{1,2,3,n\}$ | |

$R_{23-}$

Table 13: Completing the pair of pink rectangles $R_{6-}^*$ and $R_{23-}$

**Definition 11** *An incomplete Latin rectangle $R_-$ is called **pink** if $R_- \subset R_1$ (where $R_1$ is an arbitrary rectangle that includes elements $(1, j_1, 1), (2, j_2, 1)$) and any completion of $R_-$ forces some value to appear twice in the first two columns.*

In the proof that follows, to demonstrate that a particular $R_-$ is not pink, we draw a circle $\bigcirc$ (for the first row) and square $\square$ (for the second row) around selected values of the corresponding availability matrix. Such a selection shows that $R_-$ can be completed without repeating a value in the first two columns.

In summary our exhaustive, yet concise procedure, to reveal all non-equivalent circuits in $\mathcal{C}_{2,0}$ is the following: first we identify all pairwise non-equivalent pink (incomplete) Latin rectangles, then combine them to obtain pairs of incomplete orthogonal Latin rectangles in all possible ways and finally, omit any incomplete $OLR2$ that are not minimal. Since we start this expedition with a single rectangle, for convenience we will enumerate these using notation $R_{z-}$ for the incomplete Latin rectangles and $A_z$ for the corresponding availability matrices, with $z$ simply denoting the sequence in which rectangles are examined. However, not all combinations of pink rectangles give rise to a dependent (i.e., not completable) pair. For example, incomplete Latin rectangles $R_{6-}^*$ and $R_{23-}$ (Table 11) are pink but well completable to an $OLR2$ as shown in Table 13 (i.e., $R_{6-}^* \cup R_{23-}$ contains no circuit).

Before proceeding, let us note that all circuits listed in our enumeration show up for $n \geq 5$. For the sake of completeness, we list all (non-equivalent) circuits that arise for $n = 3$ and $n = 4$ in the Appendix (see Tables 29-36).

**Definition 12** *A pink Latin rectangle $R_-$ is of **type I**, **II** or **III**, if its completion forces the same value to appear*

**type I:** *only in cells* $\{(1, j_1), (2, j_2)\}$;

**type II:** *in cells* $\{(1, j_1), (2, j_2)\}$ or *in cells* $\{(1, j_2), (2, j_1)\}$*;*

**type III:** *in cells* $\{(1, j_1), (2, j_2)\}$ and *in cells* $\{(1, j_2), (2, j_1)\}$.

For example (see Table 11), $R^*_{6-}$ is of type I, $R_{23-}$ is of type II and $R_{11-}$ is of type III.

Let us emphasize that there exist alternative ways of completing a pink Latin rectangle of type II, one in which the same value appears in cells $\{(1, j_1), (2, j_2)\}$, a second in which the same value appears in cells $\{(1, j_2), (2, j_1)\}$ and, possibly, a third in which both pairs $\{(1, j_1), (2, j_2)\}$ and $\{(1, j_2), (2, j_1)\}$ contain equal values (see $R_{23-}$, for instance). Definition 12 means that, whenever the third way is available, the first two ways also are, thus completing such a rectangle *forces* (i.e., makes unavoidable) the same value to appear in only one pair of cells (and not necessarily in both). This is in contrast to pink Latin rectangles of Type III, for which only one way of completing is possible, forcing both pairs $\{(1, j_1), (2, j_2)\}$ and $\{(1, j_2), (2, j_1)\}$ to contain equal values. The following is now easy to show.

**Lemma 13** *Two pink rectangles form a pair whose completion forces a repetition of a pair of values in the first two columns if and only if both are of type I or one is of type III.*

Thus, two pink rectangles $R_-$ and $R'_-$ comply with Lemma 13 if (i) both $R_-$ and $R'_-$ are of type I or (ii) $R_-$ is of type I and $R'_-$ is of type III or (iii) $R_-$ is of type II and $R'_-$ is of type III or (iv) both $R_-$ and $R'_-$ are of type III. But then, a necessary condition for a dependent set $R_- \cup R'_-$ to be minimal is direct: there must exist no $R''_-$ such that $R''_- \subseteq R_-$ (resp. $R''_- \subseteq R'_-$) and $R_-$, $R''_-$ (resp. $R'_-$, $R''_-$) are of the same type.

**Definition 14** *A pink Latin rectangle $R_-$ is called **dominated** if there is some pink rectangle $R'_- \subset R_-$ such that $R_-$ and $R'_-$ are both of the same type (i.e., $I, II, III$).*

Let us list one last observation, to be utilized in the proof listed next.

**Remark 15** *For any two $R_{1-}$ and $R_{2-}$, $R_{1-} \subset R_{2-}$ if and only if $A_2 \subset A_1$.*

**Proposition 16** *The non-dominated pink rectangles, which share no element with $E$, are $R^*_{6-}$, $R_{10-}$, $R_{11-}$, $R_{23-}$ and $R_{24-}$.*

**Proof.** The proof proceeds by progressively emptying cells from the left rectangle of Table 12. To avoid enumerating equivalent (i.e., symmetric) cases, we assume that the number of cells emptied from the first row are less than or equal to the number of cells emptied from the second row. For each case, we illustrate the availability matrix $A_z$ of the rectangle $R_{z-}$ obtained after emptying cells. An '$*$' besides a value in $A_z$ denotes that this value is not available (e.g., due to its occurrence in a non-empty cell in the same column); we emphasize the occurrence of some '$*$' in $A_z$ by writing, instead, $A^*_z$ and $R^*_z$.

**Case 16.1** *Emptying* 1 *cell in row* 2

Without loss of generality we assume that the cell emptied is either $(2, j_3)$ or $(2, j_1)$ and that the two values missing from row 2 are $\{1, 2\}$. Emptying cell $(2, j_3)$ gives rise to $R_{1-}$ or $R^*_{1-}$ (see Table 14) depending on whether value 2 appears in cell $(1, j_2)$; emptying cell $(2, j_1)$ results, in

| $j_1$ | $j_2$ | $j_3$ | $\cdots$ $j_n$ |
|---|---|---|---|
| | $k_2$ | $k_3$ | $K_1\backslash\{1,k_2,k_3\}$ |
| $k_1$ | | | $K_1\backslash\{1,2,k_1\}$ |

$R_{1-}$

| $j_1$ | $j_2$ | $j_3$ | $\cdots$ $j_n$ |
|---|---|---|---|
| | 2 | $k_3$ | $K_1\backslash\{1,2,k_3\}$ |
| $k_1$ | | | $K_1\backslash\{1,2,k_1\}$ |

$R^*_{1-}$

| $j_1$ | $j_2$ | $\cdots$ $j_n$ |
|---|---|---|
| | $k_2$ | $K_1\backslash\{1,k_2\}$ |
| | | $K_1\backslash\{1,2\}$ |

$R_{2-}$

| $j_1$ | $j_2$ | $\cdots$ $j_n$ |
|---|---|---|
| | 2 | $K_1\backslash\{1,2\}$ |
| | | $K_1\backslash\{1,2\}$ |

$R^*_{2-}$

Table 14: Incomplete Latin rectangles for Case 16.1

$$\begin{array}{cc} j_1 & \begin{matrix} j_2 & j_3 \end{matrix} \\ (1) & \begin{bmatrix} 1 & \boxed{1} \\ \boxed{2} & 2 \end{bmatrix} \end{array}$$

$A_1$: Row 1, Col 0

$$\begin{array}{cc} j_1 & \begin{matrix} j_2 & j_3 \end{matrix} \\ (1) & \begin{bmatrix} 1 & 1 \\ 2^* & 2 \end{bmatrix} \end{array}$$

$A^*_1$: Row 1, Col 0

$$\begin{array}{c} \begin{matrix} j_1 & j_2 \end{matrix} \\ \begin{bmatrix} (1) & 1 \\ 2 & 2 \end{bmatrix} \end{array}$$

$A_2$: Row 1, Col 1

$$\begin{array}{c} \begin{matrix} j_1 & j_2 \end{matrix} \\ \begin{bmatrix} (1) & 1 \\ 2 & 2^* \end{bmatrix} \end{array}$$

$A^*_2$: Row 1, Col 1

Figure 1: Availability matrices for the rectangles of Table 14 (Case 16.1)

a similar manner, to $R_{2-}$ or $R^*_{2-}$ (also depicted at Table 14). Please notice the corresponding availability matrices in Figure 1 and the '$*$' in matrix $A^*_1$ regarding value 2 for the second row and column $j_2$ (and similarly in $A^*_2$); notice also that the caption below each matrix presents the number of rows and columns that are common to the availability matrix of both rows (at least one row is common since value 1 is missing from both rows). From these four matrices, only the last three are pink, since $A_1$ implies the completability of $R_1$ without repeating a value in the first two columns; that is, $R_1$ is completable by placing 1 in cells $(1, j_1)$ and $(2, j_3)$ and 2 in cell $(2, j_2)$ and this is illustrated (for convenience) by its circled and squared entries in Figure 1.

**Case 16.2** *Emptying* 2 *cells in row* 2

We assume the cells emptied are $(2, j_3)$ and either $(2, j_4)$ or $(2, j_1)$ and that the three values missing from row 2 are $\{1, 2, 3\}$. Emptying cells $(2, j_3)$ and $(2, j_4)$ gives rise to $R_{3-}$ or $R^*_{3-}$ (see Table 15) depending on whether values 2 and 3 appear in cells $(1, j_2)$ and $(1, j_3)$; emptying cells $(2, j_1)$ and $(2, j_3)$ results, in a similar manner, to $R_{4-}$ or $R^*_{4-}$ (also at Table 15). It becomes easy to see that $R_{3-}$ is not pink since completable by placing 1 in cells $(1, j_1)$ and $(2, j_4)$, 2 in cell $(2, j_3)$ and 3 in cell $(2, j_2)$; in fact, $R^*_{3-}$ (that is more 'restricted' than $R_{3-}$ since $A^*_3 \subset A_3$) is not pink either, since completable in exactly the same manner (see the circled entries of $A_{3-}$ and $A^*_3$ at Figure 2). This fact gives us a useful rule to avoid examining some rectangles: if $R^*_{z-}$ is non-pink so is $R_{z-}$.

The opposite does not hold. For example, $R_{4-}$ is not pink since completable by placing 1 in cells $(1, j_1)$ and $(2, j_3), 3$ in cell $(2, j_1)$ and 2 in cell $(2, j_2)$; to the contrary $R^*_{4-}$ is completable only by placing either 1 in cells $(1, j_1)$ and $(2, j_2)$ or 2 in cells $(1, j_2)$ and $(2, j_1)$ thus being pink (see the circled entries of $A_4$ at Figure 2).

