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Wallis, Alge

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Topological invariants of G_2 manifolds

submitted by

Dominic Wallis

for the degree of Doctor of Philosophy

of the

University of Bath

Department of Mathematical Sciences

September 2019

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Dominic Wallis

Abstract

The main aim of this thesis is to demonstrate new topological phenomena amongst manifolds with holonomy G_2. This splits into two complementary aims: develop invariants that detect the topological phenomena, and compute these invariants on a pool of G_2-manifolds.

To address the first aim, we discuss a general framework in which one can define invariants of structured manifolds via coboundaries. We consider how previously defined invariants are constructed from this perspective. The framework provides a transparent manner in which to generalise known invariants and define new ones. We extend invariants of polarized spin 7-manifolds, and define new invariants of almost contact 7-manifolds.

To address the second aim, we consider the Twisted Connected Sum construction for G_2-manifolds. We construct a suitably structured coboundary on which to compute invariants. Using this we: present examples of smooth 7-manifolds with disconnected G_2-moduli space; compute aforementioned invariants of polarized spin manifolds on several hundred examples; and detect formality in the sense of rational homotopy theory. To date we find only formal examples.

The TCS construction takes as input pairs of certain complex threefolds called building blocks, together with some cohomological data called a 'configuration'. Most examples and mass production methods in the literature have used simple types of configurations. Using simple types of configurations restricts the possible topology of the manifolds obtained. To demonstrate that the invariants defined can be nontrivial it is necessary we consider more sophisticated configurations. Although the theory for these configurations is available in the literature, it has not been developed.

For more sophisticated types of configurations one needs additional 'genericity conditions' on the building block. In general, this requires a greater understanding of their complex geometry. Building blocks can often be derived from weak Fano threefolds. We outline a systematic approach to producing genericity conditions for certain building blocks derived from weak Fanos.

Contents

Chapter 1

Introduction

We are motivated to better understand manifolds with holonomy G_2. We study structures on manifolds more broadly and ways to define invariants via coboundaries. We construct examples of G_2-manifolds and compute invariants for them.

Structures on manifolds. Let G be a Lie group and $\rho : G \to \mathsf{GL}(\mathbb{R}^n)$ a representation. Let M be an n-dimensional smooth manifold with frame bundle FM. Suppose that $E \to M$ is a principal G-bundle, and that we have a map $E \to FM$ such that the actions of G on E and $\mathsf{GL}(\mathbb{R}^n)$ on FM intertwine with ρ. Then we may refer to E as a *G-structure* on M. The are several related notions of specialized structure on manifolds of which this is one.

Many standard geometries are characterized, at least in part, by a specialized structure. For example: riemannian geometry is concerned with $\mathsf{O}(n)$-structures; spin geometry is concerned with $\mathsf{Spin}(n)$-structures; almost complex geometry is concerned with $\mathsf{GL}(\mathbb{C}^m)$-structures, where $2m = n$. Reductions may come with natural *integrability conditions*. For example, in complex geometry the integrability conditions are encapsulated in the 'Nijenhuis tensor'. If a $\mathsf{GL}(\mathbb{C}^m)$-structure has vanishing Nijenhuis tensor then it corresponds to a complex structure. Central to *holonomy theory* are specialized structures satisfying integrability conditions.

The holonomy group of a Riemannian manifold is the group of isometries of a tangent space generated by parallel transport around closed paths based at a point via the Levi-Civita connection. Berger [10] produced a list of all potential holonomy groups for manifolds, after certain reduction hypotheses. These are naturally all Lie groups. The list consists of four infinite families each of which has a corresponding geometry:

(i) Kähler ($\mathsf{U}(n)$)
(ii) Calabi-Yau ($\mathsf{SU}(n)$)
(iii) Hyperkähler ($\mathsf{Sp}(n)$)
(iv) Quaternionic-Kähler ($\mathsf{Sp}(n) \cdot \mathsf{Sp}(1)$).

In addition the list contains two exceptional cases: G_2 and $\mathsf{Spin}(7)$.

We are concerned primarily with manifolds with holonomy G_2. G_2 is the Lie group which can be defined either as the automorphism group of the imaginary octonians or equivalently as the stabilizer of a certain type of 3-form. A G_2-structure on a manifold M is determined by a 'nondegenerate' 3-form φ, and so $\mathrm{Hol}_g(M) < \mathsf{G}_2$ if and only if φ solves a certain nonlinear PDE. An overview of its properties and representations can be found in say Bryant [19], or Corti Haskins Nordström and Pacini [25].

G_2 manifolds are of keen interest to both geometers, and theoretical physicists (for example [1]). Compact examples were first produced by Joyce [61], and later via a different construction by Kovalev [64]. Corti Haskins Nordström and Pacini [25] generalised Kovalev's *Twisted Connected Sum* approach to give many examples of G_2-manifolds. The TCS construction allows one to keep a handle on many of the topological invariants. Although we are motivated by a desire to understand manifolds with special holonomy we pay almost no attention to the integrability conditions. We study G_2-manifolds solely via the G_2-structure associated to the metric. In doing, we consider more general questions of classification.

Invariants. Classification problems are fundamental to the theory of manifolds. The main aims of a classification can be summarized as follows. We wish to have a complete set of invariants such that:

(i) The invariants of a manifold are computable.
(ii) Two manifolds are isomorphic if and only if they have the same invariants.
(iii) Any conceivable combination of evaluations of the set of invariants is realized by some manifold.

The notion of isomorphism is subject to the context, or more precisely the category, under consideration. Generally a classification problem is broken down into incremental refinements. For example *when are two homotopy equivalent manifolds homeomorphic?* or *when are two homeomorphic manifolds diffeomorphic?* and so on. We can construct categories of manifolds with specialized structure. From a categorical perspective the incremental refinement of a classification problem is equivalent to the behaviour of the relevant forgetful functor.

The h-Cobordism Theorem (Proposition 4.2.3) of Smale demonstrates the profound link between bordism theory and the classification of smooth or topological manifolds. Smale's result relies on the 'Whitney trick', which only works in contexts where the dimension is sufficiently high. Freedman [41] proved the equivalent result in the topological case for dimension 4. In the smooth context the Whitney trick fails in dimensions < 5 and in the topological context for dimensions < 4. Roughly speaking, the classification of manifolds is partitioned into 'low' and 'high' dimension depending on whether one is able to easily employ techniques such as the Whitney trick.

Studying the properties of manifolds via a coboundary was famously used by Milnor [76] to demonstrate the existence of exotic 7-spheres. These are manifolds that are homeomorphic to the standard S^7 but not diffeomorphic. The effectiveness of studying a manifold via a coboundary is another manifestation of the link between bordism theory and classification problems.

In the course of the thesis, we discuss a framework in which to define invariants of manifolds with structure via coboundaries. We recast some known invariants, including that of Milnor's, into the framework. In some cases, the framework provides a reasonably transparent manner in which to extend these invariants. In addition we define new invariants for almost contact 7-manifolds. Let us consider some examples of invariants of manifolds with structure that are sufficiently simple that we need not get too bogged down in technicalities.

Almost contact 7-manifolds. Let M be a smooth closed 2-connected 7-manifold with torsion free cohomology. Suppose that M has an almost contact structure which we will understand to mean a $\mathsf{U}(3)$-structure on M (Example 2.5.9). As M is 2-connected the Chern classes c_1, c_3 both vanish. Let W be an almost complex coboundary to M. As $c_1(W)$ and $c_3(W)$ have compact support we can define the following value

$$\nu(W) := -\langle c_1(W)c_3(W), [W,M]\rangle - 3\sigma(W) + \chi(W) \tag{1.1}$$

where $\chi(W)$ is the Euler characteristic, and $\sigma(W)$ is the signature of W. The value of $\nu(W)$ depends on the choice of W. For a closed almost complex 8-manifold X, the corresponding value $\nu(X) = 0 \mod 48$ (see Section 5.2.5). It follows that $\nu(W) \mod 48$ is independent of choice of W, and so we have a (mod 48)-valued invariant $\nu(M)$ of M itself.

Suppose that $c_2(M)$ is nontrivial. As $c_2(W)$ does not have compact support, $\langle c_2^2(W), [W,M]\rangle$ is ill-defined, but all is not lost. We discuss how to define a product on $H^4(W)$ by making some auxiliary choices, but that certain residues of this product are independent of choices made. We can then derive a second invariant, again evaluated via a coboundary but independent of choice.

We place an additional assumption on the coboundary. We require that the greatest divisor of $c_2(M) \in H^4(M)$ is in the image of $H^4(W) \to H^4(M)$. The existence of such a coboundary is considered in Proposition 4.3.8.

Let $\alpha : H^\bullet(W) \to H^\bullet(M)$ be the restriction map. Let $s : FH^4(M) \to H^4(M)$ be a section of $F : H^\bullet(M) \to FH^\bullet(M)$, and let $\beta : FH^4(M) \to H^4(W)$ be a right inverse of $F \circ \alpha$ such that $\alpha \circ \beta$. Then we can define a product via β

$$\smile_\beta : H^4(W) \times H^4(W) \to H^8(W,M;\mathbb{Q}) \cong \mathbb{Q} \tag{1.2}$$

Using this product we can define the following

$$\begin{aligned}\xi(W,\beta) := &-\left\langle \tfrac{7}{6}c_1^4(W) + 7c_1(W)c_3(W), [W,M]\right\rangle \\ &+ \tfrac{3}{2}\left\langle (c_2(W) + \tfrac{2}{3}c_1^2(W)) \smile_\beta (c_2(W) + \tfrac{2}{3}c_1^2(W)), [W,M]\right\rangle \\ &- \tfrac{45}{2}\sigma(W) + 7\chi(W).\end{aligned} \tag{1.3}$$

Let $m \in \mathbb{N}$ be the greatest divisor of $c_2(M)$. We find that in the quotient space $\xi(W,\beta) \in \mathbb{Q}/12\ \mathrm{Num}(\frac{m}{4})\mathbb{Z}$ is independent of choice both W and β. Hence it is an invariant of M.

Suppose that M also admits a G_2-structure. If there exists an $\mathsf{SU}(3)$-reduction of the G_2-structure, such that it is isomorphic to an $\mathsf{SU}(3)$-reduction of the $\mathsf{U}(3)$-structure, then we say that the two structures are *compatible*. A compatible structure always exists, and

(ν, ξ) are independent of choice of compatible $\mathsf{U}(3)$-structure. Thus (ν, ξ) are invariants of the G_2-structure itself. Moreover, these invariants are equivalent to those of Crowley and Nordström [31] defined on G_2-structures via a spin coboundary.

To date, it has been hard to find a coboundary to TCS manifolds pleasant enough that the invariants of Crowley and Nordström can be completely computed. They ascertain that $\nu = 24 \mod 48$ for all TCS manifolds of the topological type described, but do not manage to compute ξ in any examples. [29] modifies the TCS construction to find examples with $\nu \neq 24 \mod 48$, and so with disconnected G_2-moduli space.

We construct an almost complex coboundary for TCS manifolds. By computing ξ on many TCS manifolds we find examples of pairs of G_2-manifolds with diffeomorphic structures, but with G_2-structures distinguished by ξ. It follows that these manifolds have disconnected G_2-moduli space. These are the first TCS manifolds demonstrating this phenomenon, and the first G_2-manifolds where this phenomenon is demonstrated via computing ξ alone (appearing in the preprint [105].)

Spin 7-manifolds. Let M be a smooth closed simply connected spin 7-manifold with torsion free cohomology, and with $b_2 = 1$. Throughout cohomology is assumed to have integer coefficients unless stated otherwise. A *polarization* in this context corresponds to a choice of isomorphism $H^2(M) \cong \mathbb{Z}$. Let $x \in H^2(M)$ be such that $x \mapsto 1$. Let $q_1(M) \in H^4(M)$ be the first spin class so $q_1(M) = \frac{1}{2}p_1(M)$, where $p_1(M)$ is the first Pontrjagin class of M.

Suppose first that $H^4(M)$ is trivial and W is a spin coboundary to M such that $H^2(W) \to H^2(M)$ is onto. In particular there exists a $\tilde{x} \in H^2(W)$ $\tilde{x} \mapsto x$. We define the following.

$$\begin{aligned} \mu(M) &:= \tfrac{1}{8}\left(\left\langle q_1^2(W), [W,M]\right\rangle - \sigma(W)\right) \mod 28 \\ \sigma(M) &:= \tfrac{1}{2}\left\langle q_1(W)\tilde{x}^2 - \tilde{x}^4, [W,M]\right\rangle \mod 12 \\ \tau(M) &:= \left\langle \tilde{x}^4, [W,M]\right\rangle) \mod 2 \end{aligned} \tag{1.4}$$

As in the previous example, these do not depend on choice of W. These invariants were known previously. μ is the Eells Kuiper invariant and is an extension of Milnor's invariant. The invariants $\sigma(M)$ and $\tau(M)$ are equivalent to the Kreck-Stolz invariants.

The framework discussed makes systematic which components of the invariants are defined when H^4 is nontrivial. Suppose that $H^4(M)$ is nontrivial. We have two preferred vectors in $H^4(M)$: $q = q_1(M)$ and $\tilde{x}^2$. The properties of the submodule $\left\langle q, \tilde{x}^2\right\rangle$ depend on the topology of M. Suppose that the submodule is of rank 2. We can choose a basis $\{e_i\}$ of $H^4(M)$ such that $q = me_1$ for $m > 0$ and $\langle e_1, e_2\rangle$ is an overlattice of $\left\langle q, \tilde{x}^2\right\rangle$. Moreover there is an essentially unique choice such that in this basis $\tilde{x}^2 = n(ae_1 + be_2)$, for $n > 0$, $0 \leq a < b$, $\gcd(a, b) = 1$. The 4-tuple (m, n, b, a) is an invariant of M. One may refer to this tuple as a *primary* invariant.

