Real Johnson-Wilson Theories and Computations

by

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A dissertation submitted to The Johns Hopkins University in conformity with the requirements for the degree of Doctor of Philosophy.

Baltimore, Maryland

February 18, 2016

Abstract

Our main result is a computation of $ER(n)^*(\mathbb{C}P^\infty)$, the Real Johnson-Wilson cohomology of $\mathbb{C}P^\infty$, for all n. We apply techniques from equivariant stable homotopy theory to the Bockstein spectral sequence. We produce permanent cycles, compute differentials, and solve extension problems to give an explicit description of the ring $ER(n)^*(\mathbb{C}P^\infty)$. In the case $n = 1$, our results reproduce $KO^*(\mathbb{C}P^\infty)$ as computed by Sanderson, Fujii, Yamaguchi, and Bruner and Greenlees. In the case $n = 2$, our result yields the $TMF_0(3)$-cohomology of $\mathbb{C}P^\infty$ after a suitable completion.

This thesis forms part of a program to compute the $ER(n)$-cohomology of basic spaces. We conclude with a discussion of work in progress with Kitchloo and Wilson on the $ER(n)$-cohomology of $\mathbb{C}P^k$, classifying spaces of various groups, and Eilenberg-MacLane spaces, as well as future directions and possible applications to topology and geometry.

We include an appendix which proves some lemmas in equivariant stable homotopy theory used in our computations.

Acknowledgments

I am deeply grateful to Nitu Kitchloo for being an enthusiastic and generous advisor and for teaching me most of the algebraic topology I know–I always left our meetings feeling encouraged and excited. Steve Wilson has been a sharp second advisor and collaborator, helping me chase down every last differential. My mathematical values have been shaped largely by seeing Nitu and Steve at work.

Within the algebraic topology community, I owe thanks to many people for their hospitality, encouragement, and sharing their expertise: this especially includes Mike Hill and the topologists at Wayne State University–Bob Bruner, Andrew Salch, Dan Isaksen, Mike Catanzaro, and Gabe Angelini-Knoll–who hosted me for a wonderful week last summer. I have been fortunate to have made many friends in the community as well including Eric Peterson, Paul VanKoughnett, Saul Glasman, and Sean Tilson. I also owe thanks to several professors at Lawrence University for creating a nurturing environment within which I flourished as an undergraduate, especially Scott Corry, Bruce Pourciau, Richard Sanerib, Peter John-Thomas, and Tom Ryckman.

Within the Johns Hopkins math department, I have benefited from the input and friendship of some wonderful people: Alex, Apurv, Dan, Jesse, Kalina, and Mona. I would have been deeply lonely in graduate school had it not been for Richard, Billy, Joel, Kevin, and Tavian, my second family. I have grown a lot from my involvement with Thread–it's hard to know where I would be without it. In Jon I have much more than a mathematical comrade and double at math conferences—Jon is a true and lifelong friend whose wisdom and ability to channel chaotic nonsense I will miss in equal parts next year.

I am at my happiest with Claire, whose kindness, sincerity, and love continue to inspire me. Finally, a first-grade Russian immigrant too shy to start speaking English would not be whatever he is today without the support of his family. This thesis is dedicated to my mom, Larisa, and to the memories of my uncle, Roma, and my grandparents, Emma and Abram.

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Chapter 1

Introduction

1.1 Motivation

In differential topology, a classic problem is to determine, given an n-dimensional manifold M, the smallest k such that M immerses in $\mathbb{R}^{n+k}$. For real projective spaces $\mathbb{R}P^n$, Milnor [1] proved lower bounds on k by showing that the nonvanishing of characteristic classes in singular cohomology obstructs the existence of immersions. Since then, many (see [2]) have demonstrated new nonimmersions by replacing singular cohomology by various other *complex-oriented cohomology theories*, which by definition support characteristic classes for complex vector bundles. Their values on a wide variety of spaces are immediately computable—complex projective spaces $\mathbb{C}P^k$ and $\mathbb{C}P^\infty$, the universal Grassmanian of complex q-planes $BU(q)$, and others. Complex-oriented theories often also carry symmetries induced by the underlying geometry and encoded by a group action. Taking fixed points of this action frequently yields cohomology theories which, though no longer complex-oriented, are much richer *especially in their ability to detect torsion.* This is critical, as many invariants in topology are torsion, such as all elements in the stable homotopy groups of spheres π_k^s for $k > 0$. Furthermore, such fixed point theories often detect geometric data, such as certain nonimmersions of projective spaces (see [3] and chapter 4), which go unseen by any complex-oriented theory. Very generally, computations with these fixed point theories and the torsion they uncover are the subject of this thesis.

1.2 Background and statement of results

The complex cobordism spectrum, MU, carries an action of C_2, the group of order two, coming from complex conjugation and may be constructed as a genuine (indexed on a complete universe) C_2-equivariant spectrum, $M\mathbb{R}$. At the prime 2, the Johnson-Wilson spectrum, $E(n)$, is an MU-algebra with coefficients

$$E(n)^* = \mathbb{Z}_{(2)}[v_1, \ldots, v_{n-1}, v_n^{\pm 1}]$$

where v_k is in cohomological degree $-2(2^k - 1)$. $E(n)$ may also be constructed as a genuine C_2-equivariant spectrum, Real Johnson-Wilson Theory, $E\mathbb{R}(n)$ [4]. It has the structure of an associative and commutative $M\mathbb{R}$-algebra up to homotopy and its underlying nonequivariant spectrum is $E(n)$. Let $ER(n)$ denote the fixed point spectrum $E\mathbb{R}(n)^{C_2}$. When $n = 1$, $E\mathbb{R}(1)$ is $\mathbb{KR}_{(2)}$, Atiyah's Real K-theory with underlying nonequivariant spectrum $KU_{(2)}$, and $ER(1)$ is $KO_{(2)}$, real K-theory. After a suitable completion, the spectrum $ER(2)$ is (additively) equivalent to the spectrum $TMF_0(3)$ of topological modular forms with level structure [5].

Just as there is a fibration

$$\Sigma KO \xrightarrow{\eta} KO \longrightarrow KU \ , \qquad \eta \in KO^{-1}$$

there is for each n a fibration

$$\Sigma^{\lambda(n)} ER(n) \xrightarrow{x(n)} ER(n) \longrightarrow E(n)$$

where $x(n) \in ER(n)^{-\lambda(n)}$ is $(2^{n+1}-1)$-nilpotent and $\lambda(n) = 2(2^n-1)^2 - 1 = 2^{2n+1} - 2^{n+2} + 1$. This was constructed in [6] and leads to a Bockstein spectral sequence of the form

$$E_1^{i,j} = E(n)^{i\lambda(n)+j-i}(X) \Rightarrow ER(n)^{j-i}(X)$$

This spectral sequence has been used to compute the $ER(2)$-cohomology of real projective spaces and establish some new non-immersion results [3, 7, 8]. The $ER(n)$-cohomology of $BO(q)$ has been computed for all n and q in [9].

Our main result is a computation of $ER(n)^*(\mathbb{C}P^\infty)$ for all n. $ER(n)$ is not a complex oriented theory, and so the computation is nontrivial. The Atiyah-Hirzebruch spectral sequence is unwieldy, so we use the Bockstein spectral sequence instead. Even this has nontrivial higher differentials, but what makes $ER(n)^*(\mathbb{C}P^\infty)$ computable is the fact that, after some rearranging, the only interesting differential is d_1 and the higher differentials all play out in the coefficients. In fact, this phenomenon extends for many spaces beyond $\mathbb{C}P^\infty$. This thesis forms part of a program to compute the $ER(n)$-cohomology of basic spaces which we continue in future work [10, 11].

Before stating the result, we fix some notation. For any $z \in E(n)^{2k}(\mathbb{C}P^\infty)$, let $\widehat{z}$ denote $zv_n^{k(2^n-1)}$. In particular, we have $\widehat{v}_i := v_i v_n^{-(2^i-1)(2^n-1)}$ and, letting $u \in E(n)^2(\mathbb{C}P^\infty)$ denote the complex orientation, $\widehat{u} := v_n^{2^n-1}u$. Let c denote the involution on $E(n)^*(\mathbb{C}P^\infty)$ coming from the action of C_2 on $E\mathbb{R}(n)$. We show that $c(\widehat{u}) = \widehat{u}^*$ is given by the power series $v_n^{2^n-1}[-1]_F(u)$, where F is the formal group law over $E(n)^*$ with $[2]_F(u) = v_0 u +_F v_1 u^2 +_F \cdots +_F v_n u^{2^n}$. Under the identification $\mathbb{C}P^\infty = BSO(2)$, the product $\widehat{u}\widehat{u}^* \in E(n)^*(\mathbb{C}P^\infty)$ is $v_n^{2(2^n-1)}$ times the first Pontryagin class. In Proposition 3.3, we describe a class in $ER(n)^*(\mathbb{C}P^\infty)$ which lifts $\widehat{u}\widehat{u}^* \in E(n)^*(\mathbb{C}P^\infty)$. We denote the lift by $\widehat{p}_1$. In $E(n)^*(\mathbb{C}P^\infty)$, we also have the sum $\widehat{u} + \widehat{u}^*$. We show that this lifts to $ER(n)^*(\mathbb{C}P^\infty)$ as a power series in $\widehat{p}_1$, denoted $\xi(\widehat{p}_1)$. We have the main theorem of this thesis.

Theorem 1.1. *There is a short exact sequence of modules over $ER(n)^*$*

$$0 \longrightarrow \mathrm{im}(N_*^{res}) \longrightarrow ER(n)^*(\mathbb{C}P^\infty) \longrightarrow \frac{ER(n)^*[[\widehat{p}_1]]}{(\xi(\widehat{p}_1))} \longrightarrow 0$$

where $\mathrm{im}(N_*^{res})$ is the image of the restricted norm

$$\mathbb{Z}_{(2)}[\widehat{v}_1, \ldots, \widehat{v}_{n-1}, v_n^{\pm 2}][[\widehat{u}\widehat{u}^*]]\{\widehat{u}, v_n\widehat{u}\} \subset E(n)^*(\mathbb{C}P^\infty) \xrightarrow{N_*} ER(n)^*(\mathbb{C}P^\infty)$$

such that for all z, $N_*(z)$ maps to $z+c(z)$ under the map $ER(n)^*(\mathbb{C}P^\infty) \longrightarrow E(n)^*(\mathbb{C}P^\infty)$.

Remark 1.2. Although the middle and right terms of the short exact sequence of Theorem 1.1 are rings, the $ER(n)^*$-module $\mathrm{im}(N_*^{\mathrm{res}})$ is not an ideal of $ER(n)^*(\mathbb{C}P^\infty)$ and the right hand map is not a ring homomorphism. However, we give a complete answer for $ER(n)^*(\mathbb{C}P^\infty)$ as an algebra in terms of generators and relations in Theorem 3.24. A simpler answer than either Theorem 1.1 or 3.24 is given by restricting to degrees multiples of 2^{n+2} (note that $ER(n)$ is $2^{n+2}(2^n-1)$-periodic), though this portion of it contains none of the 2-torsion:

$$ER(n)^{2^{n+2}*}(\mathbb{C}P^\infty) = ER(n)^{2^{n+2}*}(pt)[[\widehat{p}_1]]$$

Remark 1.3. Note that neither the complex orientation u nor $\widehat{u}$ lift to $ER(n)^*(\mathbb{C}P^\infty)$, but $\widehat{u\overline{u}^*}$ does. The characteristic class $\xi(\widehat{p}_1)$ by which we quotient in the right hand term above is not zero in $ER(n)^*(\mathbb{C}P^\infty)$ but is in the image of the norm, $\xi(\widehat{p}_1) = N_*(\widehat{u})$. It has geometric significance as we discuss further in Remark 3.18.

Remark 1.4. In Theorem 1.1 for $n = 1$, it turns out that $\xi(\widehat{p}_1) = -\widehat{p}_1$, and so the right hand term reduces to the coefficients $ER(1)^*$. This is not true for $n > 1$ (see Remark 3.25). In the case $n = 1$, our results reproduce $KO^*(\mathbb{C}P^\infty)$, which has some history. $KO^*(\mathbb{C}P^\infty)$ was first computed in degree zero by Sanderson [12] then in all degrees with ring structure on $KO^{\mathrm{even}}(\mathbb{C}P^\infty)$ by Fujii [13]. Yamaguchi [14] gave the first complete description of $KO^*(\mathbb{C}P^\infty)$ as a ring. All three computations use the Atiyah-Hirzebruch spectral sequence. Bruner and Greenlees [15] computed the connective real K-theory of $\mathbb{C}P^\infty$ using the Bockstein spectral sequence.

Chapter 2 gives the background for our computations: we review the construction of the Real Johnson-Wilson theories, the Kitchloo-Wilson fibration, and the Bockstein spectral sequence. In chapter 3, we carry out the computation of $ER(n)^*(\mathbb{C}P^\infty)$. In 3.1 and 3.2 we identify the key permanent cycle, $\widehat{uu}^*$, and give a convenient reformulation of the E_1-page of the Bockstein spectral sequence. In section 3.3 we compute $E_2^{*,*}$. From there, sections 3.4-3.6 break up the Bockstein spectral sequence into a short exact sequence of spectral sequences and show that the remainder of the computation happens in the coefficients via a Landweber flatness argument. In section 3.7, we prove Theorem 1.1, describe the multiplicative structure, and state the most explicit form of the answer as an algebra over $ER(n)^*$. Finally, section 3.8 describes the very clean form of the answer that occurs after a certain completion. In chapter 4, we discuss future directions suggested by our results. The appendix at the end contains some key equivariant lemmas necessary for our computations.

Chapter 2

Background

2.1 Real Johnson-Wilson theories

Before constructing the Real Johnson-Wilson theories, we need some definitions and constructions from C_2-equivariant homotopy theory (see [16] and [4] for more details). Let α denote the sign representation of C_2. By a genuine C_2-equivariant spectrum $\mathbb{E}$, we mean a collection of spaces $\mathbb{E}_V$ ranging over finite-dimensional C_2-representations $V = s + t\alpha$ together with a transitive system of based C_2-equivariant homeomorphisms

$$\mathbb{E}_V \longrightarrow \Omega^{W-V}\mathbb{E}_W, \quad \text{for } V \subseteq W$$

Such spectra represent bigraded cohomology theories $\mathbb{E}^\star(-)$ given by

$$\mathbb{E}^{s+t\alpha}(-) = [-, \Sigma^{s+t\alpha}\mathbb{E}]^{C_2}$$

The C_2-action means there is an involution on $\mathbb{E}$. Letting $\iota^*\mathbb{E}$ denote the underlying nonequivariant spectrum, there is an induced involution on the (nonequivariant) cohomology groups $(\iota^*\mathbb{E})^*(-)$, which we denote by c.

A fundamental example of a C_2-equivariant spectrum is given by Real cobordism $M\mathbb{R}$, first studied by Landweber [17] and Araki [18]. We construct it below.

Construction 2.1. We proceed as in the construction of the (nonequivariant) MU-spectrum, but keep track of the C_2-action throughout. We construct $M\mathbb{R}$ as a prespectrum and then spectrify. Recall that the spaces of a C_2-spectrum are $RO(C_2)$-graded, and it suffices to define the prespectrum on the cofinal subsequence given by representations $q(1+\alpha)$.

We do this as follows. The tautological complex q-plane bundle $\gamma_q^{\mathbb{C}}$ over $BU(q)$ carries a C_2-action induced by complex conjugation. Here, the group acts compatibly on both the base and the total space. The Thom space $MU(q)$ of this bundle inherits a C_2-action. We define the $q(1+\alpha)$ space in the prespectrum to be $MU(q)$. The map $BU(q) \longrightarrow BU(q+1)$ classifying $\gamma_q^{\mathbb{C}} \oplus \mathbb{C}$ induces a map of Thom spaces $\Sigma^{1+\alpha}MU(q) \longrightarrow MU(q+1)$, which gives the required structure map. We now spectrify to produce $M\mathbb{R}$. Explicitly, since the above definition in fact gives an inclusion prespectrum, spectrification has a straightforward description:

$$M\mathbb{R}_V = \operatorname*{colim}_{(q+q\alpha)\supseteq V} \Omega^{q+q\alpha-V} MU(q)$$

By construction, $M\mathbb{R}$ is a homotopy associative and commutative ring spectrum.

In fact, by [4], it is an E_∞-ring spectrum, though we do not use this structure here. We have that $\iota^* M\mathbb{R} = MU$.