**Case 16.3** *Emptying at least* 3 *cells in row* 2

| $j_1$ | $j_2$ | $j_3$ | $j_4$ | $j_n$ |
|---|---|---|---|---|
| | $k_2$ | $k_3$ | $k_4$ | $K_1\backslash\{1,k_2,k_3,k_4\}$ |
| $k_1$ | | | | $K_1\backslash\{1,2,3,k_1\}$ |

$R_{3-}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ | $j_n$ |
|---|---|---|---|---|
| | 2 | 3 | $k_4$ | $K_1\backslash\{1,2,3,k_4\}$ |
| $k_1$ | | | | $K_1\backslash\{1,2,3,k_1\}$ |

$R^*_{3-}$

| $j_1$ | $j_2$ | $j_3$ | $j_n$ |
|---|---|---|---|
| | $k_2$ | $k_3$ | $K_1\backslash\{1,k_2,k_3\}$ |
| | | | $K_1\backslash\{1,2,3\}$ |

$R_{4-}$

| $j_1$ | $j_2$ | $j_3$ | $j_n$ |
|---|---|---|---|
| | 2 | 3 | $K_1\backslash\{1,2,3\}$ |
| | | | $K_1\backslash\{1,2,3\}$ |

$R^*_{4-}$

Table 15: Incomplete Latin rectangles for Case 16.2

$$\begin{array}{cc} j_1 & \begin{array}{ccc} j_2 & j_3 & j_4 \end{array} \\ ((1)) & \left[\begin{array}{ccc} 1 & 1 & \boxed{1} \\ 2 & \boxed{2} & 2 \\ \boxed{3} & 3 & 3 \end{array}\right] \\ & \\ & \end{array}$$

A$_3$: Row 1, Col 0

$$\begin{array}{cc} j_1 & \begin{array}{ccc} j_2 & j_3 & j_4 \end{array} \\ ((1)) & \left[\begin{array}{ccc} 1 & 1 & \boxed{1} \\ 2^* & \boxed{2} & 2 \\ \boxed{3} & 3^* & 3 \end{array}\right] \\ & \\ & \end{array}$$

A$^*_3$: Row 1, Col 0

$$\begin{array}{c} \begin{array}{ccc} j_1 & j_2 & j_3 \end{array} \\ \left[\begin{array}{ccc} ((1)) & 1 & \boxed{1} \\ \boxed{2} & 2 & 2 \\ 3 & \boxed{3} & 3 \end{array}\right] \end{array}$$

A$_4$: Row 1, Col 1

$$\begin{array}{c} \begin{array}{ccc} j_1 & j_2 & j_3 \end{array} \\ \left[\begin{array}{ccc} (1) & 1 & 1 \\ 2 & 2^* & 2 \\ 3 & 3 & 3^* \end{array}\right] \end{array}$$

A$^*_4$: Row 1, Col 1

Figure 2: Availability matrices for the rectangles of Table 15 (Case 16.2)

Let us hereafter illustrate just the availability matrices, since the corresponding rectangles can be derived unambiguously. The list of all incomplete Latin rectangles examined hereafter (pink or not) appears in the Appendix (Tables 22-24). In addition, Tables 25-28, also in the Appendix, illustrate the completion of pink rectangles, grouped by their type.

Now, Case 16.3 yields only non-pink rectangles. To see this, observe that the most restricted rectangle is the one arising after emptying exactly 3 cells in row 2, namely $(2, j_1), (2, j_3)$ and $(2, j_4)$, and in addition, having values 2, 3 and 4 in cells $(1, j_2)$, $(1, j_3)$ and $(1, j_4)$, respectively. The availability matrix shown in Figure 3 (see also Table 22 in the Appendix) illustrates that this rectangle, namely $R^*_{5-}$ is non-pink, i.e., its circled entries show how $R^*_{5-}$ is completable without any value appearing twice in the first two columns (in fact, its completion is made similarly to $R_{3-}$ and $R^*_{3-}$ in Case 16.2).

**Case 16.4** *Emptying* 1 *cell in row* 1 *and* 1 *cell in row* 2

$$\begin{array}{c} \begin{array}{cccc} j_1 & j_2 & j_3 & j_4 \end{array} \\ \left[\begin{array}{cccc} ((1)) & 1 & \boxed{1} & 1 \\ 2 & 2^* & 2 & \boxed{2} \\ 3 & \boxed{3} & 3^* & 3 \\ \boxed{4} & 4 & 4 & 4^* \end{array}\right] \end{array}$$

A$^*_5$: Row 1, Col 1

Figure 3: Availability matrix for the rectangle of Case 16.3

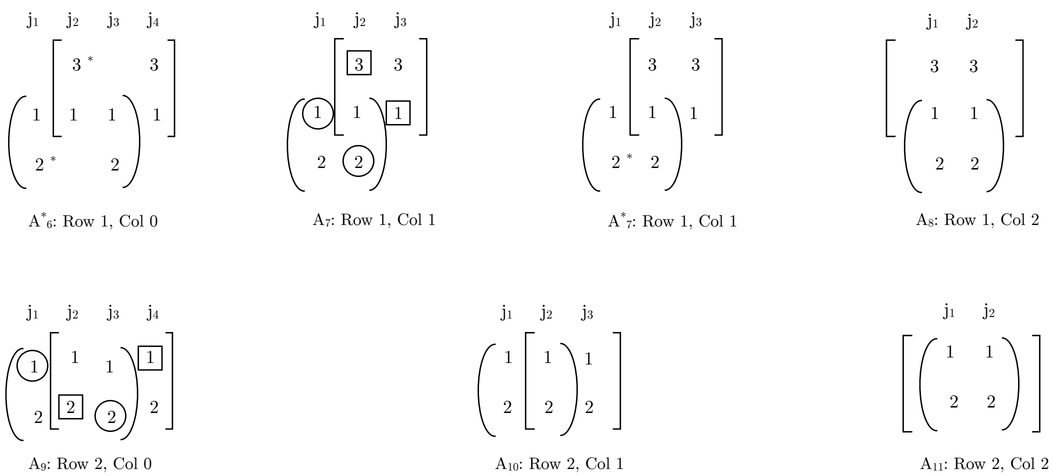


Figure 4: Availability matrices for the rectangles of Case 16.4

For all rectangles in this case, notice that row 1 has two empty cells hence two missing values $\{1, 2\}$; the same applies to row 2, apart from the fact that the second missing value may be 2 or not, i.e., the values missing from row 2 can be either $\{1, 2\}$ or $\{1, 3\}$. Thus, there is a $2 \times 2$ availability matrix per row and these two matrices share 1 or 2 rows (if the second value missing from row 2 is 2 or 3, respectively) and $0, 1$ or 2 columns (depending on which cells are empty at each row). In total, the possible availability matrices (and hence rectangles) to be examined are shown at Figure 4.

Notice that $A_6$ (i.e., $A^*_6$ without any '$*$') is not listed because $R_{6-}$ is easily completable without a value appearing twice in the first two columns; in contrast, $R^*_{6-}$ is pink (see Table 11). Based on the circled entries of $A_7$ we observe that $R_{7-}$ is not pink, whereas $R^*_{7-}$ is. There is no $R^*_{z-}$ for $t = 8, 9, 10, 11$ : observe that columns $j_1$ and $j_2$ are empty at both rows regarding $R_{8-}$ and $R_{11-}$ (thus no value is forbidden at some row because of its occurrence in the other row), while values 1 and 2 are missing from both rows regarding $R_{9-}$ and $R_{10-}$. The circled entries of $A_9$ show that $R_{9-}$ is not pink. Thus this case includes the pink rectangles $R^*_{6-}, R^*_{7-}, R_{8-}, R_{10-}$ and $R_{11-}$.

**Case 16.5** *Emptying* 1 *cell in row* 1 *and* 2 *cells in row* 2

Here, row 1 has 2 empty cells hence two missing values $\{1, 2\}$, whereas row 2 has 3 empty cells thus its missing values are either $\{1, 3, 4\}$ or $\{1, 2, 3\}$; hence there is a $2 \times 2$ availability matrix for row 1 and a $3 \times 3$ such matrix for row 2. These two matrices share 1 or 2 rows (depending on whether value 2 is missing from row 2) and 1 or 2 columns (depending on which cells are empty at each row); notice that should these matrices share 0 columns, any corresponding rectangle would not be pink.

The possible availability matrices (and hence rectangles) to be examined are shown at Figure 5, with the circled and squared entries showing that $R^*_{12-}$ (and hence $R_{12-}$) is not pink, the

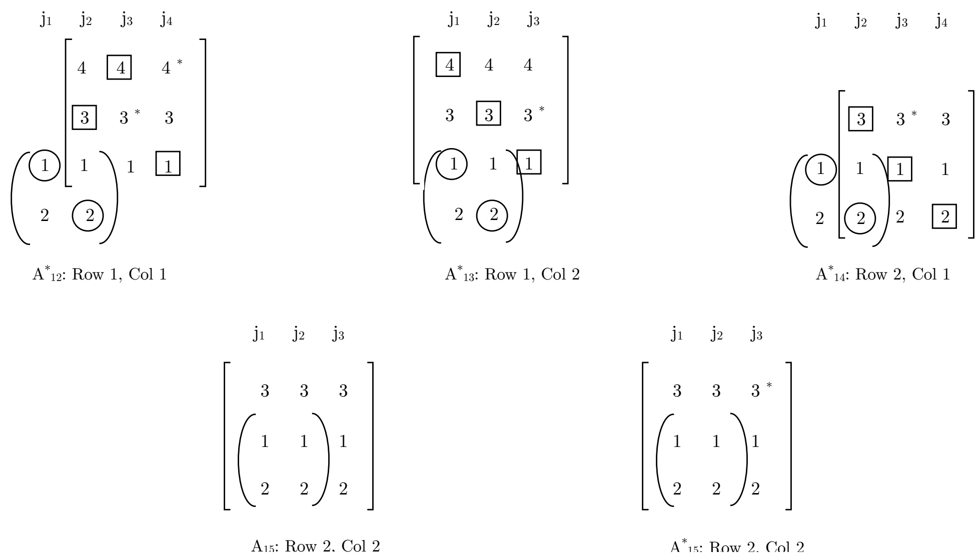


Figure 5: Availability matrices for the rectangles of Case 16.5

same applying to $R_{13-}$, $R^*_{13-}$ and to $R_{14-}$, $R^*_{14-}$. Hence this case yields the pink rectangles $R_{15-}$ and $R^*_{15-}$.

**Case 16.6** *Emptying* 2 *cells in row* 1 *and* 2 *cells in row* 2

In this case, row 1 has 3 empty cells thus its missing values are $\{1, 2, 3\}$, whereas row 2 has also 3 empty cells but its missing values can be $\{1, 4, 5\}$ or $\{1, 3, 4\}$ or $\{1, 2, 3\}$; hence there is a $3 \times 3$ availability matrix per row and the two matrices share 1 up to 3 rows and 0 up to 3 columns; notice that should these matrices share 0 columns, any corresponding rectangle would not be pink.

The possible availability matrices (and hence rectangles) to be examined are shown at Figure 5, with the circled and squared entries showing that all rectangles except for $R_{23-}$ and $R_{24-}$ are not pink. Notice also the non-applicability of an '$*$' in all matrices except for $A_{18}$, since columns $j_1, j_2, j_3$ are all empty at both rows regarding $R_{18-}$, $R_{21-}$ and $R_{24-}$ (thus no value is forbidden at some row because of its occurrence in the other row), while values $1, 2, 3$ are all missing from both rows regarding $R_{22-}$ and $R_{23-}$.

Having enumerated all pink rectangles in Cases 16.1-16.6 (see Table 16), it remains to exclude dominated ones (recall Definition 14), by utilizing Remark 15. Since $A^*_1 \subset A_{10}$ and $A_2 \subset A_{10}$, Remark 15 yields that $R^*_{1-}, R_{2-}$ are dominated. Also, $A^*_4, A^*_7$ and $A_8$ being subsets of $A_{24}$ yields $R^*_{4-}, R^*_{7-}, R_{8-}$ as dominated; the same applies to $R_{15-}, R^*_{15-}$ since $A^*_{15} \subset A_{15} \subset A_{24}$. Last, $R^*_{2-}$ is dominated since $A^*_2 \subset A_{11}$. The remaining rectangles establish the result. ■

The proof of Theorem 10 is now easy to complete.