The definitions μ, σ, and τ may be ill-defined for M, as characteristic classes on W may not have compact support. We can recover well-defined invariants in a similar fashion to ξ in the previous example. These can be described as linear combinations of μ, σ, and τ modulo some value. There is an algorithmic yet convoluted relationship between the

4-tuple (m, n, b, a) and the invariants defined. As the definition of the invariants depend on the 4-tuple, one may consider them *secondary* invariants.

For example, suppose the 4-tuple is $(4, 4, 1, 0)$. We find that the two expressions

$$\sigma \mod 2, \qquad \tau \mod 2 \tag{1.5}$$

are well-defined.

We compute the invariants for 308 TCS manifolds which have the topology described. There are 12 examples that have the 4-tuple $(4, 4, 1, 0)$, and their secondary invariants all vanish. Of these there are two triples and one pair that also share b_3. We conjecture that manifolds sharing all the listed invariants are diffeomorphic.

Other remarks. We discuss an invariant appearing in [27] dubbed the Bianchi Massey Tensor (BMT). Within certain contexts, the BMT detects a property of rational homotopy theory known as *formality*. Roughly speaking, a manifold is formal if its rational homotopy can be derived from its cohomology. The reason the BMT is of particular interest is that it is known that Kähler manifolds are necessarily formal. Thus certain manifolds with special holonomy are necessarily formal. It is not known whether special holonomy implies formality.

Our focus is mainly on invariants which distinguish classes. This corresponds to the 'only if' direction of the second of a classification aims. We consider briefly the 'if' direction.

Outline. This thesis consists of two complementary parts: developing invariants that detect the topological phenomena, and computing these invariants on a pool of G_2-manifolds

Chapters 2, 3 and 4 are mainly background. In Chapter 2 we fix notation and conventions used as we review aspects of structures on manifolds. In Chapter 3 we discuss aspects of cohomology, including our first example of boundary defects invariants. In Chapter 4 we discuss bordisms, collecting together results in the literature. In addition, we prove new bordism results required in the course of the thesis.

The threads of these introductory chapters are brought together in Chapter 5. Here we discuss a framework in which to define boundary defect invariants. We consider how this manifests in several contexts, generalizing some known invariants and defining some new ones.

The second part of the thesis begins with Chapter 6. We start with a review of parts of the TCS construction, and pay particular attention to those aspects that allow us to construct an almost complex coboundary. We describe the topology of the coboundary in terms of the data of the TCS, including expressing the boundary defect invariants in terms of this data.

In Chapter 7 we turn our attention to building blocks—the constituent parts from which a TCS is made. We focus on building blocks derived from Fano and weak Fano threefolds. A Fano threefold X is an algebraic threefold for which the anticanonical class $-K_X$ is ample. A weak Fano threefold X is an algebraic threefold for which the anticanonical class $-K_X$ satisfies the weaker condition of being nef and big. Mori and

Mukai [80, 81] famously used Mori theory (Section A.3.3) to complete the classification of Fano threefolds building on the work of many contributors. The classification of weak Fano threefolds is far from complete and is very much an area of active research. Various authors have used the same techniques of Mori and Mukai to consider certain subclasses of weak Fano threefolds with $b_2 = 2$. We summarize what has been established to date, and use this to construct building blocks. We also consider several cases of building blocks with constructions not immediate from results in the literature.

The TCS construction relies on glueing pairs of building blocks. The problem of finding a glueing can be reformulated into a problem of arranging lattices that correspond to the cohomology of the complex threefolds involved. This arrangement is called a *configuration*. Simple types of configurations automatically determine a glueing permissible in the TCS construction. However the potential topology of a manifold obtained is constrained. For more sophisticated configurations one requires additional 'genericity results'. The ξ invariant is necessarily trivial for TCS manifolds obtained via a simple configuration, and so we are interested in more sophisticated ones. Although the theory for these more sophisticated types of configurations is available in the literature, it has not been fully developed.

Genericity results required for more sophisticated glueings involve a greater understanding of the complex geometry of the relevant threefolds compared to that of simple glueings. In particular, we require results on the projective models of algebraic curves and K3 surfaces. We propose a systematic approach to genericity results for a certain class of building blocks, and begin piecing together some of these results.

In Chapter 8, we conclude with some examples that bring together themes from the first and second parts of this thesis.

Chapter 2

Structures on manifolds

This chapter consists of background material. We review some of the fundamental notions and objects encountered in the course of the thesis. In doing we shall fix notation and conventions used. Most of the content can be found in standard references and is included for context. Section 2.6 is little more off piste. We introduce the adjectives of *boundary* and *coboundary* for Lie groups, or more precisely representations of Lie groups. Unless stated otherwise, a space will be assumed to be a topological space homotopy equivalent to a CW-complex.

2.1 Some representations of Lie groups

We recall some standard representations of Lie groups as stabilizers of a general linear group. We are particularly concerned with representations in dimensions 7 and 8.

Let $\{e_i\}$ be the standard basis of $\mathbb{R}^n$ and $\{e^i\}$ be its dual in $(\mathbb{R}^n)^\vee$. We denote the standard metric and volume form by

$$g_{0,n} := \textstyle\sum_{i=1}^n e^i \otimes e^i, \qquad \mathrm{Vol}_{0,n} := e^1 \wedge \cdots \wedge e^n. \tag{2.1}$$

If $n = 2m$, then the standard symplectic form and complex volume form by

$$\omega_{0,m} := \textstyle\sum_{j=1}^m e^{2j-1} \wedge e^{2j}, \qquad \Omega_{0,m} := \bigwedge_{j=1}^m (e^{2j-1} + ie^{2j}), \tag{2.2}$$

and standard complex structure by

$$J_{0,m} := \textstyle\sum_{j=1}^m (e^{2j-1} \otimes e_{2j} - e^{2j} \otimes e_{2j-1}). \tag{2.3}$$

Then $\mathsf{O}(n) = \mathrm{Stab}(g_{0,n})$ is the n^{th} *orthogonal group*, and $\mathsf{SO}(n) = \mathrm{Stab}(g_{0,n}, \mathrm{Vol}_{0,n})$. is the n^{th} *special orthogonal group*. For $n = 2m$, $\mathsf{U}(m) = \mathrm{Stab}(\omega_{0,m}, J_{0,m})$ is the m^{th} *unitary group*, and $\mathsf{SU}(m) = \mathrm{Stab}(\omega_{0,m}, \Omega_{0,m})$ is the m^{th} *special unitary group*. Note that $\mathrm{Stab}(g_{0,2m}, \omega_{0,m}) = \mathrm{Stab}(g_{0,2m}, J_{0,m}) = \mathrm{Stab}(J_{0,m}, \omega_{0,m})$. We have

$$\mathsf{SU}(m) \hookrightarrow \mathsf{U}(m) \hookrightarrow \mathsf{SO}(2m) \hookrightarrow \mathsf{O}(2m) \tag{2.4}$$

$$\begin{tikzcd}
 & \mathsf{SO}(6) \arrow[r, "\iota_{\mathrm{std}}"] & \mathsf{SO}(7) \arrow[r, "\iota_{\mathrm{std}}"] & \mathsf{SO}(8) & \\
\mathsf{SU}(3) \arrow[ur, "\rho_{\mathbb{R}}"] \arrow[dr, "\tilde{\rho}_{\mathbb{R}}"] \arrow[dd, "\iota_{\mathrm{std}}"] & & & & \\
 & \mathsf{Spin}(6) \arrow[uu, "\rho^6_{2:1}"] \arrow[dd, "\Delta^6_+"] \arrow[r, "\iota_{\mathrm{std}}"] & \mathsf{Spin}(7) \arrow[uu, "\rho^7_{2:1}"] \arrow[dd, "\Delta^7"] \arrow[r, "\iota_{\mathrm{std}}"] & \mathsf{Spin}(8) \arrow[uu, "\rho^8_{2:1}"] \arrow[dd, "\Delta^8_+"] \arrow[r, "\lambda"] & \mathsf{Spin}(8) \arrow[uul, "\Delta^8_-"'] \arrow[ddl, "\rho^8_{2:1}"] \\
\mathsf{SU}(4) \arrow[ur, "\cong"] \arrow[dr, "\iota_{\mathrm{std}}"] & & & & \\
 & \mathsf{U}(4) \arrow[r, "\rho_{\mathbb{R}}"] & \mathsf{SO}(8) \arrow[r, "\cong"] & \mathsf{SO}(8) &
\end{tikzcd}$$

Figure 2-1: A commutative diagram of relevant Lie group homomorphisms

The inclusion $\rho_{\mathbb{R}} : \mathsf{U}(m) \to \mathsf{SO}(2m)$ is sometimes referred to as *realification*, while $\mathsf{O}(n) \to \mathsf{U}(n)$ is *complexification*. We may also treat these representations as embedded into a higher dimensional ambient space. For example, for $n = 2m + k$ $\mathsf{U}(m) \cong \mathrm{Stab}(g_{0,n}, \omega_{0,m}, e_{m+1}, \ldots, e_{m+k})$, and likewise for $\mathsf{SU}(m)$. We may instead consider the first k basis vectors fixed, rather than the last, but this shouldn't cause confusion.

In addition, we have two forms

$$\begin{aligned}
\varphi_0 :=& e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356} \in \Lambda^3(\mathbb{R}^7)^\vee \\
\psi_0 :=& e^{1234} + e^{1256} + e^{1278} + e^{1357} - e^{1368} - e^{1458} - e^{1467} - \\
& e^{2358} - e^{2367} - e^{2457} + e^{2468} + e^{3456} + e^{3478} + e^{5678} \ \in \Lambda^4(\mathbb{R}^8)^\vee
\end{aligned} \tag{2.5}$$

where $e^{ij} = e^i \wedge e^j$ etc. The first of these forms, φ_0, is the standard G_2-form since $\mathsf{G}_2 = \mathrm{Stab}(\varphi_0)$. We find that $\mathsf{G}_2 < \mathsf{SO}(7)$ (see Bryant [19, section 2]) and it acts transitively on the sphere S^6. The stabilizer of any unit vector is isomorphic to $\mathsf{SU}(3)$. The second form, ψ_0, corresponds to the image of the spinor representation $\Delta : \mathsf{Spin}(7) \to \mathsf{SO}(8)$. That is $\Delta(\mathsf{Spin}(7)) = \mathrm{Stab}(\psi_0)$. $\mathrm{Stab}(\psi_0)$ acts transitively on the sphere S^7, with the stabilizer of any unit vector isomorphic to G_2.

For groups $\mathsf{Spin}(n)$ and $\mathsf{Spin}^c(n)$ we have a slightly different approach. Recall that $\rho_{2:1} : \mathsf{Spin}(n) \to \mathsf{SO}(n)$ is a double cover, and that $\mathsf{Spin}^c(n) := \mathsf{Spin}(n) \times_{\mathbb{Z}_2} \mathsf{U}(1)$ is a double cover $\rho_{2:1} : \mathsf{Spin}^c(n) \to \mathsf{SO}(n) \times \mathsf{U}(1)$. A representation $\rho : G \to \mathsf{SO}(n)$ from a simply connected group can be lifted by $\rho_{2:1}$ to $\tilde{\rho} : G \to \mathsf{Spin}(n)$.

It is not critical although may be of assistance to some readers to note the following relations between representations. The half spin representation $\Delta^6_+ : \mathsf{Spin}(6) \to \mathsf{U}(4)$ is faithful with image $\mathsf{SU}(4)$. There is a 'triality' property characterized by an automorphism $\lambda : \mathsf{Spin}(8) \to \mathsf{Spin}(8)$ of order 3 that permutes the three $\mathsf{SO}(8)$-representations: the double cover $\rho^8_{2:1}$; and the two half spin representations $\Delta^8_\pm$. See Figure 2-1.

2.2 Special Holonomy

Let (M, g) be a connected Riemannian n-manifold, with Levi-Civita connection ∇. Fix some $x \in M$, and consider the set Γ_x of all piecewise smooth closed loops $\gamma : [0, 1] \to M$, based at x (ie $\gamma(0) = \gamma(1) = x$). For any $\gamma \in \Gamma_x$, there is an induced linear map $P_\gamma : T_xM \to T_xM$ of the tangent space T_xM of M at x, given by the parallel transport of T_xM with respect to ∇. The *holonomy group of M based at x* is $\mathrm{Hol}_{x,g}(M) := \{P_\gamma \in \mathrm{Aut}(T_xM) \colon \gamma \in \Gamma_x\}$. The holonomy group is indeed a group, and $\Gamma_x \to \mathrm{Hol}_{x,g}(M)$ is a group morphism.

Let $y \in M$, and let $\gamma : [0,1]$ be a continuous piecewise smooth curve such that $\gamma(0) = x$ and $\gamma(1) = y$. Then $P_\gamma : T_xM \to T_yM$ by parallel transport. This fixes an isomorphism $\mathrm{Aut}(T_xM) \cong \mathrm{Aut}(T_yM)$, and $\mathrm{Hol}_{x,g} \cong \mathrm{Hol}_{y,g}$. Thus the isomorphism class of the holonomy group is independent of choice of basepoint. For a connected Riemannian manifold (M, g), the *holonomy group of M* $\mathrm{Hol}(M)$ is the isomorphism class of $\mathrm{Hol}_{x,g}(M)$. We will assume that we have fixed an embedding $\mathrm{Hol}(M) \to \mathsf{O}(n)$. For a fuller explanation of holonomy and related results see Joyce [60].

For a Lie group G, a connected closed Riemannian manifold (M, g) is a *G-manifold* if it has holonomy group isomorphic to G. An oriented manifold has *special holonomy* if the holonomy group is a strict subgroup of $\mathsf{SO}(n)$.

The metric of a manifold with special holonomy satisfies a system of integro-differential equations. It is often more convenient to reformulate this as a PDE concerning sections of some bundle over the base manifold. For example in the case of G_2, we can consider a 3-form φ that is locally modelled by φ_0 as given in (2.5). If g denotes the metric determined by a 3-form φ, and φ satisfies the PDE $\nabla_g \varphi = 0$, then g has holonomy G_2. Once we have established that a G-structure (Section 2.5) corresponds to a metric with special holonomy we cease to consider any PDE constraint. The G-structure is the *topological residue* of the system of integro-differential equations, and although motivated to understand special holonomy we study only this G-structure.

