Existence and Computation of Equilibria in Games and Economies

by

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Abstract

In this dissertation we study the existence and computation of equilibria in games and economies. The results in this dissertation have been presented in the form of several papers, one of which have already been published.

The first paper is entitled "Computation of Stationary Equilibrium Payoffs in Coalitional Bargaining." In this paper, we provide an algorithm to compute the equilibrium payoffs in the coalitional bargaining model of Eraslan-McLennan (*Journal of Economic Theory*, 2013) by using recent developments in methods of numerical algebraic geometry. The Eraslan-McLennan model is a legislative bargaining model which studies weighted voting games with players that are heterogeneous in their discount factors, voting weights and in terms of the probabilities of being selected as the proposer. Eraslan-McLennan characterizes the equilibria as fixed points of a set-valued function. In this paper, we show that the equilibria of the game can be characterized by solutions to a system of polynomial equations and provide an algorithm to compute the equilibrium payoffs. As an alternative approach, we show that all equilibria of such games can be characterized by fixed points of a continuous function, and use a variety of fixed point algorithms to execute this observation. These algorithms have implications for computing equilibria of dynamic models and should be useful in other applied work.

The second paper is entitled "On the Nonemptiness of the α-core of Discontinuous Games: Transferable and Nontransferable Utilities" (published in *Journal of Economic Theory*, 2015). The nonemptiness of the α-core of games

with continuous payoff functions was proved by Scarf (1971) for nontransferable utilities and by Zhao (1999a) for transferable utilities. In this paper we present generalizations of their results to games with possibly discontinuous payoff functions. Our handling of discontinuity is based on Reny's (1999) better-reply-security concept. We present examples to show that our generalizations are nonvacuous.

The third paper is entitled "On the Nonemptiness of the Transferable Utility β-core of Discontinuous Games." Zhao (1999) proved the nonemptiness of the transferable utility β-core of games with continuous payoff functions. In this paper we present generalizations of his result to games with possibly discontinuous payoff functions by applying the concepts and methods we introduced in the second paper. We rely on Reny's (1999) concept of "better-reply-security" to handle any discontinuities that may arise. We present applications to show that our generalizations are nonvacuous.

The fourth paper is entitled "On the Existence of Equilibrium with Discontinuous Preferences: Games and Economies." In this paper, we present three fixed point theorems that can be seen as generalizations of the earlier works of Browder and Fan-Glicksberg and illustrate the use of these theorems as a 'methodological toolkit' for existence issues in a variety of economic settings. In particular, we present a synthetic treatment of the problem of the existence of an equilibrium in games and economies when the preferences of the individuals are not necessarily continuous or ordered. We also relate our results to those available in the antecedent literature.

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Chapter 1

Computation of Stationary Equilibrium Payoffs in Coalitional Bargaining

1.1 Introduction

Baron and Ferejohn (1989) study a dynamic bargaining model which has become a leading framework for the study of legislative decision making. In Baron-Ferejohn model a group of n risk neutral agents divide a fixed pie. In each period a proposer is randomly selected, the proposer offers a division of the pie, and this division is implemented if a majority of the agents vote to approve it. Otherwise the process is repeated until agreement is achieved, with payoffs discounted geometrically. In recent work, Eraslan and McLennan (2013), EM hereafter, study a substantially general model in which players are heterogeneous in discount factors, voting weights and and in terms of the probabilities of being selected as the proposer.[1] They show that the stationary subgame perfect equilibrium payoffs of this model are unique.

[1] They also allow the sum of the recognition probabilities to be less than one with the interpretation that at each period there may be no proposer with positive probability.

In this paper we study the computable aspects of the EM model and show that algorithms to compute Nash equilibria of finite games can be used to compute the stationary subgame perfect equilibrium payoffs. EM characterize the equilibria as fixed points of a set-valued function and use index theory show that the stationary subgame perfect equilibrium payoffs are unique. In this paper we provide alternative characterizations of equilibria which allow us to use the algorithms to compute Nash equilibria of finite games. Our characterizations are based on a new result that is analogous to the "best response condition" for finite games due to Nash (1951). The best-response condition states that a mixed strategy profile is a Nash equilibrium if and only if all pure strategies in the support of each player's mixed strategy gives maximal, and hence equal, payoff to that player. In particular, we characterize the equilibria of the EM model as solutions to a system of polynomial equations and as fixed points of a continuous function. We then use the state of the art algorithms to solve a system of polynomials and compute fixed points of a continuous function to compute the unique stationary subgame perfect equilibrium payoffs. The polynomial characterization has an additional attraction. Even though we are primarily focused on the equilibrium payoffs, finding the set of equilibrium winning coalitions is important in a variety of applied contexts. The recent developments in numerical algebraic geometry allows us to efficiently compute all solutions to system of polynomial equations.

Kalandrakis (2015) provided an algorithm to compute the stationary subgame perfect equilibrium of a special case of this model studied by Eraslan (2002).[2] This special model assumes k-majority winning rule for $1 \leq k \leq n$

[2]Kalandrakis show that the equilibrium payoffs of this special model can be characterized

where the agents have equal voting weights, but in many theoretical and applied settings such as legislative and corporate bankruptcy it is also natural to allow different agents to have different weights in the voting rule to evaluate a proposal. For example, this model has been recently applied to French water negotiations, Belgian railway negotiatons and the decision process of EU Council of Ministers, see Simon et al. (2007), Proost and Zaporozhets (2013) and Le Breton and Montero and Zaporozhets (2012). Therefore, the algorithms we provide for the more general model of this paper could greatly expand the range of theoretical and empirical work that can be supported computationally, and have implications for computing equilibria of dynamic models and should be useful in applied work.

The rest of the paper is organized as follows. Section 1.2 describes the model, Section 1.3 provides a three player weighted voting example, Section 1.4 presents alternative characterizations of the equilibria, Section 1.5 provides the algorithms and the computation results, and Section 1.6 concludes.

1.2 Model

In this section, we present the coalitional bargaining model of Eraslan and Mclennan (2013). Let $N = \{1, \ldots, n\}$ denote the set of agents who bargain over the allocation of a perfectly divisible pie of size 1. The allocation protocol is as follows. At the first period, agent $i \in N$ is selected as a proposer with probability $p_i \in [0, 1]$. The probability[3] p_i is referred as the recognition probability of player

as solutions to two piecewise linear equations in new variables. However, the weighted voting rule brings additional nonlinearity in the EM model, hence Kalandrakis' characterization cannot be directly applied to the EM model.

[3]We use subscripts for the components of a vector and superscripts for the players. Moreover, we lower case letters for vectors and scalars where vectors are in bold, and upper case

i. We assume $p_1 + \ldots + p_n \leq 1$. We allow there is no proposer and define $p_0 = 1 - (p_1 + \ldots + p_n)$ as the probability that there is no proposer. The set of feasible allocations (proposals) is defined as

$$X = \left\{ x \in [0,1]^n : \sum_{i=1}^{n} x_i = 1 \right\}$$

where for each $x \in X$, x_i denotes the pie share for player i. Upon recognition, agent i selects a proposal from X. All agents observe the proposal and randomly ordered to evaluate the proposal by voting for or against it. Each agent observes the votes of the agents in the ordering before selecting her own vote. The agents have heterogeneous and subjective voting weights. For each agent, a collection of agents called a *winning coalition* for herself if her proposal is accepted when they all vote for it. The set of winning coalitions of agent i is exogenously given. We assume agent i is always included in all of her winning coalitions.

If all the members of some winning coalition of the proposer i votes for her proposal, then the pie is allocated according to the proposal and the game ends. Otherwise the same process is repeated in the next period. The preferences of the agents are as follows. If the proposal $x \in \Pi$ is implemented in period t, agent i gets payoff $\delta_i^t x_i$ where $\delta_i \in (0,1)$ is the discount factor of agent i. If a proposal is never implemented, then every agent gets 0.

The equilibrium concept that we are interested is the stationary subgame perfect equilibrium. We will first introduce some useful notation. For each agent i and her winning coalition $\tilde{S} \subseteq N$, let $S \in \{0,1\}^n$ be the column vector whose jth component is 1 if $j \in \tilde{S}$ and 0 otherwise. Let

$$\mathcal{S}^i = \{S : \tilde{S} \text{ is a winning coalition of agent } i\}$$

letters for matrices.

be the collection of $n \times 1$ vectors of zeros and ones characterizing the set of winning colaitions of agent i. Let

$$\mathcal{V} = \left\{ v \in [0,1]^n : \sum_{i=1}^{n} v_i = 1 \right\}$$

be the set of possible expected payoff vectors of the game where for each $v \in \mathcal{V}$, v_i denotes the pie share for player i.

Now we define the stationary subgame perfect (SSP) equilibrium payoffs for the EM model.[4] Let $v \in \mathcal{V}$ be an SSP payoff vector. Equilibrium voting strategies are characterized as follows. Let player j is a proposer and she proposed $x \in X$. Then, player i

$$\begin{aligned} \text{accepts the proposal if } & x_i \geq \delta_i v_i, \\ \text{rejects the proposal if } & x_i < \delta_i v_i, \end{aligned}$$

for all player $i \in N$.[5] And equilibrium pure proposal strategies are characterized as follows. From the voting strategies defined above, player i must to offer at least $\delta_j v_j$ if she wants player j to approve for her proposal. Since each player's payoff is strictly increasing in her own cake share, she will offer $\delta_j v_j$ to player j in order to gain her vote. Therefore, any proposal player j makes with positive probability in equilibrium can be written as $x^j(v) = (x_1^j(v), \ldots, x_n^j(v))$ with

$$x_i^j(v) = S_i^j(v)\delta_i v_i$$

for any player $j \neq i$, and

$$x_j^j(v) = 1 - \sum_{i \neq j} S_i^j(v)\delta_i v_i$$

[4]The formal and precise definition of the SSP equilibrium for the EM model is provided in the Appendix of Eraslan and McLennan (2013).

[5]Here player i's decision when she is indifferent is not restrictive. See footnote 6 in Eraslan (2015).

where $S_i^j(v) \in \{0,1\}$ for all i, and the vector $S^j(v) = (S_1^j(v), \ldots, S_n^j(v))$ solves

$$\min_{S \in \{0,1\}^n} \sum_{i \neq j} S_i \delta_i v_i \ \text{ subject to } \ S \in \mathcal{S}^j. \tag{1.1}$$

Now, we define the SSP equilibrium proposals in *mixed* strategies. Note that any proposer indifferent among all the proposals in the support of his equilibrium proposal in pure strategies. Let $\mathcal{Y}^j = \text{conv}(\mathcal{S}^j)$ be the convex hull of the set of winning coalitions of agent j. Note that the ith component y_i^j of $y^j \in \mathcal{Y}^j$ is the probability that agent j includes agent i in her winning coalition. Then, $y^j(v) \in \mathcal{Y}^j$ is a solution to

$$\min_{y^j \in [0,1]^n} \sum_{i \neq j} y_i^j \delta_i v_i \ \text{ subject to } \ y^j \in \mathcal{Y}^j \tag{1.2}$$

if and only if it is a weighted average of the solutions to Equation 1.1. Therefore, the one-shot deviation principle implies the SSP equilibrium payoffs are characterized by

$$v_i = \left(p_0 + \sum_{j \neq i} p_j y_i^j(v) \right) \delta_i v_i + p_i \left(1 - \sum_{j \neq i} y_j^i(v) \delta_j v_j \right) \tag{1.3}$$

where $y^i(v) = (y_1^i(v), \ldots, y_n^i(v))$ solves Equation 1.2 for all i.

Eraslan and McLennan (2013) showed that the SSP equilibrium payoffs exist and are unique. In this paper, we provide different characterizations of the equilibrium payoffs which allows us to use the algorithms to compute the Nash equilibria of finite games to compute the SSP equilibrium payoffs in the EM model.

1.3 Example

In this section we provide a simple coalitional bargaining example in order to illustrate the concepts. There are three agents, each of them is selected with equal probability and each has the discount factor 0.9. Player 1 has two votes and the other players have one vote each. A proposal is approved if it gets at least three votes. Under this voting rule, in order to pass a proposal, Player 1 needs only one of the other players' vote, however each of the other players needs player 1's vote. Hence, player 1 is part of any winning coalition. Does this mean player 1 gets the whole cake in equilibrium? No, he still needs one of the other player's vote and hence, has to provide sufficient incentive. Note that, Player 1's set of winning coalitions consists of $\{(1,2),(1,3),(1,2,3)\}$ and players two and three is $\{(1,2),(1,2,3)\}$. The vector representation of the set of winning coalitions are

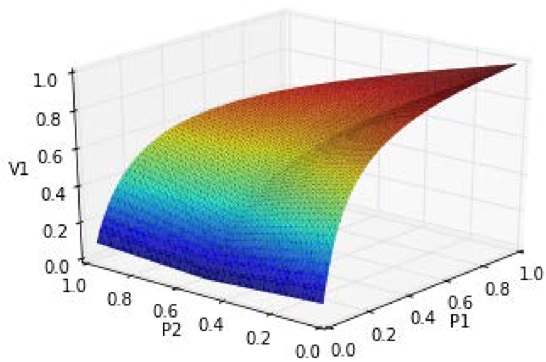
$$\mathcal{S}^1 = \left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}, \quad \mathcal{S}^2 = \left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}, \quad \mathcal{S}^3 = \left\{ \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}$$

Since each player's objective is to choose the cheapest coalition in order to pass her proposal, without loss of generality we can eliminate the grand coalition from the set of winning coalitions while computing the equilibrium payoffs. Therefore, players 2 and 3 has a unique choice, hence their decisions are trivial. For a given SSP payoff vector v, Player 1's problem given in Equation 1.2 is equivalent to choosing $\lambda \in [0,1]$ to minimize $\lambda v_2 + (1-\lambda)v_3$ where λ is the probability that she includes player 2 in her coalition. In this example this is same as probability that she chooses her first winning coalition. It is clear that if $v_3 > v_2$, then player 1 includes player 2 and if $v_3 < v_2$, then player 3 with probability 1. If $v_3 = v_2$, then she is indifferent. Then, the SSP equilibrium payoffs are characterized by

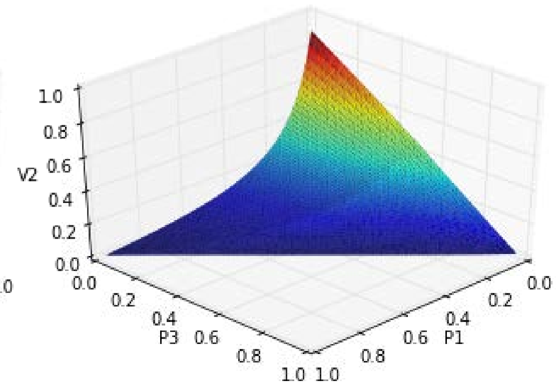
$$\begin{aligned} v_1 &= \tfrac{2}{3}0.9v_1 + \tfrac{1}{3}(1 - 0.9(\lambda v_2 + (1-\lambda)v_3)), \\ v_2 &= \tfrac{1}{3}\lambda 0.9v_2 + \tfrac{1}{3}(1 - 0.9v_1), \\ v_3 &= \tfrac{1}{3}(1-\lambda)0.9v_3 + \tfrac{1}{3}(1 - 0.9v_1) \end{aligned}$$

where λ is 1 if $v_2 < v_3$, is 0 if $v_2 > v_3$ and in $[0, 1]$ otherwise. The second and third equations have a solution if and only if $v_2 = v_3$. This implies $\lambda = 0.5$. Plugging these into the first and the second equations and a simple algebra implies the unique solution to this system is $v = (0.74, 0.13, 0.13)$. In equilibrium, double voting power of player 1 allows her to extract around three quarters of the surplus.

Now, we illustrate how the equilibrium payoffs change with different parameter values. In Figure 1.1, we set each player has the discount factor 0.9. Panel (a) and (b) illustrate the equilibrium payoffs of players 1 and 2 as a function of the recognition probabilities. It is clear that player 1's equilibrium payoffs increases with her recognition probability assuming that the second player's recognition probability is held fixed. It is easy to see from the figure that this monotonicity relation does not hold if we allow different values of player 2 and 3's recognition probabilities. As an example, for $p = (0.25, 0.05, 0.7)$ the equilibrium payoffs are $v = (0.76, 0.02, 0.24)$ and for $p' = (0.3, 0.35, 0.35)$ the equilibrium payoffs are $v = (0.70, 0.15, 0.15)$. The equilibrium payoff of player 1 does not monotonically change with player 2's recognition probability provided that her recognition probability is held constant. As player 2's recognition probability increases she is selected as the proposer more often and includes player 1 more often in her coalition but her vote become more expensive and player 1 has t pay more in order to buy her vote. There two competing effects are illustrated in Panel (a).



(a) Player 1's equilibrium payoffs



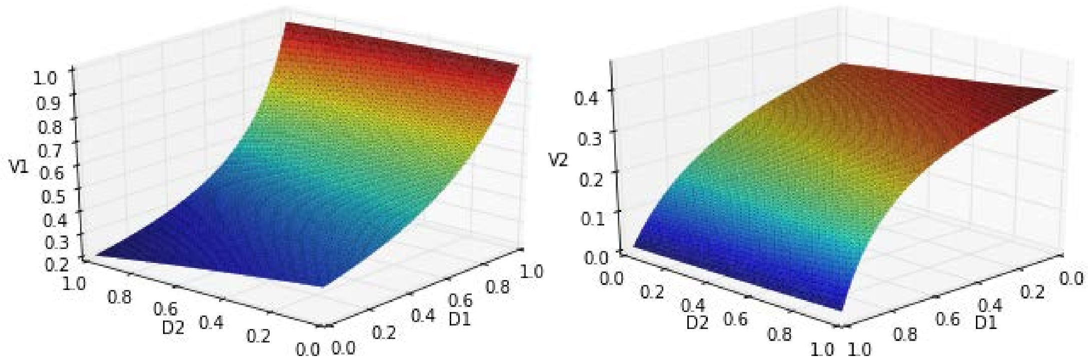
(b) Player 2's equilibrium payoffs

Figure 1.1: Equilibrium Payoffs and Recognition Probabilities

We illustrate the equilibrium payoffs as a function of discount factors in Figure 1.2. We assume players have the same recognition probability and players 2 and 3 have the same discount factor. Panels (a) and (b) illustrates that each player's equilibrium payoff is increasing in her discount factor assuming the others' discount factors are held constant. Moreover, each player's equilibrium payoff is decreasing in each of the other player's discount factor assuming the other discount factors are held constant.

1.4 Characterization of Equilibria

In this section we provide three distinct characterizations of SSP equilibrium payoffs which are useful for computation of equilibrium payoffs. The characterizations hinge on a new result provided in Lemma 1 below which is analogous to the "best-response condition" for finite games due to Nash (1951). The best-response condition states that a mixed strategy profile is a Nash equilibrium



(a) Player 1's equilibrium payoffs

(b) Player 2's equilibrium payoffs

Figure 1.2: Equilibrium Payoffs and Discount Factors

if and only if all pure strategies in the support of each player's mixed strategy gives maximal, and hence equal, payoff to that player. It is this condition that underlies the algorithms to compute the equilibria since for each player it states a finite condition about all pure strategies of the player which is easily checked, rather than about the infinite set of all mixed strategies. It is by now well-known that a Nash equilibrium can be characterized as a fixed point of a function, solution to a system of polynomial equations, solution to a non-linear complementarity problem, minimum of a function on a polytope, stationary point problem and semialgebraic set; see McKelvey and McLennan (1996).

Our result presented in Lemma 1 which is analogous to Nash's best-response condition is based on interpreting the winning coalitions as pure strategies of the players and we obtain the inclusion probabilities through randomizing over pure strategies. We first present this result. Based on this, we show that we can use the algorithms to compute the equilibria of finite games to compute the stationary subgame perfect equilibria of Eraslan-McLennan model.[6]

[6]Although we present only the three characterizations in this section, based on our results,

Recall that $\mathcal{S}^i$ represents the set of all winning coalitions for player i and $\mathcal{Y}^i = \text{conv}(\mathcal{S}^i)$ the convex hull of her set of winning coalitions. Let Λ^i be the set of all probability distributions on $\mathcal{S}^i$. We interpret $S^i \in \mathcal{S}^i$ as a pure strategy and $\lambda^i \in \Lambda^i$ a mixed strategy of player i. In particular, for the vector $S^i \in \mathcal{S}^i$, the jth component S^i_j indicates whether player j is part part of the winning coalition S^i of player i. Similarly, the scalar $\lambda^i_{S^i}$ denotes player i's probability of choosing her winning coalition S^i in her mixed strategy λ^i. Let $\Lambda = \prod_{i \in N} \Lambda^i$, λ^i and λ are typical elements of Λ^i and Λ, respectively. For each $\lambda^i \in \Lambda^i$, let $supp(\lambda^i)$ be the support of λ^i, i.e. set of all $S^i \in \mathcal{S}^i$ such that $\lambda^i_{S^i} > 0$. Define $\mathbf{y}^i : \Lambda^i \to [0, 1]^n$ as

$$\mathbf{y}^i(\lambda^i) = \sum_{S^i \in \mathcal{S}^i} \lambda^i_{S^i} S^i. \tag{1.4}$$

Specifically, given player i's mixed strategy λ^i, the vector $\mathbf{y}^i(\lambda^i)$ denotes the vector of inclusion probabilities for player i where its jth component is the probability that player i includes player j in her coalition conditional on being the proposer. Let $\mathbf{Y}(\lambda) = (\mathbf{y}^1(\lambda^1), \ldots, \mathbf{y}^n(\lambda^n))$. It is clear that $\mathbf{Y}(\lambda) \in \mathcal{Y} = \prod_{i \in N} \mathcal{Y}^i$ for all $\lambda \in \Lambda$. Define $\mathbf{m} : \Lambda \to \mathbb{R}^n$ as $\mathbf{m}(\lambda) = p_0 1 + \mathbf{Y}(\lambda)p$. Define $\Delta = \nabla(\delta)$ as the diagonal matrix whose diagonal entries are the components of δ. Define $P = \nabla(p)$ and $\mathbf{M} : \Lambda \to \mathbb{R}^{n^2}$ as $\mathbf{M}(\lambda) = \nabla(\mathbf{m}(\lambda))$. Define $\mathbf{A} : \Lambda \to \mathbb{R}^{n^2}$ as $\mathbf{A}(\lambda) = I - [\mathbf{M}(\lambda) - P\mathbf{Y}(\lambda)^T]\Delta$, and $\mathbf{v} : \Lambda \to \mathcal{V}$ as

$$\mathbf{v}(\lambda) = \mathbf{A}(\lambda)^{-1} p. \tag{1.5}$$

it is easy to prove that SSP equilibrium payoffs can be characterized as a solution to a non-linear complementarity problem, minimum of a function on a polytope, stationary point problem and semialgebraic set.

Define a correspondence $\boldsymbol{\Lambda}^i : \mathcal{Y}_i \twoheadrightarrow \mathbb{R}_+^{K_i}$for each player i as

$$\boldsymbol{\Lambda}^i(y^i) = \left\{ \lambda^i \in \Lambda^i |\ y^i = \sum_{S^i \in \mathcal{S}^i} \lambda^i_{S^i} S^i \right\}.$$

This correspondence assigns an inclusion probability vector a set of mixed strategies that support it.[7] By construction $\boldsymbol{\Lambda}^i$ has nonempty and convex values. In order to see this, recall that $\mathcal{Y}^i$ is the convex hull of $\mathcal{S}^i$ for each player i. Then, for each $y^i \in \mathcal{Y}^i$, there exists $\lambda^i \in \Lambda^i$ such that $y^i = \sum_{S^i \in \mathcal{S}^i} \lambda^i_{S^i} S^i$. Now pick $\lambda^i, \hat{\lambda}^i \in \boldsymbol{\Lambda}^i(y^i)$ and $\alpha \in (0,1)$. Then, $\sum_{S^i \in \mathcal{S}^i} (\alpha \lambda^i_{S^i} + (1-\alpha)\hat{\lambda}^i_{S^i}) S^i = \alpha \sum_{S^i \in \mathcal{S}^i} \lambda^i_{S^i} S^i + (1-\alpha) \sum_{S^i \in \mathcal{S}^i} \hat{\lambda}^i_{S^i} S^i = y^i$.

Now we characterize the minimization problem of each player in terms of her mixed and pure strategies.

Lemma 1. *Let $v \in \mathcal{V}, i \in N$ and $y^i \in \mathcal{Y}^i$. Then y^i solves*

$$\min_{y \in [0,1]^n} \sum_{j \neq i} y_j \delta_j v_j \quad \textit{subject to} \quad y \in \mathcal{Y}^i$$

if and only if there exists $\lambda^i \in \boldsymbol{\Lambda}^i(y^i)$ such that

$\sum_{j \neq i} y^i_j \delta_j v_j = \sum_{j \neq i} S^i_j \delta_j v_j$ *for all $S^i \in supp(\lambda^i)$,*

$\sum_{j \neq i} y^i_j \delta_j v_j \leq \sum_{j \neq i} S^i_j \delta_j v_j$ *for all $S^i \notin supp(\lambda^i)$.*

Proof. See Appendix. □

Lemma 1 has the following interpretation. Given SSP payoffs v, each player includes those players in her coalition with positive probability who are part of one of her least costly winning coalition.

[7]It is clear that this set need not be unique. We show in equation 1.4 that every mixed strategy induces a unique inclusion probability. Hence, we do not have a one-to-one correspondence between inclusion probabilities and the mixed strategies. However, this does not create any problem except some technical details to be clarified.

The following result is a corollary to Lemma 1 and characterizes the SSP equilibrium payoffs as a function of mixed strategies and a finite set of equalities and inequalities.

Corollary 1. *Let* $\lambda \in \Lambda$. *Then* $v = \mathbf{v}(\lambda)$ *is an SSP equilibrium payoff vector if and only if* $\sum_{j \neq i} \mathbf{y}^i_j(\lambda^i)\delta_j v_j = \sum_{j \neq i} S^i_j \delta_j v_j \leq \sum_{j \neq i} \tilde{S}^i_j \delta_j v_j$ *for all* $S^i \in supp(\lambda^i)$ *and all* $\tilde{S}^i \notin supp(\lambda^i)$.

Eraslan and McLennan (2013) showed that the SSP equilibrium payoffs of this game exist and are unique, see their Proposition 1 and Theorem 1. For the sake of completeness, we provide the existence and uniqueness result based on our new characterization.

Theorem 1. *SSP equilibrium payoffs exist and are unique.*

Proof. See Appendix. □

1.4.1 Equilibria as Fixed Points of a Continuous Function

In this section we provide a characterization of the reduced equilibria as fixed points of a continuous function defined on a convex polytope. This function is analogous to the mapping called "Nash-map" introduced by Nash in his influential 1951 paper.

First, define a gain function $g^i : \mathcal{S}^i \times \Lambda \to \mathbb{R}_+$ for each player i as

$$g^i(S^i, \lambda) = \max\left\{0, \sum_{j \neq i} \mathbf{y}^i_j(\lambda^i)\delta_j \mathbf{v}_j(\lambda) - \sum_{j \neq i} S^i_j \delta_j \mathbf{v}_j(\lambda)\right\}. \quad (1.6)$$

This function has positive values if and only if the wining coalition S^i costs less than the weighted winning coalition $\mathbf{y}^i(\lambda^i)$ induced by the mixed strategy λ^i of player i. In other words, $g^i(S^i, \lambda)$ measures the net gain of player i from choosing the winning coalition S^i instead of the weighted winning coalition $\mathbf{y}^i(\lambda^i)$.

Now we define a mapping $f : \Lambda \to \Lambda$ which is analogous to the Nash-map as $f = (f^1, \dots, f^n)$ where $f^i : \Lambda \to \Lambda^i$ is defined as

$$f^i(\lambda) = \hat{\lambda}^i \quad \text{where} \quad \hat{\lambda}^i_{S^i} = \frac{\lambda^i_{S^i} + g^i(S^i, \lambda)}{1 + \sum_{S^i \in \mathcal{S}^i} g^i(S^i, \lambda)}. \tag{1.7}$$

The mapping f assigns each mixed strategy profile to a mixed strategy profile as follows. Each player i increases her her probability of choosing less costly winning coalitions and decreases the probability of choosing more costly winning coalitions according to the rule given in Equation 1.7. The gain function plays a crucial role in this mapping –if the gain from a winning coalition is high then the player puts more weight on that winning coalition. Moreover, if there is no gain from choosing any winning coalition for each player, then it is clear that we obtain a fixed point of f. And the following result shows that the payoffs induced by such points characterize the SSP equilibrium payoffs.[8]

Theorem 2. *A mixed strategy profile λ is a fixed point of f if and only if the SSP payoffs $\mathbf{v}(\lambda)$ induced by λ are equilibrium payoffs.*

Proof of Theorem 2. Pick $\lambda \in \Lambda$ such that $\mathbf{v}(\lambda)$ is the SSP equilibrium payoff vector. Then, Corollary 1 implies $\sum_{j \neq i} \mathbf{y}^i_j(\lambda^i)\delta_j v_j = \sum_{j \neq i} S^i_j \delta_j v_j \leq \sum_{j \neq i} \tilde{S}^i_j \delta_j v_j$ for all $S^i \in supp(\lambda^i)$ and all $\tilde{S}^i \notin supp(\lambda^i)$. Therefore, $g^i(S^i, \lambda) = 0$ for all $i \in N$ and $S^i \in \mathcal{S}^i$. Hence, $\lambda = f(\lambda)$.

Now assume there exists $\lambda \in \Lambda$ such that $\lambda = f(\lambda)$ and $\mathbf{v}(\lambda)$ is not the SSP

[8] Kalandrakis (2004) provides a similar characterization for the equilibria in sequential bargaining games.

equilibrium payoff vector. Note that Equation 1.3 implies $\mathbf{v}(\lambda)$ is the SSP equilibrium payoff vector if and only if $\mathbf{y}^i(\lambda^i) \in \operatorname{argmin}_{y^i \in \mathcal{Y}^i} \sum_{j \neq i} y^i_j \delta_j \mathbf{v}_j(\lambda)$ for all $i \in N$. Therefore, $\mathbf{v}(\lambda)$ is not the SSP equilibrium payoff vector implies the weighted winning coalition $\mathbf{y}^j(\lambda^j)$ is not a solution to $\min_{y^j \in \mathcal{Y}^j} \sum_{i \neq j} y^j_i \delta_i \mathbf{v}_i(\lambda)$ for some $j \in N$. Then, there exists $y^j \in \mathcal{Y}^j$ such that $\sum_{i \neq j} y^j_i \delta_i \mathbf{v}_i(\lambda) < \sum_{i \neq j} \mathbf{y}^j_i(\lambda^j) \delta_i \mathbf{v}_i(\lambda)$. Therefore, there exists $\bar{S}^j \in \mathcal{S}^j$ such that $\sum_{i \neq j} \bar{S}^j_i \delta_i \mathbf{v}_i(\lambda) < \sum_{i \neq j} \mathbf{y}^j_i(\lambda^j) \delta_i \mathbf{v}_i(\lambda)$. Hence $g^j(\bar{S}^j, \lambda) > 0$, and therefore $\sum_{S^j \in \mathcal{S}^j} g^j(S^j, \lambda) > 0$. Then Equation 1.7 implies $\lambda^j_{S^j} \sum_{\hat{S}^j \in \mathcal{S}^j} g^j(\hat{S}^j, \lambda) = g^j(S^j, \lambda)$ for all $S^j \in \mathcal{S}^j$. Hence $\lambda^j_{S^j} = 0$ if and only if $g^j(S^j, \lambda) = 0$ for all $S^j \in \mathcal{S}^j$. Therefore $\sum_{i \neq j} S^j_i \delta_i \mathbf{v}_i(\lambda) < \sum_{i \neq j} \mathbf{y}^j_i(\lambda^j) \delta_i \mathbf{v}_i(\lambda)$ for all $S^j \in supp(\lambda^j)$. Hence

$$\begin{array}{rcl}
\sum_{i \neq j} \mathbf{y}^j_i(\lambda^j) \delta_i \mathbf{v}_i(\lambda) & = & \sum_{i \neq j} \left(\sum_{S^j \in \mathcal{S}^j} \lambda^j_{S^j} S^j \right)_i \delta_i \mathbf{v}_i(\lambda) \\
& = & \sum_{S^j \in \mathcal{S}^j} \lambda^j_{S^j} \sum_{i \neq j} S^j_i \delta_i \mathbf{v}_i(\lambda) \\
& = & \sum_{S^j \in supp(\lambda^j)} \lambda^j_{S^j} \sum_{i \neq j} S^j_i \delta_i \mathbf{v}_i(\lambda) \\
& < & \sum_{S^j \in supp(\lambda^j)} \lambda^j_{S^j} \sum_{i \neq j} \mathbf{y}^j_i(\lambda^j) \delta_i \mathbf{v}_i(\lambda) \\
& = & \sum_{i \neq j} \mathbf{y}^j_i(\lambda^j) \delta_i \mathbf{v}_i(\lambda).
\end{array}$$

This contradiction implies that $\mathbf{v}(\lambda)$ is the SSP equilibrium payoff vector. $\square$

Theorem 2 allows us to use the existing algorithms to compute the fixed points of f, hence the SSP equilibrium payoffs of the EM model. These algorithms are also the standard algorithms to compute the Nash equilibria of finite games. In Section 5 we illustrate a classic fixed point algorithm namely *Scarf's algorithm* and a state of the art fixed point algorithm namely *McLennan-Tourky algorithm*. We compare the computation results of these algorithms as well as the results of the algorithms to compute solutions a system of polynomial equations.

We now illustrate this construction with our three-player example presented

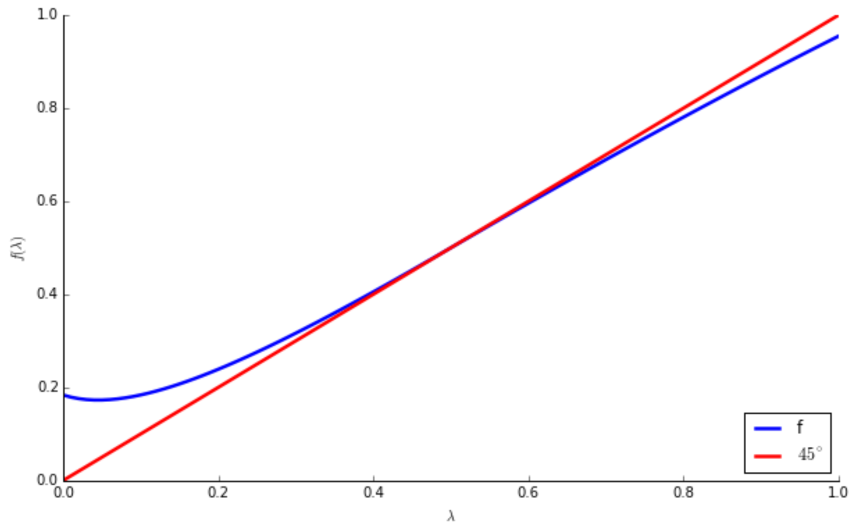


Figure 1.3: The Update Function of Player 1

above. Recall that we can eliminate the grand coalition from the set of winning coalitions of each player since it is not played with positive probability in equilibrium. Hence, the second and the third components of f are constant. Moreover, since player 1 has only two pure strategies after the elimination of the grand coalition focusing on the first component of f_1 is enough to characterize the fixed points of f. To this end, we omit the player subscript and define f on $[0, 1]$ as the first component of player 1's function. The blue line in Figure 1.3 illustrates the graph of this function. The function f has the following interpretation. If player 1 includes player 2 with zero probability in her coalition, $\lambda = 0$, then player 3's value induced by this probability is high, hence player 3 becomes costly. Therefore, it is better for player 1 to include player 2 more often in her coalition. Hence, the function f maps this zero probability to a higher value according Equation 1.7. Similar argument holds for all $\lambda < 0$ and the role of players 2 and 3 are switched for all $\lambda > 0.5$. The function f has a unique fixed point $\lambda = 0.5$ and hence the payoffs induced by λ is the SSP equilibrium payoffs $v = (0.74, 0.13, 0.13)$. Note that the uniqueness of the fixed points of f is not necessarily hold, indeed there can be continuum of fixed points. However, the value induced by each of these fixed points has to be the same.

1.4.2 Equilibria as Solutions to a System of Polynomial Equations

In this section, we provide a characterization of the SSP equilibriu payoffs as solutions to a system of polynomial equations. There has been significant developments in computing the solutions to system of polynomial equations in last two decades. The new algorithms and methods developed recently have been used in computing equilibria in a variety of economic models, especially Nash equilibria of finite games, see Datta (2010), Kubbler and Schmedders (2010) and Kubbler, Renner and Schmedders (2014). We provide the details of these algorithms in the subsequent sections.

