EFFICIENT AND PERFECT DOMINATION ON ARCHIMEDEAN LATTICES

by

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Abstract

An Archimedean lattice is an infinite graph constructed from a vertex-transitive tiling of the plane by regular polygons. A set of vertices S is said to dominate a graph $G = (V, E)$ if every vertex in V is either in the set S or is adjacent to a vertex in set S. A dominating set is a perfect dominating set if every vertex not in the dominating set is dominated exactly once. The domination ratio is the minimum proportion of vertices in a dominating set. The perfect domination ratio is the minimum proportion of vertices in a perfect dominating set. Dominating sets are provided to establish upper bounds for the domination ratios of all the Archimedean lattices. A dominating set is an efficient dominating set if every vertex is dominated exactly once. We show that seven of the eleven Archimedean lattices are efficiently dominated, which easily determine their domination ratios and perfect domination ratios. We prove that the other four Archimedean lattices cannot be efficiently dominated. For the four Archimedean lattices that cannot be efficiently dominated, we have determined their exact perfect domination ratios. Integer programming

ABSTRACT

bounds for domination ratios are provided. A perfect domination proportion is the proportion of vertices in a perfect dominating set that is not necessarily minimal. We study nonisomorphic perfect dominating sets and possible perfect domination proportions of Archimedean lattices.

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Chapter 1

Introduction

In a simple graph $G = (V_G, E_G)$, a vertex x *dominates* a vertex y if either x is adjacent to y or $x = y$. A subset $D \subseteq V_G$ is a *dominating set* if every vertex in V_G is dominated by at least one vertex in D. More formally, and to introduce useful notation and terminology, define the *closed neighborhood* of a vertex $v \in V_G$ by $N[v] = \{u \in V_G : u = v \text{ or } u \text{ is adjacent to } v\}$. Vertices in $N[v] - \{v\}$ are *neighbors* of v. A vertex v is said to dominate itself and all of its neighbors. A *dominating set* is a set $D \subseteq V_G$ such that every vertex in $V_G - D$ is dominated by a vertex in D. A *perfect dominating set* is a set $D \subseteq V_G$ such that every vertex in $V_G - D$ is dominated by exactly one vertex in D. For a finite graph G, the *domination number* $\gamma(G)$ is the minimum number of vertices in a dominating set in G. There is an extensive literature on domination in finite graphs, in which many variants of domination are defined and studied, for which the classical comprehensive reference is the two-volume series by Haynes, Hedetniemi, and Slater [1].

In this thesis, we consider domination on a class of infinite planar graphs called Archimedean lattices. A *regular tiling* is a tiling of the plane by regular polygons. Considering the vertices and edges of a regular tiling to be the vertices and edges of an infinite graph, an *Archimedean lattice* is a regular tiling which is vertex-transitive. Due to the restriction that the sum of the angles in polygons surrounding a vertex is 2π, there are only finitely many possibilities for regular polygons to surround a vertex, and only eleven of these can be continued indefinitely to form a vertex-transitive lattice. All eleven of the Archimedean lattices are illustrated in the figures in this thesis. There is a naming convention for the Archimedean lattices, in which the numbers of edges of the polygons incident to a vertex are listed in the order they appear around the vertex, with exponents indicating the number of successive polygons of a given size. The most commonly recognized Archimedean lattices are the square (4^4) lattice, the triangular (3^6) lattice, and the hexagonal (6^3) lattice. For a complete discussion, see the beautiful monograph by Grünbaum and Shephard [2, pp. 58–64].

Since the dominating set of an Archimedean lattice must be infinite, we will consider the domination ratio of an infinite graph, which is essentially the smallest proportion of vertices that constitute a dominating set. We will also consider the perfect domination ratio of an infinite graph, which is essentially the smallest proportion of vertices that constitute a perfect dominating set. The goal of this thesis is to exactly determine the domination ratio and the perfect domination ratio for as many Archimedean lattices as possible, and to find accurate bounds for those remaining.

A concept that is useful in our proofs is efficient domination. Let $|S|$ denote the cardinality of set S. A set $D \subseteq V_G$ is an *efficient dominating set* if $|N[v] \cap D| = 1$ for all $v \in V_G$. Thus, an efficient dominating set must dominate every vertex in the graph exactly once.

Each Archimedean lattice is a vertex-transitive graph, and thus is a k-regular graph, with $k = 3, 4, 5,$ or 6. If it is efficiently dominated, its domination ratio and perfect domination ratio both equal $\frac{1}{k+1}$. Chapter 3 shows that seven of the Archimedean lattices are efficiently dominated, determining their domination ratios and perfect domination ratios.

However, for a given graph, an efficient dominating set may not exist, as is proved for four of the Archimedean lattices. For those lattices, $\frac{1}{k+1}$ is a trivial lower bound, while the proportion of dominating vertices in any dominating set or perfect dominating set provides an upper bound for domination ratio and perfect domination ratio respectively. We exhibit examples to establish the best upper bounds that we have found. We prove that the perfect domination ratios for four of these graphs, the $(3,6,3,6)$, $(3,4,6,4)$, $(3^2,4,3,4)$, and $(4,6,12)$

lattices, are equal to $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{4}$ and $\frac{5}{18}$ respectively. Our results are summarized in Table 1.1.

Archimedean Lattice	Efficient Domination	γ_p
$(3, 12^2)$	Yes	$\frac{1}{4}$
$(4, 6, 12)$	No	$\frac{5}{18}$
$(4, 8^2)$	Yes	$\frac{1}{4}$
(6^3)	Yes	$\frac{1}{4}$
$(3, 4, 6, 4)$	No	$\frac{1}{4}$
$(3, 6, 3, 6)$	No	$\frac{1}{3}$
(4^4)	Yes	$\frac{1}{5}$
$(3^4, 6)$	Yes	$\frac{1}{6}$
$(3^2, 4, 3, 4)$	No	$\frac{1}{4}$
$(3^3, 4^2)$	Yes	$\frac{1}{6}$
(3^6)	Yes	$\frac{1}{7}$

Table 1.1: Results for the eleven Archimedean lattices. The column labeled "Efficient Domination" indicates whether or not there exists an efficient dominating set for the lattice. The column labeled γ_p provides the exact value of the perfect domination ratio for all of the lattices.

The remainder of the thesis is organized as follows:

In Chapter 2, periodic graphs are defined, then the domination ratio and the perfect domination ratio are defined for a periodic graph. Definitions, terminology, and lemmas that apply to all Archimedean lattices are provided.

The existence of efficient domination is determined for seven of the Archimedean lattices in Chapter 3. A proof for each of the seven of the Archimedean lattices is given in the form of a figure illustrating an efficient dominating set.

Our results on the $(3, 6, 3, 6)$ or kagome lattice are discussed in Chapter 4.

CHAPTER 1. INTRODUCTION

The kagome lattice is proved to not have an efficient dominating set. Bounds for the domination ratio of the kagome lattice are determined. The proof of the exact value of the perfect domination ratio of kagome lattice is provided. Nonisomorphic perfect dominating sets and possible perfect domination proportions are investigated.

Our results on the $(3, 4, 6, 4)$ lattice, the $(3^2, 4, 3, 4)$ lattice, and the $(4, 6, 12)$ lattice are provided in Chapter 5, Chapter 6, and Chapter 7 respectively, organized in a form similar to Chapter 4.

In Chapter 8, integer programming bounds for the domination ratio and perfect domination ratio of Archimedean lattices are discussed.

Chapter 9 breifly mentions some ongoing research and open questions.

Chapter 2

Preliminaries

2.1 Applications of Efficient and Perfect Domination

The existence of efficient dominating sets is studied in coding theory, since it is a variant of the classical problem of the existence and non-existence of perfect codes as a set in a vector space. A perfect e-error-correcting code of block length n over V is a subset $S \subseteq V^n$ such that for every $v \in V^n$ there exists a unique $u \in S$ with $d(u, v) \leq e$. A perfect 1-code in a graph is an efficient dominating set.

Perfect domination in a graph is a model for facility location problems. Consider a city represented by a graph G where vertices represent different locations or areas in the city. Every location is a potential site for a facility. Every pair of vertices representing adjacent locations are joined by an edge. Consider a company that wants to minimize the number of facilities such that each location is served by a facility in it or by a unique facility adjacent to it. The company's goal is to find a minimum perfect dominating set of G. A real world facility location problem may be of large scale and require a graph theory model with thousands of vertices. Studying perfect domination on infinite periodic graphs may provide insight into large scale facility location problems.

2.2 Periodicity

A *periodic graph* G is a locally-finite connected simple graph with a countably-infinite vertex set, which can be embedded in $\mathbb{R}^d$ for some $d < \infty$ such that G is invariant under translation by each unit vector in a coordinate axis direction in $\mathbb{R}^d$ and each compact set of $\mathbb{R}^d$ intersects only finitely many edges and vertices of G. Note that it is actually the embedding which is periodic. For convenience, we will identify a graph with its periodic embedding, although the properties of a dominating set only depend on the adjacency structure of the graph. Each of the eleven Archimedean lattices is a periodic graph in $\mathbb{R}^2$. Figures showing periodic embeddings of the Archimedean lattices are provided in [3] and throughout the thesis.

2.3 Existence of the Domination Ratio

2.3.1 Definition of the Domination Ratio

For a periodic graph G, denote the subgraph of G induced by the vertices in the rectangle $[m_1, m_2) \times [n_1, n_2) \subset \mathbb{R}^2$ by $R_G(m_1, m_2; n_1, n_2)$, where $m_1 < m_2, n_1 < n_2$ and $m_1, m_2, n_1, n_2 \in \mathbb{Z}$. Note that all induced subgraphs $R_G(m_1, m_2; n_1, n_2)$ corresponding to translations of rectangles with the same edge lengths are isomorphic. Denote the minimum size of a dominating set for $R_G(0, m; 0, n)$, known as its *domination number*, by $\gamma_{m,n}(G)$, and the number of vertices in $R(0, m; 0, n)$ by $N_{m,n}(G)$. Denote $N_{1,1}(G) = k$. We define the *domination ratio* of G by

$$\lim_{m,n\to\infty} \frac{\gamma_{m,n}(G)}{N_{m,n}(G)} = \inf_{r,s} \frac{1}{rsk} \gamma_{r,s}(G).$$

A proof that the limit exists relies on *subadditivity*. Let G_1 and G_2 be vertex-disjoint induced subgraphs of G. Since the union of dominating sets for G_1 and G_2 is a dominating set for G, but there might be a smaller dominating set for G,

$$\gamma(G_1 \cup G_2) \leq \gamma(G_1) + \gamma(G_2),$$

while

$$N(G_1 \cup G_2) = N(G_1) + N(G_2).$$

Together, these imply that, for example, doubling the length or width of the rectangle cannot increase the domination ratio of the subgraph, and may decrease it. Our literature search did not find a proof of the existence of the limit for deterministic multiparameter subadditive functions, but one may find a proof for the more difficult stochastic case in [4], which is modified appropriately in the following section.

To discuss bounds for the domination ratio, we need to consider dominating sets which are not minimum dominating sets. For a finite graph G that has a dominating set D, let its *domination proportion*, $\gamma_D(G)$, be the number of vertices in D divided by total number of vertices in G. We extend the notion of domination proportion to infinite periodic graphs. Given a dominating set, suppose the vertex set of an infinite graph can be partitioned into finite subsets such that the subgraph induced by each subset is connected and all these finite induced subgraphs have the same domination proportion. The domination proportion of the dominating set is defined as the common value of the domination proportion of the finite induced subgraphs.

For the induced subgraphs, we require the same domination proportion and connectedness to avoid ambiguity arising from one-to-one or many-to-one correspondences between subgraphs, which can be used to obtain different domination proportions for all the subgraphs.

If the same domination proportion is not required for the induced subgraphs, we will have the following issue: For simplicity, assume the domination proportion of induced subgraphs are either γ_1 or γ_2, where $\gamma_1 \neq \gamma_2$. We can pair every induced subgraph having domination proportion γ_1 with two induced subgraphs having domination proportion γ_2 to obtain $\frac{\gamma_1+2\gamma_2}{3}$ as the domination proportion of the infinite periodic graph. Similarly, we can pair every induced subgraph having domination proportion γ_1 with three induced subgraphs having domination proportion γ_2 to obtain $\frac{\gamma_1+3\gamma_2}{4}$ as the domination proportion of the infinite periodic graph. Therefore, the domination proportion of an infinite periodic graph is not well defined if the same domination proportion is not required for the induced subgraphs.

If connectedness is not required for the induced subgraphs, we will have the following issue: For simplicity, assume every induced subgraph is the disjoint union of two connected components. The two connected components may have different domination proportions, γ_1 and γ_2 respectively. The same reasoning as in the previous paragraph can be applied to show that the domination proportion of an infinite periodic graph is not defined if connectedness is not required for the induced subgraphs.

2.3.2 A Proof that Domination Ratio Exists

Let $k = N_{1,1}(G)$. Fix positive integers r and s. Any integers m and n sufficiently large may be expressed as

$$m = \alpha r + \beta, \text{ where } \alpha = \left\lfloor \frac{m}{r} \right\rfloor \text{ and } 0 \leq \beta < r$$

$$n = \rho s + \sigma, \text{ where } \rho = \left\lfloor \frac{n}{s} \right\rfloor \text{ and } 0 \leq \sigma < s$$

When we divide m by r, we obtain α as the quotient and β as the remainder. When we divide n by s, we get ρ as the quotient and σ as the remainder. The vertex set of the rectangular region $R_G(0, m; 0, n)$ is the disjoint union of vertex sets of rectangular regions listed below [4]. Figure 2.1 illustrates the reasoning.

$$R_{ij} = R_G(((i-1)r, (j-1)s), (ir, js)), \text{ where } 1 \leq i \leq \alpha, 1 \leq j \leq \rho$$

$$S_i = R_G(((i-1)r, \rho s), (ir, \rho s + \sigma)), \text{ where } 1 \leq i \leq \alpha$$

$$T_j = R_G((\alpha r, (j-1)s), (\alpha r + \beta, js)), \text{ where } 1 \leq j \leq \rho$$

$$U = R_G((\alpha r, \rho s), (\alpha r + \beta, \rho s + \sigma))$$

The rectangular regions are labeled in Figure 2.1. For simplicity, we do not label all of R_{ij} in Figure 2.1.

Using subadditivity and the fact that the domination number of a subgraph

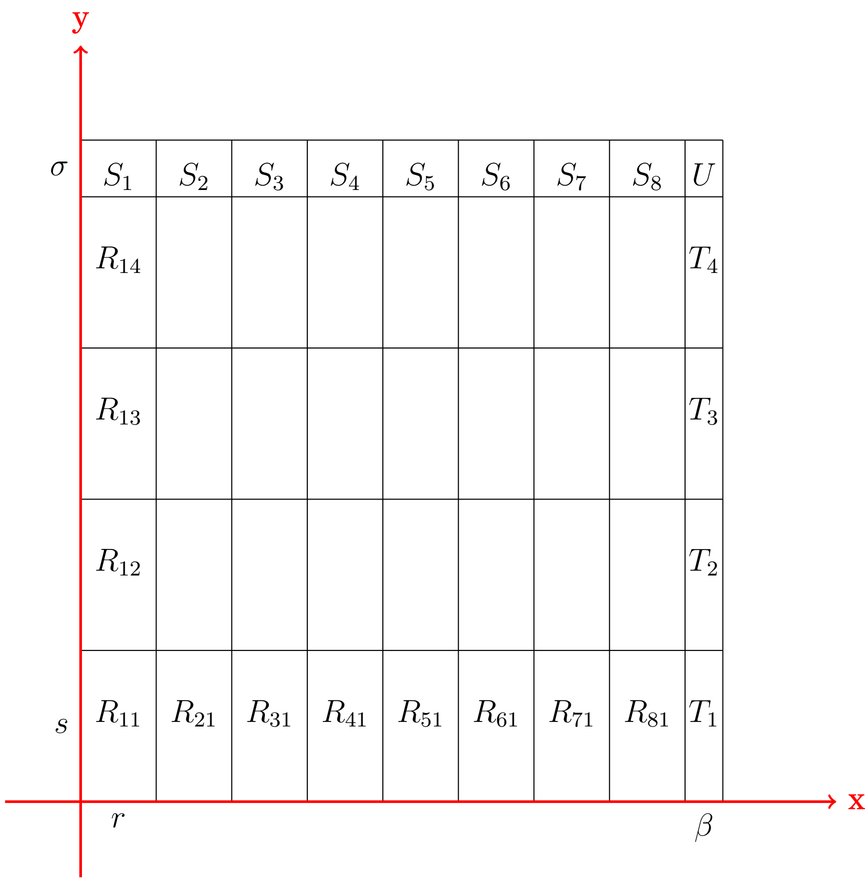

Figure 2.1: An illustration of the proof that the domination ratio exists.

is no greater than the number of vertices in the subgraph, we deduce that

$$\gamma_{m,n}(G) \leq \sum_{i=1}^{\alpha} \sum_{j=1}^{\rho} \gamma(R_{ij}) + (\alpha\sigma r + \rho\beta s + \beta\sigma)k.$$

Notice $\gamma(R_{ij})$ is the same for all R_{ij} from periodicity and the embedding of the graph. Furthermore, $\gamma(R_{ij}) = \gamma_{r,s}(G)$.

$$\gamma_{m,n}(G) \leq \alpha\rho\gamma_{r,s}(G) + (\alpha\sigma r + \rho\beta s + \beta\sigma)k.$$

$$\frac{\gamma_{m,n}(G)}{N_{m,n}(G)} \leq \frac{\alpha\rho\gamma_{r,s}(G)}{N_{m,n}(G)} + \frac{(\alpha\sigma r + \rho\beta s + \beta\sigma)k}{N_{m,n}(G)}.$$

Since $N_{m,n}(G) = mnk$, we have

$$\frac{\gamma_{m,n}(G)}{N_{m,n}(G)} \leq \frac{\alpha\rho\gamma_{r,s}(G)}{mnk} + \frac{\alpha\sigma r + \rho\beta s + \beta\sigma}{mn}.$$

$$\liminf_{m,n\to\infty} \frac{\gamma_{m,n}(G)}{N_{m,n}(G)} \leq \liminf_{m,n\to\infty} \frac{\alpha\rho\gamma_{r,s}(G)}{mnk} + \liminf_{m,n\to\infty} \frac{\alpha\sigma r + \rho\beta s + \beta\sigma}{mn}.$$

Because $\alpha \leq m$ and σ, r are fixed, as $m, n \to \infty$, we have $\frac{\alpha\sigma r}{mn} = \frac{\alpha}{m} \times \frac{\sigma r}{n} \to 0$.

Because $\rho \leq n$ and β, s are fixed, as $m, n \to \infty$, we have $\frac{\rho\beta s}{mn} = \frac{\rho}{n} \times \frac{\beta s}{m} \to 0$.

Because β, σ are fixed, as $m, n \to \infty$, we have $\frac{\beta\sigma}{mn} \to 0$.

Adding up all three terms, as $m, n \to \infty$, we have $\frac{\alpha\sigma r+\rho\beta s+\beta\sigma}{mn} \to 0$. Therefore we have

$$\liminf_{m,n\to\infty} \frac{\gamma_{m,n}(G)}{N_{m,n}(G)} \leq \liminf_{m,n\to\infty} \frac{\alpha\rho}{mnk}\gamma_{r,s}(G).$$

Since $\alpha r \leq m$ and $\rho s \leq n$, we have $\alpha r \rho s \leq mn$, so

$$\frac{\alpha\rho}{mnk}\gamma_{r,s}(G) \leq \frac{1}{rsk}\gamma_{r,s}(G).$$

$$\liminf_{m,n\to\infty} \frac{\alpha\rho}{mnk}\gamma_{r,s}(G) \leq \liminf_{m,n\to\infty} \frac{1}{rsk}\gamma_{r,s}(G).$$

Because $\frac{1}{rsk}\gamma_{r,s}(G)$ does not depend on m, n, we have

$$\liminf_{m,n\to\infty} \frac{\alpha\rho}{mnk}\gamma_{r,s}(G) \leq \frac{1}{rsk}\gamma_{r,s}(G).$$

Because the inequality above holds for any r, s, we have

$$\liminf_{m,n\to\infty} \frac{\alpha\rho}{mnk}\gamma_{r,s}(G) \leq \inf_{r,s} \frac{1}{rsk}\gamma_{r,s}(G).$$

$$\liminf_{m,n\to\infty} \frac{\gamma_{m,n}(G)}{N_{m,n}(G)} \leq \liminf_{m,n\to\infty} \frac{\alpha\rho}{mnk}\gamma_{r,s}(G) \leq \inf_{r,s} \frac{1}{rsk}\gamma_{r,s}(G).$$

Since $\frac{\gamma_{m,n}(G)}{N_{m,n}(G)} = \frac{1}{rsk}\gamma_{r,s}(G)$ when $r = m, s = n$, we have

$$\frac{\gamma_{m,n}(G)}{N_{m,n}(G)} \geq \inf_{r,s} \frac{1}{rsk}\gamma_{r,s}(G),$$

and therefore

$$\liminf_{m,n\to\infty} \frac{\gamma_{m,n}(G)}{N_{m,n}(G)} \geq \inf_{r,s} \frac{1}{rsk}\gamma_{r,s}(G).$$

Thus, we conclude the limit exists and

$$\lim_{m,n\to\infty} \frac{\gamma_{m,n}(G)}{N_{m,n}(G)} = \inf_{r,s} \frac{1}{rsk}\gamma_{r,s}(G).$$

2.3.3 Why This Definition ?

At first glance, one might think our proof that domination ratio exists has a counterexample, an infinite row of the kagome lattice shown in Figure 2.2. Even though efficient domination of the kagome lattice is not possible, the infimum definition would yield a ratio of $\frac{1}{5}$ (the domination ratio of the kagome lattice if perfect domination were possible). The infinite row of the kagome lattice is not a valid counterexample, because the definition of domination ratio is restricted to infimum over subgraphs induced by vertices in rectangles, where a rectangle must be a period of the embedding and it is not in the example. Recall from Section 2.3.1 that an Archimidean lattice can be embedded in a plane such that all induced subgraphs corresponding to translations of rectangles with the same edge lengths are isomorphic. The infinite row of the kagome lattice is not a subgraph induced by vertices in a rectangular region.

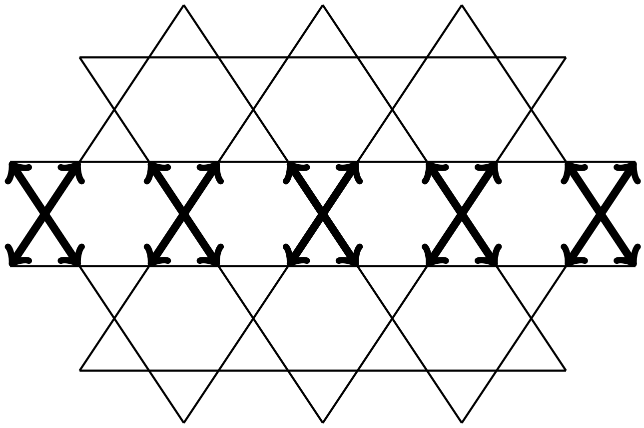

Figure 2.2: An induced row of the kagome lattice can have a domination ratio of $\frac{1}{5}$ even though efficient domination of the kagome lattice is not possible.