2.3 Principal Bundles

Let $p : E \to B$ be a map between spaces. Suppose that there exists a space F, an open covering $\{U_i\}$ of B, and homeomorphisms $h_i : p^{-1}(U_i) \to U_i \times F$ such that $p = pr_1 \circ h_i$, where pr_1 is the projection onto the first factor. Then p is a *fibre bundle*, F is the *fibre* of p, E is the *total space*, and B is the *base*. The set $\{(U_i, h_i)\}$ is a *trivializing cover* for p. Two fibre bundles $p_i : E_i \to B$, $i = 0, 1$, over the same base are *equivalent* if there exists a homeomorphism $f : E_0 \to E_1$ such that $p_0 = p_1 \circ f$.

It is common for fibre bundles to be equipped with a *structure group*. Let $p : E \to B$ be an F-fibre bundle over B with a trivializing cover $\{(U_i, h_i)\}$. For $U_{ij} = U_i \cap U_j \neq \varnothing$, we have a map

$$\begin{aligned} h_i \circ h_j^{-j} : U_{ij} \times F &\to U_{ij} \times F \\ (u, f) &\mapsto (u, g_{ij}(u)(f)) \end{aligned} \tag{2.6}$$

where $g_{ij}(u) \in \mathrm{Homeo}(F)$, the group of homeomorphisms of F. The functions $g_{ij} : U_{ij} \to \mathrm{Homeo}(F)$ are *transition functions*.

Let $G < \mathrm{Homeo}(F)$. If there exist a trivializing cover $\{(U_i, h_i)\}$ of B such that all transition functions $g_{ij} : U_{ij} \to G$ then G is a *structure group* of p. For a fibre bundle with structure group, the trivializing cover will be assumed to have this property. Two fibre bundles $p_i : E_i \to B$, $i = 0, 1$, over the same base B and same structure group G are *equivalent* if there exists an isomorphism of fibre bundles $f : E_0 \to E_1$ that intertwines with the G-action.

Any fibre bundle can be endowed with a structure group by simply taking $G = \mathrm{Homeo}(F)$. Things get more interesting when the structure group is a strict subgroup.

Example 2.3.1. (i) Let B, F be spaces, $E = F \times B$ and $p : E \to F$ be the projection onto the second factor. Then p has the trivial group as the structure group. We say that p is the *product bundle*. Any bundle equivalent to a product bundle is a *trivial bundle*.

(ii) Let $p : E \to B$ be a rank n real vector bundle over B. Then p is a fibre bundle with fibres $F = \mathbb{R}^n$, and structure group $G = \mathsf{GL}(\mathbb{R}^n)$.

Definition 2.3.2. Let G be a topological group, and X a space. A *principal G-bundle* $P \to X$ is a fibre bundle P equipped with a continuous right G-action that preserves fibres, and acts freely and transitively on them.

The structure group of a principal G-bundle is G. Thus two principal G-bundles $P_i \to X$ are equivalent if there is a fibre bundle map $f : P_0 \to P_1$ that intertwines with the G-action *ie* for all $p \in P$, $g \in G$, $f(pg) = f(p)g$. The set of equivalence classes of principal G-bundles over a space X is denoted by $\mathcal{P}_G(X)$.

Example 2.3.3. Let $G = \mathsf{GL}(\mathbb{R}^n)$ and X be a smooth n-manifold. A *frame* at $x \in X$ is an isomorphism of vector spaces $f : \mathbb{R}^n \to T_xX$. The *frame bundle* FX, as a set, is the set of all frames for all points in X. It inherits a topology, and moreover a smooth structure, from $\mathbb{R}^n$ and X. In addition, G acts on FX by $(g, f) \mapsto f \circ g$, and thus FX is a principal G-bundle.

Suppose that in addition X is equipped with a metric h. We can restrict to considering only frames $f : \mathbb{R}^n \to T_xX$ that are isometries, treating $\mathbb{R}^n$ as an inner product space with the standard metric. The set of all isometric frames $F_{\mathsf{O}(n)}X$ is again a smooth manifold, but now equipped with an $\mathsf{O}(n)$ action and by which $F_{\mathsf{O}(n)}X$ is a principal $\mathsf{O}(n)$-bundle. The analogous logic applies to oriented manifolds and almost complex manifolds etc.

Suppose $P \to X$ is a principal G-bundle and that a space F admits a G-action. Then we can construct an F-fibre bundle by $P \times_G F$, defined as the quotient of $P \times F$ by the equivalence relation

$$(p, f) \sim (p', f') \Leftrightarrow \exists g \in G, \quad (p \cdot g, g^{-1}(f)) = (p', f') \tag{2.7}$$

Conversely, suppose that $p : E \to B$ is an F-fibre bundle with structure group G. Then we can construct a principal G-bundle from P by taking a trivializing cover $\{(U_i, h_i)\}$, and replacing the F-fibres of $U_i \times F$ with G. These are 'glued' back together with the transition functions g_{ij}. This is the *associated principal G-bundle* of E.

Example 2.3.4. Let $G = \mathsf{GL}(\mathbb{R}^n)$ and X be a smooth manifold. Then G acts on the tangent bundle TX. The frame bundle FX is equivalent to the associated principal bundle of TX.

2.4 Classifying spaces

Some structures on manifolds can be encoded by maps to a particular space. This tends not to be a particularly useful description if one is working with concrete examples but studying this 'terminal' target space allows us to make general statements regarding structures of this type. The terminal target space is a classifying space. Category theory is the most natural setting in which to discuss classifying spaces. However, the main point is simply that in the contexts we consider such spaces exist (Proposition 2.4.1). More explicit constructions of some classifying space such as Example 2.4.3 aid the cohomology calculations of Section 3.3 and Bordism groups in Section 4.2.

Recall that a category $\mathcal{C}$ is *locally small* if for any pair of objects $a, b \in \mathrm{ob}(\mathcal{C})$, the class $\mathrm{Hom}(a, b)$ is a set. For a locally small category $\mathcal{C}$ and object $a \in \mathrm{ob}(\mathcal{C})$, we have a functor $\mathrm{Hom}(a, -) : \mathcal{C} \to \mathsf{Set}$ to the category of sets. A (covariant) functor $F : \mathcal{C} \to \mathsf{Set}$ is *representable* if it is naturally isomorphic $\mathrm{Hom}(a, -)$ for some $a \in \mathrm{ob}(\mathcal{C})$. Analogously, a contravariant functor F is *representable* if it is naturally isomorphic to $\mathrm{Hom}(-, a)$. a is the *representing space* of F.

The homotopy category of CW complexes $\mathcal{H}$ is locally small. A functor $F : \mathcal{H}^{op} \to \mathsf{Set}$ is a *Brown functor* if it takes coproducts to products and homotopy pushouts to weak pullbacks. Brown representablity theorem states that Brown functors are representable [17].

For a Lie group G (or more generally a topological group), let $\mathcal{P}_G$ be the functor mapping CW complex X to the set $\mathcal{P}_G X$ of principal G-bundles over X. For an abelian group Γ and $n > 1$ or any group Γ and $n = 1$, we have the singular homology functor $H^n(-; \Gamma)$.

Proposition 2.4.1. *The functors $\mathcal{P}_G$ and $H^n(-; \Gamma)$ are Brown functors.*

The representing space of $\mathcal{P}_G$ is the *classifying space* of G and is denoted by BG. The representing space of $H^n(-; \Gamma)$ is an *Eilenberg-MacLane space* and denoted by $K(\Gamma, n)$. By abuse of notation, we sometimes write BG to denote a connected pointed CW-complex that is a representative class of homotopy equivalent spaces $BG \in \mathcal{H}$. Although the existence of these universal spaces follows from Brown representablity, it is useful to have a constructive proof. For example, this can then be used to compute $H^\bullet(BG)$.

Theorem 2.4.2. *Let G be a topological group, and $P \to P/G$ be a principal G-bundle. If P is contractible then the natural transformation $\Phi : [-, P/G] \to \mathcal{P}G(-)$ sending the homotopy class of a map $f : B \to P/G$ to the equivalence class of the principal G-bundle f^*P is a natural isomorphism of functors.*

See, for example, [71, Section 23.8]. In the notation of Theorem 2.4.2, P is a *universal G-bundle*. Milnor [75] gives a general construction for P for certain topological groups G. We describe the standard constructions for some relevant Lie groups.

Example 2.4.3 (Classifying space for Orthogonal groups). Let $Gr(n, k)$ be the real Grassmanian of n-dimensional subspaces of $\mathbb{R}^k$. Let $V(n, k)$ be the real Stiefel manifold of all orthogonal n-frames of $\mathbb{R}^k$. Let $V(n, k) \to Gr(n, k)$ take an n-frame to its span. This has the structure of a principal $\mathsf{O}(n)$-bundle. Let $Gr(n) := \bigcup_k Gr(n,k)$, $V(n) := \bigcup_k V(n,k)$ understood as the nested inclusions (ie colimit). We find that $V(n)$ is weakly contractible, and thus $E\mathsf{O}(n) = V(n)$ and $B\mathsf{O}(n) = Gr(n)$.

Restricting $Gr(n,k)$ and $V(n,k)$ to the subset of fixed orientation we construct the classifying space $E\mathsf{SO}(n) \to B\mathsf{SO}(n)$. Similarly, considering instead the complex Grassmanian leads to $E\mathsf{U}(n) \to B\mathsf{U}(n)$.

Example 2.4.4 (Classifying space for any compact Lie group). Let G be a compact Lie group with an injective morphism $G \to \mathsf{O}(n)$. We have an induced action by G on $E\mathsf{O}(n)$. Thus the classifying space of G can be presented as $BG = E\mathsf{O}(n)/\mathsf{G}$, with universal bundle $EG = E\mathsf{O}(n)$.

Let $Gr(n,k) \to Gr(n+1,k+1)$ be defined by adjoining one coordinate to both the ambient space and each subspace. In the colimit, this defines the *canonical inclusion map* $i_n : B\mathsf{O}(n) \to B\mathsf{O}(n+1)$. This is part of a homotopy fibre sequence

$$S^n \to B\mathsf{O}(n) \to B\mathsf{O}(n+1). \tag{2.8}$$

See, for example, [16]. We have an analogous homotopy fibre sequence for the families of groups $\mathsf{SO}(n)$ and $\mathsf{Spin}(n)$. For $\mathsf{U}(n)$, the fibre of the homotopy fibre sequence has homotopy fibre S^{2n+1}. This is used in calculating characteristic classes inductively, as mentioned in Section 3.3.

For G_n belonging to one of the infinite families of lie groups, such as $\mathsf{O}(n)$, $\mathsf{SO}(n)$, $\mathsf{Spin}(n)$ or $\mathsf{U}(n)$, we have the homotopy colimit $BG := \lim_{n\to\infty} BG_n$ via the canonical maps. BG is the classifying space of *stable* structures corresponding to G_n. For example $B\mathsf{U}$ is the classifying space of *stable unitary structures*. These spaces classify structures up to *stability* and segues into K-theory. See for example [71, Section 24.1]. Again, we introduce these objects briefly as they make an appearance in Sections 3.3 and 4.2.

Similarly to the canonical map above, let $Gr(n_1,k_1)\times Gr(n_2,k_2) \to Gr(n_1+n_2,k_1+k_2)$ be defined by sending the ambient spaces and their subspaces to their direct sum. Again in the colimit, this defines the *canonical map* $i_{n_1,n_2} : B\mathsf{O}(n_1) \times B\mathsf{O}(n_2) \to B\mathsf{O}(n_1+n_2)$.

On Eilenberg-MacLane spaces, $K(\Gamma,n)$ are completely characterized (up to homotopy) by

$$\pi_k(K(\Gamma,n)) = \begin{cases} \Gamma & k = n \\ 0 & k \neq n \end{cases} \tag{2.9}$$

May [71, Section 16.5] describes a construction for $K(\Gamma,n)$ following the Milnor construction.

2.5 G-structures

Let G be a Lie group, X a topological space, and $P \to X$ a principal G-bundle over X. Suppose we have a Lie group homomorphism $\rho : G \to H$. Then we can define a principal H-bundle by defining an H-fibre bundle $P \times_\rho H$ defined as the quotient of $P \times H$ by the relation

$$(p,h) \sim (p',h') \Leftrightarrow \exists g \in G, \quad (p\cdot g, \rho(g^{-1})h) = (p',h'). \tag{2.10}$$

We wish to define the converse of this relation.

Definition 2.5.1. Let H be a Lie group, X a topological space, and $E \to X$ a principal H-bundle over X. Let $\rho : G \to H$ be a homomorphism of Lie groups. A *ρ-structure* is a pair (P, ι) consisting of a principal G-bundle $P \to X$ together with an H-bundle isomorphism $\iota : P \times_\rho H \xrightarrow{\sim} E$.

Two ρ-structures (P, ι) and (P', ι') are equivalent if there is an isomorphism $\Phi : P \to P'$ as principal G-bundles, such that for all $p \in P$ and $h \in H$, $\iota'([\Phi(p), h]) = \iota([p, h])$.

In the case where $H = \mathsf{GL}(V)$ and $E = FX$, the frame bundle of X, then we refer to a *ρ-structure on X*. In the case where ρ is unambiguous or implicit, such as when G is a subgroup of H, we refer to P as a *G-structure on E*. We may refer to P as a *reduction of* E in the case ρ is injective, and a *lift* when ρ is surjective.

With the exception of spin and spin^c structures, all cases of interest to us are reductions of structure *ie* $\rho : G \hookrightarrow H$.

Example 2.5.2. Let $\rho : \mathrm{O}(n) \to \mathrm{GL}(\mathbb{R}^n)$ the standard representation (2.1). Let X be a Riemannian n-manifold. The orthogonal frame bundle $F_{\mathsf{O}}X$ is a ρ-structure on X, and clearly a reduction of FX.

