View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by LJMU Research Online

# Analytical formula for leg voltage THD of a PWM multilevel inverter

**O. Dordevic, M. Jones, E. Levi**

Liverpool John Moores University, UK, o.dordevic@ljmu.ac.uk

**Keywords:** Analytical derivation, Multilevel inverter, Pulse width modulation, Total harmonic distortion.

## Abstract

In this paper derivation of an analytical formula for the leg voltage THD is presented. The considered system is a leg of a multilevel pulse width modulated (PWM) voltage source inverter (VSI). The solution is based on the Parseval's theorem. The assumption throughout the derivations is that the ratio of the switching to the fundamental frequency is high. Derivations are based on the integration of the power of the PWM signal in a single switching period over the fundamental period of the signal. Only an ideal sinusoidal reference leg voltage is analysed. Analytical expression for the leg voltage THD is given for any number of levels. Validity of the derived analytical equations is confirmed by simulations and experiments.

## 1 Introduction

THD is a common parameter for evaluation of the quality of different PWM techniques and the importance of it is highlighted in [1]. However, it is possible that two totally different waveforms that differently distribute energy in the spectrum can have the same THD. This is explained in [1]. Hence, one has to be aware of the fact that the THD is just a global parameter and that there is a risk in taking a THD indiscriminately as a figure of merit.

The problem of finding analytical formula for the THD of the leg voltage created by a PWM multilevel inverter has been analysed in the past, but no pure analytical general solution has been given. It should be also noted that there are some papers that analysed THD of the multilevel inverters that are not operating in PWM mode, and where the output is of a quasi-square-wave form [1-4] – this is not of interest here. The case analysed in the paper is as in [5] where an analytical solution for the leg voltage THD and the weighted THD of the PWM multilevel inverter is given. The solution has been developed for the most typical numbers of levels, two, three and five, individually for each. The research of [5] was extended in [6] with an attempt at generalisation for an arbitrary number of levels. However, the generalisation is given in a piece-wise integral form and the integrals are not solved analytically, since this is a difficult problem due to the dependence of the borders of integration on the modulation index value. Numerical solution of the problem is rather simple, so it appears that the curves given for the multilevel cases in [6] were obtained by a computer program that does the integration. In this paper, the research of [6] is advanced and the integrals are solved analytically. A general analytical formula for the leg voltage THD is thus derived. The results are obtained and expressed in a simple meaningful way and some important conclusions are given. Circumstances when results are valid are explained. The same idea of integration and for obtaining general formulae has been used in [7-9]. However, the results in [7-9] are unfortunately not valid for the shape of the voltage that they were aimed for.

## 2 Signal power and definition of THD

One of the common approaches in signal processing is to obtain spectrum of the periodical signal and then perform analysis in the frequency domain. For obtaining spectrum, Fourier analysis is commonly in use. Fourier transformation is closely related to the definition of the energy and the power of the signal. The instantaneous power $p(t)$ and the energy $W$ of the real continuous signal $x(t)$ can be defined as [10]:

$$W = \int_{t_1}^{t_2} p(t)\mathrm{d}t = \int_{t_1}^{t_2} x(t)^2\,\mathrm{d}t \qquad (1)$$

where $t_1$ and $t_2$ represent instants in time between which the energy is calculated. The average (active) power of the continuous periodical signal, with period $T$, is defined as:

$$P = \frac{1}{T}\int_{0}^{T} x(t)^2\,\mathrm{d}t = \mathrm{RMS}^2(x) \qquad (2)$$

All signals measured in practice are discrete (e.g. sampled by oscilloscope), thus discrete forms of (1) and (2) are convenient for the signal processing. They have practically the same form in the discrete domain, but $x(t)$ should be replaced with $x[k]$, integrals becomes sums, and averaging is obtained by division with the number of samples per period $K$, rather than division with the fundamental period $T$.

The equation that links energy in the time domain and in the frequency domain is known as the Parseval's theorem. The theorem for the periodical signal states that the energy in one period of the signal in the time domain, i.e. power $P$, is equal to the energy (power) in the spectrum [10]. For a discrete periodical signal, spectrum is also discrete and periodical, and Parseval's theorem has the following form:

$$P = \frac{1}{K}\sum_{k=1}^{K} x[k]^2 = \sum_{h=0}^{K-1} \left|\underline{X}_h\right|^2 \qquad (3)$$

where $\underline{X}_h$ are complex values of the Fourier series of the signal $x[k]$.