Following [4], we recall that $M\mathbb{R}$ is *Real-oriented*: that is, there is a C_2-equivariant map $u : \mathbb{C}P^\infty \longrightarrow \Sigma^{1+\alpha} M\mathbb{R}$ such that the following diagram commutes (up to homotopy):

$$\begin{array}{ccc} \mathbb{C}P^\infty & \xrightarrow{\ u\ } & \Sigma^{1+\alpha} M\mathbb{R} \\ \uparrow & \nearrow_{1} & \\ \mathbb{C}P^1 = S^{1+\alpha} & & \end{array}$$

As in [4], the usual skeletal filtration of $\mathbb{C}P^\infty$ (which respects the C_2-action) now produces a Real Atiyah-Hirzebruch spectral sequence with $E_2^{*,\star} = H^*(\mathbb{C}P^\infty, M\mathbb{R}_\star) \Rightarrow M\mathbb{R}^\star(\mathbb{C}P^\infty)$. As in the nonequivariant case, the spectral sequence collapses to give $M\mathbb{R}^\star(\mathbb{C}P^\infty) = M\mathbb{R}^\star[[u]]$, with $|u| = 1+\alpha$. It also follows that the multiplication map $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty$ yields a formal group law over $M\mathbb{R}_\star$ with coefficients a_{ij} in degrees $(1-i-j)(1+\alpha)$. Thus, there is a (algebraic) map of rings $MU_{2*} \longrightarrow M\mathbb{R}_{*(1+\alpha)}$ classifying this formal group law. Forgetting the C_2-action on $M\mathbb{R}$ in the above construction produces the usual universal formal group law over MU_*. Thus, the forgetful map $M\mathbb{R}_{*(1+\alpha)} \longrightarrow MU_{2*}$ is split by the map classifying the formal group.

We now localize at the prime 2 and work 2-locally for the rest of this manuscript. Let $v_k \in \pi_{2(2^k-1)} MU_{(2)}$ denote the coefficients of the 2-series of the 2-typification of the universal formal group law (i.e. the Araki v_k). We map each v_k to $\pi_{(2^k-1)(1+\alpha)} M\mathbb{R}$ along the classifying map $MU_{2k} \longrightarrow M\mathbb{R}_{k(1+\alpha)}$ and call the image of v_k by the same name. By the above paragraph, the forgetful map sends the 'diagonal' equivariant v_k

to the usual nonequivariant v_k.

We are now ready to define the Real Johnson-Wilson theories. A C_2-equivariant analog of the Quillen idempotent splits $M\mathbb{R}$ into a wedge of suspensions of the C_2-spectrum $BP\mathbb{R}$ (Real Brown-Peterson cohomology). Give a nonnegative integer n, we kill the (equivariant, diagonal) classes $v_{n+1}, v_{n+2}, \ldots$, then invert v_n to produce a C_2-spectrum we will denote $E\mathbb{R}(n)$. We define Real Johnson-Wilson theory to be its homotopy fixed points: $ER(n) := E\mathbb{R}(n)^{hC_2}$.

Note that the above construction does not guarantee that $E\mathbb{R}(n)$ is a C_2-equivariant ring spectrum. Indeed, the usual argument that nonequivariant $E(n)$ is a ring spectrum relies on $E(n)_*$ being evenly graded. The analogous argument for $E\mathbb{R}(n)$ does not work—the $RO(C_2)$-graded homotopy groups of $E\mathbb{R}(n)$ are well-understood (see [4, 9]) but do not vanish in the necessary degrees. That $E\mathbb{R}(n)$ is a homotopy commutative and associative ring spectrum is stated in [4], though the following proposition from [19] is enough for the purposes of this thesis.

Proposition 2.2. *[19] The theory $E\mathbb{R}(n)^\star(-)$ is a cohomology theory valued in commutative rings on the category of C_2-spaces and for any such space X, the forgetful map $E\mathbb{R}(n)^\star(X) \longrightarrow E(n)^*(X)$ is a ring homomorphism.*

Given a C_2-spectrum $\mathbb{E}$, we call the function spectrum $F(EC_{2+}, \mathbb{E})$ the *completion* of $\mathbb{E}$. We say that $\mathbb{E}$ is *complete* (or *cofree*) if the map $F(EC_{2+}, \mathbb{E}) \longrightarrow \mathbb{E}$ is an equivariant equivalence. If E is complete, it follows that the homotopy fixed points, defined as $\mathbb{E}^{hC_2} := F(EC_{2+}, \mathbb{E})^{C_2}$, are equivalent to the genuine fixed points $\mathbb{E}^{C_2}$. We have

Proposition 2.3. *[4] The spectra $M\mathbb{R}$ and $E\mathbb{R}(n)$ are complete.*

We close this section by describing the coefficients of $ER(n)$. These were first computed in [4], but we present a description which follows directly from [9]. First, we fix some notation. Let $\lambda = \lambda(n) := 2^{2n+1} - 2^{n+2} + 1$. Recall that $\widehat{v}_k \in E(n)^*$ denotes the class $v_k v_n^{-(2^k-1)(2^n-1)}$.

Theorem 2.4. *[4, 9] The integer-graded coefficients $\pi_* ER(n) = \pi_{*+0\alpha} E\mathbb{R}(n)$ are given by*

$$ER(n)^*(pt) = \mathbb{Z}_{(2)}[x, v_k(s), v_n^{\pm 2^{n+1}}]/I, \quad 0 \le k < n$$

where $|x| = \lambda := 2^{2n+1} - 2^{n+2} + 1$, $v_k(s)$ restricts to $\widehat{v}_k v_n^{2^{k+1}s}$ in $E(n)^*$, and $v_n^{2^{n+1}}$ maps to the class by the same name in $E(n)^*$. The ideal I encodes the relations

$$v_0(0) = 2, \quad x^{2^{k+1}-1} v_k(s) = 0, \quad x^{2^{n+1}-1} = 0$$

in addition to the ones that are detected in $E(n)^*$.

Tables of the coefficients in the cases $n = 1$ and 2 are given below in Example 2.15. The interested reader is encouraged to flip ahead.

Remark 2.5. Notice that the class $v_n^{2^{n+1}}$ in degree $2^{n+2}(2^n - 1)$ is invertible. This is our periodicity class and it makes $ER(n)$ a $2^{n+2}(2^n - 1)$-periodic cohomology theory.

Kitchloo and Wilson noticed that the bigraded homotopy $\pi_\star E\mathbb{R}(n)$ contains an invertible class $y(n)$ in bidegree $\lambda + \alpha$, which allows all bigraded classes to be shifted to integer grading by multiplying by suitable powers of $y(n)$.

Theorem 2.6. *[9] There exists an invertible class* $y(n) \in \pi_{\lambda+\alpha}(E\mathbb{R}(n))$ *and an isomorphism of bigraded rings*

$$\pi_\star(E\mathbb{R}(n)) = \pi_*(ER(n))[y(n)^{\pm 1}]$$

Remark 2.7. Let $a : S^0 \longrightarrow S^\alpha$ denote inclusion of fixed points. Note that a is not equivariantly null-homotopic, but is trivial non-equivariantly. The class x defined above is equal to $a \cdot y(n)$. Note also that when $n = 1$, the class x is precisely $\eta \in \pi_*(KO_{(2)})$ and the class v_1^4 is the Bott class in degree 4.

2.2 The Kitchloo-Wilson fibration

Much of the computational power of Real Johnson-Wilson theories comes from a fibration constructed by Kitchloo and Wilson [6]. We review their construction below.

Theorem 2.8. *[6] There is a fibration*

$$\Sigma^\lambda ER(n) \xrightarrow{x} ER(n) \longrightarrow E(n)$$

where the right hand map denotes the inclusion of fixed points map.

Proof. We apply the functor $F(-, E\mathbb{R}(n))^{C_2}$ to the cofibration

$$C_{2_+} \longrightarrow S^0 \xrightarrow{a} S^\alpha$$

to get the fibration

$$(\Sigma^{-\alpha} E\mathbb{R}(n))^{C_2} \xrightarrow{a} ER(n) \longrightarrow F(C_{2_+}, E\mathbb{R}(n)^{C_2} \simeq E(n)$$

Precomposing with the self-map $\Sigma^\lambda ER(n) = (\Sigma^\lambda E\mathbb{R}(n))^{C_2} \longrightarrow (\Sigma^{-\alpha} E\mathbb{R}(n))^{C_2}$ given by $y(n)$ yields the desired fibration. To see that the right hand map is given by inclusion of fixed points, notice that it is precisely the map $F(S^0, E\mathbb{R}(n))^{C_2} \longrightarrow F(C_{2_+}, E\mathbb{R}(n))^{C_2}$, which factors as the inclusion of fixed points $F(S^0, E\mathbb{R}(n))^{C_2} \longrightarrow \iota^* F(S^0, E\mathbb{R}(n))$ followed by the equivalence $\iota^* F(S^0, E\mathbb{R}(n)) \simeq F(C_{2_+}, E\mathbb{R}(n))^{C_2}$. □

2.3 The Bockstein spectral sequence

Applying $[X, -]$ to the fibration of Theorem 2.8 yields an exact couple

$$\begin{array}{ccc} ER(n)^*(X) & \xrightarrow{\quad x \quad} & ER(n)^*(X) \\ & \nwarrow \quad \swarrow & \\ & E(n)^*(X) & \end{array}$$

and produces the Bockstein spectral sequence (BSS).

Remark 2.9. Depending on whether one truncates the multiplication by x tower, there are two spectral sequences that can arise from the above. One converges to $ER(n)^*(X)$ (as in [9]), the other to 0 (as in [3], [7], and [10]). In the latter case, one must go back to reconstruct the answer from the differentials. Both have their advantages and ultimately contain equivalent information, but it is the truncated BSS converging to $ER(n)^*(X)$ that we use here.

The BSS is described by the following theorem.

Theorem 2.10. *[9]*

(i) There is a first and fourth quadrant spectral sequence of $ER(n)^*$-modules, $E_r^{i,j} \Rightarrow ER(n)^{j-i}(X)$. The differential d_r has bidegree $(r, r+1)$ for $r \geq 1$.

(ii) The E_1-term is given by $E_1^{i,j} = E(n)^{i\lambda+j-i}(X)$ with

$$d_1(z) = v_n^{1-2^n}(1-c)(z)$$

where $c(v_i) = -v_i$. The differential d_r increases cohomological degree by $1 + r\lambda$ between the appropriate subquotients of $E(n)^*(X)$.

(iii) $E_{2^{n+1}}(X) = E_\infty(X)$, which is described as follows. Filter $M = ER(n)^*(X)$ by $M_r = x^r M$ so that

$$M = M_0 \supset M_1 \supset M_2 \supset \cdots \supset M_{2^{n+1}-1} = \{0\}$$

Then $E_\infty^{r,*}(X)$ is canonically isomorphic to M_r/M_{r+1}.

(iv) $d_r(ab) = d_r(a)b + c(a)d_r(b)$. In particular, if $c(z) = z \in E_r(X)$ then $d_r(z^2) = 0$, $r > 1$.

Remark 2.11. As in [9], we note that when X is a space, there is a canonical class in $E_1^{1,1-\lambda}$ that corresponds to $1 \in E(n)^{\lambda+1-\lambda-1}(X) = E(n)^0(X)$ and is a permanent cycle representing $x \in ER(n)^{-\lambda}$. We abuse notation and give its representative in $E_1^{1,1-\lambda}$ the name x as well. Note that though x is a permanent cycle, $x^{2^{n+1}-1}$ does not survive the spectral sequence (as we will see, it is the target of a differential) and is equal to zero in $ER(n)^*(X)$. We may rewrite the E_1-page to index the vertical lines by powers of x

$$E_1^{*,*} = E_1^{0,*}[x] = E(n)^*(X)[x]$$

$$d_1(z) = v_n^{1-2^n}(1-c)(z)x, \qquad v_n \in E_1^{0,2(1-2^n)}$$

Remark 2.12. To make things even more confusing, the representative of $x = x(n)$ in $E_1^{1,1-\lambda}$ was previously called y in [9] and is not the same as $y(n) \in E\mathbb{R}(n)^{-\lambda(n)-\alpha}$ as described above. Since our $x \in E_1^{1,1-\lambda}$ represents $x = x(n) \in ER(n)^{-\lambda}$, we choose the lesser of two evils and henceforth use our notation instead.

2.4 The spectral sequence for $X = pt$

When $X = pt$, we have

$$E_1^{*,*} = E(n)^* = \mathbb{Z}_{(2)}[v_1, \ldots, v_{n-1}, v_n^{\pm 1}][x], \qquad |v_k| = -2(2^k-1)$$

None of the generators v_k are permanent cycles as $c(v_k) = -v_k$. However, there is a trick we can do to replace v_k for $k < n$ by permanent cycles $\widehat{v}_k$. As in [4], each class $v_k \in MU^{-2(2^k-1)}$ has an equivariant lift in $M\mathbb{R}^{-(2^k-1)(1+\alpha)}$. For $0 \le k \le n$, the $M\mathbb{R}$-algebra structure on $E\mathbb{R}(n)$ produces these classes in $E\mathbb{R}(n)^\star$. We may use $y \in E\mathbb{R}(n)^{-\alpha-\lambda}$ to shift the "diagonal" v_k classes to integer grading. For $0 \le k < n$, let $\widehat{v}_k \in E\mathbb{R}(n)^{(2^k-1)(\lambda-1)} = ER(n)^{(2^k-1)(\lambda-1)}$ denote $v_k y^{-(2^k-1)}$. Then by construction, this class restricts to $v_k v_n^{-(2^k-1)(2^n-1)} \in E(n)^{(2^k-1)(\lambda-1)}$ and represents a permanent cycle in $E_1^{0,(2^k-1)(\lambda-1)}$.

We have now shifted all of the differentials onto powers of v_n. Let $R_n = \mathbb{Z}_{(2)}[\widehat{v}_1, \dots, \widehat{v}_{n-1}]$, $I_j = (2, \widehat{v}_1, \dots, \widehat{v}_{j-1})$, and $I_0 = (0)$. The Bockstein spectral sequence computing $ER(n)^*$ goes as follows.

Theorem 2.13. *[9] In the spectral sequence $E_r(pt) \Rightarrow ER(n)^*$,*

(i) We have

$$E_1^{*,*} \cong \mathbb{Z}_{(2)}[\widehat{v}_1, \widehat{v}_2, \dots, \widehat{v}_{n-1}, v_n^{\pm 1}][x].$$

That is,

$$E_1^{m,*} = \mathbb{Z}_{(2)}[\widehat{v}_1, \dots, \widehat{v}_{n-1}, v_n^{\pm 1}] \text{ on } x^m$$

(ii) The only non-zero differentials are generated by

$$d_{2^{k+1}-1}(v_n^{-2^k}) = \widehat{v}_k v_n^{-2^{n+k}} x^{2^{k+1}-1} \text{ for } 0 \le k \le n$$

(iii) $E_{2^k}^{*,*} = E_{2^k+1}^{*,*} = \cdots = E_{2^{k+1}-1}^{*,*}$, for $0 \le k \le n$, and $E_{2^{n+1}}^{*,*} = E_\infty^{*,*}$.

(iv) For $0 \leq j < k \leq n+1$,

$$E_{2^k}^{m,*} = R_n[v_n^{\pm 2^k}]/I_j \bigoplus_{j<i<k} I_i R_n[v_n^{\pm 2^{i+1}}]v_n^{2^i}/I_j \text{ on } x^m$$

when $2^j - 1 \leq m < 2^{j+1} - 1$.

(v) For $0 < k \leq n+1$ *and* $2^k - 1 \leq m$,

$$E_{2^k}^{m,*} = R_n[v_n^{\pm 2^k}]/I_k \text{ on } x^m$$

Remark 2.14. Note that v_n does not survive the spectral sequence, but $v_n^{2^{n+1}}$ does. This is what gives $ER(n)$ the $|v_n^{-2^{n+1}}| = 2^{n+2}(2^n - 1)$-periodicity described above.

Examples 2.15. Below we record a chart of the coefficients $ER(n)^*$ for $n = 1, 2$. Because of periodicity, we present them in their $\mathbb{Z}/(2^{n+2}(2^n - 1))$-graded form by setting $v_n^{2^{n+1}} = 1$. We have that $ER(1)$ and $ER(2)$ are 8 and 48-periodic, respectively. Classes listed generate a $\mathbb{Z}/(2)$ if divisible by x and generate $\mathbb{Z}_{(2)}$ otherwise.