Suppose in addition that X is oriented. Let $\rho : \mathsf{SO}(n) \to \mathsf{O}(n)$ be the standard inclusion. The oriented orthogonal bundle $F_{\mathsf{SO}}X$ is a ρ-structure on $F_{\mathsf{O}}X$.

Example 2.5.3. Let X be a smooth complex n-manifold. Let $\rho : \mathsf{GL}(\mathbb{C}^n) \to \mathsf{GL}(\mathbb{R}^{2n})$ be the standard inclusion. The complex frame bundle $F_{\mathbb{C}}X$ is a $\mathsf{GL}(\mathbb{C}^n)$-structure on X, or equivalently, an almost complex structure.

Proposition 2.5.4. *For H, X, and E as in Definition 2.5.1, suppose $\rho : G \to H$ is injective. Any ρ-structure is equivalent to a ρ-structure (S, σ) where $S \subset E$ is a subbundle and $\sigma([s, h]) = s \cdot h$.*

Proof. Let (P, ι) a ρ-structure on E. We define a morphism $j : P \to E$, by sending $p \mapsto \iota([p, \mathrm{id}_H])$.

Firstly, j is injective. Suppose $j(p) = j(p')$, then $\iota([p, \mathrm{id}_H]) = \iota([p', \mathrm{id}_H])$. As ι is an isomorphism, so $[p, \mathrm{id}_H] = [p', \mathrm{id}_H]$. Thus there exists $g \in G$ such that $p \cdot g = p'$, and $\rho(g^{-1})\mathrm{id}_H = \mathrm{id}_H$. As ρ is injective, the latter implies $g = \mathrm{id}_G$, and so $p = p'$.

Secondly j is naturally a G-bundle morphism. Define $j(p) \cdot g = j(p) \cdot \rho(g^{-1})$.

$$\begin{aligned} j(p) \cdot g &= \iota([p, \mathrm{id}_H]) \cdot \rho(g^{-1}) = \iota([p, \mathrm{id}_H \cdot \rho(g^{-1})]) \\ &= \iota([p, \rho(g^{-1})]) = \iota([p \cdot g, \mathrm{id}_H]) = j(p \cdot g) \end{aligned} \tag{2.11}$$

Where the second equality holds since ι is an isomorphism of H-bundles. Thus j is an equivalence of principal G-bundles.

Let $S = j(P) \subset E$. Then (S, σ) is a ρ-structure equivalent to (P, ι). □

In the case that ρ is injective, a ρ-structure on E will be treated as a subbundle $S \subset E$.

Definition 2.5.5. Let H be a Lie group and $E \to Y$ be a principal H-bundle over a topological space Y. Suppose H acts on a topological set X, and fix some $x \in X$. For $y \in Y$, an element $[e, x'] \in (E \times_H X)|_p$ is *x-like* if there exists $e' \in E_p$, such that $[e, x'] = [e', x]$. A section $s \in C^\infty(E \times_H X)$ is *x-like* if for all $y \in Y$, $s(y)$ is x-like.

For example, let M be a 7-manifold. We may be motivated to find a section $\varphi \in C^\infty(\bigwedge^3 T^*M)$ that is φ_0-like (from (2.5)). Note that if s as in Definition 2.5.5 is x-like at y, then we can define nonempty set $E_{s,y} = \{e_y \in E \colon (e_y, x) \in s(y)\}$. Let $G = \mathrm{Stab}_H(x)$. Each $E_{s,y}$ has a natural, free and transitive G-action. Moreover if s is x-like, we can define subbundle E_s, given fibrewise by $E_{s,y}$ over y, and inheriting the bundle structure from H-bundle E. Thus E_s is a principal G-bundle over M.

Proposition 2.5.6. *Let H be a Lie group and $E \to B$ a principal H-bundle over a topological space B. Suppose F has an H-action, and fix some $x \in F$. Let $G = \mathrm{Stab}_H(x)$. Suppose a section $s \in C^\infty(E \times_H F)$ is x-like in the sense of Definition 2.5.5. Then E_s is a G-structure on E.*

Proof. E_s is a principal G-bundle, and $E_s \times_G H$ is a principal H-bundle. Required to show that there is an isomorphism of H-bundles $\iota : E_s \times_G H \cong E$.

Define ι by $\iota : [e, h] \mapsto e \cdot h$. For any two representatives (e, h), and (e', h') sharing a class, there exists $g \in G$, such that $e \cdot g = e'$ and $g^{-1}h = h'$, hence $e' \cdot h' = (e \cdot g) \cdot (g^{-1}h) = eh$. Thus ι is a well defined map.

If $\iota([e, h]) = \iota([e', h'])$ then $e \cdot h = e' \cdot h'$. As $e, e' \in E_{s,p}$ for some point p, then there exists $g \in G$ such that $e \cdot g = e'$, and g is the unique such element in H. Hence $g^{-1}h = h'$, and so $[e, h] = [e', h']$. Thus ι is injective.

Consider action by $h' \in H$.

$$\iota([e, h] \cdot h') = \iota([e, hh']) = e \cdot (hh') = (e \cdot h) \cdot h' = \iota([e, h]) \tag{2.12}$$

Hence ι respects action by H. Therefore ι is an isomorphism of H-bundles. □

In light of Proposition 2.5.6, we may refer to s itself as a G-structure.

Example 2.5.7. Let M be a 7-manifold. Let $s \in C^\infty(\bigwedge^3(T^*M))$ be φ_0-like. Then E_s is a G_2 structure on M.

Example 2.5.8. Let X be an 8-manifold and $(\omega_0, J_0) = (\omega_{0,4}, J_{0,4})$ as denoted in (2.2). Suppose that (ω, J) is (ω_0, J_0)-like. Then $E = E_{(\omega,J)}$ is a $\mathsf{U}(4)$-structure on X. We refer to E as an *almost complex structure* on X.

Example 2.5.9. Let M be a 7-manifold. Let (ω_0, J_0) be the standard complex structure on $\mathbb{R}^6 = \langle e_1 \rangle^\perp < \mathbb{R}^7$. Suppose that (v, ω, J) is (e_1, ω_0, J_0)-like. Then $E = E_{(v,\omega,J)}$ is a $\mathsf{U}(3)$-structure on M. We refer to E as an *almost contact structure* on M.

Note that when the action of H on F is transitive then reference to a specific $x \in F$ in saying a section is x-like is essentially vacuous. Moreover, in such case $F \cong H/G$ as an H-space.

Example 2.5.10. Let M be a manifold with a G_2-structure φ as in Example 2.5.7. Note that $\mathsf{G}_2 < \mathsf{SO}(7)$ acts on S^6 transitively, with the stabilizer of any vector being isomorphic to $\mathsf{SU}(3)$. That is we have a fibration

$$\mathsf{SU}(3) \to \mathsf{G}_2 \to S^6. \tag{2.13}$$

Suppose $v \in \Gamma(TM)$ is a unit vector field. Then (φ, v) defines an $\mathsf{SU}(3)$-reduction of the G_2-structure.

We consider now the case of ρ-structures for which $\rho : G \to H$ is surjective.

Definition 2.5.11. Let X be an oriented n-manifold and $F_{\mathsf{SO}}X$ the $\mathsf{SO}(n)$-structure. A *spin structure on* X is ρ-structure where $\rho : \mathsf{Spin}(n) \to \mathsf{SO}(n)$ is the standard double cover. It is a principal $\mathsf{Spin}(n)$-bundle over X. A *spin manifold* will mean a manifold with a spin structure.

Definition 2.5.12. Let X be an oriented n-manifold. Let $\rho' : \mathsf{Spin}^c(n) \to \mathsf{SO}(n) \times \mathrm{U}(1)$ be the standard double cover and $p_1 : \mathsf{SO}(n) \times \mathsf{U}(1) \to \mathsf{SO}(n)$ be the projection. A *spin*^{c} *structure on* X is ρ-structure $E \to X$ where $\rho = p_1 \circ \rho'$. There exists a principal $\mathsf{U}(1)$-bundle $E_{\mathsf{U}(1)} \to X$ and a $\mathsf{Spin}^c(n)$-map

$$E \to F_{\mathsf{SO}}X \times E_{\mathsf{U}(1)} \tag{2.14}$$

The principal $\mathsf{U}(1)$-bundle in Definition 2.5.12 determines a complex line bundle $L \to X$. We may refer to L as the *associated line bundle* of the spin^c structure.

Remark 2.5.13. In the case that a manifold X is *a priori* endowed with an H-structure, and $G < H$, then a G-structure on X will be assumed to be compatibble with the H-structure. For example, if X^7 is a spin manifold, any G_2-structure on X will induce the metric and orientation agreeing with that of the spin structure on X; an $\mathsf{SU}(3)$-structure on a manifold X^7 with G_2-structure is defined by a nowhere vanishing vector field and so on.

The embedding $\mathsf{G}_2 \hookrightarrow \mathsf{Spin}(7)$ induces a map from G_2-structures to spin structures. A $\mathsf{U}(m)$-structure induces a canonical spin^c structure. Explicitly, the map $i \oplus \det : \mathsf{U}(m) \to \mathsf{SO}(2m) \times \mathsf{U}(1)$ where i is the standard embedding forgetting complex structure, lifts to spin^c. The associated line bundle is then the det bundle associated to the $\mathsf{U}(m)$-structure.

Definition 2.5.14. Let M be a spin 7-manifold. Suppose that M is equipped with both a G_2-structure φ and a $\mathsf{U}(3)$-structure (ω, J). If there exist $\mathsf{SU}(3)$-reductions of both φ and (ω, J) which are equivalent when considered as reductions of the $\mathsf{SO}(7)$-structure, then φ and (ω, J) are *compatible.*

We consider the notion of homotopies of structures.

Definition 2.5.15. Let H be a Lie group, X a manifold and $E \to X$ a principal H-bundle on X. Let $G < H$ be a Lie subgroup, $F = H/G$ and $E_F = E \times_H F$. Two G-structures $s_i \in \Gamma(E_F)$ are *homotopic* if there exists map

$$S : I \times X \to E_F \tag{2.15}$$

such that $S(0, x) = s_0(x)$ and $S(1, x) = s_1(x)$, and for all $t \in I$, $S(t, \cdot) \in \Gamma(E_F)$.

2.6 Structures on boundaries

For any fibre bundle $E \to X$, a map $Y \to X$ induces the pullback of the fibre bundle. In the context of an embedding $f : Y^m \to X^n$ of smooth manifolds, an H-structure $E \to FX$ pulls back to a principal H-bundle $f^*E \to Y$. Moreover we have a decomposition of $f^*TX \cong TY \oplus f^*N_{f(Y)/X}$. Under certain conditions this decomposition induces a G-structure on Y, where $G = \mathrm{Stab}_H(\mathbb{R}^{n-m})$.

Example 2.6.1. Let X be an almost complex manifold. Let $Y \to X$ be an almost complex submanifold. Then the associated $\mathsf{U}(n)$-structure reduces to the associated $\mathsf{U}(m)$-structure on Y.

Our focus is on manifolds with boundary and the induced G-structures that occur on restricting an H-structure to the boundary.

Definition 2.6.2. Suppose $H < \mathsf{SO}(n+1)$ has transitive action on S^n. $G := \mathrm{Stab}_H(e_1)$ is the *boundary group* to H. Conversely, for $G < \mathsf{SO}(n)$, any $H < \mathsf{SO}(n+1)$ that has a transitive action on S^n, such that $G \cong \mathrm{Stab}_H(e_1)$ is a *coboundary group* of G.

Clearly for any $H < \mathsf{SO}(n+1)$ with transitive action on S^n leads to a unique boundary group. The existence and/or uniqueness of a coboundary group to a given $G < \mathsf{SO}(n)$ is perhaps less immediate.

Example 2.6.3.
(i) The obvious case is of $G = \mathsf{SO}(n)$ and $H = \mathsf{SO}(n+1)$ with $\mathsf{SO}(n) \to \mathrm{Stab}_{\mathsf{SO}(n+1)}(e_1)$.
(ii) Take the standard representation $\mathsf{U}(n) \to \mathsf{SO}(2n)$. Then $\mathsf{U}(n-1) = \mathrm{Stab}_{\mathsf{U}(n)}(e_1)$. Note that the induced representation $\mathsf{U}(n-1) \to \mathsf{SO}(2n-1)$ fixes the first basis vector. In particular G is not transitive on S^{2n-1}.
(iii) We can do as above, but replacing U with SU.
(iv) Let $\rho : \mathsf{Spin}(7) \to \mathsf{SO}(8)$ be the spin representation. Then $\mathsf{G}_2 = \mathrm{Stab}_\rho(e_1)$.

Proposition 2.6.4. *Let X be an $(n+1)$-manifold with boundary and $\partial X \cong Y$. Let $E \to X$ be an H-structure over smooth manifold X. Suppose that $H < \mathsf{SO}(n+1)$ acts transitively on S^n with boundary group G. Then Y inherits a G-structure.*

Proof. On restriction to Y the tangent bundle TX of X splits as $TX|_Y \cong TY \oplus \underline{\mathbb{R}}$. Since there is a global trivial line bundle we can choose coordinates near the boundary such that ∂e_1 corresponds to the trivial $\mathbb{R}$-line bundle. It follows that the H-structure induces a G-structure on the boundary. □

Example 2.6.5. The following are of particular interest to us. Let W be an 8-manifold with boundary M.
(i) Let $H = \mathsf{U}(4)$, and so $G = \mathsf{U}(3)$. Then W is an almost complex manifold, with almost contact boundary $M = \partial W$.
(ii) $H = \mathsf{SU}(4)$, and $G = \mathsf{SU}(3)$.
(iii) Let $H = \mathsf{Spin}(7)$, and let $\rho : \mathsf{Spin}(7) \to \mathsf{SO}(8)$ be the spinor representation. Then $G = \mathsf{G}_2$.