A *polynomial* in one variables x with complex coefficients is a function of the form

$$f = \sum_{\alpha=0}^{d} a_\alpha x^\alpha$$

where d is a non-negative integer, a_αs are the complex coefficients and x^αs are monomials. In this paper we are interested in multivariate polynomials defined as follows. A *polynomial* in n variables $x_1, \ldots, x_n$ with complex coefficients is a function of the form

$$f = \sum_{\alpha \in S} a_\alpha x^\alpha, \quad a_\alpha \in \mathbb{C},\ S \subset \mathbb{Z}_+^n \text{ finite}$$

where a_αs are the coefficients and x^αs are monomials defined as follows. A *monomial* in n variables $x_1, \ldots, x_n$ is a product of the form $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ where all of the exponents $\alpha_1, \ldots, \alpha_n$ are nonnegative integers. In general, the coefficients can be elements of any abstract field. However, in this paper we are interested polynomials with real coefficients. For computational purposes, working with complex numbers are very useful which will be clear in the subsequent sections. Hence, the polynomial f defined above is a function $f : \mathbb{C}^n \to \mathbb{C}^n$. Now consider a system of polynomial equations $f_1(x) = \ldots, f_n(x) = 0$. The *solution set* of this system of equations is the set

$$V(f_1, \ldots, f_s) = \{(a_1, \ldots, a_n) \in \mathbb{C}^n : f_i(a_1, \ldots, a_n) = 0 \text{ for all } 1 \leq i \leq s\}.$$

A *rational function* is a function which can be represented as the quotient of two polynomials.

The following result shows that the SSP payoffs is a rational function of the mixed strategy profiles. This allows us to characterize the SSP equilibrium payoffs as solutions a system of polynomial equations.

Proposition 1. *Each component of the vector valued function* $\mathbf{v}$ *defined in Equation* (1.5) *is a rational function.*

Proof of Proposition 1. It is clear that components of $\mathbf{Y}, \mathbf{m}, \mathbf{M}, \mathbf{A}$ are polynomials in K variables $\lambda_{11}, \ldots, \lambda_{nK_n}$. Since inverse of a matrix is the ratio of linear combination of its components, each component of $\mathbf{v}$ is the ratio of two polynomials, hence it is a rational function. □

Now we show that SSP equilibrium payoffs are characterized by solutions to a system of polynomial equations. The following theorem describes a system[9] of $2K$ polynomial equations in $2K$ variables $\left\{\lambda^i_{S^i}, \mu^i_{S^i}\right\}_{i\in N, S^i \in \mathcal{S}^i}$ with real

[9] Recall that $K = \sum_{i=1}^n K_i$ is the sum of the number of all players' winning coalitions.

coefficients whose nonnegative real solutions contains the equilibrium mixed strategies of the agents in the EM model. Moreover, the payoffs $\mathbf{v}(\lambda)$ induced the equilibrium mixed strategies λ are the SSP equilibrium payoffs.

Theorem 3. *Let $\lambda \in \Lambda$. Then $\mathbf{v}(\lambda)$ is the SSP equilibrium payoff vector if and only if there exists $\mu \in \mathbb{R}^K_+$ such that (λ, μ) satisfy the following system of $2K$ polynomial equations in $2K$ unknowns*

$$\sum_{j \neq i} S^i_j \delta_j \mathbf{v}_j(\lambda) - \mu^i_{S^i} = \sum_{j \neq i} \bar{S}^i_j \delta_j \mathbf{v}_j(\lambda) - \mu^i_{\bar{S}^i} \text{ for all } i \in N \text{ and } S^i \in \mathcal{S}^i, S_i \neq \bar{S}^i,$$

$$\lambda^i_{S^i} \mu^i_{S^i} = 0 \text{ for all } i \in N \text{ and } S^i \in \mathcal{S}^i, \qquad (\star)$$

$$\sum_{S^i \in \mathcal{S}^i} \lambda^i_{S^i} = 1 \text{ for all } i \in N.$$

This theorem has the following interpretation. A mixed strategy profile λ has K many components to be determined. For each component $\lambda^i_{S^i}$ we introduce a non-negative slack variable $\mu^i_{S^i}$ such that if the corresponding winning coalition is chosen with positive probability then the slack variable is zero and if the slack variable is positive than the corresponding winning coalition is zero. This property is illustrated in line two above and such relation is called *complementarity*. The first line above compares the costs of winning coalitions for each player. Together with line two, they imply that the slack variables corresponding to the least costly coalitions must be zero and the other slack variables are positive. The last line above stresses that λ is indeed a mixed strategy profile.

Proof of Theorem 3. Pick $\lambda \in \Lambda$ and assume $\mathbf{v}(\lambda)$ is an SSP equilibrium payoff vector. It is clear that $\lambda \geq 0$ and $\sum_{S^i \in \mathcal{S}^i} \lambda^i_{S^i} = 1$ for all $i \in N$. For each $i \in N$

and $S^i \in \mathcal{S}^i$, let

$$\mu^i_{S^i} = \sum_{j \neq i} S^i_j \delta_j \mathbf{v}_j(\lambda) - \sum_{j \neq i} \mathbf{y}^i_j(\lambda^i) \delta_j \mathbf{v}_j(\lambda).$$

Corollary 1 implies $\mu^i_{S^i} \geq 0$. The quantities $\sum_{j \neq i} S^i_j \delta_j \mathbf{v}_j(\lambda) - \mu^i_{S^i}$ are all equal to $\sum_{j \neq i} \mathbf{y}^i_j(\lambda^i) \delta_j \mathbf{v}_j(\lambda)$ and hence to each other. In order to show that $\lambda^i_{S^i} \mu^i_{S^i} = 0$ for all $i \in N$ and $S^i \in \mathcal{S}^i$, pick $i \in N, \hat{S}^i \in \mathcal{S}^i$.

$$\begin{aligned}
\lambda^i_{\hat{S}^i} \mu^i_{\hat{S}^i} &= \lambda^i_{\hat{S}^i} \left(\textstyle\sum_{j \neq i} \hat{S}^i_j \delta_j \mathbf{v}_j(\lambda) - \sum_{j \neq i} \mathbf{y}^i_j(\lambda^i) \delta_j \mathbf{v}_j(\lambda) \right) \\
&= \lambda^i_{\hat{S}^i} \left(\textstyle\sum_{j \neq i} \hat{S}^i_j \delta_j \mathbf{v}_j(\lambda) - \sum_{j \neq i} \left(\sum_{S^i \in \mathcal{S}^i} \lambda^i_{S^i} S^i \right)_j \delta_j \mathbf{v}_j(\lambda) \right) \\
&= \lambda^i_{\hat{S}^i} \left(\textstyle\sum_{j \neq i} \hat{S}^i_j \delta_j \mathbf{v}_j(\lambda) - \sum_{S^i \in \mathcal{S}^i} \lambda^i_{S^i} \sum_{j \neq i} S^i_j \delta_j \mathbf{v}_j(\lambda) \right) \\
&= \lambda^i_{\hat{S}^i} \left(\textstyle\sum_{j \neq i} \hat{S}^i_j \delta_j \mathbf{v}_j(\lambda) - \sum_{j \neq i} \mathbf{y}^i_j(\lambda^i) \delta_j \mathbf{v}_j(\lambda) \right) + \sum_{j \neq i} \mathbf{y}^i_j(\lambda^i) \delta_j \mathbf{v}_j(\lambda) \\
&\quad - \textstyle\sum_{S^i \in \mathcal{S}^i} \lambda^i_{S^i} \sum_{j \neq i} S^i_j \delta_j \mathbf{v}_j(\lambda) \\
&= \left(1 - \lambda^i_{\hat{S}^i} \right) \textstyle\sum_{j \neq i} \mathbf{y}^i_j(\lambda^i) \delta_j \mathbf{v}_j(\lambda) - \sum_{S^i \neq \hat{S}^i} \lambda^i_{S^i} \sum_{j \neq i} S^i_j \delta_j \mathbf{v}_j(\lambda) \\
&= \textstyle\sum_{S^i \neq \hat{S}^i} \lambda^i_{S^i} \left(\sum_{j \neq i} \mathbf{y}^i_j(\lambda^i) \delta_j \mathbf{v}_j(\lambda) - \sum_{j \neq i} S^i_j \delta_j \mathbf{v}_j(\lambda) \right).
\end{aligned}$$

Corollary 1 implies $\sum_{j \neq i} \mathbf{y}^i_j(\lambda^i) \delta_j \mathbf{v}_j(\lambda) - \sum_{j \neq i} S^i_j \delta_j \mathbf{v}_j(\lambda)$ for all $S^i \in \mathcal{S}^i$. Hence, $\lambda^i_{\hat{S}^i} \mu^i_{\hat{S}^i} \leq 0$. Since $\lambda^i_{\hat{S}^i}, \mu^i_{\hat{S}^i} \geq 0$, $\lambda^i_{\hat{S}^i} \mu^i_{\hat{S}^i} = 0$.

Pick $(\lambda, \mu) \in \Lambda \times \mathbb{R}^K_+$ which satisfy the polynomial system. Set $v = \mathbf{v}(\lambda)$. In order to show that v is the SSP equilibrium payoff vector we must show that given v, the weighted winning coalition $\mathbf{y}^i(\lambda^i)$ is the least costly weighted winning coalition for each player i. To this end, pick $i \in N$ and $\hat{y}^i \in \mathcal{Y}^i$. Then, there exists $\hat{\lambda}^i \in \Lambda^i$ such that $\hat{y}^i = \mathbf{y}^i(\hat{\lambda}^i)$. Since $\hat{\lambda}^i$ and λ^i are probability measures, $\sum_{S^i \in \mathcal{S}^i} (\hat{\lambda}^i_{S^i} - \lambda^i_{S^i}) = 0$. Hence there exists $\hat{S}^i \in \mathcal{S}^i$ such that $\hat{\lambda}^i_{\hat{S}^i} \geq \lambda^i_{\hat{S}^i}$.

Then,

$$\begin{aligned}
\textstyle\sum_{j\neq i}\mathbf{y}^i_j(\hat{\lambda}^i)\delta_j v_j-\sum_{j\neq i}\mathbf{y}^i_j(\lambda^i)\delta_j v_j &= \textstyle\sum_{S^i\in\mathcal{S}^i}\left(\hat{\lambda}^i_{S^i}-\lambda^i_{S^i}\right)\sum_{j\neq i}S^i_j\delta_j v_j\\
&= \textstyle\sum_{S^i\in\mathcal{S}^i}\left(\hat{\lambda}^i_{S^i}-\lambda^i_{S^i}\right)\sum_{j\neq i}S^i_j\delta_j v_j-\\
&\quad \textstyle\sum_{S^i\in\mathcal{S}^i}\left(\hat{\lambda}^i_{S^i}-\lambda^i_{S^i}\right)\sum_{j\neq i}\hat{S}^i_j\delta_j v_j\\
&= \textstyle\sum_{S^i\in\mathcal{S}^i}\left(\hat{\lambda}^i_{S^i}-\lambda^i_{S^i}\right)\sum_{j\neq i}\left(S^i_j-\hat{S}^i_j\right)\delta_j v_j\\
&= \textstyle\sum_{S^i\in\mathcal{S}^i}\left(\hat{\lambda}^i_{S^i}-\lambda^i_{S^i}\right)\left(\mu^i_{S^i}-\mu^i_{\hat{S}^i}\right)\\
&= \textstyle\sum_{S^i\neq\hat{S}^i}\hat{\lambda}^i_{S^i}\mu^i_{S^i}+\mu^i_{\hat{S}^i}\sum_{S^i\neq\hat{S}^i}\left(\lambda^i_{S^i}-\hat{\lambda}^i_{S^i}\right)\\
&= \textstyle\sum_{S^i\neq\hat{S}^i}\hat{\lambda}^i_{S^i}\mu^i_{S^i}+\mu^i_{\hat{S}^i}\left(\hat{\lambda}^i_{\hat{S}^i}-\lambda^i_{\hat{S}^i}\right)\\
&\geq 0
\end{aligned}$$

where the last inequality follows from $(\lambda,\mu)\in\Lambda\times\mathbb{R}^K_+$ and $\hat{\lambda}^i_{\hat{S}^i}\geq\lambda^i_{\hat{S}^i}$. Hence, $\mathbf{y}^i(\lambda^i)\in\operatorname{argmin}_{y^i\in\mathcal{Y}^i}\sum_{j\neq i}y^i_j\delta_j v_j$ for all $i\in N$ where $v=\mathbf{v}(\lambda)$. Therefore, $\mathbf{v}(\lambda)$ is the SSP equilibrium payoff vector. □

This result states that SSp equilibrium payoffs are characterized by solutions to a NLCP (nonlinear complementarity problem). Furthermore, it shows that these nonlinear functions are polynomials. We use the recent developments in the field of numerical algebraic geometry to compute the solutions of this system of polynomial equations, see Sommese and Wampler (2005).

1.4.3 Equilibria as Nash Equilibria of a 2-person Game

Let $v\in\mathcal{V}$ be a SSP payoff and consider the minimization problem defined in Equation 1.2. It is clear that each player's minimization problem depends on her choice and the payoff vector v, and does not depend on others' choices. Therefore, given v, we can assume that a fictitious player chooses an optimal strategy for all players. His set of pure strategies is the Cartesian product of all player's winning coalitions $\mathcal{S}=\prod_{i\in N}\mathcal{S}_i$. His mixed strategies is the set of all probability distributions on $\mathcal{S}$. Note that his set of mixed strategies corresponds to the set of all correlated strategies of individual players. But, in this model given v, since each player's payoff does not depend on other players' choices, players cannot gain from correlation, hence mixed strategies and correlated strategies are payoff equivalent. Player 1's objective is to minimize the aggregate cost of coalitions. Player 2's set of pure strategies is $\mathcal{V}$ and her objective is to match the image of player 1's choice, $\mathbf{v}(\lambda)$. It is easy to see that $v(\lambda)$ is an SSP equilibrium payoff if and only if $(\lambda, v(\lambda))$ is a Nash equilibrium of the two person game defined above.

At this point we do not have an algorithm for this characterization. We are currently working on an algorithm based on discretization of the action sets of each players where at each iteration we endogenously increase the number of grid points. Note that player 1's payoff function is linear in player 2's strategies but player 2's payoff function is nonlinear in player 1's mixed strategies. A useful feature of such an algorithm is that we can explicitly incorporate the structure of the model.

1.5 Computation of Equilibrium Payoffs

In this section we present and compare the algorithms we use to compute the unique stationary subgame perfect equilibrium payoffs of EM model. In the previous section, we characterized the equilibria as fixed points of a continuous function defined on a convex polytope and as solutions to a system of polynomial equations. In order to compute a fixed point of a function, we use Scarf's (1967) algorithm and the recent imitation game algorithm introduced by see McLennan and Tourky (2010). For solving a system of polynomial equations we use recently provided homotopy continuation based algorithms to solve such systems, see Datta (2010), Kubbler, Renner and Schmedders (2014) and Sommese and Wampler (2005).

We use Python and C to execute these algorithms. For the imitation games algorithm, we modify and use the C codes for the imitation games algorithm that is developed as part of the GAMBIT project.[10] The imitation games algorithm is accepted as a good algorithm in practice, particularly for high dimensional problems. Our initial results show that imitation games algorithm works fast –our tests for the number of players 3, 5, 7, 10, for various recognition probabilities, discount factors and set of winning coalitions, the algorithm converges on average less than one second for the precision error 10^{-8}. Our experience is similar for the Scarf's algorithm. McLennan-Tourky algorithm is known for its better performance for high dimensional problems, hence as the number of the players increase we expect it to have a better performance compared to Scarf's algorithm. We will present detailed comparison of the imitation algorithm, Scarf's algorithm and the homotopy continuation based algorithms to solve the system of polynomial equations for computing the EM model for different parameter values.

Before proceeding to the description of these algorithms, it is worth noting that that the simple iterative map we defined for the fixed point algorithm which is analogous to the Nash map for finite games is convergent for all the initial values that we randomly picked. This is not one of the algorithms we present

[10] The code is written by Colin Ramsay and the source is freely available in http://cupid.economics.uq.edu.au/mclennan/Software/software.html.

in this paper and at this point, we do not have a proof that it convergences. We want to stress that the algorithms we mentioned above and are using for computing equilibrium payoffs of EM model are globally convergent. Although the speed of convergence depends highly on the initial value for this method, it could be a useful benchmark algorithm if such convergence property is proved.

1.5.1 Computing Fixed Points of a Continuous Function

In this section we describe two influential algorithms to compute a fixed point of a continuous function: Scarf's algorithm and imitation games algorithm. The use of fixed point theorems in economic theory dates back to von Neumann's model of an expanding economy published in 1937. They became the main tool to show and characterize the equilibria in many economic model by the early 1950s. However, it had to wait until Scarf's (1967) influential algorithm to be able to compute fixed points. We first describe this algorithm. Second, we describe a recent algorithm introduced by McLennan and Tourky (2010) which uses game theory tools to compute fixed points of a continuous function.

1.5.1.1 McLennan-Tourky Algorithm

Let X be a nonempty and convex and compact subset of a Euclidean space and $f : X \to X$ a continuous function. Then Brouwer's fixed point theorem implies f has a fixed point. McLennan-Tourky (MT) algorithm first defines a simple two person game such that the fixed points of f as Nash equilbria of this game. Then it uses the tools to compute a Nash equilibrium of bimatrix games in order to compute an approximate fixed point of f.

Define a two person game as follows. The action set of each player is X.

Player 1's objective is to chose an action that is closest to the image of player 2's action under f. Player 2's objective is to choose an action that is closest to player 1's action. In any Nash equilibrium, player 2's action is same as player 1's action and player 1's action is same as the image of player 2's action, hence her action. Therefore, a point in X is a fixed point of f if and only if it is a Nash equilibrium of this game. This is a two person game but not a bimatrix game since each player has continuum of actions. Hence we cannot directly use the tools to compute Nash equilibria of bimatrix games.

Hence, the algorithm discretize the set X endogenously as follows. First, we randomly choose a point $x_1$from X and set $\{x_1, f(x_1)\}$ as the common pure strategy set for both players. Then we compute a mixed strategy Nash equilibrium of this game. Then we add the weighted average of player 2's pure strategies –by using her equilibrium mixed strategy as wights– as the new pure strategy for the game played in the next iteration. Then we find a Nash equilibrium of this new game. This process is repeated until we obtain an approximate fixed point of f. This process is guaranteed to convergence.

1.5.1.2 Scarf's Algorithm

Let X be a nonempty and convex and compact subset of a Euclidean space and $f : X \to X$ a continuous function. Then Brouwer's fixed point theorem implies f has a fixed point. Scarf's algorithm first divides the set X into a large number of small triangles.[11] Then based on Sperner's lemma and Scarf's lemma in order

[11] Although dividing a general convex set into small triangles (triangulation) may not be possible, we can embed X into a larger simplex and easily extend the domain of the function f to this simplex such that the continuity is preserved and the fixed points of this extended function coincide with the fixed points of f. We can then focus on the simplex and the extended function in order to compute a fixed point of f. Hence, there is no loss of generality in using triangulation.

to provide a path of triangles which is guaranteed to converge a triangle whose points are approximate fixed point of f. Several extensions and modifications of this algorithm has been provided in order to obtain efficiency, see Yang (1999). In this paper, we are using the homotopy method extension to compute fixed points, see Yang (1999) for the details.

Now we illustrate Scarf's algorithm with our three-player example presented above. Recall that the function f is mapping points in $[0, 1]$ to itself as illustrated in Figure 1.3. It has a unique fixed point $\lambda^* = 0.5$. Our objective is to find an approximation of λ^*. Let us divide the interval $[0, 1]$ into equal-length sub-intervals, say 3 sub-intervals for simplicity. We label the ends of the sub-intervals as follows. If the value of the function $f(\lambda)$ at an end point λ is greater than λ, then the label of λ is 1, otherwise 2. It is clear from the graph of the function that the end points 0 and 1/3 have label 1 and the end points 2/3 and 1 have label 2. Since f is continuous the interval $[1/3, 2/3]$ must include a fixed point. And the labeling identifies this property. Note that the end points of this interval has both labels and the end points of each of the other two sub-intervals have only one label. Indeed the other sub-intervals do not contain any fixed point. Any point in the interval $[1/3, 2/3]$ is an approximation of λ^*. Increasing the number of sub-intervals increases the precision of the approximation. Note that a limit argument also provides a proof of Brouwer fixed point theorem.

1.5.2 Solving System of Polynomial Equations

Consider a system of polynomial equations with complex coefficients[12]

$$f(x) := \begin{bmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_n(x_1, \dots, x_n) \end{bmatrix} = 0.$$

We are interested in finding a (real) solution of this system.[13] Note that this system is a general representation of the system ($\star$) defined above and that the solution set to this system can be positive dimensional. Before going into details of the algorithms to find a member (if not all) of this positive dimensional set we will first illustrate the homotopy algorithm to compute isolated solutions of a system of polynomial equations. This serves more than just an illustration since the algorithm to compute positive dimensional solution set are based on algorithms to compute isolated solutions.[14]

We use the homotopy method to find a solution to the system above defined as follows. First define a homotopy $H : \mathbb{C}^n \times [0, 1] \to \mathbb{C}^n$ as

$$H(x, t) = (1 - t)g(x) + tf(x)$$

where t is a parameter and g is a system of polynomial equations mapping

[12]Note that although we are interested in only real solutions, working with complex numbers has many advantages. First there are very nice and useful results which hold for complex numbers and does not hold for reals. For example the Fundamental Theorem of Algebra states that every one-variable degree d polynomial has d complex roots, in counting multiplicities. However, this is not true for real roots –for example x^2+1 does not have a real root. Moreover, the complex solution set of a system of polynomial equations is connected which is not true for the real solution set. More aligned with our objective, working with complex numbers has various advantages in computation, especially when homotopy methods and path tracking are considered, see Sommese and Wampler (2005).

[13]Theorems 1 and 3 show that a real solution to the system ($\star$) exists.

[14]The main references for this part are Bochnak, Coste and Roy (1998), Sturmfels (2002), Cox, Little and O'Shae (2005, 2007),Datta (2010), in addition to Bates etal. (2013) and Sommese and Wampler (2005) which provide a comprehensive survey and detailed description of the algorithms to solve both zero dimensional and positive dimensional polynomials based on homotopy continuation method.

points from $\mathbb{C}^n$ to itself. The system $H(x, 0) = g(x)$ is called the starting system whose solutions are easy to find and the system $H(x, 1) = f(x)$ is the system of polynomials whose solutions we are interested in. The homotopy method relies on starting at a solution of g and following a path of solutions parametrized by t in order to reach a solution of the target system f. We follow this path connecting solutions of a starting system to solutions of the target system by numeric predictor-corrector methods such as Euler-Newton method.

In the system $(\star)$ above, our variables are (λ, μ). In the context of our three-player example illustrated above, $(\star)$ can be represented as a system of four equations and four unknowns – player 1's probabilities of including player 2 and of including player 3 in her coalition along with the corresponding two slack variables. The target system has finitely many solutions and a unique nonnegative real solution $(\lambda^*; \mu^*) = (0.5, 0.5; 0, 0)$. Hence, choosing the starting system which has finitely many solutions and following the homotopy paths will give us the desired target solution.

1.6 Concluding Remarks

In this paper we show that algorithms to compute finite games can be used to compute stationary subgame perfect equilibrium payoffs of EM coalitional bargaining model. Our approach is based on a new result analogous to Nash's "best-response condition" for finite games. It is an open question whether a significant improvement of the algorithms that we use in this paper can be provided. Due to the weighted voting rule in this model any algorithm should based on a nonlinear method. However, in all of our characterizations of SSP equilibrium payoffs only the continuation values are represented as nonlinear functions of weighted winning coalitions and the mixed strategy profiles, whereas the optimization problem of each player is a linear function of SSP payoffs, weighted winning coalitions and the mixed strategy profiles. Hence, an algorithm which explicitly incorporates this property would be more efficient. Moreover, it would be interesting to see whether it possible to obtain a simpler alternative proof of the uniqueness result by using any of the alternative characterizations we provided in this paper.

Chapter 2

On the Nonemptiness of the α-core of Discontinuous Games: Transferable and Nontransferable Utilities

2.1 Introduction

Two major solution concepts for normal form games are the Nash equilibrium and the core. A Nash equilibrium is a noncooperative solution in which the joint interest of groups of players is not explicitly considered, whereas the core is a cooperative solution involving the behavior of coalitions of players. Motivated by economic problems that are suitably modeled by games with discontinuous payoff functions, Nash's existence result has been considerably extended following the seminal works of Dasgupta and Maskin (1986) and Reny (1999).[1] The existence of a cooperative solution for such games has yet to be provided. This note fulfills this objective.

[1] See Carmona (2011) for a symposium on the recent developments in the discontinuous games literature.

An action profile belongs to the core of a game if no group of players has an incentive to form a coalition in which each of its members are made better-off, i.e. the action profile cannot be *blocked* by any coalition. In a (normal form) game, the actions of the complementary coalition affect the payoff of the members of a coalition, and therefore the definition of "blocking" hinges crucially on what the complementary coalition does. Among various blocking concepts defined in the literature, the α-*core* due to Aumann (1961) has attracted a significant attention.[2] In this paper we study the existence of this cooperative solution for games with possibly discontinuous payoff functions without transferable utilities (α-core) and with transferable utilities (α^T-core).

An action profile is in the α-core of a game if no coalition has an alternative action which makes all of its members better off, independently of the actions of the complementary coalition. Hence it is a pessimistic solution concept regarding the actions of the complementary coalition. And a pair of an action profile and a payoff profile is in the α^T-core of a game if the action profile maximizes the grand coalition's aggregate payoff, and no coalition has an alternative action which guarantees a higher aggregate payoff, independently of the actions of the complementary coalition. Therefore, the α-core allows the coordination of the actions among the members of the coalitions, whereas the α^T-core allows payoff transfers within the coalitions in addition to the coordination of actions. Scarf (1971) and Zhao (1999a) proved the following theorems.

[2] The α-*core,* along with the *strong Nash equilibrium* and the β-*core,* are the standard cooperative solution concepts for normal form games, see, for example, Ray and Vohra (1997). These cooperative solution concepts differ on the definition of blocking. The α-core requires a blocking coalition to select a specific strategy independently of the complementary coalition's choice, the β-core allows a blocking coalition to vary its blocking strategy as a function of the complementary coalition's choice, and the strong Nash equilibrium requires the complementary coalition to stick on its choice. See Ichiishi (1993, Section 2.3, p.36) for details.

Theorem (Scarf). *Let $G = (X_i, u_i)_{i \in N}$ be a game such that for each player i,*

(i) *X_i is a nonempty, compact and convex subset of a finite dimensional Euclidean space,*

(ii) *u_i is a quasiconcave and continuous function on $X = \prod_{i \in N} X_i$.*

Then G has a nonempty α-core.

Theorem (Zhao). *Let $G = (X_i, u_i)_{i \in N}$ be a game such that for each player i,*

(i) *X_i is a nonempty, compact and convex subset of a finite dimensional Euclidean space,*

(ii) *u_i is a concave and continuous function on $X = \prod_{i \in N} X_i$,*

(iii) *G is weakly separable.*

Then G has a nonempty α^T-core.

In this paper we generalize these results to games where the continuity assumptions are weakened. And in line with the recent literature on discontinuous games pioneered by Reny (1999), the notions of *coalitionally C-secure*, *coalitionally C^T-secure* and *coalitionally C_N^T-secure* games are presented, and the existence of an imputation in the α-core and α^T-core are shown for them.

The paper is organized as follows. Section 2.2 defines the basic concepts and states the results, Section 2.3 illustrates examples, Section 2.4 provides proofs of the results, and Section 2.5 concludes.

2.2 The Model and the Results

A *(normal form) game* is a list $G = (X_i, u_i)_{i \in N}$ where

(i) $N = \{1, \dots, n\}$ is the finite set of players,

(ii) X_i is the nonempty set of actions of player $i \in N$,

(iii) $u_i : X \to \mathbb{R}$ is the utility function of player $i \in N$ defined on $X = \prod_{i \in N} X_i$.

A *quasiconcave* (*concave*) *game* is a game $G = (X_i, u_i)_{i \in N}$ such that for each player i, X_i is a nonempty, compact and convex subset of a finite dimensional Euclidean space and u_i is a quasiconcave (concave) function on X.

In his pioneering work, Reny (1999) proved the existence of a Nash equilibrium of normal form games satisfying the following weak continuity assumption.

Definition 1. A game G is *better-reply-secure*[3] (BRS) if at each $x \in X$ that is not a Nash equilibrium of G, there exist $y^x \in X$, $\delta^x > 0$, and an open neighborhood U^x of x such that for each $x' \in U^x$ there exists $i \in N$ such that for each $z \in U^x$; $u_i(y_i^x, z_{-i}) > u_i(x') + \delta^x$.

McLennan, Monteiro and Tourky (2011) provided the following continuity concept which is weaker than BRS, and used it to prove the existence of a (pure strategy) Nash equilibrium. Barelli and Meneghel (2013) extended this existence result by further weakening the continuity assumption.

Definition 2. A (quasiconcave) game G is C-*secure* if at each $x \in X$ that is not a Nash equilibrium of G, there exist $v_i^x \in \mathbb{R}$ and $y_i^x \in X_i$ for each $i \in N$ and an open neighborhood U^x of x such that

(i) $u_i(y_i^x, z_{-i}) \geq v_i^x$ for each $i \in N$ and each $z \in U^x$,

(ii) for each $x' \in U^x$ there exists $i \in N$ such that $u_i(x') < v_i^x$.

Remark 1. If an action profile x is not a Nash equilibrium, then by definition at least one player deviates. The C-security imposes the following structure on the individual deviations: (i) an open neighborhood of x contains no Nash equilibrium, i.e. at each point on the neighborhood at least one player deviates,

[3]This definition is equivalent to but different from the original definition of better-reply-security. See footnote 7 in Reny (2015) and B-security concept in McLennan et al. (2011, Definition 2.4, p.1646).

(ii) the identity of the deviant is allowed to vary, but the deviation for any deviant must be fixed on the neighborhood, and (iii) each deviant's deviation should be robust against all other players' trembles.

A *coalition* is an element S in $\mathcal{N} = 2^N \backslash \emptyset$. Let $\mathring{\mathcal{N}} = \mathcal{N} \backslash N$. The set of actions available to a coalition S is denoted as $X_S = \prod_{i \in S} X_i$, and the vector of utility functions of coalition S as $u_S = (u_i)_{i \in S}$.[4] For each coalition S, $-S$ denotes the complementary coalition $N \backslash S$. Next, we define equilibrium and continuity concepts for normal form games where each coalition is allowed to coordinate actions among its members, but payoff transfers are not allowed.

Definition 3. Let G be a game. A coalition S α*-blocks* an action profile $x \in X$ if $\exists x'_S \in X_S$ such that $u_S(x'_S, z_{-S}) \gg u_S(x)$[5] for each $z_{-S} \in X_{-S}$. An action profile $x^* \in X$ is in the α*-core* of G if it is not α-blocked by any coalition.

This equilibrium definition differs in two major aspects from the Nash equilibrium. First, an arbitrary coalition is permitted to modify its action profile. Second, the complementary coalition is permitted a subsequent modification of its action profile. Now we define a continuity concept analogous to C-security (Definition 2 above) which involves the behavior of the coalitions as follows.

Definition 4. A game G is *coalitionally C-secure* if at each $x \in X$ that is not in the α-core of G, there exist $v^x_S \in \mathbb{R}^{|S|}$ and $y^x_S \in X_S$ for each $S \in \mathcal{N}$ and an open neighborhood U^x of x such that

(i) $u_S(y^x_S, z_{-S}) \geq v^x_S$ for each $S \in \mathcal{N}$ and each $z_{-S} \in X_{-S}$,

[4]By abusing the notation, we drop the subscript N for the grand coalition, and when there it is clear from the context, we use i and $\{i\}$ interchangeably for the singleton coalition $S = \{i\}$.

[5]We use usual vector comparison symbols: $x \geq y$ represents $x^k \geq y^k$ for each index k; $x > y$ represents $x^k \geq y^k$ for each index k and inequality is strict at least for one k; $x \gg y$ represents $x^k > y^k$ for each index k. Also, $\subset$ represents subset and $\subsetneq$ proper subset.

(ii) for each $x' \in U^x$ there exists $S \in \mathcal{N}$ such that $u_S(x') \ll v_S^x$.

Remark 2. Note that coalitional C-security is conceptually analogous to the notion of C-security (see Remark 1). First, the role of individual players is taken by coalitions. Second, we impose restrictions on *coalitional blockings* instead of *individual deviations.* The main difference is that we assume $u_S(y_S^x, z_{-S}) \geq v_S^x$ for each $S \in \mathcal{N}$ and $z_{-S} \in X_{-S}$, instead of each $z \in U^x$. In other words, we assume each coalition can guarantee the payoff level independent of all actions of the complementary coalition, instead of the complementary coalition's tremble. Although this is a strong assumption, it is consistent with the definition of blocking and Reny's insight. The C-security assumes a structure on *individual deviations,* and coalitional C-security on *coalitional blockings*, and the definition of blocking, unlike that of deviation, already incorporates *all* actions of the complementary coalition. Therefore, there is no inclusion relation between the coalitional C-security and C-security concepts. Example 1 below illustrates this relation.

Now we define equilibrium and continuity concepts for normal form games where coalitions are allowed to both coordinate their actions and reallocate their aggregate payoffs, i.e. payoff transfers are allowed.

Definition 5. Let G be a game. A coalition S α^T*-blocks* a payoff profile $v \in \mathbb{R}^n$ if $\exists x'_S \in X_S$ such that $\sum_{i \in S} u_i(x'_S, z_{-S}) > \sum_{i \in S} v_i$ for each $z_{-S} \in X_{-S}$. A pair of an action profile and a payoff profile $(x^*, v^*) \in X \times \mathbb{R}^n$ with $\sum_{i \in N} u_i(x^*) = \sum_{i \in N} v_i^*$ is in the α^T*-core* of G if v^* is not α^T-blocked by any coalition.

Note that due to payoff transfers, this equilibrium definition requires two variables, an action profile x^* and a payoff profile v^*, whereas the former is enough to define the α-core. Now we define a class of games which plays an essential role for the nonemptiness of the α^T-core.

Definition 6. A game G is *bounded* if u_i is bounded for each player i.

For a bounded game G, define

$$\mathcal{X} = \{(x,v) \in X \times \mathbb{R}^n | \sum_{i \in N} v_i = \sum_{i \in N} u_i(x), \ \inf_{x \in X} u_i(x) \leq v_i \ \forall i \in N\}.$$

$\mathcal{X}$ is the set of all attainable pairs of action profiles and payoff profiles that are individually rational. Since any payoff profile which assigns player i a payoff below $\inf_{x \in X} u_i(x)$ is blocked by player i, hence $\mathcal{X}$ contains the α^T-core of G. If X is compact then it is clear that the closure of $\mathcal{X}$, denoted by $\bar{\mathcal{X}}$, is compact. Now we define a continuity concept analogous to C-security (Definition 2) for TU games as follows.

Definition 7. A bounded game G is *coalitionally C^T-secure* if at each $(x,v) \in \bar{\mathcal{X}}$ that is not in the α^T-core of G, there exist $w_S^{x,v} \in \mathbb{R}$ and $y_S^{x,v} \in X_S$ for each $S \in \mathcal{N}$ and an open neighborhood $U^{x,v}$ of (x,v) such that

(i) $\sum_{i \in S} u_i(y_S^{x,v}, z_{-S}) \geq w_S^{x,v}$ for each $S \in \mathcal{N}$ and each $z_{-S} \in X_{-S}$,

(ii) for each $(x', v') \in U^{x,v}$ there exists $S \in \mathcal{N}$ such that $\sum_{i \in S} v'_i < w_S^{x,v}$.

Note that since α^T-core consists of a pair of an action profile and a payoff profile, the coalitional C^T-security is defined on the set of pairs. Moreover, a point (x^*, v^*) in the α^T-core must satisfy the following two conditions: (i) $\sum_{i \in N} v_i^* = \sum_{i \in N} u_i(x^*) = \max_{x \in X} \sum_{i \in N} u_i(x)$, and (ii) v^* is not α^T-blocked by any coalition. In order to guarantee the existence of a solution to the maximization problem (i), we define coalitional C^T-security at points in $\bar{\mathcal{X}}$, not only in $\mathcal{X}$.[6]

[6]This property of coalitional C^T-security is similar to the *BRS* notion of Reny (1999) which takes the closure of the graph of the game into account.