(a) **Coefficients of** $ER(1) = KO_{(2)}$ (set $v_1^4 = 1$):

0	1
1	x
2	x^2
3	
4	$2v_1^2$
5	
6	
7	

(b) **Coefficients of** $ER(2)$: (set $v_2^8 = 1$) (see also appendix of [11])

0	$\widehat{v}_1^{3k}$	16	$\widehat{v}_1^{3k+1}$	32	$\widehat{v}_1^{3k+2}$
1		17		33	
2		18		34	
3		19		35	
4	$(2v_2^2)\widehat{v}_1^{3k+1}$	20	$(2v_2^2)\widehat{v}_1^{3k+2}$	36	$(2v_2^2)\widehat{v}_1^{3k}$
5		21		37	
6	$x^2\widehat{v}_1^{3k+1}v_2^4$	22	$x^2\widehat{v}_1^{3k+2}v_2^4$	38	$x^2\widehat{v}_1^{3k+3}v_2^4$
7	$x\widehat{v}_1^{3k+3}v_2^4$	23	$x\widehat{v}_1^{3k+1}v_2^4$	39	$x\widehat{v}_1^{3k+2}v_2^4$
8	$\widehat{v}_1^{3k+2}v_2^4$	24	$2v_2^4$, $\widehat{v}_1^{3k+3}v_2^4$	40	$\widehat{v}_1^{3k+1}v_2^4$
	$2(\widehat{v}_1^{3k+2}v_2^4) = (2v_2^4)\widehat{v}_1^{3k+2}$		$2(\widehat{v}_1^{3k+3}v_2^4) = (2v_2^4)\widehat{v}_1^{3k+3}$		$2(\widehat{v}_1 v_2^4) = (2v_2^4)\widehat{v}_1^{3k+1}$
9		25		41	
10		26		42	x^6
11	x^5	27		43	
12	$(2v_2^6)\widehat{v}_1^{3k}$	28	x^4, $(2v_2^6)\widehat{v}_1^{3k+1}$	44	$(2v_2^6)\widehat{v}_1^{3k+2}$
13		29		45	x^3
14	$x^2\widehat{v}_1^{3k}$	30	$x^2\widehat{v}_1^{3k+1}$	46	$x^2\widehat{v}_1^{3k+2}$
15	$x\widehat{v}_1^{3k+2}$	31	$x\widehat{v}_1^{3k}$	47	$x\widehat{v}_1^{3k+1}$

Chapter 3

The $ER(n)$-cohomology of $\mathbb{C}P^\infty$

3.1 $E_1^{*,*}$ and the action of c

The BSS for $\mathbb{C}P^\infty$ starts with

$$E_1^{*,*} = E(n)^*(\mathbb{C}P^\infty)[x] = E(n)^*[[u]][x]$$

Again, we hat off v_i, $0 \le i < n$ so that $E(n)^* = \mathbb{Z}_{(2)}[\widehat{v}_1, \ldots, \widehat{v}_{n-1}, v_n^{\pm 1}]$. As described in the introduction, in general, for a class $z \in E(n)^{2j}(X)$, we set

$$\widehat{z} := v_n^{j(2^n-1)} z \in E(n)^{j(1-\lambda)}(X)$$

However, note that for arbitrary z, $\widehat{z}$ need not be a permanent cycle. In fact, $\widehat{u} = v_n^{2^n-1}u \in E(n)^{1-\lambda}(\mathbb{C}P^\infty)$ is not. In any case, we replace u by $\widehat{u}$ as the power series generator of $E(n)^*(\mathbb{C}P^\infty)$, which is valid since v_n is a unit. We may similarly hat off the coefficients of the formal group law so that $\widehat{F}(\widehat{u}_1, \widehat{u}_2)$ is a homogenous expression of degree $1-\lambda$ and satisfies $v_n^{2^n-1}F(u_1,u_2) = \widehat{F}(\widehat{u}_1,\widehat{u}_2)$.

We now have

$$E_1^{*,*} = \mathbb{Z}_{(2)}[\widehat{v}_1, \dots, \widehat{v}_{n-1}, v_n^{\pm 1}][[\widehat{u}]][x]$$

with bidegrees

$$|\widehat{v}_k| = \left(0, \frac{1-\lambda}{2}|v_k|\right) = (0, (\lambda-1)(2^k-1)), \qquad |v_n| = (0, -2(2^n-1))$$

$$|\widehat{u}| = (0, 1-\lambda), \qquad |x| = (1, 1-\lambda)$$

To compute d_1, by Theorem 2.10(ii), we need the action of c on $E_1^{*,*}$. The classes $\widehat{v}_k$, $0 \le k < n$, as well as x are permanent cycles and in particular have trivial c-action. We have $c(v_n) = -v_n$. It remains to identify $c(\widehat{u})$.

Lemma 3.1.

$$c(\widehat{u}) = [-1]_{\widehat{F}}(\widehat{u})$$

Proof. We view u as an equivariant map $u : \mathbb{C}P^\infty \longrightarrow \Sigma^{1+\alpha}E(n)$. The diagram

$$\begin{array}{ccc} \mathbb{C}P^\infty & \xrightarrow{u} & S^{1+\alpha} \wedge E(n) \\ \big\downarrow{\scriptstyle \mathrm{inv}} & & \big\downarrow{\scriptstyle (-1)\wedge c} \\ \mathbb{C}P^\infty & \xrightarrow{u} & S^{1+\alpha} \wedge E(n) \end{array}$$

commutes, where inv denotes the involution on $\mathbb{C}P^\infty$ classifying the conjugate line bundle with $\mathrm{inv}^*(u) = [-1]_F(u)$. The above diagram shows that $c(u) = -[-1]_F(u)$.

Then on $\widehat{u} = v_n^{2^n-1}u$, we have

$$c(\widehat{u}) = c(v_n^{2^n-1}u) = -v_n^{2^n-1}c(u) = v_n^{2^n-1}[-1]_F(u) = [-1]_{\widehat{F}}(\widehat{u})$$

□

From now on, we let $\widehat{u}^*$ denote $c(\widehat{u}) = [-1]_{\widehat{F}}(\widehat{u})$. For future reference, it will be helpful to have some terms of this power series, so we pause to derive some formulas.

Lemma 3.2. *We have the following congruences in $E(n)^*(\mathbb{C}P^\infty)$:*

$$\widehat{u}^* \equiv -\widehat{u} \ \mathit{mod}\ (\widehat{u}^2)$$

$$\widehat{u}^* \equiv \widehat{u} + \widehat{v}_k\widehat{u}^{2^k} \ \mathit{mod}\ (\widehat{v}_0, \ldots, \widehat{v}_{k-1}, \widehat{u}^{2^k+1}) \qquad \mathit{for}\ 0 < k < n$$

Proof. Both follow from the formula for the 2-series

$$[2]_{\widehat{F}}(\widehat{u}) = \sum_{i=0}^{n}{}_{\widehat{F}}\ \widehat{v}_i\widehat{u}^{2^i}$$

and the equation

$$\widehat{u}^* +_{\widehat{F}} [2]_{\widehat{F}}(\widehat{u}) = \widehat{u}$$

□

3.2 A topological basis for $E_1^{*,*}$

The next step in computing d_1 is finding a convenient topological basis for $E_1^{*,*}$. To that end, we identify a large collection of permanent cycles in our spectral sequence.

Note that $E_1^{*,*}$ is a power series ring over $E(n)^*[x]$, and throughout, by a basis for $E_1^{*,*}$ we mean a topological basis.

Proposition 3.3. $\widehat{u}\widehat{u}^*$ *is a permanent cycle*

Proof. Our starting point is $\mathbb{ER}(n)^\star(BU(2))$. It may be computed using the Real Atiyah-Hirzebruch spectral sequence completely analogously to the complex-oriented case (see [4]). We have

$$\mathbb{ER}(n)^\star(BU(2)) = \mathbb{ER}(n)^\star[[c_1, c_2]]$$

with $|c_i| = i(1+\alpha)$. We hat c_2 to produce $\widehat{c}_2 = c_2 y^2$ in degree $2(1-\lambda) + 0\alpha$ which restricts to $c_2 v_n^{2(2^n-1)} \in E(n)^*(BU(2))$. When we take fixed points to land in $ER(n)^*(BO(2))$ and map over to $ER(n)^*(BSO(2)) = ER(n)^*(\mathbb{C}P^\infty)$, we claim this will produce a permanent cycle which lifts $\widehat{u}\widehat{u}^* \in E(n)^*(\mathbb{C}P^\infty)$. That is, consider the following commutative diagram.

$$\begin{array}{ccc}
[BU(2), \mathbb{ER}(n)]^{C_2} & \longrightarrow & [BSO(2), \mathbb{ER}(n)]^{C_2} \\
\downarrow & & \| \\
[BU(2), \mathbb{ER}(n)] & & [BSO(2), ER(n)] \\
\| & & \downarrow \\
[BU(2), E(n)] & \longrightarrow & [BSO(2), E(n)]
\end{array}$$

Here the two horizontal maps are induced by the inclusion $BSO(2) \longrightarrow BU(2)$. From the diagram, we conclude that the image of $\widehat{c}_2 \in \mathbb{ER}(n)^\star(BU(2))$ in $ER(n)^*(BSO(2))$ is a permanent cycle, whose representative on E_1 is given by mapping to the bottom right corner. $\widehat{c}_2$ restricts to $c_2 v_n^{2(2^n-1)}$ in $E(n)^*(BU(2))$, so it remains to show that the image of this class in $E(n)^*(BSO(2))$ is $\widehat{uu}^*$. This follows from the homotopy commutativity of the diagram

$$\begin{array}{ccccc} BU(1) & \xrightarrow{\Delta} & BU(1) \times BU(1) & \xrightarrow{1\times c} & BU(1) \times BU(1) \\ \simeq\downarrow & & & & \downarrow m \\ BSO(2) & & \longrightarrow & & BU(2) \end{array}$$

□

Remark 3.4. The above argument shows that, as an element of $E(n)^*(\mathbb{C}P^\infty) = E(n)^*(BSO(2))$, the class $\widehat{uu}^*$ is in fact $v_n^{2(2^n-1)}$ times the first Pontryagin class in $E(n)$-cohomology and lifts to $ER(n)^*(\mathbb{C}P^\infty)$. We denote its lift by $\widehat{p}_1$ as in Theorem 1.1.

Now that we have permanent cycles $(\widehat{uu}^*)^l$ for $l \geq 0$, we can use these to form half of our basis for $E_1^{*,*}$. The other half of the basis will consist of classes $\widehat{u}(\widehat{uu}^*)^l$, $l \geq 0$. Since

$$(\widehat{uu}^*)^l \equiv (-1)^l \widehat{u}^{2l} \bmod (\widehat{u}^{2l+1})$$

and

$$\widehat{u}(\widehat{uu}^*)^l \equiv (-1)^l \widehat{u}^{2l+1} \bmod (\widehat{u}^{2l+2})$$

it follows that $\{\widehat{u}^\epsilon(\widehat{uu}^*)^l : \epsilon = 0 \text{ or } 1,\ l \geq 0\}$ clearly forms a topological basis for $E_1^{*,*}$ over $E(n)^*[x]$.

3.3 Computing $E_2^{*,*}$

To determine d_1 on this basis, it is necessary to distinguish between odd and even exponents of v_n, since $c(v_n^l) = (-1)^l v_n^l$. In what follows, recall that $\widehat{v}_i$ are permanent cycles and note that $d_1(v_n^2) = 0$. We have

$$d_1(v_n^{2p}(\widehat{u}\widehat{u}^*)^l) = 0$$

$$d_1(v_n^{2p+1}(\widehat{u}\widehat{u}^*)^l) = 2v_n^{2p-2^n}(\widehat{u}\widehat{u}^*)^l x$$

$$d_1(v_n^{2p}(\widehat{u}(\widehat{u}\widehat{u}^*)^l)) = v_n^{2p-(2^n-1)}(\widehat{u} - \widehat{u}^*)(\widehat{u}\widehat{u}^*)^l x$$

$$d_1(v_n^{2p+1}(\widehat{u}(\widehat{u}\widehat{u}^*)^l)) = v_n^{2p-2^n}(\widehat{u} + \widehat{u}^*)(\widehat{u}\widehat{u}^*)^l x$$

Set $R = \mathbb{Z}_{(2)}[\widehat{v}_1, \ldots, \widehat{v}_{n-1}, v_n^{\pm 2}][x]$ so that $E_1^{*,*} = (R \oplus v_n R)[[\widehat{u}]]$. To analyze the image and kernel of d_1, we begin with a technical lemma concerning $\widehat{u}^*$.

Lemma 3.5. *$\widehat{u}^*$ is in $R[[\widehat{u}]]$.*

Proof. Consider R as a submodule of $\mathbb{Z}_{(2)}[\widehat{v}_1, \ldots, \widehat{v}_{n-1}, v_n^{\pm 1}][x]$ over the ring $\mathbb{Z}_{(2)}[\widehat{v}_1, \ldots, \widehat{v}_{n-1}]$. Notice $\widehat{v}_n = v_n^{-(2^n-1)^2+1}$ is in R. Thus, the coefficients of $\widehat{F}$, formed by hatting the coefficients of F, are also in R, as is $[2]_{\widehat{F}}(\widehat{u})$. We have

$$\widehat{u} = [2]_{\widehat{F}}(\widehat{u}) +_{\widehat{F}} \widehat{u}^*$$

Reducing modulo the submodule $R[[\widehat{u}]]$, we have

$$0 \equiv 0 +_{\widehat{F}} \widehat{u}^* \bmod R[[\widehat{u}]]$$

Thus, $\widehat{u}^* \in R[[\widehat{u}]]$. $\square$

It follows from the formulas for d_1 above together with Lemma 3.5 that d_1 interchanges classes in $R[[\widehat{u}]]$ with classes in $v_n R[[\widehat{u}]]$. We now describe a convenient (topological) basis for the kernel of d_1.

Proposition 3.6. *A basis for the kernel of d_1 over $R = \mathbb{Z}_{(2)}[\widehat{v}_1, \ldots, \widehat{v}_{n-1}, v_n^{\pm 2}][x]$ is given by*

$$\{(\widehat{u}\widehat{u}^*)^l, v_n(\widehat{u} - \widehat{u}^*)(\widehat{u}\widehat{u}^*)^l : l \geq 0\}$$

Proof. Let f be in the kernel of d_1. We may write $f = f_e + v_n f_o$ with $f_e \in R[[\widehat{u}]]$ and $v_n f_o \in v_n R[[\widehat{u}]]$. Since d_1 interchanges classes in $R[[\widehat{u}]]$ and $v_n R[[\widehat{u}]]$, we must have both $d_1(f_e) = 0$ and $d_1(v_n f_o) = 0$. We will show that $f_e \in \operatorname{span}\{(\widehat{u}\widehat{u}^*)^l\}$ and $v_n f_o \in \operatorname{span}\{v_n(\widehat{u} - \widehat{u}^*)(\widehat{u}\widehat{u}^*)^l\}$.

Let $f_e = \mu \widehat{u}^j \bmod (\widehat{u}^{j+1})$ with $\mu \in R$. By Lemma 3.2, $\widehat{u}^{*j} \equiv (-1)^j \widehat{u}^j \bmod (\widehat{u}^{j+1})$. Since

$$d_1(f_e) \equiv \mu v_n^{1-2^n}(\widehat{u}^j - \widehat{u}^{*j}) \equiv \mu v_n^{1-2^n}(\widehat{u}^j + (-1)^{j+1}\widehat{u}^j) \bmod (\widehat{u}^{j+1})$$

must be zero, j must be even. Then $f_e - (-1)^{\frac{j}{2}} \mu (\widehat{u}\widehat{u}^*)^{\frac{j}{2}}$ is in $R[[\widehat{u}]]$, in the kernel of d_1, and has $\widehat{u}$-adic valuation strictly larger than that of f_e. Thus, f_e may be approximated to any degree by polynomials in $R[\widehat{u}\widehat{u}^*]$, which proves the claim for f_e.