2.3.4 Generalized Results

Corollary 2.3.1: *If a bounded function $f(m,n)$ is subadditive, where m,n are length and width of a rectangular region in an infinite periodic graph, then $f(m,n)$ has a limit as $m,n \to \infty$, and the limit equals $\inf_{r,s} \frac{1}{rsk} f(r,s)$.*

Proof: Let $f(m,n)$ be a bounded subadditive function, where m,n are length and width of a rectangular region in an infinite periodic graph. The proof of the existence of the domination ratio in Section 2.3.2 can be applied to show that $f(m,n)$ has a limit as $m,n \to \infty$. One may replace $\gamma_{m,n}$ in the proof in Section 2.3.2 by $f(m,n)$ and obtain $\inf_{r,s} \frac{1}{rsk} f(r,s)$ as the limit. □

Corollary 2.3.2: *If a bounded function $f(m,n)$ is superadditive, where m,n are length and width of a rectangular region in an infinite periodic graph, then $f(m,n)$ has a limit as $m,n \to \infty$, and the limit equals $\sup_{r,s} \frac{1}{rsk} f(r,s)$.*

Proof: Let $f(m,n)$ be a bounded superadditive function, where m,n are length and width of a rectangular region in an infinite periodic graph. Notice that $-f(m,n)$ is subadditive. By Corollary 2.3.1, $-f(m,n)$ has a limit as $m,n \to \infty$, and the limit equals $\inf_{r,s} \frac{1}{rsk}\{-f(r,s)\}$. Thus, $f(m,n)$ has a limit as $m,n \to \infty$, and the limit equals $\sup_{r,s} \frac{1}{rsk} f(r,s)$. $\square$

2.3.5 Different Periodic Embeddings Yield the Same Domination Ratio

Let A and B be two periodic embeddings of an infinite graph G. Let $\gamma(G_A)$ and $\gamma(G_B)$ denote the domination ratio of G yielded by A and B respectively. The two periodic embeddings A and B provide two sets of (x,y) axes that may have different scales and angles between the x-axis and the y-axis. We can embed the infinite periodic graph in the plane such that the x-axis and the y-axis corresponding to periodic embedding A are orthogonal. Let coordinate-A and coordinate-B denote the coordinate system that correspond to the set of (x,y) axes provided by periodic embeddings A and B respectively. Recall that $R_G(m_1,m_2;n_1,n_2)$ denotes the subgraph of G induced by the vertices in the rectangle $[m_1,m_2) \times [n_1,n_2) \subset \mathbb{R}^2$. For simplicity, we denote $R_G(m_1,m_2;n_1,n_2)$ in coordinate-A and in coordinate-B by $R_A(m_1,m_2;n_1,n_2)$ and $R_B(m_1,m_2;n_1,n_2)$ respectively.

A rectangular region $R_B(0,m;0,n)$ is a parallelogram in coordinate-A. Figure 2.3 illustrates the reasoning. Fix positive integers r,s. The origin in coordinate-B is in a $r\times s$ rectangle whose vertices have integer coordinates in coordinate-A. Let $R_A(\alpha r,\beta s;\alpha r+r,\beta s+s)$ denote the rectangular region that contains the origin in coordinate-B, where $\alpha,\beta\in\mathbb{Z}$. Similarly, let points $(m,0),(m,n),(0,n)$ in coordinate-B be in rectangular regions:

$$R_A(\alpha r+\gamma r,\beta s+\delta s;\alpha r+\gamma r+r,\beta s+\delta s+s)$$

$$R_A(\alpha r+\gamma r+\theta r,\beta s+\delta s+\lambda s;\alpha r+\gamma r+\theta r+r,\beta s+\delta s+\lambda s+s)$$

$$R_A(\alpha r+\theta r,\beta s+\lambda s;\alpha r+\theta r+r,\beta s+\lambda s+s)$$

respectively, where $\alpha,\beta,\gamma,\delta,\theta,\lambda\in\mathbb{Z}$.

Notice a union of rectangles with length r and width s in coordinate-A has $R_B(0,m;0,n)$ as a subgraph. Let k denote the minimum number of rectangles with length r and width s in coordinate-A whose union has $R_B(0,m;0,n)$ as a subgraph. Recall that $\gamma_{m,n}(G)$ denotes the domination number of $R_G(0,m;0,n)$, and $N_{m,n}(G)$ denotes the number of vertices in $R_G(0,m;0,n)$. For simplicity, we denote $\gamma_{m,n}(G)$ and $N_{m,n}(G)$ in coordinate-A by $\gamma_{m,n}(A)$ and $N_{m,n}(A)$ respectively. Similarly, we denote $\gamma_{m,n}(G)$ and $N_{m,n}(G)$ in coordinate-B by $\gamma_{m,n}(B)$ and $N_{m,n}(B)$ respectively.

Since a union of k rectangles with length r and width s in coordinate-A has $R_B(0,m;0,n)$ as a subgraph,

$$0 \leq kN_{r,s}(A) - N_{m,n}(B).$$

Notice every rectangle in the union contains some vertices in $R_B(0,m;0,n)$, otherwise a union of less than k rectangles with length r and width s in coordinate-A has $R_B(0,m;0,n)$ as a subgraph, contradicting that k is the minimum number of $r \times s$ rectangles required. Since $2(\gamma+\theta+\delta+\lambda)$ rectangles with length r and width s can cover all vertices on the internal boundary of $R_B(0,m;0,n)$, at most $2(\gamma+\theta+\delta+\lambda)$ rectangles in the union contains vertices not in $R_B(0,m;0,n)$.

$$kN_{r,s}(A) - N_{m,n}(B) \leq 2(\gamma+\theta+\delta+\lambda)N_{r,s}(A).$$

$$0 \leq kN_{r,s}(A) - N_{m,n}(B) \leq 2(\gamma+\theta+\delta+\lambda)N_{r,s}(A).$$

$$N_{m,n}(B) \leq kN_{r,s}(A) \leq N_{m,n}(B) + 2(\gamma+\theta+\delta+\lambda)N_{r,s}(A).$$

$$1 \leq \frac{kN_{r,s}(A)}{N_{m,n}(B)} \leq 1 + \frac{2(\gamma+\theta+\delta+\lambda)N_{r,s}(A)}{N_{m,n}(B)}.$$

where $N_{m,n}(B) = \Theta(mn)$ and $\gamma+\theta+\delta+\lambda = \Theta(m+n)$. Since $2N_{r,s}(A)$ is a fixed positive integer, as $m, n \to \infty$, we have

$$\frac{2(\gamma + \theta + \delta + \lambda)N_{r,s}(A)}{N_{m,n}(B)} \to 0.$$

Therefore, as $m, n \to \infty$,

$$\frac{kN_{r,s}(A)}{N_{m,n}(B)} \to 1.$$

Using subadditivity and the fact that domination number of a graph is no smaller than the domination number of its subgraph, we deduce that

$$k\gamma_{r,s}(A) \geq \gamma_{m,n}(B).$$

$$\frac{kN_{r,s}(A)}{N_{m,n}(B)} \times \frac{k\gamma_{r,s}(A)}{kN_{r,s}(A)} \geq \frac{\gamma_{m,n}(B)}{N_{m,n}(B)}.$$

As $m, n \to \infty$, $\frac{kN_{r,s}(A)}{N_{m,n}(B)} \to 1$. Therefore we have

$$\lim_{m,n\to\infty} \frac{k\gamma_{r,s}(A)}{kN_{r,s}(A)} \geq \lim_{m,n\to\infty} \frac{\gamma_{m,n}(B)}{N_{m,n}(B)}.$$

where the existence of the limit is proved in section 2.3.2.

$$\lim_{m,n\to\infty} \frac{\gamma_{r,s}(A)}{N_{r,s}(A)} \geq \lim_{m,n\to\infty} \frac{\gamma_{m,n}(B)}{N_{m,n}(B)}.$$

Since $\lim_{m,n\to\infty} \frac{\gamma_{m,n}(B)}{N_{m,n}(B)} = \gamma(G_B)$ and $\frac{\gamma_{r,s}(A)}{N_{r,s}(A)}$ is independent of m, n, we have

$$\frac{\gamma_{r,s}(A)}{N_{r,s}(A)} \geq \gamma(G_B).$$

$$\inf_{r,s} \frac{\gamma_{r,s}(A)}{N_{r,s}(A)} \geq \gamma(G_B).$$

Since $\inf_{r,s} \frac{\gamma_{r,s}(A)}{N_{r,s}(A)} = \gamma(G_A)$, we have

$$\gamma(G_A) \geq \gamma(G_B).$$

Similarly, we can embed the infinite periodic graph on a plane such that the x-axis and the y-axis corresponding to the subgraph B are orthogonal. The same reasoning can be applied to show that $\gamma(G_A) \leq \gamma(G_B)$. Thus, $\gamma(G_A) = \gamma(G_B)$.

2.4 Existence of the Perfect Domination Ratio

Definition (Internal boundary): Given a graph G with a subgraph H, the *internal boundary* of H is the set of vertices in H which are adjacent to some vertex outside H.

Definition (Dominated for free): Given a graph $G = (V, E)$, a vertex $v \in V$

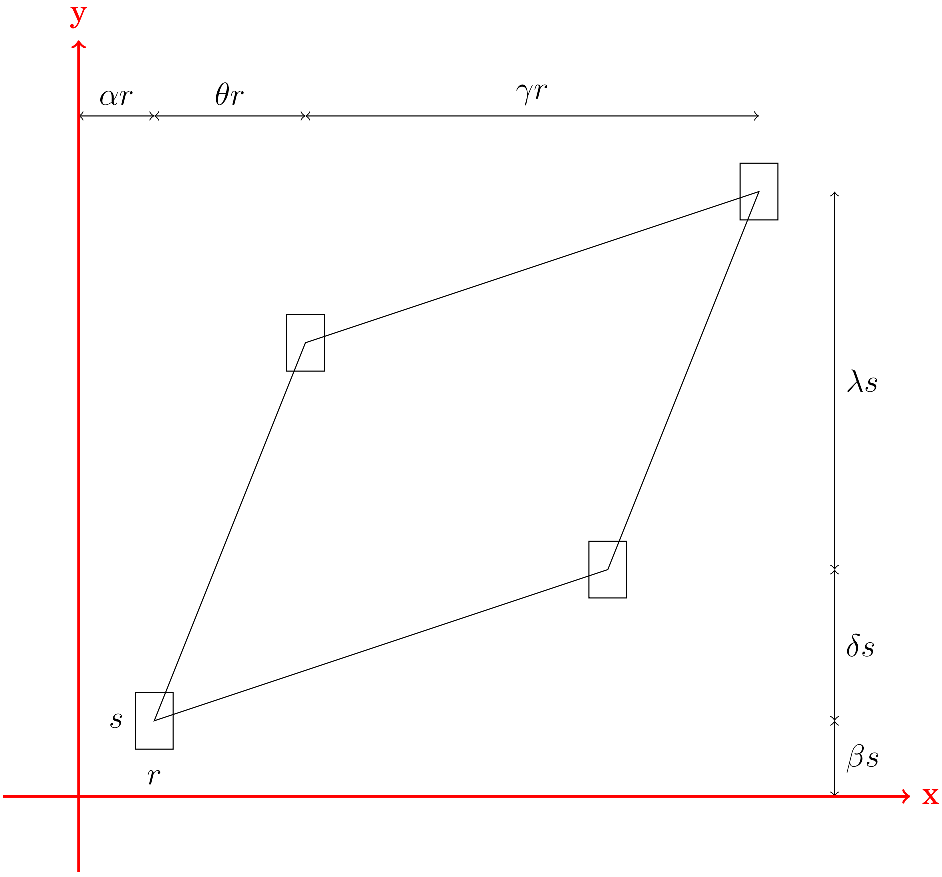

Figure 2.3: A rectangle $R_B(0, m; 0, n)$ is a parallelogram in coordinate-A.

is *dominated for free* means that we accept F as a dominating set for G if F is a dominating set for $G \setminus v$.

For a periodic graph G, let $R_G(m_1, m_2; n_1, n_2)$, where $m_1 \leq m_2, n_1 \leq n_2$, denote the subgraph of G induced by the vertices in the rectangle $[m_1, m_2) \times [n_1, n_2) \subset \mathbb{R}^2$. Note that all induced subgraphs $R_G(m_1, m_2; n_1, n_2)$ with corresponding to rectangles with the same edge lengths are isomorphic. Denote the minimum size of a perfect dominating set for $R_G(0, m; 0, n)$, known as its *perfect domination number*, by $\gamma_{p;m,n}(G)$, and the number of vertices in $R(0, m; 0, n)$ by $N_{m,n}(G)$. We define the *perfect domination ratio* of G by

$$\lim_{m,n \to \infty} \left\{ \frac{\gamma_{p;m,n}(G)}{N_{m,n}(G)} \right\},$$

To prove the limit exists, we consider a variant of the perfect domination ratio. Assume vertices in the internal boundary of graphs are dominated for free, and boundary vertices can still dominate other vertices if they are in a perfect dominating set. Denote the minimum size of a perfect dominating set under the condition above by $\gamma^B_{p;m,n}(G)$. We define the variant of the perfect domination ratio by

$$\lim_{m,n \to \infty} \left\{ \frac{\gamma^B_{p;m,n}(G)}{N_{m,n}(G)} \right\},$$

A proof that the limit exists relies on *superadditivity*. Let G_1 and G_2 denote vertex-disjoint induced subgraphs of G. Let S denote the minimum perfect dominating set of $G_1 \cup G_2$ with internal boundary dominated for free. Let $S_1 = S \cap V(G_1)$ and $S_2 = S \cap V(G_2)$. Since S_1 is a perfect dominating set of G_1, we have $|S_1| \geq \gamma_p^B(G_1)$. Similarly, $|S_2| \geq \gamma_p^B(G_2)$. Therefore,

$$\gamma_p^B(G_1 \cup G_2) \geq \gamma_p^B(G_1) + \gamma_p^B(G_2),$$

while

$$N(G_1 \cup G_2) = N(G_1) + N(G_2).$$

Together, these imply that, for example, doubling the length or width of the rectangle cannot decrease the variant of the perfect domination ratio of the subgraph, and may increase it. By Corollary 2.3.2, $\gamma^B_{p;m,n}(G)$ has a limit as $m, n \to \infty$, and the limit equals $\sup_{r,s} \frac{1}{rsk}\gamma^B_{p;r,s}(G)$.

As the length and width of the rectangle approach infinity, one may apply the same reasoning as in Section 2.3.2 to show the proportion of vertices on the internal boundary approaches zero. Therefore, the perfect domination ratio approaches a limit as $m, n \to \infty$, and the limit is

$$\lim_{m,n\to\infty} \left\{ \frac{\gamma_{p;m,n}(G)}{N_{m,n}(G)} \right\} = \lim_{m,n\to\infty} \left\{ \frac{\gamma^B_{p;m,n}(G)}{N_{m,n}(G)} \right\} = \sup_{r,s} \frac{1}{rsk}\gamma^B_{p;r,s}(G).$$

The perfect domination ratio is the same regardless of the choice of the periodic embedding. One can modify the proof in Section 2.3.4 to obtain desired result. In particular, let k denote the maximum number of disjoint $r \times s$ rectangles whose vertices have integer coordinates in coordinate-A that are subgraphs of $R_B(0, m; 0, n)$. In addition, the proof replies on superadditivity instead of subadditivity, which we used in the domination ratio case.

2.5 Definitions and Preliminaries

We now provide some definitions, terminology, and lemmas that apply to perfect domination on all the Archimedean lattices.

If a graph G has vertex set $V(G)$ and edge set $E(G)$, for simplicity we will write $v \in G$ rather than $v \in V(G)$ and write $e \in G$ rather than $e \in E(G)$.

In the remainder of this thesis, we will abbreviate perfect dominating set as "PDS." As for any graph, given a PDS D in a graph G, the subgraph of G induced by vertices in D is a disjoint union of connected components. Our proofs use certain features of the structure of the boundary of the components, described in the remainder of this section.

Definition (D_n): Given a PDS D, let D_n denote a connected component of size n in the subgraph induced by vertices in D.

Note: For a fixed positive integer n, there may exist components D_n which are not isomorphic. An example of nonisomorphic D_n is shown in Figure 5.4. A D_6 in the figure on the left is not isomorphic to a D_6 in the figure on the right.

Fortunately, in our graphs, this does not happen when n is small.

Definition (Graph distance): For two vertices v and u in a graph G, let $d_G(v, u)$ denote the number of edges in the shortest path between v and u. For a vertex v and a subgraph S of G, define $d_G(v, S) = \min_{u \in S}\{d_G(v, u)\}$. For brevity, when the graph G is clear from the context, we omit the subscript G.

Definition (External boundary): Given a subgraph S in a graph G, define the *external boundary* as the set of vertices v such that $d_G(v, S) = 1$.

Definition (Double external boundary): Given a subgraph S in a graph G, define the *double external boundary* as the set of vertices v such that $d_G(v, S) = 1$ or 2.

Lemma 2.4.1: *Given a component D_n in a PDS D, no vertex in the double external boundary of D_n is in D.*

Proof: Let v be in the double external boundary of D_n.

If $d(v, D_n) = 1$, then v is adjacent to a vertex in D_n and thus is in the component D_n, contradicting $d(v, D_n) = 1$. Therefore, no vertex in the external boundary is in D.

If $d(v, D_n) = 2$, there exists a path of length two with vertices v, w, and x, where $w \notin D_n$ and $x \in D$. If $v \in D$, then vertex w is dominated by both v and x. Thus, $w \in D$ and thus also in D_n. This implies that $v \in D_n$ also, contradicting that v is in the double external boundary of D_n. □

Lemma 2.4.2: *Given a PDS D, if $v \notin D$, u is a neighbor of v, and every other neighbor of v is not in D, then $u \in D$.*

Proof: To deduce a contradiction, suppose $u \notin D$. Then v is not dominated by any vertex in D, contradicting the assumption that D is a dominating set. □

Definition (pulls in): Let v *pulls in* u indicate that for a PDS D and a vertex $v \notin D$, u is a neighbor of v and every other neighbor of v is not in D, requiring that $u \in D$ by Lemma 3.2.

By the definition of PDS, a vertex that is not in the PDS must be dominated exactly once. Thus, given a PDS D, if a vertex v has two neighbors u and w in D, then $v \in D$.

Definition (double force in): Let u and w *double force in* v indicate that for a PDS D, if a vertex v has two neighbors u and w in D, then $v \in D$.

Lemma 2.4.3: *Given a PDS D, if a vertex $v \notin D$ has a neighbor $u \in D$, then no other neighbor of v is in D*

Proof: Suppose v has another neighbor $w \in D$. Then v is dominated by both u and w, contradicting the assumption that D is a PDS. □

Definition (forces out): Let v *and* u *force out* w indicate that if vertex $v \notin D$ has a neighbor $u \in D$, then another neighbor w of v is not in D.

Note: In each of chapters 4, 5, 6, and 7, we consider a specific Archimedean lattice. In each chapter, the notations such as PDS, γ_p, and D_n refer to only that specific lattice.

2.6 How Our Proof Uses the Definition of the Perfect Domination Ratio

In the remainder of this thesis, we determine the exact value of the perfect domination ratio for all of the Archimedean lattices. For each Archimedean lattice that is efficiently dominated, the perfect domination ratio is $\frac{1}{k+1}$ if it is a k-regular lattice. Details on efficiently dominated lattices are discussed in Chapter 3. For an Archimedean lattice G that is not efficient dominated, we exhibit a PDS D and prove that $\gamma_p(G) = \gamma_p(D)$ as follows.

To deduce a contradiction, suppose $\gamma_p(G) < \gamma_p(D)$. Then there exists a PDS D' such that $\gamma_p(D') = \gamma_p(G) < \gamma_p(D)$. We demonstrate that D' must contain a certain component D_n (typically a D_1). This D_n forces certain structure around it, which requires more vertices in D. Therefore, this D_n forces the perfect domination proportion of a large subgraph around it to be above $\gamma_p(D)$. Since the perfect domination ratio is defined as

$$\lim_{m,n\to\infty} \left\{ \frac{\gamma_{p;m,n}(G)}{N_{m,n}(G)} \right\},$$

as $m, n \to \infty$, the large subgraph around this D_n will be included in $R_{0,m;0,n}$, contradicting that $\gamma_p(D') < \gamma_p(D)$. Therefore, we conclude that $\gamma_p(G) = \gamma_p(D)$.

Chapter 3

Existence of Efficient Domination

It is well-known that for finite graphs, efficient domination is optimal domination, and all efficient dominating sets have the same cardinality [1]. Since the definition of domination ratio for infinite periodic graphs is in terms of domination numbers for finite graphs, all efficient dominating sets are optimal and have the same domination ratio.

Existence of an efficient perfect dominating set was previously proved for the three most common Archimedean lattices – the square (4^4) lattice [5,6], the triangular (3^6) lattice [7], and the hexagonal (6^3) lattice [8]. For completeness, we illustrate the efficient dominating sets in these three lattices in Figures 3.1 – 3.3. In Figures 3.4 – 3.7, we illustrate efficient dominating sets for the $(3, 12^2)$,

$(4, 8^2)$, $(3^4, 6)$, and $(3^3, 4^2)$ lattices, respectively. Each of the figures shows a subgraph of the lattice that is sufficiently large to demonstrate a periodic pattern that can be extended to efficiently dominate the infinite lattice. In each of the figures, a star with bold edges is centered at each vertex in the dominating set, with the edges with arrows pointing to vertices that are dominated by the central vertex. Notice that every non-central vertex is the endpoint of exactly one arrow, so every vertex is dominated exactly once.

Since they are vertex-transitive, each of the Archimedean lattices is a regular graph. Each is k-regular for some $k = 3, 4, 5$ or 6. For each of the seven Archimedean lattices which can be efficiently dominated, the domination ratio is $1/(k+1)$ if it is a k-regular lattice, since each vertex in the dominating set dominates itself and precisely k neighbors, and no vertex is dominated more than once. Notice an efficient dominating set is a perfect dominating set, since every vertex is dominated exactly once. Therefore, for each of the seven Archimedean lattices which can be efficiently dominated, the perfect domination ratio is $1/(k+1)$ if it is a k-regular lattice, since each vertex in the efficient dominating set dominates itself and precisely k neighbors, and no vertex is dominated more than once.

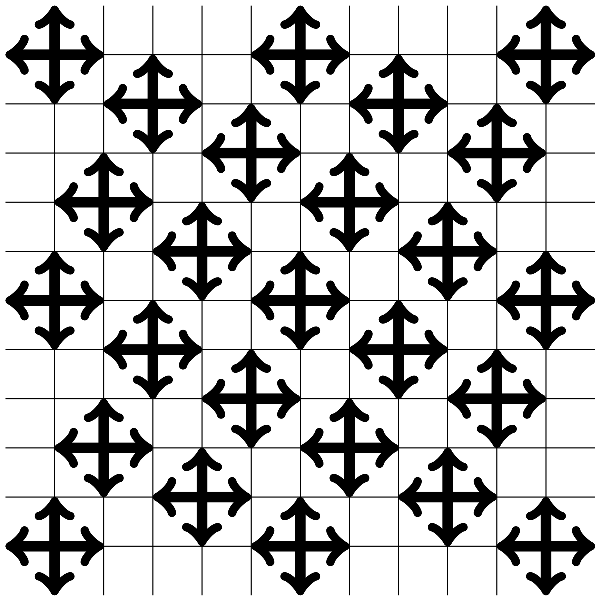

Figure 3.1: An efficient dominating set in the square lattice.

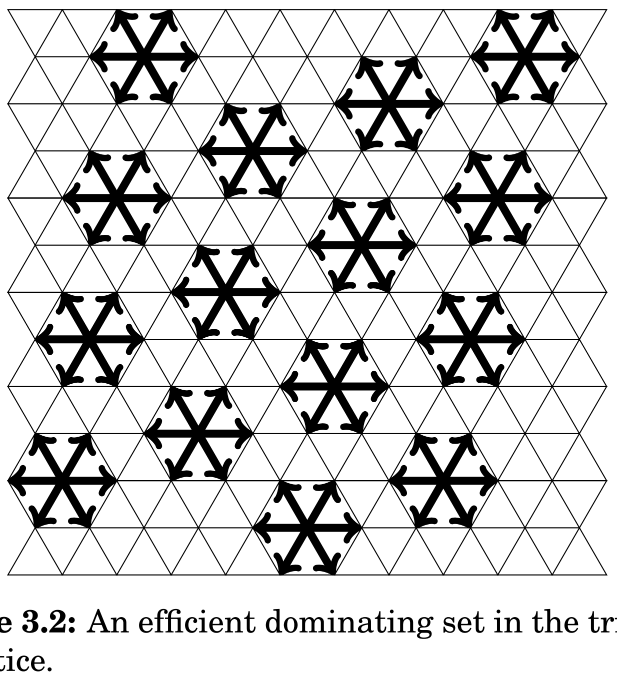

Figure 3.2: An efficient dominating set in the triangular lattice.