Although coalitional C^T-security is the natural analogue of C-security, some important games, such as Bertrand duopoly game with different marginal costs, have a nonempty α^T-core and are not coalitionally C^T-secure, see Example 1 below. Therefore, we introduce a separate continuity concept which we term coalitional C^T_N-security as follows. For a given game $G = (X_i, u_i)_{i \in N}$, let $G_N = (X, \bar{u})$ be the induced one player where $X = \prod_{i \in N} X_i$ and $\bar{u}(x) = \sum_{i \in N} u_i(x)$. If G_N is C-secure, or equivalently coalitionally C-secure which are identical for one player games, then Theorem 4 implies there exists a maximizer $\bar{x} \in X$ of $\bar{u}$. (And our proof technique provides an alternative proof of the existence of a maximal element based on Scarf's (1967) theorem.) Since any point in the α^T-core must satisfy condition (i) described above, focusing only on the redistributions of the grand coalition's maximum aggregate payoff is enough to show the nonemptiness of the α^T-core.[7] In particular, define

$$\mathcal{V} = \{v \in \mathbb{R}^n | \sum_{i \in N} v_i = \sum_{i \in N} u_i(\bar{x}),\ \inf_{x \in X} u_i(x) \leq v_i\ \forall i \in N\}.$$

Now imposing a structure on the deviations at each point in $\mathcal{V}$ will guarantee the nonemptiness of the α^T-core. Note that by construction, $\mathcal{V}$ is compact.

Definition 8. A bounded game G is *coalitionally C^T_N-secure* if the induced one player game G_N is C-secure, and at each $v \in \mathcal{V}$ such that $(\bar{x}, v)$ is not in the α^T-core of G, there exist $w^v_S \in \mathbb{R}$ and $y^v_S \in X_S$ for each $S \in \mathring{\mathcal{N}}$ and an open neighborhood U^v of v such that

(i) $\sum_{i \in S} u_i(y^v_S, z_{-S}) \geq w^v_S$ for each $S \in \mathring{\mathcal{N}}$ and each $z_{-S} \in X_{-S}$,

[7] A one player game may be helpful in illustrating the underlying structure. Consider the following example. $X = [0,1]$ and $u(0) = 1$, $u(1) = 0$, and $u(x) = x$ for all $x \in (0,1)$. This 1-player game is not coalitionally C^T-secure since 1 is not maximizer and the game is not coalitionally C^T-secure at $(x, v) = (1,1) \in \bar{\mathcal{X}}$. However, it will be clear below that the game is coalitionally C^T_N-secure.

(ii) for each $v' \in U^v$ there exists $S \in \mathring{\mathcal{N}}$ such that $\sum_{i\in S} v'_i < w^v_S$.

Note that if $\bar{u}$ is an upper semicontinuous function, then G_N is C-secure.[8] But the converse of this claim is not true, see the example presented in footnote 7. Note that although the coalitional C^T_N-security imposes restriction only on $\{\bar{x}\} \times \mathcal{V} \subset \bar{\mathcal{X}}$, it explicitly requires the grand coalition to have a well-behaved blocking behavior at each points it blocks that is not imposed by the coalitional C^T-security. Under certain conditions, coalitional C^T_N-security is weaker than coalitional C^T-security. This relation is summarized in Claims 1 and 2 at the end of this section. Moreover, there is no inclusion relation between coalitional C-security and coalitional C^T-security as well as coalitional C^T_N-security. These relations are illustrated in Examples 2 and 3 below.

Definition 9. A coalitionally C^T-secure game G is *quasiseparable* if for each $S \in \mathcal{N}\backslash N$ and $y_S \in \{y^{x,v}_S | (x,v) \in \bar{\mathcal{X}}$ is not in the α^T-core of $G\}$,

$$\inf_{z_{-S}\in X_{-S}} \sum_{i\in S} u_i(y_S, z_{-S}) = \sum_{i\in S} \inf_{z_{-S}\in X_{-S}} u_i(y_S, z_{-S}).$$

Definition 10. A coalitionally C^T_N-secure game G is *quasiseparable* if for each $S \in \mathcal{N}\backslash N$ and $y_S \in \{y^v_S | v \in \mathcal{V}$ such that $(\bar{x}, v)$ is not in the α^T-core of $G\}$,

$$\inf_{z_{-S}\in X_{-S}} \sum_{i\in S} u_i(y_S, z_{-S}) = \sum_{i\in S} \inf_{z_{-S}\in X_{-S}} u_i(y_S, z_{-S}).$$

[8]In order to see this, choose $x \in X$ such that there exists $y^x \in X$ with $\bar{u}(y^x) > \bar{u}(x)$. Then upper semicontinuity implies that for each $\varepsilon \in (0, \bar{u}(y^x) - \bar{u}(x))$, there exists an open neighborhood U^x of x such that $\bar{u}(x') \le \bar{u}(x) + \varepsilon$ for all $x' \in \bar{U}^x$. Setting $v^x_1 = \bar{u}(y^x)$ implies G_N is C-secure.

In Zhao (1999a), a game is called *weakly separable* if for each $S \in \mathring{\mathcal{N}}$ and $i \in S$,

$$u_i(x_S^*, x_{-S}^*(x_S^*)) = \min_{z_{-S} \in X_{-S}} u_i(x_S^*, z_{-S})$$

where $(x_S^*, x_{-S}^*(x_S^*)) \in X$ is a solution to the problem $\max_{z_S \in X_S} \min_{z_{-S} \in X_{-S}} \sum_{i \in S} u_i(z_S, z_{-S})$. Note that every weakly separable game is quasiseparable. To see this, if the game is weakly separable, then the maxmin problem has a solution for each coalition. Hence at each $(x, v) \in \bar{\mathcal{X}}$ and $S \neq N$, we can set $y_S^{x,v} = x_S^*$. But, even for continuous payoff functions, quasiseparability is weaker than weak separability since it imposes a restriction on the aggregate payoff functions of coalitions, not on each member of the coalitions' payoff functions.

Moreover, Zhao (1999a) briefly noted in footnote 4 that if the maxmin problem does not have a solution, one can define the induced TU game by simply replacing maxmin with supinf. This line of argumentation requires the boundedness of the payoff functions (otherwise one has to take the complications of the extended real line into consideration) and can only be used to show the nonemptiness of the epsilon α^T-core. In order to see this, first redefine the weak separability as follows. A game G is weakly separable if for each $S \neq \mathcal{N}$,

$$\sup_{z_S \in X_S} \inf_{z_{-S} \in X_{-S}} \sum_{i \in S} u_i(z_S, z_{-S}) = \sum_{i \in S} \sup_{z_S \in X_S} \inf_{z_{-S} \in X_{-S}} u_i(z_S, z_{-S}).$$

And consider the following trivial two-player game. $X_1 = X_2 = [0, 1]$, $u_1(x_1, x_2) = x_1$ for all $x_1 \in [0, 1)$ and $x_2 \in X_2$, and $u_1(1, x_2) = 0$. And $u_2(x_1, x_2) = u_1(x_2, x_1)$. It is clear that the maximum of the aggregate utility does not exist, i.e. the grand coalition always blocks, hence the α^T-core of the game is empty. Note that every two-player game trivially satisfies weak separability.

Now we are ready to state our results.

Theorem 4. *Every coalitionally C-secure, quasiconcave game has a nonempty α-core.*

Theorem 5. *Every coalitionally C^T-secure, concave,*[9] *bounded, quasiseparable game has a nonempty α^T-core.*

Theorem 6. *Every coalitionally C_N^T-secure, concave, bounded, quasiseparable game has a nonempty α^T-core.*

As is by now well-understood in the context of NTU games, the special case of a 2-player game is rather important in that the 2-player set-up allows one to dispense with a linear space structure on both the actions sets and the payoff functions, see Ichiishi (1993, Remark 2.3.2, p.37). Proposition 2 shows that this carries over *verbatim* to the discontinuous setting as is formalized and presented here. Proposition 3 shows that the analogous result holds for TU games. But more to the point the 2-player game results serve as an important backdrop to the examples we presented that validate our results as meaningful and useful generalizations of Scarf's and Zhao's results. First, define a *compact game* as a game $G = (X_i, u_i)_{i \in N}$ such that for each player i, X_i is a nonempty and compact subset of a finite dimensional Euclidean space.

Proposition 2. *Every coalitionally C-secure, compact 2-player game has a nonempty α-core.*

[9]It is well known that every real-valued concave function on a Euclidean space is continuous on its domain's relative interior. And it is easy to define a concave function that is discontinuous at every point of the relative boundary of its domain. If the domain is contained in $\mathbb{R}$, it is easy to see that such functions are lower semicontinuous at the relative boundary of their domain. However this result is not necessarily true for the higher dimensional Euclidean spaces, see Ernst (2013, Theorem 2.4, p.3672). Therefore, the minimization problems defined above may not have a solution, and hence our setup with *infimum* is crucial.

Proposition 3. *Every coalitionally C_N^T-secure, compact, bounded 2-player game has a non-empty α^T-core.*

Now we provide two results related to the relationship between coalitional C^T-security and coalitional C_N^T-security.

Claim 1. *Every bounded and coalitionally C^T-secure 2-player game is coalitionally C_N^T-secure. A bounded and coalitionally C_N^T-secure 2-player game is not necessarily coalitionally C^T-secure.*

Claim 2. *Let G be a bounded and coalitionally C^T-secure game such that $\bar{u} = \sum_{i \in N} u_i$ has a maximizer. Then G is coalitionally C_N^T-secure. A bounded and coalitionally C_N^T-secure game is not necessarily coalitionally C^T-secure.*

Note that Claim 2 shows that coalitional C_N^T-security is a necessary and sufficient condition for the existence of a maximizer of $\bar{u}$ for bounded and coalitionally C^T-secure games with compact action sets.

2.3 Examples

The first example illustrates a game which does not have a (pure strategy) Nash equilibrium, but has a nonempty α-core and α^T-core. The game satisfies all the assumptions of Propositions 1 and 2, particularly coalitional C-security and coalitional C_N^T-security. However, it neither satisfies C-security nor coalitional C^T-security.

Example 1. Consider the following Bertrand duopoly game. Each firm's action set is $P_i = [0, 10]$. The market demand function $D : [0, 10] \to \mathbb{R}$ is defined as

$$D(p) = \max\{4 - p, 0\}.$$

And the profit functions of the firms are defined as

$$\pi_1(p) = \begin{cases} p_1 D(p_1) & \text{if } p_1 < p_2, \\ p_1 D(p_1)/2 & \text{if } p_1 = p_2, \\ 0 & \text{otherwise.} \end{cases} \qquad \pi_2(p) = \begin{cases} p_2 D(p_2) - 1 & \text{if } p_2 < p_1, \\ p_2 D(p_2)/2 - 1 & \text{if } p_1 = p_2, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that this game does not have a Nash equilibrium.[10] Also, it is not hard to check that the game is not C-secure, but is coalitionally C-secure. For example, $p = (0,0)$ is not in the α-core. Consider an ε-neighborhood of p with $\varepsilon = 0.1$. Setting $\{0, 0, (1,1); 0, 0, (2, 0.5)\}$ as the collection of actions and utility levels for coalitions $\{1\}, \{2\}$ and $\{1,2\}$, respectively, proves the coalitional C-security at p. The price pair $(2,4)$ is in the α-core of this game since firm 1 gets the monopoly profit, hence it cannot be part of any blocking coalition, and firm 2 cannot block since it cannot guarantee a profit above 0 for itself. Note that $(2,2,4,0) \in \bar{\mathcal{X}}$ is not in the α^T-core of G since $\sum_{i \in N} u_i(2,2) = 3 < 4$. But the game is not coalitionally C^T-secure at $(2,2,4,0)$ since no coalition can block it. However, it is not hard to check that the game is coalitionally C_N^T secure and the α^T-core is nonempty.

Although the discontinuity of the payoff functions prevents us from applying the theorem of Scarf presented in the introduction, the induced non-transferable utility game satisfies the assumptions of Scarf's Theorem presented in the Appendix, see Figure 2.1 (a). Therefore, the nonemptiness of the α-core is guaranteed without referring to our result. A more severe problem due to discontinuity is the violation of the closedness of the set of attainable utilities of the coalitions, especially the grand coalition. The following example is more interesting from this perspective.

[10]Note that in this paper we consider only the existence of Nash equilibrium in pure-strategies. It is known that this game has a Nash equilibrium in mixed-strategies, see Blume

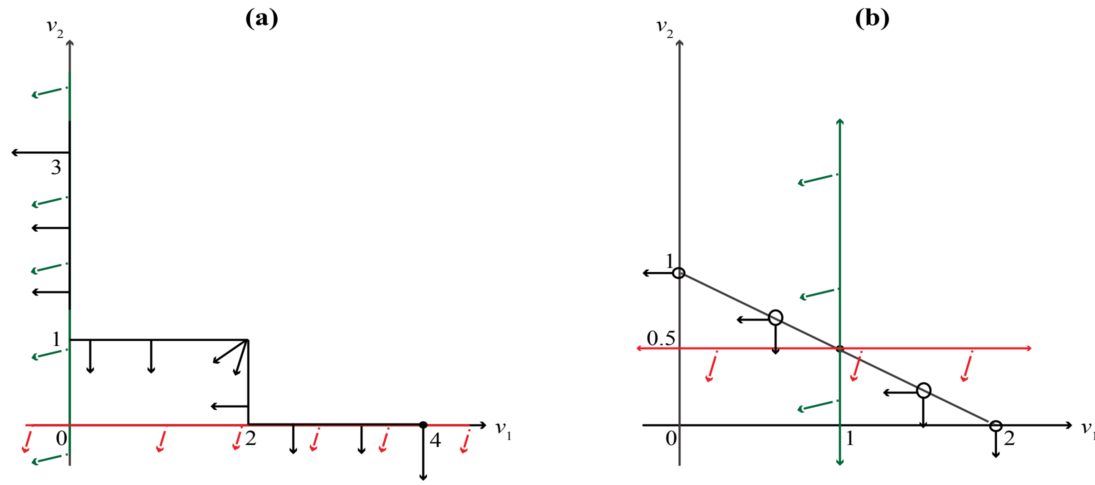


Figure 2.1: The Induced NTU Games

Panels a and b correspond to Examples 1 and 2, respectively. The boundary of the set of attainable utilities for coalition $\{1\}$ is illustrated by the green color, for coalition $\{2\}$ by the red color and for coalition $\{1,2\}$ by the black color. The arrows show which side of the boundary can be attained by the coalitions.

Example 2. Consider a two player game where the action set of each player is $X_i = [0,2]$. And, the payoff function $u_i : X = X_1 \times X_2 \to \mathbb{R}$ of player $i = 1,2$ is defined as

$$u_1(x_1,x_2) = \begin{cases} x_2 & \text{if } x_1 + x_2 < 2, \\ 1 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad u_2(x_1,x_2) = \tfrac{1}{2}u_1(x_2,x_1).$$

It is easy to check that the action profile $(1,1)$ is in the α-core of the game. Moreover, the set of attainable utilities for the grand coalition is not closed, see Figure 2.1, Panel b. Also, it is not hard to check that this game is coalitionally C-secure. Lastly, it is easy to see that this game has an empty α^T-core, and is not both coalitionally C^T-secure and coalitionally C^T_N-secure.

The following example illustrates a game which has a nonempty α^T-core and an empty α-core. The game is coalitionally C^T-secure but not coalitionally

(2003).

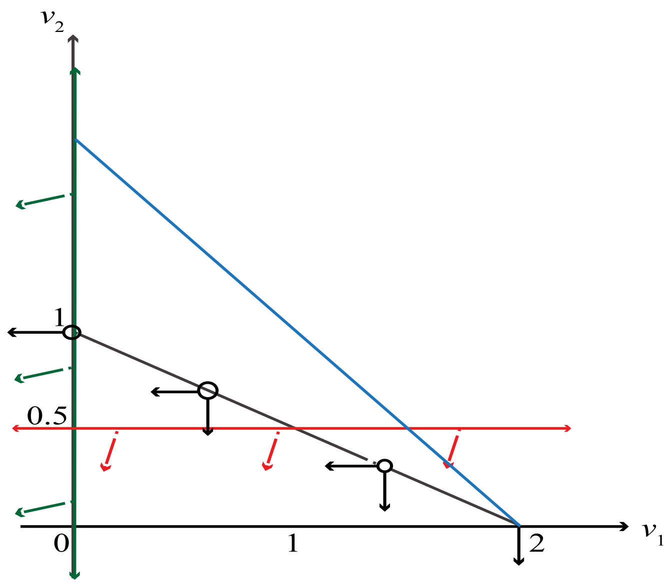


Figure 2.2: The Induced *NTU* and *TU* Games

The games presented in Example 3 are illustrated. The boundary of the set of attainable utilities for coalition $\{1\}$ is illustrated by the green color, for coalition $\{2\}$ by the red color and for coalition $\{1,2\}$ by the black color. The arrows show which side of the boundary can be attained by the coalitions. The blue line describes the set of redistributions of the grand coalition's maximum aggregate payoff. The game is coalitionally C^T-secure but not coalitionally C-secure.

C-secure.

Example 3. Consider a two player game where the action set of each player is $X_i = [0,2]$. And, the payoff function $u_i : X = X_1 \times X_2 \to \mathbb{R}$ of player $i = 1,2$ is defined as

$$u_1(x_1,x_2) = \begin{cases} x_2 & \text{if } x_1 \leq 1.5, x_1 + x_2 < 2, \\ x_2 & \text{if } x_1 > 1.5, x_1 + x_2 \leq 2, \\ 0 & \text{otherwise,} \end{cases}$$

$$u_2(x_1,x_2) = \begin{cases} \frac{1}{2}x_1 & \text{if } x_1 \leq 1.5, x_1 + x_2 < 2, \\ \frac{1}{2}x_1 & \text{if } x_1 > 1.5, x_1 + x_2 \leq 2, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Note that this game is not coalitionally C-secure and has an empty α-core.

However, it is bounded and coalitionally C^T-secure, and hence has a nonempty α^T-core.

In the context of the industrial organization theory, our results would have implications for both the formation and the outcome of the grand cartel (covert collusion) and monopoly merger (overt collusion). In line with this observation, we close this section by providing two results on Cournot oligopoly models for which we considerably relax quasiconcavity and concavity assumptions and allow discontinuity in payoffs. A stronger version of the first one is introduced in Ichiishi (1993, Example 2A.11, p.64) and the second in Zhao (1999a, Theorem 2, p.29). A Cournot oligopoly game is a game $G = (X_i, \pi_i)_{i\in N}$ where $N = \{1, \ldots, n\}$ is the set of firms, $X_i = [0, \bar{y}_i]$ is the production set of firm i where $\bar{y}_i > 0$ denotes firm i's capacity constraint, and $\pi_i : X \to \mathbb{R}$ is the profit function of firm i which is defined as

$$\pi_i(x) = f_i(x_i, \sum_{j\in N\setminus\{i\}} x_j) = p(x_i + \sum_{j\in N\setminus\{i\}} x_j)x_i - c_i(x_i),$$

where $c_i : \mathbb{R}_+ \to \mathbb{R}_+$ is firm i's cost function, $p : \mathbb{R}_+ \to \mathbb{R}_+$ is the inverse demand function and $f_i : \mathbb{R}^2_+ \to \mathbb{R}$.

Claim 3. *A coalitionally C-secure Cournot oligopoly game has a nonempty α-core if f_i is quasiconcave for each firm i.*

Claim 4. *A coalitionally C^T-secure (coalitionally C^T_N-secure), bounded Cournot oligopoly game has a nonempty α^T-core if p is decreasing and f_i is concave for each firm i.*

2.4 Proofs of the Results

The proofs of Theorems 4–6 require a delicate construction. In order to give a general overview of the proofs, we first provide a *heuristic outline* of the proof of Theorem 4. Whereas the details of the proofs of Theorems 5 and 6 are different, the construction in the proof is quite similar. Our proof of Theorem 4 is by contradiction. We assume the α-core of the game is empty. By using the compactness of the action sets and the coalitional C-security of the game, we obtain a finite selection of points. And by using these finite selections, and quasiconcavity of the payoff functions we construct an NTU game which satisfies the assumptions of Scarf's Theorem. The nonemptiness of the core of this NTU game furnishes us a contradiction with the coalitional C-security assumption. Both Scarf's and our proof are *algorithmic.* He used an algorithm to find the core, we uses an algorithm to obtain a contradiction which is in line with Reny's insight. Note that in the proof of Theorem 4, we use Scarf's Theorem provided in the Appendix as Scarf (1971) did. But our application of Scarf's Theorem is very different. In Scarf's proof, continuity of the payoff functions and compactness of the action sets trivially implies that the induced NTU game satisfies assumptions (i)-(iii) of Scarf's Theorem. The nontrivial part of his proof is to show that quasiconcavity of the payoff functions implies the balancedness of the induced NTU game. Hence, Scarf's proof is a direct proof. In our problem, due to discontinuity in the payoff functions, the induced NTU game does not necessarily satisfy the closedness assumption, hence we cannot directly use Scarf's Theorem.

Proof of Theorem 4. Assume G has an empty α-core. Then, since G is coalitionally C-secure, for each $x \in X$, there exist $v_S^x \in \mathbb{R}^{|S|}$ and $y_S^x \in X_S$ for each $S \in \mathcal{N}$ and an open neighborhood U^x of x such that $u_S(y_S^x, z_{-S}) \geq v_S^x$ for each coalition $S \in \mathcal{N}$ and each $z_{-S} \in X_{-S}$, and for each $z \in U^x$ there exists a coalition $S \in \mathcal{N}$ such that $u_S(z) \ll v_S^x$. The family $\{U^x |\ x \in X\}$ is an open covering of X which, by compactness of X, contains a finite subcovering $\{U^{x_k} |\ k = 1, \ldots, m\}$. Let $U^k = U^{x_k}$, and for all $S \in \mathcal{N}$, $v_S^k = v_S^{x_k}$ and $y_S^k = y_S^{x_k}$ for all $k \in K = \{1, \ldots, m\}$.

Now define an NTU game[11] $V : \mathcal{N} \twoheadrightarrow \mathbb{R}^n$ as follows. For all $S \in \mathcal{N} \backslash N$,

$$V(S) = \bigcup_{k \in K} \{v \in \mathbb{R}^n |\ v_S \leq v_S^k\},$$

$$V(N) = \left(\bigcup_{k \in K} \{v \in \mathbb{R}^n |\ v \leq u(y_N^k)\} \right) \bigcup \left(\bigcup_{l \in L, \mathbf{k} \in K^{|\mathcal{B}^l|}} \{v \in \mathbb{R}^n |\ v \leq u(x^{l,\mathbf{k}})\} \right),$$

where $L, \mathcal{B}^l$ and $x^{l,\mathbf{k}} \in X$ are defined as follows. Let $\mathcal{T} = \{\mathcal{B}^l\}_{l \in L}$ be the set of all balanced collections of coalitions. Since $\mathcal{N}$ is finite, the collection of all balanced coalltions is finite and we denote this collection as $\{\mathcal{B}^l\}_{l \in L}$. For each balanced collection $\mathcal{B}^l$, let $\lambda^l = \{\lambda_S^l\}_{S \in \mathcal{B}^l}$ be the balancing weights (if the balancing weights of a balanced collection of coalitions are not unique, pick and fix an arbitrary one). Now, for each $l \in L$ and $\mathbf{k} \in K^{|\mathcal{B}^l|}$, define $x^{l,\mathbf{k}} \in X$ as

$$x_i^{l,\mathbf{k}} = \sum_{S \in \mathcal{B}^l : i \in S} \lambda_S^l y_{S,i}^{k_S} \quad \text{for all } i \in N,$$

where $y_{S,i}^{k_S}$ is the action of player $i \in S$ in the joint action $y_S^{k_S}$ of coalition S. Since each $x_i^{l,\mathbf{k}}$ is a convex combination of points in X_i, and X_i is convex, $x_i^{l,\mathbf{k}} \in X_i$.

[11]See the Appendix for the definition and properties of NTU games, and TU games as well which are used in the proofs of the results.

By construction, the NTU game V is balanced. To see this, pick a balanced collection of coalitions $\mathcal{B} \in \mathcal{T}$ with the balancing weights λ, and let $v \in \bigcap_{S \in \mathcal{B}} V(S)$. If $N \in \mathcal{B}$, then $v \in V(N)$ trivially holds. Otherwise, by construction of V, for each $S \in \mathcal{B}$ there exists $k_S \in K$ such that $v_S \leq v_S^{k_S} \leq u_S(y_S^{k_S}, z_{-S})$ for all $z_{-S} \in X_{-S}$. Define $x \in X$ as

$$x_i = \sum_{S \in \mathcal{B}: i \in S} \lambda_S y_{S,i}^{k_S} \quad \text{for all } i \in N.$$

By construction of V, $u(x) \in V(N)$. Therefore, showing

$$v_i \leq u_i(x) \;\; \text{for each } i \in N$$

implies V is balanced. At this level of generality, it is sufficient to demonstrate that $v_1 \leq u_1(x)$, since by a suitable renaming of players any particular player can be made the first. We now adapt the proof technique in Scarf (1971) to our framework. Define $y^S \in X$ for each $S \in \mathcal{B}$ containing player 1 as follows. If $i \in S$, then

$$y_i^S = y_{S,i}^{k_S}.$$

If $i \notin S$, then

$$y_i^S = \frac{\sum \lambda_E y_{E,i}^{k_E}}{\sum \lambda_E}.$$

where in both the numerator and the denominator the summation is taken over all $E \in \mathcal{B}$ which contain player i but not player 1. From Scarf (1971, p.179)

$$x = \sum_{S \in \mathcal{B}: 1 \in S} \lambda_S y^S.$$

For each coalition $S \in \mathcal{B}$ containing player 1, we have defined an action profile such that each player i in S use the $y_{S,i}^{k_S}$, and each player i not in S use a specific strategy y_i^S. But since $y_S^{k_S}$ guarantees player 1 a utility of at least v_1 regardless of the strategy choices of the players not in S, we see that $u_1(y^S) \geq v_1$, for all $S \in \mathcal{B}$ which contain player 1. And, the quasiconcavity of u_1 implies

$$v_1 \leq u_1(x).$$

Therefore, V is balanced.

It is clear that conditions (i)-(ii) of Scarf's Theorem provided in the Appendix are satisfied. And since for each coalition S, the set $V(S)$ is constructed by using finitely many points, condition (iii) of Scarf's Theorem is satisfied. Hence, V has a nonempty core, i.e. there exists $v^* \in V(N)$ such that $v^* \notin \text{int}V(S)$ for all $S \in \mathcal{N}$. Since $v^* \in V(N)$, by construction there exists $x^* \in X$ such that $v^* \leq u(x^*)$. Also, since G is coalitionally C-secure and $x^* \in U^k$ for some $k \in K$, there exists $S \in \mathcal{N}$ such that $u_S(x^*) \ll v_S^k$. Since $\{v_S^k\} \times \mathbb{R}^{|-S|} \subset V(S)$, $v^* \in \text{int}V(S)$. This furnishes us a contradiction. $\square$

Proof of Theorem 5. Assume G has an empty α^T-core. Then, since G is coalitionally C^T-secure, for each $(x, v) \in \bar{\mathcal{X}}$, there exist $w_S^{x,v} \in \mathbb{R}$ and $y_S^{x,v} \in X_S$ for each $S \in \mathcal{N}$ and an open neighborhood $U^{x,v}$ of (x, v) such that $\sum_{i \in S} u_i(y_S^{x,v}, z_{-S}) \geq w_S^{x,v}$ for each coalition $S \in \mathcal{N}$ and each $z_{-S} \in X_{-S}$, and for each $(x', v') \in U^{x,v}$ there exists a coalition $S \in \mathcal{N}$ such that $\sum_{i \in S} v'_i < w_S^{x,v}$. The family $\{U^{x,v} | \ (x, v) \in \bar{\mathcal{X}}\}$ is an open covering of $\bar{\mathcal{X}}$ which, by compactness of $\bar{\mathcal{X}}$, contains a finite subcovering $\{U^{x_k,v_k} | \ k = 1, \ldots, m\}$. Let $U^k = U^{x_k,v_k}$, and for all $S \in \mathcal{N}$, $w_S^k = w_S^{x_k,v_k}$ and $y_S^k = y_S^{x_k,v_k}$ for all $k \in K = \{1, \ldots, m\}$.

Now define a TU game $W : \mathcal{N} \to \mathbb{R}$ as follows. For all $S \in \mathcal{N} \backslash N$,

$$W(S) = \max_{k \in K} w_S^k,$$

$$W(N) = \max\left\{\max_{k\in K}\sum_{i\in N} u_i(y_N^k), \max_{l\in L, \mathbf{k}\in K^{|\mathcal{B}^l|}}\sum_{i\in N} u_i(x^{l,\mathbf{k}})\right\},$$

where $L, \mathcal{B}^l$ and $x^{l,\mathbf{k}} \in X$ are defined as follows. Let $\mathcal{T} = \{\mathcal{B}^l\}_{l\in L}$ be the set of all minimally balanced collections of coalitions. Since $\mathcal{N}$ is finite, the collection of all minimally balanced coalltions is finite and we denote this collection as $\{\mathcal{B}^l\}_{l\in L}$. For each minimal balanced collection $\mathcal{B}^l$, let $\lambda^l = \{\lambda_S^l\}_{S\in\mathcal{B}^l}$ be the balancing weights (note that the balancing weights of a minimal balanced collection of coalitions are always unique). Now, for each $l \in L$ and $\mathbf{k} \in K^{|\mathcal{B}^l|}$, define $x^{l,\mathbf{k}} \in X$ as

$$x_i^{l,\mathbf{k}} = \sum_{S\in\mathcal{B}^l: i\in S} \lambda_S^l y_{S,i}^{k_S} \quad \text{for all } i \in N,$$

where $y_{S,i}^{k_S}$ is the action of player $i \in S$ in the joint action $y_S^{k_S}$ of coalition S. Since each $x_i^{l,\mathbf{k}}$ is a convex combination of points in X_i, and X_i is convex, $x_i^{l,\mathbf{k}} \in X_i$. We shall show for each minimally balanced collection of coalitions $\mathcal{B} \in \mathcal{T}$, $\sum_{S\in\mathcal{B}} \lambda_S W(S) \leq W(N)$. Pick $\mathcal{B} \in \mathcal{T}$ with the balancing weights λ. If $\mathcal{B} = \{N\}$, then $\sum_{S\in\mathcal{B}} \lambda_S W(S) \leq W(N)$ trivially holds. Otherwise, since $\mathcal{B}$ is a minimal balanced collection of coalition, it does not contain N. By construction of W, for each $S \in \mathcal{B}$ there exists $k_S \in K$ such that $W(S) = w_S^{k_S} \leq \sum_{i\in S} u_i(y_S^{k_S}, z_{-S})$ for all $z_{-S} \in X_{-S}$. Define $x \in X$ as

$$x_i = \sum_{S\in\mathcal{B}: i\in S} \lambda_S y_{S,i}^{k_S} \quad \text{for all } i \in N.$$

By construction of W, $\sum_{i\in N} u_i(x) \leq W(N)$. Therefore, showing

$$\sum_{S\in\mathcal{B}} \lambda_S W(S) \leq \sum_{i\in N} u_i(x)$$

will be sufficient. First, from the construction of W and quasiseparability,

$$\sum_{S\in\mathcal{B}} \lambda_S W(S) \leq \sum_{S\in\mathcal{B}} \lambda_S \inf_{z_{-S}\in X_{-S}} \sum_{i\in S} u_i(y_S^{k_S}, z_{-S}) = \sum_{i\in N}\sum_{S\in\mathcal{B}: i\in S} \lambda_S \inf_{z_{-S}\in X_{-S}} u_i(y_S^{k_S}, z_{-S}).$$

Hence, showing the following inequality implies the desired result.

$$\sum_{S\in\mathcal{B}:i\in S} \lambda_S \inf_{z_{-S}\in X_{-S}} u_i(y_S^{k_S}, z_{-S}) \le u_i(x) \text{ for each } i \in N.$$

At this level of generality, it is sufficient to demonstrate that the above inequality holds for player 1, since by a suitable renaming of players any particular player can be made the first. We now define $y^S \in X$ for each $S \in \mathcal{B}$ containing player 1 as follows. If $i \in S$, then

$$y_i^S = y_{S,i}^{k_S}.$$

If $i \notin S$, then

$$y_i^S = \frac{\sum \lambda_E y_{E,i}^{k_E}}{\sum \lambda_E}.$$

where in both the numerator and the denominator the summation is taken over all $E \in \mathcal{B}$ which contain player i but not player 1. From Scarf (1971, p.179)

$$x = \sum_{S\in\mathcal{B}:1\in S} \lambda_S y^S.$$

Pick a coalition $S \in \mathcal{B}$ containing player 1. Then by construction of y^S,

$$\inf_{z_{-S}\in X_{-S}} u_1(y_S^{k_S}, z_{-S}) \le u_1(y^S).$$

Therefore, from the concavity of u_1,

$$\sum_{S\in\mathcal{B}:1\in S} \lambda_S \inf_{z_{-S}\in X_{-S}} u_1(y_S^{k_S}, z_{-S}) \le \sum_{S\in\mathcal{B}:1\in S} \lambda_S u_1(y^S) \le u_1(x).$$

Therefore, since $\mathcal{B}$ is arbitrarily chosen, for each $\mathcal{B} \in \mathcal{T}$, $\sum_{S\in\mathcal{B}} \lambda_S W(S) \le W(N)$.

Bondareva-Shapley Theorem provided in the Appendix implies W has a nonempty core, i.e. there exists $v^* \in \mathbb{R}^n$ such that $\sum_{i\in N} v_i^* = W(N)$ and

$\sum_{i \in S} v_i^* \geq W(S)$ for all $S \in \mathcal{N}$. By construction of $W(N)$, there exists $x^* \in X$ such that $\sum_{i \in N} v_i^* = \sum_{i \in N} u_i(x^*)$. In particular, $x^* = y_N^{k'}$, or $x^{l,\mathbf{k}}$ for some $k' \in K, l \in L, \mathbf{k} \in K^{|\mathcal{B}^l|}$. Hence, $(x^*, v^*) \in \bar{\mathcal{X}}$. Since G is coalitionally C^T-secure and $(x^*, v^*) \in U^k$ for some $k \in K$, there exists $S \in \mathcal{N}$ such that $\sum_{i \in S} v_i^* < w_S^k$. By construction, $w_S^k \leq W(S)$. This furnishes us a contradiction. □

Proof of Theorem 6. Since G_N is C-secure, there exists $\bar{x} \in X$ which maximizes the aggregate payoff function $\bar{u}$ of the grand coalition. Hence, $\mathcal{V}$ is a well-defined compact set. Now assume G has an empty α^T-core. Then, since G is coalitionally C_N^T-secure, for each $v \in \mathcal{V}$, there exist $w_S^v \in \mathbb{R}$ and $y_S^v \in X_S$ for each $S \in \mathring{\mathcal{N}}$ and an open neighborhood U^v of v such that $\sum_{i \in S} u_i(y_S^v, z_{-S}) \geq w_S^v$ for each coalition $S \in \mathring{\mathcal{N}}$ and each $z_{-S} \in X_{-S}$, and for each $z \in U^v$ there exists a coalition $S \in \mathring{\mathcal{N}}$ such that $\sum_{i \in S} v_i' < w_S^v$. The family $\{U^v |\ v \in \mathcal{V}\}$ is an open covering of $\mathcal{V}$ which, by compactness of $\mathcal{V}$, contains a finite subcovering $\{U^{v_k} |\ k = 1, \ldots, m\}$. Let $U^v = U^{v_k}$, and for all $S \in \mathring{\mathcal{N}}$, $w_S^k = w_S^{v_k}$ and $y_S^k = y_S^{v_k}$ for all $k \in K = \{1, \ldots, m\}$.

Now define a TU game $W : \mathcal{N} \to \mathbb{R}$ as follows. For all $S \in \mathcal{N} \backslash N$,

$$W(S) = \max_{k \in K} w_S^k, \quad \text{and } W(N) = \bar{w} = \max_{x \in X} \sum_{i \in N} u_i(x).$$

We shall show for each minimally balanced collection of coalitions $\mathcal{B}$, $\sum_{S \in \mathcal{B}} \lambda_S W(S)$ $\leq W(N)$. Pick a minimal balanced collection of coalitions $\mathcal{B}$ with the balancing weights $\lambda = \{\lambda_S\}_{S \in \mathcal{B}}$ (note that the balancing weights of a minimal balanced collection of coalitions are always unique). If $\mathcal{B} = \{N\}$, then $\sum_{S \in \mathcal{B}} \lambda_S W(S) \leq W(N)$ trivially holds. Otherwise, since $\mathcal{B}$ is a minimal balanced collection of coalition, it does not contain N. We shall show that there exists $x' \in X$ such that

$$\sum_{S\in\mathcal{B}}\lambda_S W(S) \leq \sum_{i\in N} u_i(x') \leq \bar{w} = W(N).$$

where the last inequality follows from the definition of $\bar{w}$. By construction of W, for each $S \in \mathcal{B}$ there exists $k_S \in K$ such that $W(S) = w_S^{k_S} \leq \sum_{i\in S} u_i(y_S^{k_S}, z_{-S})$ for all $z_{-S} \in X_{-S}$. Define $x \in X$ as

$$x_i = \sum_{S\in\mathcal{B}:i\in S}\lambda_S y_{S,i}^{k_S} \quad \text{for all } i \in N.$$

By construction of W, $\sum_{i\in N} u_i(x) \leq W(N) = \bar{w}$. Therefore, showing

$$\sum_{S\in\mathcal{B}}\lambda_S W(S) \leq \sum_{i\in N} u_i(x)$$

will be sufficient. The remainder of the proof now follows the argument of Theorem 5 verbatim. In particular, we conclude that $\sum_{S\in\mathcal{B}}\lambda_S W(S) \leq W(N)$ for each $\mathcal{B} \in \mathcal{T}$.