We need to consider the first two terms in the case of f_o. Let $v_n f_o \equiv \mu v_n \widehat{u}^j + \nu v_n \widehat{u}^{j+1} \bmod (\widehat{u}^{j+2})$. Applying d_1 modulo $(\widehat{u}^{j+1})$ shows that j must now be odd. Next we apply d_1 modulo $(\widehat{u}^{j+2})$. The congruences in Lemma 3.2 give

$$\widehat{u}^j + \widehat{u}^{*j} \equiv \widehat{v}_1^j \widehat{u}^{j+1} \bmod (2, \widehat{u}^{j+2})$$

$$\widehat{u}^{j+1} + \widehat{u}^{*j+1} \equiv 0 \bmod (2, \widehat{u}^{j+2})$$

Thus,

$$0 = d_1(v_n f_o) \equiv v_n^{2-2^n} \mu(\widehat{u}^j + \widehat{u}^{*j}) + v_n^{2-2^n} \nu(\widehat{u}^{j+1} + \widehat{u}^{*j+1})$$

$$\equiv \widehat{v}_1^j v_n^{2-2^n} \mu \widehat{u}^{j+1} \bmod (2, \widehat{u}^{j+2})$$

It follows that $\mu = 2\gamma$ for some $\gamma \in R$. Then

$$\gamma v_n(\widehat{u} - \widehat{u}^*)(\widehat{u}\widehat{u}^*)^{\frac{j-1}{2}} \equiv \mu \widehat{u}^j \bmod(\widehat{u}^{j+1})$$

Thus, $v_n f_o - (-1)^{\frac{j-1}{2}} \gamma v_n(\widehat{u} - \widehat{u}^*)(\widehat{u}\widehat{u}^*)^{\frac{j-1}{2}}$ is in $v_n R[[\widehat{u}]]$, is in $\ker(d_1)$, and has $\widehat{u}$-adic valuation strictly larger than that of $v_n f_o$. This shows that f_o may be approximated to any degree by elements of $\text{span}\{v_n(\widehat{u} - \widehat{u}^*)(\widehat{u}\widehat{u}^*)^l\}$, which proves the claim for f_o.

That the above set of elements is linearly independent follows from inspecting their leading terms. □

The next step is to relate the image of d_1 to its kernel. This consists of analyzing the class $\widehat{u} + \widehat{u}^*$ in terms of the above basis for the kernel.

Lemma 3.7. *$\widehat{u} + \widehat{u}^*$ is in $\mathbb{Z}_{(2)}[\widehat{v}_1, \ldots, \widehat{v}_{n-1}, v_n^{\pm 2}][[\widehat{u}\widehat{u}^*]]$. In other words, there is a power series ξ with coefficients in $\mathbb{Z}_{(2)}[\widehat{v}_1, \ldots, \widehat{v}_{n-1}, v_n^{\pm 2}]$ such that $\widehat{u} + \widehat{u}^* = \xi(\widehat{u}\widehat{u}^*)$.*

Proof. It follows from the proof of Proposition 3.6 that $\widehat{u} + \widehat{u}^* \in R[[\widehat{u}\widehat{u}^*]]$, as it shows that any class that is in $R[[\widehat{u}]]$ and in $\ker(d_1)$ is also in $R[[\widehat{u}\widehat{u}^*]]$. Lemma 3.5 shows that $\widehat{u}+\widehat{u}^* \in R[[\widehat{u}]]$ and $c(\widehat{u}+\widehat{u}^*) = \widehat{u}+\widehat{u}^*$ shows it is in $\ker(d_1)$. Since x does not divide $\widehat{u}+\widehat{u}^*$, the coefficients of ξ lie in $\mathbb{Z}_{(2)}[\widehat{v}_1, \ldots, \widehat{v}_{n-1}, v_n^{\pm 2}]$. □

Remark 3.8. In fact, something stronger is true. In the proof of Proposition 3.6, if we replace R by the ring $\mathbb{Z}_{(2)}[\widehat{v}_1, \ldots, \widehat{v}_n]$, the same argument applies to show that $\widehat{u}+\widehat{u}^*$ is in $\mathbb{Z}_{(2)}[\widehat{v}_1, \ldots, \widehat{v}_n][[\widehat{u}\widehat{u}^*]]$.

We will say more about the power series expansion of $\widehat{u}+\widehat{u}^*$ in $\widehat{\widehat{u}\widehat{u}^*}$ in section 3.8. For now, we describe the E_2-page. We will present the result as a module over $E_2^{*,*}(pt)$. Recall that

$$E_2^{0,*}(pt) = \mathbb{Z}_{(2)}[\widehat{v}_1, \ldots, \widehat{v}_{n-1}, v_n^{\pm 2}]x^0$$

$$E_2^{s,*}(pt) = \mathbb{Z}/2[\widehat{v}_1, \ldots, \widehat{v}_{n-1}, v_n^{\pm 2}]x^s \qquad \text{for } s > 0$$

We then have

Theorem 3.9. *The E_2-page is given by*

$$E_2^{0,*} = E_2^{0,*}(pt)[[\widehat{\widehat{u}\widehat{u}^*}]]\{1, v_n(\widehat{u}-\widehat{u}^*)\}$$

$$E_2^{s,*} = E_2^{s,*}(pt)[[\widehat{\widehat{u}\widehat{u}^*}]]/(\widehat{u}+\widehat{u}^*) \qquad \textit{for } s > 0$$

Proof. In the basis for the kernel given in Proposition 3.6, for $s > 0$, the classes $v_n(\widehat{u}-\widehat{u}^*)(\widehat{\widehat{u}\widehat{u}^*})^l x^s$ are targets of differentials as are the classes $2(\widehat{\widehat{u}\widehat{u}^*})^l x^s$. Thus, these classes only survive on the zero line. Away from the zero line, we just have $R[[\widehat{\widehat{u}\widehat{u}^*}]]x^s$

modulo the image of d_1. Lemma 3.7 shows that the ideal generated by $(\widehat{u}+\widehat{u}^*)x$ in $E_1^{*,*}$ is contained in $R[[\widehat{u}\widehat{u}^*]][x]$. This proves the theorem.

□

3.4 The image of the norm

We now find ourselves in a very nice place. We know how the differentials act on the coefficients, and we have a large collection of permanent cycles. The remaining permanent cycles live on the zero line and the next step is to find representatives for them in $ER(n)^*(\mathbb{C}P^\infty)$. We do this using the norm described by Proposition 5.1 in the appendix. We let N_* denote the map

$$N_* : E(n)^*(\mathbb{C}P^\infty) \longrightarrow ER(n)^*(\mathbb{C}P^\infty)$$

and $\mathcal{N}_*$ denote the map resulting from postcomposing N_* with the inclusion of fixed points map $ER(n)^*(\mathbb{C}P^\infty) \to E(n)^*(\mathbb{C}P^\infty)$,

$$\mathcal{N}_* : E(n)^*(\mathbb{C}P^\infty) \longrightarrow ER(n)^*(\mathbb{C}P^\infty) \longrightarrow E(n)^*(\mathbb{C}P^\infty)$$

Thus, for any $z \in E(n)^*(\mathbb{C}P^\infty)$, $\mathcal{N}_*(z)$ is a permanent cycle on the zero line represented in $ER(n)^*(\mathbb{C}P^\infty)$ by $N_*(z)$. From the appendix, we have $\mathcal{N}_*(z) = z + c(z)$. For any w such that $c(w) = w$, we have $\mathcal{N}_*(wz) = w\mathcal{N}_*(z)$. Let $S = \mathbb{Z}_{(2)}[\widehat{v}_1, \ldots, \widehat{v}_{n-1}, v_n^{\pm 2}]$ (so that R above is $S[x]$). As a module over $S[[\widehat{u}\widehat{u}^*]] = \mathbb{Z}_{(2)}[\widehat{v}_1, \ldots, \widehat{v}_{n-1}, v_n^{\pm 2}][[\widehat{u}\widehat{u}^*]]$, we may write

$$E(n)^*(\mathbb{C}P^\infty) = S[[\widehat{u}\widehat{u}^*]]\{1, v_n, \widehat{u}, v_n\widehat{u}\}$$

Since c fixes $S[[\widehat{u}\widehat{u}^*]]$, $\mathcal{N}_*$ is a map of modules over $S[[\widehat{u}\widehat{u}^*]]$. We restrict $\mathcal{N}_*$ to the submodule $S[[\widehat{u}\widehat{u}^*]]\{\widehat{u}, v_n\widehat{u}\} \subset E(n)^*(\mathbb{C}P^\infty)$ and let $\mathrm{im}(\mathcal{N}_*^{\mathrm{res}})$ denote the image. (We restrict to the submodule because we do not want the coefficients, in particular 2, to be in $\mathrm{im}(\mathcal{N}_*^{\mathrm{res}})$. This is because we will mod out by $\mathrm{im}(\mathcal{N}_*)$ later, and we will want multiplication by 2 to be injective on the quotient.) We have

$$\mathcal{N}_*(\widehat{u}) = \widehat{u} + \widehat{u}^*$$

$$\mathcal{N}_*(v_n\widehat{u}) = v_n(\widehat{u} - \widehat{u}^*)$$

so

$$\mathrm{im}(\mathcal{N}_*^{\mathrm{res}}) = S[[\widehat{u}\widehat{u}^*]]\{\widehat{u} + \widehat{u}^*, v_n(\widehat{u} - \widehat{u}^*)\}$$

This is a submodule of $E_1^{0,*}$, and furthermore, since elements of $\mathrm{im}(\mathcal{N}_*^{\mathrm{res}})$ are permanent cycles, it is contained in $\ker(d_1)$. Since no differentials have their targets in the zero line, it follows that $\mathrm{im}(\mathcal{N}_*^{\mathrm{res}})$ is a submodule of $E_2^{0,*}$. We have a short exact sequence

$$0 \longrightarrow \mathrm{im}(\mathcal{N}_*^{\mathrm{res}}) \longrightarrow E_2^{*,*} \longrightarrow \widetilde{E}_2^{*,*} \longrightarrow 0$$

where $\widetilde{E}_2^{*,*}$ is by definition the quotient. Since all differentials on $\mathrm{im}(\mathcal{N}_*^{\mathrm{res}})$ are zero, we may further view it as a sub-spectral sequence. Furthermore, since $\mathrm{im}(\mathcal{N}_*^{\mathrm{res}})$ injects into $E_r^{*,*}$ at each stage, it follows that the above short exact sequence is in fact a short exact sequence of spectral sequences. From Theorem 3.9 we conclude

$$\widetilde{E}_2^{*,*} = \frac{E_2^{*,*}(pt)[[\widehat{u}\widehat{u}^*]]}{(\widehat{u}+\widehat{u}^*)}$$

In the short exact sequence above $\mathrm{im}(\mathcal{N}_*^{\mathrm{res}})$ collapses immediately, so it remains to compute the spectral sequence $\widetilde{E}_2^{*,*}$.

3.5 Landweber flatness

Let $\widehat{E}(n)^* = \mathbb{Z}_{(2)}[\widehat{v}_1, \ldots, \widehat{v}_{n-1}, \widehat{v}_n^{\pm 1}]$. There is an isomorphism of rings (but not graded rings) between $E(n)^*$ and $\widehat{E}(n)^*$ sending v_k to $\widehat{v}_k$. $\widehat{E}(n)^*$ consists entirely of permanent cycles. Thus $\widehat{E}(n)^* \subset ER(n)*$ and $ER(n)^*$ is a module over $\widehat{E}(n)^*$. We may view $E_r^{*,*}(pt)$ as a spectral sequence of $\widehat{E}(n)^*$-modules. The E_2-page of the spectral sequence of interest, $\widetilde{E}_2^{*,*}$, may be written as

$$\widetilde{E}_2^{*,*} = E_2^{*,*}(\mathrm{pt})[[\widehat{u}\widehat{u}^*]]/(\widehat{u}+\widehat{u}^*) = E_2^{*,*}(pt) \otimes_{\widehat{E}(n)^*} \widehat{E}(n)^*[[\widehat{u}\widehat{u}^*]]/(\widehat{u}+\widehat{u}^*)$$

The right hand coordinate of the tensor product consists entirely of permanent cycles, so we will be done if we can show that we can commute taking homology past the tensor product at each stage. That is, we need to know that tensoring with $\widehat{E}(n)^*[[\widehat{u}\widehat{u}^*]]/(\widehat{u}+\widehat{u}^*)$ over $\widehat{E}(n)^*$ is exact. This would be true if $\widehat{E}(n)^*[[\widehat{u}\widehat{u}^*]]/(\widehat{u}+\widehat{u}^*)$ were flat over $\widehat{E}(n)^*$, but we can in fact show that a weaker condition holds.

If we identify $E(n)^*$ with $\widehat{E}(n)^*$ as above, then we may view $E_r^{*,*}(pt)$ as a spectral sequence of $E(n)^*$-modules. Starting with this observation, it is shown in [9] that $E_r^{*,*}(pt)$ in fact lives in the category of $E(n)_*E(n)$-comodules that are finitely presented as $E(n)^*$-modules.

Thus, to solve our problem we only need to show that $\widehat{E}(n)^*[[\widehat{u}\widehat{u}^*]]/(\widehat{u}+\widehat{u}^*)$ is flat on the category of finitely presented $E(n)_*E(n)$-comodules, i.e. that it is *Landweber flat*.

In [20], Hovey and Strickland prove an $E(n)$-version of the Landweber filtration theorem. We state and prove an $E(n)$-exact functor theorem, which follows as a corollary of Hovey and Strickland's work. To be consistent with the literature, we prove the result for $E(n)_*$-modules, keeping in mind that $E(n)_*$ is formally isomorphic (as rings but not graded rings) to $E(n)^*$ so everything below holds for $E(n)^*$-modules as well.

Proposition 3.10. *Let M be an $E(n)_*$-module. The functor $(-)\otimes_{E(n)_*} M$ is exact on the category of $E(n)_*E(n)$-comodules that are finitely presented as $E(n)_*$-modules if and only if for each $k \geq 0$ multiplication by v_k is monic on $M/(v_0,\dots,v_{k-1})M$, i.e. $(v_0, v_1, v_2, \dots)$ is a regular sequence on M.*

Remark 3.11. Since $E(n)_*$ is height n in the sense of Hovey-Strickland, there is only something to check for $0 \leq k \leq n$. For $k > n$, $M/(v_0,\dots,v_k) = 0$ so multiplication by v_k is trivially monic.

Proof. We follow Landweber's original proof over MU_* in [21]. Applying $(-)\otimes_{E(n)_*} M$

to the sequence

$$0 \longrightarrow E(n)_* \xrightarrow{p} E(n)_* \longrightarrow E(n)_*/(p) \longrightarrow 0$$

shows that $p : M \longrightarrow M$ is monic if and only if $\mathrm{Tor}_1^{E(n)_*}(E(n)_*/(p), M) = 0$. For $k > 0$, applying $(-) \otimes_{E(n)_*} M$ to the sequence

$$0 \longrightarrow E(n)_*/(v_0, \ldots, v_{k-1}) \xrightarrow{v_k} E(n)_*/(v_0, \ldots, v_{k-1}) \longrightarrow E(n)_*/(v_0, \ldots, v_n) \longrightarrow 0$$

shows that multiplication by v_k is monic if and only if

$$\mathrm{Tor}_1^{E(n)_*}(E(n)_*/(v_0, \ldots, v_{k-1}), M) \longrightarrow \mathrm{Tor}_1^{E(n)_*}(E(n)_*/(v_0, \ldots, v_k), M)$$

is surjective. It follows that multiplication by v_k is monic on $M/(v_0, \ldots, v_{k-1})$ for all k if and only if $\mathrm{Tor}_1^{E(n)_*}(E(n)_*/(v_0, \ldots, v_k), M)$ is zero for all k. In [20] it is shown that every $E(n)_*E(n)$-comodule N that is finitely presented over $E(n)_*$ admits a finite filtration by subcomodules

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_s = N$$

for some s with $N_r/N_{r-1} \equiv \Sigma^t E(n)_*/(v_0, \ldots, v_j)$ for some $j \leq n$ and some t, both depending on r. In view of this, $\mathrm{Tor}_1^{E(n)_*}(E(n)_*/(v_0, \ldots, v_k), M) = 0$ for all k is equivalent to $\mathrm{Tor}_1^{E(n)_*}(N, M) = 0$ for all finitely presented $E(n)_*E(n)$-comodules, N. Finally, this is equivalent to $(-) \otimes_{E(n)_*} M$ being an exact functor on the category of $E(n)_*E(n)$-comodules finitely presented over $E(n)_*$. □

We now show that $M = \widehat{E}(n)^*[[\widehat{u}\widehat{u}^*]]/(\widehat{u}+\widehat{u}^*)$ satisfies the algebraic criterion given above.