**Proof (Theorem 10).** Since, by definition, a pair of rectangles $R_{1-} \cup R_{2-}$ is a circuit of $\mathcal{C}_{2,0}$ only if $|R_{1-} \cap E| = |R_{2-} \cap E| = 0$, to obtain all non-equivalent circuits of $\mathcal{C}_{2,0}$ one must examine only pink rectangles sharing no element with $E$. In fact, it suffices to examine only

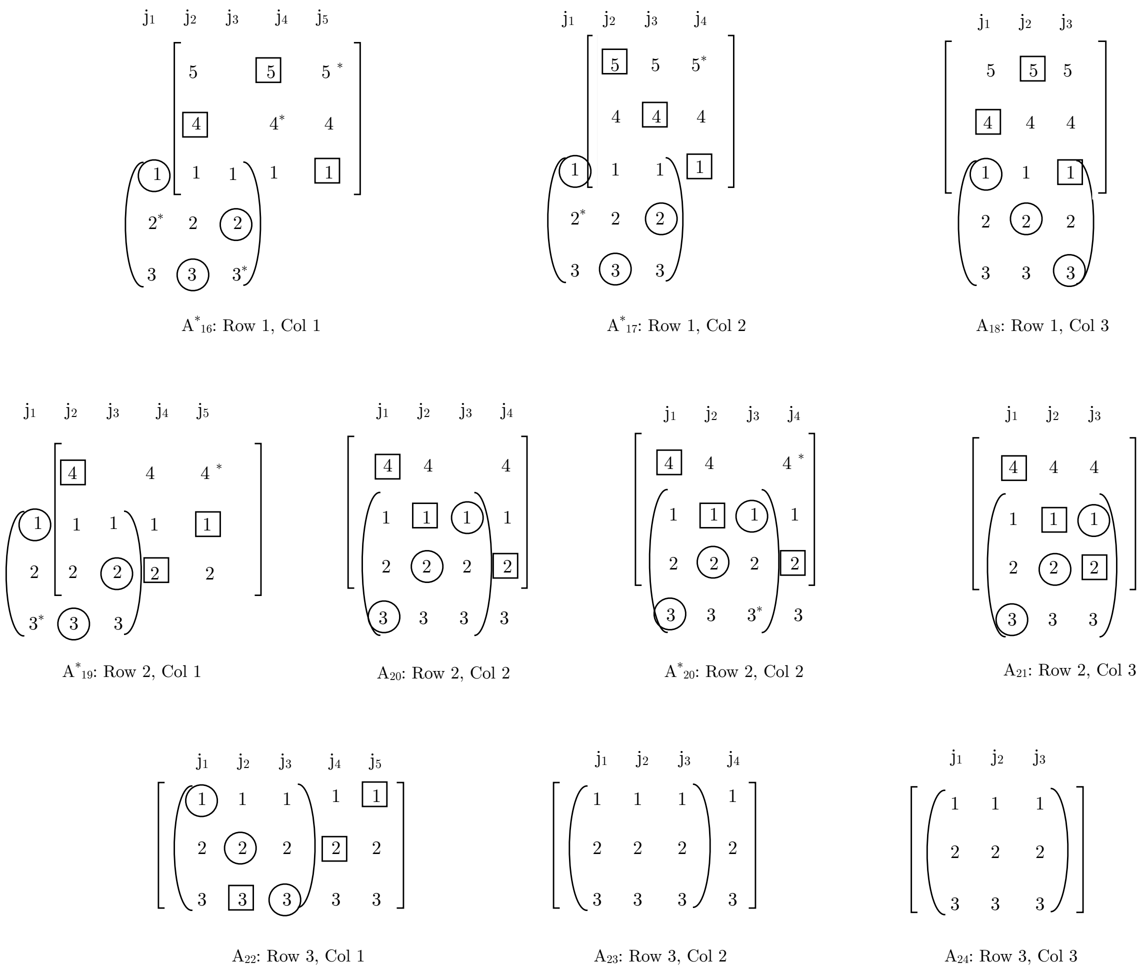


Figure 6: Availability matrices for the rectangles of Case 16.6

| *Type* | *Rectangles* |
|---|---|
| $I$ | $R^*_{1-}, R_{2-}, R^*_{6-}, R_{10-}$ |
| $II$ | $R^*_{4-}, R^*_{7-}, R_{8-}, R_{15-}, R^*_{15-}, R_{23-}, R_{24-}$ |
| $III$ | $R^*_{2-}, R_{11-}$ |

Table 16: All pink rectangles that share no element with $E$

non-dominated such rectangles, i.e., the five rectangles stipulated by Proposition 16, since a circuit is by definition inclusion-wise minimal. Recall from Table 16 that $R^*_{6-}$ and $R_{10-}$ are of type I, $R_{23-}$ and $R_{24-}$ are of type II and $R_{11-}$ is of type III.

By Lemma 13, $R^*_{6-}$ can be paired by itself or with $R_{10-}$, the same applying for $R_{10-}$; hence the first three circuits of Table 11. The same lemma yields that $R_{23-}$ and $R_{24-}$ can be paired neither with one another nor with any of $R^*_{6-}$ and $R_{10-}$. The last implication of Lemma 13 is that any of $R^*_{6-}, R_{10-}, R_{23-}, R_{24-}$ can be paired with the single type III pink rectangle $R_{11-}$. However notice that $R_{23-} \subset R^*_{6-}$ and $R_{24-} \subset R_{10-}$, i.e., $R_{23-} \cup R_{11-} \subset R^*_{6-} \cup R_{11-}$ and $R_{24-} \cup R_{11-} \subset R_{10-} \cup R_{11-}$. It follows that the only remaining pairs, which are inclusion-wise minimal, are exactly the ones defining the last two circuits of Table 11. ■

Let us now proceed to examine classes $\mathcal{C}_{2,1} - \mathcal{C}_{2,3}$; to avoid distracting the reader, the proof of some intermediate results appear at the Appendix.

**Theorem 17** *The pairs of incomplete Latin rectangles of Table 17 comprise, up to equivalence, the complete list of members of* $\mathcal{C}_{2,1}$.

<table>
<tr><td></td><td>$j_1$</td><td>$j_2$</td><td>$j_3$</td><td>$j_4$</td><td>...</td><td>$j_n$</td></tr>
<tr><td rowspan="2">$R^*_{6-}$</td><td></td><td>3</td><td></td><td colspan="3">$K_1\backslash\{1,2,3\}$</td></tr>
<tr><td>2</td><td></td><td>$n$</td><td></td><td colspan="2">$K_1\backslash\{1,2,3,n\}$</td></tr>
</table>

<table>
<tr><td></td><td>$j_1$</td><td>$j_2$</td><td>...</td><td>$j_n$</td></tr>
<tr><td rowspan="2">$R^*_{25-}$</td><td></td><td></td><td colspan="2">$K_2\backslash\{1,2\}$</td></tr>
<tr><td></td><td>1</td><td colspan="2"></td></tr>
</table>

<table>
<tr><td></td><td>$j_1$</td><td>$j_2$</td><td>$j_3$</td><td>...</td><td>$j_n$</td></tr>
<tr><td rowspan="2">$R_{10-}$</td><td></td><td></td><td colspan="3">$K_1\backslash\{1,2\}$</td></tr>
<tr><td>$n$</td><td></td><td></td><td colspan="2">$K_1\backslash\{1,2,n\}$</td></tr>
</table>

<table>
<tr><td></td><td>$j_1$</td><td>$j_2$</td><td>...</td><td>$j_n$</td></tr>
<tr><td rowspan="2">$R^*_{25-}$</td><td></td><td></td><td colspan="2">$K_2\backslash\{1,2\}$</td></tr>
<tr><td></td><td>1</td><td colspan="2"></td></tr>
</table>

<table>
<tr><td></td><td>$j_1$</td><td>$j_2$</td><td>...</td><td>$j_n$</td></tr>
<tr><td rowspan="2">$R_{11-}$</td><td></td><td></td><td colspan="2">$K_1\backslash\{1,2\}$</td></tr>
<tr><td></td><td></td><td colspan="2">$K_1\backslash\{1,2\}$</td></tr>
</table>

<table>
<tr><td></td><td>$j_1$</td><td>$j_2$</td><td>...</td><td>$j_n$</td></tr>
<tr><td rowspan="2">$R^*_{25-}$</td><td></td><td></td><td colspan="2">$K_2\backslash\{1,2\}$</td></tr>
<tr><td></td><td>1</td><td colspan="2"></td></tr>
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<table>
<tr><td></td><td>$j_1$</td><td>$j_2$</td><td>$j_3$</td><td>$j_4$</td><td>...</td><td>$j_n$</td></tr>
<tr><td rowspan="2">$R_{23-}$</td><td></td><td></td><td></td><td colspan="3">$K_1\backslash\{1,2,3\}$</td></tr>
<tr><td></td><td></td><td>$n$</td><td></td><td colspan="2">$K_1\backslash\{1,2,3,n\}$</td></tr>
</table>

<table>
<tr><td></td><td>$j_1$</td><td>$j_2$</td><td>...</td><td>$j_n$</td></tr>
<tr><td rowspan="2">$R^*_{26-}$</td><td></td><td></td><td colspan="2">$K_2\backslash\{1,2\}$</td></tr>
<tr><td>2</td><td>1</td><td colspan="2"></td></tr>
</table>

<table>
<tr><td></td><td>$j_1$</td><td>$j_2$</td><td>$j_3$</td><td>...</td><td>$j_n$</td></tr>
<tr><td rowspan="2">$R_{24-}$</td><td></td><td></td><td></td><td colspan="2">$K_1\backslash\{1,2,3\}$</td></tr>
<tr><td></td><td></td><td></td><td colspan="2">$K_1\backslash\{1,2,3\}$</td></tr>
</table>

<table>
<tr><td></td><td>$j_1$</td><td>$j_2$</td><td>...</td><td>$j_n$</td></tr>
<tr><td rowspan="2">$R^*_{26-}$</td><td></td><td></td><td colspan="2">$K_2\backslash\{1,2\}$</td></tr>
<tr><td>2</td><td>1</td><td colspan="2"></td></tr>
</table>

<table>
<tr><td></td><td>$j_1$</td><td>$j_2$</td><td>...</td><td>$j_n$</td></tr>
<tr><td rowspan="2">$R_{11-}$</td><td></td><td></td><td colspan="2">$K_1\backslash\{1,2\}$</td></tr>
<tr><td></td><td></td><td colspan="2">$K_1\backslash\{1,2\}$</td></tr>
</table>

<table>
<tr><td></td><td>$j_1$</td><td>$j_2$</td><td>$j_3$</td><td>...</td><td>$j_n$</td></tr>
<tr><td rowspan="2">$R^*_{27-}$</td><td></td><td></td><td></td><td colspan="2">$K_2\backslash\{1,2,3\}$</td></tr>
<tr><td>2</td><td>1</td><td colspan="3"></td></tr>
</table>

Table 17: Circuits in $\mathcal{C}_{2,1}$

To prove Theorem 17 we need the following result, whose proof we illustrate at the Appendix.

**Proposition 18** *The non-dominated pink rectangles, which share one element with $E$, are $R^*_{25-}$ (type I), $R^*_{26-}$ (type III) and $R^*_{27-}$ (type II).*

**Proof (Theorem 17).** By definition, a pair of rectangles $R_{1-} \cup R_{2-}$ is a circuit of $\mathcal{C}_{2,1}$ only if $|(R_{1-} \cup R_{2-}) \cap E| = 1$, thus assume without loss of generality that $|R_{1-} \cap E| = 0$ and $|R_{2-} \cap E| = 1$. To maintain inclusion-wise minimality, we restrict ourselves to the five rectangles listed in Proposition 16 (that can play the role of $R_{1-}$) and the three rectangles listed in Proposition 18 (that can play the role of $R_{2-}$). However, not all 15 combinations give rise to minimal incompletable *pairs* of rectangles, as explained next.

Recall from Table 16 that $R^*_{6-}$ and $R_{10-}$ are of type I, $R_{23-}$ and $R_{24-}$ are of type II and $R_{11-}$ is of type III. By Lemma 13, $R^*_{6-}$ or $R_{10-}$ can be paired with $R^*_{25-}$ (since all are of type I), hence the first two circuits of Table 17. By the same Lemma, $R_{11-}$ can be paired with $R^*_{25-}$ since the former is of type III, hence the third circuit of Table 17.