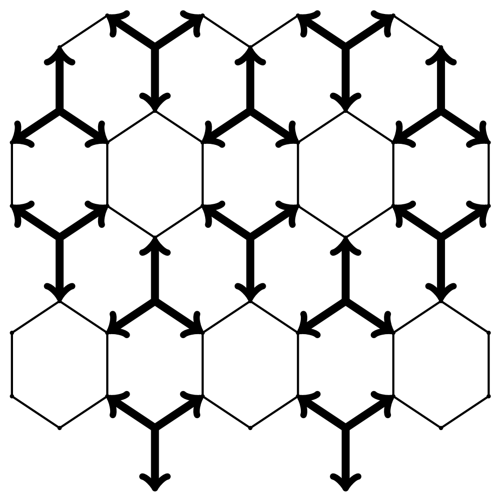

Figure 3.3: An efficient dominating set in the hexagonal lattice.

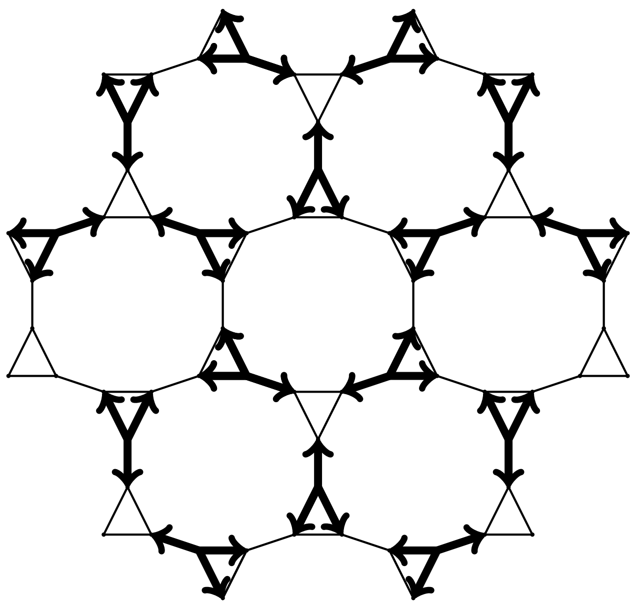

Figure 3.4: An efficient dominating set in the $(3, 12^2)$ lattice.

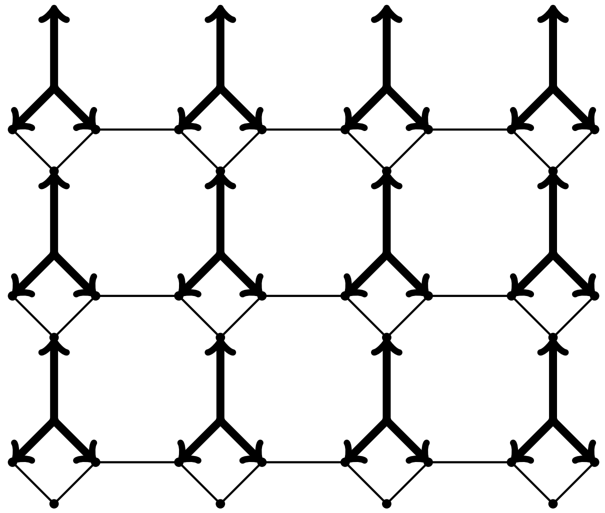

Figure 3.5: An efficient dominating set in the $(4, 8^2)$ lattice.

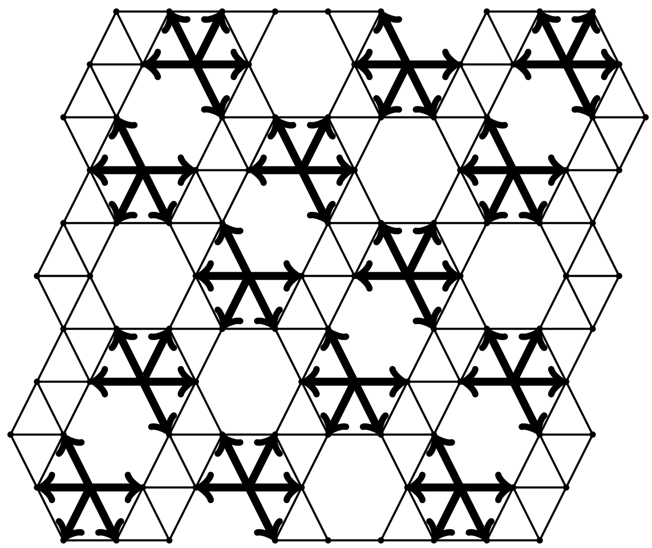

Figure 3.6: An efficient dominating set in the $(3^4, 6)$ lattice.

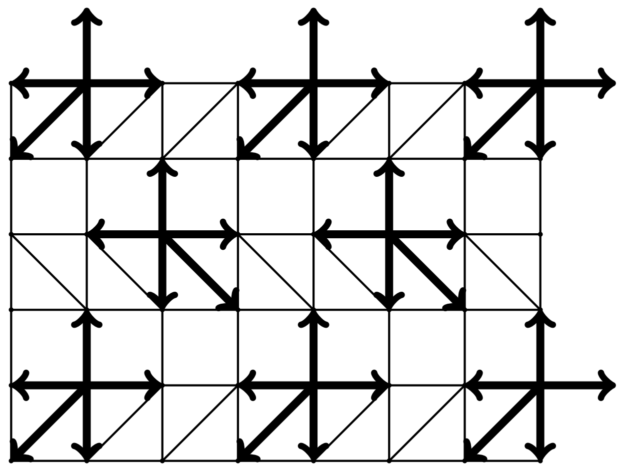

Figure 3.7: An efficient dominating set in the $(3^3, 4^2)$ lattice. Note that, for convenience, the lattice is drawn in a periodic rectangular structure, rather than using regular polygons.

Chapter 4

The $(3, 6, 3, 6)$ or Kagome Lattice

4.1 Nonexistence of Efficient Domination

Lemma 4.1.1: *There does not exist an efficient dominating set in the* $(3, 6, 3, 6)$ *lattice.*

Proof: The proof is by contradiction. Assume that there exists an efficient dominating set D. Since $D \neq \emptyset$, there exists a vertex $v_1 \in D$. Figure 4.1 illustrates the reasoning. By vertex-transitivity, any vertex may be chosen to represent v_1.

Vertex v_2 is adjacent to a vertex in $N[v_1]$, so $v_2 \notin D$ or the adjacent vertex would be dominated by both v_1 and v_2. Therefore, v_2 must be dominated by one

of its neighbors. The only neighbors v for which $N[v] \cap N[v_1] = \emptyset$ are v_3 and v_4. So if D is to be an efficient dominating set, either $v_3 \in D$ or $v_4 \in D$, but not both.

Consider the case $v_3 \in D$. Vertex v_5 is adjacent to a vertex in $N[v_1]$, so $v_5 \notin D$. However, every neighbor v of v_5 satisfies either $N[v] \cap N[v_1] \neq \emptyset$ or $N[v] \cap N[v_3] \neq \emptyset$, so there does not exist any vertex $v \in D$ such that $v \in N[v_5]$. Since there is no $v \in D$ which dominates v_5, D is not a dominating set, and thus not an efficient dominating set, contradicting our original assumption.

Consider the case $v_4 \in D$. Vertex v_6 is adjacent to a vertex in $N[v_1]$, so $v_6 \notin D$. The only neighbors v for which $N[v] \cap N[v_1] = \emptyset$ are v_8 and v_9. So if D is to be an efficient dominating set, either $v_8 \in D$ or $v_9 \in D$, but not both.

If $v_8 \in D$, then v_{10} is adjacent to a vertex in $N[v_8]$, so $v_{10} \notin D$. However, every neighbor v of v_{10} satisfies either $N[v] \cap N[v_1] \neq \emptyset$ or $N[v] \cap N[v_8] \neq \emptyset$, so there does not exist any vertex $v \in D$ such that $v \in N[v_{10}]$. Thus, v_{10} cannot be dominated.

If $v_9 \in D$, then v_7 is adjacent to a vertex in $N[v_9]$, so $v_7 \notin D$. However, every neighbor v of v_7 satisfies either $N[v] \cap N[v_4] \neq \emptyset$ or $N[v] \cap N[v_9] \neq \emptyset$, so there does not exist any vertex $v \in D$ such that $v \in N[v_7]$. Thus, v_7 cannot be dominated.

Thus, every case leads to the contradication that D cannot be a dominating set. $\square$

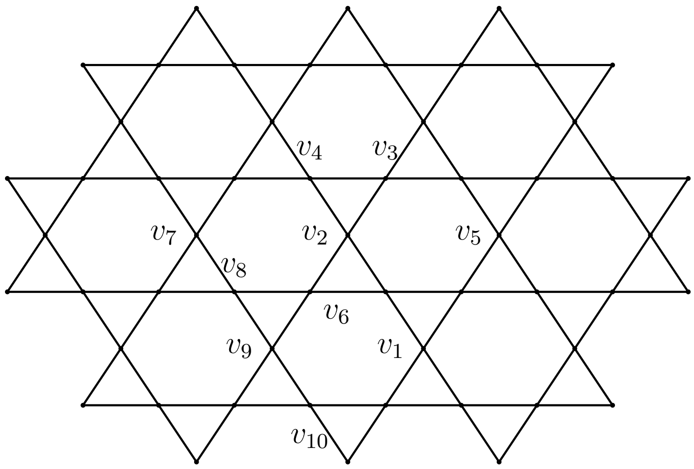

Figure 4.1: An illustration of the proof of non-existence of an efficient dominating set in the $(3, 6, 3, 6)$ lattice.

4.2 Bounds for the Domination Ratio

Lemma 4.2.1: $\gamma_r(3, 6, 3, 6) \leq \frac{2}{9}$.

Proof: Figure 4.2 illustrates a periodic dominating set in the $(3, 6, 3, 6)$ lattice. There is an infinite connected component of edges in the closed neighborhoods of dominating vertices. For convenience in counting, delete the edges with rightward-pointing arrows in the infinite component. The set of dominating vertices and dominated vertices are unchanged by the deletions. Now pair in a one-to-one correspondence adjacent connected components of five vertices and four vertices (as in the figure), and use the pattern to dominate the entire graph with isomorphic, disjoint, connected subgraphs.

Considering a representative subgraph which consists of one component of each type, the dominating proportion of the dominating set illustrated is $\frac{2}{9}$.

□

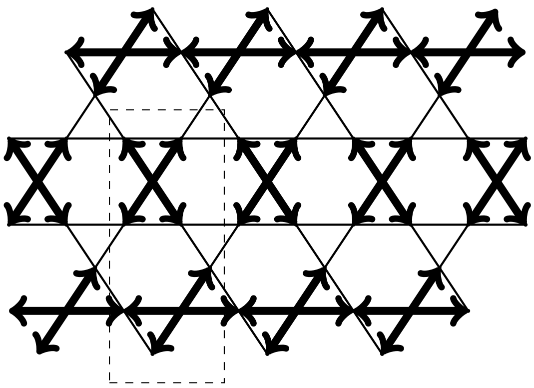

Figure 4.2: An induced subgraph of the kagome lattice.

4.3 Perfect Domination Ratio

Definition (a row of D_1s): A row of D_1s is a sequence (possibly doubly-infinite) of at least two consecutive D_1s such that every two consecutive D_1s in the sequence are distance three apart in a 6-cycle.

Lemma 4.3.1: $\gamma_p(3,6,3,6) \leq \frac{1}{3}$

Proof: A periodic PDS D with $\gamma_p(D) = \frac{1}{3}$ is shown in Figure 4.3, establishing $\frac{1}{3}$ as an upper bound. □

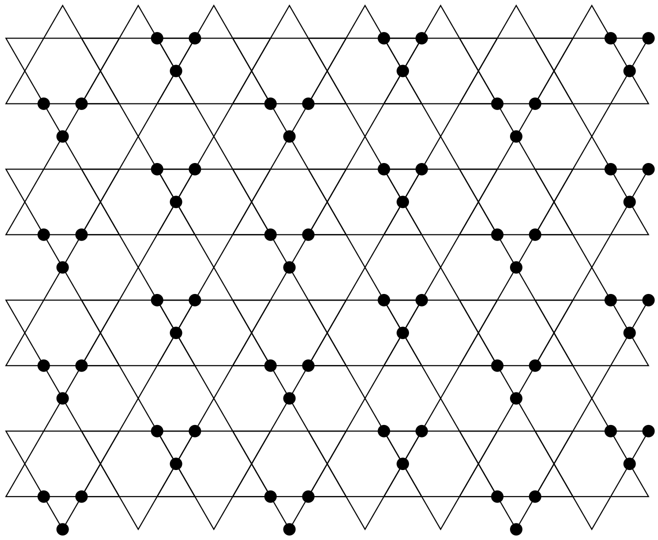

Figure 4.3: A PDS D of the $(3,6,3,6)$ lattice with $\gamma_p(D) = \frac{1}{3}$.

Lemma 4.3.2: *A D_1 must appear in an infinite row of D_1s.*

Proof: Suppose $v_1 \in D$ is a D_1. Figure 4.4 (left) illustrates the following reasoning. By vertex-transitivity, any vertex may be chosen to represent v_1. Notice that v_2 and v_5 are not in D since they are in the double external boundary of v_1. Thus, v_2 pulls in either v_3 or v_4, and v_5 pulls in either v_4 or v_6.

Suppose $v_3 \in D$. Then $v_2 \notin D$ and $v_3 \in D$ forces out v_4. Thus, v_5 pulls in v_6, and consequently v_4 is dominated by both v_3 and v_6, contradicting that D is a perfect dominating set. Therefore $v_3 \notin D$. The same reasoning can be applied to show $v_6 \notin D$.

Next, v_2 pulls in v_4. Notice that v_4 is a D_1, and the same reasoning regarding v_1 can be applied to v_4 to show v_9 is a D_1. Thus, one can show by induction that any vertex v on the line (extending infinitely in both directions) going through

v_1 and v_4 must be a D_1. □

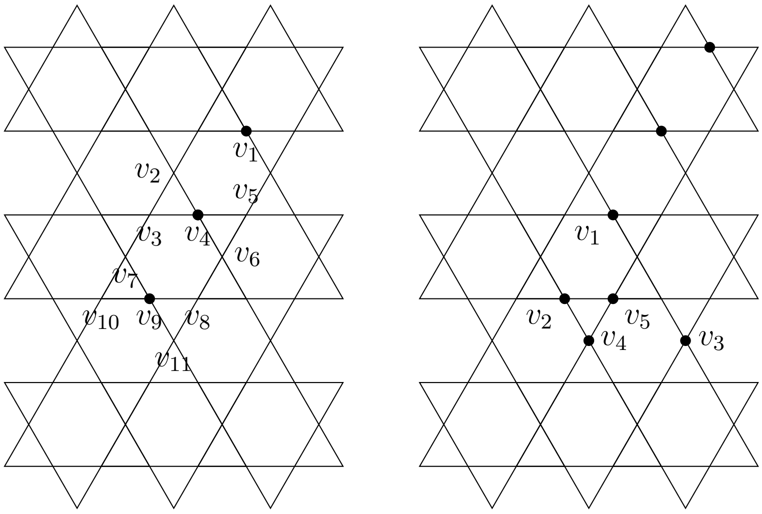

Figure 4.4: The figure on the left illustrates the proof of Lemma 4.3.2. The figure on the right illustrates the proof of Lemma 4.3.3.

Lemma 4.3.3: *Two rows of D_1s must be parallel.*

Proof: To deduce a contradiction, suppose there exist two rows of D_1s that are not parallel. By Lemma 4.3.2, the two rows of D_1s must extend infinitely and therefore must intersect. There are only three possible directions for a row of D_1s, so these two rows of D_1s must form an angle of $\frac{\pi}{3}$. Figure 4.4 (right) illustrates the reasoning. Notice that v_1 and v_2 are in a row of D_1s, and v_3 and v_4 are in another row of D_1s. Thus, v_2 and v_4 are in the same D_n. Then, v_2 is in a D_2 or larger D_n, contradicting that v_2 is a D_1. □

Lemma 4.3.4: *A D_2 cannot exist.*

Proof: To deduce a contradiction, suppose there exists a PDS D that contains a D_2. Let u and v be vertices in this D_2. Since any edge in the Kagome lattice is in a 3-cycle, u and v are in a 3-cycle $\{u, v, w\}$. Then $w \notin D$ is dominated by both u and v, contradicting that D is a PDS. □

Lemma 4.3.5: *If a PDS D of an induced subgraph of the kagome lattice does not contain a D_1, then the perfect domination proportion of D is at least $\frac{1}{3}$.*

Proof: Suppose there exists a PDS D that does not contain a D_1. By Lemma 4.3.4, any vertex $v \in D$ must be in a D_3 or larger D_n. Observe that a vertex v in a D_3 or larger D_n has at least two neighbors in D. Thus, v dominates at most two vertices not in D, which implies that the perfect domination proportion of D is greater than or equal to $\frac{1}{3}$.

The same reasoning can be applied to any induced subgraph to show that if D is a PDS that does not contain a D_1, then any vertex $v \in D$ dominates at most two vertices not in D. Thus, the domination proportion of the induced subgraph is at least $\frac{1}{3}$. □

Lemma 4.3.6: *A PDS D with perfect domination proportion strictly less than $\frac{1}{3}$ must include infinitely many rows of D_1s.*

Proof: Suppose there exists a PDS D that includes only finitely many rows of D_1s. Let W denote the set of vertices that are neither D_1s nor dominated by D_1s. Consider the subgraph H induced by W, which by Lemma 4.3.5 has a perfect domination proportion at least $\frac{1}{3}$. Since the effect of finitely many rows of D_1s is negligible, the perfect domination proportion of D is at least $\frac{1}{3}$. Thus, any PDS D with a perfect domination proportion strictly less than $\frac{1}{3}$ must include infinitely many rows of D_1s. □

Lemma 4.3.7: *If a D_n U contains a D_9 W, then the perfect domination proportion of U is greater than or equal to that of W.*

Proof: It is easily verified that all D_9s are isomorphic to the D_9 formed by v_{26}, v_{27}, v_{28}, v_{29}, v_{30}, v_{31}, v_{32}, v_{33}, and v_{34} shown in Figure 4.5.

Let W denote a D_9. Observe that W contains 9 vertices and dominates 21 vertices. Thus, the perfect domination proportion of W equals $\frac{3}{7}$. Since U contains W, we can add vertices to W to obtain U. Each time a vertex v is added to W, v must have a neighbor $u \in W$. There are two possible cases.

Case 1: Suppose that v has exactly one neighbor in W. Since any edge is in a 3-cycle, u and v are in a 3-cycle $\{u, v, w\}$, where $w \notin W$. Thus, u and v double force in w, so v actually has two neighbors in W, which is a contradiction.

Case 2: Thus, v has at least 2 neighbors in W. Then v has at most 2 neighbors not in W, so v dominates at most two neighbors not in D that have not been previously dominated.

Thus, if $n \geq 9$, the perfect domination proportion of U is at least

$$\frac{n}{2(n-9)+21} = \frac{n}{2n+3} \geq \frac{3}{7}.$$ □

Lemma 4.3.8: $\gamma_p(3,6,3,6) \geq \frac{1}{3}$.

Proof: The proof is by contradiction. Let V be the vertex set of the kagome lattice. Assume there exists a PDS D with perfect domination proportion strictly less than $\frac{1}{3}$. By Lemma 4.3.6, D must contain infinitely many rows of D_1s. By Lemma 4.3.3, the rows of D_1s in D must be parallel. Let W be a row of D_1s. Figure 4.5 illustrates the reasoning.

Let v_1 be a D_1 in W. Notice that $v_2 \notin D$ since it is in the double external boundary of v_1. Thus, v_2 pulls in either v_3 or v_4. The two cases are equivalent by symmetry. Without loss of generality, let $v_3 \in D$ and $v_4 \notin D$. Notice that v_3 is not a D_1, since otherwise by Lemma 4.3.2, v_3 and v_5 form a row of D_1s that intersects W. This contradicts Lemma 4.3.3. By Lemma 4.3.4, v_3 is in a D_3 or larger D_n. Thus, v_6 and v_7 are in D.

Notice that $v_8 \notin D$ since it is in the double external boundary of v_1. Thus $v_8 \notin D$ and $v_6 \in D$ force out v_9. Then $v_9 \notin D$ and $v_6 \in D$ force out v_{10} and v_{11}. A similar argument on v_2 can be applied to v_{12} to show that $v_{12} \notin D$ and v_{13}, v_{14}, and v_{15} are in D. Then v_{10} pulls in either v_{16} or v_{17}. The two cases are equivalent by symmetry. Without loss of generality, let $v_{16} \in D$, and $v_{17} \notin D$. Then v_7 and v_{16} double force in v_{18}. Thus, v_7 and v_{18} double force in v_{19}, and v_{16} and v_{18} double force in v_{20}.

Next, $v_4 \notin D$ and $v_3 \in D$ force out v_{21} and v_{22}. Then $v_{23} \notin D$, since otherwise v_{19} and v_{23} double force in v_{24} and consequently v_{23} and v_{24} double force in

v_{21}, contradicting our previous argument that $v_{21} \notin D$. Thus, v_{21} pulls in v_{24}. Finally, v_{24} and v_{19} in D double force in v_{25}.

The same reasoning can be applied to show that v_{26}, v_{27}, v_{28}, v_{29}, v_{30}, v_{31}, v_{32}, v_{33}, and v_{34} are in D.

Next, we calculate a lower bound for the perfect domination proportion of such a PDS D, given the reasoning above. Refer to Figure 3, in which we define the following subgraphs. Let W denote the line of D_1s containing v_1, and let H_1 denote $W \cup N(W)$. Let H_2 denote the set of alternating D_3s and D_ns with $n \geq 9$, together with the vertices they dominate, just above $W \cup N(W)$. Let H_3 denote the isomorphic subgraph obtained by reflecting H_2 through the line corresponding to W. Within $H_1 \cup H_2 \cup H_3$ we can form connected subgraphs consisting of four D_1s, one D_3 on each side, and one D_9 (or larger) on each side, and the vertices that they dominate.

Number of Components	D_n	Vertices in D	Vertices Dominated
4	D_1	1	5
2	D_3	3	9
2	D_9	9	21

Table 4.1: Data for calculation of the perfect domination proportion.

Denoting the vertex sets of H_1, H_2, H_3 by $V_{H_1}, V_{H_2}, V_{H_3}$ respectively we have

$$\frac{|D \cap V(H_1 \cup H_2 \cup H_3)|}{|V(H_1 \cup H_2 \cup H_3)|} = \frac{4 \times 1 + 2 \times 3 + 2 \times 9}{4 \times 5 + 2 \times 9 + 2 \times 21} = \frac{7}{20} > \frac{1}{3}$$

In this calculation, we assume that v_3, v_6, v_7, v_{16}, v_{18}, v_{19}, v_{20}, v_{24}, and v_{25} are a D_9. Otherwise, they are in an even larger D_n, so by Lemma 4.3.7 the perfect domination proportion is even higher. The same reasoning can be applied to every row of D_1s.