The usual method of the THD determination is in practice based on the signal spectrum and on the FFT calculation. THD of an arbitrary real periodic signal can be calculated as:

$$\mathrm{THD}(x) = \sqrt{\left(\sum_{h=2}^{\lfloor K/2 \rfloor} X_{rms,h}^2\right) \Big/ X_{rms,1}^2} \tag{4}$$

where $X_{rms,h}$ represents RMS value in the single-sided (asymmetrical) spectrum of the $h^{th}$ component. One can see that the THD is a square root of the ratio of the distorting power over the useful power. Dc voltage can be easily filtered out; hence it is not considered as a distortion, and is excluded in (4). THD in (4) can be expressed in a different way, if Parseval's theorem (3) for RMS values of the asymmetrical spectrum and the fact that $X_{rms} = \sqrt{P}$ from (2) are applied:

$$\mathrm{THD}(x) = \sqrt{\left(X_{rms}^2 - X_{rms,1}^2 - X_{dc}^2\right) \Big/ X_{rms,1}^2} \tag{5}$$

From (5) THD can be easily numerically calculated from the time domain without full spectrum calculation.

## 3 Average power and THD of the PWM signal obtained by *l*-level PWM VSI during $T_s$

In this section, using the Parseval's theorem and the idea from [7, 8], the average power of the obtained PWM signal for a constant reference (i.e. for the sampled reference signal during one switching period $T_s$) is determined. Consider a general case when the PWM output is obtained using a multilevel modulator with $l$ levels. Normalisation is used, so that the signal is in the range from 0 to 1 (0 to $V_{dc}$). For obtaining real values in Volts, normalised values $u$ should be multiplied by $V_{dc}$, i.e. $v = uV_{dc}$. In this way comparison of obtained results is made easier. If a natural sampling is used, normalised reference leg voltage, $u_{LEG}^*(t)$, is sampled and held as constant $u_{LEG}^*(kT_s)$ during the whole switching period $T_s$. PWM process is shown in Fig. 1a. The obtained PWM signal $u_{LEG}[t]$ takes discrete values in time and is within a particular switching period $kT_s$ denoted as $u_{LEG}[kT_s]$.

Let us introduce parameters $i$ and $f$ as:

$$\begin{aligned} i &= \left\lfloor u_{LEG}^*(kT_s)\cdot(l-1)\right\rfloor \\ f &= u_{LEG}^*(kT_s)\cdot(l-1) - i_{I,J} \end{aligned} \tag{6}$$

Integer parameter $i$ takes values $0, 1, 2, \ldots, (l-2)$, while the fractional part $f$ is in the range $0 \le f < 1$.

The average power of the PWM signal $u_{LEG}[kT_s]$ during the switching period $T_s$ can be calculated using the integration in time as in (2). The process of integration is represented graphically in Fig. 1b. One gets that the average power of the produced PWM signal $u_{LEG}[kT_s]$, which corresponds to the dc reference signal $u_{LEG}^*(kT_s)$, is:

$$P_{T_s}(u_{LEG}[kT_s]) = \frac{1}{T_s}\int_0^{T_s} u_{LEG}^2[kT_s]\,\mathrm{d}t = P_{T_s}(u_{LEG}^*(kT_s)) + \frac{f - f^2}{(l-1)^2} \tag{7}$$

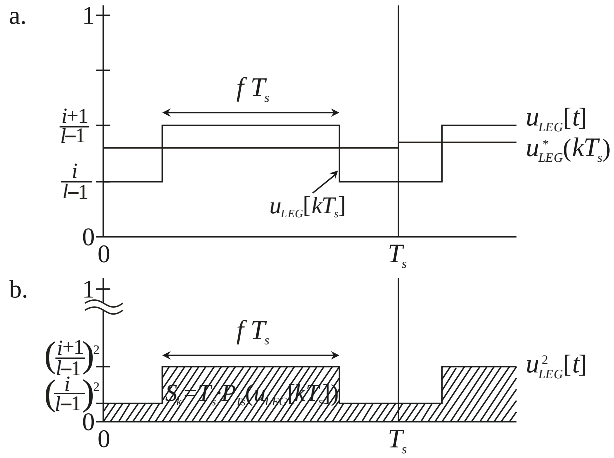


Fig. 1: a. Multilevel PWM for a constant signal $u_{LEG}^*(kT_s)$ during one switching period, $T_s$. b. Graphical determination of the active power of the multilevel PWM signal $u_{LEG}[t]$.

where $P_{T_s}(u_{LEG}^*(kT_s)) = (u_{LEG}^*(kT_s))^2$, and it represents the average power of the reference leg voltage $u_{LEG}^*(kT_s)$ within one switching period. Symbol $T_s$ in subscript is used to highlight that the calculation of the average power takes place during the period of $T_s$.