Lemma 3.12. *$(\widehat{v}_0, \ldots, \widehat{v}_{n-1}, \widehat{v}_n)$ is a regular sequence in $\widehat{E}(n)^*[[\widehat{u}\widehat{u}^*]]/(\widehat{u}+\widehat{u}^*)$.*

Proof. Recall our notation $I_k = (\widehat{v}_0, \widehat{v}_1, \ldots, \widehat{v}_{k-1})$ and $I_0 = (0)$. Suppose $f(\widehat{u}\widehat{u}^*) \in \widehat{E}(n)^*[[\widehat{u}\widehat{u}^*]]/(I_k, \widehat{u}+\widehat{u}^*)$ is such that $\widehat{v}_k f(\widehat{u}\widehat{u}^*) = 0$. Then

$$\widehat{v}_k f(\widehat{u}\widehat{u}^*) = g(\widehat{u}\widehat{u}^*)(\widehat{u}+\widehat{u}^*) \text{ mod } I_k$$

Further modding out by $\widehat{v}_k$, we have

$$0 = g(\widehat{u}\widehat{u}^*)(\widehat{u}+\widehat{u}^*) \text{ mod } I_{k+1}$$

By Lemma 3.2, we have that

$$\widehat{u}+\widehat{u}^* = \widehat{v}_{k+1}(\widehat{u}\widehat{u}^*)^{2^{k+1}} \text{ mod } (I_{k+1}, (\widehat{u}\widehat{u}^*)^{2^{k+1}+1})$$

which means $\widehat{u}+\widehat{u}^* \neq 0$ mod I_{k+1}. Since I_{k+1} is prime in $\widehat{E}(n)^*$, it follows that $g(\widehat{u}\widehat{u}^*) = 0$ mod I_{k+1}. Then $g(\widehat{u}\widehat{u}^*) = \widehat{v}_k h(\widehat{u}\widehat{u}^*)$ mod I_k for some $h(\widehat{u}\widehat{u}^*)$. Hence,

$$f(\widehat{u}\widehat{u}^*) = h(\widehat{u}\widehat{u}^*)(\widehat{u}+\widehat{u}^*) \text{ mod } I_k$$

so that $f(\widehat{u}\widehat{u}^*) = 0$ in $\widehat{E}(n)^*[[\widehat{u}\widehat{u}^*]]/(I_k, \widehat{u}+\widehat{u}^*)$. Thus, multiplication by $\widehat{v}_k$ is injective.

□

Thus, M is Landweber flat. That is, tensoring with M over $\widehat{E}(n)^*$ is an exact functor on the category of finitely presented $E(n)_*E(n)$-comodules. Since $E_r^{*,*}(pt)$ lives in this category, we may commute homology past the tensor product at each stage of $\widetilde{E}_{r,}^{*,*} = E_r^{*,*}(pt) \otimes_{\widehat{E}(n)^*} M$. Furthermore, since M consists entirely of permanent cycles by Proposition 3.3, the entire spectral sequence can be evaluated on the coefficients. In other words, Theorem 4.3 in [9] applies to show that $\widetilde{E}_r^{*,*}$ is isomorphic to $(E_r^{*,*}(pt) \otimes_{\widehat{E}(n)^*} M, d_r \otimes_{\widehat{E}(n)^*} \mathrm{id}_M)$ as spectral sequences of $ER(n)^*$-modules and converges to $ER(n)^* \otimes_{\widehat{E}(n)^*} M$. We conclude

Proposition 3.13.

$$\widetilde{E}_\infty^{*,*} = E_\infty^{*,*}(pt) \otimes_{\widehat{E}(n)^*} M = E_\infty^{*,*}(pt)[[\widehat{u}\widehat{u}^*]]/(\widehat{u} + \widehat{u}^*)$$

3.6 The E_∞-page

We will now put all of the pieces together. Let us return to the short exact sequence of E_2-terms we had in Section 3.4.

$$0 \longrightarrow \mathrm{im}(\mathcal{N}_*^{\mathrm{res}}) \longrightarrow E_2^{*,*} \longrightarrow \widetilde{E}_2^{*,*} \longrightarrow 0$$

Lemma 3.14. *Upon taking homology the induced long exact sequence collapses into short exact sequences at each stage. In particular, we have a short exact sequence*

$$0 \longrightarrow \mathrm{im}(\mathcal{N}_*^{res}) \longrightarrow E_\infty^{*,*} \longrightarrow E_\infty^{*,*}(pt)[[\widehat{u}\widehat{u}^*]]/(\widehat{u} + \widehat{u}^*) \longrightarrow 0$$

Proof. Since the connecting homomorphism of the long exact sequence must increase filtration degree (because the differentials do), yet $\mathrm{im}(\mathcal{N}_*^{\mathrm{res}})$ is concentrated in filtration degree zero, the connecting homomorphism is zero at each stage. Thus we have a short exact sequence at E_∞. Recall that the left hand spectral sequence collapses immediately. The right hand spectral sequence was computed in the previous section. □

We analyze this short exact sequence, starting away from the zero line. In strictly positive filtration degree, the left hand term is zero and so we have

$$E_\infty^{s,*} = E_\infty^{s,*}(pt)[[\widehat{u}\widehat{u}^*]]/(\widehat{u} + \widehat{u}^*) \qquad \text{for } s > 0$$

The zero line is more involved. We begin by giving a "polite" answer.

Proposition 3.15. *The zero-line $E_\infty^{0,*}$ injects into its rationalization, and the rationalization may be computed as the algebraic invariants of the rationalization of $E_1^{0,*}$:*

$$E_\infty^{0,*} \hookrightarrow E_\infty^{0,*} \otimes \mathbb{Q} \xrightarrow{\cong} (E_1^{0,*} \otimes \mathbb{Q})^{C_2}$$

Proof. Since no classes on the zero line are targets of differentials, $E_\infty^{0,*}$ contains no torsion and so injects into its rationalization. $E_2^{0,*}$ is exactly the invariants in $E_1^{0,*}$. Away from the zero line, everything is 2-torsion from E_2 onward. Thus, for any class in $E_r^{0,*}$, twice it is in the kernel of d_r. Since $d_{2^{n+1}-1}$ is the last possible differential, we have that $2^{2^{n+1}-1}$ times any class on $E_2^{0,*}$ survives to E_∞. After we rationalize, the isomorphism

$$E_\infty^{0,*} \otimes \mathbb{Q} \cong (E_1^{0,*} \otimes \mathbb{Q})^{C_2}$$

follows. □

We now describe the zero line explicitly. Recall that

$$E_2^{0,*}(pt) = \mathbb{Z}_{(2)}[\widehat{v}_1, \ldots, \widehat{v}_{n-1}, v_n^{\pm 2}]$$

$$E_\infty^{0,*}(pt) = \mathbb{Z}_{(2)}[\widehat{v}_1, \ldots, \widehat{v}_{n-1}, v_n^{\pm 2^{n+1}}] \bigoplus_{0<i<n+1} (\widehat{v}_0, \ldots, \widehat{v}_{i-1})\mathbb{Z}_{(2)}[\widehat{v}_1, \ldots, \widehat{v}_{n-1}, v_n^{\pm 2^{i+1}}]v_n^{2^i}$$

We have a short exact sequence of E_2-terms

$$0 \longrightarrow \mathrm{im}(\mathcal{N}_*^{\mathrm{res}}) \longrightarrow E_2^{0,*} \longrightarrow \widetilde{E}_2^{0,*} \longrightarrow 0$$

where, as modules over $E_2^{0,*}(pt)$, we have

$$\mathrm{im}(\mathcal{N}_*^{\mathrm{res}}) = E_2^{0,*}(pt)[[\widehat{u}\widehat{u}^*]]\{\widehat{u} + \widehat{u}^*, v_n(\widehat{u} - \widehat{u}^*)\}$$

$$\widetilde{E}_2^{0,*} = E_2^{0,*}(pt)[[\widehat{u}\widehat{u}^*]]/(\widehat{u} + \widehat{u}^*)$$

and

$$E_2^{0,*} = E_2^{0,*}(pt)[[\widehat{u}\widehat{u}^*]]\{1, v_n(\widehat{u} - \widehat{u}^*)\}$$

Recall that $\widehat{u} + \widehat{u}^* = \xi(\widehat{u}\widehat{u}^*) \cdot 1$. When we pass to the E_∞-page, we have, as a module over $E_\infty^{0,*}(pt)$,

$$\mathrm{im}(\mathcal{N}_*^{\mathrm{res}}) = E_\infty^{0,*}(pt)[[\widehat{u}\widehat{u}^*]]\{v_n^{2p}(\widehat{u} + \widehat{u}^*), v_n^{2p+1}(\widehat{u} - \widehat{u}^*)\}/J, \quad 0 \le p < 2^n$$

where J is the submodule generated by the following relations (which come from writing $E_2^{0,*}(pt)$ as a module over $E_\infty^{0,*}(pt)$). Write $2p = 2^{i+1}m + 2^i$. For $0 \le j < i$, we have

$$\widehat{v}_j \cdot [v_n^{2p}(\widehat{u} + \widehat{u}^*)] = (\widehat{v}_j v_n^{2p}) \cdot (\widehat{u} + \widehat{u}^*)$$

$$\widehat{v}_j \cdot [v_n^{2p+1}(\widehat{u} - \widehat{u}^*)] = (\widehat{v}_j v_n^{2p}) \cdot v_n(\widehat{u} - \widehat{u}^*)$$

Over $E_\infty^{0,*}(pt)$, we also have

$$\widetilde{E}_\infty^{0,*} = E_\infty^{0,*}(pt)[[\widehat{u}\widehat{u}^*]]/(\widehat{u} + \widehat{u}^*)$$

Thus, we conclude

Theorem 3.16. *As a module over $E_\infty^{0,*}(pt)$, we have*

$$E_\infty^{0,*} = \frac{E_\infty^{0,*}(pt)[[\widehat{u}\widehat{u}^*]]\{1, v_n^{2p}(\widehat{u} + \widehat{u}^*), v_n^{2p+1}(\widehat{u} - \widehat{u}^*)\}}{K}, \qquad 0 \le p < 2^n$$

where K encodes the relations generating J in $\mathrm{im}(\mathcal{N}_*^{res})$ over $E_\infty^{0,*}$ above together with the relation $v_n^0(\widehat{u} + \widehat{u}^*) = \xi(\widehat{u}\widehat{u}^*) \cdot 1$.

3.7 Extension problems and multiplicative structure

Most of the hard work in solving extension problems is already done by Propositions 3.3 and 5.1 as they provide canonical lifts to $ER(n)^*(\mathbb{C}P^\infty)$ of our generators of $E_\infty^{*,*}$. Proposition 3.3 produces a class $\widehat{p}_1 \in ER(n)^*(\mathbb{C}P^\infty)$ whose image in $E(n)^*(\mathbb{C}P^\infty)$ is $\widehat{u}\widehat{u}^*$. On the zero line, we also have classes $v_n^k(\widehat{u} + (-1)^k\widehat{u}^*)$ which are the images under the norm of classes $v_n^k\widehat{u} \in E(n)^*(\mathbb{C}P^\infty)$. Since the norm factors through the map $ER(n)^*(\mathbb{C}P^\infty) \longrightarrow E(n)^*(\mathbb{C}P^\infty)$, these classes have canonical lifts in $ER(n)^*(\mathbb{C}P^\infty)$ as well.

In Lemma 3.7 we showed that in $E(n)^*(\mathbb{C}P^\infty)$, $\widehat{u} + \widehat{u}^*$ may be written as a power series $\xi(\widehat{u}\widehat{u}^*)$, with the coefficients of ξ in $\widehat{E}(n)^*$. We have independently constructed lifts $N_*(\widehat{u})$ of $\widehat{u} + \widehat{u}^*$ and $\widehat{p}_1$ of $\widehat{u}\widehat{u}^*$, so we must verify this equality lifts to $ER(n)^*(\mathbb{C}P^\infty)$. This turns out to be true for degree reasons:

Lemma 3.17. *$N_*(\widehat{u}) = \xi(\widehat{p}_1)$ in $ER(n)^*(\mathbb{C}P^\infty)$.*

Proof. The two classes have the same image in $E(n)^*(\mathbb{C}P^\infty)$, so their difference is a multiple of x. If $N_*(\widehat{u}) - \xi(\widehat{p}_1) \neq 0$ in $ER(n)^*(\mathbb{C}P^\infty)$, let r be the maximal power of x that divides $N_*(\widehat{u}) - \xi(\widehat{p}_1)$. Suppose $r \geq 1$. Then $N_*(\widehat{u}) - \xi(\widehat{p}_1)$ is represented by a nonzero class $z \in E_\infty^{r,1-\lambda+r}$. Since $r \geq 1$, we have $2^j - 1 < r \leq 2^{j+1} - 1$. Since $E_\infty^{r,*} = 0$ for $r \geq 2^{n+1}$, we must have $j < n+1$. By inspection of degrees, we have that $E_\infty^{r,l} = 0$ unless $l = 0 \bmod 2^{j+1}$. Then

$$1 - \lambda + r = 0 \bmod 2^{j+1}$$

Since $1 - \lambda = -2^{n+2}(2^{n-1} - 1)$, it follows that

$$r = 0 \bmod 2^{j+1}$$

which is impossible. Thus, $N_*(\widehat{u}) - \xi(\widehat{p}_1) = 0$ in $ER(n)^*(\mathbb{C}P^\infty)$. □

Before we go on to solve the remaining extension problems, we pause to remark on the significance of the class $N_*(\widehat{u}) = \xi(\widehat{p}_1)$ above which appears in the denominator of the right hand term of the short exact sequence of Theorem 1.1.

Remark 3.18. $N_*(\widehat{u})$ is a canonical class in $ER(n)^*(\mathbb{C}P^\infty)$ in the following sense. The "hatted" orientation $\widehat{u}$ induces a map

$$B(U(1) \rtimes C_2)_+ = \mathbb{C}P^\infty_+ \wedge_{C_2} EC_{2+} \xrightarrow{\widehat{u}\wedge 1} \Sigma^{1-\lambda} E\mathbb{R}(n) \wedge_{C_2} EC_{2+}$$

Postcomposing with the Adams isomorphism $E\mathbb{R}(n) \wedge_{C_2} EC_{2+} \simeq E\mathbb{R}(n)^{hC_2} = ER(n)$ and precomposing with $BU(1)_+ \to B(U(1) \rtimes C_2)_+$ yields the class $N_*(\widehat{u})$ in the $ER(n)$-cohomology of $BU(1) = \mathbb{C}P^\infty$ whose image in $E(n)^*(\mathbb{C}P^\infty)$ is the $\widehat{u} + \widehat{u}^*$ described above. This description also gives an alternate argument that $N_*(\widehat{u})$ is a power series on $\widehat{p}_1$ as follows. Identifying $B(U(1) \rtimes C_2)$ above with $BO(2)$, it is shown in [9] that $ER(n)^*(BO(2))$ is the quotient of a power series ring over $ER(n)^*$ on two classes, $\widehat{c}_1$ and $\widehat{c}_2$. The above description of $N_*(\widehat{u}) \in ER(n)^*(\mathbb{C}P^\infty)$ shows that it is the image of a class in $ER(n)^*(BO(2))$, i.e. some power series in $\widehat{c}_1$ and $\widehat{c}_2$. Identifying $BU(1)$ with $BSO(2)$, it can be shown that the map $ER(n)^*(BO(2)) \longrightarrow ER(n)^*(BSO(2))$ above sends $\widehat{c}_1$ to zero and $\widehat{c}_2$ to $\widehat{p}_1$ (see Proposition 3.3). It follows that $N_*(\widehat{u})$ is a power series on $\widehat{p}_1$ over $ER(n)^*$. Note that this argument does not identify the power series explicitly—to do that, we still need Lemmas 3.7 and 3.17.

With canonically determined lifts in hand, we now solve all extension problems. The relations of Theorem 3.16 need to be lifted to $ER(n)^*(\mathbb{C}P^\infty)$. Additionally, we describe how classes in $\text{im}(N_*^{\text{res}})$ multiply together. In both cases, we use Proposition 5.1 in the appendix. In $1(a)$ and (b) of the following lemma, recall that classes in

$E(n)^*$ of the form

$$\widehat{v}_j v_n^{2^{i+1}m+2^i} \qquad \text{with } 0 \le j < i$$

are permanent cycles and lift to classes of the same name in $ER(n)^*$.