Bondareva-Shapley Theorem provided in the Appendix implies W has a nonempty core, i.e. there exists $v^* \in \mathcal{V}$ such that $\sum_{i\in N} v_i^* \leq W(N)$ and $\sum_{i\in S} v_i^* \geq W(S)$ for all $S \in \mathcal{N}$. Since G is coalitionally C_N^T-secure and $v^* \in U^k$ for some $k \in K$, there exists $S \in \mathring{\mathcal{N}}$ such that $\sum_{i\in S} v_i^* < w_S^k$. By construction, $w_S^k \leq W(S)$. This furnishes us a contradiction. □

Proof of Proposition 2. Assume G has an empty α-core. Then, since G is coalitionally C-secure, for each $x \in X$, there exist $v_S^x \in \mathbb{R}^{|S|}$ and $y_S^x \in X_S$ for each $S \in \mathcal{N}$ and an open neighborhood U^x of x such that $u_S(y_S^x, z_{-S}) \geq v_S^x$ for each coalition $S \in \mathcal{N}$ and each $z_{-S} \in X_{-S}$, and for each $x' \in U^x$ there exists a coalition $S \in \mathcal{N}$ such that $u_S(x') \ll v_S^x$. The family $\{U^x |\ x \in X\}$ is an open covering of X which, by compactness of X, contains a finite subcovering

$\{U^{x_k} | \ k = 1, \ldots, m\}$. Let $U^k = U^{x_k}$, and for all $S \in \mathcal{N}$, $v_S^k = v_S^{x_k}$ and $y_S^k = y_S^{x_k}$ for all $k \in K = \{1, \ldots, m\}$.

Now define an NTU game $V : \mathcal{N} \twoheadrightarrow \mathbb{R}^2$ as follows. For each $i \in N$,

$$V(\{i\}) = \bigcup_{k \in K} \{v \in \mathbb{R}^2 | \ v_i \leq v_i^k\},$$

$$V(N) = \bigcup_{k,k' \in K} \left(\{v \in \mathbb{R}^2 | \ v \leq u(y_N^k)\} \cup \{v \in \mathbb{R}^2 | \ v \leq u(y_1^k, y_2^{k'})\} \right).$$

By construction, the NTU game V is balanced. To see this, note that there are three balanced collections of coalitions for this game of which two contain N. For these collections, there is nothing to prove. The only balanced collection of coalitions which does not include N is $\mathcal{B} = \{\{1\}, \{2\}\}$. Pick $v \in V(S)$ for all $S \in \mathcal{B}$. Then, there exists $k, k' \in K$ such that[12] $v_1 \leq v_1^k$ and $v_2 \leq v_2^{k'}$. By construction, $v_1^k \leq u_1(y_1^k, z_2)$ for all $z_2 \in X_2$ and $v_2^k \leq u_2(z_1, y_2^{k'})$ for all $z_1 \in X_1$. Therefore $(v_1^k, v_2^{k'}) \leq u(y_1^k, y_2^{k'})$, and hence $v \in V(N)$. Therefore V is balanced.

It is clear that conditions (i)-(ii) of Scarf's Theorem provided in the Appendix are satisfied. And since for each coalition S, the set $V(S)$ is constructed by using finitely many points, condition (iii) of Scarf's Theorem is satisfied. Hence, V has a nonempty core, i.e. there exists $v^* \in V(N)$ such that $v^* \notin \text{int}V(S)$ for all $S \in \mathcal{N}$. Since $v^* \in V(N)$, by construction there exists $x^* \in X$ such that $v^* \leq u(x^*)$. Also, since G is coalitionally C-secure and $x^* \in U^k$ for some $k \in K$, there exists $S \in \mathcal{N}$ such that $u_S(x^*) \ll v_S^k$. Since $\{v_S^k\} \times \mathbb{R}^{|-S|} \subset V(S)$, $v^* \in \text{int}V(S)$. This furnishes us a contradiction. □

Proof of Proposition 3. Since G_N is C-secure, there exists $\bar{x} \in X$ which maximizes the aggregate payoff function $\bar{u}$ of the grand coalition. Hence, $\mathcal{V}$ is a

[12] Recall that we abuse the notation here, and refer to the singleton coalitions without using curly brackets, see footnotes 4 and 5 for details about the notation.

well-defined compact set. Now assume G has an empty α^T-core. Then, since G is coalitionally C_N^T-secure, for each $v \in \mathcal{V}$, there exist $w_i^v \in \mathbb{R}$ and $y_i^v \in X_S$ for each $i \in N$ and an open neighborhood U^v of v such that $u_i(y_i^v, z_j) \geq w_i^v$ for each player $i \neq j$ and each $z_j \in X_j$, and for each $z \in U^v$ there exists a player $i \in N$ such that $v_i' < w_i^v$. The family $\{U^v |\ v \in \mathcal{V}\}$ is an open covering of $\mathcal{V}$ which, by compactness of $\mathcal{V}$, contains a finite subcovering $\{U^{v_k} |\ k = 1, \ldots, m\}$. Let $U^v = U^{v_k}$, and for all $i \in N$, $w_i^k = w_i^{v_k}$ and $y_i^k = y_i^{v_k}$ for all $k \in K = \{1, \ldots, m\}$.

Now define a TU game $W : \mathcal{N} \to \mathbb{R}$ as follows. For each $i \in N$,

$$W(\{i\}) = \max_{k \in K} w_i^k, \quad \text{and } W(N) = \bar{w} = \max_{x \in X} \sum_{i \in N} u_i(x).$$

We shall show for each minimally balanced collection of coalitions $\mathcal{B}$, $\sum_{S \in \mathcal{B}} \lambda_S W(S) \leq W(N)$. Note that there are only two minimal balanced collections of coalitions: $\mathcal{B} = \{N\}$, or $\{\{1\}, \{2\}\}$. For $\mathcal{B} = \{N\}$, there is nothing to prove. Let $\mathcal{B} = \{\{1\}, \{2\}\}$. By construction, there exists $k, k' \in K$ such that $W(\{1\}) = w_1^k$ and $W(\{2\}) = w_2^{k'}$. By construction, $w_1^k \leq u_1(y_1^k, z_2)$ for all $z_2 \in X_2$ and $w_2^k \leq u_2(z_1, y_2^{k'})$ for all $z_1 \in X_1$. Therefore, $(w_1^k, w_2^{k'}) \leq u(y_1^k, y_2^{k'})$ and $\sum_{i \in N} u_i(y_1^k, y_2^{k'}) \leq \bar{w}$ imply $W(\{1\}) + W(\{2\}) \leq W(N)$.

Bondareva-Shapley Theorem provided in the Appendix implies W has a nonempty core, i.e. there exists $v^* \in \mathcal{V}$ such that $\sum_{i \in N} v_i^* = W(N)$ and $v_i^* \geq W(\{i\})$ for all $i \in N$. Since G is coalitionally C_N^T-secure and $v^* \in U^k$ for some $k \in K$, there exists $i \in N$ such that $v_i^* < w_i^k$. By construction, $w_i^k \leq W(\{i\})$. This furnishes us a contradiction. □

Proof of Claim 1. Let G be a bounded and coalitionally C^T-secure 2-player game. Showing the induced one player game G_N is C-secure is enough to prove that G is coalitionally C_N^T-secure. Now pick $x \in X$ that is not a maximizer of

$\bar{u} = u_1 + u_2$. Then for each $v \in \mathbb{R}^2$ such that $(x, v) \in \bar{\mathcal{X}}$ cannot be in the α^T-core of G. Then there exist $w_S^{x,v} \in \mathbb{R}$ and $y_S^{x,v} \in X_S$ for each $S \in \mathcal{N}$ and an open neighborhood $U^{x,v}$ of (x, v) such that $\sum_{i \in S} u_i(y_S^{x,v}, z_{-S}) \geq w_S^{x,v}$ for each coalition $S \in \mathcal{N}$ and each $z_{-S} \in X_{-S}$, and for each $(x', v') \in U^{x,v}$ there exists a coalition $S \in \mathcal{N}$ such that $\sum_{i \in S} v_i' < w_S^{x,v}$. Let U^x be the projection of $U^{x,v}$ on X. Since the projection is an open map, U^x is an open neighborhood of x. Define y^x as the maximizer of $\bar{u}$ over the set $\{(y_1^{x,v}, y_2^{x,v}), y_N^{x,v}\}$. Pick $x' \in U^x$. Then $(x', v') \in U^{x,v}$ for each $v' \in \mathcal{V}^{x'} = \{v'' \in \mathbb{R}^2 | v_1'' + v_2'' = \bar{u}(x'),\ \inf_{x'' \in X}\ u_i(x'') \leq v_i''\ \ \forall i \in N\}$. If for some $v' \in \mathcal{V}^{x'}$, $v_1' + v_2' < w_N^{x,w}$ then $\bar{u}(x') < w_N^{x,w} \leq \bar{u}(y^x)$. Otherwise, for each $v' \in \mathcal{V}^{x'}$, either $v_1' < w_1^{x,v}$ or $v_2' < w_2^{x,v}$. If for some player $i \in N$, $v_i' < w_i^{x,v}$ for each $v' \in \mathcal{V}^{x'}$, then $\bar{u}(x') < w_i^{x,v} \leq \bar{u}(y^x)$. Otherwise, let $v_i' = w_i^{x,v}$ for some $v' \in \mathcal{V}^{x'}$. Then $w_j^{x,v} > \bar{u}(x') - v_i'$ for player $j \neq i$. Hence $\bar{u}(x') = v_1' + v_2' < w_1^{x,v} + w_2^{x,v} \leq \bar{u}(y^x)$. Therefore, for each $x'' \in U^x$, $\bar{u}(x'') < w^x = \bar{u}(y^x)$.

Example 1 illustrates a bounded and coalitionally C_N^T-secure 2-player game that is not coalitionally C^T-secure. □

Proof of Claim 2. Let G be a bounded and coalitionally C^T-secure game such that $\bar{u}$ has a maximizer $\bar{x} \in X$, and let $\bar{w} = \bar{u}(\bar{x})$. Note that since G is coalitionally C^T-secure, showing the induced one player game $G_N = (X, \bar{u})$ is C-secure is equivalent to showing G is coalitionally C_N^T-secure. Now assume G is not C-secure. Then, there exists $x \in X$ such that $\bar{u}(x) < \bar{w}$ and for each open neighborhood U^x of x there exists $x' \in X$ such that $\bar{u}(x') \geq \bar{u}(y)$ for each $y \in X$. Therefore there exists a sequence $x^m \in X$ such that $x^m \to x$ and for each $m \in \mathbb{N}$, $\bar{u}(x^m) \geq \bar{u}(y)$ for each $y \in X$. Hence $\bar{u}(x^m) \to \bar{w}$. But since x is not a maximizer of $\bar{u}$, G cannot be coalitionally C^T-secure.

Now we illustrate a bounded and coalitionally C^T_N-secure game that is not C^T-secure. Let $X_i = [0,1]$, $u_i(x_i, x_{-i}) = x_i$ for $x_i \in (0,1)$, $u_i(0, x_{-i}) = 1$ and $u_i(1, x_{-i}) = 0$ for each $i \in N$ and $x_{-i} \in X_{-i}$. It is clear that $\{((0,\dots,0),(1,\dots,1))\}$ is the α^T-core of this game. For each $S \in \mathcal{N}$ and $x \in X\backslash\{(0,\dots,0)\}$, set $w^v_S = |S|$, y^x_S is the vector of zeros in $\mathbb{R}^{|S|}$, and the ε-neighborhood of x by choosing $\varepsilon = (1-u_i(x))/4$ for some $i \in N$. Hence this game is coalitionally C^T_N-secure. However, since $((1, ldots, 1),(1,\dots,1))$ is in $\bar{\bar{\mathcal{X}}}$, this game is not coalitionally C^T-secure. □

Proof of Claim 3. It is enough to show that π_i is quasiconcave on X if and only if f_i is quasiconcave in the two argument for each $i \in N$. Pick $i \in N, x, y \in X$ and $\delta \in (0,1)$. Then, $\pi_i(\delta x + (1-\delta)y) \geq \min\{\pi_i(x), \pi_i(y)\}$ if and only if $p(\delta x_i + (1-\delta)y_i + \delta\sum_{j\in N\backslash\{i\}} x_j + (1-\delta)\sum_{j\in N\backslash\{i\}} x_j)(\delta x_i + (1-\delta)y_i) - c_i(\delta x_i + (1-\delta)y_i) \geq \min\{p(x_i + \sum_{j\in N\backslash\{i\}} x_j)x_i - c_i(x_i),\ p(y_i + \sum_{j\in N\backslash\{i\}} y_j)y_i - c_i(y_i)\}$ if and only if $f_i(\delta x + (1-\delta)y) \geq \min\{f_i(x), f_i(y)\}$. □

Proof of Claim 4. We shall first show that π_i is concave on X if and only if f_i is concave in the two argument for each $i \in N$. Pick $i \in N, x, y \in X$ and $\delta \in (0,1)$. Then, $\pi_i(\delta x+(1-\delta)y) \geq \delta\pi_i(x)+(1-\delta)\pi_i(y)$ if and only if $p(\delta x_i+(1-\delta)y_i + \delta\sum_{j\in N\backslash\{i\}} x_j + (1-\delta)\sum_{j\in N\backslash\{i\}} x_j)(\delta x_i + (1-\delta)y_i) - c_i(\delta x_i + (1-\delta)y_i) \geq \delta(p(x_i+\sum_{j\in N\backslash\{i\}} x_j)x_i - c_i(x_i)) + (1-\delta)(p(y_i+\sum_{j\in N\backslash\{i\}} y_j)y_i - c_i(y_i))\}$ if and only if $f_i(\delta x+(1-\delta)y) \geq \delta f_i(x) + (1-\delta) f_i(y)$. Next, since p is a decreasing function, for each $S \neq N$, $\min_{z_{-S}\in X_{-S}} \sum_{i\in S} \pi_i(x_S, z_{-S})$ is well defined for each $x_S \in X_S$ and $\bar{y}_T$ is a minimizer for both this problem and for $\sum_{i\in S} \min_{z_{-S}\in X_{-S}} \pi_i(x_S, z_{-S})$. Therefore, the game is quasiseparable. □

2.5 Concluding Remarks

This paper provides sufficient conditions for the nonemptiness of the TU and NTU α-cores of games with possibly discontinuous payoff functions. We end this paper with three remarks. First, although the α-core is widely applied cooperative solution concept for normal form games, a number of different solution concepts, such as β-core, strong equilibrium and hybrid solution of Zhao (1999a) are also of interest for analyzing specific problems. It will be of interest to discuss the existence of such solutions for discontinuous games.

Second, a generalization of Scarf (1971) to games with nonordered preferences is provided by Kajii (1992), and a generalization to nonordered and discontinuous preferences by Martins-da-Rocha and Yannelis (2011). Although their models capture nonordered preferences on infinite dimensional spaces, our coalitional-security condition is substantially weaker than the continuity assumption they imposed when the set of actions are finite dimensional and the preferences are represented by payoff functions. They worked with correspondences $P_S : X \twoheadrightarrow X_S$ which map each action of the grand coalition to the set of blocking actions of coalition S. Martins-da-Rocha and Yannelis assumed that the correspondence P_S has open fibers on X which implies if a coalition S blocks an action profile x by using action y_S, then it blocks all actions around an open neighborhood of x by using y_S. Whereas, coalitional-security implies each point in some neighborhood of x is blocked by some coalition by using a fixed action profile. Hence, the blocking coalition may alternate for different points in the neighborhood. It is possible to generalize the state of the art result of Martins-da-Rocha and Yannelis (2011) in this line of literature by weakening the open fibers assumption and using the concepts and methods defined in Uyanık (2014). Moreover, the nonemptiness of the α^T-core of has not been studied even for nonordered and continuous preferences.

Third, the unbounded payoff functions, especially the logarithmic functions, are essential for many economic models. Our results in this paper require compactness of the action sets and boundedness of the payoff functions. It may be of interest to provide existence results for games with unbounded payoff functions and noncompact action sets.

Chapter 3

On the Nonemptiness of the Transferable Utility β-core of Discontinuous Games

3.1 Introduction

A pair of an action profile and a payoff profile is in the α-core of a game if the grand coalition's aggregate payoff from the action profile is equivalent to the aggregate payoff profile, and no coalition has an alternative action which makes all of its members better off, independently of the actions of the other players. The α-core is often criticized to be too pessimistic. In an alternative solution concept designated as the β-core,[1] the blocking coalition is permitted to counteract to each action of the complementary coalition so as to achieve a higher aggregate payoff. It is as if a blocking coalition announces its intention to block, forces the complementary coalition to move first, and then responds, rather than the reverse order of moves.

The existence of these cooperative solutions has direct applications many

[1] Both the α-core and β-core concepts are due to Aumann (1961). An element of the β-core was once called an acceptable payoff vector in Aumann (1959).

economic environments. In particular, cooperation with side payments can be interpreted as overt collusion in oligopoly markets. Higher monopoly profits provides firms an incentive to act cooperatively. However, this profitable merger will not take place unless firms could split the monopoly profits without any objection. The β-core describes an allocation of the monopoly profits in a monopoly merger. Therefore, the nonemptiness of β-core provides a necessary condition for monopoly merger. In other words, the monopoly merger can only take place if the original market has a nonempty core.

Zhao (1999) proved the following theorem.

Theorem (Zhao). *Let $G = (X_i, u_i)_{i \in N}$ be a game such that for each player i,*

(i) *X_i is a nonempty, compact and convex subset of a finite dimensional Euclidean space of actions,*

(ii) *u_i is a concave and continuous utility function on $X = \prod_{i \in N} X_i$,*

(iii) *G is strongly separable.*[2]

Then G has a nonempty β-core.

Motivated by economic problems that are suitably modeled by games with discontinuous payoff functions, a rich and evolving literature has emerged following the seminal works of Dasgupta and Maskin (1986) and Reny (1999).[3] Recently, an existence result for the α-core of a game with possibly discontinuous payoff functions has been provided by Uyanik (2015). The purpose of this paper is to generalize Zhao's result to games with possibly discontinuous payoff functions. In line with the recent literature on discontinuous games, the notions of *coalitionally* CS and *coalitionally* CS_N games are presented, and the existence of an imputation in the β-core is shown for such games.

[2]See Definition 17 and Remark 5 for the definition of strong separability.

[3]See Carmona (2011) for a symposium on the recent developments in the discontinuous games literature.

The paper is organized as follows. Section 3.2 defines the basic concepts and states the main results, Section 3.3 discusses two-player games, Section 3.4 illustrates the applications of our results to Bertrand duopoly and Cournot oligopoly games, Section 3.5 provides proofs of the results, and Section 3.6 concludes.

3.2 The Model and the Results

A *(normal form) game* is a list $G = (X_i, u_i)_{i \in N}$ where

(i) $N = \{1, \ldots, n\}$ is the finite set of players,

(ii) X_i is the nonempty set of actions of player $i \in N$,

(iii) $u_i : X \to \mathbb{R}$ is the utility function of player $i \in N$ defined on $X = \prod_{i \in N} X_i$.

A *concave game* is a game $G = (X_i, u_i)_{i \in N}$ such that for each player i, X_i is a nonempty, compact and convex subset of a finite dimensional Euclidean space and u_i is a concave function on X.

In his pioneering work, Reny (1999) proved the existence of a Nash equilibrium of normal form games satisfying the following weak continuity assumption.

Definition 11. A game G is *better-reply-secure*[4] (BRS) if at each $x \in X$ that is not a Nash equilibrium of G, there exist $y^x \in X$, $\delta^x > 0$, and an open neighborhood U^x of x such that for each $x' \in U^x$ there exists $i \in N$ such that for each $z \in U^x$; $u_i(y_i^x, z_{-i}) > u_i(x') + \delta^x$.

And Barelli and Meneghel (2013) provided the following continuity concept which is weaker than BRS, and used it to prove the existence of a (pure strategy) Nash equilibrium.

[4]This definition is equivalent to but different from the original definition of better-reply-security. See Footnote 7 in Reny (2015) and B-security concept in McLennan et al. (2011, Definition 2.4, p.1646).

Definition 12. A game G is *correspondence-secure* (CS) if for each $x \in X$ that is not a Nash equilibrium of G, there exist an open neighborhood U^x of x, and for each player $i \in N$, $v_i^x \in \mathbb{R}$ and a nonempty-valued upper semicontinuous, u.s.c. hereafter, correspondence $\delta_i^x : U^x \twoheadrightarrow X_i$ such that

(i) $u_i(z'_i, z_{-i}) \geq v_i^x$ for each $i \in N$, $z \in U^x$ and $z'_i \in \delta_i^x(z)$,

(ii) for each $x' \in U^x$ there exists $i \in N$ such that $u_i(x') < v_i^x$.

Remark 3. If an action profile x is not a Nash equilibrium of the game G, then by definition at least one player deviates at it. The CS imposes the following structure on the deviations: (a) an open neighborhood of x does not contain a Nash equilibrium, i.e. at each point in the neighborhood at least one player deviates at it, (b) the identity of the deviant is allowed to vary, but his deviation strategy must change upper semicontinuously in response to the remaining players' tremble on the neighborhood.

A *coalition* is an element S in $\mathcal{N} = 2^N \backslash \emptyset$. Let $\mathcal{N}^p$ denote the set of all nonempty and proper subsets of N, i.e. $\mathcal{N}^p = \mathcal{N} \backslash N$. The set of actions available to a coalition S is denoted as $X_S = \prod_{i \in S} X_i$, and the vector of utility functions of coalition S as $u_S = (u_i)_{i \in S}$.[5] For each coalition S, let $-S$ denote the complementary coalition $N \backslash S$. Next, we define equilibrium and continuity concepts for games where coalitions are allowed to both coordinate their actions and reallocate their aggregate payoffs, i.e. payoff transfers are allowed.

Definition 13. Let G be a game. A coalition S *blocks* a payoff profile $v \in \mathbb{R}^n$ if $\forall z_{-S} \in X_{-S}$, $\exists x'_S \in X_S$ such that $\sum_{i \in S} u_i(x'_S, z_{-S}) > \sum_{i \in S} v_i$. A pair of an action profile and a payoff profile $(x^*, v^*) \in X \times \mathbb{R}^n$ with $\sum_{i \in N} u_i(x^*) = \sum_{i \in N} v_i^*$ is in the *β-core* of G if v^* is not blocked by any coalition.

[5] By abusing the notation, we drop the subscript N for the grand coalition, and when it is clear from the context, we use i and $\{i\}$ interchangeably for the singleton coalition $S = \{i\}$.

Definition 14. A game G is *bounded* if for each player i, u_i is bounded.

For a bounded game G, define

$$\mathcal{X} = \{(x, v) \in X \times \mathbb{R}^n | \sum_{i \in N} v_i = \sum_{i \in N} u_i(x), \ \inf_{x \in X} u_i(x) \leq v_i \ \forall i \in N\}.$$

Note that $\mathcal{X}$ is the set of all individually rational action-payoff pairs, hence it contains the β-core of G.[6] Moreover, if X is compact then it is clear that the closure of $\mathcal{X}$, denoted by $\bar{\mathcal{X}}$, is compact. Now we define a continuity concept similar to CS (Definition 12) which includes the behavior of the coalitions as follows.

Definition 15. A bounded game G is *coalitionally correspondence-secure* (*coalitionally CS*) if for each $(x, v) \in \bar{\mathcal{X}}$ that is not in the β-core of G, there exist an open neighborhood $U^{x,v}$ of (x, v), $y_N^{x,v} \in X$, $w_N^{x,v} \in \mathbb{R}$, and for each $S \in \mathring{\mathcal{N}}$ there exist $w_S^{x,v} \in \mathbb{R}$ and a function $\delta_S^{x,v} : X_{-S} \to X_S$ such that

(i) $\sum_{i \in N} u_i(y_N^{x,v}) \geq w_N^{x,v}$,

(ii) $\sum_{i \in S} u_i(\delta_S^{x,v}(z_{-S}), z_{-S}) \geq w_S^{x,v}$ for each $S \in \mathring{\mathcal{N}}$ and $z_{-S} \in X_{-S}$,

(iii) for each $(x', v') \in U^{x,v}$ there exists $S \in \mathcal{N}$ such that $\sum_{i \in S} v'_i < w_S^{x,v}$.

Remark 4. Coalitional CS is conceptually analogous to the CS notion, the role of individual players is taken by coalitions; see Remark 3. However, there are some significant differences. First, coalitional CS does not require the securing strategies of proper coalition to have a continuity property, hence assuming they are single valued will not put any restriction. Second, in part (ii) of the Definition 15, 'for each $z \in U^x$' is replaced by 'for each $z_{-S} \in X_{-S}$.' Although this is a strong assumption, it is consistent with the definition of

[6]In order to see this, note that any payoff profile which assigns player i a payoff below $\inf_{x \in X} u_i(x)$ is blocked by player i, therefore it cannot be part of β-core.

blocking and Reny's insight. The CS assumes a structure on *deviations,* and coalitional CS on *blockings*, and the definition of blocking, in contrast to deviation, already incorporates *all* actions of the complementary coalition.[7] Hence, there is no inclusion relation between the coalitional CS and CS concepts, Example 1 below illustrates this relation. Lastly, it is remarkable that the C-security concept of McLennan et al. (2011) naturally fits into α-core and CS of Barelli and Meneghel (2013) into β-core, see Uyanik (2015, Remark 2, p.3).

Note that since β-core consists of a pair of action and payoff profile, the coalitional CS is defined on the set of pairs. Moreover, a point (x^*, v^*) in the β-core must satisfy the following three conditions: (i) $x^* \in \underset{x \in X}{\text{argmax}} \sum_{i \in N} u_i(x)$, (ii) $\sum_{i \in N} v_i^* = \max_{x \in X} \sum_{i \in N} u_i(x)$, and (iii) v^* is not blocked by any coalition. In order to guarantee a solution to the maximization problem defined in (i), we define coalitional CS at points in $\bar{\mathcal{X}}$, not only in $\mathcal{X}$.[8] Although coalitional CS is the *natural analogue* of CS, some important games which have a nonempty β-core are not coalitionally CS, such as Bertrand duopoly game with different marginal costs, see Example 1 below.[9] Therefore, we introduce a separate continuity concept which we term coalitional CS_N as follows.

Let G be a game. Define a one player game $G_N = (X, \bar{u})$ where $X =$

[7]It is possible to generalize the securing strategy of the grand coalition (condition (i) of Definition 15) as a nonempty-valued u.s.c. correspondence $\delta_N^{x,v} : X \twoheadrightarrow X$ such that $\sum_{i \in N} u_i(z) \geq w_N^{x,v}$ for each $(x', v') \in U^{x,v}$ and $z \in \delta_N^{x,v}$. This will require a long and careful construction of a well-behaved correspondence and a fixed point argument in the proof of our results. Since the grand coalition's problem is a pure optimization problem, i.e. there is no interaction, this new setup would yield a merely technical contribution.

[8]This property of coalitional CS is similar to the BRS notion of Reny (1999) which takes the closure of the graph of the game into account.

[9]The following one player game may be helpful in illustrating the underlying structure. $X = [0, 1]$ and $u(0) = 1$, $u(1) = 0$, and $u(x) = x$ for all $x \in (0, 1)$. This 1-player game is not coalitionally CS since 1 is not maximizer and $(x, v) = (1, 1) \in \bar{\mathcal{X}}$. However, it will be clear below that the game is coalitionally C_N-secure.

$\prod_{i\in N} X_i$ and $\bar{u}(x) = \sum_{i\in N} u_i(x)$. If G_N is CS, then the main theorem of Barelli and Meneghel (2013) implies there exists a maximizer $\bar{x} \in X$ of $\bar{u}$. Therefore, this result implies focusing only on the redistributions of the grand coalition's maximum aggregate payoff is enough to show the nonemptiness of the β-core. In particular, define

$$\mathcal{V} = \{v \in \mathbb{R}^n | \sum_{i\in N} v_i = \sum_{i\in N} u_i(\bar{x}),\ \inf_{x\in X} u_i(x) \le v_i\ \forall i \in N\}.$$

Now imposing a structure on the deviations at each point in $\mathcal{V}$ will guarantee the nonemptiness of the β-core. Note that by construction, $\mathcal{V}$ is compact.

Definition 16. A bounded game $G = (X_i, u_i)_{i\in N}$ is *coalitionally* CS_N if the induced one player game G_N is CS, and at each $v \in \mathcal{V}$ such that $(\bar{x}, v)$ is not in the β-core of G, there exist an open neighborhood U^v of v, and for each $S \in \mathring{\mathcal{N}}$ there exist $w_S^v \in \mathbb{R}$ and a function $\delta_S^v : X_{-S} \to X_S$ such that

(i) $\sum_{i\in S} u_i(\delta_S^v(z_{-S}), z_{-S}) \ge w_S^v$ for each $S \in \mathring{\mathcal{N}}$ and $z_{-S} \in X_{-S}$,

(ii) for each $v' \in U^v$ there exists $S \in \mathring{\mathcal{N}}$ such that $\sum_{i\in S} v'_i < w_S^v$.

Note that there is no inclusion relation between coalitional CS and coalitional CS_N. Example 1 illustrates this relation. Although the coalitional CS_N does not take the point in the closure of $\mathcal{X}$ into account, it focuses only on $\{\bar{x}\}\times\mathcal{V} \subset \bar{\mathcal{X}}$, it puts assumption on the grand coalition's blocking behavior which is not imposed by the coalitional CS. Hence, there is no inclusion relation among them.

Let G be a bounded coalitionally CS game. For each $S \in \mathring{\mathcal{N}}$ and $(x, v) \in \bar{\mathcal{X}}$ that is not in the β-core of G define the set of all securing reaction functions as

$$\Delta_S(x, v) = \{\delta_S^{x',v'} : X_{-S} \to X_S \mid (x', v') \in \bar{\mathcal{X}} \text{ is not in the } \beta\text{-core of } G \text{ and } w_S^{x',v'} \ge w_S^{x,v}\},$$

Now let G be a bounded coalitionally CS_N game. For each $S \in \mathring{\mathcal{N}}$ and $v \in \mathcal{V}$ such that $(\bar{x}, v)$ is not in the β-core of G define the set of all securing reaction functions as

$$\Delta_S^N(v) = \{\delta_S^{v'} : X_{-S} \to X_S \mid v' \in \mathcal{V} \text{ such that } (\bar{x}, v') \text{ is not in the } \beta\text{-core of } G \text{ and } w_S^{v'} \geq w_S^v\}.$$

Definition 17. A bounded coalitionally CS game G is *strongly separable* if for each $S \in \mathring{\mathcal{N}}$ there exists $(x, v) \in \bar{\mathcal{X}}$ such that for each $\delta_S \in \Delta_S(x, v)$ there exists $\tilde{z}_{-S} \in X_{-S}$ such that

$$\sum_{i \in S} u_i(\delta_S(\tilde{z}_{-S}), \tilde{z}_{-S}) = \sum_{i \in S} \inf_{z_{-S} \in X_{-S}} u_i(\delta_S(\tilde{z}_{-S}), z_{-S}).$$

Definition 18. A bounded coalitionally CS_N game G is *strongly separable* if for each $S \in \mathring{\mathcal{N}}$ there exists $v \in \mathcal{V}$ such that for each $\delta_S \in \Delta_S^N(v)$ there exists $\tilde{z}_{-S} \in X_{-S}$ such that

$$\sum_{i \in S} u_i(\delta_S(\tilde{z}_{-S}), \tilde{z}_{-S}) = \sum_{i \in S} \inf_{z_{-S} \in X_{-S}} u_i(\delta_S(\tilde{z}_{-S}), z_{-S}).$$

Remark 5. Note that the strong separability definition we provide here is weaker than the original one defined in Zhao (1999b, Definition 3, p.157) for games with continuous payoff functions as follows. A game G is *strongly separable* if for each $S \in \mathring{\mathcal{N}}$ and each $i \in S$,

$$u_i(x^*(\hat{z}_{-S}), \hat{z}_{-S}) = \min_{z_{-S} \in X_{-S}} u_i(x^*(\hat{z}_{-S}), z_{-S}),$$

where for each $z_{-S} \in X_{-S}$, $x_S^*(z_{-S})$ is a solution to $\max_{z_S \in X_S} \sum_{i \in S} u_i(z_S, z_{-S})$, and for given $x_S^*(\cdot)$, $\hat{z}_{-S}$ is a solution to $\min_{z_{-S} \in X_{-S}} \sum_{i \in S} u_i(x^*(z_{-S}), z_{-S})$. First, if payoff functions are continuous, setting $\delta_S = x^*_S$ and $\tilde{z}_{-S} = \hat{z}_{-S}$ for each S and δ_S shows Zhao's definition implies our definition. However, we do not require $\tilde{z}_{-S} = \hat{z}_{-S}$ and we put restriction only on the aggregate utility of each coalition, not all of its members.

Now we are ready to state our main results.

Theorem 7. *Every coalitionally CS, concave, bounded, strongly separable game has a nonempty β-core.*

Theorem 8. *Every coalitionally CS_N, concave, bounded, strongly separable game has a nonempty β-core.*

For some game theoretic situations only some subsets of the players can behave cooperatively, due to factors such as transaction costs, social and legal restrictions. A solution concept for such models is called hybrid solution which assumes that the players are partitioned into coalitions and that they will cooperate within each coalition but compete (in the Nash sense) among coalitions, see Zhao (1999b). Now, let $G = (X_i, u_i)_{i \in N}$ be a game. A given coalition structure (a partition of N) $\Delta = \{S_1, \dots, S_m\}$ induces a game $G_\Delta = \{X_S, \sum_{i \in S} u_i\}_{S \in \Delta}$ among partition members and m parametric games

$$G_S(x_{-S}) = \{X_i, u_i(\cdot, x_{-S})\}_{i \in S}.$$

Definition 19. Let $G = (X_i, u_i)_{i \in N}$ be a game and $\Delta = \{S_1, \dots, S_m\}$ a coalition structure. A *hybrid solution* for G is a pair of an action profile and a payoff profile $(x^*, v^*) \in X \times \mathbb{R}^n$ such that for each $S \in \Delta$,

(i) x^* is a Nash equilibrium of G_Δ,

(ii) (x^*_S, v^*_S) is in the β-core of $G_S(x^*_{-S})$.

The following result is a corollary to Theorem 8 and Barelli and Meneghel (2013, Theorem 2.2, p.816).

Corollary 2. *A bounded game G has a hybrid solution if G_Δ is CS and for each $S \in \Delta$ and $x_{-S} \in X_{-S}$, $G_S(x_{-S})$ is concave and strongly separable.*

Remark 6. The existence of hybrid solution result of Zhao (1999b) is more general than Corollary 2 in the sense that each coalition in Δ is allowed to use other distribution rules such as α-core. It is easy to generalize our result to this setup by using the results presented in Uyanik (2015).

It is well known that every real-valued concave function on a Euclidean space is continuous at each point of its domain's relative interior. Hence, discontinuities can occur only at the relative boundary of the domain. And, it is easy to define a concave function that is discontinuous at every point of the relative boundary of its domain. Ernst (2013) provided a nice characterization of the continuity properties of a concave function on the relative boundary of its domain. Before stating this result, we shall introduce some concepts. A subset X of $\mathbb{R}^m$ is called a *polytope* provided that it is the convex hull of a finite set of points And X is said to be *boundedly polyhedral* provided that its intersection with any polytope is a polytope. It is clear that any compact boundedly polyhedron is a polytope. Ernst (2013, Theorem 2.4, p.3672) stated that *given a convex and compact subset X of $\mathbb{R}^m$, every concave function on X is lower semicontinuous if and only if X is a polytope.* Since any convex and compact subset X of $\mathbb{R}$ is a polytope, Ernst's theorem implies that any concave function on X is lower semicontinuous. However, for $\mathbb{R}^m$, $m \geq 2$, this result is not true. Given a convex and compact subset X of $\mathbb{R}^m$ which is not polytope (such as unit ball), it is always possible to find a concave function on X which is not lower semicontinuous. Hence, our setup which eliminates *max* and replaces *min* with *inf* is crucial for games with possibly discontinuous payoff functions. And as Carter (2001, p.334) stated “This is not a mere curiosity. Economic life often takes place at the boundaries of convex sets, where the possibility of discontinuities must be taken into account.”