Representation in (7) shows that the power of the obtained PWM signal is equal to the sum of the power of the reference dc voltage plus power that is the consequence of the PWM. This additional power in the multilevel PWM signal $u_{LEG}[kT_s]$ is contained in the square waveform in order to get $u_{LEG}^*(kT_s)$ on average, and is manifested through the additional harmonics in spectrum.

The variation of the average power (7) of the multilevel (2, 3, 4 and 5 level) PWM produced signal $u_{LEG}[kT_s]$, using adjacent levels, for dc reference value $u_{LEG}^*(kT_s)$, is shown in Fig. 2. Because of the introduced normalisation, $u_{LEG}^*(kT_s)$ is in the range from 0 to 1. The average power of the reference signal $u_{LEG}^*(kT_s)$ is shown with thick grey dashed line.

Note that the average power of the PWM signal $u_{LEG}[kT_s]$ is independent of the PWM strategy, i.e. position and number of pulses inside $T_s$, which can be different. This will not affect the value of the average power $P_{T_s}(u_{LEG}[kT_s])$, i.e. the shaded area $S_k$ in Fig. 1b remains the same.

## 4 Average power and THD of the PWM signal obtained by *l*-level PWM VSI for sinusoidal reference signal

The assumed reference in this section is a sinusoidal signal and the results from the previous section will be used for determination of the average power of the PWM generated

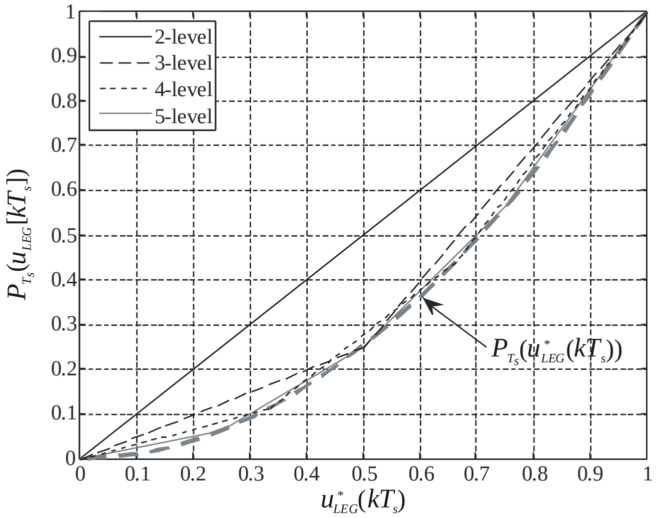


Fig. 2: The average power of the signal $u_{LEG}[kT_s]$, created using a multilevel PWM, whose average value is $u^*_{LEG}(kT_s)$.

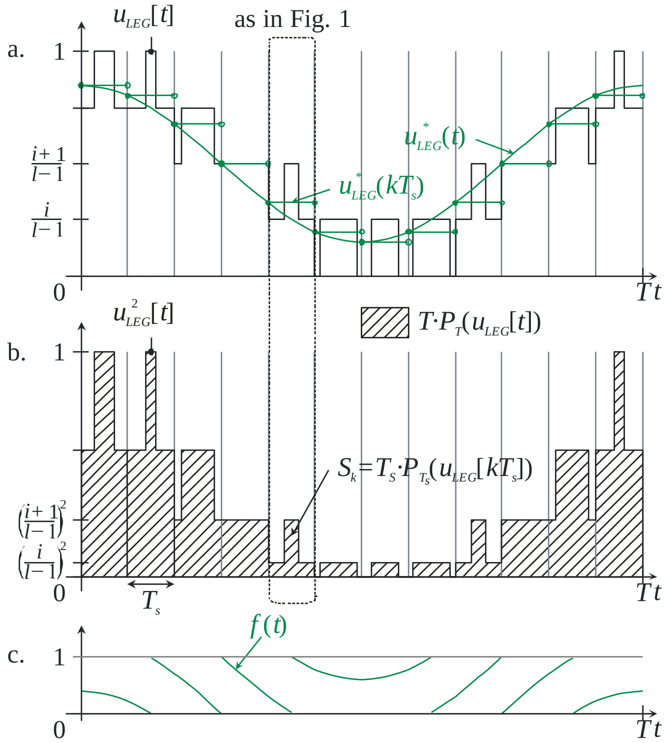


Fig. 3: a. Normalised reference leg voltage $u^*_{LEG}(t)$ and $l$-level VSI generated output leg voltage $u_{LEG}[t]$. b. Graphical determination of the average power of the obtained leg voltage. c. Reference fractional part $f(t)$. Five-level case for $m=0.7$ is shown.