All five rectangles of Proposition 16 can be paired with $R^*_{26-}$ since the latter is of type III. Notice, however, that $R^*_{25-} \subset R^*_{26-}$ hence the pairs $R^*_{6-} \cup R^*_{25-}$, $R_{10-} \cup R^*_{25-}$ and $R_{11-} \cup R^*_{25-}$ dominate, respectively, the pairs $R^*_{6-} \cup R^*_{26-}$, $R_{10-} \cup R^*_{26-}$ and $R_{11-} \cup R^*_{26-}$, which are therefore omitted. It follows that the only remaining pairs containing $R^*_{26-}$ are $R_{23-} \cup R^*_{26-}$ and $R_{24-} \cup R^*_{26-}$, i.e., the fourth and the fifth circuit, respectively, of Table 17.

Last, $R_{11-}$ (being of type III) can be paired with all rectangles of Proposition 18; as above, $R^*_{25-} \subset R^*_{26-}$ yields that the $R_{11-} \cup R^*_{26-}$ is not a circuit, hence the remaining two pairs containing $R_{11-}$ are the third and the last circuits of Table 17. ■

**Theorem 19** *The pairs of incomplete Latin rectangles of Table 18 comprise, up to equivalence, the complete list of members of $\mathcal{C}_{2,2}$.*

The proof of the following appears in the Appendix.

**Proposition 20** *The non-dominated pink rectangles, which share two elements with $E$, are $R_{28-}$ (type I) and $R_{29-}$ (type III).*

**Proof (Theorem 19).** By definition, a pair of rectangles $R_{1-} \cup R_{2-}$ is a circuit of $\mathcal{C}_{2,2}$ only if $|(R_{1-} \cup R_{2-}) \cap E| = 2$, thus we may consider that either $|R_{1-} \cap E| = 0$ and $|R_{2-} \cap E| = 2$ or $|R_{1-} \cap E| = |R_{2-} \cap E| = 1$.

For the former case ($|R_{1-} \cap E| = 0$, $|R_{2-} \cap E| = 2$) we restrict ourselves to the five rectangles listed in Proposition 16 (in the role of $R_{1-}$) and the two rectangles listed in Proposition 20 (in the role of $R_{2-}$). By Lemma 13, $R^*_{6-}$ or $R_{10-}$ can be paired with $R_{28-}$ (since all are of type I), hence the first two circuits of Table 18. By the same Lemma 13, $R_{11-}$ can be paired with $R_{28-}$ since the former is of type III, hence the third circuit of Table 18. All five rectangles of Proposition 16 can be paired with $R_{29-}$ since the latter is of type III; however, since $R_{28-} \subset R_{29-}$, the pairs $R^*_{6-} \cup R_{29-}$, $R_{10-} \cup R_{29-}$ and $R_{11-} \cup R_{29-}$ are dominated, respectively, by the first three pairs of Table 18. The only remaining pairs containing $R_{29-}$ are $R_{23-} \cup R_{29-}$ and $R_{24-} \cup R_{29-}$, i.e., the fourth and the fifth circuit, respectively, of Table 18.

| | $j_1$ | $j_2$ | $j_3$ | $j_4$ | $\cdots$ $j_n$ |
|---|---|---|---|---|---|
| $R^*_{6-}$ | | 3 | | $K_1\setminus\{1,2,3\}$ | |
| | 2 | | $n$ | | $K_1\setminus\{1,2,3,n\}$ |

| | $j_1$ | $j_2$ | $\cdots$ $j_n$ |
|---|---|---|---|
| $R_{28-}$ | 1 | | |
| | | 1 | |

| | $j_1$ | $j_2$ | $j_3$ | $\cdots$ $j_n$ |
|---|---|---|---|---|
| $R_{10-}$ | | | $K_1\setminus\{1,2\}$ | |
| | $n$ | | | $K_1\setminus\{1,2,n\}$ |

| | $j_1$ | $j_2$ | $\cdots$ $j_n$ |
|---|---|---|---|
| $R_{28-}$ | 1 | | |
| | | 1 | |

| | $j_1$ | $j_2$ | $\cdots$ $j_n$ |
|---|---|---|---|
| $R_{11-}$ | | | $K_1\setminus\{1,2\}$ |
| | | | $K_1\setminus\{1,2\}$ |

| | $j_1$ | $j_2$ | $\cdots$ $j_n$ |
|---|---|---|---|
| $R_{28-}$ | 1 | | |
| | | 1 | |

| | $j_1$ | $j_2$ | $j_3$ | $j_4$ | $\cdots$ $j_n$ |
|---|---|---|---|---|---|
| $R_{23-}$ | | | | $K_1\setminus\{1,2,3\}$ | |
| | | | $n$ | | $K_1\setminus\{1,2,3,n\}$ |

| | $j_1$ | $j_2$ | $\cdots$ $j_n$ |
|---|---|---|---|
| $R_{29-}$ | 1 | 2 | |
| | 2 | 1 | |

| | $j_1$ | $j_2$ | $j_3$ | $\cdots$ $j_n$ |
|---|---|---|---|---|
| $R_{24-}$ | | | | $K_1\setminus\{1,2,3\}$ |
| | | | | $K_1\setminus\{1,2,3\}$ |

| | $j_1$ | $j_2$ | $\cdots$ $j_n$ |
|---|---|---|---|
| $R_{29-}$ | 1 | 2 | |
| | 2 | 1 | |

| | $j_1$ | $j_2$ | $\cdots$ $j_n$ |
|---|---|---|---|
| $R^*_{25-}$ | | | $K_1\setminus\{1,2\}$ |
| | | 1 | |

| | $j_1$ | $j_2$ | $\cdots$ $j_n$ |
|---|---|---|---|
| $R^*_{25-}$ | | | $K_2\setminus\{1,2\}$ |
| | | 1 | |

| | $j_1$ | $j_2$ | $\cdots$ $j_n$ |
|---|---|---|---|
| $R^*_{26-}$ | | | $K_1\setminus\{1,2\}$ |
| | 2 | 1 | |

| | $j_1$ | $j_2$ | $j_3$ | $\cdots$ $j_n$ |
|---|---|---|---|---|
| $R^*_{27-}$ | | | | $K_2\setminus\{1,2,3\}$ |
| | 2 | 1 | | |

Table 18: Circuits in $\mathcal{C}_{2,2}$

For the latter case ($|R_{1-} \cap E| = |R_{2-} \cap E| = 1$), it suffices to examine the three rectangles of Proposition 18. By Lemma 13, $R^*_{25-}$ (of type I) can be paired with itself and with $R^*_{27-}$ (of type III); however, $R^*_{25-} \cup R^*_{27-} \subset R^*_{25-} \cup R^*_{25-}$ since $R^*_{27-} \subset R^*_{25-}$; for the same reason $R^*_{27-} \cup R^*_{27-}$ (plausible by Lemma 13 since $R^*_{27-}$ is of type III) is omitted since it is a subset of $R^*_{25-} \cup R^*_{25-}$. Last, $R^*_{26-}$ can be paired with $R^*_{27-}$. Overall, the case of $|R_{1-} \cap E| = |R_{2-} \cap E| = 1$ leads to the last two circuits of Table 18. ■

**Theorem 21** *The rectangles of Table 19 comprise, up to equivalence, the complete list of members of $\mathcal{C}_{2,3}$.*

**Proof.** By definition, a pair of rectangles $R_{1-} \cup R_{2-}$ is a circuit of $\mathcal{C}_{2,3}$ only if $|(R_{1-} \cup R_{2-}) \cap E| = 3$, thus we may consider without loss of generality that $|R_{1-} \cap E| = 1$ and $|R_{2-} \cap E| = 2$. That is, we may restrict ourselves to the three rectangles of Proposition 18 and the two rectangles of Proposition 20 for the roles of $R_{1-}$ and $R_{2-}$, respectively.

By Lemma 13, $R^*_{25-}$ can be paired with $R_{28-}$ (since both are of type I), hence the first circuit of Table 19. By the same Lemma, all three rectangles of Proposition 18 can be paired with $R_{29-}$ since the latter is of type III; observe, however, that $R_{28-} \subset R_{29-}$ yields $R^*_{25-} \cup R_{28-} \subset R^*_{25-} \cup R_{29-}$ and $R^*_{27-} \subset R^*_{26-}$ yields $R^*_{27-} \cup R_{29-} \subset R^*_{26-} \cup R_{29-}$, the only non-omitted such pair is the second circuit in Table 19. Last, although $R^*_{26-}$ (since of type III) can also be paired with $R_{28-}$, $R^*_{26-} \cup R_{28-}$ includes $R^*_{25-} \cup R_{28-}$ since $R^*_{25-} \subset R^*_{26-}$. ■

$$
R^*_{25-} \quad \begin{array}{c|c|c|c|}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{j_1} & \multicolumn{1}{c}{j_2} & \multicolumn{1}{c}{\cdots \quad j_n} \\ \cline{2-4}
 & & & K_1\backslash\{1,2\} \\ \cline{2-4}
 & & 1 & \\ \cline{2-4}
\end{array}
\qquad
R_{28-} \quad \begin{array}{c|c|c|c|}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{j_1} & \multicolumn{1}{c}{j_2} & \multicolumn{1}{c}{\cdots \quad j_n} \\ \cline{2-4}
 & 1 & & \\ \cline{2-4}
 & & 1 & \\ \cline{2-4}
\end{array}
$$

$$
R^*_{27-} \quad \begin{array}{c|c|c|c|c|}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{j_1} & \multicolumn{1}{c}{j_2} & \multicolumn{1}{c}{j_3} & \multicolumn{1}{c}{\cdots \quad j_n} \\ \cline{2-5}
 & & & & K_1\backslash\{1,2,3\} \\ \cline{2-5}
 & 2 & 1 & \multicolumn{2}{c|}{} \\ \cline{2-5}
\end{array}
\qquad
R_{29-} \quad \begin{array}{c|c|c|c|}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{j_1} & \multicolumn{1}{c}{j_2} & \multicolumn{1}{c}{\cdots \quad j_n} \\ \cline{2-4}
 & 1 & 2 & \\ \cline{2-4}
 & 2 & 1 & \\ \cline{2-4}
\end{array}
$$

Table 19: Circuits in $\mathcal{C}_{2,3}$

Overall, Tables 11, 17, 18, 19 and 9 list, up to equivalence, all circuits in $\mathcal{C}_2 \backslash \mathcal{C}_1$.

## 5 Implications

The results of the previous section, the sole such results in related literature except for [6] and [5], essentially address the question on whether a given pair of 2-row rectangles is completable to an $OLR2$. To answer that, it suffices to examine all non-equivalent circuits listed above. Checking whether a given pair of incomplete 2-row Latin rectangles contains a circuit of a specific class can be done in time polynomial with respect to $n$, hence the completability problem for pairs of rectangles is polytime solvable (although $\mathcal{NP}$-hard in general).