Lemma 2.6.6. *In the context of Proposition 2.6.4, the homotopy class of the G-structure on Y is uniquely determined by the homotopy class of the H-structure on X.*

Proof. Let $h_t : X \to BH$ be a homotopy for $t \in [0,1]$. Define $g_t : Y \to BG$ to be the induced restriction of h_t to Y. It is immediate that g_t is a homotopy of G-structures on Y. □

Definition 2.6.7. Let Y be smooth closed n-manifold with G-structure. Let $G \to H < \mathsf{SO}(n+1)$ be a coboundary group to G. Let $X = Y \times [0,1]$ be the manifold with boundary. Endow X with the H-structure such that the G-structure on $i : Y \to Y \times \{1\}$ is an isomorphism of G-structures, and such that the H-structure is translation invariant in its final coordinate. The *H-orientation reversal* of $Y = Y \times \{1\}$ is $\iota_H : Y \to Y \times \{0\}$ with the induced G-structure.

Any boundary group has a notion of orientation reversal, and on fixing a coboundary group this is unique. In the context of $G = \mathsf{SO}(n)$ and $H = \mathsf{SO}(n+1)$, then $\iota_H(Y) = -Y$ in the usual sense. In the context of $G = \mathsf{U}(k-1)$ and $H = \mathsf{U}(k)$, we describe the G-structure by (v, ω, J) (see Example 2.5.9). The orientation reversal ι_H maps the structure to $(-v, \omega, J)$.

Chapter 3

Cohomology

In this chapter we collect together a variety of results that fit under the broad label of cohomology. Almost all of the content presented can be found in the literature and has been included for context.

In Section 3.1, we discuss a form of products of which we make extensive use in the course of the thesis. Products on the cohomology on a manifold with boundary are well understood, provided the classes involved have compact support. We can take products more generally, but recover only a residue dependent on the divisibility of the class restricted to the boundary.

Section 3.2 introduces the quadratic refinement of a torsion linking form. It is an important invariant and is our first example of a boundary defect invariant. This provides a precursor for the setup of Chapter 5.

The final three sections of this chapter review content relevant to the examples discussed in Section 5.2. The examples we consider do not utilize fully all the results included in this chapter on characteristic classes and cohomology operations. Had we applied our framework to other contexts, these additional results would be required. We have included them here for this reason and additional context.

3.1 Products on compact manifolds with boundary

For topological pairs (X, U) and (X, V), there is a cup product

$$\smile : H^\bullet(X, U; \Lambda) \times H^\bullet(X, V; \Lambda) \to H^\bullet(X, U \cup V; \Lambda) \tag{3.1}$$

where Λ is a ring of coefficients. The cup product is a graded bilinear product, and if $U = V$ then it is a graded commutative product. For $U = V = \varnothing$ this is the cup product on the absolute cohomology of X. For $U = V$, this is the cup product on the relative cohomology of (X, U). Setting $V = \varnothing$ we have a product between relative and absolute cohomology. These cup products commute with one another. Moreover, the cup product is functorial. See for example [49, Section 3.2]. We denote all of these by $\smile$. The domain and codomain should be sufficiently clear from the context.

Lemma 3.1.1. *Let $\smile$ be the cup product of* (3.1). *Let $\delta : H^\bullet(U;\Lambda) \to H^{\bullet+1}(X,U;\Lambda)$ be the connecting homomorphism in the long exact sequence of cohomology associated to (X,U), and suppose that $V = \varnothing$. Then for all $u \in H^\bullet(U;\Lambda)$ and $x \in H^\bullet(X,U;\Lambda)$, $\delta(u) \smile x = 0 \in H^\bullet(X,U;\Lambda)$.*

Corollary 3.1.2. *Let $H_0^\bullet(X;\Lambda) := \mathrm{im}(j : H^\bullet(X,U;\Lambda) \to H^\bullet(X;\Lambda))$. Then the cup product determines a natural graded commutative product*

$$\begin{aligned} H_0^\bullet(X;\Lambda) \times H_0^\bullet(X;\Lambda) &\to H^\bullet(X,U;\Lambda) \\ (a,b) &\mapsto \widetilde{a} \smile b \end{aligned} \tag{3.2}$$

where $\widetilde{a} \in H^\bullet(X,U;\Lambda)$ such that $j(\widetilde{a}) = a$.

A graded symmetric multilinear map is equivalent to a linear map on the graded symmetric product of its domain.

Definition 3.1.3. Let Λ be a commutative ring and M a $\mathbb{Z}$-graded Λ-module. For $x \in M$, let $|x|$ denote the degree of x. The *graded symmetric algebra* $\mathcal{P}^\bullet(M)$ of M is the quotient of tensor algebra of M by the two sided ideal generated by

$$x \cdot y - (-1)^{|x||y|} y \cdot x, \quad z \cdot z \tag{3.3}$$

where $x, y, z \in M$ are homogeneous and $|z|$ is odd.

See [40, Appendix A.2] for a fuller discussion of this object. The cup product of (3.2) commutes with the relative and absolute cup products. Thus the cup product can be considered unambiguously as a morphism

$$\smile : \mathcal{P}^{\geq 2}(H_0^\bullet(X)) \to H^\bullet(X,U) \tag{3.4}$$

We wish to extend the domain of the cup product of (3.2). We are concerned only with the case that the topological pair is an oriented compact $(n+1)$-manifold with boundary (W,M), and will restrict our notation to this context.

We denote the torsion of $H^\bullet(M)$ by $TH^\bullet(M)$. Let $F : H^\bullet(M) \to FH^\bullet(M) := H^\bullet(M)/TH^\bullet(M)$ be the quotient onto the free part of $H^\bullet(M)$. Recall that $\mathbb{Z} \to \mathbb{Q}$ induces a functor $r : H^\bullet(\cdot;\mathbb{Z}) \to H^\bullet(\cdot;\mathbb{Q})$. The cup product commutes with this functor. Recall also that we have an identification $FH^\bullet(M) \otimes \mathbb{Q} \cong H^\bullet(M;\mathbb{Q})$.

For a compact oriented manifold with boundary (W,M), the fundamental class induces an isomorphism $H^{n+1}(W,M) \cong \mathbb{Z}$. Thus we can treat the cup product as a linear functional for integral and rational cohomologies, ie $\smile \in \mathrm{Hom}([\mathcal{P}^{\geq 2} H_0^\bullet(W)]^{n+1}, \mathbb{Z})$ and $\smile \in \mathrm{Hom}([\mathcal{P}^{\geq 2} H_0^\bullet(W;\mathbb{Q})]^{n+1}, \mathbb{Q})$. Under the map $H^\bullet(W) \to H^\bullet(W;\mathbb{Q})$, the preimage of $H_0^\bullet(W;\mathbb{Q})$ is precisely $\ker(F \circ \alpha)$. Thus we can also treat $\smile$ as an element of $\mathrm{Hom}([\mathcal{P}^{\geq 2}\ker(F \circ \alpha)]^{n+1}, \mathbb{Q})$.

It will be helpful later that we accommodate restricting attention to a submodule of $H^\bullet(W)$. Let $C < H^\bullet(W)$ be some submodule of interest. Let $\alpha : H^\bullet(W) \to H^\bullet(M)$, $\alpha|_C$ its restriction to C, $H^\bullet(M)_C := \mathrm{im}(\alpha|_C)$, and $FH^\bullet(M)_C := \mathrm{im}(F \circ \alpha|_C)$.

For a section $s : FH^\bullet(M) \to H^\bullet(M)$ of F, define

$$R(C,s) := \{\beta : FH^\bullet(M)_C \to C \colon \alpha \circ \beta = s|_{FH^\bullet(M)_C}\} \tag{3.5}$$

ie the set of right inverses of $F \circ \alpha|_C$ such that on its domain $\alpha \circ \beta$ agrees with s. We fix a section s such that this set is nonempty.

For $\beta \in R(C,s)$, let $\Pi_\beta := \mathrm{id} - \beta \circ F \circ \alpha|_C$. Note that $\mathrm{im}(\Pi_\beta) < \ker(F \circ \alpha)$. Define $\smile_\beta$ as

$$\begin{aligned} \smile_\beta \colon [\mathcal{P}^{\geq 2}C]^{n+1} &\to \mathbb{Q} \\ (u_1 \cdot \ldots \cdot u_k) &\mapsto \Pi_\beta u_1 \smile \ldots \smile \Pi_\beta u_k \end{aligned} \tag{3.6}$$

for $u_i \in C$. Thus $\smile_\beta$ is an extension to C of the cup product. However, it depends on the choice of $\beta \in R(C,s)$. The remainder of the section is concerned with determining to what extent $\smile_\beta$ is independent of choice.

A natural next step is to consider $\mathrm{Hom}(FH^\bullet(M)_C, \ker(\alpha|_C))$, its action on $R(\alpha, s)$, and the associated action on $\mathrm{Hom}(\mathcal{P}^{\geq 2}C, \mathbb{Q})$. However this action seems intractable to compute in practice. We consider something coarser but computable.

Proposition 3.1.4. *Let $C_0 := \ker(F \circ \alpha|_C)$. For $\beta, \beta' \in R(C,s)$, $\smile_\beta - \smile_\beta$ vanishes on $[\mathcal{P}^{\geq 2}C_0]^{n+1}$.*

Proof. The set of monomials of $\mathcal{P}^{\geq 2}$ are $u \in [\mathcal{P}^{\geq 2}C]^{n+1}$ such that $u = u_1 \cdot \ldots \cdot u_k$ and each $u_i \in C$. These form a spanning set of $[\mathcal{P}^{\geq 2}C]^{n+1}$. Thus it is sufficient to prove the statement for elements of this form.

It follows from the definition of $u_i \in C_0$ that $u_i = \Pi_\beta(u_i) = \Pi_{\beta'}(u_i)$. The result is then immediate. □

Proposition 3.1.4 can also be considered immediate from noting that the cup product on C_0 is completely determined by the rational cup product, and has no dependence on β.

Proposition 3.1.5. *For $\beta, \beta' \in R(C,s)$, the difference of the associated cup products is such that*

$$\smile_\beta - \smile_\beta \in \mathrm{Hom}([\mathcal{P}^{\geq 2}C]^{n+1}, \mathbb{Z}). \tag{3.7}$$

Proof. As above, it is sufficient to consider monomials $u \in [\mathcal{P}^{\geq 2}C]^{n+1}$. For $\beta, \beta' \in R(C,s)$ and $u_i \in C$

$$\Pi_\beta u_i = \Pi_{\beta'} u_i - ((\beta - \beta') \circ F \circ \alpha) u_i \tag{3.8}$$

Note that $\mathrm{im}(\beta - \beta') < \ker(\alpha|_C)$.

Suppose that for some i $u_i \in \ker(\alpha|_C)$ and let $\widetilde{u}_1 \in H^\bullet(W, M)$ be a lift. For any $\beta \in R(C,s)$, $\smile_\beta (u)$ is equal to the cup product of $\widetilde{u}_1$ with the cup product of $(u_2 \cdot \ldots \cdot u_k)$. In particular, $\smile_\beta (u) \in \mathbb{Z}$.

Consider then

$$(\smile_\beta - \smile_{\beta'})(u) = \Pi_\beta u_1 \smile \ldots \smile \Pi_\beta u_k - \Pi_{\beta'} u_1 \smile \ldots \smile \Pi_{\beta'} u_k \tag{3.9}$$

Using (3.11) and expanding the products, we see that two terms cancel out and all remaining products contain at least one element in $\ker(\alpha|_C)$. In particular this product is integral. □

For $m \in \mathbb{N}$, let $q_m : FH^\bullet(M)_C \to FH^\bullet(M)_C \otimes \mathbb{Z}/m\mathbb{Z}$ and $C_m := \ker(q_m \circ F \circ \alpha|_C)$.

Proposition 3.1.6. *For $\beta, \beta' \in R(C, s)$, the difference of the associated cup products is such that*

$$\smile_\beta - \smile_\beta \in \mathrm{Hom}([\mathcal{P}^{\geq 2} C_m]^{n+1}, \mathbb{Z}/m\mathbb{Z}) \tag{3.10}$$

is trivial.

Proof. Proposition 3.1.5 ensures that treating $\smile_\beta - \smile_\beta$ as a map to $\mathbb{Z}/m\mathbb{Z}$ is sensible. The logic is analogous to that of Proposition 3.1.5.

As above, it is sufficient to consider monomials $u \in [\mathcal{P}^{\geq 2} C_m]^{n+1}$. It follows from the definition of $u_i \in C_m$ that there exists $\overline{u}_i \in \ker(\alpha|_C)$, such that $m\overline{u}_i = (\beta - \beta') \circ F \circ \alpha(u_i)$. Then

$$\Pi_\beta u_i = \Pi_{\beta'} u_i - m\overline{u}_i \tag{3.11}$$

Substituting and expanding this into

$$(\smile_\beta - \smile_{\beta'})(u) = \Pi_\beta u_1 \smile \ldots \smile \Pi_\beta u_k - \Pi_{\beta'} u_1 \smile \ldots \smile \Pi_{\beta'} u_k \tag{3.12}$$

As before, we see that two terms cancel out and all remaining products contain at least one element of $\ker(\alpha|_C)$ times m. Thus it vanishes modulo m. □

We can make analogous statements to this result concerning powers. It is sufficient for our needs to consider only squares, and we restrict our attention to this case.

We recall the following. Let S_k be the symmetric group on k letters. For a $\mathbb{Z}$-graded Λ-module N, Λ a commutative ring with involution, S_k acts on $N^{\oplus k}$ by signed permutation of factors. This induces an action on $N^{\otimes k}$ via $N^{\oplus k} \to N^{\otimes k}$. Let $\widehat{\mathcal{P}}^k N < \mathcal{P}^k N$ be the image of the fix set of $N^{\otimes k}$ under action of S_k. Note that if $\Lambda = \mathbb{Q}$ then $\widehat{\mathcal{P}}^k(N) = \mathcal{P}^k(N)$. However if $\Lambda = \mathbb{Z}$, N is free, and $a, b \in N$ are primitive, then in general $ab \in \mathcal{P}^2(N) \setminus \widehat{\mathcal{P}}^2(N)$.