sinusoidal signal. As in Section 3, output leg voltage is examined, but now the whole fundamental period of the sinusoidal reference is considered rather than just one switching period. Average power is obtained by integration of the PWM signal squared value in each switching period, throughout the whole fundamental period. The same idea and the same result as in this section have already been used and reported in [5], individually for 2, 3 and 5 level single-phase PWM inverter. As noted, the work of [5] was extended and generalised in [6]. However, the final integrals were actually not solved analytically, in contrast to the situation here. Hence the analytical results obtained in this section are much more convenient for use and they also encompass results of [5] in a generalised manner.

Normalised leg voltage, used as the reference for production of the PWM signal, is defined as:

$$u^*_{LEG}(t)=\frac{1}{2}+\frac{m}{2}\cos(\omega t) \tag{8}$$

where $m$ is a modulation index ($0\le m\le 1$), and dc value of 1/2 is used to centre reference signal between dc bus rails (0 and 1 in normalised form).

Process of PWM for the sinusoidal reference signal of (8) is shown in Fig. 3a. Graphical calculation of the average power of the obtained leg voltage is shown in Fig. 3b. According to (2), $P_T(u_{LEG}[t])$ can be calculated as a shaded area in Fig. 3c, divided by the period of the signal $T$. Thus:

$$P_T(u_{LEG}[t])=\frac{\sum_{k=1}^{K}S_k}{T}=\frac{1}{K}\sum_{k=1}^{K}P_{T_s}(u_{LEG}[kT_s]) \tag{9}$$

$K$ is the number of samples during one period of the signal, $K$=[$T/T_s$]. Replacing the value of $P_{T_s}(u_{LEG}[kT_s])$ with (7), and using $u^*_{LEG}(kT_s)=\ 1/2+(m/2)\cos(\omega kT_s)$, one gets:

$$P_T(u_{LEG}[t])=\frac{1}{K}\sum_{k=1}^{K}\left(u^*_{LEG}(kT_s)+\frac{f(kT_s)-f^2(kT_s)}{(l-1)^2}\right) \tag{10}$$

If the sampling period $T_s$ is much smaller than the period of the signal $T$, then (9) i.e. (10) can be written in integral form:

$$P_T(u_{LEG}[t])=P_T(u^*_{LEG}(t))+\frac{1}{T}\int_{t=0}^{T}\left(\frac{f(t)-f^2(t)}{(l-1)^2}\right)\mathrm{d}t \tag{11}$$

Here $1/T\int_0^T u^*_{LEG}(t)\mathrm{d}t$ has been already replaced with $P_T(u^*_{LEG}(t))$, where reference leg voltage $u^*_{LEG}(t)$ is as in (8) and its fractional part $f(t)$ can be calculated using (6).

Reference leg voltage $u^*_{LEG}(t)$ and $f(t)$ are shown in Fig. 3a and c. It is obvious that (11) has to be solved in a piece-wise manner. Note that (11) is valid for any shape of the reference leg voltage $u^*_{LEG}(t)$ whose PWM signal $u_{LEG}[t]$ is obtained using adjacent levels, as explained in Section 3. It is easy to conclude that the position of the pulse inside the switching period $T_s$ is still unimportant. This practically means that if in an $l$-level inverter modulation strategy uses only adjacent voltage levels to switch, and if the step of the output voltage is constant, then power of the obtained leg voltage will be the same regardless of the particular applied modulation strategy. Of course, this only holds true if the ratio of $f_s/f$ is high.

To simplify calculations, one can see in Fig. 3c that quarter-wave symmetry exists. Thus the integration will be done in the first quarter of the signal period, $T/4$. If $\omega t$ is denoted with $\theta$, that means that integration will be done between 0 and $\pi/2$. The value of $P_T(u^*_{LEG}(t))$ in (11) can be determined directly by using (2) for reference of (8) and it comes to:

$$P_T(u^*_{LEG}(t)) = 1/4 + m^2/8 \tag{12}$$

For the determination of the second part of (11) Fig. 4 can be used, where seven-level case for $m = 0.8$ is shown. One can see that the borders of integration are not constant and depend on the value of $m$. If $l$ is an odd number, the borders between these integration segments, for example in the seven-level case, are when modulation index becomes greater than $m_1 = 1/3$ and when it becomes greater than $m_2 = 2/3$. Generally, this means that these borders for odd $l$ appear at $m_k = 2k/(l-1)$, and similarly for even $l$ at $m_k = (2k-1)/(l-1)$. Value of index $k$ is in the range from $k = 1$ to $\lfloor l/2 \rfloor - 1$. One can see that when $m \le m_{k+1}$ angles for piece-wise integration $\theta_k$ can be expressed as $\theta_k = \arccos(m_k/m)$.