Beyond that, these results could also be fruitful in terms of optimization, i.e., regarding the design of an algorithm finding whether a specific pair of incomplete Latin rectangles is completable to an $OLS$. To show that, let us first illustrate how to formulate the OLR completion problem as an Integer Program (IP). Considering that the two rectangles are $R$ and $R'$, let variable $x_{ijk}$ (resp. $y_{ijk}$) be 1 if value $k$ appears in cell $(i, j)$ of $R$ (resp. $R'$).

$$\max \sum \{x_{ijk} : i \in I, j \in J, k \in K_1\} + \sum \{y_{ijk} : i \in I, j \in J, k \in K_2\}$$

$$\sum \{x_{ijk} : i \in I\} \leq 1, j \in J, k \in K_1, \tag{1}$$

$$\sum \{x_{ijk} : j \in J\} \leq 1, i \in I, k \in K_1, \tag{2}$$

$$\sum \{x_{ijk} : k \in K_1\} \leq 1, i \in I, j \in J, \tag{3}$$

$$\sum\{y_{ijk} : i \in I\} \leq 1,\, j \in J,\, k \in K_2, \tag{4}$$
$$\sum\{y_{ijk} : j \in J\} \leq 1,\, i \in I,\, k \in K_2, \tag{5}$$
$$\sum\{y_{ijk} : k \in K_2\} \leq 1,\, i \in I,\, j \in J, \tag{6}$$
$$x_{i_1 j_1 k_1} + x_{i_2 j_2 k_1} + y_{i_1 j_1 k_2} + y_{i_2 j_2 k_2} \leq 3,\ \{i_1, i_2\} \subseteq I,\, \{j_1, j_2\} \subset J,\, k_1 \in K_1,\, k_2 \in K_2 \tag{7}$$
$$x_{ijk} \in \{0,1\},\, i \in I,\, j \in J,\, k \in K_1$$
$$y_{ijk} \in \{0,1\},\, i \in I,\, j \in J,\, k \in K_2$$

Constraints (1) and (4) ensure that no value is repeated per column of $R$ and $R'$, constraints (2) and (5) ensure the same per each row and constraints (3) and (6) ensure that every cell of $R$ and $R'$ contains at most one value; constraints (7) ensure that $R$ and $R'$ are orthogonal, i.e., it forbids a pair of values to occur twice. Therefore, integer vectors $(x, y)$ that are feasible with respect to (1)-(7) are in $1-1$ correspondence with pairs of (possibly incomplete) Latin rectangles, each not violating the Latin rectangle structure; clearly these vectors include also $OLS$ for all orders other than 6. Also, a pair of incomplete Latin rectangles can be modelled via the above IP simply by setting to 1 one variable per non-empty cell; then, the pair is completable to an $OLS$ if and only if there is a solution having exactly $2n^2$ variables at value 1 (i.e., if the optimum value for the IP is $2n^2$).

Any $C \in \mathcal{C}_2 \backslash \mathcal{C}_1$ induces a circuit inequality, stating that not all variables indexed by $C$ should receive value 1. That inequality is

$$\sum\{x_c : c \in C \cap G_1\} + \sum\{y_c : c \in C \cap G_2\} \leq |C| - 1,\ C \in \mathcal{C}_2 \backslash \mathcal{C}_1 \tag{8}$$

or, if $C$ is denoted as $R \cup R'$,

$$\sum\{x_c : c \in R\} + \sum\{y_c : c \in R'\} \leq |R \cup R'| - 1,\ (R \cup R') \in \mathcal{C}_2 \backslash \mathcal{C}_1.$$

For example, (7) is the set of all inequalities arising from $\mathcal{C}_{2,4}$. Clearly, adding all circuit inequalities restricts the set of feasible integer vectors $(x, y)$ to those corresponding to pairs of rectangles in which any two rows are completable in a way that no pair of values occurs more than once (in these two rows only); hence our interest in actually obtaining these inequalities. Since, however, the number of circuit inequalities is prohibitively large, it would be far more useful to employ such inequalities in a cutting-plane algorithm, i.e., generating them only if violated by the current LP-solution.

Even better, one would be interested in generating *lifted* circuit-inequalities, i.e., circuit inequalities in which further variables are included one-by-one in their left-hand side with the largest (positive) coefficient such that the augmented inequality remains valid. In our setting, an inequality is valid if not excluding integer feasible vectors, i.e., vectors associated with completable pairs of rectangles. This process is known as *sequential lifting* [14]. A lifted circuit inequality has the form

$$\begin{aligned}&\sum\{a_s x_s : s \in S \cap G_1\} + \sum\{a_s y_s : s \in S \cap G_2\} + \\ &+ \sum\{x_c : c \in C \cap G_1\} + \sum\{y_c : c \in C \cap G_2\} \leq |C| - 1, \;\; C \in \mathcal{C}_2 \backslash \mathcal{C}_1, \end{aligned} \tag{9}$$

where $S \subseteq (G_1 \cup G_2) \backslash C$ and $a_s > 0, s \in S$. To avoid a lengthy presentation, we only list here some general properties of these inequalities, along with the inequalities arising from $\mathcal{C}_{2,3}$ and $\mathcal{C}_{2,4}$. Our aim is to indicatively show that the results of the previous section have some interesting consequences, along with presenting the diversity of inequalities arising from 2-row circuits.

**Proposition 22** *No lifted circuit inequality can have a left-hand side coefficient greater than* 2.

**Proof.** Consider that the inequality (8) is augmented by introducing variable $x_s$ with coefficient $a_s$ where $s \in G_1 \backslash C$, i.e.,

$$a_s x_s + \sum\{x_c : c \in C \cap G_1\} + \sum\{y_c : c \in C \cap G_2\} \leq |C| - 1. \tag{10}$$

Define $C(s) = \{c \in C \cap G_1 : |s \cap c| = 2\}$ and let us show that $|C(s)| \leq 3$. For $s = (i_s, j_s, k_s), c \in C(s)$ implies $c \in \{(i_c, j_s, k_s), (i_s, j_c, k_s), (i_s, j_s, k_c)\}$ where $i_c \in I \backslash \{i_s\}, j_c \in J \backslash \{j_s\}$ and $k_c \in K \backslash \{k_s\}$. Clearly, $|C(s)| > 3$ only if there are two elements $c, d$ in $C(s)$ sharing the same two indices with $s$, e.g., $(i_c, j_s, k_s)$ and $(i_d, j_s, k_s)$; but then, circuit $C$ including both $(i_c, j_s, k_s)$ and $(i_d, j_s, k_s)$ implies that value $k_s$ appears twice in column $j_s$, a contradiction to the fact that no $C \in \mathcal{C}_2 \backslash \mathcal{C}_1$ violates the Latin rectangle structure.

Next, notice that constraints (1)-(3) yield that any $s \in G_1$ appears in the same constraint with some $c \in G_1 \backslash \{s\}$ if and only if $c$ and $s$ share two among the indices $i, j, k$. It follows that setting $x_s = 1$ implies $x_c = 0$ for all $c \in C(s)$, in which case (10) becomes,

$$a_s + \sum\{x_c : c \in (C \cap G_1) \backslash C(s)\} + \sum\{y_c : c \in C \cap G_2\} \leq |C| - 1$$

or, using that $|C(s)| \leq 3$,

$$\begin{aligned} a_s &\leq |C| - 1 - \Big(\sum\{x_c : c \in (C \cap G_1) \backslash C(s)\} + \sum\{y_c : c \in C \cap G_2\}\Big) \leq \\ &\leq |C| - 1 - (|C| - |C(s)|) \leq |C| - 1 - (|C| - 3) = 2. \end{aligned}$$

■

The following is shown in a pretty similar fashion.

**Corollary 23** $a_s \geq 1$ *only if* $|C_s| \geq 2$.

Let us now examine the inequalities arising from class $\mathcal{C}_{2,4}$. Recall that $\mathcal{C}_{2,4}$ has a single circuit, up to equivalence (recall Table 9) hence (7) represents the circuit inequalities in this class. By Corollary 23, it is easy to see that $a_s \geq 1$ only if $s$ is one of $(i_1, j_2, k_1), (i_2, j_1, k_1)$ in $G_1$

| $j_1$ | $j_2$ | $\cdots$ | $j_n$ |
|---|---|---|---|
| | $k_1$ | $\cdots$ | |
| $k_1$ | | $\cdots$ | |

| $j_1$ | $j_2$ | $\cdots$ | $j_n$ |
|---|---|---|---|
| $k_2$ | | $\cdots$ | |
| | $k_2$ | $\cdots$ | |

Table 20: A pair of rectangles containing no circuit

or one of $(i_1, j_2, k_2), (i_2, j_1, k_2)$ in $G_2$. Notice however that including both $(i_1, j_2, k_1), (i_2, j_1, k_1)$ yields the inequality

$$x_{i_1 j_1 k_1} + x_{i_2 j_2 k_1} + y_{i_1 j_1 k_2} + y_{i_2 j_2 k_2} + x_{i_1 j_2 k_1} + x_{i_2 j_1 k_1} \leq 3$$

which is invalid since setting the last four variables to value 1 is not allowed (because of the left-hand side becoming 4), although the corresponding pair of rectangles (see Table 20) contains no circuit; it follows that only one of $(i_1, j_2, k_1), (i_2, j_1, k_1)$ can be included, the same applying to $(i_1, j_2, k_2), (i_2, j_1, k_2)$ of $G_2$. Hence the lifted circuit inequalities arising from class $\mathcal{C}_{2,4}$ are

$$\begin{aligned}
x_{i_1 j_1 k_1} + x_{i_2 j_2 k_1} + y_{i_1 j_1 k_2} + y_{i_2 j_2 k_2} + x_{i_1 j_2 k_1} + y_{i_1 j_2 k_2} &\leq 3, \\
x_{i_1 j_1 k_1} + x_{i_2 j_2 k_1} + y_{i_1 j_1 k_2} + y_{i_2 j_2 k_2} + x_{i_1 j_2 k_1} + y_{i_2 j_1 k_2} &\leq 3, \\
x_{i_1 j_1 k_1} + x_{i_2 j_2 k_1} + y_{i_1 j_1 k_2} + y_{i_2 j_2 k_2} + x_{i_2 j_1 k_1} + y_{i_1 j_2 k_2} &\leq 3, \text{ and} \\
x_{i_1 j_1 k_1} + x_{i_2 j_2 k_1} + y_{i_1 j_1 k_2} + y_{i_2 j_2 k_2} + x_{i_2 j_1 k_1} + y_{i_2 j_1 k_2} &\leq 3,
\end{aligned}$$

where $\{i_1, i_2\} \subseteq I, \{j_1, j_2\} \subseteq J, k_1 \in K_1$ and $k_2 \in K_2$.

To provide one more example, recall that class $\mathcal{C}_{2,3}$ contains two non-equivalent circuits listed in Table 19. The first one gives rise to the circuit inequality

$$\sum_{j \in J \setminus \{j_1, j_2\}} x_{i_1 j \pi(j)} + x_{i_2 j_2 k_1} + y_{i_1 j_1 k_3} + y_{i_2 j_2 k_3} \leq n,$$

where $\{i_1, i_2\} \subseteq I, \{j_1, j_2\} \subseteq J, \{k_1, k_2\} \subseteq K_1, k_3 \in K_2$ and $\pi : J \setminus \{j_1, j_2\} \rightarrow K_1 \setminus \{k_1, k_2\}$ is bijective. This inequality yields two different families of lifted circuit inequalities, namely:

$$\begin{aligned}
\sum_{\substack{j \in J \setminus \{j_1, j_2\} \\ k \in K_1 \setminus \{k_1, k_2\}}} x_{i_1 j k} + x_{i_2 j_2 k_1} + y_{i_1 j_1 k_3} + y_{i_2 j_2 k_3} + x_{i_2 j_2 k_2} + y_{i_1 j_2 k_3} &\leq n, \text{ and} \\
\sum_{\substack{j \in J \setminus \{j_1, j_2\} \\ k \in K_1 \setminus \{k_1, k_2\}}} x_{i_1 j k} + x_{i_2 j_2 k_1} + y_{i_1 j_1 k_3} + y_{i_2 j_2 k_3} + x_{i_2 j_2 k_2} + y_{i_2 j_1 k_3} &\leq n.
\end{aligned}$$

# 6 Concluding Remarks

The results presented here could motivate an analogous study for either two Latin rectangles having more than 2 rows (as in [5] for a single 3-row rectangle) or for more than two 2-row Latin rectangles. An exhaustive enumeration of non-equivalent circuits in these cases remains an open question. As a motivating example, Table 21 presents a set of three incomplete Latin rectangles that are not completable to three $MOLR2$, although any two are completable to an $OLR2$. This is because the completion of $R_{3-}$ forces the same value to occur in cells $\{(1,j_1),(2,j_2)\}$ (hence violating the orthogonality condition with $R_{2-}$) or in cells $\{(1,j_3),(2,j_2)\}$ (hence violating the orthogonality condition with $R_{1-}$); observe that emptying any of the non-empty cells yields a completable set.

| $j_1$ | $j_2$ | $j_3$ | |
|---|---|---|---|
| | | 1 | |
| | 1 | | |

$R_{1-}$

| $j_1$ | $j_2$ | $j_3$ | |
|---|---|---|---|
| 1 | | | |
| | 1 | | |

$R_{2-}$

| $j_1$ | $j_2$ | $j_3$ | |
|---|---|---|---|
| | | | $K_3 \backslash \{1,2,3\}$ |
| | | | $K_3 \backslash \{1,2,3\}$ |

$R_{3-}$

Table 21: A circuit in $\mathcal{C}_3 \backslash \mathcal{C}_2$.