3.3 Two-Player Games

The special case of a two-player game is rather important in that the two-player set-up allows one to substantially weaken the convexity and separability assumptions. We also show that the strong separation assumption compensates the convexity and continuity assumptions for these games. But more to the point the two-player game results serve as an important backdrop to the examples we presented that validate our results as meaningful and useful generalizations of Zhao's results. Before presenting the results we shall introduce some concepts. A game $G = (X_i, u_i)_{i\in N}$ is *strongly coalitionally CS* if for each $S \in \mathcal{N}$ and $(x, v) \in \bar{\mathcal{X}}$, the function $\delta_S^{x,v}$ is a nonempty-valued u.s.c. correspondence. If the payoff functions are continuous, then the *best response correspondences* satisfies this assumption. Hence, our strong coalitional CS is substantially weaker than continuity assumption. Indeed, it is direct analogue of CS. And the coalitionally CS is substantially weaker than strongly coalitionally CS. And, *strongly coalitionally* CS_N is defined analogously.

Proposition 4. *A strongly coalitionally CS, bounded 2-player game has a nonempty β-core if for each player i, X_i is a nonempty, convex and compact subset of a finite dimensional Euclidean space and u_i is quasiconcave on X_i.*

Proposition 5. *A strongly coalitionally CS_N, bounded 2-player game has a nonempty β-core if for each player i, X_i is a nonempty, convex and compact subset of a finite dimensional Euclidean space and u_i is quasiconcave on X_i.*

Proposition 6. *A coalitionally CS, bounded, strongly separable 2-player game has a nonempty β-core if for each player i, X_i is a nonempty and compact subset of a finite dimensional Euclidean space.*

Proposition 7. *A coalitionally CS_N, bounded, strongly separable 2-player game has a nonempty β-core if for each player i, X_i is a nonempty and compact subset of a finite dimensional Euclidean space.*

Now, for the sake of completeness, we provide a continuity concept analogous to CS and a nonemptiness result for the β-core of two-player games with possibly discontinuous payoff functions with nontransferable utilities. An analogous result for continuous payoff functions is provided in Ichiishi (1993, Remark 2.3.2, p.37).

Definition 20. Let G be a game. A coalition S *β^N-blocks* a payoff profile $x \in X$ if $\forall z_{-S} \in X_{-S}$, $\exists x'_S \in X_S$ such that $u_S(x'_S, z_{-S}) \gg u_S(x)$. An action profile $x^* \in X$ is in the *β^N-core* of G if x^* is not β^N-blocked by any coalition.

Definition 21. A two-player game G is *coalitionally CS^N* if for each $x \in X$ that is not in the β^N-core of G, there exist an open neighborhood U^x of x, $y^x_N \in X$, $v^x_N \in \mathbb{R}^2$, and for each $i \in N$ there exist $v^x_i \in \mathbb{R}$ and a nonempty valued, u.s.c. correspondence $\delta^x_i : X_j \twoheadrightarrow X_i$ for $i \neq j$ such that

(i) $u_i(z_i, z_j) \geq v^x_i$ for each $i \neq j$, $z_j \in X_j$ and $z_i \in \delta^x_i(z_j)$, and $u_N(y^x_N) \geq v^x_N$,

(ii) for each $x' \in U^x$ there exists $S \in \mathcal{N}$ such that $x' \ll v^x_S$.

Proposition 8. *A strongly coalitionally CS^N, 2-player game has a nonempty β^N-core if for each player i, X_i is a nonempty, convex and compact subset of a finite dimensional Euclidean space and u_i is quasiconcave on X_i.*

3.4 Applications

Example 1. Consider the following Bertrand duopoly game. Firm i's action set is $P_i = [0, 10]$ for $i = 1, 2$. Define $P = P_1 \times P_2$. The market demand function $D : [0, 10] \to \mathbb{R}$ is defined as

$$D(p) = \max\{4 - p, 0\},$$

and the profit function $\pi_1 : P \to \mathbb{R}$ of firm 1 and $\pi_2 : P \to \mathbb{R}$ of firm 2 are defined as

$$\pi_1(p) = \begin{cases} p_1 D(p_1) & \text{if } p_1 < p_2, \\ p_1 D(p_1)/2 & \text{if } p_1 = p_2, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\pi_2(p) = \begin{cases} p_2 D(p_2) - 1 & \text{if } p_2 < p_1, \\ p_2 D(p_2)/2 - 1 & \text{if } p_1 = p_2, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that this game does not have a Nash equilibrium.[10] Now we show that it has a nonempty β-core by verifying the assumptions of Proposition 5. The nontrivial part of our verification is to show that this game is strongly coalitionally CS_N. First, the price pair $\bar{p} = (2, 0)$ is a maximizer of the aggregate profit function $\pi_1 + \pi_2$ in P and the aggregate profit at $\bar{p}$ is $\bar{w} = 4$. Second, a profit vector $v \in \mathbb{R}^2$ such that $(\bar{p}, v)$ is in the β-core of G if and only if $v_1 + v_2 = \bar{w}$ and $v_i \geq 0$ for $i = 1, 2$. This implies that any $v \in \mathcal{V} = \{v' | v'_1 + v'_2 = 4, v'_1 \geq$

[10] Note that in this paper we consider only the existence of Nash equilibrium in pure-strategies. It is known that this game has a Nash equilibrium in mixed-strategies, see Blume (2003).

$0, v_2' \geq -1\}$ that is not in the β-core must satisfy $v_2 < 0$. One can check that the game is strongly coalitionally CS_N by defining, for each $v \in V$ that is not the β-core, $\varepsilon^v = -\frac{v_2}{2}$, $U^v = \{v' \in V | \; \|v' - v\| < \varepsilon^v\}$, $\delta_1(z_2) = \{0\}, \delta_2(z_1) = \{1\}$ for every $z \in X$ and $w_1 = w_2 = 0$.

Next we show that this game is neither CS nor coalitionally CS.[11] In order to see that it is not CS, this game satisfies all the assumptions of the main theorem of Barelli and Meneghel (2013) except CS and it does not have a Nash equilibrium, hence it cannot be CS. In order to see it is not coalitionally CS, note that $((2,2),(4,0)) \in \bar{\mathcal{X}}$ is not in the β-core of G since $\sum_{i \in N} u_i(2,2) = 3 < 4$. But, no coalition can block $((2,2),(4,0))$.

The strong separability assumption is a strong assumption, and generally not easy to verify. However, the next proposition shows that oligopoly games with monotonically decreasing demand functions satisfy this property. A Cournot oligopoly game is a game $G = (X_i, \pi_i)_{i \in N}$ where $N = \{1, \ldots, n\}$ is the set of firms, $X_i = [0, \bar{y}_i]$ is the production set of firm i where $\bar{y}_i > 0$ denotes firm is capacity constraint, and $\pi_i : X \to \mathbb{R}$ is the profit function of firm i which is defined as

$$\pi_i(x) = f_i(x_i, \bar{x}_{-i}) = p(x_i + \bar{x}_{-i})x_i - c_i(x_i),$$

where $\bar{x}_{-i} = \sum_{j \in N \setminus \{i\}} x_j$, $p : \mathbb{R}_+ \to \mathbb{R}_+$ is the inverse demand function and $c_i : \mathbb{R}_+ \to \mathbb{R}_+$ is firm i's cost function.

Proposition 9. *A coalitionally CS (coalitionally CS_N), bounded Cournot oligopoly game has a nonempty β-core if p is decreasing and f_i is concave for each firm i.*

[11] Recall that CS and coalitional CS are strictly weaker than their strong counterparts, see the discussion in the first paragraph of Section 3.3.

3.5 Proofs of the Results

We first present a Lemma which is used in the proofs of Theorems 7 and 8.

Lemma 2. *If a bounded coalitionally CS (coalitionally CS_N) game G is strongly separable, then for each $S \in \mathring{\mathcal{N}}$ there exists $(x, v) \in \bar{\mathcal{X}}$ ($v \in \mathcal{V}$) such that for each $\delta_S \in \Delta_S(x, v)$ ($\delta_S \in \Delta_S^N(v)$) there exists $\tilde{z}_{-S} \in X_{-S}$ such that*

$$\inf_{z_{-S} \in X_{-S}} \sum_{i \in S} u_i(\delta_S(z_{-S}), z_{-S}) \leq \sum_{i \in S} \inf_{z_{-S} \in X_{-S}} u_i(\delta_S(\tilde{z}_{-S}), z_{-S}).$$

Proof of Lemma 2. Let G be a bounded, coalitionally CS and strongly separable game. Then for each $S \in \mathring{\mathcal{N}}$, there there exists $(x, v) \in \bar{\mathcal{X}}$ such that for each $\delta_S \in \Delta_S(x, v)$ there exists $\tilde{z}_{-S} \in X_{-S}$ such that

$$\sum_{i \in S} u_i(\delta_S(\tilde{z}_{-S}), \tilde{z}_{-S}) = \sum_{i \in S} \inf_{z_{-S} \in X_{-S}} u_i(\delta_S(\tilde{z}_{-S}), z_{-S}),$$

and, from the definition of infimum,

$$\inf_{z_{-S} \in X_{-S}} \sum_{i \in S} u_i(\delta_S(z_{-S}), z_{-S}) \leq \sum_{i \in S} u_i(\delta_S(\tilde{z}_{-S}), \tilde{z}_{-S}).$$

Therefore,

$$\inf_{z_{-S} \in X_{-S}} \sum_{i \in S} u_i(\delta_S(z_{-S}), z_{-S}) \leq \sum_{i \in S} \inf_{z_{-S} \in X_{-S}} u_i(\delta_S(\tilde{z}_{-S}), z_{-S}).$$

The proof is analogous for coalitionally CS_N games. □

Proof of Theorem 7. Assume G has an empty β-core. Then, since G is coalitionally CS, for each $(x, v) \in \bar{\mathcal{X}}$, there exist an open neighborhood $U^{x,v}$ of (x, v), $y_N^{x,v} \in X$, $w_N^{x,v} \in \mathbb{R}$, and for each $S \in \mathring{\mathcal{N}}$ there exist $w_S^{x,v} \in \mathbb{R}$ and a function $\delta_S^{x,v} : X_{-S} \to X_S$ such that $\sum_{i \in N} u_i(y_N^{x,v}) \geq w_N^{x,v}$, $\sum_{i \in S} u_i(\delta_S^{x,v}(z_{-S}), z_{-S}) \geq w_S^{x,v}$

for each $S \in \mathring{\mathcal{N}}$ and $z_{-S} \in X_{-S}$, and for each $(x', v') \in U^{x,v}$ there exists $S \in \mathcal{N}$ such that $\sum_{i \in S} v'_i < w_S^{x,v}$. The family $\{U^{x,v} | \ (x,v) \in \bar{\mathcal{X}}\}$ is an open covering of $\bar{\mathcal{X}}$ which, by compactness of $\bar{\mathcal{X}}$, contains a finite subcovering $\{U^{x_k,v_k} | \ k = 1, \ldots, m\}$. And for each $S \in \mathring{\mathcal{N}}$, $(x_S, v_S) \in \bar{\mathcal{X}}$ is identified by strong separability. Let $U^k = U^{x_k,v_k}$, and for all $S \in \mathcal{N}$, $w_S^k = w_S^{x_k,v_k}$ and $\delta_S^k = \delta_S^{x_k,v_k}$ for $S \neq N$, and $y_N^k = y_N^{x_k,v_k}$, for all $k \in K = \{1, \ldots, m, m+1, \ldots, m+|\mathcal{N}|-1\}$.

Now define a TU game $W : \mathcal{N} \to \mathbb{R}$ as follows. For all $S \in \mathring{\mathcal{N}}$,

$$W(S) = \max_{k \in K} w_S^k,$$

$$W(N) = \max \left\{ \max_{k \in K} \sum_{i \in N} u_i(y_N^k), \ \max_{l \in L} \sum_{i \in N} u_i(x^l) \right\},$$

where $L, \mathcal{B}^l$, $y^* \in X$, and $x^l \in X$ are defined as follows. First, define for each coalition $S \in \mathring{\mathcal{N}}$,

$$k_S \in \operatorname*{argmax}_{k \in K} w_S^k, \quad \text{and} \quad \delta_S = \delta_S^{k_S}.$$

Let $\mathcal{T} = \{\mathcal{B}^l\}_{l \in L}$ be the set of all minimal balanced collection of coalitions which does not include N. Since number of all collection of coalitions is $2^{|\mathcal{N}|}$ and $\mathcal{N}$ is a finite set, L is finite. For each $\mathcal{B}^l \in \mathcal{T}$, let $\lambda^l = \{\lambda_S^l\}_{S \in \mathcal{B}^l}$ be the balancing weights (note that the balancing weights of a minimal balanced collection of coalitions are always unique). By construction, for each $S \in \mathcal{B}^l$, $W(S) = w_S^{k_S} \leq \sum_{i \in S} u_i(\delta_S(z_{-S}), z_{-S})$ for each $z_{-S} \in X_{-S}$. Define $x^l \in X$ as

$$x_i^l = \sum_{S \in \mathcal{B}^l : i \in S} \lambda_S^l \delta_{S,i}(\tilde{z}_{-S}) \quad \text{for all } i \in N,$$

where $\tilde{z}_{-S}$ is determined by strong separation.

We shall show for each minimally balanced collection of coalitions $\mathcal{B}$, $\sum_{S \in \mathcal{B}} \lambda_S W(S)$

$\leq W(N)$. For $\mathcal{B} = \{N\}$, there is nothing to prove. Since the only minimal balanced collection of coalition which includes N is $\mathcal{B} = \{N\}$, we shall prove the above inequality for all $\mathcal{B} \in \mathcal{T}$. Pick $\mathcal{B} \in \mathcal{T}$ with the balancing weights $\lambda = \{\lambda_S\}_{S\in\mathcal{B}}$ (note that the balancing weights of a minimal balanced collection of coalitions are always unique). We shall show that there exists $x' \in X$ such that

$$\sum_{S\in\mathcal{B}} \lambda_S W(S) \leq \sum_{i\in N} u_i(x') \leq W(N).$$

Recall that in defining $W(N)$, we define an action profile for each balanced collection. Now, pick the corresponding $x^* = x^l$ where

$$x_i^* = \sum_{S\in\mathcal{B}:i\in S} \lambda_S \delta_{S,i}(\tilde{z}_{-S}) \quad \text{for all } i \in N.$$

By construction of W, $\sum_{i\in N} u_i(x^*) \leq W(N)$. Therefore, showing

$$\sum_{S\in\mathcal{B}} \lambda_S W(S) \leq \sum_{i\in N} u_i(x^*)$$

will be sufficient. From the construction of W,

$$\sum_{S\in\mathcal{B}} \lambda_S W(S) \leq \sum_{S\in\mathcal{B}} \lambda_S \inf_{z_{-S}\in X_{-S}} \sum_{i\in S} u_i(\delta_S(z_{-S}), z_{-S}),$$

and since W is strongly separable, Lemma 2 implies there exists $\tilde{z}_{-S} \in X_{-S}$ such that

$$\sum_{S\in\mathcal{B}} \lambda_S \inf_{z_{-S}\in X_{-S}} \sum_{i\in S} u_i(\delta_S(z_{-S}), z_{-S}) \leq \sum_{S\in\mathcal{B}} \lambda_S \sum_{i\in S} \inf_{z_{-S}\in X_{-S}} u_i(\delta_S(\tilde{z}_{-S}), z_{-S}).$$

Therefore,

$$\sum_{S\in\mathcal{B}} \lambda_S W(S) \leq \sum_{i\in N} \sum_{S\in\mathcal{B}:i\in S} \lambda_S \inf_{z_{-S}\in X_{-S}} u_i(\delta_S(\tilde{z}_{-S}), z_{-S}),$$

hence showing the following inequality implies the desired result.

$$\sum_{S\in\mathcal{B}:i\in S}\lambda_S \inf_{z_{-S}\in X_{-S}} u_i(\delta_S(\tilde{z}_{-S}),z_{-S}) \le u_i(x^*) \text{ for each } i\in N.$$

At this level of generality, it is sufficient to demonstrate that the above inequality holds for player 1, since by a suitable renaming of players any particular player can be made the first. We now define $y^S\in X$ for each $S\in\mathcal{B}$ containing player 1 as follows. If $i\in S$, then

$$y_i^S = \delta_{S,i}(\tilde{z}_{-S}).$$

If $i\notin S$, then

$$y_i^S = \frac{\sum \lambda_E \delta_{E,i}(\tilde{z}_{-E})}{\sum \lambda_E}.$$

where in both the numerator and the denominator the summation is taken over all $E\in\mathcal{B}$ which contain player i but not player 1. From Scarf (1971, p.179)

$$x^* = \sum_{S\in\mathcal{B}:1\in S}\lambda_S y^S.$$

Pick a coalition $S\in\mathcal{B}$ containing player 1. Then by construction of y^S and the strong separability,

$$\inf_{z_{-S}\in X_{-S}} u_1(\delta_S(\tilde{z}_{-S}),z_{-S}) \le u_1(y^S).$$

Therefore, from the concavity of u_1,

$$\sum_{S\in\mathcal{B}:i\in S}\lambda_S \inf_{z_{-S}\in X_{-S}} u_1(\delta_S(\tilde{z}_{-S}),z_{-S}) \le \sum_{S\in\mathcal{B}:1\in S}\lambda_S u_1(y^S) \le u_1(x^*).$$

Therefore, since $\mathcal{B}$ is arbitrarily chosen, for each $\mathcal{B}\in\mathcal{T}$, $\sum_{S\in\mathcal{B}}\lambda_S W(S)\le W(N)$.

Bondareva-Shapley Theorem provided in the Appendix implies W has a nonempty core, i.e. there exists $v^*\in\mathbb{R}^n$ such that $\sum_{i\in N} v_i^* = W(N)$ and

$v_i^* \geq W(S)$ for all $S \in \mathcal{N}$. By construction of $W(N)$, there exists $\bar{x} \in X$ such that $\sum_{i\in N} v_i^* = \sum_{i\in N} u_i(\bar{x})$. In particular, $\bar{x} = y_N^k$, or x^l for some $k \in K, l \in L$. Hence, $(\bar{x}, v^*) \in \bar{\mathcal{X}}$. Since G is coalitionally CS and $(\bar{x}, v^*) \in U^k$ for some $k \in K$, there exists $S \in \mathcal{N}$ such that $\sum_{i\in S} v_i^* < w_S^k$. By construction, $w_S^k \leq W(S)$. This furnishes us a contradiction. □

Proof of Theorem 8. Since G_N is CS, there exists $\bar{x} \in X$ which maximizes the aggregate payoff function $\bar{u}$ of the grand coalition. Hence, $\mathcal{V}$ is a well-defined compact set. Now, assume G has an empty β-core. Then, since G is coalitionally CS_N, for each $v \in \mathcal{V}$, there exist an open neighborhood U^v of v, and for each $S \in \mathring{\mathcal{N}}$ there exist $w_S^v \in \mathbb{R}$ and a function $\delta_S^v : X_{-S} \to X_S$ such that $\sum_{i\in S} u_i(\delta_S^v(z_{-S}), z_{-S}) \geq w_S^v$ for each $S \in \mathring{\mathcal{N}}$, $z_{-S} \in X_{-S}$, and for each $v' \in U^v$ there exists $S \in \mathring{\mathcal{N}}$ such that $\sum_{i\in S} v_i' < w_S^v$. The family $\{U^v |\ v \in \mathcal{V}\}$ is an open covering of $\mathcal{V}$ which, by compactness of $\mathcal{V}$, contains a finite subcovering $\{U^{v_k} |\ k = 1, \ldots, m\}$. And for each $S \in \mathring{\mathcal{N}}$, $v_S \in \mathcal{V}$ is identified by strong separability. Let $U^v = U^{v_k}$, and for all $S \in \mathcal{N}$, $w_S^k = w_S^{v_k}$ and $\delta_S^k = \delta_S^{v_k}$ for all $k \in K = \{1, \ldots, m, m+1, \ldots, m + |\mathcal{N}| - 1\}$.

Now define a TU game $W : \mathcal{N} \to \mathbb{R}$ as follows. For all $S \in \mathcal{N} \backslash N$,

$$W(S) = \max_{k\in K} w_S^k, \quad \text{and} \quad W(N) = \bar{w} = \max_{x\in X} \sum_{i\in N} u_i(x).$$

First, define for each coalition $S \in \mathring{\mathcal{N}}$,

$$k_S \in \operatorname*{argmax}_{k\in K} w_S^k, \quad \text{and} \quad \delta_S = \delta_S^{k_S}.$$

We shall show for each minimally balanced collection of coalitions $\mathcal{B}$, $\sum_{S\in\mathcal{B}} \lambda_S W(S) \leq W(N)$. Pick a minimal balanced collection of coalitions $\mathcal{B}$ with the balancing weights $\lambda = \{\lambda_S\}_{S\in\mathcal{B}}$ (note that the balancing weights of a minimal balanced collection of coalitions are always unique). For $\mathcal{B} = \{N\}$, there is nothing to prove. Otherwise, since $\mathcal{B}$ is a minimal balanced collection of coalition, it does not contain N. We shall show that there exists $x' \in X$ such that

$$\sum_{S \in \mathcal{B}} \lambda_S W(S) \leq \sum_{i \in N} u_i(x') \leq \bar{w} = W(N).$$

where the last inequality follows from the definition of $\bar{w}$. By construction, for each $S \in \mathcal{B}$, $W(S) = w_S^{ks} \leq \sum_{i \in S} u_i(\delta_S(z_{-S}, z_{-S})$ for each $z_{-S} \in X_{-S}$. Define $x^* \in X$ as

$$x_i^* = \sum_{S \in \mathcal{B}: i \in S} \lambda_S \delta_{S,i}(\tilde{z}_{-S}),$$

where $\tilde{z}_{-S}$ is determined by strong separation. By construction, $\sum_{i \in N} u_i(x^*) \leq W(N) = \bar{w}$. Therefore, showing

$$\sum_{S \in \mathcal{B}} \lambda_S W(S) \leq \sum_{i \in N} u_i(x^*)$$

will be sufficient. From the construction of W,

$$\sum_{S \in \mathcal{B}} \lambda_S W(S) \leq \sum_{S \in \mathcal{B}} \lambda_S \inf_{z_{-S} \in X_{-S}} \sum_{i \in S} u_i(\delta_S(z_{-S}), z_{-S}),$$

and since W is strongly separable, Lemma 2 implies there exists $\tilde{z}_{-S} \in X_{-S}$ such that

$$\sum_{S \in \mathcal{B}} \lambda_S \inf_{z_{-S} \in X_{-S}} \sum_{i \in S} u_i(\delta_S(z_{-S}), z_{-S}) \leq \sum_{S \in \mathcal{B}} \lambda_S \sum_{i \in S} \inf_{z_{-S} \in X_{-S}} u_i(\delta_S(\tilde{z}_{-S}), z_{-S}).$$

Therefore,

$$\sum_{S \in \mathcal{B}} \lambda_S W(S) \leq \sum_{i \in N} \sum_{S \in \mathcal{B}: i \in S} \lambda_S \inf_{z_{-S} \in X_{-S}} u_i(\delta_S(\tilde{z}_{-S}), z_{-S}),$$

hence showing the following inequality implies the desired result.

$$\sum_{S \in \mathcal{B}: i \in S} \lambda_S \inf_{z_{-S} \in X_{-S}} u_i(\delta_S(\tilde{z}_{-S}), z_{-S}) \leq u_i(x^*) \text{ for each } i \in N.$$

At this level of generality, it is sufficient to demonstrate that the above inequality holds for player 1, since by a suitable renaming of players any particular player can be made the first. We now define $y^S \in X$ for each $S \in \mathcal{B}$ containing player 1 as follows. If $i \in S$, then

$$y_i^S = \delta_{S,i}(\tilde{z}_{-S}).$$

If $i \notin S$, then

$$y_i^S = \frac{\sum \lambda_E \delta_{E,i}(\tilde{z}_{-E})}{\sum \lambda_E}.$$

where in both the numerator and the denominator the summation is taken over all $E \in \mathcal{B}$ which contain player i but not player 1. From Scarf (1971, p.179)

$$x^* = \sum_{S \in \mathcal{B}: 1 \in S} \lambda_S y^S.$$

Pick a coalition $S \in \mathcal{B}$ containing player 1. Then by construction of y^S and the strong separability,

$$\inf_{z_{-S} \in X_{-S}} u_1(\delta_S(\tilde{z}_{-S}), z_{-S}) \leq u_1(y^S).$$

Therefore, from the concavity of u_1,

$$\sum_{S \in \mathcal{B}: i \in S} \lambda_S \inf_{z_{-S} \in X_{-S}} u_1(\delta_S(\tilde{z}_{-S}), z_{-S}) \leq \sum_{S \in \mathcal{B}: 1 \in S} \lambda_S u_1(y^S) \leq u_1(x^*).$$

Therefore, since $\mathcal{B}$ is arbitrarily chosen, for each $\mathcal{B} \in \mathcal{T}$, $\sum_{S \in \mathcal{B}} \lambda_S W(S) \leq W(N)$.

Bondareva-Shapley Theorem provided in the Appendix implies W has a nonempty core, i.e. there exists $v^* \in \mathcal{V}$ such that $\sum_{i \in N} v_i^* = W(N)$ and $\sum_{i \in S} v_i^* \geq W(S)$ for all $S \in \mathcal{N}$. Since G is coalitionally CS_N and $v^* \in U^k$ for some $k \in K$, there exists $S \in \mathcal{N}$ such that $\sum_{i \in S} v_i^* < w_S^k$. By construction, $w_S^k \leq W(S)$. This furnishes us a contradiction. □

Proof of Corollary 2. Since $G_S(x_{-S})$ is a concave game for each $S \in \Delta$ and $x_{-S} \in X_{-S}$, u_i is concave for each player $i \in S$, hence $\sum_{i \in S} u_i(\cdot, x_{-S})$ is concave. Then, since G_Δ is CS, Barelli and Meneghel (2013, Theorem 2.2, p.816) implies there exists $x^* \in X$ such that $x^*_S \in \operatorname{argmax}_{x_S \in X_S} \sum_{i \in S} u_i(x_S, x^*_{-S})$ for each $S \in \Delta$. And since all conditions of Theorem 8 are satisfied, for each $S \in \Delta$ there exists $v^*_S \in \mathbb{R}^{|S|}$ such that (x^*_S, v^*_S) is in the β-core of $G_S(x^*_{-S})$. □

Proof of Proposition 4. Assume G has an empty β-core. Then, since G is strongly coalitionally CS, for each $(x, v) \in \bar{\mathcal{X}}$, there exist an open neighborhood $U^{x,v}$ of (x, v), $y^{x,v}_N \in X$, $w^{x,v}_N \in \mathbb{R}$, and for each $i \in N$ there exist $w^{x,v}_i \in \mathbb{R}$ and a nonempty-valued u.s.c. correspondence $\delta^{x,v}_i : X_j \to X_i$ for $i \neq j$ such that $\sum_{i \in N} u_i(y^{x,v}_N) \geq w^{x,v}_N$, $u_i(\delta^{x,v}_i(z_j), z_j) \geq w^{x,v}_i$ for each $i \neq j$ and $z_j \in X_j$, and $z \in \delta^{x,v}_N(x')$, and for each $(x', v') \in U^{x,v}$ there exists $S \in \mathcal{N}$ such that $\sum_{i \in S} v'_i < w^{x,v}_S$. The family $\{U^{x,v} | \, (x, v) \in \bar{\mathcal{X}}\}$ is an open covering of $\bar{\mathcal{X}}$ which, by compactness of $\bar{\mathcal{X}}$, contains a finite subcovering $\{U^{x_k,v_k} | \, k = 1, \ldots, m\}$. And for each $i \in N$, $(x_i, v_i) \in \bar{\mathcal{X}}$ is identified by strong separability. Let $U^k = U^{x_k,v_k}$, and for all $S \in \mathcal{N}$, $w^k_S = w^{x_k,v_k}_S$, $y^k_N = y^{x_k,v_k}_N$, and for $i = 1, 2$, $\delta^k_i = \delta^{x_k,v_k}_i$ for all $k \in K = \{1, \ldots, m, \ldots, m + 2\}$.

Now define a TU game $W : \mathcal{N} \to \mathbb{R}$ as follows. For all $i \in N$,

$$W(\{i\}) = \max_{k \in K} w^k_i,$$

$$W(N) = \max \left\{ \max_{k \in K} \sum_{i \in N} u_i(y^k_N), \; \sum_{i \in N} u_i(x^*) \right\},$$

where $x^* \in X$ is defined as follows. First, define for each player $i \in N$,

$$k_i \in \operatorname*{argmax}_{k \in K} w^k_i, \quad \text{and} \quad \delta_i = \operatorname{co}(\delta^{k_i}_i),$$

where $\text{co}(\delta_i^{k_i})$ is the convex hull of $\delta_i^{k_i}$. Since δ_i is u.s.c. and X_i is subset of a finite dimensional Euclidean space for each $i \in N$, δ_i is *usc*. And since u_i is quasiconcave for each $i \in N$, $u_i(z_i, z_j) \geq w_i^{k_i}$ for each $z_j \in X_j$ and $z_i \in \delta_i(z_j)$ for $i \neq j$. Now, define a correspondence $\delta : X \twoheadrightarrow X$ as $\delta(z) = (\delta_1(z_2), \delta_2(z_1))$. Since δ is *usc*, and has nonempty and convex values, Kakutani's fixed point theorem implies there exists $x^* \in X$ such that

$$x_i^* \in \delta_i(x_j^*),\ i \neq j.$$

Hence, $w_i^{k_i} \leq u_i(z_i, z_j)$ for all $z_j \in X_j$ and $z_i \in \delta_2(z_j)$ for $i \neq j$ implies $w_i^{k_i} \leq u_i(x^*)$ for $i = 1, 2$.

We shall show for each minimally balanced collection of coalitions $\mathcal{B}$, $\sum_{S \in \mathcal{B}} \lambda_S W(S) \leq W(N)$. Note that there are only two minimal balanced collection of coalitions: $\mathcal{B} = \{N\}$, or $\{\{1\}, \{2\}\}$. For $\mathcal{B} = \{N\}$, there is nothing to prove. Let $\mathcal{B} = \{\{1\}, \{2\}\}$. Recall that, $w_1^{k_1} \leq u_1(x^*)$ and $w_2^{k_2} \leq u_2(x^*)$. Therefore, $\sum_{i \in N} u_i(x^*) \leq W(N)$ implies $W(1) + W(2) \leq W(N)$.

Bondareva-Shapley Theorem provided in the Appendix implies W has a nonempty core, i.e. there exists $v^* \in \mathbb{R}^2$ such that $\sum_{i \in N} v_i^* = W(N) = \sum_{i \in N}$ and $v_i^* \geq W(\{i\})$ for all $i \in N$. By construction of $W(N)$, there exists $\bar{x} \in X$ such that $\sum_{i \in N} v_i^* = \sum_{i \in N} u_i(\bar{x})$. In particular, $\bar{x} = y_N^k$ for some $k \in K$, or $\bar{x} = x^*$. Hence, $(\bar{x}, v^*) \in \bar{\mathcal{X}}$. Since G is coalitionally CS and $(\bar{x}, v^*) \in U^k$ for some $k \in K$, there exists $S \in \mathcal{N}$ such that $\sum_{i \in S} v_i^* < w_S^k$. By construction, $w_S^k \leq W(S)$. This furnishes us a contradiction. □

Proof of Proposition 5. Since G_N is CS, there exists $\bar{x} \in X$ which maximizes the aggregate payoff function $\bar{u}$ of the grand coalition. Hence, $\mathcal{V}$ is a well-defined compact set. Now, assume G has an empty β-core. Then, since G is coalitionally CS_N, for each $v \in \mathcal{V}$, there exist an open neighborhood U^v of v, and for each $i \in N$ there exist $w_i^v \in \mathbb{R}$ and u.s.c. correspondence $\delta_i^v : X_j \twoheadrightarrow X_i$ with nonempty values such that $u_i(z_i, z_j) \geq w_i^v$ for each $i \neq j$, $z_j \in X_j$ and $z_i \in \delta_i^v(z_j)$, and for each $v' \in U^v$ there exists $i \in N$ such that $v'_i < w_i^v$. The family $\{U^v |\ v \in \mathcal{V}\}$ is an open covering of $\mathcal{V}$ which, by compactness of $\mathcal{V}$, contains a finite subcovering $\{U^{v_k} |\ k = 1, \ldots, m\}$. And for each $i \in N$, $v_i \in \mathcal{V}$ is identified by strong separability. Let $U^v = U^{v_k}$, and for all $i \in N$, $w_i^k = w_i^{v_k}$ and $\delta_i^k = \delta_i^{v_k}$ for all $k \in K = \{1, \ldots, m, \ldots, m+2\}$.

Now define a TU game $W : \mathcal{N} \rightarrow \mathbb{R}$ as follows. For all $i \in N$,

$$W(\{i\}) = \max_{k \in K} w_i^k, \quad \text{and } W(N) = \bar{w} = \max_{x \in X} \sum_{i \in N} u_i(x).$$

We shall show for each minimally balanced collection of coalitions $\mathcal{B}$, $\sum_{S \in \mathcal{B}} \lambda_S W(S) \leq W(N)$. Note that there are only two minimal balanced collection of coalitions: $\mathcal{B} = \{N\}$, or $\{\{1\}, \{2\}\}$. For $\mathcal{B} = \{N\}$, there is nothing to prove. Let $\mathcal{B} = \{\{1\}, \{2\}\}$. Define for each player $i \in N$,

$$k_i \in \operatorname*{argmax}_{k \in K} w_i^k, \quad \text{and} \quad \delta_i = \text{co}(\delta_i^{k_i}),$$

where $\text{co}(\delta_i^{k_i})$ is the convex hull of $\delta_i^{k_i}$. Since δ_i is u.s.c. and X_i is subset of a finite dimensional Euclidean space for each $i \in N$, δ_i is *usc*. And since u_i is quasiconcave for each $i \in N$, $u_i(z_i, z_j) \geq w_i^{k_i}$ for each $z_j \in X_j$ and $z_i \in \delta_i(z_j)$ for $i \neq j$. Now, define a correspondence $\delta : X \twoheadrightarrow X$ as $\delta(z) = (\delta_1(z_2), \delta_2(z_1))$. Since δ is *usc*, and has nonempty and convex values, Kakutani's fixed point theorem implies there exists $x^* \in X$ such that

$$x_i^* \in \delta_i(x_j^*),\ i \neq j.$$

Hence, $w_i^{k_i} \leq u_i(z_i, z_j)$ for all $z_j \in X_j$ and $z_i \in \delta_2(z_j)$ for $i \neq j$ implies $w_i^{k_i} \leq u_i(x^*)$ for $i = 1, 2$. Therefore, $\sum_{i \in N} u_i(x^*) \leq \bar{w}$ implies $W(1) + W(2) \leq W(N)$.

Bondareva-Shapley Theorem provided in the Appendix implies W has a nonempty core, i.e. there exists $v^* \in \mathcal{V}$ such that $\sum_{i \in N} v_i^* = W(N)$ and $v_i^* \geq W(\{i\})$ for all $i \in N$. Since G is coalitionally CS_N and $v^* \in U^k$ for some $k \in K$, there exists $i \in \mathcal{N}$ such that $v_i^* < w_i^k$. By construction, $w_i^k \leq W(\{i\})$. This furnishes us a contradiction. □

Proof of Proposition 6. Assume G has an empty β-core. Then, since G is coalitionally CS, for each $(x, v) \in \bar{\mathcal{X}}$, there exist an open neighborhood $U^{x,v}$ of (x, v), $y_N^{x,v} \in X$, $w_N^{x,v} \in \mathbb{R}$, and for each $i \in N$ there exist $w_i^{x,v} \in \mathbb{R}$ and a function $\delta_i^{x,v} : X_j \to X_i$ for $i \neq j$ such that $\sum_{i \in N} u_i(y_N^{x,v}) \geq w_N^{x,v}$, $u_i(\delta_i^{x,v}(z_j), z_j) \geq w_i^{x,v}$ for each $i \neq j$ and $z_j \in X_j$, and for each $(x', v') \in U^{x,v}$ there exists $S \in \mathcal{N}$ such that $\sum_{i \in S} v_i' < w_S^{x,v}$. The family $\{U^{x,v} | \ (x, v) \in \bar{\mathcal{X}}\}$ is an open covering of $\bar{\mathcal{X}}$ which, by compactness of $\bar{\mathcal{X}}$, contains a finite subcovering $\{U^{x_k, v_k} | \ k = 1, \ldots, m\}$. And for each $i \in N$, $(x_i, v_i) \in \bar{\mathcal{X}}$ is identified by strong separability. Let $U^k = U^{x_k, v_k}$, and for all $S \in \mathcal{N}$, $w_S^k = w_S^{x_k, v_k}$, $y_N^k = y_N^{x_k, v_k}$, and for $i = 1, 2$, $\delta_i^k = \delta_i^{x_k, v_k}$ for all $k \in K = \{1, \ldots, m, \ldots, m+2\}$.