Let G denote the union of all rows of D_1s and their corresponding H_1, H_2, H_3. We have shown above that the perfect domination proportion of G is strictly larger than $\frac{1}{3}$. Since $V \setminus V_G$ does not contain any D_1, by Lemma 4.3.5, the perfect domination proportion of the rest of the lattice is greater than or equal to $\frac{1}{3}$. Combining these, we conclude that the perfect domination proportion of the lattice is at least $\frac{1}{3}$, contradicting our original assumption. □

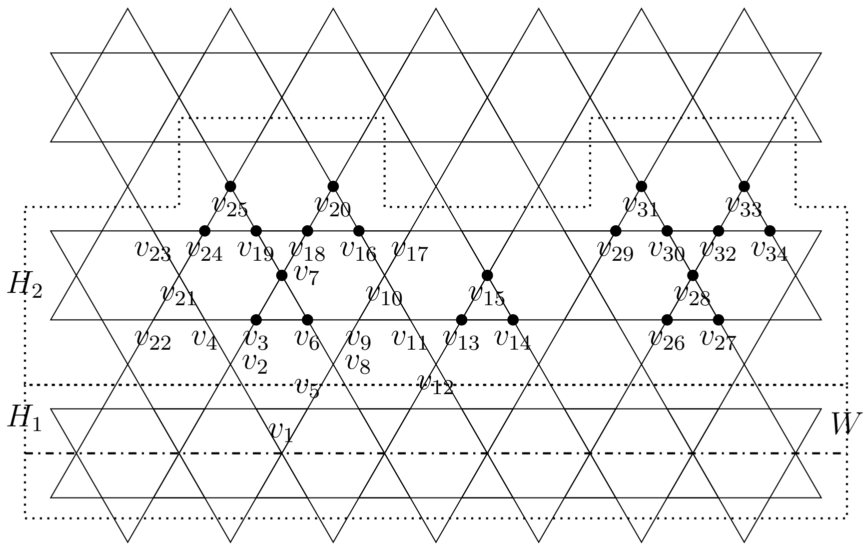

Figure 4.5: An illustration of the proof of Lemma 4.3.8.

Theorem 4.3.9: $\gamma_p(3, 6, 3, 6) = \frac{1}{3}$

Proof: The result is immediate from Lemma 4.3.1 and Lemma 4.3.8.

4.4 Possible Perfect Domination Proportions

Fact: *The kagome lattice has infinitely many non-isomorphic PDSs that achieve distinct perfect domination proportions.*

Proof: A periodic PDS D with $\gamma_p(D)$ =$\frac{1}{3}$ is shown in Figure 4.3.

A periodic PDS D consisting of only D_9s is shown in Figure 4.6. Because each D_9 has 9 vertices and dominates 21 vertices (including vertices in D_9), perfect domination proportion =$\frac{9}{21} = \frac{3}{7}$.

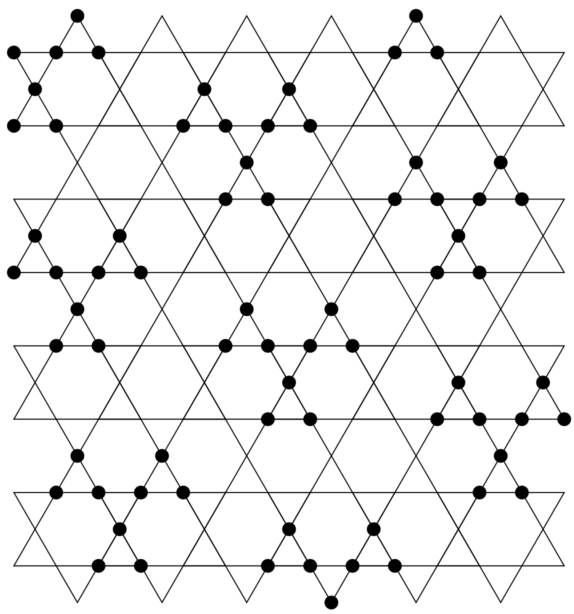

Figure 4.6: A PDS D with perfect domination proportion $\frac{3}{7}$.

A periodic PDS D consisting of only D_{18}s is shown in Figure 4.7. Since each D_{18} has 18 vertices and dominates 36 vertices (including the vertices in D_{18}), the perfect domination proportion is $\frac{1}{2}$.

Similarly, there exists a periodic PDS D consisting of only D_ns, where D_n is a triangular arrangement of 3-cycles. For example, we can add a row of three 3-cycles on the top of a D_9 and obtain a D_{18}. We can add a row of four 3-cycles on the top of a D_{18} and obtain a D_{30}. By repeatedly adding a row of 3-cycles, we can obtain a $D_n(k)$ that has $(1+2+3+...+k)$ 3-cycles.

Therefore, for any positive integer k, there exists a periodic PDS D consisting of only $D_{n(k)}$s, where $n(k) = 3(1+2+3+...+k) = \frac{3}{2}(k^2+k)$. As k approaches infinity, $\gamma_p(D)$ approaches 1, because the proportion of vertices on the external boundary of D_n approaches 0. □

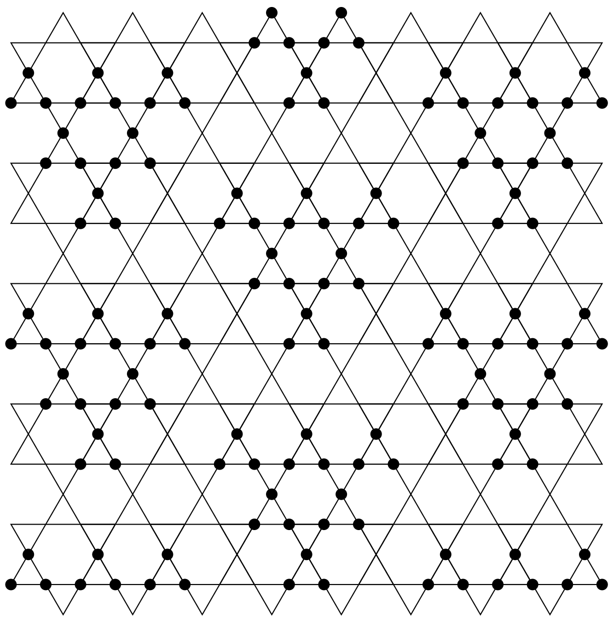

Figure 4.7: A PDS D with perfect domination proportion $\frac{1}{2}$.

Chapter 5

The $(3, 4, 6, 4)$ Lattice

5.1 Nonexistence of Efficient Domination

Lemma 5.1.1: *There does not exist an efficient dominating set in the* $(3, 4, 6, 4)$ *lattice.*

Proof: The proof is by contradiction. Assume that there exists an efficient dominating set D. Since $D \neq \emptyset$, there exists a vertex $v_1 \in D$. Figure 5.1 illustrates the reasoning. By vertex-transitivity, any vertex may be chosen to represent v_1.

Vertex v_2 is adjacent to a vertex in $N[v_1]$, so $v_2 \notin D$ or the adjacent vertex would be dominated by both v_1 and v_2. Therefore, v_2 must be dominated by one

of its neighbors. The only neighbor v for which $N[v] \cap N[v_1] = \emptyset$ is v_3, so $v_3 \in D$ if D is to be an efficient dominating set.

Similarly, $v_4 \notin D$ and must be dominated by $v_5 \in D$.

Continuing, $N[v_6] \cap N[v_5] \neq \emptyset$ and $N[v_6] \cap N[v_3] \neq \emptyset$, so $v_6 \notin D$. However, every neighbor v of v_6 satisfies either $N[v] \cap N[v_5] \neq \emptyset$ or $N[v] \cap N[v_3] \neq \emptyset$, so there does not exist any vertex $v \in D$ such that $v \in N[v_6]$. Since there is no $v \in D$ which dominates v_6, D is not a dominating set, and thus not an efficient dominating set, contradicting our original assumption. □

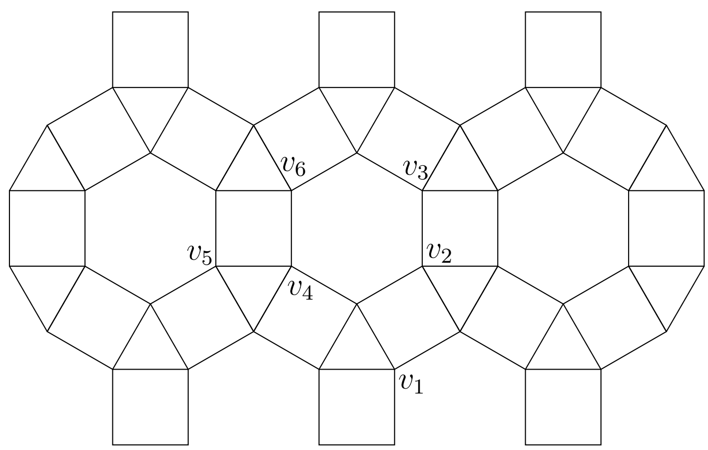

Figure 5.1: An illustration of the proof of non-existence of an efficient dominating set in the $(3,4,6,4)$ lattice.

5.2 Bounds for the Domination Ratio

Lemma 5.2.1: $\gamma_r(3,4,6,4) \leq \frac{2}{9}$.

Proof: Figure 5.4 illustrates a dominating set D in the $(3,4,6,4)$ lattice. The set D is periodic, so its dominating proportion may be computed based on the domination number of a single representative subgraph. Notice that the edges in the closed neighborhoods of vertices $v \in D$ form two types of connected components. One type consists of a single $v \in D$ with dominated vertices. The other consists of three vertices in D together with ten dominated vertices. Pair such adjacent components with a one-to-one correspondence (as in the figure), and use the pattern to dominate the entire graph with isomorphic, disjoint, connected subgraphs. Letting the representative subgraph be the union of one component of each type, we have a dominating set of size four for a graph with 18 vertices, and thus a dominating proportion of $\frac{2}{9}$. □

5.3 Perfect Domination Ratio

Definition (row of D_1s): A row of D_1s is a sequence (possibly doubly-infinite) of at least two consecutive D_1s such that every two consecutive D_1s in the sequence are distance three apart in a 6-cycle and lie on a line which bisects

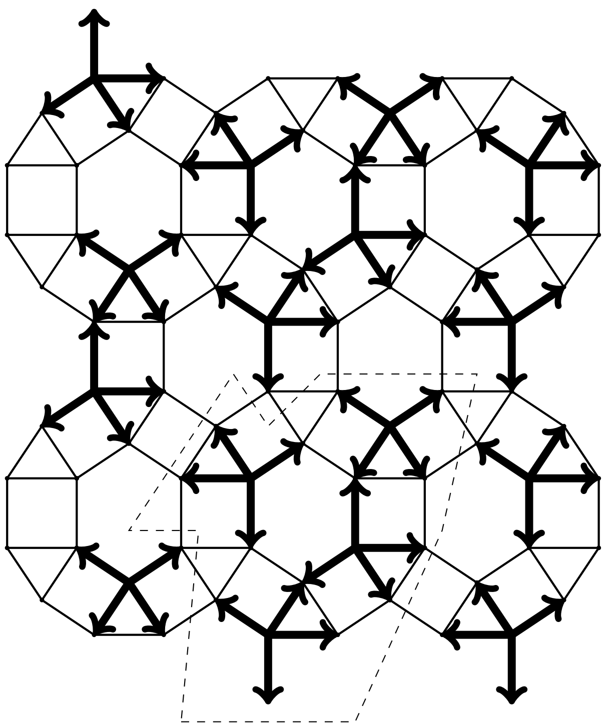

Figure 5.2: An induced subgraph of the $(3, 4, 6, 4)$ lattice.

hexagonal faces of the lattice.

Note: In Figure 5.4, the vertices v_1, v_2, and v_3 are in a row of D_1s.

Lemma 5.3.1: $\gamma_p(3,4,6,4) \leq \frac{1}{4}$

Proof: A periodic PDS D with perfect domination proportion $\frac{1}{4}$ is shown in Figure 5.3, establishing $\frac{1}{4}$ as a upper bound. □

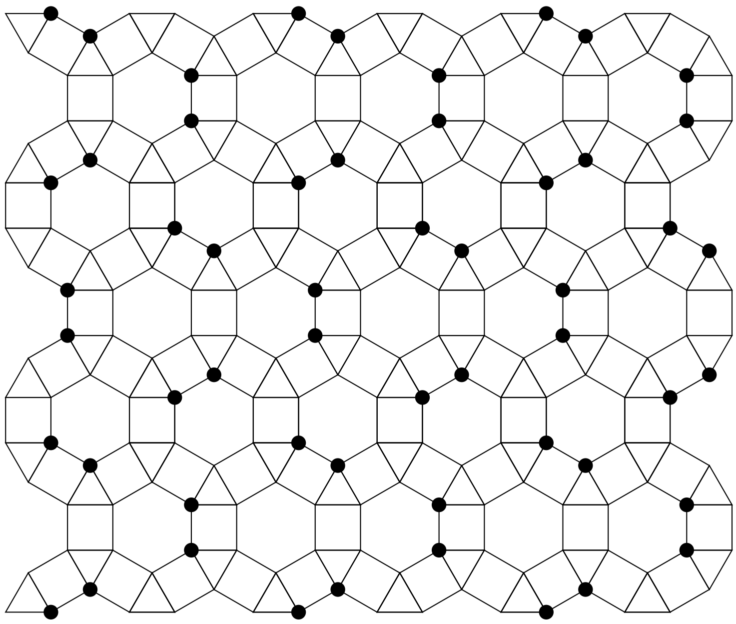

Figure 5.3: A PDS D on the $(3,4,6,4)$ lattice with $\gamma_p(D)$ =$\frac{1}{4}$.

Lemma 5.3.2: *If a D_1 is not in a row of D_1s, the perfect domination proportion of its closed neighborhood is at least $\frac{1}{4}$.*

Proof: Suppose there exists a PDS D in which v_1 is a D_1 and is not in a row of D_1s. We consider three cases.

Case 1: v_2 and v_3 are not in D.

Figure 5.4 illustrates the reasoning. Notice that v_7, v_{11}, v_{13}, v_{21},v_{23}, and v_{32} are not in D since they are in the double external boundary of v_1. Consequently, v_2 pulls in either v_4 or v_5, but not both. The two cases are equivalent by symmetry. Without loss of generality, let $v_4 \in D$, but $v_5 \notin D$. Then $v_5 \notin D$ and $v_4 \in D$ force out v_6, so v_7 pulls in v_8. As a result, $v_9 \notin D$ and $v_8 \in D$ force out v_{10}.

Notice that v_{11} pulls in v_{12} and that v_{13} pulls in v_{14}. Then, together, v_{12} and v_{14} double force in v_{15}.

Next $v_{10} \notin D$ and $v_{12} \in D$ force out v_{16}. Consequently, a sequence of vertices, v_{17}, v_{18}, v_{19}, and v_{20} are forced out.

Continuing similar reasoning, v_{21} pulls in v_{22}, and v_{23} pulls in v_{24}. Thus, v_{22} and v_{24} double force in v_{25}. Notice that v_3 pulls in v_{26}, and then v_{24} and v_{26} double force in v_{27}. The sequence of vertices v_{28}, v_{29}, v_{30}, and v_{31} are then double forced in.

We now calculate the perfect domination proportion of a resulting subgraph. Let V_1 denote the set of vertices dominated by v_1. Let V_2 denote vertices dominated by v_{12}, v_{15}, and v_{14}, and let V_3 denote the set of vertices dominated by v_{22}, v_{24}, v_{25}, v_{26}, v_{27}, v_{28}, v_{29}, v_{30}, and v_{31}.

In this calculation, we assume that v_{22}, v_{24}, v_{25}, v_{26}, v_{27}, v_{28}, v_{29}, v_{30}, and v_{31} form a D_9. If not, then they are part of a larger D_n and using similar reasoning as Lemma 4.7, the perfect domination proportion is even higher. For the same reason, assume that there is a D_1 on the opposite side of v_{22}. Let V_4 denote the set of vertices dominated by this D_1.

Then

$$\frac{|D \cap V(V_1 \cup V_2 \cup V_3 \cup V_4)|}{|V(V_1 \cup V_2 \cup V_3 \cup V_4)|} = \frac{1+3+9+1}{5+11+19+5} = \frac{7}{20} > \frac{1}{4} \qquad \square$$

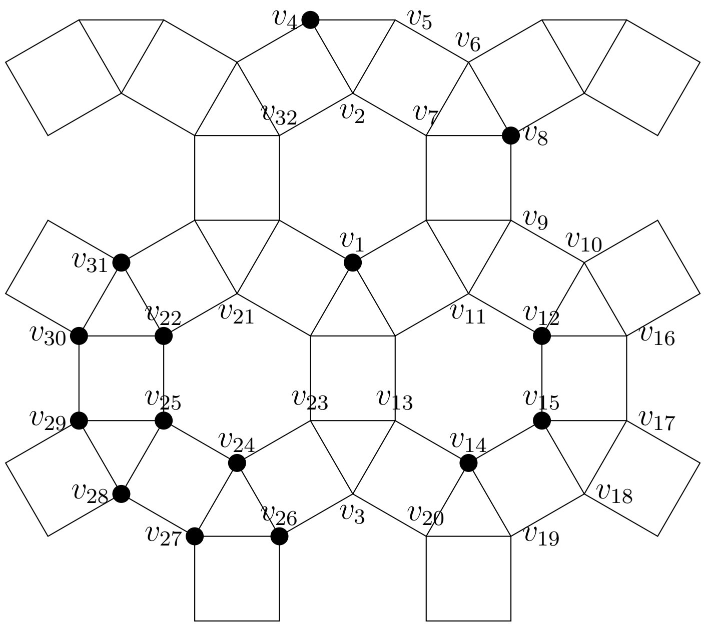

Figure 5.4: An illustration of the proof of Lemma 5.3.2.

Case 2: $v_2 \in D$

Figure 5.5 illustrates the reasoning. Since v_1 is not in a row of D_1s by assumption, v_2 is not a D_1 and is in a D_2 or larger D_n. Note that $v_4, v_6, v_7, v_8,$ and v_9 are not in D since they are either in external boundary or in double external boundary of v_1. Then $v_4 \notin D$ and $v_2 \in D$ force out v_5. Consequently, v_9 pulls in v_{10}, and v_7 pulls in v_{11}. Thus, v_{10} and v_{11} double force in v_{12}.

The same reasoning can be applied to show v_{13}, v_{14}, v_{15} in D. Therefor, v_1 would not reduce the perfect domination proportion to be below $\frac{1}{4}$, as can easily be verified by calculation as in Case 1.

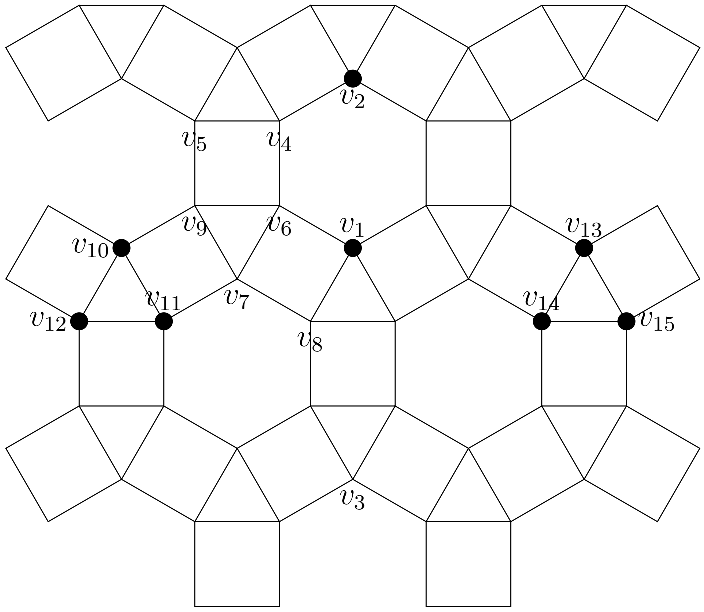

Figure 5.5: An illustration of the proof of Lemma 5.3.2.

Case 3: $v_2 \notin D$, and $v_3 \in D$

Figure 5.6 illustrates the reasoning. Since v_1 is not in a row of D_1s by assumption, v_3 is not a D_1 and is in a D_2 or larger D_n. Notice $v_4, v_5, v_6, v_7, v_8,$ and v_9 are not in D since they are either in external boundary or in double external boundary of v_1. Then v_2 pulls in either v_{10} or v_{11}, but not both. The two cases are equivalent by symmetry. Without loss of generality, let $v_{10} \in D$, but $v_{11} \notin D$. Note $v_{12} \notin D$, otherwise v_{12} and v_{10} double force in v_{13}, and therefore v_{12} and v_{13} double force in v_4, contradicting that $v_4 \notin D$. Consequently, v_6 pulls in v_{14}, and v_8 pulls in v_{15}. As a result, v_{14} and v_{15} double force in v_{16}.

Therefore, v_1 would not reduce the perfect domination proportion to be below $\frac{1}{4}$, as can easily be verified by calculation as in Case 1.

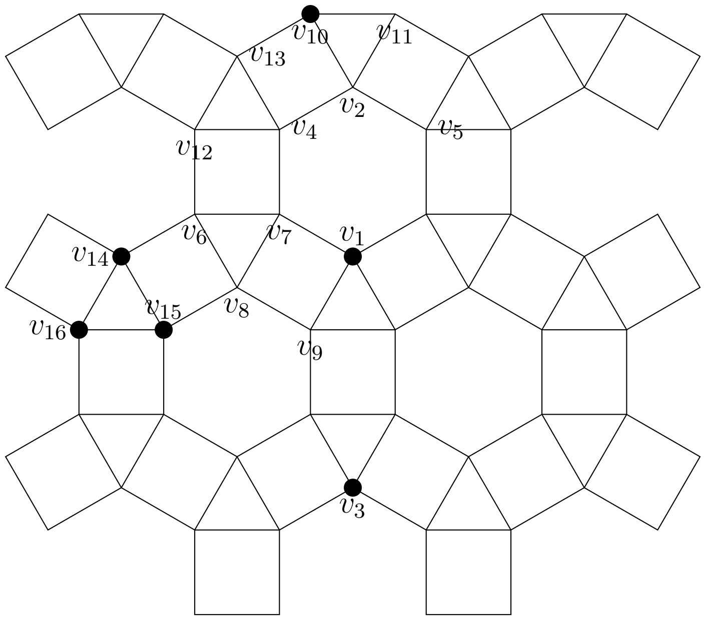

Figure 5.6: An illustration of the proof of Lemma 5.3.2.

Thus, in every case, the perfect domination proportion of the closed neighborhood of v_1 is at least $\frac{1}{4}$.

Lemma 5.3.3: *If there is a row of exactly two D_1s which are not on the same hexagonal face, then the perfect domination proportion of the union of their closed neighborhoods is at least $\frac{1}{4}$.*

Proof: Figure 5.7 illustrates the reasoning. Suppose there exists a PDS D such that v_1 and v_2 in D are two D_1s distance 3 apart, but not in the same hexagon. We assume v_3 and v_4 are not in D. Otherwise v_1 and v_2 would not reduce the perfect domination proportion to be below $\frac{1}{4}$, as can easily be verified by reasoning as in Lemma 5.3.2, Case 2.

Note that v_5, v_7, v_9, v_{11}, v_{17}, v_{18}, v_{24}, and v_{26} are not in D since they are either in double external boundary of v_1 or in double external boundary of v_2. Then v_5 pulls in v_6, v_7 pulls in v_8, v_9 pulls in v_{10}, and v_{11} pulls in v_{12}.

In additon, v_4 pulls in either v_{13} or v_{14} but not both. The two cases are equivalent by symmetry. Without loss of generality let $v_{14} \in D$, and $v_{13} \notin D$. Consequently $v_{15} \notin D$, since otherwise v_{14} and v_{15} double force in v_{16}, and then v_{15} and v_{16} double force in v_{17}, contradicting that $v_{17} \notin D$. Therefore, v_{17} pulls in v_{16}, and v_{18} pulls in v_{19}. This implies that the sequence of vertices v_{20}, v_{21}, and v_{22} are double forced in.