Lemma 3.19. *The following relations hold in $ER(n)^*(\mathbb{C}P^\infty)$:*

1. *Write $2p = 2^{i+1}m + 2^i$ and suppose $0 \le j < i$. Then, as a module over $ER(n)^*$, $ER(n)^*(\mathbb{C}P^\infty)$ satisfies*

   (a) $\widehat{v}_j \cdot N_*(v_n^{2p}\widehat{u}) = (\widehat{v}_j v_n^{2p}) \cdot N_*(\widehat{u})$

   (b) $\widehat{v}_j \cdot N_*(v_n^{2p+1}\widehat{u}) = (\widehat{v}_j v_n^{2p}) \cdot N_*(v_n\widehat{u})$

   (c) $x \cdot N_*(v_n^{2p}\widehat{u}) = 0$

   (d) $x \cdot N_*(v_n^{2p+1}\widehat{u}) = 0$

2. *Let $0 \le 2p, 2l < 2^{n+1}$ and write $2p + 2l = 2^{n+1}q + 2r$ with $0 \le 2r < 2^{n+1}$ and $q = 0, 1$. As an algebra over $ER(n)^*$, $ER(n)^*(\mathbb{C}P^\infty)$ satisfies*

   (a) $N_*(v_n^{2p}\widehat{u})N_*(v_n^{2l}\widehat{u}) = v_n^{2^{n+1}q} N_*(v_n^{2r}\widehat{u})N_*(\widehat{u})$

   (b) $N_*(v_n^{2p+1}\widehat{u})N_*(v_n^{2l}\widehat{u}) = v_n^{2^{n+1}q} N_*(v_n^{2r+1}\widehat{u})N_*(\widehat{u})$

   (c) $N_*(v_n^{2p+1}\widehat{u})N_*(v_n^{2l+1}\widehat{u}) = v_n^{2^{n+1}q}\left(N_*(v_n^{2r+2}\widehat{u})N_*(\widehat{u}) - 4v_n^{2r+2}\widehat{p}_1\right)$.

Proof. The key facts are that N_* is a map of modules over $ER(n)^*(\mathbb{C}P^\infty)$ and that the $ER(n)^*(\mathbb{C}P^\infty)$-module (really, algebra) structure on $E(n)^*(\mathbb{C}P^\infty)$ comes from the quotient by x-map $ER(n)^*(\mathbb{C}P^\infty) \longrightarrow E(n)^*(\mathbb{C}P^\infty)$. The module structure is described by the diagram

$$\begin{array}{ccc} ER(n)^*(\mathbb{C}P^\infty) \otimes E(n)^*(\mathbb{C}P^\infty) & \xrightarrow{1\otimes N_*} & ER(n)^*(\mathbb{C}P^\infty) \otimes ER(n)^*(\mathbb{C}P^\infty) \\ \downarrow & & \downarrow \\ E(n)^*(\mathbb{C}P^\infty) & \xrightarrow{\quad N_* \quad} & ER(n)^*(\mathbb{C}P^\infty) \end{array}$$

We prove $1(a), 1(c)$ and $2(a)$; the other relations are proved similarly. For $1(a)$, note that in the upper left corner, the classes $\widehat{v}_j \otimes v_n^{2p}\widehat{u}$ and $\widehat{v}_j v_n^{2p} \otimes \widehat{u}$ have as their images in the bottom right corner the classes $\widehat{v}_j \cdot N_*(v_n^{2p}\widehat{u})$ and $(\widehat{v}_j v_n^{2p}) \cdot N_*(\widehat{u})$, respectively. But both $\widehat{v}_j \otimes v_n^{2p}\widehat{u}$ and $\widehat{v}_j v_n^{2p} \otimes \widehat{u}$ map to the same class, $\widehat{v}_j v_n^{2p}\widehat{u}$, in the bottom left corner, which proves $1(a)$. In $1(c)$, $x \otimes v_n^{2p}\widehat{u}$ maps to zero in the bottom left corner (since x maps to zero in $E(n)^*$); thus, $x \cdot N_*(v_n^{2p}\widehat{u}) = 0$. Finally, for $2(a)$, note that the classes $v_n^{2p}\widehat{u} \otimes N_*(v_n^{2l}\widehat{u})$ and $v_n^{2(p+l)}\widehat{u} \otimes N_*(\widehat{u})$ map to the same class under the left vertical map. Thus, their images in $ER(n)^*(\mathbb{C}P^\infty)$ are equal. But these are exactly $N_*(v_n^{2p}\widehat{u})N_*(v_n^{2l}\widehat{u})$ and $N_*(v_n^{2(p+l)}\widehat{u})N_*(\widehat{u})$, respectively. For any $\alpha \in E(n)^*(\mathbb{C}P^\infty)$ which admits a lift to $ER(n)^*(\mathbb{C}P^\infty)$ and any $z \in E(n)^*(\mathbb{C}P^\infty)$, the above diagram shows that $N_*(\alpha z) = \alpha N_*(z)$. Applying this to $\alpha = v_n^{2^{n+1}q}$ and $z = v_n^{2r}\widehat{u}$ completes the proof of $2(a)$. $\square$

We now prove Theorem 1.1 as stated in the introduction.

Proof of Theorem 1.1. $ER(n)^*(\mathbb{C}P^\infty)$ is (topologically) generated over $ER(n)^*$

by $\mathrm{im}(N_*^{\mathrm{res}})$ and the classes $\widehat{p}_1^l$. To see the isomorphism (of $ER(n)^*$-modules)

$$ER(n)^*(\mathbb{C}P^\infty)/\mathrm{im}(N_*^{\mathrm{res}}) \cong ER(n)^*[[\widehat{p}_1]]/(\xi(\widehat{p}_1))$$

we must show that the intersection of $\mathrm{im}(N_*^{\mathrm{res}})$ and the submodule of $ER(n)^*(\mathbb{C}P^\infty)$ generated over $ER(n)^*$ by $\{\widehat{p}_1^l | l \geq 0\}$ is precisely the ideal of $ER(n)^*(\mathbb{C}P^\infty)$ generated by $\xi(\widehat{p}_1)$. Clearly, $(\xi(\widehat{p}_1))$ is in this intersection. We now prove the reverse inclusion. First recall that the domain of N_*^{res} is by definition the submodule

$$\mathbb{Z}_{(2)}[\widehat{v}_1, \ldots, \widehat{v}_{n-1}, v_n^{\pm 2}][[\widehat{u}\widehat{u}^*]]\{\widehat{u}, v_n\widehat{u}\}$$

of $E(n)^*(\mathbb{C}P^\infty)$. Choose some

$$z = \sum_i (a_i\widehat{u} + b_i v_n \widehat{u})(\widehat{u}\widehat{u}^*)^i \qquad \text{with } a_i, b_i \in \mathbb{Z}_{(2)}[\widehat{v}_1, \ldots, \widehat{v}_{n-1}, v_n^{\pm 2}]$$

so that (since N_* is a map of modules over $ER(n)^*(\mathbb{C}P^\infty)$)

$$N_*(z) = \sum_i (N_*(a_i\widehat{u}) + N_*(b_i v_n \widehat{u}))\widehat{p}_1^i \tag{3.20}$$

Suppose that $N_*(z)$ is also in $\mathrm{span}\{\widehat{p}_1^l | l \geq 0\}$ over $ER(n)^*$, i.e. that

$$N_*(z) = \sum_i \lambda_i \widehat{p}_1^i \qquad \text{with } \lambda_i \in ER(n)^* \tag{3.21}$$

We claim that $\xi(\widehat{p}_1) = N_*(\widehat{u})$ divides $N_*(z)$ in $ER(n)^*(\mathbb{C}P^\infty)$. This will follow from two claims: for all i, (i) $b_i = 0$ and (ii) a_i is in the subalgebra $E_\infty^{0,*}(pt)$ of $\mathbb{Z}_{(2)}[\widehat{v}_1, \ldots, \widehat{v}_{n-1}, v_n^{\pm 2}]$. Together, they imply that $N_*(z) = \sum_i N_*(a_i\widehat{u})\widehat{p}_1^i$ with each a_i having a representative in $ER(n)^*$. Since N_* is a map or $ER^*(\mathbb{C}P^\infty)$-modules, it follows that $N_*(a_i\widehat{u}) = a_i N_*(\widehat{u})$ and so $N_*(\widehat{u})$ divides $N_*(z)$ in $ER(n)^*(\mathbb{C}P^\infty)$.

To prove both claims, we map into $E(n)^*(\mathbb{C}P^\infty)$. Then (3.20) becomes

$$\mathcal{N}_*(z) = \sum_i (a_i(\widehat{u} + \widehat{u}^*) + b_i v_n(\widehat{u} - \widehat{u}^*))(\widehat{u}\widehat{u}^*)^i \tag{3.22}$$

and (3.21) becomes

$$\mathcal{N}_*(z) = \sum_i \lambda_i (\widehat{u}\widehat{u}^*)^i \tag{3.23}$$

with λ_i now in $E_\infty^{0,*}(pt)$ (by abuse of notation, we denote the image of λ_i in $E_\infty^{0,*}(pt)$ by the same name). Recall that $\widehat{u} - \widehat{u}^* = 2\widehat{u} + \ldots$ and $\widehat{u} + \widehat{u}^*$ is the span of $\{(\widehat{u}\widehat{u}^*)^l | l \geq 0\}$ over $E_\infty^{0,*}(pt)$ (since $\widehat{v}_i \in E_\infty^{0,*}(pt)$ for all i). The collection $\{(\widehat{u}\widehat{u}^*)^l, \widehat{u}(\widehat{u}\widehat{u}^*)^l\}$ is clearly linearly independent over $E(n)^*$. Hence, from inspecting the right hand sides of (3.22) and (3.23), it follows that $b_i = 0$ for all i.

To prove the second claim, suppose that i_1 is the first index such that $a_{i_1} \notin E_\infty^{0,*}(pt)$. Let l be the maximal such that $a_{i_1} = f(v_n^{2^l})$ with the coefficients of f in $\mathbb{Z}_{(2)}[\widehat{v}_1, \ldots, \widehat{v}_{n-1}]$. Note that f cannot be a constant polynomial since a_{i_1} is not in $E_\infty^{0,*}$ by assumption. Then inspection of $E_\infty^{0,*}(pt)$ shows that $\widehat{v}_k a_{i_1} \in E_\infty^{0,*}(pt)$ for $0 \leq k < l$ and $\widehat{v}_k a_{i_1} \notin E_\infty^{0,*}(pt)$ for $k \geq l$. Recall that

$$\widehat{u} + \widehat{u}^* \equiv \widehat{v}_l (\widehat{u}\widehat{u}^*)^{2^l} + \ldots \mod (2, \widehat{v}_1, \ldots, \widehat{v}_{l-1})$$

Since we know that

$$\sum_i a_i(\widehat{u} + \widehat{u}^*)(\widehat{u}\widehat{u}^*)^i = \sum_i \lambda_i (\widehat{u}\widehat{u}^*)^i \quad \text{with } \lambda_i \in E_\infty^{0,*}$$

it follows from looking at lowest degree terms mod $(2, \widehat{v}_1, \ldots, \widehat{v}_{l-1})$ that $\widehat{v}_l a_{i_1}$ is in $E_\infty^{0,*}$, a contradiction. Thus, $a_i \in E_\infty^{0,*}$ for all i. This proves Theorem 1.1.

□

Theorem 1.1 is the nice form of the answer, but we have in fact shown something stronger. The following theorem presents $ER(n)^*(\mathbb{C}P^\infty)$ explicitly as an algebra over $ER(n)^*$.

Theorem 3.24.

$$ER(n)^*(\mathbb{C}P^\infty) = ER(n)^*(pt)[[\widehat{p}_1, N_*(v_n^j \widehat{u})]]/K$$

where $0 \leq j < 2^{n+1}$ *and* K *is the ideal generated by the relation* $N_*(\widehat{u}) = \xi(\widehat{p}_1)$ *of Lemma 3.17 together with the relations of Lemma 3.19*

Proof. This is a consequence of Lemmas 3.17 and 3.19 together with the description of $E_\infty^{*,*}$ in section 3.6. □

Remark 3.25. We conclude by returning to the case $n = 1$. It is somewhat degenerate, as $\mathcal{N}_*(\widehat{u}) = \widehat{u} + \widehat{u}^* = -\widehat{u}\widehat{u}^*$ in $KU^*(\mathbb{C}P^\infty)$, so $N_*(\widehat{u}) = -\widehat{p}_1$. It follows that $x \cdot \widehat{p}_1 = 0$ and powers of $\widehat{p}_1$ do not generate any 2-torsion. This is *not* true for $n > 1$. In general, $\widehat{p}_1$ supports higher powers of x up to and including $x^{2^{n+1}-2}$. In light of this, when $n = 1$, $\widehat{p}_1$ is redundant and it suffices to take $\{1, N_*(\widehat{u}), N_*(v_1\widehat{u}), N_*(v_1^2\widehat{u}), N_*(v_1^3\widehat{u})\}$ as a set of algebra generators with $x \cdot N_*(v_1^k \widehat{u}) = 0$ for all k. The relations come from Lemma 3.19 and our answer matches exactly the answer in Corollary 2.13 of [14].

3.8 Completing at I

We obtain an especially nice form of the answer if we complete at $I := I_n = (2, \widehat{v}_1, \ldots, \widehat{v}_{n-1})$. It turns out that the right hand side of the short exact sequence of Theorem 1.1 is free after completion. To see this, we need a technical lemma concerning $\widehat{u} + \widehat{u}^*$.

Lemma 3.26. *The following congruence holds in $\widehat{E}(n)[[\widehat{u}\widehat{u}^*]]$:*

$$\widehat{u} + \widehat{u}^* \equiv \widehat{v}_n(\widehat{u}\widehat{u}^*)^{2^{n-1}} \ \mathit{mod}\ ((\widehat{u}\widehat{u}^*)^{2^{n-1}+1}, I)$$

Since $\widehat{E}(n) \subset ER(n)^*$, this lifts to

$$\xi(\widehat{p}_1) \equiv \widehat{v}_n \widehat{p}_1^{2^{n-1}} \ \mathit{mod}\ (\widehat{p}_1^{2^{n-1}+1}, I)$$

in $ER(n)^*[[\widehat{p}_1]]$.

Proof. By Lemma 3.7, $\widehat{u} + \widehat{u}^* \in \widehat{E}(n)[[\widehat{u}\widehat{u}^*]]$. By Lemma 3.2,

$$\widehat{u} + \widehat{u}^* \equiv \widehat{v}_n \widehat{u}^{2^n} \ \mathrm{mod}\ (\widehat{u}^{2^n+1}, I)$$

It follows that

$$\widehat{u} + \widehat{u}^* \equiv \widehat{v}_n(\widehat{u}\widehat{u}^*)^{2^{n-1}} \ \mathrm{mod}\ ((\widehat{u}\widehat{u}^*)^{2^{n-1}+1}, I)$$

□

We will use the following version of the Weierstrass Preparation Theorem in [22].

Lemma 3.27. *[22] Let A be a graded commutative ring, complete in the topology* *defined by powers of an ideal I. Suppose $\alpha(x) \in A[[x]]$ satisfies $\alpha(x) \equiv \omega x^d$ mod (x^{d+1}, I), with $\omega \in A$ a unit. Then the ring $A[[x]]/(\alpha(x))$ is a free A-module with basis $\{1, x, x^2, \ldots, x^{d-1}\}$*

In what follows, we apologize for the poor notation: the hat of completion and the hat in $\widehat{p}_1$ are unrelated. Recall the short exact sequence of Theorem 1.1:

$$0 \longrightarrow \mathrm{im}(N_*^{\mathrm{res}}) \longrightarrow ER(n)^*(\mathbb{C}P^\infty) \longrightarrow \frac{ER(n)^*[[\widehat{p}_1]]}{(\xi(\widehat{p}_1))} \longrightarrow 0$$

After completing at I, if we set $A = ER(n)^{*\wedge}_{I}$ we find that the Weierstrass Preparation Theorem applies to the right hand term.

Proposition 3.28. *$ER(n)^{*\wedge}_{I}[[\widehat{p}_1]]/(\xi(\widehat{p}_1))$ is a free module over $ER(n)^{*\wedge}_{I}$ with basis $\{1, \widehat{p}_1, \widehat{p}_1^2, \ldots, \widehat{p}_1^{2^{n-1}-1}\}$.*

Chapter 4

Future directions

Historically, complex-oriented cohomology theories have been extraordinarily amenable to computations. If E is complex-oriented, its value on many spaces comes for free via the collapse of the Atiyah-Hirzebruch spectral sequence: $\mathbb{C}P^\infty$, $\mathbb{CP}^k$, $BU(q)$, and $B\mathbb{Z}/2^q$ are a few important examples. Other computations with complex-oriented theories on various fundamental or geometrically significant spaces have been the result of comprehensive and elaborate work which has uncovered deep structure in these theories. Examples include Wilson and Ravenel's computations of the Morava K-theory [23] and BP-cohomology [24] of Eilenberg MacLane spaces and work of Kitchloo, Laures, and Wilson on BO and its connective covers [25].

This thesis is a step towards extending the plethora of complex-oriented computations to a non-complex-oriented context. Many of the above examples have also proved amenable to computations with Real Johnson-Wilson theories, and we discuss some work in progress in this direction, joint with Kitchloo and Wilson, below.