Now define a TU game $W : \mathcal{N} \to \mathbb{R}$ as follows. For all $i \in N$,

$$W(\{i\}) = \max_{k \in K} w_i^k,$$

$$W(N) = \max \left\{ \max_{k \in K} \sum_{i \in N} u_i(y_N^k), \ \sum_{i \in N} u_i(x^*) \right\},$$

where $x^* \in X$ is defined as follows. First, define for each coalition $S \in \mathcal{N}$,

$$k_S \in \operatorname*{argmax}_{k \in K} w_S^k, \quad \text{and} \quad \delta_S = \delta_S^{k_S}.$$

By construction, for each $i \neq j$, $W(\{i\}) = w_i^{k_i} \leq u_i(\delta_i(z_j), z_j)$ for each $z_j \in X_j$. Define $x^* \in X$ as

$$x_i^* = \delta_i(\tilde{z}_j) \quad \text{for } i \neq j,$$

where $\tilde{z}_j$ is determined by strong separation.

We shall show for each minimally balanced collection of coalitions $\mathcal{B}$, $\sum_{S \in \mathcal{B}} \lambda_S W(S) \leq W(N)$. Note that there are only two minimal balanced collection of coalitions: $\mathcal{B} = \{N\}$, or $\{\{1\}, \{2\}\}$. For $\mathcal{B} = \{N\}$, there is nothing to prove. Let $\mathcal{B} = \{\{1\}, \{2\}\}$. The definition of infimum and strong separation imply for each $i \neq j$,

$$\inf_{z_j \in X_j} u_i(\delta_i(z_j), z_j) \leq u_i(\delta_i(\tilde{z}_j), \tilde{z}_j) = \inf_{z_j \in X_j} u_i(\delta_i(\tilde{z}_j), z_j).$$

Hence,

$$w_i^{k_i} \leq u_i(\delta_i(\tilde{z}_j), \delta_j \tilde{z}_i) = u_i(x^*).$$

Therefore, $\sum_{i \in N} u_i(x^*) \leq W(N)$ implies $W(1) + W(2) \leq W(N)$.

Bondareva-Shapley Theorem provided in the Appendix implies W has a nonempty core, i.e. there exists $v^* \in \mathbb{R}^2$ such that $\sum_{i \in N} v_i^* = W(N) = \sum_{i \in N}$ and $v_i^* \geq W(\{i\})$ for all $i \in N$. By construction of $W(N)$, there exists $\bar{x} \in X$ such that $\sum_{i \in N} v_i^* = \sum_{i \in N} u_i(\bar{x})$. In particular, $\bar{x} = y_N^k$ for some $k \in K$, or $\bar{x} = x^*$. Hence, $(\bar{x}, v^*) \in \bar{\mathcal{X}}$. Since G is coalitionally CS and $(\bar{x}, v^*) \in U^k$ for some $k \in K$, there exists $S \in \mathcal{N}$ such that $\sum_{i \in S} v_i^* < w_S^k$. By construction, $w_S^k \leq W(S)$. This furnishes us a contradiction. □

Proof of Proposition 7. Since G_N is CS, there exists $\bar{x} \in X$ which maximizes the aggregate payoff function $\bar{u}$ of the grand coalition. Hence, $\mathcal{V}$ is a well-defined compact set. Now, assume G has an empty β-core. Then, since G is coalitionally CS_N, for each $v \in \mathcal{V}$, there exist an open neighborhood U^v of v, and for each $i \in N$ there exist $w_i^v \in \mathbb{R}$ and a function $\delta_i^v : X_j \to X_i$ such that $u_i(\delta_i^v(z_j), z_j) \geq w_i^v$ for each $i \neq j$, and $z_j \in X_j$, and for each $v' \in U^v$ there exists $i \in N$ such that $v'_i < w_i^v$. The family $\{U^v |\ v \in \mathcal{V}\}$ is an open covering of $\mathcal{V}$ which, by compactness of $\mathcal{V}$, contains a finite subcovering $\{U^{v_k} |\ k = 1, \ldots, m\}$. And for each $i \in N$, $v_i \in \mathcal{V}$ is identified by strong separability. Let $U^v = U^{v_k}$, and for all $i \in N$, $w_i^k = w_i^{v_k}$ and $\delta_i^k = \delta_i^{v_k}$ for all $k \in K = \{1, \ldots, m, \ldots, m+2\}$.

Now define a TU game $W : \mathcal{N} \to \mathbb{R}$ as follows. For all $i \in N$,

$$W(\{i\}) = \max_{k \in K} w_i^k, \quad \text{and } W(N) = \bar{w} = \max_{x \in X} \sum_{i \in N} u_i(x).$$

We shall show for each minimally balanced collection of coalitions $\mathcal{B}$, $\sum_{S \in \mathcal{B}} \lambda_S W(S) \leq W(N)$. Note that there are only two minimal balanced collection of coalitions: $\mathcal{B} = \{N\}$, or $\{\{1\}, \{2\}\}$. For $\mathcal{B} = \{N\}$, there is nothing to prove. Let $\mathcal{B} = \{\{1\}, \{2\}\}$. By construction, $W(\{1\}) = w_1^{k_1} \leq u_1(\delta_1(z_2), z_2)$ for all $z_2 \in X_2$ and $W(\{2\}) = w_2^{k_2} \leq u_2(z_1, \delta_2(z_1))$ for all $z_1 \in X_1$. And strong separability implies for each $i \in N$, there exists $\tilde{z}_{-i} \in X_{-i}$ such that

$$w_i^{k_i} \leq u_i(\delta_i(\tilde{z}_{-i}), \tilde{z}_{-i}) = \inf_{z_{-i} \in X_{-i}} u_i(\delta_i(\tilde{z}_{-i}), z_{-i}) \ \text{ for } i = 1, 2.$$

Hence,

$$w_1^{k_1} \leq u_1(\delta_1(\tilde{z}_2), \delta_2(\tilde{z}_1)) \ \text{ and } w_2^{k_2} \leq u_2(\delta_1(\tilde{z}_2), \delta_2(\tilde{z}_1)).$$

Therefore, $\sum_{i \in N} u_i(\delta_1(\tilde{z}_2), \delta_2(\tilde{z}_1))) \leq \bar{w}$ implies $W(1) + W(2) \leq W(N)$.

Bondareva-Shapley Theorem provided in the Appendix implies W has a nonempty core, i.e. there exists $v^* \in \mathcal{V}$ such that $\sum_{i \in N} v_i^* = W(N)$ and $v_i^* \geq W(\{i\})$ for all $i \in N$. Since G is coalitionally CS_N and $v^* \in U^k$ for some $k \in K$, there exists $i \in N$ such that $v_i^* < w_i^k$. By construction, $w_i^k \leq W(\{i\})$. This furnishes us a contradiction. □

Proof of Proposition 8. Assume G has an empty β^N-core. Then, since G is coalitionally CS^N, for each $x \in X$, there exist an open neighborhood U^x of x, $y_N^x \in X$, $v_N^x \in \mathbb{R}^2$, and for each $i \in N$ there exist $v_i^x \in \mathbb{R}$ and a nonempty valued, u.s.c. correspondence $\delta_i^x : X_j \twoheadrightarrow X_i$ for $i \neq j$ such that $u_i(z_i, z_j) \geq v_i^x$ for each $i \neq j$, $z_j \in X_j$ and $z_i \in \delta_i^x(z_j)$, and $u_N(y_N^x) \geq v_N^x$, and for each $x' \in U^x$ there exists $S \in \mathcal{N}$ such that $x' \ll v_S^x$. The family $\{U^x | \, x \in X\}$ is an open covering of X which, by compactness of X, contains a finite subcovering $\{U^{x_k} | \, k = 1, \ldots, m\}$. Let $U^k = U^{x_k}$, and for all $S \in \mathcal{N}$, $v_S^k = v_S^{x_k}$, $y_N^k = y_N^{x_k}$ and for each $i \in N$, $\delta_i^k = \delta_i^{x_k}$, for all $k \in K = \{1, \ldots, m\}$.

Now define an NTU game $V : \mathcal{N} \twoheadrightarrow \mathbb{R}^2$ as follows. For all $i \in N$,

$$V(\{i\}) = \bigcup_{k \in K} \{v \in \mathbb{R}^2 | \, v_i \leq v_i^k\},$$

$$V(N) = \bigcup_{k,k' \in K} \left(\{v \in \mathbb{R}^2 | \, v \leq v_N^k\} \cup \{v \in \mathbb{R}^2 | \, v \leq u(x_1^k, x_2^{k'})\} \right),$$

where $(x_1^k, x_2^{k'}) \in X$ is defined as follows. First, define for each player $i \in N$ and $k \in K$,

$$\delta_i^k = \text{co}(\delta_i^k),$$

where $\text{co}(\delta_i^k)$ is the convex hull of δ_i^k. Since δ_i^k is u.s.c. and X_i is subset of a finite dimensional Euclidean space for each $i \in N$, and $k \in K$, δ_i^k is *usc*. And since u_i is quasiconcave for each $i \in N$, $u_i(z_i, z_j) \geq v_i^k$ for each $z_j \in X_j$ and $z_i \in \delta_i^k(z_j)$ for $i \neq j$, $k \in K$. Now, for each $k, k' \in K$, define a correspondence $\delta^{k,k'} : X \twoheadrightarrow X$ as $\delta^{k,k'}(z) = (\delta_1^k(z_2), \delta_2^{k'}(z_1))$. Since $\delta^{k,k'}$ is *usc*, and has nonempty and convex values, Kakutani's fixed point theorem implies there exists $x^* \in X$ such that

$$x_1^k \in \delta_1^k(x_2^*) \text{ and } x_2^{k'} \in \delta_2^{k'}(x_1^*).$$

Hence, $v_1^k \leq u_1(x_1^k, x_2^{k'})$ and $v_2^{k'} \leq u_2(x_1^k, x_2^{k'})$.

By construction, the *NTU* game V is balanced. To see this, note that there are three balanced collection of coalitions for this game of which two contain N. For these collections, there is nothing to prove. The only balanced collection of coalitions which does not include N is $\mathcal{B} = \{\{1\}, \{2\}\}$. Pick $v \in V(S)$ for all $S \in \mathcal{B}$. Then, there exists $k_1, k_2 \in K$ such that $v_1 \leq v_1^{k_1}$ and $v_2 \leq v_2^{k_2}$. We showed above that $(v_1^{k_1}, v_2^{k_2}) \leq u(x_1^{k_1}, x_2^{k_2})$, and hence $v \in V(N)$. Therefore V is balanced.

It is clear that conditions (i)-(ii) of Scarf's Theorem provided in the Appendix are satisfied. And since for each coalition S, the set $V(S)$ is constructed by using finitely many points, condition (iii) of Scarf's Theorem is satisfied. Hence, V has a nonempty core, i.e. there exists $v^* \in V(N)$ such that $v^* \notin \text{int}V(S)$ for all $S \in \mathcal{N}$. Since $v^* \in V(N)$, by construction there exists $x^* \in X$ such that $v^* \leq u(x^*)$. Also, since G is coalitionally C-secure and $x^* \in U^k$ for some $k \in K$, there exists $S \in \mathcal{N}$ such that $u_S(x^*) \ll v_S^k$. Since $\{v_S^k\} \times \mathbb{R}^{|-S|} \subset V(S)$, $v^* \in \text{int}V(S)$. This furnishes us a contradiction. □

Proof of Proposition 9. Let G be a coalitionally CS bounded Cournot oligopoly game and p is a decreasing function. Pick $S \in \mathring{\mathcal{N}}$, $\delta_S \in \{\delta_S^{x,v} | (x, v) \in \bar{\mathcal{X}}$ is not in the β-core of $G\}$, and $z_{-S} \in X_{-S}$. Note that $\bar{y}_{-S} \geq z_{-S}$ implies $p(\sum_{i \in S} \delta_{S,i}(\bar{y}_{-S}) + \sum_{i \in -S} \bar{y}_{-S,i}) \leq p(\sum_{i \in S} \delta_{S,i}(\bar{y}_{-S}) + \sum_{i \in -S} z_{-S,i})$. And since $\pi_i(\delta_S(\bar{y}_{-S}), x_{-S}) = p(\sum_{i \in S} \delta_{S,i}(\bar{y}_{-S}) + \sum_{i \in -S} x_{-S,i})\delta_{S,i}(\bar{y}_{-S}) - c_i(\delta_{S,i}(\bar{y}_{-S}))$ for $x_{-S} \in X_{-S}$,

$$\pi_i(\delta_S(\bar{y}_{-S}), \bar{y}_{-S}) \leq \pi_i(\delta_S(\bar{y}_{-S}), z_{-S}).$$

Since $z_{-S} \in X_{-S}$ is arbitrarily chosen,

$$\sum_{i\in S} \pi_i(\delta_S(\bar{y}_{-S}), \bar{y}_{-S}) = \sum_{i\in S} \inf_{z'_{-S}\in X_{-S}} \pi_i(\delta_S(\bar{y}_{-S}), z'_{-S}).$$

Hence G is strongly separable. The proof is analogous for coalitionally CS_N games.

It remains to show that π_i is concave on X if and only if f_i is concave in the two argument for each $i \in N$. Pick $i \in N, x, y \in X$ and $\delta \in (0,1)$. Then, $\pi_i(\delta x + (1-\delta)y) \geq \delta\pi_i(x) + (1-\delta)\pi_i(y)$ if and only if $p(\delta x_i + (1-\delta)y_i + \delta\sum_{j\in N\setminus\{i\}} x_j + (1-\delta)\sum_{j\in N\setminus\{i\}} x_j)(\delta x_i + (1-\delta)y_i) - c_i(\delta x_i + (1-\delta)y_i) \geq \delta(p(x_i + \sum_{j\in N\setminus\{i\}} x_j)x_i - c_i(x_i)) + (1-\delta)(p(y_i + \sum_{j\in N\setminus\{i\}} y_j)y_i - c_i(y_i))\}$ if and only if $f_i(\delta x + (1-\delta)y) \geq \delta f_i(x) + (1-\delta)f_i(y)$. $\square$

3.6 Conclusion

This paper provides sufficient conditions for the nonemptiness of the β-core of games with transferable utilities and possibly discontinuous payoff functions. In our results, we assume concave payoff functions. They are continuous in the relative interior of their domain, but can have discontinuities at their relative boundaries. We explain in Section 3.2 that equilibria often exist at the boundaries of the domain and the possibility of discontinuities therefore be taken into account. However, there are many interesting economic problems that are modeled by payoff functions which are not concave. A challenging topic is to extend our results by weakening the concavity assumption. This problem would require us to replace the separability assumption with another one, thereby opening up another perspective into the problem. In addition, although the β-core is a widely applied cooperative solution concept for normal form games, a variety of different solution concepts, such as strong Nash equilibrium, nucleolus, Shapley value are also of interest for analyzing specific problems. It will surely be of interest to investigate how these solution concepts behave in a setting with discontinuities. Coalitional bargaining games, as for example in Eraslan and McLennan (2013) and Okada (2011)[12], but now with discontinuous payoffs offer another domain of investigation.

[12] See Ray and Vohra (2014) for a survey on coalition formation.

Chapter 4

On the Existence of Equilibrium with Discontinuous Preferences: Games and Economies

4.1 Introduction

Motivated by a variety of applied problems, especially those in auction theory, and following Dasgupta and Maskin (1986), Reny's (1999, 2015) work has led to a resurgence of interest in models with discontinuous preferences.[1] It bears emphasis that his work contributes a novel methodology to handle the issues: rather than applying fixed point theorems to best response correspondences and making suitable assumptions on the primitives of the problem that lead to these correspondences, Reny (1999) formulates a property on the model as a whole. This property manifests itself in the so-called *better-reply-security* property of the game itself, and it is this property that allows one to conclude that the preferences behave "nicely" around any element that is *not* a solution. A question arises whether Reny's insights extend to Walrasian general equilibrium theory in

[1]See 2011 and the forthcoming symposia in Economic Theory on the recent developments in the discontinuous games literature.

the large and to existence issues arising in models that constitute the canonical settings for the new household economics. This work answers this question in the affirmative by appealing to a portmanteau fixed point theorem that yields those of Browder and Kakutani-Fan-Glicksberg as special cases.

The principal fixed point theorem we present in this paper assumes a continuity property which we call *continuous neighborhood selection property*, *CNS property* hereafter,[2] that is motivated by Reny's better-reply-security concept. The continuous neighborhood selection property is weaker than both the upper semicontinuity, *usc* hereafter, assumption of Kakutani-Fan-Glicksberg fixed point theorem and the assumption of open fibers in that of Browder. Hence, it *generalizes* and *unifies* the two well-known fixed point theorems for locally convex Hausdorff topological vector spaces, *tvs* hereafter. Moreover, for a special case of our continuity concept, we provide two additional fixed point theorems for larger class of spaces. These fixed point theorems as a composite can be considered as a 'methodological toolkit' for existence issues in a variety of economic settings.

The paper is organized as follows. In Section 4.2 we present the fixed point theorems. In Section 4.3 we provide applications that constitute the essential contribution of the paper and in Section 4.4 we conclude. For the convenience of the reader, we provide a brief elaboration of the organization of Section 4.3. First, we generalize the existence results on individual decision problems when the individual's preferences satisfy a weak convexity assumption. Moreover, we

[2] A precise definition of the CNS property, and of other italicized terms in this introduction, will be offered in the sequel.

provide an existence of a maximal element result that unifies the existing generalizations of the classical Weierstrass's theorem. Second, we provide a direct proof of the state of the art result of Reny (2015) which assumes preferences are complete and transitive but not necessarily continuous. There is a rich literature on the existence of equilibrium in games without continuous, transitive or complete preferences,[3] and the generalization of Reny's result to this setup has been an open problem at least since his 1999 paper. We resolve this problem by providing a *counterexample.* Third, we generalize the state of the art results of McKenzie (2002) on household demand. Moreover, we provide two generalizations to the well-known Gale-Nikaido-Debreu lemma by *substantially* weakening the upper semicontinuity assumption. The first generalization assumes the excess demand correspondence has well-behaved *downward jumps* and the second assumes it has well-behaved *local selections.*

4.2 Fixed Point Theorems

In this section, we present three fixed point theorems. First, we presents a continuity concept which underlies these fixed point theorems.

Definition 22. *Let X, Y be nonempty subsets of a topological vector space and $P : X \twoheadrightarrow Y$ a correspondence. The correspondence P has the continuous neighborhood selection, CNS, property if for all $x \in X$ such that $P(x) \neq \emptyset$ there exist an open neighborhood U^x of x and an usc correspondence $F^x : X \twoheadrightarrow Y$ which has nonempty, convex and closed values and $F^x(y) \subset P(y)$ for all $y \in U^x$.*

The following fixed point theorem generalizes and unifies the fixed point

[3] See for example Borglin-Keiding (1976), Yannelis-Prabhakar (1983), Toussaint (1984), Tarafdar (1991), Wu-Shen (1996), Park (1999), Barelli-Soza (2011) and Prokopovych (2013).

theorems of Browder and Kakutani-Fan-Gliksberg for locally convex Hausdorff *tvs*.

Theorem 9. *Let X be a nonempty, convex and compact subset of a locally convex Hausdorff tvs, and $P : X \twoheadrightarrow X$ a correspondence with nonempty, convex values and have the CNS property. Then there exists $x^* \in X$ such that $x^* \in P(x^*)$.*

Proof of Theorem 9 . For each $x \in X$ there exist an open neighborhood U^x of x and a nonempty and closed valued, *usc* correspondence $F^x : X \twoheadrightarrow X$ such that $F^x(y) \subset P(y)$ for all $y \in U^x$. The family $\{U^x | \ x \in X\}$ is an open covering of X which, by compactness of X, contains a finite subcovering $\{U^{x_k} | \ k = 1, \dots, m\}$. Denote $F^k \equiv F^{x_k}$ and $U^k \equiv U^{x_k}$ for all $k = 1, \dots, m$. Let $\{\beta^k | \ k = 1, \dots, m\}$ be a partition of unity corresponding to this covering, i.e. each β^k is a continuous function of X into $\mathbb{R}$ which vanishes outside of U^k, and $0 \leq \beta^k(x) \leq 1$ for all x in X and all $k = 1, \dots, m$, while $\sum_{k=1}^{m} \beta^k(x) = 1$ for all x in X.

Now define a correspondence $Q : X \twoheadrightarrow X$ by setting $Q(x) = \sum_{k=1}^{m} \beta^k(x)$ co$F^k(x)$ where co$F^k(x)$ is the convex hull of $F^k(x)$. It is clear that Q is nonempty valued. Since for each $ik = 1, \dots, m$ and $x \in X$, $F^k(x) \subset X$ and $Q(x)$ is a convex linear combination of the sets $F^k(x)$, $Q(x)$ lies in X. Moreover, for each k such that $\beta^k(x) \neq 0$, $F^k(x) \subset P(x)$. Hence $Q(x)$ is a convex linear combination of convex sets contained in the convex set $P(x)$ and therefore, $Q(x)$ is a convex subset of $P(x)$ for each x in X.

Now, pick a net x_n in X converging to $x \in X$. Since β^k is continuous, $\beta^k(x_n) \to \beta^k(x)$. And since F^k is *usc*[4] there exists $y_n^k \in F^k(x_n)$ such that

[4]See Aliprantis-Border (2006, Sections 17.2-3) for a discussion on different definitions of an *usc* correspondence.

$y_n^k \to y^k \in F^k(x)$. Define $y_n \equiv \sum_{k=1}^n \beta^k(x_n) y_n^k$. Then $y_n \in Q(x_n)$ and $y_n \to y \in Q(x)$. Hence, Q is *usc*. And a convex linear combination of closed sets is closed, hence Q is closed valued. Applying Kakutani-Fan-Glicksberg fixed point theorem implies Q has a fixed point $x^* \in X$. For this point we have $x^* \in Q(x^*) \subset P(x^*)$. □

Remark 7. Theorem 9 is a generalization and unification of the fixed point theorems of Kakutani-Fan-Glicksberg and Browder for locally convex *tvs* since the CNS property is weaker than the continuity assumptions of these two theorems, see Proposition 11 provided in Appendix. In particular, Kakutani-Fan-Glicksberg fixed point theorem assumes the relevant mapping is upper semicontinuous whereas its fibers are open in Browder's theorem where fiber of a mapping $P : X \twoheadrightarrow Y$ is defined as $P^{-1}(y) = \{x \in X \mid y \in P(x)\}$. Note that neither of these two continuity concepts implies the other. In order to see that upper semicontinuity does not imply having open fibers consider the following example. Let $[0,1]$ has the usual topology and $P : [0,1] \twoheadrightarrow [0,1]$ is defined as $P(x) = \{x\}$ for all $x \in [0,1]$. It is clear that P is upper semicontinuous and it does not have open fibers. In order to see that having open fibers does not imply upper semicontinuity consider the following example. Let $[0,1]$ has the usual topology and $Q : [0,1] \twoheadrightarrow [0,1]$ is defined as $Q(x) = \{x' | x' > x\} \cup \{1\}$ for all $x \in [0,1]$. It is clear that Q has open fibers and is not upper semicontinuous. Figure 4.1 below illustrates this relation.

Remark 8. The CNS property assumes the relevant correspondence has 'nice' local selections. The continuity concepts based on local selections has been used in the literature, Tarafdar (1977), Wu-Shen (1996) proposed fixed point theorems that are based on constant selections and Park (1999) proposed fixed point theorems that are based on single-valued continuous selections. The CNS property assumes the local selections are upper semicontinuous. In Proposition 11 in the Appendix we show that the CNS property is weaker than these continuity concepts along with open fibers assumption of Browder and upper semicontinuity assumption of Kakutani-Fan-Glicksberg fixed point theorems. Moreover, Barelli-Soza (2011) provided a selection theorem by using a condition similar to the CNS property. Here, we provide a fixed point theorem and show that it not only generalizes and unifies the fixed point theorems of Browder and Kakutani-Fan-Glicksberg, it also generalizes the fixed point theorems provided in Tarafdar, Wu-Shen and Park based on a 'local selection' assumption. Furthermore, He-Yannelis (2015a) independently provided a fixed point theorem which assumes a continuity concept which they call 'continuous inclusion property' that is analogous to the CNS property.[5] Lastly, Wu (1997) proved a fixed point theorem for *lsc* correspondences described in Definition 28 **(C7)**. Hence, Theorem 9 is also a generalization of this result.

Theorem 9 is a generalization of Browder's theorem for locally convex Hausdorff *tvs*. But Browder proved his theorem for Hausdorff *tvs*, hence it is not a full generalization. The following result provides a full generalization.

Theorem 10. *Let X be a nonempty, convex and compact subset of a tvs, and $P : X \twoheadrightarrow X$ a correspondence with nonempty, convex values and have the*

[5] Moreover, they apply their fixed point theorem for the existence of a Walrasian equilibrium in economies. We highlight the similarities and differences in our applications section above.

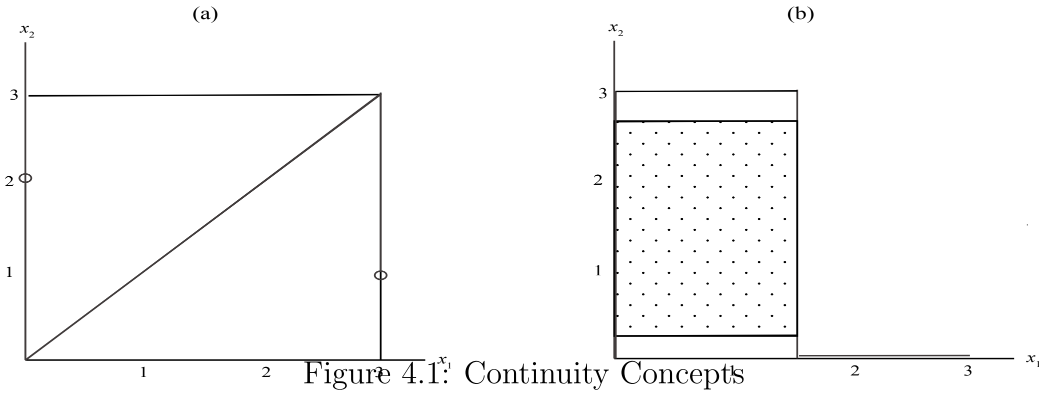


Figure 4.1: Continuity Concepts

Both correspondences have the CNS property, but they are neither usc and have closed values, nor have open fibers, see Appendix for the definition of a fiber. Moreover, the correspondence presented in panel (b) does not satisfy any of the properties (i)-(iii) of Proposition 11 provided in Appendix.

constant CNS property. Then there exists $x^* \in X$ such that $x^* \in P(x^*)$.

Proof of Theorem 10 . We present the proof in two steps: (1) proof of the theorem for Hausdorff tvs, (2) proof of the theorem for arbitrary tvs.

Step 1. Assume X is equipped with Hausdorff topology. For each $x \in X$ there exist $y^x \in X$ and an open neighborhood U^x of X such that $y^x \in \bigcap_{z \in U^x} P(z)$. The family $\{U^x | \; x \in X\}$ is an open covering of X which, by compactness of X, contains a finite subcovering $\{U^{x_k} | \; k = 1, \ldots, m\}$. Mark the corresponding y^{x_k}s and denote $y^k = y^{x_k}$ for all $k = 1, \ldots, m$. Let $\{\beta^k | \; k = 1, \ldots, m\}$ be a partition of unity corresponding to this covering, i.e. each β^k is a continuous function of X into $\mathbb{R}$ which vanishes outside of U^{x_k}, and $0 \leq \beta^k(x) \leq 1$ for all x in X and all $k = 1, \ldots, m$, while $\sum_{k=1}^m \beta^k(x) = 1$ for all x in X.

Now define a continuous function $p : X \to X$ by setting $p(x) = \sum_{k=1}^m \beta^k(x) y^k$. Since each y^k lies in X and $p(x)$ is a convex linear combination of the points y^k, $p(x)$ lies in X. Moreover, for each k such that $\beta^k(x) \neq 0$, $y^k \in P(x)$. Hence $p(x)$

is a convex linear combination of points in the convex set $P(x)$ and therefore, $p(x) \in P(x)$ for each x in X.

Let X^0 be the finite dimensional simplex spanned by the n points $\{y^1, \ldots, y^k\}$. Since the topology induced on any finite dimensional subspace of X by the topological structure of X coincides with the usual Euclidean topology, X^0 is homeomorphic to an Euclidean ball. p maps X^0 into X^0, and by the Brouwer fixed point theorem, p has a fixed point x^* in X^0. For this point, we have $x^* = p(x^*) \in P(x^*)$.

Step 2. Assume X is an arbitrary *tvs*. Let $[x]$ is the closure of $x \in X$ and $\tilde{X} \equiv X/[0]$ the quotient space. Since $[0]$ is a closed set, $\tilde{X}$ is a Hausdorff *tvs*. Let $\pi : X \to \tilde{X}$ be the quotient projection. Define a correspondence $\bar{P} : X \twoheadrightarrow X$ as $\bar{P}(x) \equiv \bigcap_{z\in[x]} P(z)$. For all $x, z \in X$, if $\pi z = \pi x$, i.e. $[x] = [z]$, then $\bar{P}(x) = \bar{P}(z)$. Since any open neighborhood of $x \in X$ contains $[x]$, $\bar{P}$ has the constant CNS property.

Now, define a correspondence $\tilde{P} : \tilde{X} \twoheadrightarrow \tilde{X}$ as $\tilde{P}(\pi x) \equiv \pi(\bar{P}(x))$. $\tilde{P}$ is well defined since π is surjective, and has nonempty and convex values since $\bar{P}$ has nonempty and convex values. Now pick $\tilde{x} \in \tilde{X}$. Then there exists $x \in X$ such that $\tilde{x} = \pi x$. Since $\bar{P}$ has the constant CNS property, there exist an open neighborhood U of $[x]$ and $y \in \bar{P}(x)$ such that $y \in \bigcap_{z\in U} \bar{P}(z)$. Since π is an open map, $\pi(U)$ is an open neighborhood of $\tilde{x}$ and $\pi y \in \bigcap_{\tilde{z}\in\pi(U)} \tilde{P}(\tilde{z})$. Hence $\tilde{P}$ has the constant CNS property. Since $\tilde{X}$ is equipped with Hausdorff topology there exists $\tilde{x} \in \tilde{X}$ such that $\tilde{x} \in \tilde{P}(\tilde{x})$. Therefore there is $x \in X$ such that $\pi x \in \pi\big(\bar{P}(x)\big)$. This implies that there exists $z \in X$ such that $\pi x = \pi z$ and $z \in \bar{P}(x)$. And $\bar{P}(x) = \bar{P}(z)$ implies $z \in \bar{P}(z)$, and in particular $z \in P(z)$. $\square$

Remark 9. Theorem 10 may be of special interest since it is a fixed point theorem for *non-Hausdorff tvs*. In particular, Browder (1968) proved his theorem for Hausdorff *tvs*. Later, Tarafdar (1977) generalized Browder's theorem by assuming correspondences satisfies a continuity property which is equivalent to the constant CNS property. Also, Wu-Shen (1996) proved Theorem 10 for locally convex Hausdorff *tvs*. Here we generalize their results to *arbitrary tvs* and provide a direct and alternative proof. Moreover, this theorem also implies the main lemma of McLennan et al. (2011, Lemma 7.1, p 1657) which is essential for the proof of their main existence result. The constant CNS property is also called *local intersection property*, see Wu-Shen (1996), and *transfer open lower sections*, see Tian-Zhou (1995).

The two fixed point theorems presented above use the linear and topological structures. An alternative setup is to use the order structure alone such as Tarski's fixed point theorem. The following fixed point theorem is an intermediate step between these two classes of fixed point theorems since it keeps topological structure and replaces the linear structure with a weak order structure on the relevant correspondence as follows.

Definition 23. A correspondence $P : X \twoheadrightarrow X$, X a nonempty set, is *transitive* if for any $x, y, z \in X$, $z \in P(y)$ and $y \in P(x)$ implies $z \in P(x)$.[6]

Theorem 11. *Let X be a nonempty and compact subset of a topological space, and $P : X \twoheadrightarrow X$ a transitive correspondence with nonempty values and have the constant CNS property. Then there exists $x^* \in X$ such that $x^* \in P(x^*)$.*

[6] The transitivity of the correspondence P is not an unusual property if we interpret the graph of P as a binary relation on X.

Proof of Theorem 11 . For each $x \in X$ there exist $y^x \in X$ and an open neighborhood U^x of X such that $y^x \in \bigcap_{z \in U^x} P(z)$. The family $\{U^x |\ x \in X\}$ is an open covering of X which, by compactness of X, contains a finite subcovering $\{U^{x_k} |\ k = 1, \ldots, m\}$. Mark the corresponding y^{x_k}s and denote $y^k \equiv y^{x_k}$ for all $k = 1, \ldots, m$, and define $Y \equiv \{y^1, \ldots, y^m\} \subset X$.

First, note that, since $\{U^{x_k} |\ k = 1, \ldots, m\}$ is a covering of X, for each $y^k \in Y$ there exists $y^l \in Y$ such that $y^l \in P(y^k)$. Pick $y^1 \in Y$. There exists $y^{k_1} \in Y$ such that $y^{k_1} \in P(y^1)$. If $y^{k_1} = y^1$, then we have a fixed point. Otherwise, there exists $y^{k_2} \in Y$ such that $y^{k_2} \in P(y^{k_1})$. If $y^{k_2} \in \{y^1, y^{k_1}\}$, then transitivity implies we have a fixed point. Otherwise, there exists $y^{k_3} \in Y$ such that $y^{k_3} \in P(y^{k_2})$. If $y^{k_3} \in \{y^1, y^{k_1}, y^{k_2}\}$, then transitivity implies we have a fixed point. We repeat this process until $y^{k_{n-1}}$. For $y^{k_{n-1}}$ there exists $y^i \in Y$ such that $y^i \in P(y^{k_{n-1}})$. And, if we reach up to $y^{k_{n-1}}$ we must have already used each y^j in Y. Hence $y^i \in \{y^1, y^{k_1}, \ldots, y^{k_{n-1}}\}$. Then, transitivity implies we have a fixed point. □

4.3 Applications

In this section, we provide three applications of our fixed pint theorems: maximal elements, games, and market equilibrium theorems.

4.3.1 Maximal Elements

An *individual decision problem* is an ordered set $(X, \succ)$ where X is a set and $\succ$ is a binary relation on X. Let $P_\succ : X \twoheadrightarrow X$ be a correspondence defined as $P_\succ(x) \equiv \{y \in X |\ y \succ x\}$. A *maximal element* of X is $x^* \in X$ such that $P_\succ(x^*) = \emptyset$. A useful feature of this definition of maximal element is its capability of covering problems when the binary relation is not represented by a real valued function, or not ordered. Specifically, the set of maximizers of a real valued function $u : X \to \mathbb{R}$ on a set X is equivalent to the set of maximal elements of X when the relation $\succ$ on X is defined as $y \succ x$ if and only if $u(y) > u(x)$ for $x, y \in X$. We do not need to express the importance of the existence of a maximal element in economics; see Yannelis-Prabhakar (1983); Toussaint (1984); Bergstrom et al. (1976); Bergstrom (1975); Walker (1977); Tian-Zhou (1995); Alcantud (2002).

First, we show the existence of a maximal element when the binary relation is nonordered in a Hausdorff locally convex *tvs* by assuming the binary relation satisfies a convexity property. The following result is a simple corollary of Theorem 9.

Corollary 3. *If $(X, \succ)$ is an individual decision problem such that X is a nonempty, convex, compact subset of a locally convex Hausdorff tvs, $P_{\succ} : X \twoheadrightarrow X$ has the CNS property and for all $x \in X$ $x \notin \text{co}P_{\succ}(x)$, then there exists $x^* \in X$ such that $P_{\succ}(x^*) = \emptyset$.*

Proof of Corollary 3. Assume $P_{\succ}(x) \neq \emptyset$ for each $x \in X$. Then, $\text{co}P_{\succ}$ has nonempty values. And, since $P_{\succ}$ has the CNS property and $P_{\succ}(x) \subset \text{co}P_{\succ}(x)$ for each $x \in X$, $\text{co}P$ has the CNS property. Hence, Theorem 9 implies there exists $x^* \in X$ such that $x^* \in \text{co}P_{\succ}(x^*)$. Since $x \notin \text{co}P_{\succ}(x)$ for all $x \in X$, the proof is finished. □

Remark 10. Corollary 3 generalizes Yannelis-Prabhakar (1983, Theorem 5.1, p. 239), Anderson (1981, Theorem 1, p. 1459) and McKenzie (2002, Theorem 3, p. 14). The assumption that the correspondence P has the CNS property in Corollary 3 is replaced by the open fiber property in those results. Moreover, He-Yannelis (2015a) provides an analogous result to our corollary.

The existence of a maximal element when the binary relation is not necessarily convex but satisfies some order property is well characterized; see Bergstrom (1975); Walker (1977); Tian-Zhou (1995); Alcantud (2002). Here, for completeness and understanding the implications of the constant CNS property better we prove those results as a corollary to our fixed point theorem.[7] The following result is a simple corollary of Theorem 11.