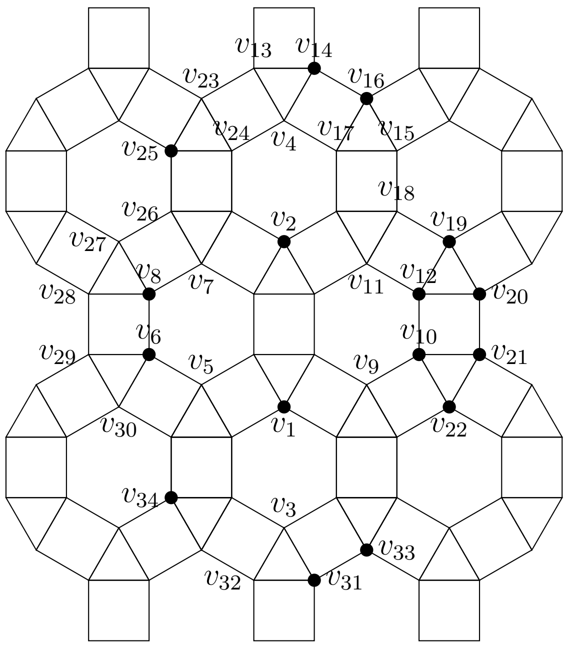

Figure 5.7: An illustration of the proof of Lemma 5.3.3.

Next we see that $v_{23} \notin D$, since otherwise v_{23} and v_{14} double force in v_{13}, contradicting that $v_{13} \notin D$. Consequently v_{24} pulls in v_{25}, so $v_{26} \notin D$ and $v_{25} \in D$ force out v_{27}. Then the sequence of vertices v_{28}, v_{20}, and v_{30} are forced out.

Notice that v_3 pulls in v_{31} so $v_{32} \notin D$. Otherwise the same reasoning that shows $v_{19} \in D$ can be applied to show $v_{30} \in D$, contradicting the previous determination. Then the same reasoning that shows that v_{16} and v_{25} are in D can be applied to show that v_{33} and v_{34} are in D.

Next we calculate the perfect domination proportion of a resulting subgraph. Let V_1 denote the set of vertices dominated by v_1 and v_2. Let V_2 denote the set of vertices dominated by v_{10}, v_{12}, v_{19}, v_{20}, v_{21}, and v_{22}. We assume that $v_{10}, v_{12}, v_{19}, v_{20}, v_{21}$ and v_{22} are a D_6. Otherwise, they are in a larger D_n, and reasoning similar to that in Lemma 4.7 shows that the perfect domination proportion is even higher. For the same reason, assume there are two other D_1s on the opposite side of $v_{10}, v_{12}, v_{19}, v_{20}, v_{21}$ and v_{22}. Let V_3 denote the set of vertices dominated by these two D_1s. Finally, we see that the perfect domination proportion of the $V_1 \cup V_2 \cup V_3$ satisfies

$$\frac{|D \cap V(V_1 \cup V_2 \cup V_3)|}{|V(V_1 \cup V_2 \cup V_3)|} = \frac{2+6+2}{10+14+10} = \frac{5}{17} > \frac{1}{4} \qquad \square$$

Lemma 5.3.4: *If there exists an infinite row of D_1s, there exist two infinite rows of D_6s or larger D_ns along the sides of the row of D_1s.*

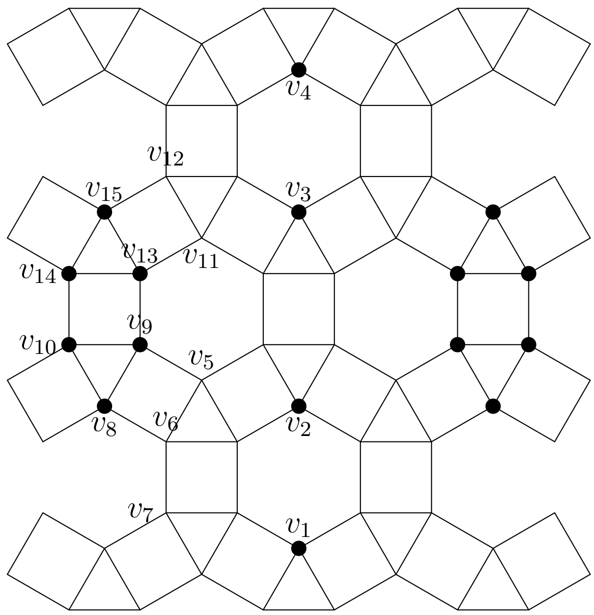

Figure 5.8: An illustration of the proof of Lemma 5.3.4.

Proof: Figure 5.8 illustrates the reasoning. Let v_1, v_2, v_3, and v_4 be in an infinite row of D_1s.

Notice that v_5 and v_6 are not in D since they are in the double external boundary of v_2, and that $v_7 \notin D$ since it is in the double external boundary of v_1. Consequently, v_6 pulls in v_8, and v_5 pulls in v_9. Then v_8 and v_9 double force in v_{10}.

Similarly v_{11} and v_{12} are not in D since they are in the double external boundary of v_3, so v_{11} pulls in v_{13}. Consequently, v_{10} and v_{13} double force in v_{14}, and v_{13} and v_{14} double force in v_{15}.

The same reasoning can be applied inductively to other vertices in the row of D_1s to show that the row of D_1s is bordered by two infinite rows of D_6s or larger D_ns. □

Lemma 5.3.5: *If a row of D_1s contains two D_1s distance 3 apart in a hexagonal face, it must be either in a row of at least 4 D_1s, or end at a D_n, $n \geq 2$.*

Proof: Figure 5.9 illustrates the reasoning. To deduce a contradiction, suppose there exists a PDS D such that v_1 and v_2 in D are two D_1s distance 3 apart in a hexagonal face and $v_3 \notin D$.

Notice that $v_4 \notin D$ since v_4 is in the double external boundary of v_2. Thus, v_4 pulls in v_5. Similarly, $v_6 \notin D$, since v_6 is in the double external boundary of v_1, so v_8 pulls in v_9.

Next, v_{10} pulls in v_{11}, since $v_3 \notin D$ by assumption. Thus, v_5 and v_{11} double force $v_{12} \in D$. Consequently, the sequence of vertices v_{13}, v_{14}, v_{15}, v_{16}, and v_{17} are double forced in.

By symmetry, we have the same structure on the opposite side of the row of D_1s, so $v_{18} \in D$. However, v_3 is then dominated by both v_{17} and v_{18}, contradicting that D is a PDS.

Therefore, if v_1 and v_2 are in D, then $v_3 \in D$. Again, by symmetry, the same reasoning can be applied to show if v_1 and v_2 are in D, then $v_{19} \in D$. Notice that v_{19} is either a D_1 or in a D_n with $n \geq 2$.

Note that if the row of D_1s ends with a $D_n, n \geq 2$, it would not reduce the perfect domination proportion to be below $\frac{1}{4}$. We can verify this by reasoning as in Lemma 5.3.2: In particular, it is still true that v_5, v_9, and v_{13} are in D when v_3 is in a D_2 or larger D_n. □

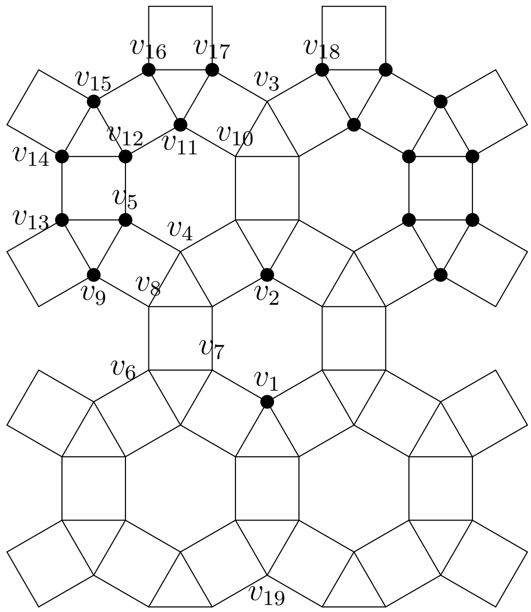

Figure 5.9: An illustration of the proof of Lemma 5.3.5.

Lemma 5.3.6: *A row of at least 3 consecutive D_1s must either be doubly-infinite or end at a $D_n, n \geq 2$.*

Proof: Figure 5.10 illustrates the reasoning. To deduce a contradiction, suppose there exsits a PDS D such that v_1, v_2, and v_3 in D are a row of D_1s and $v_4 \notin D$.

Notice that v_5 and v_6 are not in D since they are in the double external boundary of v_2. Similarly, $v_7 \notin D$ since v_7 is in the double external boundary of v_1. Thus, v_6 pulls in v_8, and v_5 pulls in v_9. Together, v_8 and v_9 double force in v_{10}.

Next, v_{11} and v_{12} are not in D since they are in the double external boundary of v_3. Then v_{11} pulls in v_{13}, in turn v_{10} and v_{13} double force in v_{14}, and continuing, v_{13} and v_{14} double force in v_{15}. Since $v_{12} \notin D$ and $v_{15} \in D$, they force out v_{16}.

Therefore, v_{17} forces in v_{18}.

The same reasoning can be applied to the opposite side of the row of D_1s to show that $v_{19} \in D$.

Thus v_4 pulls in either v_{20} or v_{21}. The two cases are equivalent by symmetry. Without loss of generality, let $v_{20} \in D$. Then v_{19} and v_{20} double force in v_{21}. Continuing, v_{20} and v_{21} double force in v_4, contradicting our assumption that $v_4 \notin D$.

Therefore, if v_1, v_2, and v_3 are in D, then $v_4 \in D$. Notice that v_4 is either a D_1 or in a $D_n, n \geq 2$. If the row of D_1s ends with a larger D_n, it would not reduce the perfect domination proportion to be below $\frac{1}{4}$, as can easily be verified by reasoning as in Lemma 5.3.2. (In particular, it is still true that v_8, v_9, v_{10}, v_{13}, v_{14}, and v_{15} are in D when v_4 is in a D_2 or larger D_n.) Otherwise, the row of D_1s does not end with a $D_n, n \geq 2$, so by Lemma 5.3.5 the row of D_1s must extend infinitely in both directions. □

Theorem 5.3.7: $\gamma_p(3, 4, 6, 4) = \frac{1}{4}$.

Proof: Suppose there exists a PDS D with perfect domination proportion strictly less than $\frac{1}{4}$. We know that D must contain D_1s. The only possibilities are that a D_1 can occur as a D_1 that is not in a row of D_1s (discussed in Lemma 5.3.2), or is in a row of only two D_1s (discussed in Lemma 5.3.3), or is in a row of more than two D_1s (discussed in Lemma 5.3.5 and Lemma 5.3.6),

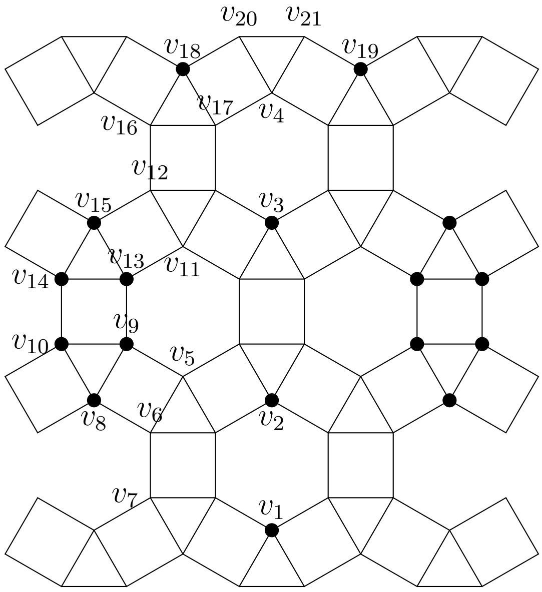

Figure 5.10: An illustration of the proof of Lemma 5.3.6.

or is in an infinite row of D_1s (discussed in Lemma 5.3.4). For each possibility, we have shown that a D_1 cannot reduce the perfect domination proportion to be strictly less than $\frac{1}{4}$. Therefore, $\gamma_p(3, 4, 6, 4) \geq \frac{1}{4}$. However, by Lemma 5.3.1, we have $\gamma_p(3, 4, 6, 4) \leq \frac{1}{4}$. $\square$

5.4 Non-isomorphic Perfect Dominating Sets

Fact: *There exist two non-isomorphic PDSs for the (3,4,6,4) lattice with equal perfect domination proportions.*

Proof: Figure 5.4 shows two non-isomorphic PDSs with perfect domination proportion $\frac{1}{3}$. □

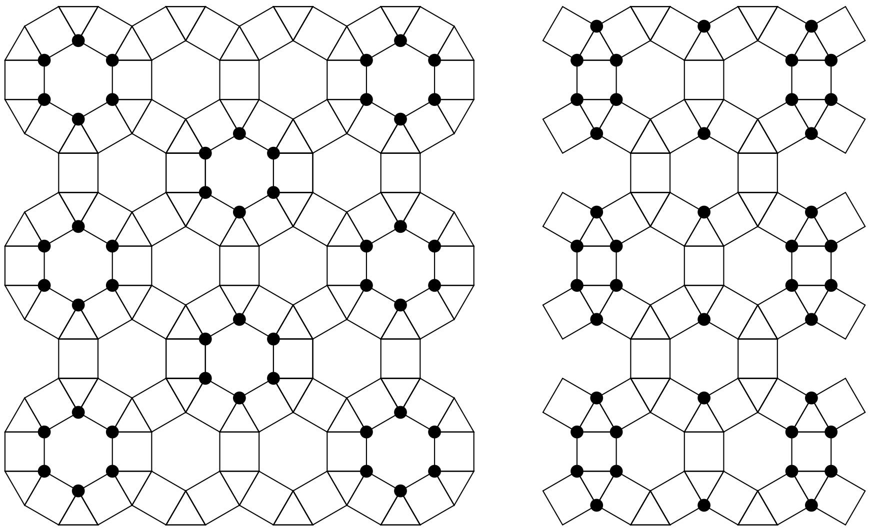

Figure 5.11: Two non-isomorphic PDSs with $\gamma_p(D) = \frac{1}{3}$.

Chapter 6

The $(3^2, 4, 3, 4)$ Lattice

6.1 Nonexistence of Efficient Domination

Lemma 6.1.1: *There does not exist an efficient dominating set in the* $(3^2, 4, 3, 4)$ *lattice.*

Proof: The proof is by contradiction. Assume that there exists an efficient dominating set D. Since $D \neq \emptyset$, there exists a vertex $v_1 \in D$. Figure 6.1 illustrates the reasoning. By vertex-transitivity, any vertex may be chosen to represent v_1.

Vertex v_2 is adjacent to a vertex in $N[v_1]$, so $v_2 \notin D$ or the adjacent vertex would be dominated by both v_1 and v_2. Therefore, v_2 must be dominated by one

of its neighbors. The only neighbor v for which $N[v] \cap N[v_1] = \emptyset$ is v_3, so $v_3 \in D$ if D is to be an efficient dominating set.

Continuing, $N[v_4] \cap N[v_3] \neq \emptyset$, so $v_4 \notin D$. However, every neighbor v of v_4 satisfies either $N[v] \cap N[v_1] \neq \emptyset$ or $N[v] \cap N[v_3] \neq \emptyset$, so there does not exist any vertex $v \in D$ such that $v \in N[v_4]$. Since there is no $v \in D$ which dominates v_4, D is not a dominating set, and thus not an efficient dominating set, contradicting our original assumption. □

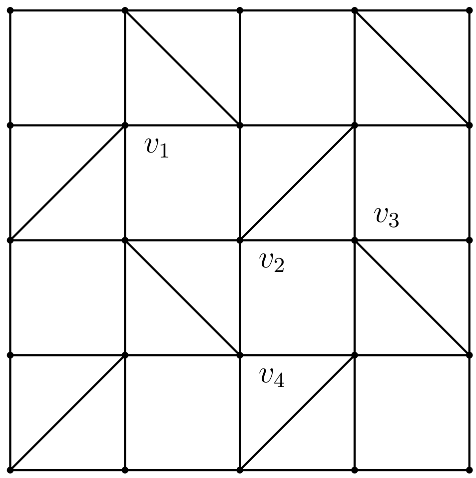

Figure 6.1: An illustration of the proof of non-existence of an efficient dominating set in the $(3^2, 4, 3, 4)$ lattice.

6.2 Bounds for the Domination Ratio

Lemma 6.2.1: $\gamma_r(3^2, 4, 3, 4) \leq \frac{1}{5}$.

Proof: The $(3^2, 4, 3, 4)$ lattice contains a square lattice, obtained by deleting the diagonal edges. (See Figure 6.1.) A dominating set for the square lattice is also a dominating set for the $(3^2, 4, 3, 4)$ lattice. Thus, since the square lattice is efficiently dominated, $\gamma_r(3^2, 4, 3, 4) \leq \gamma_r(4^4) = \frac{1}{5}$. □

6.3 Perfect Domination Ratio

We first provide a PDS that establishes an upper bound, then prove this PDS is actually the minimal PDS, to conclude that $\gamma_p(3^2, 4, 3, 4) = \frac{1}{4}$.

Lemma 6.3.1: $\gamma_p(3^2, 4, 3, 4) \leq \frac{1}{4}$

Proof: Figure 6.2 shows a periodic PDS D on the $(3^2, 4, 3, 4)$ lattice. To calculate the domination ratio of this PDS, note that there are pairs of D_1s which are distance three apart. In the figure, there are D_4s above and below each such pair of D_1s. These four components of D and their external boundaries induce a subgraph with 40 vertices which are dominated by 10 vertices, giving a domination proportion of $\frac{1}{4}$. The lattice may be decomposed into disjoint isomorphic connected subgraphs, so $\gamma_p(D)$ =$\frac{1}{4}$. Thus, $\frac{1}{4}$ is an upper bound for $\gamma_p(3^2, 4, 3, 4)$.□

Lemma 6.3.2: *A PDS of the* $(3^2, 4, 3, 4)$ *lattice cannot contain a* D_2.

Proof: To deduce a contradiction, suppose there exists a PDS D that contains a D_2. Let x and y denote the vertices in this D_2. Since every edge is in a 3-

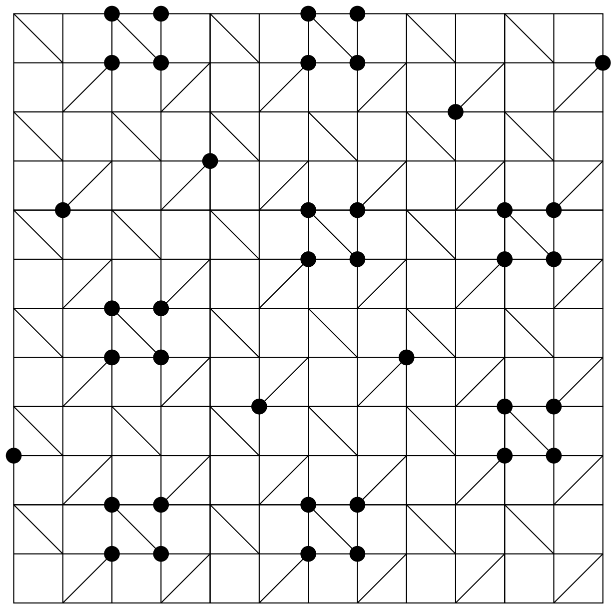

Figure 6.2: A PDS D on the $(3^2, 4, 3, 4)$ lattice with $\gamma_p(D) = \frac{1}{4}$.

cycle, there exists vertex $z \notin D$ that is a common neighbor of x and y. Then z is dominated by both x and y, contradicting the assumption that D is a perfect dominating set. □

Lemma 6.3.3: *A PDS of the* $(3^2, 4, 3, 4)$ *lattice cannot contain a* D_3.

Proof: To deduce a contradiction, suppose there exists a PDS D that contains a D_3. Let x, y and z denote vertices in this D_3. There are 2 possible types of D_3s: a 3-path and a 3-cycle.

If the subgraph induced by $\{x, y, z\}$ is a 3-cycle, then the adjacent 3-cycle must be in D, and therefore $\{x, y, z\}$ must be in a D_4 or a larger D_n.

If the subgraph induced by $\{x, y, z\}$ is a 3-path, then the subgraph induced by $\{x, y, z\}$ includes an edge of a 3-cycle, and the 3-cycle must be in D. Thus,

$\{x, y, z\}$ must be in a D_4 or a larger D_n.

In either case, we reach the contradiction that $\{x, y, z\}$ is not a D_3. □

Lemma 6.3.4: *If a PDS D contains a D_1, the PDS must be a union of D_1s and D_4s. Such a PDS is unique up to isomorphism.*

Proof: Figure 6.3 illustrates the reasoning, which is rather long and intricate.

Suppose there exists a PDS D that contains a D_1. Let v_1 denote this D_1. The vertices in the double external boundary of v_1 are shown in Figure 6.3 as open circles. Therefore, v_2 pulls in v_3.

We show that $v_4 \notin D$ by contradiction: If $v_4 \in D$, then v_3 and v_4 double force $v_5 \in D$, and consequently v_4 and v_5 double force $v_6 \in D$. This contradicts the fact that $v_6 \notin D$ because it is in double external boundary of v_1.

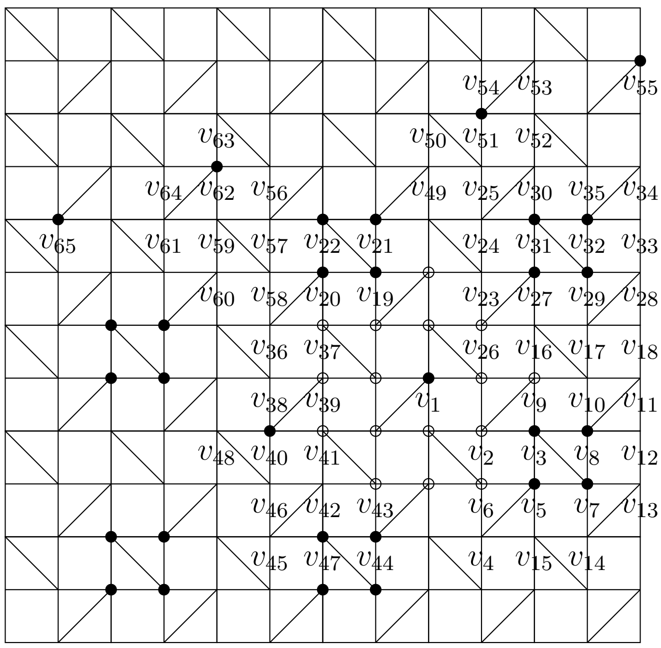

Figure 6.3: An illustration of the proof of Lemma 6.3.4.

Since it is not in D, v_6 pulls in v_5. Then v_3 and v_5 double force $v_7 \in D$, and consequently v_3 and v_7 double force $v_8 \in D$. Since $v_9 \notin D$ and $v_3 \in D$, the vertex v_{10} cannot double-dominate v_9, so v_{10} is forced out. Similarly, $v_{10} \notin D$ and $v_8 \in D$ forces out v_{11}, and by repeating this reasoning v_{12}, v_{13}, v_{14}, and v_{15} are forced out. Thus, v_3, v_5, v_7, and v_8 form a D_4. Furthermore, the double external boundary of this D_4 contains v_{16}, v_{17}, and v_{18}, so they are not in D.

By a rotation by 180^o around v_1, the same reasoning applies to show that v_{19}, v_{20}, v_{21}, and v_{22} are a D_4, and, being in its double external boundary, v_{23}, v_{24}, and v_{25} are not in D.

Next, v_{26} pulls in v_{27}, and we show that $v_{28} \notin D$ by contradiction: Otherwise v_{27} and v_{28} would double force $v_{29} \in D$, and consequently v_{28} and v_{29} would double force $v_{17} \in D$, contradicting our previous conclusion that $v_{17} \notin D$ since it is in the double external boundary of a D_4.

Thus, v_{17} pulls in v_{29}. Vertices v_{27} and v_{29} then force in v_{31} which helps double force $v_{32} \in D$. Since $v_{24} \notin D$, it forces out v_{30}. Similarly, in sequence, the vertices v_{35}, v_{34}, and v_{33} are forced out. We conclude that v_{27}, v_{29}, v_{31}, and v_{32} are a D_4.