Also, the IP listed in the previous section can be solved using an arbitrary linear function, thus addressing the issue of finding the maximum-weight $OLR2$ or the maximum-weight such $OLR2$ arising from the completion (if any) of a given pair of incomplete 2-row Latin rectangles.

Our last remark is that there are circuits in distinct sets $C_{2,d}$ that are equivalent, at least for $n \leq 4$. For instance, $R_{10-} \cup R_{10-} \in C_{2,0}$ is equivalent to $R_{28-} \cup R_{28-} \in C_{2,4}$ when $n = 3$, or $R_{23-} \cup R_{23-} \in C_{2,0}$ is equivalent to $R_{28-} \cup R_{28-}$ when $n = 4$. It is indeed interesting to study this aspect for the general case $n \geq 5$, because it could be used to determine the corresponding distribution of circuits into equivalence classes.

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## Appendix

**Proof (Proposition 18).** Let us assume without loss of generality, that the element shared with $E$ is $(2, j_2, 1)$. Our goal is to enforce a repetition of a pair in the first two columns, therefore since value 1 already appears in cell $(2, j_2)$ we can start by emptying all other cells in row 2.

**Case 18.1** *Emptying* 1 *cell in row* 1 *and* $n-1$ *cells in row* 2

Here, by definition cell $(1, j_1)$ is emptied in the first row resulting in an incomplete Latin rectangle of type $I$ which is clearly dominated by the pink rectangle $R^*_{25-}$ (type I).

**Case 18.2** *Emptying* 2 *cells in row* 1 *and* $n-1$ *cells in row* 2

In the second row, all cells but $(2, j_2)$ are emptied and in the first row we can assume that the cells emptied are $(1, j_1)$ and $(1, j_2)$ with missing values $\{1, 2\}$. This gives rise to $R^*_{25-}$ which is pink (type I), since value 1 is forbidden for the cell $(1, j_2)$ hence appearing with a '$*$' in the availability matrix of row 1, leaving value 1 as the only option for cell $(1, j_1)$. Emptying any cell other than $(1, j_2)$, e.g., $(1, j_3)$ will give rise to a non-pink rectangle, since value 1 can then be placed in cell $(1, j_3)$ and value 2 in cell $(1, j_1)$. Emptying additional cells in row 1 will result in a similar non-pink structure, unless an additional cell is filled in row 2 to enforce the selection of value 1 in cell $(1, j_1)$, hence Case 18.4.

**Case 18.3** *Emptying* 2 *cells in row* 1 *and* $n-2$ *cells in row* 2

In the second row, all cells but $(2, j_1)$ and $(2, j_2)$ can be emptied and in the first row we can assume that the cells emptied are $(1, j_1)$ and $(1, j_2)$ with missing values $\{1, 2\}$. Emptying cells $(1, j_1)$ and $(1, j_2)$ gives rise to $R^*_{26-}$ which is pink (type III) since values 1 and 2 are forced to appear in cells $(1, j_1)$ and $(1, j_2)$ due to the '$*$' appearing in the availability matrices, indicating that cells $(2, j_1)$ and $(2, j_2)$ contain values 2 and 1, respectively.

**Case 18.4** *Emptying* 3 *cells in row* 1 *and* $n-2$ *cells in row* 2

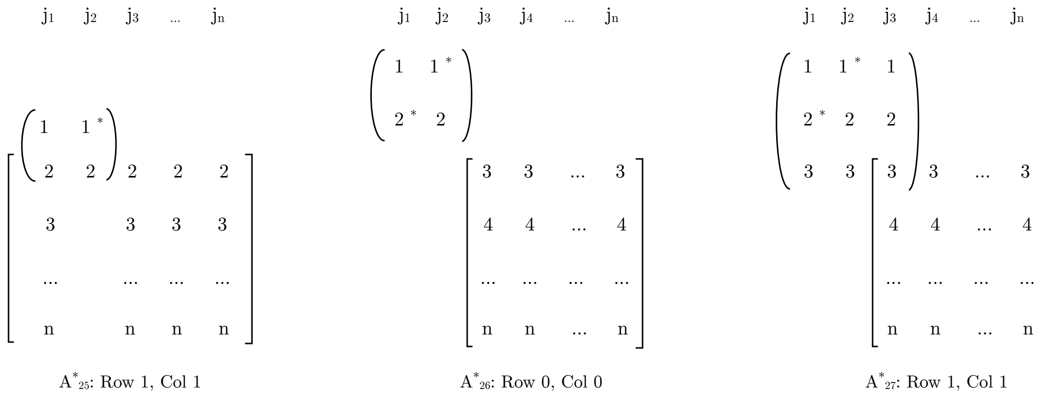


Figure 7: Availability matrices for the rectangles of Proposition 18

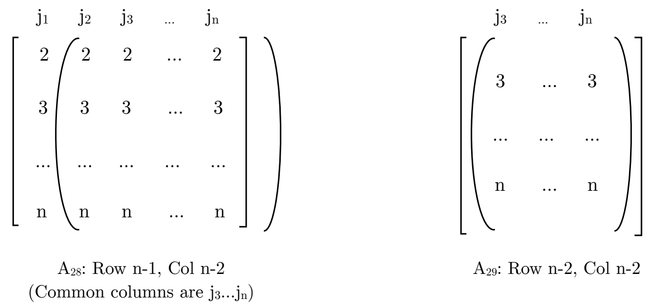


Figure 8: Availability matrices for the rectangles of Proposition 20

Here, emptied cells in the second row remain as per previous case, while in row 1 cells $(1, j_1)$, $(1, j_2)$ and $(1, j_3)$ are emptied with missing values $\{1, 2, 3\}$. This gives rise to $R^*_{27}$ which is pink (type II) and which is completable either by placing value 1 in $(1, j_1)$ or value 2 in $(1, j_3)$ or both. Notice that emptying any other cell from row 1 leads to a non-pink rectangle.

Incomplete rectangles $R^*_{25}$, $R^*_{26}$, $R^*_{27}$, are shown in Table 17, while their availability matrices are shown in Figure 7. Notice that although $A^*_{26} \subset A^*_{25}$ and $A^*_{26} \subset A^*_{27}$, by Remark 15 $R^*_{26-}$ is not dominated (recall Definition 14) since being of type III, whereas $R^*_{25-}$ is of type I and $R^*_{27-}$ is of type II. ■

**Proof (Proposition 20).** The two elements shared with $E$ are $(1, j_1, 1)$ and $(2, j_2, 1)$; keeping only these two elements yields the first availability matrix of Figure 8, i.e., $R_{28-}$ which is of type I and clearly dominates any other incomplete rectangle of the same type that shares two elements with $E$ but has fewer empty cells. However, we may also include elements $(1, j_2, 2)$ and $(2, j_1, 2)$, thus yielding the second matrix of Figure 8 that corresponds to $R_{29-}$. Although $R_{28-} \subset R_{29-}$, $R_{29-}$ is not dominated since it is of type III, whereas $R_{28-}$ is of type I. ■

| $j_1$ | $j_2$ | $j_3$ | $j_4$ … $j_n$ |
|---|---|---|---|
| | $k_2$ | $k_3$ | $K_1\setminus\{1,k_2,k_3\}$ |
| $k_1$ | | | $K_1\setminus\{1,2\}$ |

$R_{1-}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ | $j_n$ |
|---|---|---|---|---|
| | $k_2$ | $k_3$ | $k_4$ | $K_1\setminus\{1,k_2,k_3,k_4\}$ |
| $k_1$ | | | | $K_1\setminus\{1,2,3,k_1\}$ |

$R_{3-}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ | $j_n$ |
|---|---|---|---|---|
| | 2 | 3 | $k_4$ | $K_1\setminus\{1,2,3,k_4\}$ |
| $k_1$ | | | | $K_1\setminus\{1,2,3,k_1\}$ |

$R^*_{3-}$

| $j_1$ | $j_2$ | $j_3$ | $j_n$ |
|---|---|---|---|
| | $k_2$ | $k_3$ | $K_1\setminus\{1,k_2,k_3\}$ |
| | | | $K_1\setminus\{1,2,3\}$ |

$R_{4-}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ | $j_n$ |
|---|---|---|---|---|
| | 2 | 3 | 4 | $K_1\setminus\{1,2,3,4\}$ |
| | | | | $K_1\setminus\{1,2,3,4\}$ |

$R^*_{5-}$

| $j_1$ | $j_2$ | $j_3$ … | $j_n$ |
|---|---|---|---|
| | | $K_1\setminus\{1,2\}$ | |
| $n$ | | | $K_1\setminus\{1,3,n\}$ |

$R_{7-}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ | $j_n$ |
|---|---|---|---|---|
| | $k_1$ | | $K_1\setminus\{1,2,k_1\}$ | |
| $k_2$ | | $k_3$ | | $K_1\setminus\{1,2,k_2,k_3\}$ |

$R_{9-}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ | $j_n$ |
|---|---|---|---|---|
| | | 3 | 4 | $K_1\setminus\{1,2,3,4\}$ |
| $k_1$ | | | | $K_1\setminus\{1,3,4,k_1\}$ |

$R^*_{12-}$

| $j_1$ | $j_2$ | $j_3$ | $j_n$ |
|---|---|---|---|
| | | 3 | $K_1\setminus\{1,2,3\}$ |
| | | | $K_1\setminus\{1,3,4\}$ |

$R^*_{13-}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ | $j_n$ |
|---|---|---|---|---|
| | | 3 | $K_1\setminus\{1,2,3\}$ | |
| $k_1$ | | | | $K_1\setminus\{1,2,3,k_1\}$ |

$R^*_{14-}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ | $j_5$ | $j_n$ |
|---|---|---|---|---|---|
| | | | 4 | 5 | $K_1\setminus\{1,2,3,4,5\}$ |
| 2 | | 3 | | | $K_1\setminus\{1,2,3,4,5\}$ |

$R^*_{16-}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ | $j_n$ |
|---|---|---|---|---|
| | | | 5 | $K_1\setminus\{1,2,3,5\}$ |
| 2 | | | | $K_1\setminus\{1,2,4,5\}$ |

$R^*_{17-}$

| $j_1$ | $j_2$ | $j_3$ | $j_n$ |
|---|---|---|---|
| | | | $K_1\setminus\{1,2,3\}$ |
| | | | $K_1\setminus\{1,4,5\}$ |

$R_{18-}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ | $j_5$ | $j_n$ |
|---|---|---|---|---|---|
| | | | 4 | $K_1\setminus\{1,2,3,4\}$ | |
| 3 | | $k_1$ | | | $K_1\setminus\{1,2,3,4\}$ |

$R^*_{19-}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ | $j_n$ |
|---|---|---|---|---|
| | | | 4 | $K_1\setminus\{1,2,3,4\}$ |
| | | 3 | | $K_1\setminus\{1,2,3,4\}$ |

$R^*_{20-}$

| $j_1$ | $j_2$ | $j_3$ | $j_n$ |
|---|---|---|---|
| | | | $K_1\setminus\{1,2,3\}$ |
| | | | $K_1\setminus\{1,2,4\}$ |