Detailed derivation for the seven-level case is given in Appendix. The solution for the even number of levels differs only in the first term. One gets that a general analytical solution for the power of the leg voltage signal $P_T(u_{LEG}[t])$, created by an $l$-level PWM inverter, is:

$$P_T(u_{LEG}[t]) = \begin{cases} A_0 & 0 \le m \le m_1 \\ A_0 + A_1 & m_1 \le m \le m_2 \\ A_0 + A_1 + A_2 & m_2 \le m \le m_3 \\ \vdots & \vdots \\ A_0 + A_1 + \ldots + A_{\left\lfloor \frac{l}{2} \right\rfloor - 1} & m_{\left\lfloor \frac{l}{2} \right\rfloor - 1} \le m \le 1 \end{cases} \tag{13}$$

where introduced variables are defined as:

$$A_0 = \begin{cases} 1/4 + m/(\pi(l-1)) & \text{for odd } l \\ 1/2 - l(l-2)/(4(l-1)^2) & \text{for even } l \end{cases}$$
$$A_k = \frac{2}{\pi(l-1)} \cdot (m \sin\theta_k - m_k \theta_k) \tag{14}$$

and $\theta_k = \arccos(m_k/m)$ and $\sin\theta_k$ can be expressed using a basic trigonometry as $\sqrt{1-(m_k/m)^2}$.

Plot of the $P_T(u_{LEG}[t])$ for sinusoidal reference voltage (8) versus $m$, obtained by analytical formula (13), is given in Fig. 5.

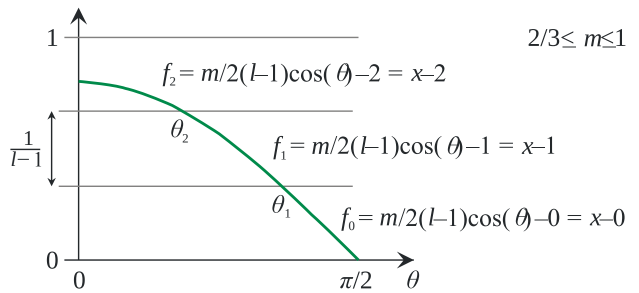


Fig. 4: The first quarter of the period for calculation of the integral in (11). Seven-level case is used as an example.

One can see that the expression (13) is general and applicable to any number of levels. This is in contrast to [5] where only particular solutions for two, three and five levels were given. In [6] the analytical derivations ended with integral equation, with defined borders for piece-wise integration $\theta_k$, so that no general analytical expression of the form of (13) was derived.

Note that, for an even number of levels $l$, the term $A_0$ that represents $P_T(u_{LEG}[t])$ for small modulation index values (less than $m_1 = 1/(l-1)$) is independent of $m$ (Fig. 5, four-level case for $m \le 1/3$). This is expected since the signal stays in the two-level zone, and, as shown for the two-level case, it has the constant value that is now <1/2, since the two level zone is $1/(l–1)$.

THD can be expressed using (5). Since dead-time effect is not considered, one can say that $X_{rms,1}$ is equal to the fundamental RMS value $1/\sqrt{2} \cdot (m/2)$. Dc value of the signal is 1/2, (8). $X^2_{rms}$ is defined by (13). Using (13) and (12), the THD can also be expressed as:

$$\mathrm{THD}_T(u_{LEG}[t]) = \sqrt{\left(P_T(u_{LEG}[t]) - P_T(u^*_{LEG}(t))\right)} \Big/ U_{LEG,rms,1} \tag{15}$$

where $U_{LEG,rms,1} = 1/\sqrt{2} \cdot (m/2)$. Plot of $\mathrm{THD}_T(u_{LEG}[t])$ versus modulation index $m$ is shown in Fig. 6 (solid lines). Results for THD (Fig. 6) are in very good agreement with those presented in [6].

The five-level case for different carrier-based modulation strategies was covered in [11]. The THD values given in [11] are slightly lower than in Fig. 6, which could be a consequence of not taking all the harmonics into consideration. Also, a low $f_s/f$ ratio was analysed, so for some values of $f_s/f$ the difference between analysed leg voltage THDs appears. However, it is confirmed here that if the ratio of $f_s/f$ is high enough, the same THD will be produced by any PWM strategy that uses two nearest levels.