Next we consider vertices in the lower left part of the figure, where the reasoning proceeds somewhat differently. The double external boundary of the D_4 formed by v_{19}, v_{20}, v_{21} and v_{22} contains v_{37} and v_{38}, and therefore v_{37} and v_{38} are not in D. Therefore, v_{39} pulls in v_{40}.

Reason by contradiction that $v_{42} \notin D$: Otherwise v_{42} and v_{40} double force

$v_{41} \in D$, contradicting the fact that $v_{41} \notin D$ because it is in double external boundary of v_1.

With no alternative, v_{43} pulls in v_{44}. By contradiction $v_{45} \notin D$: Otherwise v_{40} and v_{45} double force $v_{46} \in D$, and consequently v_{45} and v_{46} double force $v_{42} \in D$, contradicting our previous conclusion that $v_{42} \notin D$.

Since $v_{42} \notin D$, it pulls in either v_{46} or v_{47}. The two cases are equivalent by symmetry. Without loss of generality, let $v_{47} \in D$ and $v_{46} \notin D$. Then $v_{40} \in D$ and $v_{46} \notin D$ force out v_{48}, and we conclude that v_{40} is a D_1. On the other hand, v_{47} and v_{44} double force in two neighbors to form a possible D_4, and reasoning as in the previous cases forces out the boundary to confirm that it must be a D_4. (Note that if we had chosen $v_{46} \in D$ and $v_{47} \notin D$, the resulting PDS would be isomorphic, but rotated by 90^o.)

In the remainder of the proof, we show that the reasoning above can be extended to the entire $(3^2, 4, 3, 4)$ lattice. First, the entire argument so far can be repeated starting from on v_{40} instead of v_1, to show that there are four D_4s around v_{40}, as shown in the figure.

Next, notice that the double external boundary of the D_4 formed by v_{19}, v_{20}, v_{21}, and v_{22} contains v_{49} and v_{50}, and therefore $v_{49}, v_{50} \notin D$. Consequently, v_{25} pulls in v_{51}. Similarly, the double external boundary of the D_4 formed by v_{27}, v_{29}, v_{31}, and v_{32} contains v_{52}, and therefore $v_{52} \notin D$. Thus, $v_{51} \in D$ and $v_{52} \notin D$ force out v_{53}, and then $v_{51} \in D$ and $v_{53} \notin D$ force out v_{54}. We conclude that v_{51} is a D_1.

The same reasoning as starting from v_1 can be applied to v_{51} to show that v_{55} is a D_1. Similarly, both v_{62} and v_{65} can be shown to be D_1s. Thus, such an arrangement of D_1s and D_4s must extend periodically in all directions, so the PDS D is a union of only D_1s and D_4s. □

Theorem 6.3.5: $\gamma_p(3^{,}4, 3, 4) = \frac{1}{4}$

Proof: Lemma 6.3.4 shows that any PDS that contains a D_1 must be a union of D_1s and D_4s, and there is a unique such PDS. Since D_2s and D_3s do not exist by Lemma 6.3.2 and Lemma 6.3.3, any PDS that consists of only D_4s and larger D_ns are less efficient than a union of D_1s and D_4s. Thus, the PDS given in Lemma 6.3.1 is the minimal PDS, and $\gamma_p(3^2, 4, 3, 4) = \frac{1}{4}$. □

6.4 Possible Perfect Domination Proportions

We provide a proof that $(3^2, 4, 3, 4)$ lattice has only two possible perfect domination proportions, 1 and $\frac{1}{4}$.

Definition (1-square): A 1-square is a D_4 that contains two 3-cycles sharing an edge.

Note: A 1-square is shown in Figure 6.4 as v_1, v_2, v_3, v_4.

Definition ((2k+1)-square): A (2k+1)-square is a $(2k+1) \times (2k+1)$ square whose four corners are 1-squares. k is a positive integer and $k \geq 1$.

Note: A 3-square is shown in Figure 6.4 as $v_1, v_2, v_3, ..., v_{16}$.

Lemma 6.4.1: *Any D_n with $n > 4$ must contain a 1-square.*

Proof: Let W be a D_n with $n > 4$. Notice that W must contain an edge. Since any edge is in a 3-cycle, the third vertex in the 3-cycle is forced in. Thus, W contains a 3-cycle. Since every 3-cycle is in a 1-square, the fourth vertex in the 2-square is forced in. Thus, W contains a 1-square. □

Lemma 6.4.2: *If W is a D_n with $n > 4$, then W must contain a 3-square.*

Proof: By Lemma 6.4.1, W must contain a 1-square. Figure 6.4 represents such reasoning. Let v_1, v_2, v_3, v_4 denote the 1-square. Since $n > 4$, W must contain a vertex that is adjacent to one of v_1, v_2, v_3, v_4.

Consider the case $v_5 \in W$. Notice v_5 and v_1 double force in v_6. Similarly, a sequence of vertices, $v_7, v_8, ..., v_{14}$ are double forced in. Therefore, v_8 and v_9 double force in v_{15}. Similarly, v_{13} and v_{14} double force in v_{16}.

The same reasoning can be applied to show no matter which vertex adjacent to one of v_1, v_2, v_3, v_4 is in W, all of $v_5, v_6, ..., v_{16}$ are in W. Vertices $v_1, v_2, ..., v_{16}$

together form a 3-square in W. □

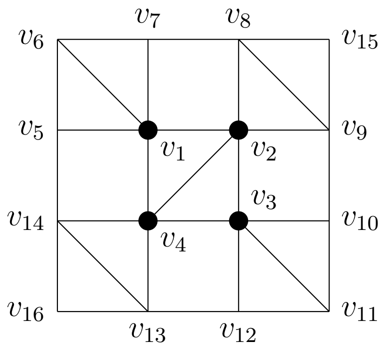

Figure 6.4: An illustration of the proof of Lemma 6.4.2.

Lemma 6.4.3: *If W is a D_n that contains a (2k+1)-square, where $k \geq 1$ is a positive integer, then W must contain a (2k+3)-square.*

Proof: Figure 6.5 represents the reasoning. Let U denote a (2k+1)-square contained in W. Since $k > 1$ and corners of U are 1-squares, vertices u_1 and u_2 are in a 3-cycle. Let v_1 be the third vertex in the 3-cycle. Notice u_1 and u_2 double force in v_1. Similarly, a sequence of vertices, $v_2, v_3, ..., v_{8k+2}$ are double forced in. Therefore, v_{2k-1} and v_{2k} double force in v_{8k+3}. Similarly, v_{6k} and v_{6k+1} double force in v_{8k+4}. Vertices $v_1, v_2, ..., v_{8k+4}$ together form a (2k+3)-square in W. □

Lemma 6.4.4: *If a PDS D contains W, a D_n with $n > 4$, then D is the entire vertex set.*

Proof: Let W be a D_n with $n > 4$. The proof is by induction on the size of W.

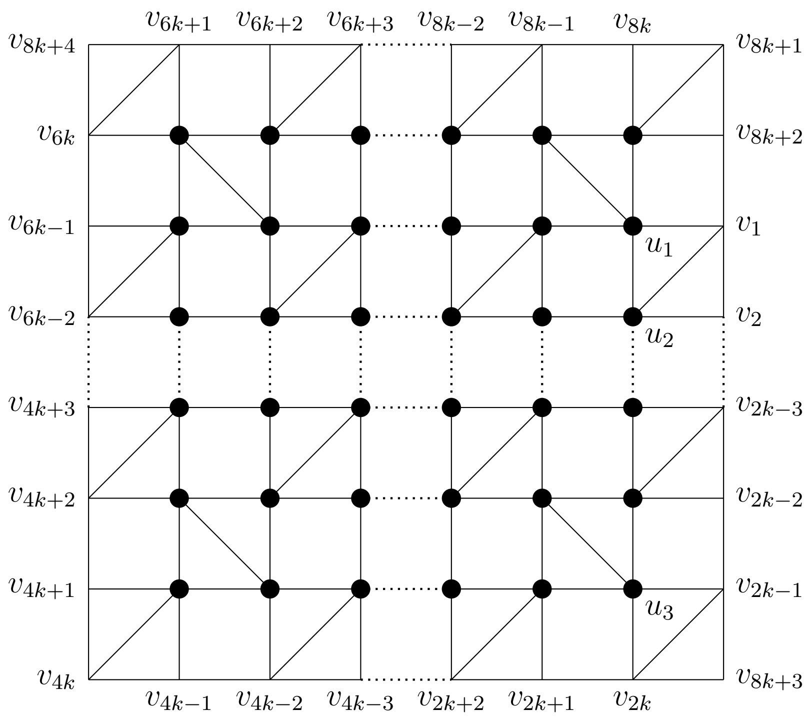

Figure 6.5: An illustration of the proof of Lemma 6.4.3.

Base case: Since W is a D_n with $n > 4$, by Lemma 6.4.1, W must contain a 3-square.

Induction step: Assume W contains a (2k+1)-square. By Lemma 6.4.3, W must contain a (2k+3)-square.

Therefore, W must extend infinitely in both directions. Thus, D is the entire vertex set. □

Lemma 6.4.5: *A PDS that contains only D_4 cannot exist.*

Proof: Figure 6.6 represents the reasoning. To deduce a contradiction, assume there exists a PDS D that contains only D_4. Let v_1, v_2, v_3, v_4 denote a D_4 in D.

Notice v_5, v_6, v_7 are not in D since they are in the double external boundary of the D_4 formed by v_1, v_2, v_3, v_4. To dominate v_6, one of v_8, v_9, v_{10} must be in D.

If $v_8 \in D$, then we must have $v_9 \in D$ for v_8 to be in a D_4, since $v_5, v_6 \notin D$. But v_8, v_9 double force in v_6, contradicting that $v_6 \notin D$. Thus, $v_8 \notin D$.

A similar argument can be applied to show that $v_{10} \notin D$.

Thus, we must have $v_9 \in D$ to dominate v_6. For v_9 to be in a D_4, we must have $v_{11}, v_{12}, v_{13} \in D$, since $v_8, v_{10} \notin D$.

Notice $v_5 \notin D$ is not dominated. But every neighbor of v_5 is either in the external boundary or in the double external boundary of the two D_4s formed by v_1, v_2, v_3, v_4 and $v_9, v_{11}, v_{12}, v_{13}$. Thus, no neighbor of v_5 is in D. So v_5 cannot be dominated, contradicting that D is a PDS. □

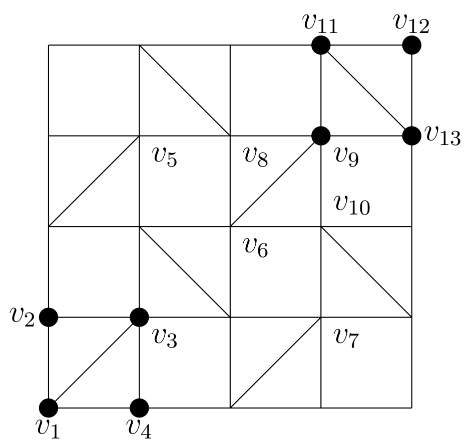

Figure 6.6: An illustration of the proof of Lemma 6.4.5.

Theorem 6.4.6: $(3^2, 4, 3, 4)$ *lattice has only two possible perfect domination proportions, 1 and* $\frac{1}{4}$.

Proof: The perfect domination proportion of 1 is achieved by taking the entire vertex set as a perfect dominating set. The perfect domination proportion of $\frac{1}{4}$ is achieved by a minimal perfect dominating set. By Lemma 6.4.4, any PDS containing a D_n with $n > 4$ is the entire vertex set. Since D_2 and D_3 do not exist, any PDS that is not the entire vertex set can only contain D_1 and D_4. But a PDS that contains only D_4 cannot exist. Therefore, any PDS that is not the entire vertex set must contain D_1. By Lemma 6.3.4, a PDS that contains a D_1 must be a union of D_1 and D_4, and such a PDS is unique up to isomprhism. Therefore, there exist only two nonisomorphic PDS (the minimal PDS and the entire vertex set). □

Chapter 7

The $(4, 6, 12)$ Lattice

7.1 Nonexistence of Efficient Domination

Lemma 7.1.1: *There does not exist an efficient dominating set in the* $(4, 6, 12)$ *lattice.*

Proof: The proof is by contradiction. Assume that there exists an efficient dominating set D. Since $D \neq \emptyset$, there exists a vertex $v_1 \in D$. Figure 7.1 illustrates the reasoning. By vertex-transitivity, any vertex may be chosen to represent v_1.

Vertex v_2 is adjacent to a vertex in $N[v_1]$, so $v_2 \notin D$ or the adjacent vertex would be dominated by both v_1 and v_2. Therefore, v_2 must be dominated by one

of its neighbors. The only neighbor v for which $N[v] \cap N[v_1] = \emptyset$ is v_3, so $v_3 \in D$ if D is to be an efficient dominating set.

Similarly, vertex v_4 is adjacent to a vertex in $N[v_3]$, so $v_4 \notin D$ or the adjacent vertex would be dominated by both v_4 and v_3. Therefore, v_4 must be dominated by one of its neighbors. The only neighbor v for which $N[v] \cap N[v_1] = \emptyset$ and $N[v] \cap N[v_3] = \emptyset$ is v_5, so $v_5 \in D$ if D is to be an efficient dominating set.

Continuing, $N[v_6] \cap N[v_1] \neq \emptyset$ and $N[v_6] \cap N[v_4] \neq \emptyset$, so $v_6 \notin D$. However, every neighbor v of v_6 satisfies $N[v] \cap N[v_4] \neq \emptyset$, so there does not exist any vertex $v \in D$ such that $v \in N[v_6]$. Since there is no $v \in D$ which dominates v_6, D is not a dominating set, and thus not an efficient dominating set, contradicting our original assumption. □

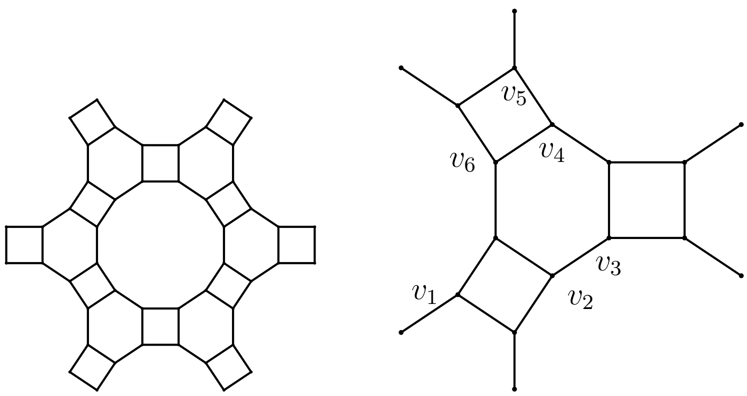

Figure 7.1: The left figure is a subgraph of the $(4,6,12)$ lattice. The right figure is an illustration of the proof of non-existence of an efficient dominating set in the $(4,6,12)$ lattice.

7.2 Domination Ratio

For domination number problems, the generic integer programming method requires an integral variable for every vertex of the graph. The vertex set of an infinite periodic graph is infinite. Therefore, the generic integer program will have infinitely many variables and contraints.

To solve the minimum dominating set problem on the $(4, 6, 12)$ lattice, we introduce a linear programming relaxation on an infinite periodic graph. The relaxation is a minimization problem on a particular polytope (A polyhedron is the solution set of a finite system of linear inequalities. A polytope is a polyhedron that contains no infinite half-line. An inequality $w^T x \leq t$ is valid for a polyhedron P if $P \subseteq \{x : w^T x \leq t\}$.). Furthermore, the relaxation has finitely many constraints and the number of constraints does not depend on the number of vertices. Therefore, the relaxation can be solved in polynomial time by any linear programming solver. Formulating the relaxation requires choosing a subgraph of the infinite periodic graph and examining the properties of the subgraph.

One can use the relaxation to compute compute a lower bound for the domination ratio of an infinite periodic graph. One can also use the relaxation to compute a lower bound for the domination number of a finite subgraph of an infinite periodic graph.

Using the relaxation, we computed a lower bound for the domination ratio of the $(4,6,12)$ lattice. The lower bound equals an upper bound we obtained from a dominating set. Therefore, we obtain the exact value of the domination ratio of the $(4,6,12)$ lattice.

Lemma 7.2.1: $\gamma(4,6,12) \leq \gamma_p(4,6,12) \leq \frac{5}{18}$.

Proof: A periodic PDS D with $\gamma_p(D)$ =$\frac{5}{18}$ is shown in Figure 7.2, establishing $\frac{5}{18}$ as a upper bound. The vertex set of the $(4,6,12)$ lattice can be partitioned into subsets of size 36 such that the subgraph induced by vertices in every subset is isomorphic to G' as shown in Figure 7.2.

To calculate the domination proportion, notice that every subgraph isomorphic to G' has 10 vertices in D. Thus,

$$\gamma_p(D) = \frac{10}{36} = \frac{5}{18}$$

Since any PDS is a dominating set, we have $\gamma(4,6,12) \leq \gamma_p(4,6,12) \leq \frac{5}{18}$.

Note: The vertex set of the (4,6,12) lattice can be partitioned into disjoint subsets such that the subgraph induced by vertices in every subset is isomorphic to H, as shown in Figure 7.3.

Note: The internal boundary of H is illustrated by $\{v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$. Throughout Section 7.2, we do not consider ends of half-edges to be vertices.

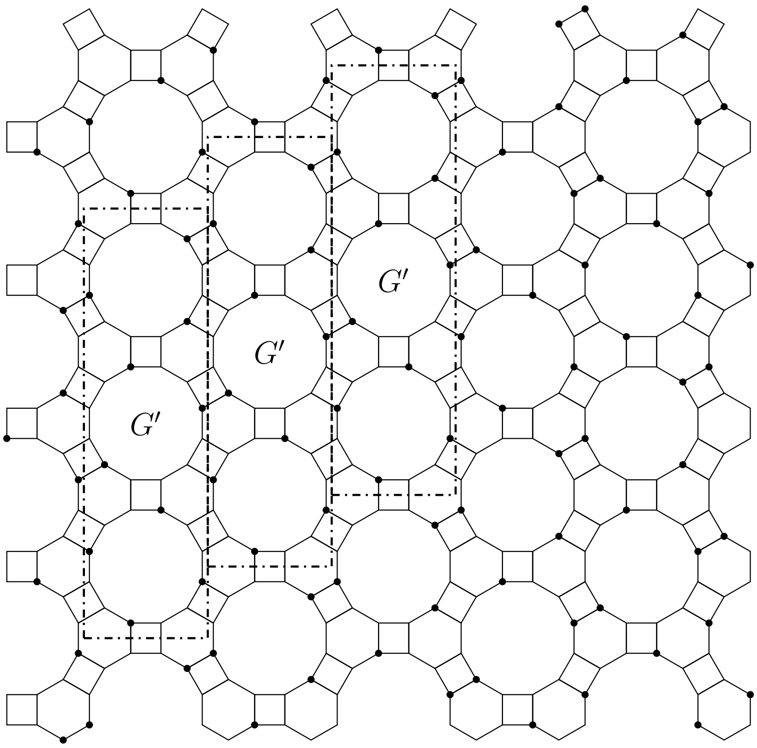

Figure 7.2: A PDS D of (4,6,12) lattice with $\gamma_p(D) = \frac{5}{18}$

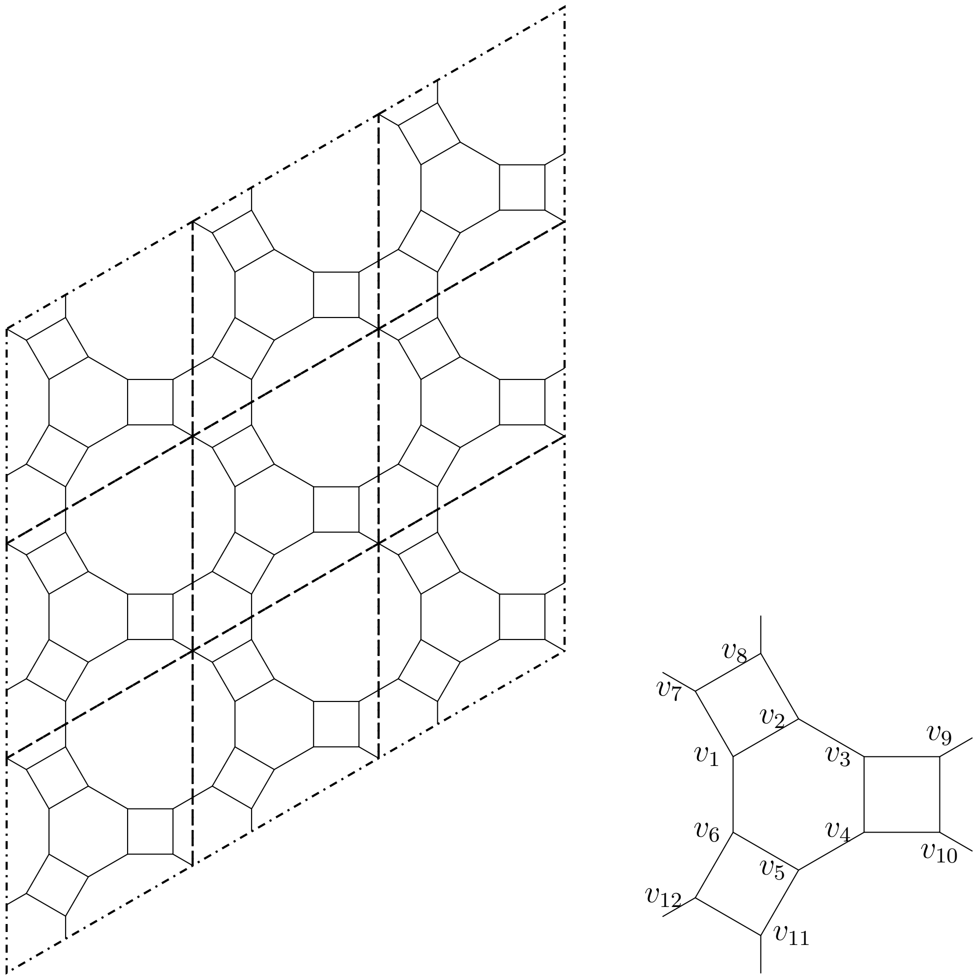

Figure 7.3: Left: A subgraph of the $(4, 6, 12)$ lattice; right: H

Definition (H_n): An H_n is a pair (G, D), where G is a graph isomorphic to H, and D is a dominating set of G assuming boundary vertices of G are dominated for free.

Note: The definition of dominated for free is povided in Section 2.4.

Definition (isomorphic H_n): Let $H^{(1)} = (G^{(1)}, D^{(1)})$ and $H^{(2)} = (G^{(2)}, D^{(2)})$ be two H_ns. We create a loop edge in $G^{(1)}$ for every vertex in $D^{(1)}$ and a loop edge in $G^{(2)}$ for every vertex in $D^{(2)}$. If the resulting $G^{(1)}$ and $G^{(2)}$ are isomorphic, then $H^{(1)}$ and $H^{(2)}$ are isomorphic.

Definition ($\mathcal{H}_n$): For a given n, $\mathcal{H}_n$ is the set of all non-isomorphic H_n.

Figure 7.4 illustrates the following definitions. Let (G, D) be a H_n, where G is an induced subgraph of the $(4,6,12)$ lattice. We have the following definitions:

Definition (V_G): Let V_G denote the set of vertices in G.

Definition (C_G): Graph G contains a unique 6-cycle, illustrated by $\{v_1, v_2, v_3, v_4, v_5, v_6\}$. We denote the set of vertices in the unique 6-cycle in G by C_G.

Definition (B_G): Graph G has six vertices on its internal boundary, illustrated by $\{v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$. We denote the set of vertices on the internal boundary of G by B_G.

Definition ($lend(G, D)$): Let $lend(G, D)$ denote the number of vertices in

$(4,6,12) \setminus G$ dominated by a vertex in D.

Definition ($borrow(G,D)$**):** Let $borrow(G,D)$ denote the number of vertices in G not dominated by a vertex in D.