Corollary 4. *If $(X, \succ)$ is an individual decision problem such that X is a nonempty, compact subset of a topological space, $\succ$ is an irreflexive and transitive relation on X, and $P_\succ : X \twoheadrightarrow X$ has the constant CNS property, then there exists $x^* \in X$ such that $P_\succ(x^*) = \emptyset$.*

Proof of Corollary 4. Assume $P_\succ(x) \neq \emptyset$ for each $x \in X$. It is clear that transitivity of $\succ$ implies $P_\succ$ is transitive. Then, Theorem 11 implies there exists $x^* \in X$ such that $x^* \in P_\succ(x^*)$. Since irreflexivity of $\succ$ implies $x \notin P_\succ(x)$ for all $x \in X$. This furnishes a contradiction. □

We illustrate the constant CNS property and our generalization with two examples in Figure ??. Panel (a) illustrates a function $u : [0, 4] \to \mathbb{R}$ which is not *usc* and does not have a maximizer. And note that the correspondence $P_u : [0, 4] \twoheadrightarrow [0, 4]$ defined as $P_u(x) = \{y \in [0, 4] | u(y) > u(x)\}$ does not have the constant CNS property at $x = 2$. Panel (b) illustrates a function $v : [0, 4] \to \mathbb{R}$ which is not *usc* but has a unique maximum at $x^* = 4$. Note that the correspondence $P_v : [0, 4] \twoheadrightarrow [0, 4]$ defined as $P_v(x) = \{y \in [0, 4] | v(y) > v(x)\}$ has the constant CNS property at each $x \in [0, 4]$. In order to see this, for each

[7] Here, we provide results for transitive and irreflexive binary relations. Such relations are acyclic and it is easy to prove our results for acyclic relations as in Alcantud (2002).

$x \in [0,4)$, choose $y = 4$ and $U^x = [0,4)$. Since $u(4) = 2 > u(x)$ for each $x \in [0,4)$, $y \in \bigcap_{z \in U^x} P_v(z)$ for each $x \in [0,4)$.

[See the Supplementary Figure 1]

4.3.2 Discontinuous Games

An *ordinal game* is a list $G = (X_i, R_i)_{i \in N}$ where (i) N is the finite set of players, (ii) X_i is the set of actions of player i, (iii) R_i is the preference relation of player i on $X \equiv \prod_{i \in N} X_i$. Let $X_{-i} \equiv \prod_{j \in N \setminus \{i\}} X_j$. If for each payer i, the preference relation $R_i \equiv \succeq_i$ is a reflexive, complete and transitive relation, such games are called *ordinal games*. Whereas, if for each payer i, the preference relation $R_i \equiv \succ_i$ is an irreflexive relation, such games are called *qualitative games*. It is clear that every ordinal game is a qualitative game since the weak preference relation $\succeq_i$ induces an irreflexive strict preference relation $\succ_i$. A *Nash equilibrium* in an ordinal, or qualitative, game is $x^* \in X$ such that there does not exist $i \in N$ and $x_i \in X_i$ such that $(x_i, x^*_{-i}) \succ_i x^*$.

For the individual decision problems, i.e. 1-person games, the continuity property is well-established. For multi-player decision problems, many different continuity concepts are introduced. We can classify those into two groups. The first is characterized by weaker continuity assumptions on the preferences of the players; see for example Dasgupta and Maskin (1986), Yannelis-Prabhakar (1983) and Reny (1999). The second is characterized by weaker continuity assumptions on a game itself; see for example Borglin-Keiding (1976), Toussaint (1984), Tarafdar (1991), Baye et al. (1993), Reny (1999), McLennan et al. (2011), Barelli and Meneghel (2013) and Prokopovych (2013).[8] In particular, the second assumes a certain robustness of the behavior of the player(s) around each action profile which is not a Nash equilibrium. In this section, as an application of our fixed point theorems, we provide a direct proof of the state of the art result of Reny (2015) on the existence of a Nash equilibrium in ordinal games. Also, we show that Reny's result cannot be extended to qualitative games by providing a counterexample.

Definition 24. An ordinal game G is *point-secure at* $x \in X$ if there exist an open neighborhood $U^x \subset X$ of x and $y^x \in X$ such that for each $x' \in U^x$, there exists $i \in N$ such that $(y_i^x, z_{-i}) \succ_i x'$ for each $z \in U^x$. And G is *point-secure* if it is point-secure at every $x \in X$ that is not a Nash equilibrium.[9]

Definition 25. An ordinal game G is *correspondence-secure at* $x \in X$ if there exist an *usc* correspondence $d^x : X \twoheadrightarrow X$ with nonempty, convex and closed values and an open neighborhood $U^x \subset X$ of x such that for each $x' \in U^x$, there exists $i \in N$ such that $(y_i, z_{-i}) \succ_i x'$ for each $z \in U^x$ and each $y_i \in d_i^x(z)$ where $d_i^x(z)$ is the i^{th} component of $d^x(z)$. And G is *correspondence-secure* if it is correspondence-secure at every $x \in X$ that is not a Nash equilibrium of G.

The point-security and correspondence-security notions are topological properties and characterize the continuity of the preferences in a game. If a point

[8] Also see Carmona (2011) for a symposium on the recent developments on discontinuous games.

[9] When the set of players in a game is a singleton it becomes a individual decision problem. In Section 4.3.1 above, we discuss the existence of a solution for individual decision problems. For such problems, the continuity of the individual's preferences is characterized by assuming the correspondence $P_\succ$, derived from the agent's strict preference relation $\succ$, has the constant CNS property (or the CNS property). It is easy to see that assuming a one player game $G \equiv (X, \succeq)$ is point-secure is equivalent to assuming $P_\succ$ has the constant CNS property. Hence, point-security is an extension of the constant CNS property to multi-player games.

$x \in X$ is not a Nash equilibrium of G, then, by definition, there exist a player who deviates. Point-security strengthen this property in two directions. First, it does not allow G to have a Nash equilibrium around some open neighborhood U^x of x, i.e, for all $z \in U^x$ there exists a player who deviates by using a fixed action. Second, it further assumes at each $z \in U^x$ at least one of those deviants will still deviate by using a fixed action even if other players tremble.

Theorem 12 (Reny 2013, Theorem 4.3)**.** *If $G = (X_i, \succeq_i)_{i \in N}$ is a correspondence-secure ordinal game such that for each player i,*

(i) *X_i is a nonempty, convex and compact subset of a locally convex Hausdorff tvs,*

(ii) *for each $z, x \in X$ the set $\{y \in X_i | (y, z_{-i}) \succeq_i x\}$ is convex.*

Then G has a Nash equilibrium.

Remark 11. In his paper, Reny (2015) showed that every correspondence-secure game can be represented by an auxiliary point-secure game and proved this theorem as a corollary to his result on point-secure games. For completeness, we provide a direct proof in the Appendix. In the statement of his theorem, Reny used a convexity assumption weaker than assumption (ii) of Theorem 12. But, he used the stronger version stated here in his proof. Also, for the point-secure games, one can prove the proposition of Theorem 12 for *any tvs* by using the quotient spaces as illustrated in the proof of Theorem 10.

Now we show that point-security is not sufficient to show the existence of a Nash equilibrium of a game when preferences are not necessarily complete or transitive, i.e. when we have a qualitative game, by providing the following counterexample.[10]

[See the Supplementary Figure 2]

The Counterexample. Let $G = (X_i, \succ_i)_{i=1}^2$ be a qualitative game where $X_1 = X_2 = [0, 3]$. The preferences are defined as follows. For each player i, define $Q_i : X \twoheadrightarrow X$ as $Q_i(x) \equiv \{y \in X |\ y \succ_i x\}$. For all $x \in [0, 1] \times [0, 3]$, if $x_2 = 0$, $Q_1(x) = \{3\} \times [0, 3]$, and $Q_2(x) = \emptyset$, otherwise $Q_1(x) = \emptyset$, and $Q_2(x) = [0, 2) \times \{0\}$. For all $x \in [2, 3] \times [0, 3]$, if $x_2 = 0$, $Q_1(x) = \emptyset$ and $Q_2(x) = (1, 3] \times \{3\}$, otherwise $Q_1(x) = \{0\} \times [0, 3]$, and $Q_2(x) = \emptyset$. For all $x \in (1, 2) \times [0, 3]$, if $x_2 = 0$, $Q_1(x) = \{3\} \times [0, 3]$ and $Q_2(x) = (1, 3] \times \{3\}$, otherwise $Q_1(x) = \{0\} \times [0, 3]$, and $Q_2(x) = [0, 2) \times \{0\}$.

Define $U = [0, 2) \times [0, 3]$ and $V = (1, 3] \times [0, 3]$, $y^1 = (3, 0)$ and $y^2 = (0, 3)$. For all $x \in U$, either $(3, z_2) \succ_1 x$ for each $z \in U$, or $(z_1, 0) \succ_2 x$ for each $z \in U$. And for all $x \in V$, either $(0, z_2) \succ_1 x$ for each $z \in V$, or $(z_1, 3) \succ_2 x$ for each $z \in V$. Therefore, the game G is point-secure at each $x \in X$, hence it has no Nash equilibrium. Note that the (strong) convexity of preferences defined in Theorem 12 part (ii) is trivially satisfied in this game.

4.3.2.1 Qualitative Games

Let $G = (X_i, \succ_i)_{i \in I}$ be a qualitative game. Define for each player i a correspondence $P_i : X \twoheadrightarrow X_i$ as $P_i(x) \equiv \{y_i \in X_i |\ (y_i, x_{-i}) \succ_i x\}$ consisting of the set of actions of player i which are 'strictly preferred to' x assuming the rest of the

[10]Note that, on a Hausdorff topological space, every point-secure game is correspondence-secure. Hence our counterexample also holds for correspondence-secure games.

players stick to their actions x_{-i}. Therefore, $x^* \in X$ is a Nash equilibrium of G if and only if $P_i(x^*) = \emptyset$ for each player $i \in I$.

Theorem 13. *If $G = (X_i, \succ_i)_{i\in I}$ be a qualitative game such that for each $i \in I$,* (i) *X_i is a nonempty, convex and compact subset of a Hausdorff locally convex tvs,* (ii) *$x_i \notin \text{co}P_i(x)$ for each $x \in X$,* (iii) *P_i has the single-valued CNS property, and* (iv) *$\{x \in X|\ P_i(x) \neq \emptyset\}$ is paracompact, then G has a Nash equilibrium.*

Remark 12. Note that, every subset of a metric space is paracompact, hence assumption (iv) of Theorem 13 is not very restrictive. Also, it is easy to see that assuming a one-player game $G \equiv (X, \succ)$ satisfies assumption (iii) is equivalent to assuming $P_\succ$, for the individual decision problem, has the single-valued CNS property. Hence, assumption (iii) is an extension of the CNS property to social decision problems. Recall that in Appendix D we show that point-security concept is also an extension of the CNS property to social decision problems. Note that these two extensions do not imply each other. In order to see this, consider a game G and pick $x \in X$ which is not a Nash equilibrium of G. If G satisfies "P_i has the constant CNS property for all $i \in I$," then each player who deviates at x must deviate by using a fixed action around a neighborhood of x. Whereas, if G is point-secure, then for all action profiles in a neighborhood of x, there is a player who deviates by using a fixed action independent of what others do in that neighborhood. In other words, first extension of the CNS property requires each player who deviates at a point should deviate around a neighborhood of that point assuming others will stick on their action. Whereas,point-security requires that at each point around a neighborhood which is blocked by a player at least one player should deviate independent of what others do in that neighborhood.

Lemma 3. [Park, 1999, Theorem 5, p. 577] *Let X be a nonempty and paracompact subset of a Hausdorff topological space, Y a nonempty and convex subset of a Hausdorff tvs and $P : X \twoheadrightarrow Y$ a nonempty and convex valued correspondence with the single-valued CNS property. Then P has a continuous selection on X.*

Proof of Theorem 13. Let $Z_i \equiv \{x \in X|\ P_i(x) \neq \emptyset\}$. Define for each player $i \in I$ a correspondence $Q_i : X \twoheadrightarrow X_i$ as

$$Q_i(x) = \begin{cases} \text{co}P_i(x) & \text{if } x \in Z_i, \\ X_i & \text{otherwise.} \end{cases}$$

It is clear that Q_i is nonempty and convex valued, and $x_i \in Q_i(x)$ only if $P_i(x) = \emptyset$ for each $i \in I$. Since Z_i is paracompact, Lemma 3 implies Q_i has a continuous selection on Z_i for each $i \in I$. And since X_i is compact, Q_i has an *usc* selection on X, hence it has the *CNS* property. Therefore, Corollary 7 implies there exists $x^* \in X$ such that $x_i^* \in Q_i(x^*)$ for each $i \in I$. Hence, $P_i(x^*) = \emptyset$ for all $i \in I$. □

A *generalized game* (or abstract economy) is a list $\mathcal{G} = (X_i, A_i, \succ_i)_{i \in I}$ where (i) I is the set of players, (ii) X_i is the set of actions of player i, (iii) $A_i : X \twoheadrightarrow X_i$ is the constraint correspondence of player i, and (iv) $\succ_i$ is the irreflexive preference relation of player i on X. For each player i define a correspondence $P_i : X \twoheadrightarrow X_i$ as $P_i(x) \equiv \{y_i \in X_i|\ (y_i, x_{-i}) \succ_i x\}$, and $\bar{A}_i : X \twoheadrightarrow X_i$ as $\bar{A}_i(x) \equiv \text{cl}A_i(x)$. An *equilibrium* of a generalized game $\mathcal{G}$ is $x^* \in X$ such that $x_i^* \in \bar{A}_i(x^*)$ and $P_i(x^*) \cap A_i(x^*) = \emptyset$ for each player $i \in I$.

Corollary 5. *If $\mathcal{G} = (X_i, A_i, \succ_i)_{i\in I}$ be a generalized game such that for each $i \in I$,* (i) *X_i is a nonempty, convex and compact subset of a Hausdorff locally convex tvs,* (ii) *$x_i \notin \text{co}P_i(x)$ for each $x \in X$,* (iii) *$P_i \cap A_i$ has the single-valued CNS property,* (iv) *$\{x \in X|\ P_i(x) \cap A_i(x) \neq \emptyset\}$ is paracompact, and* (v) *A_i is nonempty, convex valued, has the CNS property, then $\mathcal{G}$ has an equilibrium.*

Remark 13. Assumption (iii) is crucial. In most of the papers, there is no joint assumption on P_i and A_i. But since the CNS property imposes very weak continuity on the correspondences, it will not imply the intersection correspondence has the CNS property. The assumptions in earlier papers imply this. Note that (He-Yannelis, 2015b, Theorem 1, p. 6) provide an existence result by using the CNS property.

Proof of Corollary 5. Let $Z_i \equiv \{x \in X|\ P_i(x) \cap A_i(x) \neq \emptyset\}$. Since A_i has nonempty and convex values, and has the CNS property, and X is compact, A_i has an *usc* selection $F_i : X \twoheadrightarrow X_i$. Define for each player $i \in I$ a correspondence $Q_i : X \twoheadrightarrow X_i$ as

$$Q_i(x) = \begin{cases} \text{co}P_i(x) \cap A_i(x) & \text{if } x_i \in Z_i, \\ F_i(x) & \text{otherwise.} \end{cases}$$

It is clear that Q_i is nonempty and convex valued, and $x_i \in Q_i(x)$ only if $P_i(x) \cap A_i(x) = \emptyset$ for each $i \in I$. Since Z_i is paracompact, Lemma 3 implies Q_i has a continuous selection on Z_i for each $i \in I$. And since F_i has closed graph, Q_i has an *usc* selection on X, hence it has the CNS property. Therefore, Corollary 7 implies there exists $x^* \in X$ such that $x^*_i \in Q_i(x^*)$ for each $i \in I$. Hence, $x^*_i \in \bar{A}_i(X^*)$ and $P_i(x^*) \cap A_i(x^*) = \emptyset$ for all $i \in I$. □

4.3.3 Economies

In this section, we prove existence of the household demand and a Walrasian equilibrium under weak assumptions on consumers preferences.

4.3.3.1 Household Demand

The two foundations of the theory of competitive economies are the theory of demand and the theory of production. The basic question of the demand theory is the existence of an optimal choice under budget constraints of a consumer acting independently of the choices of other economic agents. And, there is no problem of the existence of aggregate demand functions so long as the individual demand functions do not depend on the consumptions of other individuals. However, it is not obvious that aggregate demand functions exist when the choices of different consumers are interdependent. This framework can also be used to model the household demand. In a household, typically, consumption of an individual depends on the consumption of the other members of the households. In this model, the decision of the members of the household are independent, rather than cooperative.

In this section, we prove existence for the household and the market demand correspondence under weak assumptions on consumers preferences.[11]

An (exchange) economy is a list

$$\mathcal{E} \equiv (X_i, w_i, \succ_i)_{i \in I}$$

where (i) $I \equiv \{1, \dots, n\}$ is the set of consumers, (ii) $X_i \subset \mathbb{R}^m$ is the consumption set of consumer i, (iii) $w_i \in \mathbb{R}_+$ is the initial holdings of consumer i and (iv) $\succ_i$

[11] See Chapter 1 of McKenzie (2002) for a discussion of the state of the art results on the existence of the aggregate demand correspondences.

is preferences[12] of consumer i on X where $X \equiv \prod_{i \in I} X_i$.

The *budget correspondence* $H_i : \mathbb{R}^m_+ \times \mathbb{R}_+ \twoheadrightarrow X_i$ of consumer i is defined as

$$H_i(p, w_i) \equiv \{x_i \in X_i |\ px_i \leqq w_i\}, \text{ and let } H(p, w) \equiv \prod_{i \in I} H_i(p, w_i).$$

The *demand correspondence* $f_i : \mathbb{R}^m_+ \times \mathbb{R}_+ \times X \twoheadrightarrow X_i$ of consumer i is defined as

$$f_i(p, w_i; x) \equiv \{y_i \in H_i(p, w_i)|\ y_i \in P_i(x) \ \text{ implies } \ y_i \notin H_i(p, w_i)\}$$

where the correspondence $P_i : X \twoheadrightarrow X_i$ is defined as $P_i(x) \equiv \{y_i \in X_i |\ (y_i, x_{-i}) \succ_i x\}$. The *market demand correspondence* $f : \mathbb{R}^m_+ \times \mathbb{R}^n_+ \twoheadrightarrow \mathbb{R}^m$ is defined as

$$f(p, w) \equiv \sum_{i \in I} f_i(p, w_i; x)$$

where $x_i \in f_i(p, w_i; x)$ for all $x \in X$ and $i \in I$.

We are interested in the nonemptiness of the market demand correspondence f. In the technical register, one can formulate this problem as a game. And the solution concept is Nash equilibrium.

Theorem 14. (McKenzie, 2002, Theorem 9 , p. 32) *Let $\mathcal{E} = (X_i, w_i, \succ_i)_{i \in I}$ be the economy defined above such that for each consumer $i \in I$,* (i) *X_i is nonempty, convex, closed and bounded from below,* (ii) *$x_i \notin coP_i(x)$ for each $x \in X$,* (iii) *P_i has open fibers. Then the market demand correspondence $f(p, w)$ is nonempty for each $(p, w) \in \mathbb{R}^m_{++} \times \mathbb{R}^n_+$.*

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[12] In McKenzie (2002), and in general in the consumer theory with externalities literature, consumer $i's$ preference relation $\succ_i$ is defined on his own consumption set X_i for each $x_{-i} \in X_{-i}$ considered as a different state. Hence, a consumer does not necessarily have a preference ordering for different states. In the current setup, since preference relation of a consumer does not necessarily complete or transitive, hence defining preference relation of each consumer on the product space X does not put any additional structure on the problem, but makes the setup of the problem clear.

McKenzie claimed that the loss of transitivity of the preferences has required a new and somewhat more difficult proof that the demand correspondence is well defined for $p > 0$. Here we show that our fixed point theorem not only provides a short proof for this claim it substantively weakens the continuity of the preferences. In particular, our continuity assumption requires preferences to contain a selection which is *locally* well behaved.

Theorem 15. *Let $\mathcal{E} = (X_i, w_i, \succ_i)_{i\in I}$ be the economy defined above such that for each consumer $i \in I$,* (i) *X_i is nonempty, convex, closed and bounded from below,* (ii) *$x_i \notin coP_i(x)$ for each $x \in X$,* (iii) *the correspondence $\bar{P}_i : H(p, w) \twoheadrightarrow H_i(p, w)$ defined as $\bar{P}_i(x) \equiv P_i(x) \cap H_i(p, w)$ has the single-valued CNS property for $(p, w) \in \mathbb{R}^m_{++} \times \mathbb{R}^n_+$. Then the market demand correspondence $f(p, w)$ is nonempty for $(p, w) \in \mathbb{R}^m_{++} \times \mathbb{R}^n_+$.*

Proof of Theorem 15. Pick $(p, w) \in \mathbb{R}^m_{++} \times \mathbb{R}^n$. Then for each consumer i, $H_i(p, w_i)$ is a nonempty, convex and compact subset of X_i. Assumption (iii) implies $\bar{P}_i$ has the single-valued CNS property. And it is clear that $x_i \notin \text{co}\bar{P}_i(x)$ for each $x \in H_i(p, w_i)$. Since any metric space is paracompact, the set $\{x \in X | \ \bar{P}_i(x) \neq \emptyset\}$ is paracompact. Therefore, Theorem 13 implies there exists $x^* \in X$ such that $x^*_i \in H_i(p, w)$ and $\bar{P}_i(x^*) = \emptyset$ for each $i \in I$. □

The difference between Theorem 14 and Theorem 15 is assumption (iii). In order to see that (iii) in the latter is weaker than (iii) in the former, note that if a correspondence $Q : X \twoheadrightarrow X$ has open fibers, then for each nonempty, convex, compact $Y \subset X$, the correspondence $\bar{Q} : Y \twoheadrightarrow Y$ defined as $\bar{Q}(x) = Q(x) \cap Y$ has open fibers. In order to see that Q has the single-valued CNS property does not imply $\bar{Q}$ has consider the following example. $X = [0, 2]^2$, $Y = [0, 1]^2$,

$Q(x) = \text{co}\{(1,1),(2,2)\}$ for all $x \in Y\backslash\{(1,1)\}$, $Q(1,1) = \text{co}\{(0,1),(2,2)\}$, $Q(x) = \{(2,2)\}$ for all $x \in X\backslash Y$. Since Q has a constant selection $(2,2)$ on X, it has the single-valued CNS property. But at $(1,1)$, $\bar{Q}$ has discontinuity.

Remark 14. When the externalities are eliminated, then assuming preferences are transitive and assuming constant CNS property will be enough to show that the market demand correspondence is nonempty without any linearity structure on the space of commodities and preferences; see Section 4.3.1.

Now we provide a result on the existence of household demand where preferences satisfy correspondence-security. This result combines Reny (2015) and McKenzie (2002). Consider an exchange economy

$$\mathcal{E} = (X_i, w_i, \succeq_i)_{i \in I}$$

where the preference relation of each consumer is reflexive, complete and transitive. $\mathcal{E}$ is *correspondence-secure* with respect to $(p, w) \in \mathbb{R}^m_{++} \times \mathbb{R}^n_+$ if for all $x \in H(p, w)$ at which $P_j(x) \cap H_j(p, w) \neq \emptyset$ for some $j \in I$ there exist an open neighborhood $U^x \subset X$ of x and a nonempty, convex and closed valued, *usc* correspondence $d^x : H(p, w) \twoheadrightarrow H(p, w)$ such that for each $x' \in U^x \cap H(p, w)$, there exists $i \in I$ such that $(y_i, z_{-i}) \succ_i x'$ for each $z \in U^x \cap H(p, w)$ and each $y_i \in d^x_i(z)$ where $d^x_i(z)$ is the i^{th} component of $d^x(z)$.

Theorem 16. *Let $\mathcal{E} = (X_i, w_i, \succeq_i)_{i \in I}$ be the economy defined above such that for each consumer $i \in I$,* (i) *X_i is nonempty, convex, closed and bounded from below,* (ii) *$x_i \notin coP_i(x)$ for each $x \in X$, and* (iii) *$\mathcal{E}$ is correspondence-secure with respect to $(p, w) \in \mathbb{R}^m_{++} \times \mathbb{R}^n_+$. Then the market demand correspondence $f(p, w)$ is nonempty for $(p, w) \in \mathbb{R}^m_{++} \times \mathbb{R}^n_+$.*

Proof of Theorem 16. Pick $(p, w) \in \mathbb{R}^m_{++} \times \mathbb{R}^n$. Then for each consumer i, $H_i(p, w_i)$ is a nonempty, convex and compact subset of X_i. And assumption (iii) implies $(H_i(p, w_i), \succeq_i)_{i \in I}$ is a correspondence-secure game. Theorem 12 implies it has a Nash equilibrium. □

4.3.3.2 Market Equilibrium

The classical results in the excess demand approach to existence of a competitive equilibrium assume (or derive) that the excess demand correspondence is *usc*; see Debreu (1982). The main theorem in this approach is the well known Gale-Nikaido-Debreu (GND) lemma. The classical proofs of this lemma use either a fixed point theorem, or KKM lemma. McCabe (1981) provided a simple alternative proof by using the separating hyperplane theorem and a fixed point theorem. He introduced a 'nice' mapping, *Browder-McCabe map*,[13] which assigns for each price p, the set of prices at which the value of the excess demand is strictly greater than zero. It also makes it easier to work with infinite dimensional commodity spaces; see Yannelis (1985); Mehta-Tarafdar (1987). McCabe (1981) assumed the excess demand correspondence is *usc* which implies the Browder-McCabe map has open fibers. Later, Mehta-Tarafdar (1987) extended Yannelis (1985) by directly imposing a weaker continuity assumption on the Browder-McCabe map, rather than the excess demand correspondence. Inspired by the recent development in discontinuous games, we propose two generalizations of the GND Lemma. First, we impose very weak continuity assumption on Browder-McCabe map, the CNS property, which generalizes McCabe (1981), Yannelis (1985) and Mehta-Tarafdar (1987). Then, we show

[13] The Browder-McCabe map is related to quasivariational inequalities and the traces can be found in the proof of Theorem 3 in Browder (1968).

that a continuity assumption on the excess demand correspondence which put restrictions only on the *downward jumps,* hence weaker than *usc*, which implies the Browder-McCabe map has the CNS property. Second, we generalize the GND lemma to the excess demand correspondences which have the CNS property which is weaker than *usc* and neither implies nor is implied by the Browder-McCabe map has the CNS property.

An economy $E \equiv (Z, \zeta)$ with m commodities is defined as follows: $Z \subset \mathbb{R}^m$ is the set of possible excess demands which is nonempty, convex and compact, and $\zeta : \Delta \twoheadrightarrow Z$ the excess demand correspondence where $\Delta \equiv \{p \in \mathbb{R}^m_+ | \sum_{k=1}^m p^k = 1\}$ is the set of prices. Let $-\Omega \equiv \{x \in \mathbb{R}^m | \ x \leqq 0\}$. The Browder-McCabe map is a correspondence $\Psi : \Delta \twoheadrightarrow \Delta$ defined as

$$\Psi(p) \equiv \{q \in \Delta : q \cdot \zeta(p) > 0\}.$$

Theorem 17. *Let* $E \equiv (Z, \zeta)$ *be the economy defined above such that*

(i) *the Browder-McCabe map* Ψ *has the* CNS *property,*

(ii) ζ *has nonempty, convex and closed values,*

(iii) *for each* $p \in \Delta$ *there exists* $x \in \zeta(p)$ *such that* $p \cdot x \leqq 0$.

Then there exists $p^* \in \Delta$ *such that* $\zeta(p^*) \cap -\Omega \neq \emptyset$.

Remark 15. In Theorem 17, we use a weaker version of the classical Walras law, '$p \cdot \zeta(p) \leqq 0$ for each $p \in \Delta$,' as in McCabe (1981), Yannelis (1985) and Mehta-Tarafdar (1987). This weaker version is helpful to analyze consumer's problem with general budget constraints. McCabe (1981, Theorem 1, p. 169) and Yannelis (1985, Theorem 3.1, p. 597) assumed the excess demand correspondence is *usc*, hence the Browder-McCabe map Ψ has open fibers.[14] And, Mehta-Tarafdar (1987, Theorem 8, p. 337) assumed Ψ has the local intersection property.[15] Since open fibers and local intersection property implies the CNS property, Theorem 17 generalizes these results as well as previous results cited in these papers.

Proof of Theorem 17. Assume the conclusion of Theorem 17 is false. Then $\zeta(p) \cap -\Omega = \emptyset$ for each $p \in \Delta$. Also, since $\zeta(p)$ is nonempty, compact, convex set and $-\Omega$ is a closed, convex set, there exists $q \in \Delta$ that strictly separates $\zeta(p)$ and $-\Omega$ for each $p \in \Delta$. Hence, the correspondence Ψ has nonempty values. It is clear that Ψ has convex values. And since Ψ is assumed to have the CNS property, Theorem 9 implies there exists $p^* \in \Delta$ such that $p^* \in \Psi(p^*)$, i.e. $p^* \cdot x > 0$ for all $x \in \zeta(p^*)$. This furnishes a contradiction with (iii). □

In Theorem 17, we assume the correspondence Ψ has the CNS property which is a technical assumption and does not have direct economic implications. Now, we propose a weak continuity assumption on excess demand correspondence which puts restriction only on downward jumps and implies the Browder-McCabe map has the CNS property.

Definition 26. Let X, Y be nonempty subsets of $\mathbb{R}^m$ and $P : X \twoheadrightarrow Y$ a correspondence. P is *continuous from below*, cfb, *at* $x \in X$ if for all open set

[14] McCabe provided and used a selection theorem to prove his result. Yannelis showed that Ψ has open fibers and then used Yannelis-Prabhakar selection theorem to prove his result. One can alternatively use the Browder fixed point theorem to prove their result.

[15] Note that both Yannelis (1985) and Mehta-Tarafdar (1987) worked with infinite dimensional commodity spaces. Here we refer to the finite dimensional version of their theorems. And by using their setup, it is routine exercise to generalize our results to infinite dimension.

V such that for all $y' \in P(x)$ there exists $y \in V$ such that $y \leqq y'$, there exists an open neighborhood U of x such that for all $x' \in U$, for all $z' \in P(x')$ there exists $z \in V$ such that $z \leqq z'$. And P is called cfb if it is cfb at each $x \in X$.

Note that, cfb only requires P to continuously move at the negative direction whereas usc requires P to continuously move. Hence, usc implies cfb. Now, let X, Y be nonempty subsets of $\mathbb{R}^m$, $P : X \twoheadrightarrow Y$ a correspondence, and π the usual projection map. Define $\pi^k P : X \twoheadrightarrow \mathbb{R}$ as $\pi^k P(x) \equiv \pi^k(P(x))$ and $\underline{x}^k \equiv \min \pi^k P(x)$ for each $k = 1, \dots, m$.

Definition 27. Let X, Y be nonempty subsets of $\mathbb{R}^m$ and $P : X \twoheadrightarrow Y$ a correspondence. P is *weakly continuous from below, $wcfb$, at $x \in X$* if there exists $k \in \{1, \dots, m\}$ such that $\underline{x}^k > 0$ and $\pi^k P$ is cfb at x, otherwise P is cfb at x. And P is called $wcfb$ if it is $wcfb$ at each $x \in X$.

Note that, if at some price p, the excess demand of at least one commodity is positive and its excess demand remains positive around a neighborhood of p, then we say the excess demand correspondence is $wcfb$ at p, irrespective of the behavior of the excess demand of other commodities around p.

Proposition 10. *Let $E \equiv (Z, \zeta)$ be the economy defined above such that ζ is $wcfb$ and has nonempty, closed values Then the Browder-McCabe map Ψ has the CNS property.*

Proof of Proposition 10. First, since Z is compact and ζ has nonempty and closed values $\pi^k \zeta$ has nonempty and compact values, and hence, $\underline{p}^k \equiv \min \pi^k \zeta(p)$ is well defined for each $k = 1, \dots, m$ and $\Delta \in \Delta$.

Now pick $p \in \Delta$ such that $\Psi(p) \neq \emptyset$, hence there exists $q \in \Delta$ such that $q \cdot \zeta(p) > 0$. First, assume for some $k = 1, \dots, m$, $\underline{p}^k > 0$ and $\pi^k \zeta$ is cfb at p.

Since $\zeta(p)$ and Z are compact, $\pi^k\zeta(p) \times \pi^{-k}(Z)$ is compact. Then, there exists $q' \in \Delta$ that strictly separates $\pi^k\zeta(p) \times \pi^{-k}(Z)$ and $-\Omega$. And, since $\pi^k\zeta$ is cfb at $p \in \Delta$, for sufficiently small $\varepsilon > 0$, there exists an open neighborhood U^p of p such that for all $p' \in U^p$, q' still strictly separates $\{\pi^k\zeta(p) - \varepsilon\} \times \pi^{-k}(Z)$. Therefore, $q' \cdot \zeta(p') > 0$ for all $p' \in U^p$, hence ζ has the constant CNS property at p. And Proposition 11 implies Ψ has the CNS property at p. Second, assume for each commodity $k = 1, \ldots, m$, $\underline{p}^k \leqq 0$. Therefore, ζ is $wcfb$ at p implies ζ is cfb at p. And since $q \cdot \zeta(p) > 0$, q determines an open half space H^q in $\mathbb{R}^m$ containing $\zeta(p)$. And since ζ is cfb, there exists an open neighborhood U^p of p such that for all $p' \in U^p$, for all $z' \in \zeta(p')$ there exists $z \in H^q$ such that $z \leqq z'$. Therefore, $q \cdot \zeta(p') > 0$ for all $p' \in U^p$, hence Ψ has the constant CNS property at p. And Proposition 11 implies Ψ has the CNS property at p. □

The following result is of particular interest since it replaces upper semicontinuity of the excess demand correspondence with the CNS property.

Theorem 18. *Let $E \equiv (Z, \zeta)$ be the economy defined above such that*

(i) *ζ has nonempty, convex values and the CNS property,*

(ii) *for each $p \in \Delta$, $p \cdot \zeta(p) \leqq 0$.*

Then there exists $p^* \in \Delta$ such that $\zeta(p^*) \cap -\Omega \neq \emptyset$.

Remark 16. First, note that in Theorem 18, we do not assume ζ has closed values. Also we use the classical Walras law; see Debreu (1982). Second, one can routinely extend Theorems 17 and 18 to infinite dimensional commodity spaces by following Yannelis (1985); Mehta-Tarafdar (1987) and to approximate equilibrium by following Anderson et al. (1982). Moreover, He-Yannelis (2015a) provided a generalization of the GND lemma that is similar to Theorem 18 for infinite dimensional commodity spaces.

Proof of Theorem 18. Assume the conclusion of Theorem 18 is false. Since ζ has the CNS property, and has nonempty and convex values, it has an usc selection $F : \Delta \twoheadrightarrow Z$ which has nonempty, convex and closed values. Then $F(p) \cap -\Omega = \emptyset$ for each $p \in \Delta$. And, since $F(p)$ is convex and compact, and $-\Omega$ is a closed, convex set, there exists $q \in \Delta$ that strictly separates $F(p)$ and $-\Omega$ for each $p \in \Delta$. Hence, the Browder-McCabe map $\Psi : \Delta \twoheadrightarrow \Delta$ defined as $\Psi(p) = \{q \in \Delta : q \cdot F(p) > 0\}$ is nonempty valued. It is clear that Ψ is convex valued. And since F is usc, Ψ has open fibers. In order to see this, pick $q \in \Delta$. If $\Psi^{-1}(q) \equiv \{p \in \Delta | \ q \in F(p)\} = \emptyset$, then it is open in Δ. Otherwise, pick $p \in \Psi^{-1}(q)$. Since F is usc, and q determines an open half space H^q in $\mathbb{R}^m$ containing $F(p)$, there exists an open neighborhood U^p of p such that $F(p') \subset H^q$, hence $q \in \Psi(p')$, for each $p' \in U^p$. Hence, $U^p \subset \Psi^{-1}(q)$. Therefore, Ψ has open fibers. Proposition 11 implies Ψ has the CNS property at p. Theorem 9 implies there exists $p^* \in \Delta$ such that $p^* \in \Psi(p^*)$, i.e. $p^* \cdot x > 0$ for all $x \in F(p^*) \subset \zeta(p^*)$. This furnishes a contradiction with (ii). □

4.4 Concluding Remarks

In this paper we present, we present three fixed point theorems for correspondences which satisfies weak continuity properties. We show these fixed point theorems can be used as a toolkit to prove existence of an equilibrium by providing applications in existence of a maximal elements and an equilibrium in games and economies. Moreover, we show that the state of the art result of

Reny (2015) cannot be generalized to games without ordered preferences.