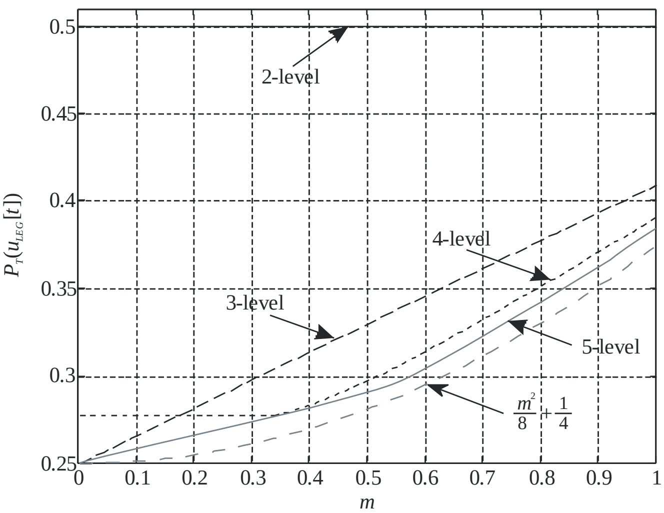


Fig. 5: Average power of the leg voltage $u_{LEG}[t]$, created by an ideal PWM $l$-level VSI, for sinusoidal reference of (8).

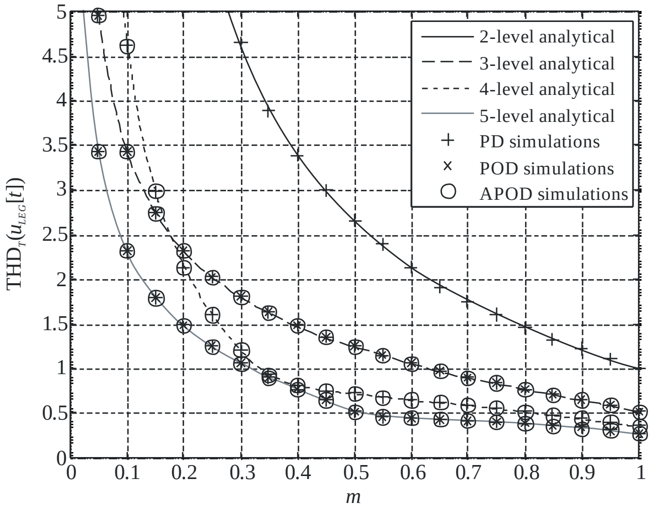


Fig. 6: THD of the leg voltage $u_{LEG}[t]$, generated by an ideal PWM *l*-level VSI, for sinusoidal reference leg voltage of (8): analytical curves (solid lines) and simulation results from PLECS scope (values shown with markers).

## 5 Comparison of the theoretical curves with simulation and experimental results

To validate the theoretically obtained analytical results, simulations and experiments have been done. Simulation software PLECS has been used. Scope in this software has a built-in function for RMS and THD calculation. The built-in functions use exact numerical approach for calculation of (5), which is also adapted to work with a variable simulation step time [12].

Theoretical curves for the leg voltage THD, for the two- to five-level cases for pure sinusoidal references, are compared with the values obtained by simulation using PLECS scope (discrete markers) in Fig. 6. Simulations are done for the constant $V/f$ ratio ($m/f$ = 1/50), and for the switching frequency of $f_s$ = 2 kHz. Excellent agreement between simulation and analytical results is obvious. This means that the used switching frequency, i.e. ratio $f_s/f$, is high enough for all the modulation indices. Also, one can see that the different carrier-based dispositions (PD, POD and APOD) do produce the same THD in a leg voltage.

Theoretical (analytical) curves for the leg voltage THD of Fig. 6 are compared next with the experimentally obtained THD values in Fig. 7. Because of the lack of availability of the hardware for more than three-level operation, only results for the two-level and three-level cases are shown. As in simulations, $V/f$ ratio was kept constant and equal to $m/f$ = 1/50. Custom-made two-level and three-level NPC inverters were supplied from Sorensen SGI 600/25 dc source. Dc bus voltage was set to 600 V. To obtain the THD from the experimental results, spectrum has been calculated first. The THDs when full spectrum has been used and when only components from the first ten side-bands (up to 21kHz) are taken into consideration were calculated using (4), and are shown in Fig. 7. One can see that, because of the dead time, which is 6μs for both inverters, the THD from the experiments is very slightly higher (worse) than the one predicted by the theoretical curves. This is visible in Fig. 7 only for low modulation indices, where the dead-time effect is more pronounced. Also, if the finite number of samples is taken into consideration (the value used here is 21 kHz and is much higher than many that are usually in use in practice), the calculated THD can be easily lower than the correct analytical value, Fig. 7. This problem is well known and has also been analysed in [2, 3].