Note: If a vertex $v \in V_G$ is not dominated by vertices in D, then we must have $v \in B_G$ for D to be a dominating set of G assuming boundary vertices of G are dominated for free.

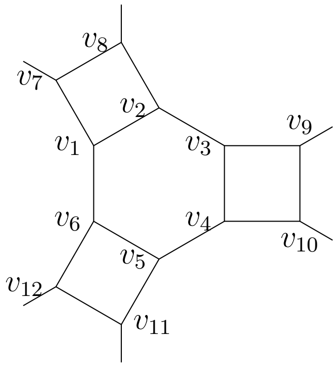

Figure 7.4: An illustration of definitions.

Definition ($netlend(\mathcal{H}_n)$**):** For a fixed n,

$$netlend(\mathcal{H}_n) = \max_{(G,D)\in\mathcal{H}_n} \Big(lend(G,D) - borrow(G,D)\Big).$$

Lemma 7.2.2: *If* (G,D) *is a* H_n*, then* $lend(G,D) = |D \cap B_G|$.

Proof: No vertex in C_G could dominate any vertex in $(4,6,12)\setminus G$. Every vertex in B_G could dominate one vertex in $(4,6,12) \setminus G$. Thus, $lend(G,D) = |D \cap B_G|$.

Lemma 7.2.3: *If* (G, D) *is a* H_n*, then* $borrow(G, D) \geq 6 - |D \cap C_G| - 2 \times |D \cap B_G|$.

Proof: Every vertex in C_G could dominate one vertex in B_G. Every vertex in B_G could dominate two vertices in B_G. Since some vertices in B_G may be dominated twice, at most $|D \cap C_G| + 2 \times |D \cap B_G|$ vertices in B_G are dominated by vertices in $V_G \cap D$. Thus, at least $6 - |D \cap C_G| - 2 \times |D \cap B_G|$ vertices in B_G are not dominated by vertices in V_G. Thus, $borrow(G, D) \geq 6 - |D \cap C_G| - 2 \times |D \cap B_G|$.

Fact 7.2.4: *If* (G, D) *is a* H_n*, then* $|D \cap B_G| = n - |D \cap C_G|$.

Proof: Since (G, D) is a H_n, $|D \cap B_G| + |D \cap C_G| = |D| = n$. Thus, $|D \cap B_G| = n - |D \cap C_G|$.

Lemma 7.2.5: *If* (G, D) *is a* H_n*, then* $|D \cap C_G| \geq \lceil \frac{6-n}{2} \rceil$.

Proof: Every vertex in C_G could dominate three vertices in C_G. Every vertex in B_G could dominate one vertex in C_G. To dominate all six vertices in C_G, we must have

$$3 \times |D \cap C_G| + |D \cap B_G| \geq 6.$$

By Fact 7.2.4, $|D \cap B_G| = n - |D \cap C_G|$. Thus,

$$3 \times |D \cap C_G| + (n - |D \cap C_G|) \geq 6,$$

so

$$|D \cap C_G| \geq \frac{6-n}{2}.$$

Since $|D \cap C_G|$ is an integer, we have

$$|D \cap C_G| \geq \lceil \frac{6-n}{2} \rceil.$$

Lemma 7.2.6: *$netlend(\mathcal{H}_2)$ = -4.*

Proof: Assume (G, D) is a H_2. By Lemma 7.2.5, $|D \cap C_G| \geq \lceil \frac{6-2}{2} \rceil = 2$. Since (G, D) is a H_2, by Fact 7.2.4, $|D \cap B_G| = 2 - |D \cap C_G| = 0$. By Lemma 7.2.2, $lend(G, D) = |D \cap B_G| = 0$. By Lemma 7.2.3,

$$borrow(G, D) \geq 6 - |D \cap C_G| - 2 \times |D \cap B_G| = 6 - 2 - 0 = 4.$$

Thus,

$$netlend(\mathcal{H}_2) = \max_{(G,D) \in \mathcal{H}_2} \Big(lend(G, D) - borrow(G, D) \Big) \leq 0 - 4 = -4.$$

Figure 7.5 demonstrates a pair (G', D') that is a H_2 such that $borrow(G', D') - lend(G', D') = -4$. Thus, $netlend(\mathcal{H}_2) = -4$.

Lemma 7.2.7: $netlend(\mathcal{H}_3) = -1$.

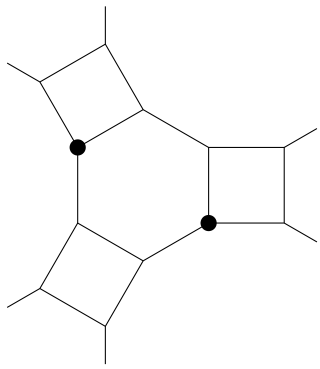

Figure 7.5: An illustration of the proof of Lemma 7.2.6.

Proof: Assume (G, D) is a H_3. By Lemma 7.2.5, $|D \cap C_G| \geq \lceil \frac{6-3}{2} \rceil = 2$. We consider a few cases, depending on the number of vertices of D in C_G.

Case 1: $|D \cap C_G| = 3$. Since (G, D) is a H_3, by Fact 7.2.4, $|D \cap B_G| = 3 - |D \cap C_G| = 0$. By Lemma 7.2.2, $lend(G, D) = |D \cap B_G| = 0$. By Lemma 7.2.3,

$$borrow(G, D) \geq 6 - |D \cap C_G| - 2 \times |D \cap B_G| \geq 6 - 3 = 3.$$

Thus,

$$lend(G, D) - borrow(G, D) \leq 0 - 3 = -3.$$

Case 2: $|D \cap C_G| = 2$. Since (G, D) is a H_3, by Fact 7.2.4, $|D \cap B_G| = 3 - |D \cap C_G| = 1$. By Lemma 7.2.2, $lend(G, D) = |D \cap B_G| = 1$. By Lemma 7.2.3,

$$borrow(G, D) \geq 6 - |D \cap C_G| - 2 \times |D \cap B_G| \geq 6 - 2 - 2 = 2.$$

Thus,

$$lend(G, D) - borrow(G, D) \leq 1 - 2 = -1$$

.

In every case,

$$netlend(\mathcal{H}_3) = \max_{(G,D)\in\mathcal{H}_3} \Big(lend(G, D) - borrow(G, D)\Big) \leq -1.$$

Figure 7.6 demonstrates a pair (G', D') that is a H_3 such that $borrow(G', D') - lend(G', D') = -1$. Thus, $netlend(\mathcal{H}_3) = -1$.

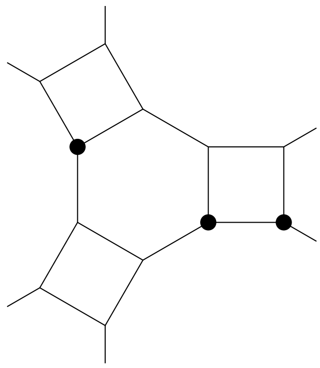

Figure 7.6: An illustration of the proof of Lemma 7.2.7.

Lemma 7.2.8: $netlend(\mathcal{H}_4) = 2$.

Proof: Assume (G, D) is a H_4. By Lemma 7.2.5, $|D \cap C_G| \geq \lceil\frac{6-4}{2}\rceil = 1$. We consider a few cases, depending on the number of vertices of D in C_G.

Case 1: $|D \cap C_G| = 1$. Since (G, D) is a H_4, by Fact 7.2.4, $|D \cap B_G| = 4 - |D \cap C_G| = 3$. Figure 7.7 represents the reasoning. Since $|D \cap C_G| = 1$ and choices of vertex in $|D \cap C_G|$ are equivalent by symmetry, let $v_1 \in |D \cap C_G|$. To dominate v_3, v_4, v_5, we must have $v_9, v_{10}, v_{11} \in D$. Since v_7 is not dominated by a vertex in $V_G \cap D$, $borrow(G, D) = 1$. By Lemma 7.2.2, $lend(G, D) = |D \cap B_G| = 3$. Thus,

$$lend(G, D) - borrow(G, D) \leq 3 - 1 = 2.$$

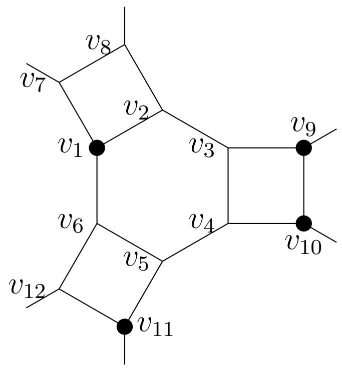

Figure 7.7: An illustration of the proof of Lemma 7.2.8.

Case 2: $|D \cap C_G| = 2$. Since (G, D) is a H_4, by Fact 7.2.4, $|D \cap B_G| = 4 - |D \cap C_G| = 2$. By Lemma 7.2.2, $lend(G, D) = |D \cap B_G| = 2$. By Lemma 7.2.3,

$$borrow(G, D) \geq 6 - |D \cap C_G| - 2 \times |D \cap B_G| \geq 6 - 2 - 4 = 0.$$

Thus,

$$lend(G, D) - borrow(G, D) \leq 2 - 0 = 2.$$

Case 3: $|D \cap C_G| = 3$. Since (G, D) is a H_4, by Fact 7.2.4, $|D \cap B_G| = 4 - |D \cap C_G| = 1$. By Lemma 7.2.2, $lend(G, D) = |D \cap B_G| = 1$. By Lemma 7.2.3,

$$borrow(G, D) \geq 6 - |D \cap C_G| - 2 \times |D \cap B_G| \geq 6 - 3 - 2 = 1.$$

Thus,

$$lend(G, D) - borrow(G, D) \leq 1 - 1 = 0.$$

Case 4: $|D \cap C_G| = 4$. Since (G, D) is a H_4, by Fact 7.2.4, $|D \cap B_G| = 4 - |D \cap C_G| = 0$. By Lemma 7.2.2, $lend(G, D) = |D \cap B_G| = 0$. By Lemma 7.2.3,

$$borrow(G, D) \geq 6 - |D \cap C_G| - 2 \times |D \cap B_G| \geq 6 - 4 - 0 = 2.$$

Thus,

$$lend(G, D) - borrow(G, D) \leq 0 - 2 = -2.$$

In every case,

$$netlend(\mathcal{H}_4) = \max_{(G,D) \in \mathcal{H}_4} \Big(lend(G, D) - borrow(G, D) \Big) \leq 2.$$

Figure 7.7 demonstrates a pair (G', D') that is a H_4 such that $borrow(G') - lend(G') = 2$. Thus, $netlend(\mathcal{H}_4) = 2$.

Lemma 7.2.9: $netlend(\mathcal{H}_5) = 4$.

Proof: Assume (G, D) is a H_5. By Lemma 7.2.5, $|D \cap C_G| \geq \lceil \frac{6-5}{2} \rceil = 1$. Since (G, D) is a H_5, by Fact 7.2.4, $|D \cap B_G| = 5 - |D \cap C_G| \leq 4$. By Lemma 7.2.2, $lend(G, D) = |D \cap B_G| \leq 4$. Notice $borrow(G, D) \geq 0$. Thus,

$$netlend(\mathcal{H}_5) = \max_{(G,D) \in \mathcal{H}_5} \Big(lend(G, D) - borrow(G, D) \Big) \leq 4 - 0 = 4.$$

Figure 7.8 demonstrates a pair (G', D') that is a H_5 such that $borrow(G', D') - lend(G', D') = 4$. Thus, $netlend(\mathcal{H}_5) = 4$.

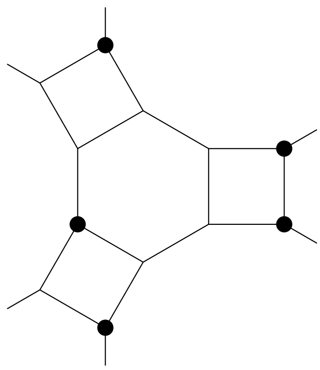

Figure 7.8: An illustration of the proof of Lemma 7.2.9.

Lemma 7.2.10: *For* $n \geq 6$*,* $netlend(\mathcal{H}_n) = 6$.

Proof: Assume (G, D) is a H_n, where $n \geq 6$. Notice $lend(G, D) \leq 6$ and

$borrow(G, D) \geq 0$. Thus,

$$netlend(\mathcal{H}_n) = \max_{(G,D)\in\mathcal{H}_n} \Big(lend(G, D) - borrow(G, D)\Big) \leq 6.$$

Since $n \geq 6$, we can choose all vertices in B_G to be in D such that D is a dominating set of G. In this case, $lend(G, D) = 6$ and $borrow(G, D) = 0$. Consequently, $lend(G, D) - borrow(G, D) = 6$. Thus, $netlend(\mathcal{H}_n) = 6$.

Definition ($p_n(G), p_n$): Let D be a dominating set of the $(4,6,12)$ lattice. Let G be a subgraph of the $(4,6,12)$ lattice whose vertex set can be partitioned into disjoint subsets $S_1, S_2, ..., S_m$ such that for every subset S_i, the pair $(G_i, D \cap S_i)$ is a H_n, where G_i is the subgraph induced by vertices in S_i. For $n = 2, 3, 4, ..., 12$, let $p_n(G)$ denote the proportion of H_n in the vertex disjoint subgraphs of G.

Note: We can embed the $(4,6,12)$ lattice in the plane such that the subgraph induced by vertices in every unit square with integer coordinates is isomorphic to H as shown in Figure 7.3. In Lemma 7.2.11 and Theorem 7.2.12, we consider such embedding.

Lemma 7.2.11: *Let $R_{l,m}$ denote a rectangular region $R_G(0, l; 0, m)$, where $l, m > 0$. We have*

$$\sum_{k=2,...,12} p_k \times netlend(\mathcal{H}_k) \geq -\epsilon_{l,m},$$

where $\epsilon_{l,m} \to 0^+$ as $l, m \to \infty$.

Proof: Let D be any dominating set of the $(4,6,12)$ lattice. The vertex set of $R_{l,m}$ can be partitioned into disjoint subsets $S_1, S_2, ..., S_{lm}$ such that for every subset S_i, the pair $(G_i, D \cap S_i)$ is an H_n, where G_i is the subgraph induced by vertices in S_i. For any $i = 1, ..., lm$, let $D_i = D \cap S_i$. Let $D^{(l,m)} = \sum_{i=1,...,lm} D_i$.

Let $N_k(R_{l,m})$ denote the number of H_k in $(G_1, D_1), (G_2, D_2), ..., (G_{lm}, D_{lm})$. Let a, b, c, d denote the number of vertices in the upper, lower, left and right internal boundary of $R_{l,m}$ respectively.

Notice that

$$p_k(R_{l,m}) = \frac{N_k(R_{l,m})}{lm}.$$

Therefore,

$$\sum_{k=2,...,12} p_k(R_{l,m}) \times netlend(\mathcal{H}_k) = \sum_{k=2,...,12} \frac{N_k(R_{l,m})}{lm} netlend(\mathcal{H}_k).$$

Since $netlend(\mathcal{H}_k) = \max_{(G,D)\in\mathcal{H}_k} \Big(lend(G,D) - borrow(G,D)\Big)$, for $i = 1, ..., m$, if (G_i, D_i) is a H_k, then $netlend(\mathcal{H}_k) \geq lend(G_i, D_i) - borrow(G_i, D_i)$. Therefore,

$$\sum_{k=2,...,12} \frac{N_k(R_{l,m})}{lm} netlend(\mathcal{H}_k) \geq \sum_{i=1,...,lm} \frac{lend(G_i, D_i) - borrow(G_i, D_i)}{lm},$$

which can be rewritten as

$$\sum_{k=2,...,12} p_k(R_{l,m}) \times netlend(\mathcal{H}_k) \geq \sum_{i=1,...,lm} \frac{lend(G_i, D_i) - borrow(G_i, D_i)}{lm}.$$

For D to be a dominating set of the $(4,6,12)$ lattice, every vertex $v \in B_{G_i}$ not dominated by a vertex in D_i must be dominated by a vertex in $D \setminus D_i$. In addition, a vertex $v \in B_{G_i}$ may be dominated both by a vertex in D_i and by a vertex in $D \setminus D_i$. Therefore,

$$\sum_{i=1,\ldots,lm} \Big(lend(G_i, D_i) - borrow(G_i, D_i)\Big) \geq lend(R_{l,m}, D^{(l,m)}) - borrow(R_{l,m}, D^{(l,m)}).$$

Since $lend(R_{l,m}, D^{(l,m)}) \geq 0$ and $borrow(R_{l,m}, D^{(l,m)}) \leq a+b+c+d$, we have

$$lend(R_{l,m}, D^{(l,m)}) - borrow(R_{l,m}, D^{(l,m)}) \geq 0 - (a+b+c+d).$$

Consequently,

$$\sum_{i=1,\ldots,lm} \Big(lend(G_i, D_i) - borrow(G_i, D_i)\Big) \geq -(a+b+c+d).$$

Since $l, m > 0$, we divide both sides by lm and obtain

$$\sum_{i=1,\ldots,lm} \frac{lend(G_i, D_i) - borrow(G_i, D_i)}{lm} \geq -\frac{a+b+c+d}{lm},$$

so

$$\sum_{k=2,\ldots,12} p_k(R_{l,m}) \times netlend(\mathcal{H}_k) \geq -\frac{a+b+c+d}{lm}.$$

Letting $\epsilon_{l,m} = \frac{a+b+c+d}{lm}$, we have

$$\sum_{k=2,\ldots,12} p_k(R_{l,m}) \times netlend(\mathcal{H}_k) \geq -\epsilon_{l,m}.$$

Since $a + b = O(l)$ and $c + d = O(m)$, as $m, n \to \infty$, we have

$$\epsilon_{l,m} = \frac{a+b+c+d}{lm} \to 0^+.$$

Theorem 7.2.12: $\gamma(4, 6, 12) = \gamma_p(4, 6, 12) = \frac{5}{18}$.

Proof: We prove that both the domination ratio and the perfect domination ratio of the $(4, 6, 12)$ lattice are equal to $\frac{5}{18}$.

Consider a rectangular region $R_{l,m}$ as above. We formulate the domination ratio problem in $R_{l,m}$ as a linear program. The set of all feasible solutions is described by a polytope. Lemma 7.2.11 provides a valid inequality for the polytope, which is a constraint for the LP. We describe the constraints, objective function, linear program, dual program in parts 1,2,3, and 4 of the proof respectively. The optimal solution to the linear program provides a lower bound for the domination ratio of $R_{l,m}$, as described in part 3.

In part 5, we prove the optimal solution to the linear program is a continuous function of $\epsilon_{l,m}$, where $\epsilon_{l,m} \to 0^+$ as $l, m \to \infty$. Recall the domination ratio is defined as

$$\lim_{m,n\to\infty} \frac{\gamma_{m,n}(G)}{N_{m,n}(G)}.$$

Since $\epsilon_{l,m} \to 0^+$ as $l,m \to \infty$, the optimal objective function value when $\epsilon_{l,m} = 0$ is a lower bound for the domination ratio of the $(4,6,12)$ lattice.

In part 6, we demonstrate that optimal objective function value when $\epsilon_{l,m} = 0$ is $\frac{5}{18}$. Thus, we have $\frac{5}{18}$ as a lower bound for the domination ratio. Combined with Lemma 7.2.1, we conclude that $\gamma(4,6,12) = \gamma_p(4,6,12) = \frac{5}{18}$.

1. Constraints

Let $x = [p_2, p_3, p_4, p_5, p_{other}]^T$, where $p_{other} = \sum_{k\geq 6} p_k$.

By Lemma 7.2.11, we have

$$\sum_{k=2,\ldots,12} p_k \times netlend(\mathcal{H}_k) \geq -\epsilon_{l,m},$$

where $\epsilon_{l,m} \to 0^+$ as $l,m \to \infty$.

By Lemma 7.2.10, for $n \geq 6$, $netlend(\mathcal{H}_n) = 6$. Therefore,

$$\left(\sum_{n=2,3,4,5} p_n \times netlend(\mathcal{H}_n)\right) + p_{other} \times 6 \geq -\epsilon_{l,m}.$$

For $n = 2,3,4,5$, $netlend(\mathcal{H}_n)$ is calculated in Lemma 7.2.6, Lemma 7.2.7, Lemma 7.2.8, and Lemma 7.2.9. Thus,

$$[-4,-1,2,4,6]x \geq [-4,-1,2,4,6][p_2,p_3,p_4,p_5,p_{other}]^T \geq -\epsilon_{l,m}.$$

where $\epsilon_{l,m} \to 0^+$ as $l,m \to \infty$.

Notice that we also have constraints $\sum_k p_k = 1$ and $0 \leq p_k \leq 1$ for any p_k.

2. Objective function

Let $c = \frac{1}{12}[2,3,4,5,6]^T$. Notice c is multiplied by $\frac{1}{12}$ because H_n has 12 vertices. For any dominating set D of $R_{l,m}$,

$$\gamma(D) = \sum_{k=2,3,\ldots,12} \frac{k}{12} p_k \geq \frac{1}{12}[2,3,4,5,6][p_2,p_3,p_4,p_5,p_{other}]^T = c^T x.$$

3. Linear program (LP)

The linear program below provides a lower bound for the domination ratio of $R_{l,m}$.

$$\min c^T x \text{ subject to}$$

$$[-4,-1,2,4,6]x \geq -\epsilon_{l,m}$$

$$\sum_i x_i = 1 \text{ and for any i, } 0 \leq x_i \leq 1$$

The linear program provides a lower bound for the domination ratio of $R_{l,m}$

because a minimum dominating set D with associated vector x^* satisfies the constraints above and $\gamma(D) \geq c^T x^*$.

Writing the LP explicitly in matrix form:

$$\min c^T x = \frac{1}{12}[2, 3, 4, 5, 6]x \text{ subject to } x \geq \vec{0} \text{ and}$$

$$Ax = \begin{bmatrix} -4 & -1 & 2 & 4 & 6 \\ 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} x \geq \begin{bmatrix} 0 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} = b$$

4. Dual program (DP)

The dual program is

$$\max b^T y = [0, 1, -1, -1, -1, -1, -1, -1]y \text{ subject to } y \geq \vec{0} \text{ and}$$

$$A^T y = \begin{bmatrix} -4 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 2 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 4 & 1 & -1 & 0 & 0 & 0 & -1 & 0 \\ 6 & 1 & -1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} y \leq \frac{1}{12} \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} = c$$

5. $c^T x^*$ **is a continuous funtion of** $\epsilon_{l,m}$

By Lemma 7.2.11,

$$\sum_{k=2,\ldots,12} p_k \times netlend(\mathcal{H}_k) \geq -\epsilon_{l,m}$$

where $\epsilon_{l,m} \to 0^+$ as $l, m \to \infty$.

The inequality above is a constraint in the LP. We want to show that $c^T x^*$ is a continuous funtion of $\epsilon_{l,m}$, where $\epsilon_{l,m} \to 0^+$ and x^* is the primal optimal solution.

Consider the dual objective function value $b^T y^*$, where y^* is the dual optimal solution. Notice $b_1 = -\epsilon_{l,m}$ and other entries of b are fixed real numbers. Thus, $b^T y^*$ is a function of $\epsilon_{l,m}$. All entries in A and c are fixed real numbers.

Let $P = \{y : A^T y \leq c\}$ be a polytope. Let $v^{(1)}, .., v^{(n)}$ denote extreme points of the polytope P. Since the dual program is linear, the dual optimal objective function value is achieved at one of the extreme points. Therefore,

$$\max b^T y = \max_{i=1,\ldots,n} b^T v^{(i)}.$$

Since for any i, $b^T v^{(i)}$ is a linear function of $\epsilon_{l,m}$, $\max_{i=1,\ldots,n} b^T v^{(i)}$ is a convex function. Since convex functions are continuous, $\max_{i=1,\ldots,n} b^T v^{(i)}$ is a continuous function of $\epsilon_{l,m}$. Thus, $b^T y^*$ is a continuous function of $\epsilon_{l,m}$. By the Strong Duality Theorem, $c^T x^* = b^T y^*$. Therefore, $c^T x^*$ is a continuous function of $\epsilon_{l,m}$.