4.1 Truncated projective spaces and non-immersions

In the case $n = 2$, the above computation of $ER(n)^*(\mathbb{C}P^\infty)$ can be extended to the $ER(2)$-cohomology of $\mathbb{C}P^k$ for all $k \leq \infty$, including multiplicative structure. This is carried out in [10]. Recall that $ER(2)$ is 48-periodic. The most interesting part of the answer occurs in degrees multiples of 16. The value of $ER(2)^{16*}(\mathbb{C}P^k)$ depends on the congruence class of k mod 8.

Theorem 4.1. *[10] We have*

$$ER(2)^{16*}(\mathbb{C}P^{8m+1}) = ER(2)^{16*}[\widehat{p}_1]/(\widehat{p}_1^{4m+2}, 2\widehat{p}_1^{4m+1})$$

There is an analogous result for $\mathbb{C}P^{8m+i}$ for all i, $0 \leq i < 8$.

It is instructive to keep the (complex oriented) $E(2)$-cohomology of truncated projective spaces in mind to contrast with the non complex-oriented case: $E(2)^*(\mathbb{C}P^k) = E(2)^*[u]/(u^{k+1})$ where u denotes the first Chern class of the canonical line bundle. The image of $\widehat{p}_1$ in $E(2)$-cohomology is a unit multiple of u^2, and thus the image of $\widehat{p}_1^{4k+1}$ is a unit multiple of u^{8k+2}. This is zero in $E(2)^*(\mathbb{C}P^{8m+1})$, but $\widehat{p}_1^{4k+1} \neq 0$ in $ER(2)$-cohomology! *Extra powers of $\widehat{p}_1$ survive beyond what one would expect* *in a complex-oriented theory!* (beyond the top cell of $\mathbb{C}P^{8m+i}$) This also occurs in $ER(2)^*(\mathbb{C}P^{8m+i})$, for $i = 1, 2, 3, 4$, and 5.

Computations of Banerjee, Kitchloo, and Wilson [3, 7, 8] reveal this phenomenon in the $ER(2)$-cohomology of certain real projective spaces as well. They were able to leverage this exotic multiplicative structure to produce new non-immersions of real projective spaces. The fact that these extra classes show up in the $ER(2)$-cohomology of complex projective spaces and are undetected in complex-oriented theories is a promising step toward filling the gaps between known immersions and nonimmersions (see [26]).

4.2 Odd Eilenberg-MacLane spaces and connective covers of BO

For certain spaces, the Bockstein spectral sequence is as simple as possible in the sense that it can be generated by a collection of permanent cycles over the coefficients such that all differentials may be computed entirely on the coefficients (where they are already known). In such cases, the $ER(n)$-cohomology may be computed from $E(n)$-cohomology by means of base change. Note that, as we have shown above, this is *not* the case for $\mathbb{C}P^\infty$. One example where this does work out, computed by Kitchloo and Wilson [9], is $ER(n)^*(BO(q))$:

Theorem 4.2. *[9] There is a canonical isomorphism*

$$ER(n)^*(BO(q)) = ER(n)^*[[\widehat{c}_1, \widehat{c}_2, \dots]]/(\widehat{c}_1 - \widehat{c}_1^*, \widehat{c}_2 - \widehat{c}_2^*, \dots)$$

where the generators $\widehat{c}_i$ restrict to $c_i v_n^{i(2^n-1)}$ in $E(n)$-cohomology.

Note that $E(n)^*(BO(q)) = E(n)^*[[c_1, c_2, \dots]]/(c_1 - c_1^*, c_2 - c_2^*, \dots)$ [27], and the proof of Kitchloo and Wilson's result gives a road map for how to compute $ER(n)$-cohomology from $E(n)$-cohomology by a completely formal procedure. The following theorem from [11] is an extension of Kitchloo and Wilson's result for BO to a wide collection of spaces.

Theorem 4.3. *[11] There is a class of spaces which includes $X = BO, BSO, BSpin$, and $BString$ (the last for $n \leq 2$ only) as well as Eilenberg-MacLane spaces $K(\mathbb{Z}, 2m+1)$, $K(\mathbb{Z}/(2^q), 2m)$, and $K(\mathbb{Z}/2, m)$ for which*

$$ER(n)^*(X) = E(n)^*(X) \otimes_{\widehat{E(n)}} ER(n)^*$$

This theorem extends both the results of [9] on $ER(n)^*(BO)$ as well as work of Laures and Olbermann [28] on $L_{K(2)}TMF_0(3)^*(BString)$. Its proof involves identifying a key property of certain spaces X (that of being part of *Landweber flat real pair*) which entails that the above isomorphism holds. The results of [11] also construct certain short exact sequences, worked out in BP-cohomology in [25], in $ER(n)$-cohomology.

Note that the above result does not hold for $\mathbb{C}P^\infty = K(\mathbb{Z}, 2)$. We will call spaces of the form $K(\mathbb{Z}, 2k+1)$ and $K(\mathbb{Z}/2^q, 2k)$ *odd* Eilenberg-MacLane spaces, and we will denote the other half of Eilenberg MacLane spaces, those of the form $K(\mathbb{Z}, 2k)$ and $K(\mathbb{Z}/2^q, 2k-1)$, *even* Eilenberg-MacLane spaces. We discuss the even case in the next section.

4.3 Even Eilenberg-MacLane spaces and connective covers of BU

The computation of $ER(n)^*(K(\mathbb{Z}, 2))$ in this thesis is the first computation of the $ER(n)$-cohomology of an even Eilenberg-MacLane space. As we have seen, it requires computing certain differentials on classes other than the coefficients. However, there is hope that the other even Eilenberg-MacLane spaces as well as $BU(q)$ and its connective covers, fit a similar mold. The results below for $B\mathbb{Z}/2^q = K(\mathbb{Z}/2^q, 1)$ and $BU(q)$ have at present been established only for $n = 2$.

Proposition 4.4. *[10] There is a short exact sequence*

$$0 \longrightarrow \mathrm{im}(N_*) \longrightarrow ER(2)^*(B\mathbb{Z}/2^q) \longrightarrow ER(2)^*[[\widehat{p}_1, z]]/J \longrightarrow 0$$

where J is the ideal generated by the following.

(1) The relations (in degrees $16*$) encoded by the kernel of the composite map

$$ER(2)^{16*}[[\widehat{p}_1, z]] \longrightarrow E(2)^{16*}[[\widehat{p}_1, z]] \longrightarrow E(2)^{16*}(B\mathbb{Z}/2^q)$$

which, since the first map is an injection (in these degrees), are detected entirely within $E(2)$-cohomology.

(2) *The series $\xi(\widehat{u}) = N_*(\widehat{u})$ as in the computation of $\mathbb{C}P^\infty$ above, which maps to $\widehat{u} + \widehat{u}^*$ in $E(2)$-cohomology*

Proposition 4.5. *The restrictions of the Pontryagin classes in $E(2)^*(BSO(2q))$ to $E(2)^*(BU(q))$ lift to classes $\widehat{p}_i \in ER(2)^*(BU(q))$. Modulo the image of the norm (see $ER(n)^*(\mathbb{C}P^\infty)$ above), $ER(2)^*(BU(q))$ is generated as an $ER(2)^*$-algebra by $\{\widehat{p}_1, \ldots, \widehat{p}_q\}$.*

These two propositions provide some hope that there is a class of spaces for which the computations of $ER(n)^*(\mathbb{C}P^\infty)$, $ER(2)^*(B\mathbb{Z}/2^q)$, and $ER^*(BU)$ prescribe a certain mold. We conclude with a conjecture.

Conjecture 4.6. *Let X be one of the even Eilenberg-MacLance spaces, one of BU, BSU, or $BU\langle 6\rangle$, or more generally a space $\underline{BP\langle k\rangle}_q$ in the Ω-spectrum representing $BP\langle k\rangle$ for $2(2^k - 1) \leq q \leq 2(2^{k+1} - 1)$. Modulo the image of the norm, $ER(n)^*(X)$ is a quotient of a certain power series ring over $ER(n)^*$ described as follows. The ring is generated by classes represented by permanent cycles in the Bockstein spectral sequence, formed by taking usual generators z of the $E(n)$-cohomology of the above spaces, hatting them, and multiplying them by their conjugates to form $\widehat{z}c(\widehat{z})$. In the case of Eilenberg-MacLane spaces of the form $K(\mathbb{Z}/2^q, 2j+1)$ we also include the images of generators of $K(\mathbb{Z}/2, 2j+1)$ (as computed for $n = 2$ in [11]).*

$ER(n)^*(X)/\mathrm{im}(N_*)$ *is obtained by taking a quotient of this power series ring by an ideal generated by classes in degrees divisible by* 2^{n+2}. *In these degrees, the map* $ER(n)^{2^{n+2}*}(X) \longrightarrow E(n)^{2^{n+2}*}(X)$ *is an injection, and the relations which generate the ideal may be detected in* $E(n)$*-cohomology.*

Chapter 5

Appendix

5.1 Statement and discussion of results

The goal of this section is to establish some technical lemmas that enable us to solve extension problems in the Bockstein spectral sequence. Everything we need is summed up in the following proposition.

Proposition 5.1.

(a) There is a norm map

$$N : F(X, E(n)) \longrightarrow F(X, ER(n))$$

(b) Give $F(X, ER(n))$ *and* $F(X, E(n))$ *module structures over* $F(X, ER(n))$ *via the multiplication on* $ER(n)$ *and the map of ring spectra* $ER(n) \longrightarrow E(n)$*, respectively. Then* N *is a map of modules over* $F(X, ER(n))$*. On homotopy, it*

is given as a map of modules over $ER(n)^*(X)$ by

$$N_* : E(n)^*(X) \longrightarrow ER(n)^*(X)$$

(c) *After composing with the inclusion of fixed points $ER(n)^*(X) \longrightarrow E(n)^*(X)$ (which is one of the maps in our exact couple) we denote the composite by $\mathcal{N}_*$,*

$$\mathcal{N}_* : E(n)^*(X) \longrightarrow ER(n)^*(X) \longrightarrow E(n)^*(X)$$

It is given in homotopy by $\mathcal{N}_*(z) = z + c(z)$.

(d) *In the Bockstein spectral sequence, every class $\mathcal{N}_*(z)$ in $E(n)^*(\mathbb{C}P^\infty) = E_1^{0,*}$ is a permanent cycle represented by $N_*(z) \in ER(n)^*(X)$.*

Intuitively (and imprecisely), the norm is constructed as the following composite

$$E \longrightarrow E/C_2 \xrightarrow[\simeq]{A} E^{C_2}$$

where A denotes the Adams isomorphism and to get the above proposition, we hope to set $E = F(X, E(n))$. Both source and target are modules over E^{C_2} (the source via the inclusion of fixed points $E^{C_2} \longrightarrow E$), and we want to show that the composite is a map of modules over E^{C_2}. However, as we are working both stably and equivariantly, there are several subtleties that we must take into account:

(i) The Adams isomorphism above does not make sense unless E is free. As such, we will need to work with $\widetilde{E} := EC_{2+} \wedge E$ and transition between E and $\widetilde{E}$ via the map $\widetilde{E} \longrightarrow E$ induced by the map $EC_{2+} \longrightarrow S^0$, which is an equivalence nonequivariantly.

(ii) The construction of the Adams map requires working with several groups. In fact, we set things up in a slightly more general framework of a group G and a subgroup H, and at some point must expand to the group $\Gamma = H \rtimes G$. We follow the treatment of the Adams isomorphism given in [16].

(iii) Several constructions (the Adams isomorphism, orbits, fixed points) require moving between the G-complete, H-fixed, and trivial universes. As such, we must keep track of the change of universe functors.

The above considerations make the diagrams necessary to prove Proposition 5.1, especially part (b), somewhat complicated. We carry these out in the next section.

Warning 5.2. What follows is something of an exorcism. Proposition 5.1 gives everything we need for the computations in the rest of the paper and what follows is only for those interested in abstract nonsense.

5.2 Background

We begin with some general setup. Let G be a finite group, H a normal subgroup. Let $p : G \longrightarrow G/H$ the projection onto the quotient. Let $\iota : \{e\} \longrightarrow G$ be the inclusion of the trivial subgroup. Let U a complete G-universe. Let $i : U^H \longrightarrow U$ denote the inclusion of universes. Let GSU denote the category of (genuine) G-equivariant spectra indexed over U.

Let $E \in GSU$. Recall that H-fixed points are defined by first passing to the H-trivial universe U^H,

$$E^H := (i^*E)^H \in (G/H)SU^H$$

The H-fixed points functor is right adjoint to the functor $p^* : (G/H)SU^H \longrightarrow GSU^H$, where $p : G \longrightarrow G/H$ is projection onto the quotient. For a spectrum $D \in GSU^H$, we have the H-orbits

$$D/H \in (G/H)SU^H$$

The H-orbits functor is left adjoint to p^*. In the case of an H-free genuine G-spectrum $EH_+ \wedge E \in GSU$, we have the fact that free spectra are induced up from the H-trivial universe, $EH_+ \wedge E = i_*(EH_+ \wedge D)$ where $D = i^*E \in GSU^H$, which allows us to define the homotopy orbits of E by

$$E_{hH} := (EH_+ \wedge D)/H \in (G/H)SU^H$$

Fix $\widetilde{E} := EH_+ \wedge E$ to be an H-free genuine G-spectrum with E a ring spectrum such that $\widetilde{E} = i_*\widetilde{D}$ with $\widetilde{D} = EH_+ \wedge i^*E \in GSU^H$. Set $D = i^*E$. In the first part of this section, we will construct a norm map

$$\widetilde{N} : \widetilde{D} \longrightarrow p^*(\widetilde{D}^H)$$

Since D is a ring spectrum, $\widetilde{D} = EH_+ \wedge D$ is a D-module via the map $EH_+ \wedge D \longrightarrow D$ induced by $EH_+ \longrightarrow S^0$. Both source and target are $p^*(D^H)$-modules (via the counit $p^*(D^H) \longrightarrow D$). We will first show that $\widetilde{N}$ is a map of $p^*(D^H)$ modules. We will then apply ι^* to move to the nonequivariant context and construct from $\widetilde{N}$ a norm map on D itself:

$$N : \iota^* D \longrightarrow \iota^* D^H$$

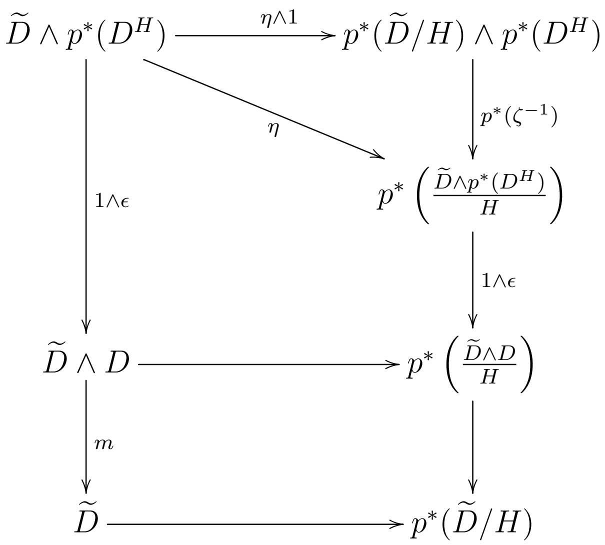
We will then examine the effect of N on homotopy groups. We conclude by describing how this construction specializes to give Proposition 5.1.

5.3 Construction of the norm

We begin with the unit of the $(-/H, p^*)$-adjunction,

$$\eta : \widetilde{D} \longrightarrow p^*(\widetilde{D}/H)$$

The following diagram shows that this is a map of $p^*(D^H)$-modules:



The map $\zeta : (\widetilde{D} \wedge p^*(D^H))/H \longrightarrow (\widetilde{D}/H) \wedge D^H$ is adjoint to $\eta \wedge 1$ and is an equivalence by Lemma 2.4.9 of [16]. The triangle commutes by adjoint functor identities.

The norm map is the composite of η with the Adams isomorphism

$$A : p^*(\widetilde{D}/H) \longrightarrow p^*(\widetilde{D}^H)$$

It is constructed as the adjoint of a map

$$A' : i_*p^*\widetilde{D}/H) \longrightarrow i_*\widetilde{D}$$

Since the functor i_* preserves smash products, both the source and target are modules over $i_*p^*(D^H)$. We will write out the construction of A', following [16] and in the process, show that it is a map of $i_*p^*(D^H)$-modules. It will then follow that the adjoint is a map of $p^*(D^H)$-modules.