$R_{21-}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ | $j_5$ | $j_n$ |
|---|---|---|---|---|---|
| | | | $K_1\setminus\{1,2,3\}$ | | |
| $k_1$ | | $k_2$ | | | $K_1\setminus\{1,2,3\}$ |

$R_{22-}$

Table 22: Non-pink rectangles

| $j_1$ | $j_2$ | $j_3$ | $\cdots$ | $j_n$ |
|---|---|---|---|---|
| | 2 | $k_3$ | $K_1\backslash\{1,2,k_3\}$ | |
| $k_1$ | | | $K_1\backslash\{1,2\}$ | |

$R^*_{1-}$

, $A^*_1 = \{(1,j_1,1),(2,j_2,1),(2,j_3,1)(2,j_3,2)\}$

| $j_1$ | $j_2$ | $\cdots$ | $j_n$ |
|---|---|---|---|
| | $k_2$ | $K_1\backslash\{1,k_2\}$ | |
| | | $K_1\backslash\{1,2\}$ | |

$R_{2-}$

, $A_2 = \{(1,j_1,1),(2,j_1,1),(2,j_1,2),(2,j_2,1),(2,j_2,2)\}$

| $j_1$ | $j_2$ | $\cdots$ | $j_n$ |
|---|---|---|---|
| | 2 | $K_1\backslash\{1,2\}$ | |
| | | $K_1\backslash\{1,2\}$ | |

$R^*_{2-}$

, $A^*_2 = \{(1,j_1,1),(2,j_1,1),(2,j_1,2),(2,j_2,1)\}$

| $j_1$ | $j_2$ | $j_3$ | $j_n$ |
|---|---|---|---|
| | 2 | 3 | $K_1\backslash\{1,2,3\}$ |
| | | | $K_1\backslash\{1,2,3\}$ |

$R^*_{4-}$

, $A^*_4 = \{(1,j_1,1),(2,j_1,1),(2,j_1,2),(2,j_1,3),(2,j_2,1),$
$(2,j_2,3),(2,j_3,1),(2,j_3,2)\}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ | $\cdots$ | $j_n$ |
|---|---|---|---|---|---|
| | 3 | | $K_1\backslash\{1,2,3\}$ | | |
| 2 | | $n$ | | $K_1\backslash\{1,2,3,n\}$ | |

$R^*_{6-}$

, $A^*_6 = \{(1,j_1,1),(1,j_3,1),(1,j_3,2),(2,j_2,1),$
$,(2,j_4,1),(2,j_4,3)\}$

| $j_1$ | $j_2$ | $j_3$ | $\cdots$ | $j_n$ |
|---|---|---|---|---|
| | | $K_1\backslash\{1,2\}$ | | |
| 2 | | | $K_1\backslash\{1,2,3\}$ | |

$R^*_{7-}$

, $A^*_7 = \{(1,j_1,1),(1,j_2,1),(1,j_2,2),(2,j_2,1),(2,j_2,3),$
$(2,j_3,1),(2,j_3,3)\}$

| $j_1$ | $j_2$ | $\cdots$ | $j_n$ |
|---|---|---|---|
| | | $K_1\backslash\{1,2\}$ | |
| | | $K_1\backslash\{1,3\}$ | |

$R_{8-}$

, $A_8 = \{(1,j_1,1),(1,j_1,2),(1,j_2,1),(1,j_2,2),(2,j_1,1)$
$(2,j_1,3),(2,j_2,1),(2,j_2,3)\}$

| $j_1$ | $j_2$ | $j_3$ | $\cdots$ | $j_n$ |
|---|---|---|---|---|
| | | $K_1\backslash\{1,2\}$ | | |
| $n$ | | | $K_1\backslash\{1,2,n\}$ | |

$R_{10-}$

, $A_{10} = \{(1,j_1,1),(1,j_1,2),(1,j_2,1),(1,j_2,2),(2,j_2,1),$
$(2,j_2,2),(2,j_3,1),(2,j_3,2)\}$

| $j_1$ | $j_2$ | $\cdots$ | $j_n$ |
|---|---|---|---|
| | | $K_1\backslash\{1,2\}$ | |
| | | $K_1\backslash\{1,2\}$ | |

$R_{11-}$

, $A_{11} = \{(1,j_1,1),(1,j_1,2),(1,j_2,1),(1,j_2,2),(2,j_1,1)$
$(2,j_1,2),(2,j_2,1),(2,j_2,2)\}$

| $j_1$ | $j_2$ | $j_3$ | $\cdots$ | $j_n$ |
|---|---|---|---|---|
| | | $k_1$ | $K_1\backslash\{1,2,k_1\}$ | |
| | | | $K_1\backslash\{1,2,3\}$ | |

$R_{15-}$

, $A_{15} = \{(1,j_1,1),(1,j_1,2),(1,j_2,1),(1,j_2,2),(2,j_1,1),$
$(2,j_1,2),(2,j_1,3),(2,j_2,1),(2,j_2,2),(2,j_2,3),$
$(2,j_3,1),(2,j_3,2),(2,j_3,3)\}$

Table 23: Pink rectangles and set notation

| $j_1$ | $j_2$ | $j_3$ | ... $j_n$ |
|---|---|---|---|
| | | 3 | $K_1\setminus\{1,2,3\}$ |
| | | | $K_1\setminus\{1,2,3\}$ |

$R^*_{15-}$

$$A^*_{15} = \{(1,j_1,1),(1,j_1,2),(1,j_2,1),(1,j_2,2),(2,j_1,1),(2,j_1,2),(2,j_1,3),(2,j_2,1),(2,j_2,2),(2,j_2,3),(2,j_3,1),(2,j_3,2)\}$$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ | ... $j_n$ |
|---|---|---|---|---|
| | | | $K_1\setminus\{1,2,3\}$ | |
| | | $n$ | | $K_1\setminus\{1,2,3,n\}$ |

$R_{23-}$

$$A_{23} = \{(1,j_1,1),(1,j_1,2),(1,j_1,3),(1,j_2,1),(1,j_2,2),(1,j_2,3),(1,j_3,1),(1,j_3,2),(1,j_3,3),(2,j_1,1),(2,j_1,2),(2,j_1,3),(2,j_2,1),(2,j_2,2),(2,j_2,3),(2,j_4,1),(2,j_4,2),(2,j_4,3)\}$$

| $j_1$ | $j_2$ | $j_3$ | ... $j_n$ |
|---|---|---|---|
| | | | $K_1\setminus\{1,2,3\}$ |
| | | | $K_1\setminus\{1,2,3\}$ |

$R_{24-}$

$$A_{24} = \{(1,j_1,1),(1,j_1,2),(1,j_1,3),(1,j_2,1),(1,j_2,2),(1,j_2,3),(1,j_3,1),(1,j_3,2),(1,j_3,3),(2,j_1,1),(2,j_1,2),(2,j_1,3),(2,j_2,1),(2,j_2,2),(2,j_2,3),(2,j_3,1),(2,j_3,2),(2,j_3,3)\}$$

| $j_1$ | $j_2$ | ... $j_n$ |
|---|---|---|
| | | $K_1\setminus\{1,2\}$ |
| | 1 | |

$R^*_{25-}$

$$A^*_{25} = \{(1,j_1,1),(1,j_1,2),(1,j_2,2),(2,j_1,2),(2,j_1,3),...,(2,j_1,n),(2,j_3,2),...,(2,j_3,n),...,(2,j_n,2),...,(2,j_n,n)\}$$

| $j_1$ | $j_2$ | ... $j_n$ |
|---|---|---|
| | | $K_1\setminus\{1,2\}$ |
| 2 | 1 | |

$R^*_{26-}$

$$A^*_{26} = \{(1,j_1,1),(1,j_2,2),(2,j_3,3),...,(2,j_3,n),...,(2,j_n,3),...,(2,j_n,n)\}$$

| $j_1$ | $j_2$ | $j_3$ | ... $j_n$ |
|---|---|---|---|
| | | | $K_1\setminus\{1,2,3\}$ |
| 2 | 1 | | |

$R^*_{27-}$

$$A^*_{27} = \{(1,j_1,1),(1,j_2,2),(1,j_3,1),(1,j_3,2),(1,j_3,3)\ (2,j_3,3),...,(2,j_3,n),...,(2,j_n,3),...,(2,j_n,n)\}$$

| $j_1$ | $j_2$ | ... $j_n$ |
|---|---|---|
| 1 | | |
| | 1 | |

$R_{28-}$

$$A_{28} = \{(1,j_2,2),...,(1,j_2,n),...,(1,j_n,2),...,(1,j_n,n)\ (2,j_1,2),...,(2,j_1,n),(2,j_3,2),...,(2,j_3,n)...,(2,j_n,3),...,(2,j_n,n)\}$$

| $j_1$ | $j_2$ | ... $j_n$ |
|---|---|---|
| 1 | 2 | |
| 2 | 1 | |

$R_{29-}$

$$A_{29} = \{(1,j_3,3),...,(1,j_3,n),...,(1,j_n,3),...,(1,j_n,n),(2,j_3,3),...,(2,j_3,n),...,(2,j_n,3),...,(2,j_n,n)\}$$

Table 24: Pink rectangles and set notation (Continued)

| $j_1$ | $j_2$ | $j_3$ | ... $j_n$ |
|---|---|---|---|
| $\mathbf{1}$ | $2$ | $k_3$ | $K_1\backslash\{1,2,k_3\}$ |
| $k_1$ | $\mathbf{1}$ | $\mathbf{2}$ | $K_1\backslash\{1,2,k_1\}$ |

$R^*_{1-}$

| $j_1$ | $j_2$ | ... $j_n$ |
|---|---|---|
| $\mathbf{1}$ | $k_2$ | $K_1\backslash\{1,k_2\}$ |
| $\mathbf{2}$ | $\mathbf{1}$ | $K_1\backslash\{1,2\}$ |

$R_{2-}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ | ... $j_n$ |
|---|---|---|---|---|
| $\mathbf{1}$ | $3$ | $\mathbf{2}$ | $K_1\backslash\{1,2,3\}$ | |
| $2$ | $\mathbf{1}$ | $n$ | $\mathbf{3}$ | $K_1\backslash\{1,2,3,n\}$ |

$R^*_{6-}$

| $j_1$ | $j_2$ | $j_3$ | ... $j_n$ |
|---|---|---|---|
| $\mathbf{1}$ | $\mathbf{2}$ | $K_1\backslash\{1,2\}$ | |
| $n$ | $\mathbf{1}$ | $\mathbf{2}$ | $K_1\backslash\{1,2,n\}$ |

$R_{10-}$

,

| $j_1$ | $j_2$ | $j_3$ | ... $j_n$ |
|---|---|---|---|
| $\mathbf{2}$ | $\mathbf{1}$ | $K_1\backslash\{1,2\}$ | |
| $n$ | $\mathbf{2}$ | $\mathbf{1}$ | $K_1\backslash\{1,2,n\}$ |

$R_{10-}$

| $j_1$ | $j_2$ | ... $j_n$ |
|---|---|---|
| $\mathbf{1}$ | $\mathbf{2}$ | $K_1\backslash\{1,2\}$ |
| $k_1$ | $1$ | ... |

$R^*_{25-}$

| $j_1$ | $j_2$ | ... $j_n$ |
|---|---|---|
| $1$ | $\mathbf{k_1}$ | ... |
| $\mathbf{k_2}$ | $1$ | ... |

$R_{28-}$

Table 25: Completion of type I pink rectangles

| $j_1$ | $j_2$ | $j_3$ | $j_n$ |
|---|---|---|---|
| **1** | 2 | 3 | $K_1\backslash\{1,2,3\}$ |
| **3** | **1** | **2** | $K_1\backslash\{1,2,3\}$ |

,

| $j_1$ | $j_2$ | $j_3$ | $j_n$ |
|---|---|---|---|
| **1** | 2 | 3 | $K_1\backslash\{1,2,3\}$ |
| **2** | **3** | **1** | $K_1\backslash\{1,2,3\}$ |