Proposition 3.1.7. *Assume that n is odd, and let $n' = \frac{1}{2}(n+1)$. For $m \in \mathbb{N}$, $\beta, \beta' \in R(C, s)$, the difference of the associated cup products is such that*

$$\smile_\beta - \smile_\beta \in \mathrm{Hom}(\widehat{\mathcal{P}}^2 [C_{2m}]^{n'}, \mathbb{Z}/4m\mathbb{Z}) \tag{3.13}$$

is trivial.

Proof. The logic is analogous to that of the previous propositions. It is sufficient to consider $u \in [\mathcal{P}^{\geq 2} C_{2m}]^{n+1}$, $u = u_1 \cdot u_1$. Adopting the notation above

$$(\smile_\beta - \smile_{\beta'})u = 2m(u_1 \smile \overline{u}_1 + \overline{u}_1 \smile u_1) - 4m^2 \overline{u}_1 \smile \overline{u}_1 \tag{3.14}$$

In either of the two cases, n' is odd or even, this vanishes modulo $4m$. □

There is another, subtle refinement. Let $H^\bullet(M)_{C_m} = \mathrm{im}(\alpha|_{C_m})$ and $\alpha_\mathbb{Q} : H^\bullet(W; \mathbb{Q}) \to H^\bullet(M; \mathbb{Q})$. Let r_M and r_W be the map from integer to rational cohomology of M and W respectively. Let $C_{\mathbb{Q},m} := \alpha_\mathbb{Q}^{-1}(\mathrm{im}(r_M|_{FH^\bullet(M)_{C_M}}))$. Note that $C_{\mathbb{Q},m} = r_W(C_m) + r_W(C_0) \otimes \mathbb{Q}$.

Corollary 3.1.8. *Propositions 3.1.6, 3.1.7 both hold when exchanging the role of C_m with $C_{\mathbb{Q},m}$.*

The notation of this result is a bit fiddly, and is required only at one point in our examples. For the most part it is sufficient for us to work exclusively with integral cohomology. We continue using C_m over $C_{\mathbb{Q},m}$ for notational and conceptual simplicity.

We can now define the *shearing submodule* $\mathrm{Sh}(C)$. Let

$$\begin{aligned}\mathrm{Sh}(C)_0 &:= \left\{\Phi \in \mathrm{Hom}([\mathcal{P}^{\geq 2}C]^{n+1}, \mathbb{Z}) \colon \Phi|_{[\mathcal{P}^{\geq 2}C_0]^{n+1}} = 0\right\} \\ \mathrm{Sh}(C)_m &:= \left\{\Phi \in \mathrm{Hom}([\mathcal{P}^{\geq 2}C]^{n+1}, \mathbb{Z}) \colon \Phi|_{[\mathcal{P}^{\geq 2}C_m]^{n+1}} = 0 \mod m\right\} \\ \mathrm{Sh}(C)^2_{2m} &:= \left\{\Phi \in \mathrm{Hom}([\mathcal{P}^{\geq 2}C]^{n+1}, \mathbb{Z}) \colon \Phi|_{\widehat{\mathcal{P}}^2[C_{2m}]^{n'}} = 0 \mod 4m\right\}\end{aligned} \tag{3.15}$$

The shearing submodule associate to C is

$$\mathrm{Sh}(C) := \mathrm{Sh}(C)_0 \cap \left(\bigcap_{m\in\mathbb{N}} \mathrm{Sh}(C)_m\right) \cap \left(\bigcap_{m\in\mathbb{N}} \mathrm{Sh}(C)^2_{2m}\right) \tag{3.16}$$

The action of $\Pi_\beta - \Pi_{\beta'}$ corresponds to a shear on C, and motivates the name chosen. As mentioned, in other applications it may be preferable to continue refining this object. For our purposes we deem this sufficient. From the propositions above, the following is immediate.

Corollary 3.1.9. *The following class*

$$[\smile_\beta] \in \mathrm{Hom}([\mathcal{P}^{\geq 2}C]^{n+1}, \mathbb{Q})/\mathrm{Sh}(C) \tag{3.17}$$

is independent of choice of $\beta \in R(C, s)$.

Note also that for $C_{\mathbb{Q}} := C \otimes \mathbb{Q} < H^\bullet(W, \mathbb{Q})$,

$$[\smile_\beta] \in \mathrm{Hom}([\mathcal{P}^{\geq 2}C_{\mathbb{Q}}]^{n+1}, \mathbb{Q})/\mathrm{Sh}(C) \tag{3.18}$$

is independent of choice of $\beta \in R(C, s)$.

We have an additional result that assists in computing the shearing module.

Proposition 3.1.10. *Suppose that we can select bases such that $F \circ \alpha$ has a diagonal form with non-zero entries $(a_1, \ldots, a_k)$. Define*

$$K := \left\{m \in \mathbb{N} \colon \gcd\left(\mathrm{Num}\left(\frac{m}{a_1}\right), \ldots, \mathrm{Num}\left(\frac{m}{a_k}\right)\right) = 1\right\}. \tag{3.19}$$

Then

$$\mathrm{Sh}(C) := \mathrm{Sh}(C)_0 \cap \left(\bigcap_{m\in K} \mathrm{Sh}(C)_m\right) \cap \left(\bigcap_{m\in K} \mathrm{Sh}(C)^2_{2m}\right) \tag{3.20}$$

Proof. Let $e_1, \ldots, e_l$ be a basis of C with respect to which $F \circ \alpha$ has the diagonal form described. Then C_m has a basis $\mathrm{Num}(\frac{m}{a_1})e_1, \ldots \mathrm{Num}(\frac{m}{a_k})e_k, e_{k+1}, \ldots, e_l$.

Suppose that $m \in \mathbb{N} \setminus K$ so that

$$g := \gcd\left(\mathrm{Num}\left(\frac{m}{a_1}\right), \ldots, \mathrm{Num}\left(\frac{m}{a_k}\right)\right) > 1 \tag{3.21}$$

Then for $m' = m/g$, $\mathrm{Sh}(C)_{m'} < \mathrm{Sh}(C)_m$. The analogous holds for $\mathrm{Sh}(C)^2_{2m}$, and the result follows. □

Members of K can be determined in an elementary fashion from the prime decomposition of a_i. It is a finite set, and from which we can compute $\mathrm{Sh}(C)$ by exhaustion.

3.2 Linking forms

We first recall the definition of a linking form defined on odd dimensional closed oriented manifolds. This can be computed via a coboundary with suitable properties. In addition, we introduce quadratic refinements which provide a further invariant.

Let M be a closed oriented n-manifold. Let $\delta : H^\bullet(M; \mathbb{Q}/\mathbb{Z}) \to H^{\bullet+1}(M)$ be the connecting homomorphism in the Bockstein sequence associated to the short exact sequence of coefficients

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0 \tag{3.22}$$

It is immediate that $\mathrm{im}(\delta) = TH^\bullet(M)$. We have a cup product

$$\smile : H^k(M; \mathbb{Q}/\mathbb{Z}) \times H^l(M) \to H^{k+l}(M; \mathbb{Q}/\mathbb{Z}) \tag{3.23}$$

See, for example, [95, Theorem 5.5.11]. The *linking form* of M is then

$$\begin{aligned} b_M : TH^{n-k}(M) \times TH^{k+1}(M) &\to \mathbb{Q}/\mathbb{Z} \\ (x, y) &\mapsto \langle \widetilde{x} \smile y, [M] \rangle \end{aligned} \tag{3.24}$$

where $\widetilde{x} \in H^{n-k-1}(M; \mathbb{Q}/\mathbb{Z})$ is such that $\delta(\widetilde{x}) = x$. This is independent of choice of $\widetilde{x}$.

The linking form is a graded symmetric bilinear product. If $n = 2k - 1$, then k is the *middle* dimension. The linking form restricted to the middle dimension is symmetric if k is even and antisymmetric if k is odd. In either case it is *nondegenerate* (ie $\widehat{b}_M : TH^k(M) \to \mathrm{Hom}(TH^k, \mathbb{Q}/\mathbb{Z})$ is a bijection). The linking form is an oriented homotopy equivalence invariant of M.

We are primarily concerned with the case that $n = 7$, but the arguments rely only on assuming that $n = 4k - 1$. Thus the linking form is a nondegenerate symmetric bilinear pairing on $TH^{2k}(M)$. We follow closely the exposition presented in [32, Section 2]. There is a cosmetic divergence in our accounts: while they define a family of invariants, we introduce additional choices determining a single invariant. See [32, Remark 2.26].

Remark 3.2.1. Throughout [32], it is assumed that the coboundary used is highly connected. This is unnecessarily strong for the purposes of defining and computing the quadratic refinements. It is sufficient to assume that $TH^{2k}(W) = 0$. By Lefschetz duality and the universal coefficient theorem $TH^{2k+1}(W, M) = 0$. By exactness in the cohomology sequence associated to (W, M), $TH^{2k}(M) < \operatorname{im}(H^{2k}(W) \to H^{2k}(M))$.

Let us first consider the algebraic story. A *torsion form* $b : T \times T \to \mathbb{Q}/\mathbb{Z}$ is a nondegenerate symmetric bilinear form on a finite abelian group T. A map $q : T \to S$ of abelian groups is a *quadratic function* if the map $b_q : (x, y) \mapsto q(x + y) - q(x) - q(y)$ is $\mathbb{Z}$-bilinear. b_q is the *associated bilinear form* of the quadratic function q. We consider only $S = \mathbb{Q}/\mathbb{Z}$ and T a torsion group, q is *nondegenerate* if b_q is nondegenerate and in which case q is a *quadratic form* on T. If $f : T' \to T$ is a morphism of groups, then it defines a pullback quadratic form f^*q. A quadratic form q' on T' is *isomorphic* to q if there exists an isomorphism $f : T' \to T$, such that $q' = f^*q$.

Let T be a torsion group and let $b : T \times T \to \mathbb{Q}/\mathbb{Z}$ be a symmetric bilinear pairing. A quadratic form $q : T \to \mathbb{Q}/\mathbb{Z}$ is a *quadratic refinement* of b if $b_q = b$. The set of all quadratic refinements of b is denoted by $\operatorname{Quad}(b)$.

The *defect* of a quadratic refinement q is $d_q \in 2T$ such that for all $x \in T$

$$q(x) - q(-x) = b_q(d_q, x). \tag{3.25}$$

q is *homogeneous* if $d_q = 0$. The classification of homogeneous quadratic refinements of nondegenerate symmetric bilinear forms $b : T \times T \to \mathbb{Q}/\mathbb{Z}$ was given by [84]. The general case was considered in [36] and [28].

We define two quantities for quadratic forms on finite groups. Let q be a quadratic form on finite group T. Following [18], the *Gauss sum* $\operatorname{GS}(q)$ and the *Arf invariant* $\operatorname{Arf}(q)$ are

$$\operatorname{GS}(q) := \sum_{x \in T} \exp(2\pi i q(x)) \in \mathbb{C}, \qquad \operatorname{Arf}(q) := \tfrac{1}{2\pi} \arg(GS(q)) \in \mathbb{Q}/\mathbb{Z}. \tag{3.26}$$

A *lattice* (L, λ) is a finitely generated free abelian group, equipped with a symmetric bilinear pairing $\lambda : L \times L \to \mathbb{Z}$. Denote the adjoint by $\widehat{\lambda} : L \to L^\vee$. Let $K := \ker(\widehat{\lambda})$, $N := \operatorname{coker}(\widehat{\lambda})$, and $\alpha : L^\vee \to N$. We have an exact sequence

$$0 \to K \to L \xrightarrow{\widehat{\lambda}} L^\vee \xrightarrow{\alpha} N \to 0 \tag{3.27}$$

In particular, λ is nondegenerate if and only if $K = 0$ if and only if N is finite. λ is *unimodular* if and only if $N = 0$. We have a natural symmetric $\mathbb{Z}$-valued bilinear pairing λ' on $\operatorname{im}(\widehat{\lambda})$ given by $\lambda'(x, y) := \lambda(\widetilde{x}, \widetilde{y})$ where $\widehat{\lambda}(x) = x$ and $\widehat{\lambda}(y) = y$. As discussed in Section 3.1 we wish to extend the product.

Definition 3.2.2. Let A be a free $\mathbb{Z}$-module, and $B < A$ a submodule. Let $\operatorname{ps}(B) < A$ be the maximal submodule of A such that the index $|\operatorname{ps}(B) : B|$ is finite. Then $\operatorname{ps}(B)$ is the *primitive supermodule* of B. Equivalently $\operatorname{ps}(B)$ is the minimal primitive submodule of A containing B.

We can extend λ' to the *primitive supermodule* $L' := \mathrm{ps}(\widehat{\lambda}(L))$

$$\lambda' : L' \times L' \to \mathbb{Q} \tag{3.28}$$

Note that $L' = \ker(F \circ \alpha)$ for $F : N \to FN$. We can then define a nondegenerate symmetric bilinear form on TN. Let

$$\begin{aligned} b_\lambda : TN \times TN &\to \mathbb{Q}/\mathbb{Z} \\ (t_1, t_2) &\mapsto -\lambda'(\widetilde{t}_1, \widetilde{t}_2) \end{aligned} \tag{3.29}$$

where $\widetilde{t}_i$ is a lift via α of t_i. b_λ is the *associated torsion form* of (L, λ). For a torsion form (T, b), a lattice (L, λ) is a *presentation* of (T, b) if $(T, b) \cong (TN, b_\lambda)$.