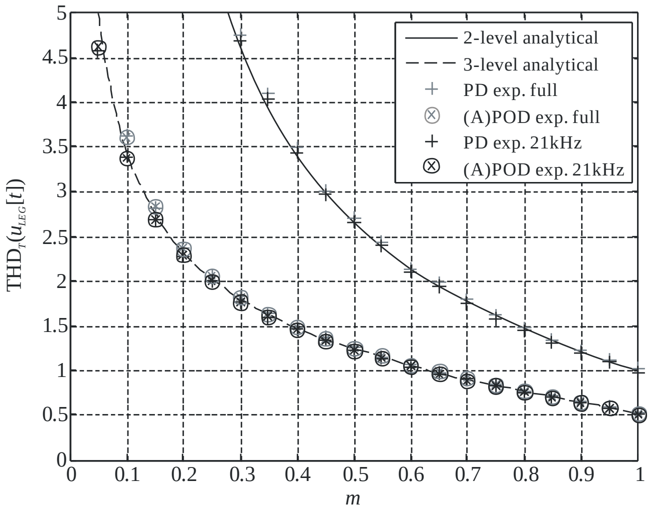


Fig. 7: Comparison of the analytical curves for the leg voltage THD of Fig. 6 (continuous lines) with experimental results using full and up to 21kHz spectrum for the THD calculation (discrete values labelled with corresponding markers) for the two-level and three-level operation.

## 6 Conclusion

Analytical expression for the average power and THD for the PWM produced leg voltage is derived in this paper. Considered reference voltage is purely sinusoidal. Derived analytical formulae for the leg voltage average power ($RMS^2$) and THD are for any number of levels. It is shown that the average power ($RMS^2$) and THD are independent of the modulation strategy if the nearest levels are used and if $f_s/f$ is high enough. Analytical curves are compared with simulation and experimental results, and an excellent agreement is demonstrated.

## References


[1] D. G. Holmes, T. A. Lipo, "*Pulse width modulation for power converters: principles and practice,*" IEEE Press, Piscataway, NJ, 2003.

[2] N. Farokhnia, S. H. Fathi, N. Yousefpoor, M. K. Bakhshizadeh, "Minimisation of total harmonic distortion in a cascaded multilevel inverter by regulating voltages of dc sources," *IET Power Electronics,* vol. 5, no. 1, pp. 106-114, 2011.

[3] N. Farokhnia, H. Vadizadeh, S. H. Fathi, F. Anvariasl, "Calculating the formula of line-voltage THD in multilevel inverter with unequal DC sources," *IEEE*

*Trans. on Industrial Electronics*, vol. 58, no. 8, pp. 3359-3372, 2011.

[4] N. Yousefpoor, S. H. Fathi, N. Farokhnia, H. A. Abyaneh, "THD minimization applied directly on the line-to-line voltage of multilevel inverters," *IEEE Trans. on Industrial Electronics*, vol. 59, no. 1, pp. 373-380, 2012.

[5] H. W. Van der Broeck, "Analytical calculation of the harmonic effects of single phase multilevel PWM inverters," in *Proc. IEEE Industrial Electronics Society Annual Conf. IECON*, Roanoke, VA, 2003, pp. 243-248.

[6] J. I. Leon, L. G. Franquelo, E. Galvan, M. M. Prats, J. M. Carrasco, "Generalized analytical approach of the calculation of the harmonic effects of single phase multilevel PWM inverters," in *Proc. IEEE Industrial Electronics Society Annual Conf. IECON*, Busan, South Korea, 2004, pp. 1658-1663.

[7] A. Ruderman, B. Reznikov, "Time domain evaluation of filterless grid-connected multilevel PWM converter voltage quality," in *Proc. IEEE Int. Symposium on Industrial Electronics ISIE*, Bari, Italy, 2010, pp. 2940-2945.

[8] A. Ruderman, B. Reznikov, "PWM power converter voltage quality bounds and their applicability to non-PWM control schemes," in *Proc. Int. Conf. Optimization of Electrical and Electronic Equipment OPTIM*, Brasov, Romania, 2010, pp. 618-624.

[9] A. Ruderman, B. Reznikov, S. Busquets-Monge, "Asymptotic time domain evaluation of a multilevel multiphase PWM converter voltage quality," *IEEE Trans. on Industrial Electronics*, vol. 60, no. 5, pp. 1999-2009, 2013.