Intuitively, the Adams map comes from smashing with a map $\tau : S^0 \longrightarrow \Sigma^\infty H_+$

$$\widetilde{D}/H \longrightarrow \frac{H_+ \wedge \widetilde{D}}{H} = \widetilde{D}$$

A subtlety that needs to be taken into account with the above is that since the copy of $\widetilde{D}$ on the right hand side needs to carry an action of H even after taking the quotient of $H_+ \wedge \widetilde{D}$ by H, we need to work with a larger group. This is remedied by introducing the following framework.

Let G act on H by conjugation. Let $\Gamma = H \rtimes G$. Let π be the subgroup $H \rtimes 1$.

We have the commutative diagram

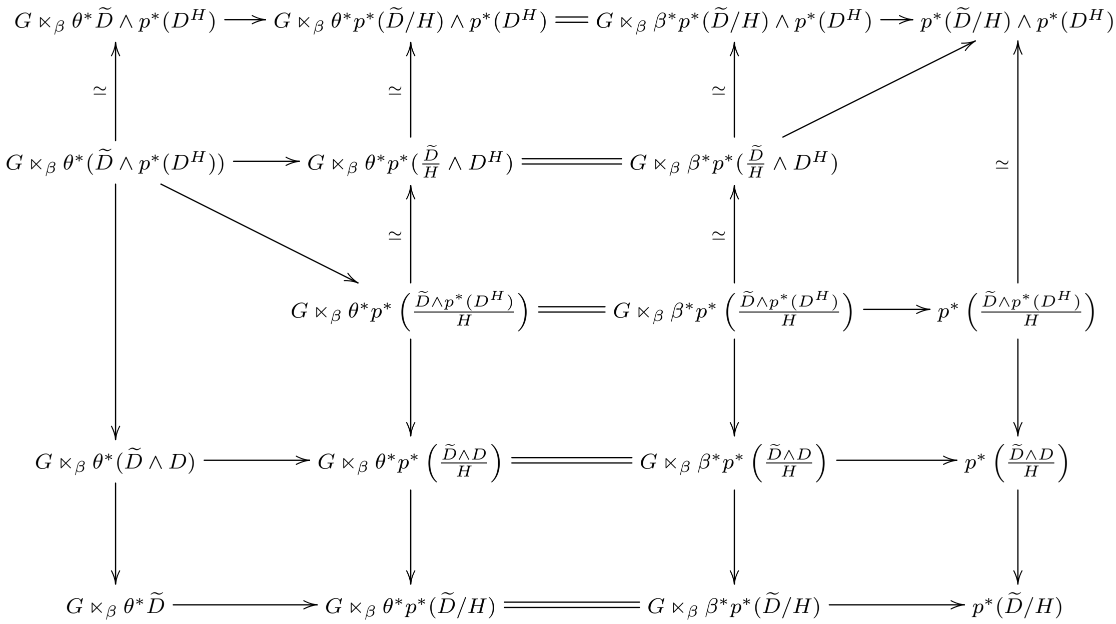
$$\begin{array}{ccc} \Gamma & \xrightarrow{\theta} & G \\ \big\downarrow{\scriptstyle \beta} & & \big\downarrow{\scriptstyle p} \\ G & \xrightarrow{p} & G/H \end{array}$$

where $\theta(n,g) = ng$, $\beta : \Gamma \longrightarrow \Gamma/\pi \cong G$ is the projection. Let $\alpha : G \cong 1 \rtimes G \longrightarrow \Gamma$ be the inclusion. Notice that $\theta \circ \alpha = \beta \circ \alpha = 1$.

We begin constructing A' by first transitioning to the larger group Γ. The first step is to note that $p^*(\widetilde{D}/H)$ is equivalent to $(\theta^*\widetilde{D})/\pi = G \ltimes_\beta \theta^*\widetilde{D}$ via the composite

$$s : G \ltimes_\beta \theta^*\widetilde{D} \longrightarrow G \ltimes_\beta \theta^*p^*(\widetilde{D}/H) = G \ltimes_\beta \beta^*p^*(\widetilde{D}/H) \longrightarrow G \ltimes_\beta p^*(\widetilde{D}/H)$$

It is an equivalence by Lemma 2.7.4 of [16]. The following diagram shows that s is a map of $p^*(D^H)$-modules.



Since s is an equivalence, s^{-1} is also a map of $p^*(D^H)$ modules. We take $i_*(s^{-1})$

to be the first component of A'.

The second subtle point is that the map $\tau : S^0 \longrightarrow \Sigma^\infty N_+$ is only defined over a G-complete universe. Since we are working with the larger group Γ, we let U' be a Γ-complete universe and take the G-complete universe U to be $(U')^\pi$. Let j denote the inclusion. Then the map τ is defined in the category $\Gamma SU'$. We thus have a map

$$j_*i_*\theta^*\widetilde{D} \longrightarrow H_+ \wedge j_*i_*\theta^*\widetilde{D} = j_*i_*(H_+ \wedge \theta^*\widetilde{D}) = j_*i_*(\Gamma \ltimes_\alpha \alpha^*\theta^*\widetilde{D})$$

Note that $\alpha^*\theta^*\widetilde{D} = \widetilde{D} = \alpha^*\beta^*\widetilde{D}$, but we keep the former notation for now.

Our first step is thus to define

$$j_*i_*(G \ltimes_\beta \theta^*\widetilde{D}) = G \ltimes_\beta (j_*i_*\theta^*\widetilde{D}) \longrightarrow G \ltimes_\beta \Gamma \ltimes_\alpha j_*i_*\alpha^*\theta^*\widetilde{D} = j_*i_*(G \ltimes_\beta \Gamma \ltimes_\alpha \alpha^*\theta^*\widetilde{D})$$

and show it is a map of $j_*i_*p^*(D^H)$-modules.

$$\begin{array}{ccc}
j_*i_*(G \ltimes_\beta \theta^*\widetilde{D}) \wedge j_*i_*p^*(D^H) & \xrightarrow{\tau\wedge 1} & j_*i_*(G \ltimes_\beta \Gamma \ltimes_\alpha \alpha^*\theta^*\widetilde{D}) \wedge j_*i_*p^*(D^H) \\
\simeq \uparrow & & \simeq \uparrow \\
j_*i_*G \ltimes_\beta \theta^*(\widetilde{D} \wedge p^*(D^H)) & \xrightarrow{\tau} & j_*i_*G \ltimes_\beta \Gamma \ltimes_\alpha \alpha^*\theta^*(\widetilde{D} \wedge p^*(D^H)) \\
\downarrow & & \downarrow \\
j_*i_*G \ltimes_\beta \theta^*(\widetilde{D} \wedge D) & \longrightarrow & j_*i_*G \ltimes_\beta \Gamma \ltimes_\alpha \alpha^*\theta^*(\widetilde{D} \wedge D) \\
\downarrow & & \downarrow \\
j_*i_*G \ltimes_\beta \theta^*\widetilde{D} & \longrightarrow & j_*i_*G \ltimes_\beta \Gamma \ltimes_\alpha \alpha^*\theta^*\widetilde{D}
\end{array}$$

Note that the entire diagram lives in the category of π-free $\Gamma SU'$-spectra. Thus, the above map is induced (via j_*) from a map

$$t : i_*G \ltimes_\beta \theta^*\widetilde{D} \longrightarrow i_*G \ltimes_\beta \Gamma \ltimes_\alpha \alpha^*\theta^*\widetilde{D}$$

in GSU. The entire diagram above lives in the category of π-free $\Gamma SU'$-spectra as well, so it is also induced up. From this, we conclude that t is a map of $i_*p^*(D^H)$-modules.

Finally, we have the action map

$$\xi : i_*G \ltimes_\beta \Gamma \ltimes_\alpha \alpha^*\theta^*\widetilde{D} = G \ltimes_\beta \Gamma \ltimes_\alpha \alpha^*\beta^*\widetilde{D} \longrightarrow i_*\widetilde{D}$$

which is constructed as i_* of the adjoint of the identity on $\widetilde{D}$.

The following diagram shows this is a map of $i_*p^*(D^H)$-modules

$$\begin{array}{ccccc}
i_*G \ltimes_\beta \Gamma \ltimes_\alpha \alpha^*\theta^*\widetilde{D} \wedge i_*p^*(D^H) & = & i_*G \ltimes_\beta \Gamma \ltimes_\alpha \alpha^*\beta^*\widetilde{D} \wedge i_*p^*(D^H) & \longrightarrow & i_*\widetilde{D} \wedge i_*p^*(D^H) \\
\simeq \uparrow & & \simeq \uparrow & \nearrow & \downarrow \\
i_*G \ltimes_\beta \Gamma \ltimes_\alpha \alpha^*\theta^*(\widetilde{D} \wedge p^*(D^H)) & = & i_*G \ltimes_\beta \Gamma \ltimes_\alpha \alpha^*\beta^*(\widetilde{D} \wedge p^*(D^H)) & & \\
\downarrow & & \downarrow & & \\
i_*G \ltimes_\beta \Gamma \ltimes_\alpha \alpha^*\theta^*(\widetilde{D} \wedge D) & = & i_*G \ltimes_\beta \Gamma \ltimes_\alpha \alpha^*\beta^*(\widetilde{D} \wedge D) & \longrightarrow & i_*(\widetilde{D} \wedge D) \\
\downarrow & & \downarrow & & \downarrow \\
i_*G \ltimes_\beta \Gamma \ltimes_\alpha \alpha^*\theta^*\widetilde{D} & = & i_*G \ltimes_\beta \Gamma \ltimes_\alpha \alpha^*\beta^*\widetilde{D} & \longrightarrow & i_*\widetilde{D}
\end{array}$$

We have the composite A'

$$i_*p^*(\widetilde{D}/H) \xrightarrow{i_*(s^{-1})} i_*G \ltimes_\beta \theta^*\widetilde{D} \xrightarrow{t} i_*G \ltimes_\beta \Gamma \ltimes_\alpha \alpha^*\theta^*\widetilde{D} \xrightarrow{\xi} i_*\widetilde{D}$$

The $(i_* p^*(-), (i^*(-))^H)$-adjunction gives the adjoint

$$A : \widetilde{D}/H \longrightarrow (i_* \widetilde{D})^H$$

Finally, we apply p^* and precompose with η to get

$$\widetilde{N} : \widetilde{D} \longrightarrow p^*(\widetilde{D}/H) \longrightarrow p^* \left((i_* \widetilde{D})^H \right)$$

We have proved

Lemma 5.3. *$\widetilde{N}$ is a map of $p^*(D^H)$-modules.*

We now forget to underlying nonequivariant spectra by applying ι^*:

$$\iota^* \widetilde{N} : \iota^* \widetilde{D} \longrightarrow \iota^*(i_* \widetilde{D})^H$$

and this is now a map of $\iota^* D^H$-modules. Throughout, we have been working with $\widetilde{D} = EH_+ \wedge D$. On the right, we know now compose with the map $\widetilde{D} \longrightarrow D$. On the left, we have an equivalence $\iota^* \widetilde{D} \longrightarrow \iota^* D$ and we precompose with its inverse to obtain

$$N : \iota^* D \longrightarrow \iota^* D^H$$

Both of these are evidently maps of $\iota^* D^H$-modules, so we conclude

Corollary 5.4. *$N : \iota^* D \longrightarrow \iota^* D^H$ is a map of $\iota^* D^H$-modules.*

5.4 Effect on homotopy groups

We may now postcompose with the inclusion of fixed points to obtain an endomorphism of $\iota^* D$:

$$\mathcal{N} : \iota^* D \longrightarrow \iota^* D^H \longrightarrow \iota^* D$$

This map has a particularly nice expression on the level of homotopy groups: on $\pi_*^u D$, $\mathcal{N}$ behaves as the group theoretic norm. In the category of equivariant orthogonal spectra, this is proved in [29].

Consider

$$\begin{array}{ccccccc}
i_*\theta^*\widetilde{D} & \xrightarrow{\tau} & i_*\Gamma \ltimes_\alpha \alpha^*\theta^*\widetilde{D} & = & i_*\Gamma \ltimes_\alpha \alpha^*\beta^*\widetilde{D} & \xrightarrow{\xi} & i_*\beta^*\widetilde{D} \\
\downarrow & & \downarrow & & \downarrow & \nearrow & \\
i_*\beta^*(G \ltimes_\beta \theta^*\widetilde{D}) & \longrightarrow & i_*\beta^*(G \ltimes_\beta \Gamma \ltimes_\alpha \alpha^*\theta^*\widetilde{D} & = & i_*\beta^*(G \ltimes_\beta \Gamma \ltimes_\alpha \alpha^*\beta^*\widetilde{D} & &
\end{array}$$

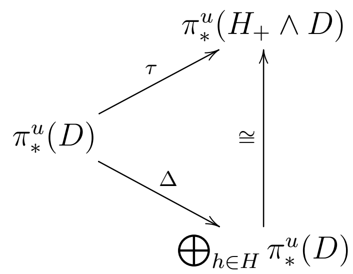
We apply α^* and use the fact that $\beta\alpha = id = \theta\alpha$ to get

$$\begin{array}{ccccccc}
i_*D & \longrightarrow & i_*\alpha^*\Gamma \ltimes_\alpha \alpha^*\theta^*\widetilde{D} & = & i_*\alpha^*\Gamma \ltimes_\alpha \alpha^*\beta^*\widetilde{D} & \longrightarrow & i_*D \\
\downarrow & & \downarrow & & \downarrow & \nearrow & \\
i_*(G \ltimes_\beta \theta^*\widetilde{D}) & \longrightarrow & i_*(G \ltimes_\beta \Gamma \ltimes_\alpha \alpha^*\theta^*\widetilde{D}) & = & i_*(G \ltimes_\beta \Gamma \ltimes_\alpha \alpha^*\beta^*\widetilde{D}) & &
\end{array}$$

Proposition 5.5. *For $z \in \pi_*^u(D)$, $\mathcal{N}_*$ is given by*

$$\mathcal{N}_*(z) = \sum_{h \in H} h \cdot z$$

Proof. We examine the top row of the above diagram on underlying nonequivariant spectra. Recall that the underlying nonequivariant spectra of $\iota^*\widetilde{D}$ and $\iota^* D$ are equivalent. Nonequivariantly, we have $\Gamma \ltimes_\alpha \alpha^*\theta^* D = H_+ \wedge D = \bigvee_{h \in H} D$ and the first horizontal map is given by smashing with $\tau : S^0 \longrightarrow \Sigma^\infty H_+$. From the construction (via the Pontryagin-Thom collapse map) it follows that, $\tau : \widetilde{D} \longrightarrow H_+ \wedge \widetilde{D}$ is nonequivariantly the diagonal map on homotopy, i.e. the following diagram commutes.



The final map is the action and sends an element in a wedge summand $\pi_*^u(D)$ in $\pi_*^u(H_+ \wedge D) \cong \bigoplus_{h \in H} \pi_*^u(D)$ indexed by $h \in H$ to the action of h on that element. Putting the two maps together proves the claim. $\square$

5.5 Proof of Proposition 5.1

Proposition 5.1 now follows as a specialization of the above results. Set $G = H = C_2$. Let X be a space with trivial C_2-action (in our present applications, $X = \mathbb{C}P^\infty$ with C_2 acting trivially and not by complex conjugation). Set $E := F(X, E\mathbb{R}(n))$. Then $\iota^* E = F(X, E(n))$, and $\pi_*^u E = E(n)^*(X)$ is the source of the norm, N_*. Since X has trivial C_2-action, we have

$$E^{C_2} = F(X, E\mathbb{R}(n))^{C_2} = F(X, E\mathbb{R}(n)^{C_2}) = F(X, ER(n))$$

and the target of N_* is $\pi_*^u F(X, ER(n)) = ER(n)^*(X)$. Upon postcomposing with the inclusion of fixed points $F(X, ER(n)) \longrightarrow F(X, E(n))$, we have that $\mathcal{N}$ is an endomorphism of $E(n)^*(X)$ which sends any element $z \in E(n)^*(X)$ to $z + c(z)$. The final part of Proposition 5.1 follows immediately from the exact couple generating the Bockstein spectral sequence. This completes the proof.

Remark 5.6. It is possible to modify the above proof of Proposition 5.1 to avoid using that $ER(n)$ is a homotopy commutative and associative ring spectrum and instead use the weaker statement that $ER(n)^*(-)$ is a cohomology theory on spaces valued in commutative and associative rings (as proved in [19]). One simply modifies the proof of (b) by working up to phantom maps and noting that after applying $F(X, -)$ to the diagrams for X any space and $\pi_*(-)$, all phantom maps are trivial. See [19] for more details.

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Vita

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