Returning to (3.27), the lattice descends to a nondegenerate lattice on the quotient $L_{/K}$, denoted by $\lambda_{/K}$. The dual of the quotient $L \to L_{/K}$ is an injective morphism $L^\vee_{/K} \to L^\vee$, the image of which is precisely L'. We have the split exact sequence of free modules

$$0 \to L^\vee_{/K} \to L^\vee \to FN \to 0 \tag{3.30}$$

Moreover, this identification is an isomorphism of $\mathbb{Q}$-valued symmetric bilinear forms $(L', \lambda') \cong (L^\vee_{/K}, \lambda_{/K})$. Note that the signature of λ and $\lambda_{/K}$ agree ie $\sigma(\lambda) = \sigma(\lambda_{/K})$. Let $N_{/K}$ denote the cokernel of $\widehat{\lambda}_{/K}$. We have an isomorphism $\varphi : TN \to N_{/K}$, and $(TN, b_\lambda) = \varphi^*(N_{/K}, b_{\lambda_{/K}})$.

Lemma 3.2.3. *Let (L_i, λ_i) be two lattices. Suppose that (L, λ) is a unimodular lattice equipped with two primitive isometric homomorphisms $e_i : L_i \to L$ such that $\mathrm{im}(e_0)^{\perp_\lambda} = \mathrm{im}(e_1)$. Let $N_i' = \mathrm{im}(\alpha_i \circ e_i^*) < N_i$. Then $TN_0 < N_0'$, and λ determines an isomorphism*

$$\psi_\lambda : N_0' \to N_1' \tag{3.31}$$

which on restriction is an isomorphism of the associated torsion forms of (L_i, λ_i) up to sign.

Proof. The map ψ_λ is defined by lifting $x \in N_0'$ via $\alpha_0 \circ e_0$, and mapping back down to N_1'. Thus it is sufficient to prove that $\ker(\alpha_0 \circ e_0^*) = \ker(\alpha_1 \circ e_1^*)$. The two assumptions of the isometric homomorphisms e_i being primitive and orthogonal imply this. See [32, Lemma 2.21]. $\square$

We consider the *discriminant construction* of quadratic forms. An element $c \in L^\vee$ is *characteristic* of a lattice (L, λ) if for all $x \in L$

$$\lambda(x, x) + c(x) = 0 \mod 2. \tag{3.32}$$

A *characteristic form* is a triple (L, λ, c) where c is characteristic for the lattice (L, λ). Consider first nondegenerate characteristic forms so that $L' = L^\vee$. The *associated quadratic form* of a nondegenerate characteristic form (L, λ, c) is

$$q_{\lambda,c} : x \mapsto -\tfrac{1}{2}(\lambda'(\widetilde{x}, \widetilde{x}) + \lambda'(c, \widetilde{x})) \mod \mathbb{Z} \tag{3.33}$$

Note that $q_{\lambda,c} \in \mathrm{Quad}(b_\lambda)$ and it has defect $\alpha(c)$. We wish to extend this notion to degenerate cases.

For a section $\beta : FN \to L^\vee$, let $\Pi_\beta : L^\vee \to L^\vee_{/K}$ be the corresponding projection, while treating $L^\vee_{/K}$ as a subspace of $L^\vee$. An *extended characteristic form* is a quadruple (L, λ, c, β), where (L, λ, c) is a characteristic form, and $\beta : FN \to L^\vee$ is a right inverse of $F \circ \alpha$. We use λ'_β to denote the extension λ' determined by β, just like we did with $\smile_\beta$. The *associated quadratic form* of the extended characteristic form (L, λ, c, β) is $q_{\lambda,c,\beta} := \varphi^* q_{\lambda_{/K},\Pi_\beta(c)}$.

Proposition 3.2.4. *Let (L, λ, c) be a characteristic form. In the above notation, fix a section $s : FN \to N$. Then for $\beta, \beta' \in R(\alpha, s)$, $q_{\lambda,c,\beta} = q_{\lambda,c,\beta'}$.*

Proof. By the classification of $\mathbb{Z}_2$-valued bilinear forms, $\alpha(c) \in N$ is even. Then so too is $\beta \circ F \circ \alpha(c)$ for $\beta \in R(\alpha, s)$. For $\beta, \beta' \in R(\alpha, s)$, $\mathrm{im}(\Pi_\beta - \Pi_{\beta'}) < \ker(\alpha)$. It follows then that there exists an $r \in L$ such that $2\widehat{\lambda}(r) = (\Pi_\beta - \Pi_\beta)(c)$. For $x \in TN$, $(q_{\lambda,c,\beta} - q_{\lambda,c,\beta'})(x) = \frac{1}{2}(2\langle r, \widetilde{x}\rangle) \in \mathbb{Q}/\mathbb{Z}$, yet $2\langle r, \widetilde{x}\rangle \in 2\mathbb{Z}$. Thus $q_{\lambda,c,\beta}$ is independent of $\beta \in R(\alpha, s)$. $\square$

A consequence of Proposition 3.2.4 is that the quadratic refinement $q_{\lambda,c,\beta}$ depends on β only via $s : FN \to N$. Thus we may specify a quadratic refinement by $q_{\lambda,c,s}$. We extend Lemma 3.2.3 to characteristic forms.

Corollary 3.2.5. *Let (L_i, λ_i, c_i) be characteristic forms for $i = 0, 1$. Suppose that (L, λ, c) is a unimodular characteristic form, equipped with primitive isometric homomorphisms $e_i : L_i \to L$ such that $\mathrm{im}(e_0)^{\perp_\lambda} = \mathrm{im}(e_1)$ and $e_i^* : c \mapsto c_i$. Fix sections $s_i : FN_i \to N_i$, such that they commute with ψ_λ.*

Then $\bar{c}_i \in N'_i$, and for $\beta_i \in R(\alpha_i, s_i)$, ψ_λ is an isomorphism of the associated quadratic forms $q_{\lambda_i,c_i,\beta_i}$ up to sign.

We can define an equivalence relation on characteristic forms. In the notation of Corollary 3.2.5, the existence of (L, λ, c) and e_i imply that (L_0, λ_0, c_0) and (L_1, λ_1, c_1) are *complements*. Two lattices (L_i, λ_i, c_i), $i = 1, 2$ are *similar* if there exist (L_0, λ_0, c_0) that complements both. We close the relation transitively (if necessary) into equivalence classes. Corollary 3.2.5 essentially says that the associated quadratic form $q_{\lambda_i,c_i,s}$ depends on (L_i, λ_i, c_i) only via its *similarity class*. Where it is not ambiguous, we will denote the associated quadratic form by $q_{c,s}$.

Let us define some objects that will act as algebraic models of boundaries. We use the word *extended* to imply some choice of right inverse or section. An *extended torsion form* is a quadruple (N, b, c_N, s) consisting of a finitely generated abelian group N, a torsion form b on TN, an element $c_N \in 2N$, and a section $s : FN \to N$. An *extended quadratic refinement* is a quadruple (N, q, c_N, s) such that (N, b_q, c_N, s) is an extended torsion form.

The *associated extended torsion form* of the extended characteristic form (L, λ, c, β) is the quadruple $(\mathrm{coker}(\widehat{\lambda}), b_\lambda, \alpha(c), \alpha \circ \beta)$. This is indeed an extended torsion form. The *associated extended quadratic form* of the extended characteristic form (L, λ, c, β) is the quadruple $(\mathrm{coker}(\widehat{\lambda}), q_{\lambda,c,\beta}, \alpha(c), \alpha \circ \beta)$.

The following is then a rephrasing of some of the results of [32, Section 2.4]. It is an extension of a result of van der Blij, [79, Lemma 5.2]. The second claim is an extension of a result of Milgram. For a proof, see [32, Proposition 2.15], and the ensuing discussion.

Proposition 3.2.6. *Let (L, λ, c) be a characteristic form, and $s : FN \to N$ a section of $F : N \to FN$. Then*

$$\mathrm{Arf}(q_{\lambda,c,\beta}) = \tfrac{1}{8}\left(\lambda'_\beta(c,c) - \sigma(\lambda)\right) \in \mathbb{Q}/\mathbb{Z} \tag{3.34}$$

where σ denotes the signature.

We now consider our intended application. Let M be a closed oriented manifold of dimension $(4k-1)$. Suppose that W is a coboundary to M and let $M \xrightarrow{\simeq} \partial W$. In addition suppose that $H^{2k}(W)$ is free. Then

$$H^{2k}(W, M) \to H^{2k}(W) \xrightarrow{\alpha^{2k}} H^{2k}(M) \tag{3.35}$$

is exact. By Lefschetz-Poincaré duality and the universal coefficient theorem $H^{2k}(W) \cong FH^{2k}(W, M)^\vee$. Let $L := FH^{2k}(W, M)$. Let $\lambda_W : L \times L \to \mathbb{Z}$ be the lattice determined by the cup product and fundamental class $[W, M]$. The lattice (L, λ_W) is the *intersection form* of W. The intersection form of W is a presentation of $(TH^{2k}(M), b_M)$.

Lemma 3.2.7. *Let M be a closed oriented $(4k-1)$-manifold. Suppose that W_0 is a coboundary to $M_0 := -M$, and W_1 is a coboundary to $M_1 := M$. In addition suppose that $TH^{2k}(W_i) = 0$, and $H^{2k}(W_i) \to H^{2k}(M_i)$ is surjective. Let $X = W_0 \cup_M W_1$.*

Then $(FH^{2k}(X), \lambda_X)$ is unimodular, and $e_i : FH^{2k}(W_i, M_i) \to FH^{2k}(X)$ are primitive isometric homomorphisms such that $\mathrm{im}(e_0)^{\perp_{\lambda_X}} = \mathrm{im}(e_1)$.

The proof follows from considering the relevant exact sequences. See Lemma 2.4 and Remark 2.5 of [32]. Thus the topological setup meets the hypotheses of Lemma 3.2.3.

Proposition 3.2.8. *Let M be a closed oriented $(4k-1)$-manifold. Fix a section $s : FH^{2k}(M) \to H^{2k}(M)$. Let W be a coboundary to M such that $TH^{2k}(W) = 0$ and $\alpha^{2k} : H^{2k}(W) \to H^{2k}(M)$ is surjective. Suppose $c \in H^{2k}(W)$ is a characteristic element of λ_W.*

Then for $\beta \in R(\alpha, s)$, $(FH^{2k}(W), \lambda_W, c, \beta)$ is an extended characteristic form with extended quadratic refinement $(H^{2k}(M), q_{\lambda_W,c,s}, \alpha(c), s)$.

Jumping ahead, in Section 3.3 we discuss characteristic classes. Characteristic classes are functorial and certain characteristic classes are necessarily characteristic elements for the intersection form of coboundaries (see Corollary 3.4.6). 'Integral Wu classes' are precisely the characteristic classes with this property. Manifolds with, say, an H-structure then have 'integral H-Wu classes', denoted $\mathrm{Wu}(H)$ (Definition 3.4.5). As stated in the proof of Proposition 3.2.6, a characteristic element restricted to the boundary is necessarily even modulo torsion. That is, for $c \in \mathrm{Wu}(H)$, there exists a class $u \in FH^{2k}(M)$ such that $F(c(M)) - 2u = 0$.

Corollary 3.2.9. *Let M be a closed oriented $(4k-1)$-manifold. Fix a section $s : FH^{2k}(M) \to H^{2k}(M)$. Let $c \in \mathrm{Wu}(H)$ and integral Wu class of H. Assume that there exists an H-coboundary W_0 of $M_0 := -M$ and that $TH^{2k}(W) = 0$ and $H^{2k}(W_0) \to H^{2k}(M_0)$ is surjective.*

Suppose that W is an H-coboundary to M such that $u \in \mathrm{im}(F \circ \alpha_1^{2k} : H^{2k}(W) \to FH^{2k}(M))$. Let λ be the intersection form on W and fix $\beta \in R(\alpha, s)$. Then

$$\mathrm{Arf}(q_{c,s}) = \tfrac{1}{8}(\lambda'_\beta(c(W), c(W)) - \sigma(\lambda)) \in \mathbb{Q}/\mathbb{Z} \tag{3.36}$$

Proof. Let $W_1 := W$, $\lambda'_1 := \lambda'_\beta$ and let $u_1\beta(u)$ to $H^{2k}(W_1)$. Let $s_0 = \iota_H^* s$. Fix $\beta_0 \in R(\alpha_0, s_0)$, let λ'_0 be the extension of λ'_{W_0} by β_0, and $u_0 := \beta_0(u)$.

Let $X = W_0 \cap_M W_1$. Then X has an H-structure inherited from W_0 and W_1. By exactness of sequences of cohomology there exists $u_X \in H^{2k}(X)$, such that $u_X \mapsto u_i$. Let $c_X = C(X)$, and $c_i = c(W_i)$. Then

$$\begin{aligned} \lambda'_X(c_X, c_X) &= \lambda'_X(c_X - 2u_X, c_X - 2u_X) + 2\lambda'_X(c_X - 2u_X, 2u_X) + \lambda'_X(2u_X, 2u_X) \\ &= \lambda_1(c_1, c_1) - \lambda_0(c_0, c_0) + 4((c_X - 2u_X, u_X) + (u_X, u_X)). \end{aligned} \tag{3.37}$$

c_X is a characteristic element by assumption. Hence $c_X - 2u_X$ is also a characteristic element and so

$$\lambda'_X(c_X, c_X) = \lambda_1(c_1, c_1) - \lambda_0(c_0, c_0) \in \mathbb{Q}/8\mathbb{Z}. \tag{3.38}$$

The characteristic form on W_0 with characteristic element c_0 determines the associated quadratic refinement q_{c_0,s_0} on M_0. This corresponds to the quadratic refinement $-q_{c_1,s}$ on M. In particular $\mathrm{Arf}(q_{c_0,s}) = -\mathrm{Arf}(q_{c_1}, s)$.

By Proposition 3.2.6

$$\begin{aligned} \tfrac{1}{8}(\lambda'_X(c_X, c_X) - \sigma(X)) = 0 \in \mathbb{Q}/\mathbb{Z} \\ \tfrac{1}{8}(\lambda'_0(c_0, c_0) - \sigma(W_0)) = -\mathrm{Arf}(q_{c_1,s}). \end{aligned} \tag{3.39}$$

The final step requires the Novikov additivity theorem (Theorem 3.5.2) which implies that $\sigma(X) = \sigma(W_0) + \sigma(W_1)$. The result is then immediate. □

The moral is that as long as we know a coboundary exists meeting the hypotheses of Proposition 3.2.8, we can compute the Arf invariant via any coboundary (with H-structure) provided that we can lift the class u.