[10] A. V. Oppenheim, A. S. Willsky, H. S. Nawab, "*Signals and systems*," 2nd ed., Prentice Hall Press, Upper Saddle River, NJ, 1997.

[11] V. G. Agelidis, M. Calais, "Application specific harmonic performance evaluation of multicarrier PWM techniques," in *Proc. IEEE Power Electronics Specialists Conf. PESC*, Fukuoka, Japan, 1998, pp. 172-178.

[12] J. Allmeling, W. Hammer, "PLECS - User Manual," 2013.

## Appendix

Derivation of (13) is shown here for a seven-level case for high value of modulation index, $2/3 \le m \le 1$. Fig. 4 has been used for the derivation. Values of $\theta_k$ are $\arccos(m_k/m)$ while $m_k = 2k/(l-1)$ since $l$ = 7 is an odd number. Using values for fractional parts $f_0$, $f_1$, $f_2$ (given in Fig. 4), and quarter-wave symmetry, starting with (11), one can write:

$$P_{T,LEG} = P_T(u^*_{LEG}(t)) + \frac{1}{\pi/2}\frac{1}{(l-1)^2}\int_{\theta=0}^{\pi/2}(f - f^2)\mathrm{d}\theta = \frac{1}{4} + \frac{m^2}{8} +$$

$$+\frac{2}{\pi(l-1)^2}\left(\int_0^{\theta_2}(f_2 - f_2^2)\mathrm{d}\theta + \int_{\theta_2}^{\theta_1}(f_1 - f_1^2)\mathrm{d}\theta + \int_{\theta_1}^{\pi/2}(f_0 - f_0^2)\mathrm{d}\theta\right)$$

Value of $P_T(u_{LEG}[t])$ is here denoted just with $P_{T,LEG}$. Using $x = m/2\cdot(l-1)\cos(\theta)$ and after simplifications one gets:

$$P_{T,LEG} = \frac{1}{4} + \frac{m^2}{8} + \frac{2}{\pi(l-1)^2}\int_0^{\pi/2}(x - x^2)\mathrm{d}\theta + \frac{2}{\pi(l-1)^2}\cdot$$

$$\left(\int_0^{\theta_2}(2x2 - 2^2 - 2)\mathrm{d}\theta + \int_{\theta_2}^{\theta_1}(2x1 - 1^2 - 1)\mathrm{d}\theta + \int_{\theta_1}^{\pi/2}(2x0 - 0^2 - 0)\mathrm{d}\theta\right)$$

$$= \frac{1}{4} + \frac{m^2}{8} + \frac{m}{\pi(l-1)} - \frac{m^2}{8} + \frac{2}{\pi(l-1)^2}\cdot$$

$$\left(\begin{array}{l}\boxed{\int_0^{\theta_2}(2x\cdot 2 - 2^2 - 2)\mathrm{d}\theta} \\ \boxed{+\int_0^{\theta_1}(2x\cdot 1 - 1^2 - 1)\mathrm{d}\theta} \boxed{-\int_0^{\theta_2}(2x\cdot 1 - 1^2 - 1)\mathrm{d}\theta} \\ +\int_0^{\pi/2}(2x\cdot 0 - 0^2 - 0)\mathrm{d}\theta \; \mathbf{0} \boxed{-\int_0^{\theta_1}(2x\cdot 0 - 0^2 - 0)\mathrm{d}\theta}\end{array}\right)$$

$$= \frac{1}{4} + \frac{m}{\pi(l-1)} + \frac{2}{\pi(l-1)^2}\left(\int_0^{\theta_2}(2x - 2\cdot\underline{2})\mathrm{d}\theta + \int_0^{\theta_1}(2x - 2\cdot\underline{1})\mathrm{d}\theta + 0\right)$$

After replacing $x$ with $m/2\cdot(l-1)\cos(\theta)$, subsequent integration, and after replacing underlined values of $k = \underline{1}, \underline{2}$ with $m_k(l-1)$, one finally gets:

$$P_{T,LEG} = \frac{1}{4} + \frac{m}{\pi(l-1)} + \frac{2}{\pi(l-1)}(m\sin\theta_2 - m_2\theta_2 + m\sin\theta_1 - m_1\theta_1)$$

This expression is of the general form of (13) for $2/3 \le m \le 1$.

## Acknowledgement

This paper was made possible by NPRP grant 4-152-02-053 from the Qatar National Research Fund (a member of Qatar Foundation). The statements made herein are solely the responsibility of the authors.