$R^*_{4-}$ $R^*_{4-}$

| $j_1$ | $j_2$ | $j_3$ | ... $j_n$ |
|---|---|---|---|
| **1** | **2** | $K_1\backslash\{1,2\}$ | |
| 2 | **1** | **3** | $K_1\backslash\{1,2,3\}$ |

,

| $j_1$ | $j_2$ | $j_3$ | ... $j_n$ |
|---|---|---|---|
| **1** | **2** | $K_1\backslash\{1,2\}$ | |
| 2 | **3** | **1** | $K_1\backslash\{1,2,3\}$ |

$R^*_{7-}$ $R^*_{7-}$

| $j_1$ | $j_2$ | ... $j_n$ |
|---|---|---|
| **1** | **2** | $K_1\backslash\{1,2\}$ |
| **3** | **1** | $K_1\backslash\{1,3\}$ |

,

| $j_1$ | $j_2$ | ... $j_n$ |
|---|---|---|
| **2** | **1** | $K_1\backslash\{1,2\}$ |
| **1** | **3** | $K_1\backslash\{1,3\}$ |

$R_{8-}$ $R_{8-}$

| $j_1$ | $j_2$ | $j_3$ | ... $j_n$ |
|---|---|---|---|
| **1** | **2** | $k_1$ | $K_1\backslash\{1,2,k_1\}$ |
| **2** | **1** | **3** | $K_1\backslash\{1,2,3\}$ |

,

| $j_1$ | $j_2$ | $j_3$ | ... $j_n$ |
|---|---|---|---|
| **1** | **2** | $k_1$ | $K_1\backslash\{1,2,k_1\}$ |
| **3** | **1** | **2** | $K_1\backslash\{1,2,3\}$ |

$R_{15-}$ $R_{15-}$

| $j_1$ | $j_2$ | $j_3$ | ... $j_n$ |
|---|---|---|---|
| **2** | **1** | $k_1$ | $K_1\backslash\{1,2,k_1\}$ |
| **1** | **2** | **3** | $K_1\backslash\{1,2,3\}$ |

,

| $j_1$ | $j_2$ | $j_3$ | ... $j_n$ |
|---|---|---|---|
| **2** | **1** | $k_1$ | $K_1\backslash\{1,2,k_1\}$ |
| **3** | **2** | **1** | $K_1\backslash\{1,2,3\}$ |

$R_{15-}$ $R_{15-}$

| $j_1$ | $j_2$ | $j_3$ | ... $j_n$ |
|---|---|---|---|
| **1** | **2** | 3 | $K_1\backslash\{1,2,3\}$ |
| **3** | **1** | **2** | $K_1\backslash\{1,2,3\}$ |

,

| $j_1$ | $j_2$ | $j_3$ | ... $j_n$ |
|---|---|---|---|
| **1** | **2** | 3 | $K_1\backslash\{1,2,3\}$ |
| **2** | **3** | **1** | $K_1\backslash\{1,2,3\}$ |

$R^*_{15-}$ $R^*_{15-}$

| $j_1$ | $j_2$ | $j_3$ | ... $j_n$ |
|---|---|---|---|
| **2** | **1** | 3 | $K_1\backslash\{1,2,3\}$ |
| **1** | **3** | **2** | $K_1\backslash\{1,2,3\}$ |

,

| $j_1$ | $j_2$ | $j_3$ | ... $j_n$ |
|---|---|---|---|
| **2** | **1** | 3 | $K_1\backslash\{1,2,3\}$ |
| **3** | **2** | **1** | $K_1\backslash\{1,2,3\}$ |

$R^*_{15-}$ $R^*_{15-}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ | ... $j_n$ |
|---|---|---|---|---|
| **1** | **2** | **3** | $K_1\backslash\{1,2,3\}$ | |
| **2** | **1** | $n$ | **3** | $K_1\backslash\{1,2,3,n\}$ |

,

| $j_1$ | $j_2$ | $j_3$ | $j_4$ | ... $j_n$ |
|---|---|---|---|---|
| **1** | **2** | **3** | $K_1\backslash\{1,2,3\}$ | |
| **2** | **3** | $n$ | **1** | $K_1\backslash\{1,2,3,n\}$ |

$R_{23-}$ $R_{23-}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ | ... $j_n$ |
|---|---|---|---|---|
| **1** | **2** | **3** | $K_1\backslash\{1,2,3\}$ | |
| **3** | **1** | $n$ | **2** | $K_1\backslash\{1,2,3,n\}$ |

,

Similar completion for $R_{23-}$ if values $1,2,3$ are placed in different cells in first row.

$R_{23-}$

| $j_1$ | $j_2$ | $j_3$ | ... $j_n$ |
|---|---|---|---|
| **1** | **2** | **3** | $K_1\backslash\{1,2,3\}$ |
| **2** | **3** | **1** | $K_1\backslash\{1,2,3\}$ |

,

| $j_1$ | $j_2$ | $j_3$ | ... $j_n$ |
|---|---|---|---|
| **1** | **2** | **3** | $K_1\backslash\{1,2,3\}$ |
| **3** | **1** | **2** | $K_1\backslash\{1,2,3\}$ |

$R_{24-}$ $R_{24-}$

Similar completion for $R_{24-}$ if values $1,2,3$ are placed in different cells in first row.

Table 26: Completion of type II pink rectangles

| $j_1$ | $j_2$ | $j_3$ | $\dots$ $j_n$ |
|---|---|---|---|
| **1** | **2** | **3** | $K_1\backslash\{1,2,3\}$ |
| 2 | 1 | ... | |

$R^*_{27-}$ ,

| $j_1$ | $j_2$ | $j_3$ | $\dots$ $j_n$ |
|---|---|---|---|
| **1** | **3** | **2** | $K_1\backslash\{1,2,3\}$ |
| 2 | 1 | ... | |

$R^*_{27-}$

| $j_1$ | $j_2$ | $j_3$ | $\dots$ $j_n$ |
|---|---|---|---|
| **3** | **2** | **1** | $K_1\backslash\{1,2,3\}$ |
| 2 | 1 | ... | |

$R^*_{27-}$

Table 27: Completion of type II pink rectangles (Continued)

| $j_1$ | $j_2$ | $\dots$ $j_n$ |
|---|---|---|
| **1** | 2 | $K_1\backslash\{1,2\}$ |
| **2** | **1** | $K_1\backslash\{1,2\}$ |

$R^*_{2-}$

| $j_1$ | $j_2$ | $\dots$ $j_n$ |
|---|---|---|
| **1** | **2** | $K_1\backslash\{1,2\}$ |
| 2 | 1 | ... |

$R^*_{26-}$

| $j_1$ | $j_2$ | $\dots$ $j_n$ |
|---|---|---|
| 1 | 2 | ... |
| 2 | 1 | ... |

$R_{29-}$

| $j_1$ | $j_2$ | $\dots$ $j_n$ |
|---|---|---|
| **1** | **2** | $K_1\backslash\{1,2\}$ |
| **2** | **1** | $K_1\backslash\{1,2\}$ |

$R_{11-}$ ,

| $j_1$ | $j_2$ | $\dots$ $j_n$ |
|---|---|---|
| **2** | **1** | $K_1\backslash\{1,2\}$ |
| **1** | **2** | $K_1\backslash\{1,2\}$ |

$R_{11-}$

Table 28: Completion of type III pink rectangles

## Complete list of non-equivalent $\mathcal{C}_2\backslash\mathcal{C}_1$ circuits for $n = 3$

| $j_1$ | $j_2$ | $j_3$ |
|---|---|---|
| | | 3 |
| | 1 | |

$R^*_{25-}$

| $j_1$ | $j_2$ | $j_3$ |
|---|---|---|
| | | 3 |
| | 1 | |

$R^*_{25-}$

Table 29: Circuit in $\mathcal{C}_{2,2}$

| $j_1$ | $j_2$ | $j_3$ |
|---|---|---|
| | | 3 |
| | 1 | |

$R^*_{25-}$

| $j_1$ | $j_2$ | $j_3$ |
|---|---|---|
| 1 | | |
| | 1 | |

$R_{28-}$

Table 30: Circuit in $\mathcal{C}_{2,3}$

| $j_1$ | $j_2$ | $j_3$ |
|---|---|---|
| 1 | | |
| | 1 | |

$R_{28-}$

| $j_1$ | $j_2$ | $j_3$ |
|---|---|---|
| 1 | | |
| | 1 | |

$R_{28-}$

Table 31: Circuit $E \in \mathcal{C}_{2,4}$

## Complete list of non-equivalent $\mathcal{C}_2 \backslash \mathcal{C}_1$ circuits for $n = 4$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ |
|---|---|---|---|
| | 3 | | 4 |
| 2 | | 4 | |

$R^*_{6-}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ |
|---|---|---|---|
| | 3 | | 4 |
| 2 | | 4 | |

$R^*_{6-}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ |
|---|---|---|---|
| | 3 | | 4 |
| 2 | | 4 | |

$R^*_{6-}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ |
|---|---|---|---|
| | | 3 | 4 |
| 4 | | | 3 |

$R_{10-}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ |
|---|---|---|---|
| | | 3 | 4 |
| 4 | | | 3 |

$R_{10-}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ |
|---|---|---|---|
| | | 3 | 4 |
| 4 | | | 3 |

$R_{10-}$

Table 32: Circuits in $\mathcal{C}_{2,0}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ |
|---|---|---|---|
| | 3 | | 4 |
| 2 | | 4 | |

$R^*_{6-}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ |
|---|---|---|---|
| | | 3 | 4 |
| | 1 | | |

$R^*_{25-}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ |
|---|---|---|---|
| | | 3 | 4 |
| 4 | | | 3 |

$R_{10-}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ |
|---|---|---|---|
| | | 3 | 4 |
| | 1 | | |

$R^*_{25-}$

Table 33: Circuits in $\mathcal{C}_{2,1}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ |
|---|---|---|---|
| | 3 | | 4 |
| 2 | | 4 | |

$R^*_{6-}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ |
|---|---|---|---|
| 1 | | | |
| | 1 | | |

$R_{28-}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ |
|---|---|---|---|
| | | 3 | 4 |
| 4 | | | 3 |

$R_{10-}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ |
|---|---|---|---|
| 1 | | | |
| | 1 | | |

$R_{28-}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ |
|---|---|---|---|
| | | 3 | 4 |
| | | 4 | 3 |

$R_{11-}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ |
|---|---|---|---|
| 1 | | | |
| | 1 | | |

$R^*_{28-}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ |
|---|---|---|---|
| | | 3 | 4 |
| | 1 | | |

$R^*_{25-}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ |
|---|---|---|---|
| | | 3 | 4 |
| | 1 | | |

$R^*_{25-}$

Table 34: Circuits in $\mathcal{C}_{2,2}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ |
|---|---|---|---|
| | | 3 | 4 |
| | 1 | | |

$R^*_{25-}$

| $j_1$ | $j_2$ | $j_3$ | $j_3$ |
|---|---|---|---|
| 1 | | | |
| | 1 | | |

$R_{28-}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ |
|---|---|---|---|
| | | | 4 |
| 2 | 1 | | |

$R^*_{27-}$

| $j_1$ | $j_2$ | $j_3$ | $j_3$ |
|---|---|---|---|
| 1 | 2 | | |
| 2 | 1 | | |

$R_{29-}$

Table 35: Circuits in $\mathcal{C}_{2,3}$

| $j_1$ | $j_2$ | $j_3$ | $j_4$ |
|---|---|---|---|
| 1 | | | |
| | 1 | | |

$R^*_{28-}$

| $j_1$ | $j_2$ | $j_3$ | $j_3$ |
|---|---|---|---|
| 1 | | | |
| | 1 | | |

$R^*_{28-}$

Table 36: Circuit $E \in \mathcal{C}_{2,4}$