6. Optimal solution

By part 5, the optimal objective function value of the LP is a continuous funtion of $\epsilon_{l,m}$. Recall the domination ratio is defined as

$$\lim_{m,n\to\infty} \frac{\gamma_{m,n}(G)}{N_{m,n}(G)}.$$

Since $\epsilon_{l,m} \to 0^+$ as $l, m \to \infty$, the optimal objective function value when $\epsilon_{l,m} = 0$ is a lower bound for the domination ratio of the $(4, 6, 12)$ lattice.

For the linear program, by letting $\epsilon_{l,m} = 0$, we obtain $x^* = [1/3, 0, 2/3, 0, 0]^T$ as an optimal solution with optimal objective function value $\frac{5}{18}$.

For the dual program, by letting $\epsilon_{l,m} = 0$, we obtain $y^* = [5/180, 5/18, 0, 0, 0, 0, 0, 0]^T$ as an optimal solution with optimal objective function value $\frac{5}{18}$.

To check that x^* is the optimal solution, one can verify that x^* is primal feasible and y^* is dual feasible. One can also verify that the primal objective function value at x^* and dual objective function value at y^* are both equal to $\frac{5}{18}$. By Strong Duality Theorem, x^* and y^* are optimal solutions of primal and dual respectively.

Therefore, $\frac{5}{18} \leq \gamma(4, 6, 12)$. By Lemma 7.2.2, $\gamma(4, 6, 12) \leq \gamma_p(4, 6, 12) \leq \frac{5}{18}$. Combining the two inequalities, we get

$$\frac{5}{18} \leq \gamma(4, 6, 12) \leq \gamma_p(4, 6, 12) \leq \frac{5}{18}.$$

Therefore, $\gamma(4, 6, 12) = \gamma_p(4, 6, 12) = \frac{5}{18}$.

7.3 Perfect Domination Ratio

In Theorem 7.2.12, we proved that $\gamma_p(4, 6, 12) = \frac{5}{18}$. A periodic PDS D with $\gamma_p(D) = \frac{5}{18}$ is shown in Figure 7.2.

7.4 Possible Perfect Domination Proportions

Fact: *The* $(4, 6, 12)$ *lattice has three non-isomorphic PDS that achieve the perfect domination proportion of* $\frac{1}{3}$.

Proof: Three non-isomorphic PDS are shown in Figure 7.10, Figure 7.11, and Figure 7.12 respectively. Notice for each PDS, each dodecagon has 4 vertices in D. Thus, $\gamma_p(D) = \frac{4}{12} = \frac{1}{3}$. □

Definition (A row of D_1s): A row of D_1s is a sequence (possibly doubly-infinite) of at least two consecutive D_1s such that every two consecutive D_1s in the sequence are distance three apart and lie in a line which bisects hexagonal faces of the lattice.

Note: A row of D_1s is shown in Figure 7.9.

Lemma 7.4.1: *The* $(4, 6, 12)$ *lattice has infinitely many non-isomorphic PDS that achieve distinct perfect domination proportions. Furthermore, the perfect dominination proportion can be any rational number between* $\frac{5}{18}$ *and 1.*

Proof: A PDS D with $\gamma_p(D) = \frac{5}{18}$ is shown in Figure 7.9. Let V denote the entire vertex set of the lattice. Let W denote the set of vertices that consists of v_1, v_2, v_3, v_4, and all such vertices in 4-cycles bordered by two parallel rows of

D_1s of minimum distance apart. Let H denote the subgraph induced by W.

Let $D' = W \cup D$. Notice D' is a also PDS. Out of three dodecagons, two has three vertices in D' and one has eight vertices in D'. Thus, $\gamma_p(D') = \frac{3+3+8}{12+12+12} = \frac{7}{18}$.

Next, consider adding vertices in every other four cycle in H to D. Let D'' denote the resulting set of vertices. Notice D'' is still a PDS because adding vertices in H that are in the same 4-cycle to D does not affect the other vertices. We calculate $\gamma_p(D'') = \frac{1}{2} * (\frac{5}{18} + \frac{7}{18}) = \frac{1}{3}$.

Similarly, given any rational number between $\frac{5}{18}$ and $\frac{7}{18}$, we can add a corresponding proportion of vertices in H to D and create a PDS D''' such that $\gamma_p(D''')$ equals the given number.

The same reasoning can be applied to vertices in $V \setminus W$ to show that perfect domination proportion can take any rational number between $\frac{5}{18}$ and 1. Because adding vertices on one side of a row of D_1s to D does not affect the other side. $\square$

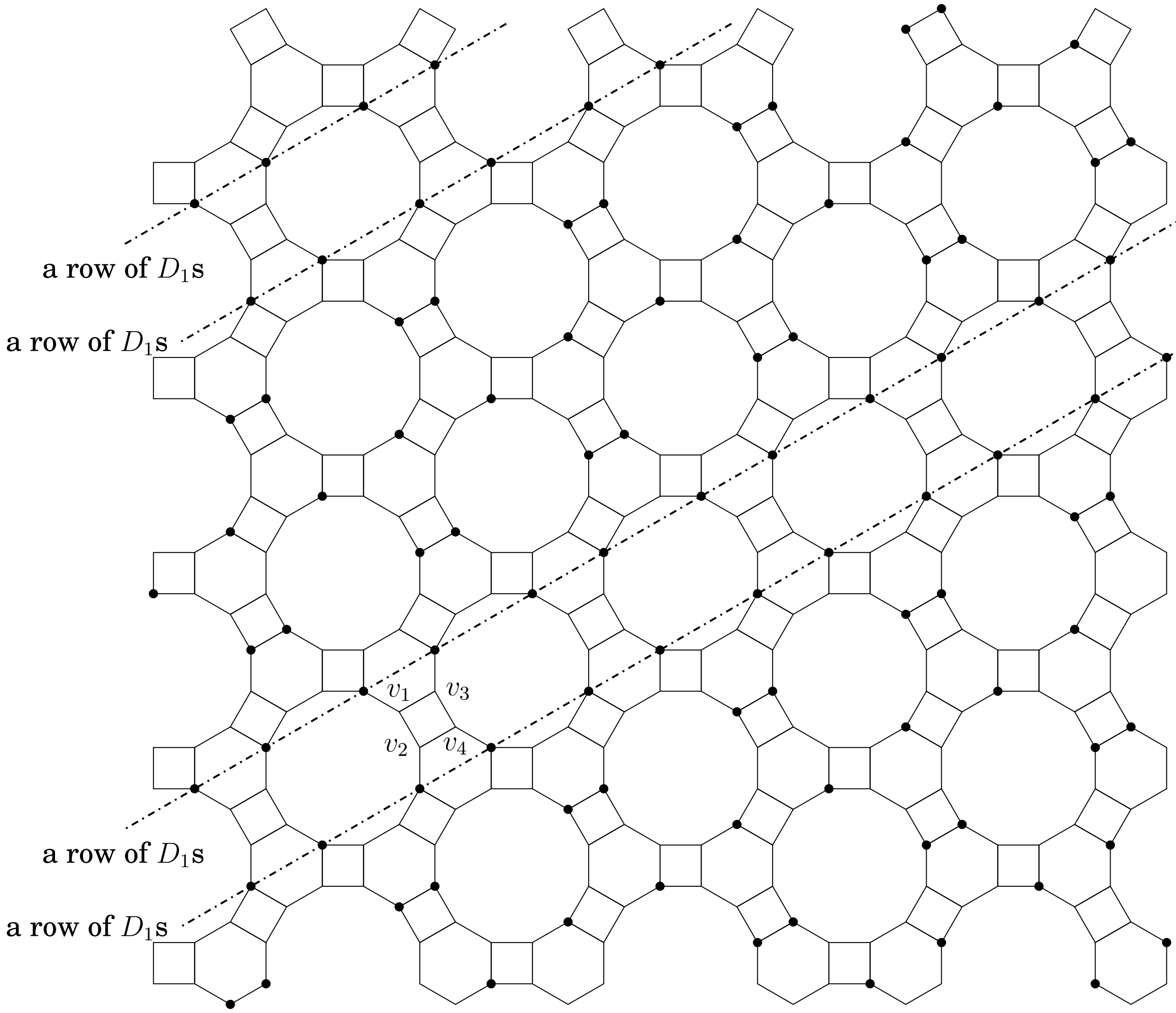

Figure 7.9: An illustration of the proof of Lemma 7.4.1.

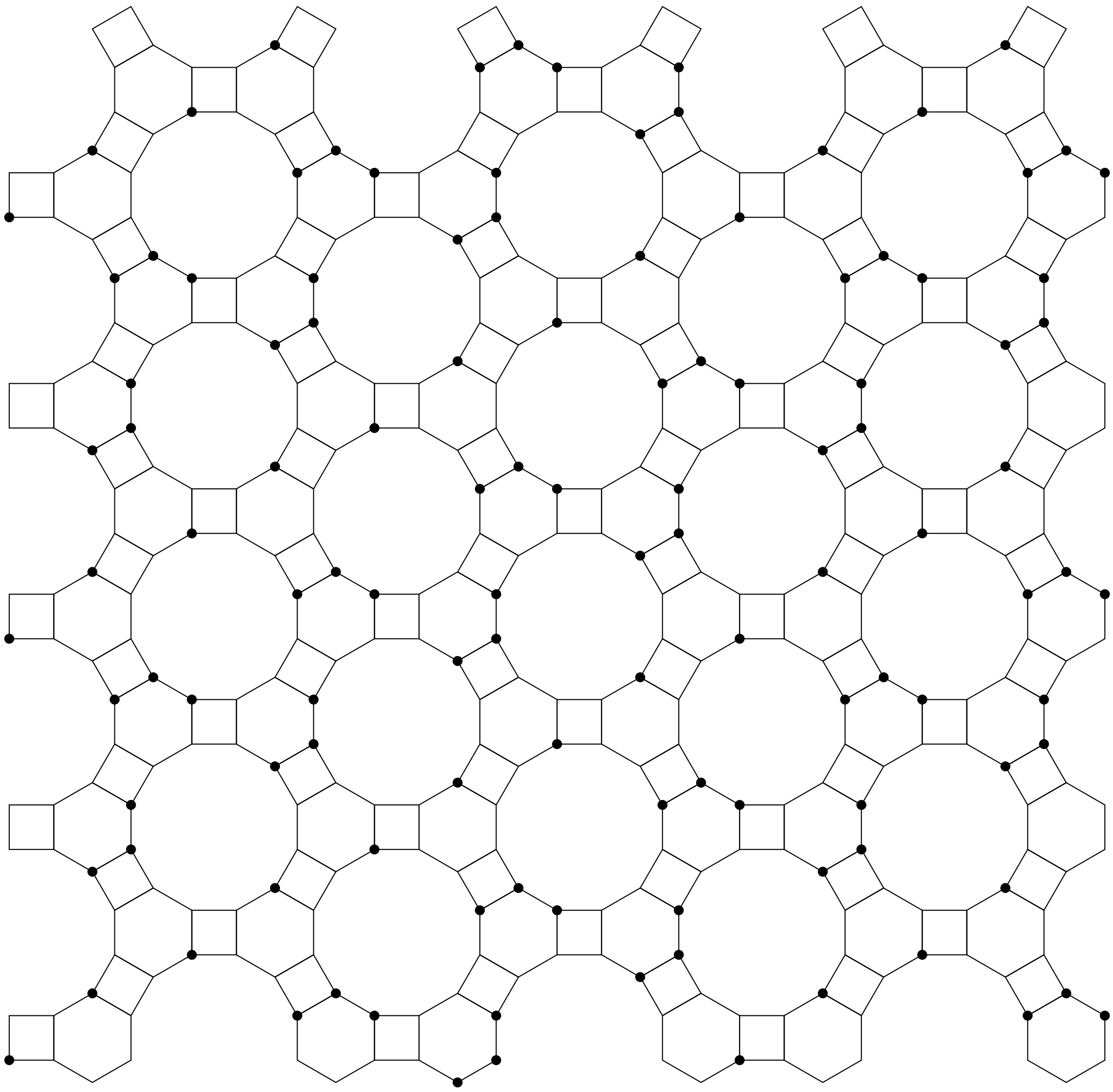

Figure 7.10: A PDS D of $(4, 6, 12)$ lattice with $\gamma_p(D) = \frac{1}{3}$.

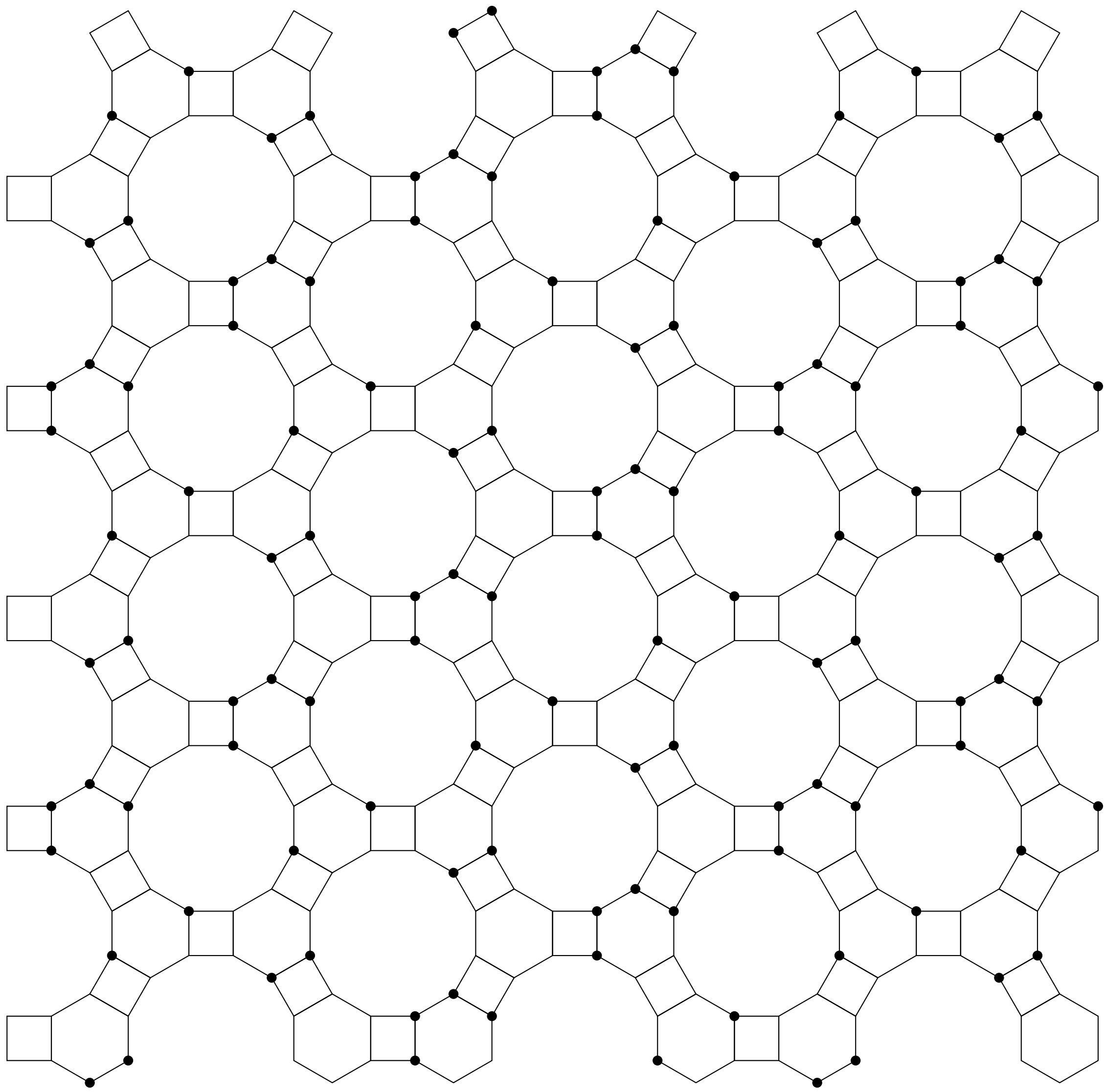

Figure 7.11: A PDS D of $(4, 6, 12)$ lattice with $\gamma_p(D) = \frac{1}{3}$.

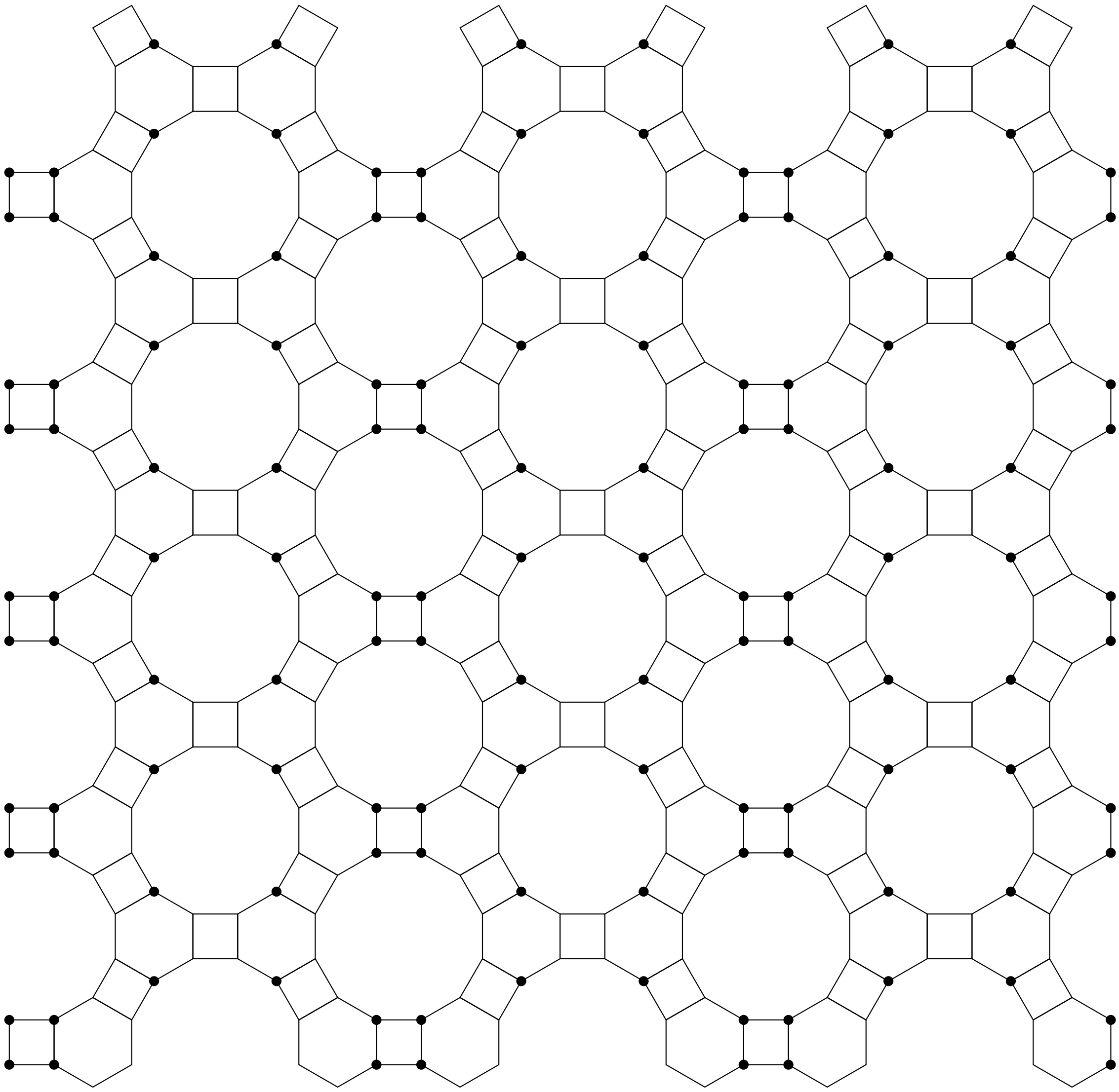

Figure 7.12: A PDS D of $(4, 6, 12)$ lattice with $\gamma_p(D) = \frac{1}{3}$.

Chapter 8

Integer Programming

We use an integer program to compute an upper bound and a lower bound for the domination ratio of the kagome lattice, which does not have an efficient dominating set. First choose a finite subgraph G such that the entire vertex set of the kagome lattice can be partitioned into subsets and the subgraph induced by the subsets are connected and isomorphic to G. Let x be a binary vector representing vertices in S, a subset of the vertex set of G. The closed neighborhood matrix N of G is the sum of the adjacency matrix of G and the identity matrix. [1]

An integer program to compute an upper bound for the domination ratio is as follows:

$$\min \frac{1}{n}(\vec{1})^T x \text{ s.t. } x \epsilon \{0,1\}^n \text{ and } Nx \geq \vec{1}$$

Notice the constraint $Nx \geq \vec{1}$ ensures that S is a dominating set of G. Let x

be any feasible solution. The objective function value $\frac{1}{n}(\vec{1})^T x$ provides an upper bound for the domination ratio, because we can obtain a dominating set D of the entire lattice by taking the minimal dominating set of every subgraph, and the minimal dominating set of the entire lattice may be smaller than D. One can find more details of the integer programming method in two-volume series by Haynes, Hedetniemi, and Slater. [1]

For an integer program to compute a lower bound for the domination ratio, we let b be a binary vector with zero entries corresponding to vertices on the external boundary and other entries are ones. We replace the constraint $Nx \geq \vec{1}$ in the integer program above with $Nx \geq \vec{b}$ and keep the rest the same. An integer program to compute a lower bound for the domination ratio is as follows:

$$\min \frac{1}{n}(\vec{1})^T x \text{ s.t. } x \epsilon \{0,1\}^n \text{ and } Nx \geq \vec{b}$$

Notice the constraint $Nx \geq \vec{b}$ ensures that S is a dominating set of G, assuming boundary vertices of G are dominated for free. Let the optimal solution be x^*. The optimal objective function value $\frac{1}{n}(\vec{1})^T x^*$ provides a lower bound for the domination ratio, because the domination ratio of the entire lattice can only decrease when we assume some vertices are dominated for free.

We wrote an integer program for the kagome lattice and obtained the nontrivial lower bound $\frac{94}{462} > 0.2034632$. Note the trivial lower bound is 0.2 and the best upper bound we have is $\frac{2}{9} < 0.2222223$.

Chapter 9

Conclusion

We have shown that seven of the eleven Archimedean lattices are efficiently dominated and the other four are not efficiently dominated. We have determined exact perfect domination ratios for all of the eleven Archimedean lattices. Tight bounds for domination ratios are obtained using integer programming.

For some ideas about future research on this problem, one might consider solving for the exact domination ratio of the $(3, 6, 3, 6)$, $(3, 4, 6, 4)$, and $(3^2, 4, 3, 4)$ lattices. Domination ratios and perfect domination ratios of the other classes of infinite lattices such as 2-uniform lattices, or three dimensional lattices, such as the cube, face-centered cube, and body centered cube, may be investigated.

For the kagome lattice, we have shown the number of possible perfect domination proportion values is infinite. For the $(3^2, 4, 3, 4)$ lattice, we have proved there are only two possible perfect domination proportions. It would be interesting to consider nonisomorphic perfect dominating sets of and possible perfect domination proportions for all Archimedean lattices. In particular, for each lattice, to determine whether the number of possible perfect domination proportion values is finite or infinite. Furthermore, it would be interesting to determine whether perfect domination proportions can be irrational for Archimedean lattices and for infinite periodic graphs in general.

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Vita

Yunfan comes from Hangzhou, China. He is a graduate student in the combined Bachelor's Master's program in Applied Mathematics and Statistics at Johns Hopkins. He is particularly fascinated by discrete optimization problems such as finding a minimum vertex cover and using graph theory and integer programming tools to solve this class of problems. When he is not busy researching, he likes running and cooking. He recently learned to make creme brulee and chicken curry. After graduation, he plans to continue to do research and pursue a Ph.D. degree.