Appendix A

Appendix for Chapter 1

Proof of Lemma 1. Pick $v \in \mathcal{V}, i \in N$ and $y^i \in \mathcal{Y}^i$. Assume $y^i \in \operatorname{argmin}_{Y \in \mathcal{Y}^i} \sum_{j \neq i} Y_j \delta_j v_j$. Pick $\lambda^i \in \mathbf{\Lambda}(y^i)$. Pick $\lambda^i \in \mathbf{\Lambda}(y^i)$. First, assume $\sum_{j \neq i} y^i_j \delta_j v_j > \sum_{j \neq i} \bar{S}^i_j \delta_j v_j$ for some $\bar{S}^i \in supp(\lambda^i)$. Then $\bar{S}^i \in \mathcal{Y}^i$ contradicts with $y^i \in \operatorname{argmin}_{\hat{y}^i \in \mathcal{Y}^i} \sum_{j \neq i} \hat{y}^i_j \delta_j v_j$. Second, assume $\sum_{j \neq i} y^i_j \delta_j v_j < \sum_{j \neq i} \bar{S}^i_j \delta_j v_j$ for some $\bar{S}^i \in supp(\lambda^i)$. Define $\hat{\lambda}^i \in \Lambda^i$ as

$$\hat{\lambda}^i_{S^i} = \frac{\lambda^i_{S^i}}{1 - \lambda^i_{\bar{S}^i}} \quad \text{for} \quad S^i \neq \bar{S}^i \quad \text{and} \quad \hat{\lambda}^i_{\bar{S}^i} = 0.$$

Define

$$\hat{y}^i = \sum_{S^i \in \mathcal{S}^i} \hat{\lambda}^i_{S^i} S^i = \sum_{S^i \neq \bar{S}^i} \frac{\lambda^i_{S^i}}{1 - \lambda^i_{\bar{S}^i}} S^i.$$

Since $0 \leq \sum_{j \neq i} y^i_j \delta_j v_j < \sum_{j \neq i} \bar{S}^i_j \delta_j v_j$ we can write $\sum_{j \neq i} \bar{S}^i_j \delta_j v_j = \sum_{j \neq i} y^i_j \delta_j v_j + \varepsilon$ where $\varepsilon > 0$. Therefore

$$\begin{aligned}
\sum_{j \neq i} \hat{y}^i_j \delta_j v_j &= \sum_{j \neq i} \left(\sum_{S^i \neq \bar{S}^i} \tfrac{\lambda^i_{S^i}}{1-\lambda^i_{\bar{S}^i}} S^i \right)_j \delta_j v_j \\
&= \tfrac{1}{1-\lambda^i_{\bar{S}^i}} \left(\sum_{j \neq i} y^i_j \delta_j v_j - \lambda^i_{\bar{S}^i} \sum_{j \neq i} \bar{S}^i_j \delta_j v_j \right) \\
&= \tfrac{1}{1-\lambda^i_{\bar{S}^i}} \left(\sum_{j \neq i} y^i_j \delta_j v_j - \lambda^i_{\bar{S}^i} \left(\sum_{j \neq i} y^i_j \delta_j v_j + \varepsilon \right) \right) \\
&< \sum_{j \neq i} y^i_j \delta_j v_j.
\end{aligned}$$

This inequality contradicts with $y^i \in \operatorname{argmin}_{\hat{y}^i \in \mathcal{Y}^i} \sum_{j \neq i} \hat{y}^i_j \delta_j v_j$. Now assume there exists $\bar{S}^i \notin supp(\lambda^i)$ such that $\sum_{j \neq i} y^i_j \delta_j v_j > \sum_{j \neq i} S^i_j \delta_j v_j$. We have just shown that $\sum_{j\neq i} y^i_j\delta_j v_j = \sum_{j\neq i} S^i_j\delta_j v_j$ for all $S^i \in supp(\lambda^i)$. Hence $\sum_{j\neq i} S^i_j\delta_j v_j > \sum_{j\neq i} \bar{S}^i_j\delta_j v_j$ for all $S^i \in supp(\lambda^i)$. Pick $S^i \in \sup(\lambda^i)$ and define

$$\bar{y}^i = y^i - \lambda^i_{S^i} S^i + \lambda^i_{S^i}\bar{S}^i.$$

Then

$$\sum_{j\neq i} \bar{y}^i_j\delta_j v_j = \sum_{j\neq i} y^i_j\delta_j v_j - \lambda^i_{S^i}\left(\sum_{j\neq i} S^i_j\delta_j v_j - \sum_{j\neq i} \bar{S}^i_j\delta_j v_j\right) < \sum_{j\neq i} y^i_j\delta_j v_j.$$

This inequality contradicts with $y^i \in \operatorname{argmin}_{\hat{y}^i\in\mathcal{Y}^i} \sum_{j\neq i} \hat{y}^i_j\delta_j v_j$.

Pick $v \in \mathcal{V}, i \in N$ and $y^i \in \mathcal{Y}^i$. Assume there exists $\lambda^i \in \mathbf{\Lambda}^i(y^i)$ such that $\sum_{j\neq i} \mathbf{y}^i_j(\lambda^i)\delta_j v_j = \sum_{j\neq i} S^i_j\delta_j v_j \leq \sum_{j\neq i} \tilde{S}^i_j\delta_j v_j$ for all $S^i \in supp(\lambda^i)$ and all $\tilde{S}^i \notin supp(\lambda^i)$. We first show that properties (i) and (ii) hold for all $\hat{\lambda}^i \in \mathbf{\Lambda}^i(y^i)$. Note that the assumption above implies that $\sum_{j\neq i} \mathbf{y}^i_j(\hat{\lambda}^i)\delta_j v_j = \sum_{j\neq i} y^i_j\delta_j v_j \leq \sum_{j\neq i} S^i_j\delta_j v_j$ for all $\hat{\lambda}^i \in \mathbf{\Lambda}^i(y^i)$ and all $S^i \in \mathcal{S}^i$. Hence, property (ii) holds for all $\hat{\lambda}^i \in \mathbf{\Lambda}^i(y^i)$. Now assume property (i) does not hold for some $\hat{\lambda}^i \in \mathbf{\Lambda}^i(y^i)$, i.e. there exists $\hat{\lambda}^i \in \mathbf{\Lambda}^i(y^i)$ such that $\sum_{j\neq i} y^i_j\delta_j v_j \neq \sum_{j\neq i} \hat{S}^i_j\delta_j v_j$ for some $\hat{S}^i \in supp(\hat{\lambda}^i)$. Then $\hat{S}^i \notin supp(\lambda^i)$, hence $\sum_{j\neq i} S^i_j\delta_j v_j < \sum_{j\neq i} \hat{S}^i_j\delta_j v_j$ for all $S^i \in supp(\lambda^i)$. Pick $S^i \in supp(\lambda^i)$. Recall that $\sum_{j\neq i} S^i_j\delta_j v_j \leq \sum_{j\neq i} \tilde{S}^i_j\delta_j v_j$ for all $\tilde{S}^i \in \mathcal{S}^i$. Then

$$\sum_{j\neq i} y^i_j\delta_j v_j = \sum_{j\neq i} S^i_j\delta_j v_j < \hat{\lambda}^i_{\hat{S}^i}\sum_{j\neq i} \hat{S}^i_j\delta_j v_j + \sum_{\tilde{S}^i\neq\hat{S}^i} \hat{\lambda}^i_{\tilde{S}^i}\sum_{j\neq i} \tilde{S}^i_j\delta_j v_j = \sum_{j\neq i} y^i_j\delta_j v_j.$$

This inequality shows that property (i) also holds for all $\hat{\lambda}^i \in \mathbf{\Lambda}^i(y^i)$. Now, we show that y^i is the least costly weighted winning coalition for player i. Note that $\hat{y}^i_j, \delta_j$ and v_j are non-negative for all $\hat{y}^i \in \mathcal{Y}^i$ and $j \in \{1, \ldots, n\}$. Then for all $\hat{y}^i \in \mathcal{Y}^i, \lambda^i \in \mathbf{\Lambda}^i(y^i)$ and $\hat{\lambda}^i \in \mathbf{\Lambda}^i(\hat{y}^i)$,

(i) $\sum_{j\neq i} \hat{y}^i_j\delta_j v_j = \sum_{j\neq i} y^i_j\delta_j v_j$ if $supp(\lambda^i) = supp(\hat{\lambda}^i)$,

(ii) $\sum_{j\neq i} \hat{y}^i_j \delta_j v_j \geq \sum_{j\neq i} y^i_j \delta_j v_j$ if $supp(\lambda^i) \neq supp(\hat{\lambda}^i)$.

Therefore, $\sum_{j\neq i} y^i_j \delta_j v_j \leq \sum_{j\neq i} \hat{y}^i_j \delta_j v_j$ for all $\hat{y}^i \in \mathcal{Y}^i$, i.e. $y^i \in \text{argmin}_{\tilde{y}^i \in \mathcal{Y}^i} \sum_{j\neq i} \tilde{y}^i_j \delta_j v_j$. □

Proof of Corollary 1. Pick $\lambda \in \Lambda$. Then the definition of equilibrium payoffs provided in Equation 1.3 implies $v = \mathbf{v}(\lambda)$ is an SSP equilibrium payoff vector if and only if $\mathbf{y}^i(\lambda^i) \in \text{argmin}_{Y \in \mathcal{Y}^i} \sum_{j\neq i} Y_j \delta_j v_j$ for all player i. Therefore Lemma 1 implies $v = \mathbf{v}(\lambda)$ is an SSP equilibrium payoff vector if and only if $\sum_{j\neq i} \mathbf{y}^i_j(\lambda^i)\delta_j v_j = \sum_{j\neq i} S^i_j \delta_j v_j \leq \sum_{j\neq i} \tilde{S}^i_j \delta_j v_j$ for all $S^i \in supp(\lambda^i)$ and all $\tilde{S}^i \notin supp(\lambda^i)$. □

Proof of Theorem 1. Note that the function $\mathbf{y}^i$ that maps mixed strategies of each player i to her weighted winning coalitions defined in Equation 1.4, the SSP payoff function that maps mixed strategy profiles to the set of SSP payoffs defined in Equation 1.5 and the gain function g^i that maps winning coalitions of each player i and the mixed strategy profiles to a non-negative real number are all continuous. Therefore, the function f which maps mixed strategy profiles to mixed strategy profiles defined in 1.7 is continuous. Since the set of mixed strategy profiles Λ is a nonempty, convex and compact set Brouwer's fixed point theorem implies f has a fixed point λ. Theorem 2 implies $v(\lambda)$ is an SSP payoff vector.

For the uniqueness of the SSP equilibrium payoffs note that Eraslan and McLennan(Definition 1, p. 2201) define a *reduced equilibrium* concept which is equivalent to SSP equilibrium –they show that reduced equilibrium induces and is induced by an SSP equilibrium– as follows. A reduced equilibrium is a pair of SSP payoffs and weighted winning coalitions $(v, (y^1, \dots, y^n)) \in \mathcal{V} \times (\mathcal{Y}^1 \times \cdots \times \mathcal{Y}^n)$ such that for each i

(a) $v_i = \left(p_0 + \sum_{j=1}^n p_j y_i^j\right) \delta_i v_i + p_i \left(1 - \sum_{j=1}^n y_j^i \delta_j v_j\right)$ where

(b) $y^i \in \underset{\hat{y}^i \in \mathcal{Y}^i}{\operatorname{argmin}} \sum_{j=1}^n \hat{y}_j^i \delta_j v_j$.

Therefore the pair $(\mathbf{v}(\lambda), \mathbf{Y}(\lambda))$ is a reduced equilibrium if and only if $\mathbf{y}^i(\lambda^i)$ is a solution to $\min_{\hat{y}^i \in \mathcal{Y}^i} \sum_{j=1}^n \hat{y}_j^i \delta_j \mathbf{v}_j(\lambda)$ for each player i where $\mathbf{Y}(\lambda) = (\mathbf{y}^1(\lambda^1), \dots, \mathbf{y}^n(\lambda^n))$. Recall that Corollary 1 implies $\mathbf{v}(\lambda)$ is an SSP equilibrium payoff vector if and only if $\mathbf{y}^i(\lambda^i) \in \operatorname{argmin}_{y^i \in \mathcal{Y}^i} \sum_{j \neq i} y_j^i \delta_j \mathbf{v}_j(\lambda)$ for all player i. Hence $\mathbf{v}(\lambda)$ is an SSP equilibrium payoff vector if and only if $(\mathbf{v}(\lambda), \mathbf{Y}(\lambda))$ is a reduced equilibrium. Theorem 1 of EM on page 2201 shows that the fist component v of the reduced equilibria are unique, and hence the SSP equilibrium payoffs are unique. □

Appendix B

Appendix for Chapter 2

A *nontransferable utility (NTU) game* is a nonempty-valued correspondence $V : \mathcal{N} \twoheadrightarrow \mathbb{R}^n$, where $N = \{1, \ldots, n\}$ is the set of players and $\mathcal{N} = 2^{N \setminus \emptyset}$ the set of coalitions. The *core* of an *NTU* game V is defined as $\text{Core}(V) = V(N) \setminus \left(\bigcup_{S \in \mathcal{N}} \text{int}V(S) \right)$. where $\text{int}V(S)$ is the (topological) interior of the set $V(S)$. A collection of coalitions $\mathcal{B} \subset 2^{\mathcal{N}}$ is *balanced* if for each coalition S, there exists a nonnegative scalar λ_S with $\lambda_S = 0$ if $S \notin \mathcal{B}$ such that for each $i \in N$, $\sum_{S:i \in S} \lambda_S = 1$. An *NTU* game V is *balanced* if for all balanced collections of coalitions $\mathcal{B}$, $\bigcap_{S \in \mathcal{B}} V(S) \subset V(N)$. Now, we state the beautiful theorem of Scarf (1967) which is used to prove our results.

Theorem (**Scarf**). *A balanced NTU game V has a nonempty core if for each coalition S,*

(i) *$V(S)$ is closed,*

(ii) *$v' \in \mathbb{R}^n$, $v \in V(S)$ and $v'_S \leq v_S$ imply $v' \in V(S)$,*

(iii) *there exists $M_S \in \mathbb{R}^{|S|}$ such that $v \in V(S)$ implies $v_S \leq M_S$.*

A *transferable utility (TU) game* is a function $W : \mathcal{N} \to \mathbb{R}$. The *core* of a *TU* game W is defined as $\text{Core}^T(W) = \{v \in \mathbb{R}^n | \sum_{i \in N} v_i \leq W(N) \text{ and } \sum_{i \in S} v_i \geq W(S) \ \forall S \in \mathcal{N}\}$. A balanced collection of coalitions $\mathcal{B} \subset 2^{\mathcal{N}}$ is *minimal* if it does not have a balanced proper subcollection. Note that the balancing weights λ of every minimally balanced collection of coalitions are unique. We end this paper by stating the influential theorem of Bondareva (1962) and Shapley (1967) which is used to prove our results.

Theorem (Bondareva-Shapley). *A TU game has a nonempty core if and only if for every minimally balanced collection of coalitions $\mathcal{B}$, $\sum_{S \in \mathcal{B}} \lambda_S W(S) \leq W(N)$.*

Appendix C

Appendix for Chapter 3

A *transferable utility (TU) game* is a function $W : \mathcal{N} \to \mathbb{R}$, where $N = \{1, \dots, n\}$ is the set of players and $\mathcal{N} = 2^{N \setminus \emptyset}$ the set of coalitions. The *core* of a *TU* game W is defined as $\text{Core}(W) = \{v \in \mathbb{R}^n | \sum_{i \in N} v_i \leq W(N)$ and $\sum_{i \in S} v_i \geq W(S)\ \forall S \in \mathcal{N}\}$. A collection of coalitions $\mathcal{B} \subset 2^{\mathcal{N}}$ is *balanced* if for each coalition S, there exists a nonnegative scalar λ_S with $\lambda_S = 0$ if $S \notin \mathcal{B}$ such that for each $i \in N$, $\sum_{S: i \in S} \lambda_S = 1$. A balanced collection of coalitions $\mathcal{B} \subset 2^{\mathcal{N}}$ is *minimal* if it does not have a balanced proper subcollection. Note that the balancing weights of every minimally balanced collection of coalitions are unique. Next, we state the influential theorem of Bondareva (1962) and Shapley (1967) which is used to prove our results.

Theorem (Bondareva-Shapley). *A TU game has a nonempty core if and only if for all minimally balanced collection of coalitions* $\mathcal{B}$, $\sum_{S \in \mathcal{B}} \lambda_S W(S) \leq W(N)$.

A *nontransferable utility (NTU) game* is a nonempty-valued correspondence $V : \mathcal{N} \twoheadrightarrow \mathbb{R}^n$. The *core* of an *NTU* game V is defined as $\text{Core}(V) = V(N) \setminus \left(\bigcup_{S \in \mathcal{N}} \text{int} V(S) \right)$. where $\text{int}V(S)$ is the (topological) interior of the set $V(S)$. An *NTU* game V is *balanced* if for all balanced collection of coalitions $\mathcal{B}$, $\bigcap_{S\in\mathcal{B}} V(S) \subset V(N)$. We end this paper by stating the beautiful theorem of Scarf (1967) which is used to prove our results.

Theorem (Scarf). *A balanced NTU game V has a nonempty core if for each coalition S,*

(i) *$V(S)$ is closed,*

(ii) *$v' \in \mathbb{R}^n$, $v \in V(S)$ and $v'_S \leq v_S$ imply $v' \in V(S)$,*

(iii) *there exists $M_S \in \mathbb{R}^{|S|}$ such that $v \in V(S)$ implies $v_S \leq M_S$.*

Appendix D

Appendix for Chapter 4

We first introduce new continuity concepts for correspondences and their relation to other important continuity concepts.

Definition 28. Let X, Y be nonempty subsets of a topological vector space and $P : X \twoheadrightarrow Y$ a correspondence. The correspondence P

(C1) has the *continuous neighborhood selection, CNS, property* if for all $x \in X$ such that $P(x) \neq \emptyset$ there exist an open neighborhood U^x of x and an *usc* correspondence $F^x : X \twoheadrightarrow Y$ which has nonempty, convex and closed values and $F^x(y) \subset P(y)$ for all $y \in U^x$. [We call F^x an *usc* selection of P on U^x.]

(C2) has the *single-valued CNS property* if for all $x \in X$ such that $P(x) \neq \emptyset$ there exist an open neighborhood U^x of x and a continuous function $f^x : X \to Y$ such that $f^x(y) \in P(y)$ for all $y \in U^x$. [We call f^x a continuous selection of P on U^x.]

(C3) has the *constant CNS property* (or, *local intersection property*) if for all $x \in X$ such that $P(x) \neq \emptyset$ there exists an open neighborhood U^x of x such that $\bigcap_{z \in U^x} P(z) \neq \emptyset$.

(C4) has *open fibers* if for all $y \in Y$, $\{x \in X | \ y \in P(x)\}$ is open in X.

(C5) has *open graph* if for all $x \in X$, $\{(x, y) \in X \times Y | \ y \in P(x)\}$ is open in $X \times Y$.

(C6) is *upper semicontinuous*, *usc*, if for all $x \in X$, $\{x \in X | \ P(x) \subset V\}$ is open in X for all $V \subset Y$ open.

(C7) is *lower semicontinuous*, *lsc*, if for all $x \in X$, $\{x \in X | \ P(x) \cap V \neq \emptyset\}$ is open in X for all $V \subset Y$ open.

It is clear that **C2** and **C3** are special cases of **C1**. The following result summarizes the relation of **C1** to the other continuity concepts defined above.

Proposition 11. *Let X be a nonempty Hausdorff topological vector space. A correspondence $P : X \twoheadrightarrow X$ has the CNS property,* **C1***, if it satisfies any of the following properties:*

(i) *one of* **C2-C5***,*

(ii) *has convex and closed values and* **C6***,*

(iii) *has convex values and contains the closure of a correspondence which satisfies* **C7** *and X is a metrizable.*

Proof of Proposition 11 . First, there is a strict ordering among **C1-C5**: **C5** $\Rightarrow$ **C4** $\Rightarrow$ **C3** $\Rightarrow$ **C2** $\Rightarrow$ **C1**. First, **C5** $\Rightarrow$ **C4** is well known, see Bergstrom et al. (1976). In order to see that **C4** $\Rightarrow$ **C3**, pick $x \in X$, and $y \in P(x)$. Since $P^{-1}(y)$ is open in X and $x \in P^{-1}(y)$, there exists an open neighborhood U of x such that $U \subset P^{-1}(y)$. Hence $y \in \bigcap_{z \in U} P(z)$. And a transparent example in which X is the unit interval and P a correspondence such that $P(x) = 1$

for all $x \in [0,1)$ and $P(1) = [0,1]$, shows that **C3** $\not\Rightarrow$ **C4**. And **C3** $\Rightarrow$ **C2** is clear since every constant function is continuous. In order to see that **C2** $\Rightarrow$ **C1**, assume **C2** holds. Pick $x \in X$ such that $P(x) \neq \emptyset$, then there exist an open neighborhood U^x of x and $y_x \in X$ such that $y^x \in \bigcap_{z \in U} P(z)$. Define $F^x : X \twoheadrightarrow X$ as $F^x(z) = \{y^x\}$. Since any constant function has nonempty, closed values and *usc* as a correspondence, and $y^x \in P(z)$ for all $z \in U^x$, F^x is an *usc* selection of P on U^x. Since x is arbitrarily chosen, P has the CNS property. And, setting the selection $F^x = P$ for each $x \in X$ is enough to show that (ii) $\Rightarrow$ **C1**. Finally, (iii) $\Rightarrow$ **C1** directly follows from Michael (1959, Theorem 1.1, p. 647). □

Remark 17. We can define the CNS property alternatively by using *lsc* correspondences with nonempty and closed values. Then, it is easy to show that there exists a *lsc* selection, with nonempty and closed values, of the original correspondence P. And this is equivalent to condition (iii) in Proposition 11 which is stronger than the continuity concept **C1** when X is metrizable.

The following two results are corollaries to our fixed point theorems. These are useful for the existence of Nash equilibrium in games.

Corollary 6. *Let I be an arbitrary index set and for each $i \in I$, X_i a nonempty, convex and compact subset of a tvs, $P_i : X \twoheadrightarrow X_i$ a correspondence with nonempty, convex values and the constant CNS property, where $X \equiv \prod_{i \in I} X_i$. Then there exists $x^* \in X$ such that $x_i^* \in P_i(x^*)$ for each $i \in I$.*

Proof of Corollary 6 . Tychonoff's theorem implies arbitrary product of nonempty and compact subsets of *tvs* is nonempty and compact. And arbitrary product of convex sets is convex. Hence, X is a nonempty, convex and compact subset of a *tvs*. Now, define $\tilde{X}_i, \tilde{P}_i : \tilde{X} \twoheadrightarrow \tilde{X}_i$ and $\bar{P}_i : X \twoheadrightarrow X_i$ analogous to Step 2 of the proof of Theorem 10 for each $i \in I$. Recall that $\tilde{P}_i$ has nonempty and convex valued and has the constant CNS property.

Pick $i \in I$. Following Step 1 of the proof of Theorem 10, there exists $\{U_i^k, \tilde{y}_i^k, \beta_i^k | \ k = 1, \ldots, m_i\}$. Define a continuous function $p_i : \tilde{X} \to \tilde{X}_i$ as $p_i(\tilde{x}) \equiv \sum_{k=1}^{m_i} \beta_i^k(\tilde{x})\tilde{y}_i^k$. Also define a function $p : \tilde{X} \twoheadrightarrow \tilde{X}$ as $p(\tilde{x}) \equiv \prod_{i \in I} p_i(\tilde{x})$ and a correspondence $\tilde{P} : \tilde{X} \twoheadrightarrow \tilde{X}$ as $\tilde{P}(\tilde{x}) \equiv \prod_{i \in I} \tilde{P}_i(\tilde{x})$. Then p is continuous and $p(\tilde{x}) \in \tilde{P}(\tilde{x})$ for all $\tilde{x} \in \tilde{X}$. Let $\tilde{X}_i^0$ be the finite dimensional simplex spanned by the m_i points $\{\tilde{y}_i^1, \ldots, \tilde{y}_i^{m_i}\}$. Define $\tilde{X}^0 \equiv \prod_{i \in I} \tilde{X}_i^0$. Since the topology induced on $\tilde{X}^0$ coincides with the Euclidean topology and p is continuous function, by the Brouwer fixed point theorem, p has a fixed point $\tilde{x}^*$ in $\tilde{X}^0$. Hence, $\tilde{x}^* = p(\tilde{x}^*) \in \tilde{P}(\tilde{x}^*)$. Therefore there is $x \in X$ such that $\pi x \in \pi\big(\bar{P}(x)\big)$. This implies that there exists $z \in X$ such that $\pi x = \pi z$ and $z \in \bar{P}(x)$. And $\bar{P}(x) = \bar{P}(z)$ implies $z \in \bar{P}(z)$, and in particular $z \in P(z)$. Therefore $z_i \in P_i(z)$ for all $i \in I$. □

Corollary 7. *Let I be an arbitrary index set and for each $i \in I$, X_i a nonempty, convex and compact subset of a locally convex Hausdorff tvs, $P_i : X \twoheadrightarrow X_i$ a correspondence with nonempty, convex values and the CNS property, where $X \equiv \prod_{i \in I} X_i$. Then there exists $x^* \in X$ such that $x_i^* \in P_i(x^*)$ for each $i \in I$.*

Proof of Corollary 7 . First note that arbitrary product of nonempty, convex and compact sets are nonempty, convex and compact. Also, arbitrary product of locally convex Hausdorff *tvs* is a locally convex Hausdorff *tvs*. Hence, X is a nonempty, convex and compact subset of a locally convex Hausdorff *tvs*.

We showed in the proof of Theorem 9 that for each $i \in I$, there exists an *usc* correspondence $Q_i : X \twoheadrightarrow X_i$ with nonempty, convex and closed values such that $Q_i(x) \subset P_i(x)$ for all $x \in X$. Define $Q : X \twoheadrightarrow X$ as $Q(x) \equiv \prod_{i \in I} Q_i(x)$, and P analogously. Then, (Fan, 1952, Lemma 3, p. 124) implies Q is an *usc* correspondence with nonempty, convex and closed values. And $Q(x) \subset P(x)$ for all $x \in X$. Hence, Theorem 9 implies P has a fixed point. Therefore, $x_i^* \in P_i(x^*)$ for each $i \in I$. □

We end this section with the proof of the existence of a Nash equilibrium for correspondence-secure games.

Proof of Theorem 12. We present the proof in 6 steps.

<u>Step 1.</u> *Select a finite collection of open sets U^k and securing strategies y^k.* Assume G does not have a Nash equilibrium. Then, G is correspondence-secure at each $x \in X$, i.e. there exist an *usc* correspondence $d^x : X \twoheadrightarrow X$ with nonempty, convex and closed values and an open neighborhood $U^x \subset X$ of x such that for each $x' \in U^x$, there exists $i \in N$ such that $(y_i, z_{-i}) \succ_i x'$ for each $z \in U^x$ and each $y_i \in d_i^x(z)$. The family $\{U^x |\ x \in X\}$ is an open covering of X which, by compactness of X, contains a finite subcovering $\{U^{x_k} |\ k = 1, \ldots, m\}$. Let $U^k \equiv U^{x_k}$ and $d^k \equiv d^{x_k}$ for all $k \in K \equiv \{1, \ldots, m\}$.

<u>Step 2.</u> *Define a relation on the index set K.* For each $i \in I$, define a binary relation $\mathcal{D}_i$ on K as for each $k, l \in K$,

$$k \mathcal{D}_i l \ \text{ if } \ \forall x \in U^k \, \forall y_i \in d_i^k(x) \, \exists z^x \in U^l \, \exists y_i^x \in d_i^l(z^x) \text{ such that } (y_i, x_{-i}) \succeq_i (y_i^x, z_{-i}^x).$$

Claim 5. *For each $i \in I$, $\mathcal{D}_i$ is reflexive, complete and transitive.*

<u>*Step 3.*</u> *For each $x \in X$, construct an open neighborhood $W(x)$ of x from* $\{U^k|\ k \in K\}$. Since the topology on X is Hausdorff, there exists a closed refinement $\{C^k|\ k \in K\}$ of $\{U^k|\ k \in K\}$, that is to say C^k is closed and $C^k \subset U^k$ for all $k \in K$, and $\bigcup_{k \in K} C^k = X$.

Define correspondences $J^U : X \twoheadrightarrow K$ and $J^C : X \twoheadrightarrow K$ as,

$$J^U(x) \equiv \{k \in K|\ x \in U^k\} \ \text{ and } \ J^C(x) \equiv \{k \in K|\ x \in C^k\},$$

and for each player $i \in I$, a function $k_i : X \to K$ as,

$$k_i(x) \in \{k \in J^U(x)|\ kD_ik' \text{ for all } k' \in J^U(x)\}.$$

Now, define for each $x \in X$, correspondences $V : X \twoheadrightarrow X$ and $W : X \twoheadrightarrow X$ as

$$V(x) \equiv \bigcap_{k \in K \setminus J^C(x)} \left(C^k\right)^c \ \text{ and } \ W(x) \equiv V(x) \cap \left(\bigcap_{i \in I} U^{k_i(x)}\right).$$

And, if $J^C(x) = K$, define $V(x) \equiv X$.

Claim 6. *For each $x \in X$ and $k \in K$, $C^k \cap W(x) \neq \emptyset$ if and only if $x \in C^k$.*

Claim 7. *For each $i \in I, x \in X, k \in J^C(x), x' \in W(x)$ and $y_i \in d_i^{k_i(x)}(x')$ there exists*[1] *$z^{x'} \in U^k$ and $y_i^{x'} \in d_i^l(z^{x'})$ such that*

$$(y_i, x'_{-i}) \succeq_i (y_i^{x'}, z_{-i}^{x'}).$$

<u>*Step 4.*</u> *W is a simple correspondence.* Note that, the image of W is characterized by the intersection of a finitely many open subsets of X. Let this finite family be $\{W^1, \dots, W^l\}$, which are determined by $\{x^1, \dots, x^l\}$, and $T \equiv \{1, \dots, l\}$. Also, for each $i \in I$ and $t \in T$, we have $k_i(x^t) \in K$.

[1]Note that we suppress the dependence of z on i, x and $k_i(x)$.

Claim 8. *For each* $i \in I, t \in T, k \in J^C(x^t), x' \in W^t$ *and* $y_i \in d_i^{k_i(x^t)}(x')$ *there exists* $z^{x'} \in U^k$ *and* $y_i^{x'} \in d_i^l(z^{x'})$ *such that*

$$(y_i, x'_{-i}) \succeq_i (y_i^{x'}, z_{-i}^{x'}).$$

<u>*Step 5.*</u> *For each* $i \in I$*, define a correspondence.* First, define a correspondence $J^W : X \twoheadrightarrow T$ as,

$$J^W(x) \equiv \{t \in T | x \in W^t\}.$$

Now for each $i \in I$, define a correspondence $P_i : X \twoheadrightarrow X_i$ as,

$$P_i(x) \equiv \text{co} \left\{ \bigcup_{t \in J^W(x)} d_i^{k_i(x^t)}(x) \right\}.$$

Claim 9. *For each* $i \in I$*,* P_i *has nonempty and convex values, and the CNS property.*

<u>*Step 6.*</u> *Contradiction step.* By Claim 9, Corollary 6 implies there exists $x^* \in X$ such that $x_i^* \in P_i(x^*)$ for each $i \in I$. Then, $x_i^* \in \text{co}\{d_i^{k_i(x^t)}(x^*)\}_{t \in J^W(x^*)}$ for each $i \in I$. And, since $\{C^k | \ k \in K\}$ covers X, there exists $k \in K$ such that $x^* \in C^k \subset U^k$. By Claim 8, for each $i \in I, t \in J^W(x^*)$ and $y_i \in d_i^{k_i(x^t)}(x^*)$ there exists $z^{x^*,i,t} \in U^k$ and $y_i^{x^*,i,t} \in d_i^k(z^{x^*,i,t})$ such that

$$(y_i, x_{-i}^*) \succeq_i (y_i^{x^*,i,t}, z_{-i}^{x^*,i,t}).$$

For each $i \in I$, let $(y_i^{x^*,i}, z_{-i}^{x^*,i}) \in \{(y_i^{x^*,i,t}, z_{-i}^{x^*,i,t}) | \ t \in J^W(x^*)\}$ such that $(y_i^{x^*,i}, z_{-i}^{x^*,i}) \succeq_i (y_i^{x^*,i,t}, z_{-i}^{x^*,i,t})$ for all $t \in J^W(x^*)$. Hence, for each $i \in I, t \in J^W(x^*)$ and $y_i \in d_i^{k_i(x^t)}(x^*)$

$$(y_i, x_{-i}^*) \succeq_i (y_i^{x^*,i}, z_{-i}^{x^*,i}).$$

Then, since $x_i^* \in \text{co}\{y_i\}$ for each $i \in I$, convexity of preferences (assumption (ii) above) implies $x^* \succeq_i (y_i^{x^*,i}, z_{-i}^{x^*,i})$ for each $i \in I$. Since $x^* \in U^k$ and $z^{x^*,i} \in U^k$ for each $i \in I$, correspondence-security implies there exists $j \in I$ such that $(y_j', z_{-j}^{x^*,j}) \succ_j x^*$ for all $y_j' \in d_j^k(z^{x^*,j})$. This furnishes a contradiction.

Thus, all remains is to prove Claims 5–9.

Proof of Claim 5. Pick $i \in I$. Reflexivity is clear. In order to show completeness, pick $k, l \in K$. Let $\neg k\mathcal{D}_i l$. Then there exists $z \in U^k$ and $y_i \in d_i^k(z)$ such that for all $x \in U^l$ and $y_i' \in d_i^l(x)$, $(y_i', x_{-i}) \succ_i (y_i, z_{-i})$, which implies $l\mathcal{D}_i k$. In order to show transitivity, pick $k, k', k'' \in K$ and let $k\mathcal{D}_i k'$ and $k'\mathcal{D}_i k''$. Pick $x \in U^k$ and $y_i \in d_i^k(x)$. Then there exists $z^x \in U^{k'}$ and $y_i^x \in d_i^{k'}(z^x)$ such that $(y_i, x_{-i}) \succeq_i (y_i^x, z_{-i}^x)$. Since $k'\mathcal{D}_i^x k''$ there exists $z^{z^x} \in U^{k''}$ and $y_i^{x'}$ such that $(y_i^x, z_{-i}^x) \succeq_i (y_i^{x'}, z_{-i}^{z^x})$. Hence, $(y_i, x_{-i}) \succeq_i (y_i^{x'}, z_{-i}^{z^x})$. □

Proof of Claim 6. Pick $x \in X$ and $k \in K$. If $C^k \cap W(x) \neq \emptyset$, then $x \notin C^k$ is obvious. Now assume $x \in C^k$. By construction $x \in V(x)$ and by assumption $x \in U^{k_i(x)}$ for each $i \in I$. Hence $x \in W(x)$. □

Proof of Claim 7. Pick $i \in I, x \in X, k \in J^C(x), x' \in W(x)$ and $y_i \in d_i^{k_i(x)}(x')$. First, since $C^k \subset U^k$, $J^C(x) \subset J^U(x)$. Hence, $k_i(x)\mathcal{D}_i k$ for all $k \in J^C(x)$. Second, by construction $W(x) \subset U^k$. This together with the definition of $\mathcal{D}_i$ imply there exist $z^{x'} \in U^k$ and $y_i^{x'} \in d_i^{k_i(x)}(x')$ such that $(y_i, x_{-i}') \succeq_i (y_i^{x'}, z_{-i}^{x'})$. □

Proof of Claim 8. For each $t \in T$, observing that W^t is an open neighborhood of x^t and applying Claim 7 finishes the proof. □

Proof of Claim 9. Pick $x \in X$ and $i \in I$. The nonemptiness and convexity of $P_i(x)$ are clear. In order to show P_i has the CNS property at x recall that

$\{W^1, \ldots, W^l\}$ is an open cover of X. Hence, $x \in W^t$ for some $t \in T$. By construction of P_i, $d_i^{k_i(x^t)}(x') \subset P_i(x')$ for all $x' \in W^t$. $\square$

Therefore, the proof is finished. $\square$

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Curriculum Vitae

Metin Uyanık received a B.A. in economics in 2008 from Boğaziçi University, Turkey and an M.A. in economics in 2010 from Koç University, Turkey. He then enrolled the Ph.D. program in the Department of Economics at the Johns Hopkins University, United States in 2010. He accepted a tenure-track faculty position in the School of Economics at the University of Queensland, Australia. Before starting at the University of Queensland in July 2017, he will spend one year as a postdoctoral fellow at the W. Allen Wallis Institute of Political Economy at the University of Rochester.

His primary fields of academic interest are Economic Theory, Computational Economics, Game Theory and, Political Economy. In his computational research, he provides algorithms to compute stationary equilibrium payoffs in a dynamic coalitional bargaining model which has become a leading framework for the study of legislative decision making. In his theoretical research, he focuses on existence and characterization of equilibria in games and Walrasian economies. Moreover, throughout his graduate studies he had the opportunity to accumulate extensive teaching experience ranging from Elements of Microeconomics, Game Theory and Intermediate Microeconomics to graduate courses such as Microeconomic Theory and Optimization.