3.3 Characteristic classes

An H-structure on a space X induces a classifying map $\varphi_X : X \to BH$. For a ring of coefficients Λ and a class $c \in H^\bullet(BH; \Lambda)$, $\varphi_X^*(c) \in H^\bullet(X; \Lambda)$ is an invariant of the H-structure up to homotopy. This motivates us to understand the structure of $H^\bullet(BH; \Lambda)$. Characteristic classes are simply elements of the cohomology of a classifying space. A comprehensive description of the structure of each cohomology algebra $H^\bullet(BH; \Lambda)$ and each morphism $B\rho : H^\bullet(BH; \Lambda) \to H^\bullet(BG; \Lambda)$ induced by Lie group morphism $\rho : G \to H$

would be nice. Unfortunately, these algebras are complicated and a treatment of this sort is not possible here.

We recall the definitions of Chern, Stiefel-Whitney, and Pontrjagin classes which are prototypical examples of characteristic classes. We describe the behaviour of some morphisms induced by Lie group homomorphisms. In particular, we show that there are no characteristic classes to distinguish G_2-structures on a spin 7-manifold.

The image on X of a characteristic class $c \in H^\bullet(BH;\Lambda)$ is referred to as a characteristic class of X, and often denoted $c(X) = \varphi_X^*(c)$. Some authors refer to characteristic classes $c \in H^\bullet(BH)$ as a *universal characteristic classes*, in order to distinguish them from a pullback $c(X)$.

For the families of Lie groups such as $\mathsf{O}(n)$ and $\mathsf{U}(n)$ we have descriptions of representatives of their classifying spaces discussed in 2.4. To prove properties of the characteristic classes of these spaces, one can induct on n via the canonical inclusions. Each step considers the induced Thom-Gysin sequence with its corresponding *Euler class*. (See, for example, [16, Section 1].) We begin with Chern classes.

Definition 3.3.1. For $n \geq 1$, the k^{th} *Chern class* $c_k \in H^{2k}(B\mathsf{U}(n))$ is characterized by the following

(i) $c_0 = 1$, and $c_k = 0$ for $k > n$;
(ii) for $n = 1$, c_1 is the canonical generator of $H^2(B\mathsf{U}(1)) \cong \mathbb{Z}$;
(iii) $i_n^* c_k = c_k$ for the canonical map $i_n : B\mathsf{U}(n) \to B\mathsf{U}(n+1)$;
(iv) $i_{r,s}^* c_k = \sum_{j=0}^k c_j \smile c_{k-j}$ for the canonical map $i_{r,s} : B\mathsf{U}(r) \times B\mathsf{U}(s) \to B\mathsf{U}(r+s)$.

The *total Chern class* is $c := \sum_{j=0}^n c_j$.

Chern classes exist and form a basis for the integral characteristic classes of $B\mathsf{U}(n)$. That is

$$H^\bullet(B\mathsf{U}(n)) = \mathbb{Z}[c_1, \ldots, c_n] \tag{3.40}$$

See, for example, [78, Theorem 14.5]. The n^{th} Chern class corresponds to the Euler class from the Thom-Gysin sequence. They are *stable* in the sense that we can identify $H^\bullet(B\mathsf{U}(n))$ as

$$H^\bullet(B\mathsf{U}) = \mathbb{Z}[c_1, c_2 \ldots] \tag{3.41}$$

We often omit to mention which value of n we are referring in a given context.

Consider the Bockstein sequence corresponding to the coefficient sequence $\mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \to \mathbb{Z}_p$. As $H^\bullet(B\mathsf{U}(n)) \xrightarrow{\cdot p} H^\bullet(B\mathsf{U}(n))$ is injective the connecting homomorphism vanishes. Thus $H^\bullet(B\mathsf{U}(n);\mathbb{Z}_p) \cong \mathbb{Z}_p[c_1, \ldots, c_n]$.

The standard inclusion $\rho : \mathsf{SU}(n) \to \mathsf{U}(n)$ induces a surjective morphism $B\rho^* : H^\bullet(B\mathsf{U}(n)) \to H^\bullet(\mathsf{SU}(n))$. The kernel of $B\rho^*$ is generated by c_1.

Definition 3.3.2. For $n \geq 1$, the k^{th} *Stiefel-Whitney class* $w_k \in H^k(B\mathsf{O}(n);\mathbb{Z}_2)$ is characterized by the following

(i) $w_0 = 1$, and $c_k = 0$ for $k > n$;
(ii) for $n = 1$, $w_1 \neq 1$;
(iii) $i_n^* w_k = w_k$ for the canonical map $i_n : B\mathsf{O}(n) \to B\mathsf{O}(n+1)$

(iv) $i^*_{r,s}w_k = \sum_{j=0}^k w_j \smile w_{k-j}$ for the canonical map $i_{r,s} : B\mathsf{O}(r) \times B\mathsf{O}(l) \to B\mathsf{O}(r+s)$
The *total Stiefel-Whitney class* is $w := \sum_{j=0}^n w_j$.

Stiefel-Whitney classes exist and form a basis for the $\mathbb{Z}_2$ characteristic classes of $B\mathsf{O}(n)$. That is

$$H^\bullet(B\mathsf{O}(n); \mathbb{Z}_2) = \mathbb{Z}_2[w_1, \ldots, w_n] \tag{3.42}$$

See, for example, [78, Theorem 7.1]. The Stiefel-Whitney classes are stable so

$$H^\bullet(B\mathsf{O}; \mathbb{Z}_2) = \mathbb{Z}_2[w_1, w_2 \ldots] \tag{3.43}$$

The inclusion $\mathsf{SO}(n) \to \mathsf{O}(n)$ induces a surjective morphism $H^\bullet(B\mathsf{O}(n); \mathbb{Z}_2) \to H^\bullet(B\mathsf{SO}(n); \mathbb{Z}_2)$. The kernel is generated by w_1.

The realification $\rho_n : \mathsf{U}(n) \to \mathsf{O}(2n)$ induces the morphism $B\rho_n^* : H^\bullet(B\mathsf{O}(2n); \mathbb{Z}_2) \to H^\bullet(B\mathsf{U}(n); \mathbb{Z}_2)$. Let $\bar{c} \in H^\bullet(B\mathsf{U}(n); \mathbb{Z}_2)$ denote the total Chern class with $\mathbb{Z}_2$ coefficients. Then $B\rho_n^* : w \mapsto \bar{c}$. In particular, as $H^\bullet(B\mathsf{U}(n); \mathbb{Z}_2)$ is generated in even degrees, so $B\rho_n^*(w_i) = 0$ for i odd.

Definition 3.3.3. Let $\rho_n : \mathsf{O}(n) \to \mathsf{U}(n)$ be the complexification morphism. Ignoring two-torsion, the k^{th} *Pontrjagin class* $p_k \in H^{4k}(B\mathsf{O}(n))$ is $p_k := B\rho_n^* c_k$.

Two-torsion in $H^\bullet(B\mathsf{O}(n))$ makes things more complicated than the unitary case. We consider first $B\mathsf{SO}(n)$. The Pontrjagin classes together with the Euler class e form a basis for the cohomology of $B\mathsf{SO}(n)$ up to 2-torsion. By considering the coefficient ring $\Lambda = \mathbb{Z}[\frac{1}{2}]$, we can ignore the torsion. That is

$$\begin{aligned} H^\bullet(B\mathsf{SO}(2m+1); \Lambda) &= \Lambda[p_1, \ldots, p_m] \\ H^\bullet(B\mathsf{SO}(2m); \Lambda) &= \Lambda[e, p_1, \ldots, p_m]/(e^2 - p_m). \end{aligned} \tag{3.44}$$

See, for example, [78, Theorem 15.9]. $B\mathsf{SO}(n) \to B\mathsf{O}(n)$ is a double cover so $B\mathsf{SO}(n)$ has a corresponding involution. We can identify $H^\bullet(B\mathsf{O}(n); \Lambda)$ with the fixed set in $H^\bullet(B\mathsf{SO}(n); \Lambda)$ of the morphism induced by the involution. The Pontrjagin classes are stable ie

$$FH^\bullet(B\mathsf{SO}(n); \Lambda) \cong \Lambda[p_1, p_2 \ldots]. \tag{3.45}$$

A complete description of the integral cohomology of $B\mathsf{O}(n)$ and $B\mathsf{SO}(n)$ is given in [16].

In the Bockstein sequence corresponding to the coefficient sequence $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}_2$ we have that $p_k \mapsto w_{2k}^2$. The realification $\rho_n : \mathsf{U}(n) \to \mathsf{O}(2n)$ induces the morphism $B\rho_n^* : H^\bullet(B\mathsf{O}(2n); \Lambda) \to H^\bullet(B\mathsf{U}(n); \Lambda)$

$$p_k \mapsto c_k^2 - 2c_{k-1}c_{k+1} + \cdots - (-1)^k 2c_1 c_{2k-1} + (-1)^k 2c_{2k}. \tag{3.46}$$

The structure of $H^\bullet(B\mathsf{Spin}(n))$ are also complicated, as is explained in [9, Section 1]. Some insight can be gained by considering the homotopy fibration $B\mathsf{Spin}(n) \to B\mathsf{SO}(n) \to K(\mathbb{Z}_2, 2)$ induced by the double cover $\rho_{2:1} : \mathsf{Spin}(n) \to \mathsf{SO}(n)$ $(n \geq 3)$. The Poincare polynomial of $B\mathsf{Spin}(n)$ are

$$Q_{2m}(t) = \left(\prod_{j=1}^{n-1}(1-t^{4j})\right)^{-1} \cdot (1-t^{2m})^{-1}, \quad Q_{2m+1}(t) = \left(\prod_{j=1}^{n-1}(1-t^{4j})\right)^{-1} \tag{3.47}$$

where $n = 2m$ or $2m+1$. (In other words the k^{th} Betti number of $B\mathsf{Spin}(n)$ corresponds to the coefficient of t^k.)

It is sufficient for our particular interest to restrict to cases where $n \geq 8$, for which we can be more explicit. We follow the exposition given in [44, Section 1]. Let $\varphi = B\rho_{2:1} : B\mathsf{Spin}(8) \to B\mathsf{SO}(8)$. According to Thomas [102], there exists stable characteristic classes $q_k \in H^{4k}(B\mathsf{Spin})$ such that via φ^*

$$p_1 \mapsto 2q_1, \quad p_2 \mapsto 2q_2 + q_1^2 \quad p_3 \mapsto q_3, \quad p_4 \mapsto 2q_4 + q_2^2. \tag{3.48}$$

Similarly, there exists $u_k \in H^k(\mathsf{Spin}(8); \mathbb{Z}_2)$ such that via φ^*

$$w_i \mapsto 0, \quad i \in \{2,3,5\}; \quad w_j \mapsto u_j, \quad j \in \{4,6,7,8\}. \tag{3.49}$$

Let β_2 be the Bockstein homomorphism corresponding to the coefficient sequence $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}_2$. Then via β_2

$$q_1 \mapsto u_4, \quad q_2 \mapsto u_8, \quad q_3 \mapsto u_6^2, \quad q_4 \mapsto u_{16}. \tag{3.50}$$

These classes map to characteristic classes of $B\mathsf{Spin}(8)$ which we denote by the same symbols.

Proposition 3.3.4 ([44, Theorem 2.1]). *(i) There exists $y \in H^\bullet(B\mathsf{Spin}(8))$ and $v \in H^\bullet(B\mathsf{Spin}(8); \mathbb{Z}_2)$ such that*

$$\begin{aligned} H^\bullet(B\mathsf{Spin}(8)) &= \mathbb{Z}[q_1, q_2, q_3, q_4, y] + \textit{2-torsion}, \\ H^\bullet(B\mathsf{Spin}(8); \mathbb{Z}_2) &= \mathbb{Z}_2[u_4, u_6, u_7, u_8, v]. \end{aligned} \tag{3.51}$$

(ii) Let $e \in H^8(\mathsf{SO}(8)$ be the Euler class. Then $\varphi^*(e) = 2y - q_2$ and $\beta_2(y) = v$.

(iii) Recall the triality automorphism $\lambda : \mathsf{Spin}(8) \to \mathsf{Spin}(8)$. The induced action λ^* on characteristic classes is determined by

$$q_1 \mapsto q_1, \quad y \mapsto y - q_2, \quad q_2 \mapsto 3y - 2q_2, \quad q_3 \mapsto q_3 + 2q_1(y - q_2), \tag{3.52}$$

and for $i \in \{4,6,7\}$

$$u_i \mapsto u_i, \quad v \mapsto v_i + u_8, \quad u_8 \mapsto v. \tag{3.53}$$

We refer to the q_i as *spin classes*. Only q_1 plays a role of any significance in our applications. By Proposition 3.3.4, we can compute the action of the spin representation $\Delta^7 : \mathsf{Spin}(7) \to \mathsf{SO}(8)$, since $\Delta^7 = \rho^8_{2:1} \circ \lambda \circ \iota_{\text{std}}$ in the notation of Figure 2-1.

Proposition 3.3.5 ([44, Theorem 3.2]). *(i) Let $\gamma = B\iota^7_{\text{std}}$. Identify $y, q_j \in H^\bullet(B\mathsf{Spin}(7))$ with $\gamma^* y$ and $\gamma^* q_j$ for $j = 1,3$. Likewise identify $v, u_j \in H^\bullet(B\mathsf{Spin}(7); \mathbb{Z}_2)$ with $\gamma^* v$ and $\gamma^* u_j$ for $j = 4,6,7$. Then*

$$\begin{aligned} H^\bullet(B\mathsf{Spin}(7)) &= \mathbb{Z}[q_1, q_3, y] + \textit{2-torsion}, \\ H^\bullet(B\mathsf{Spin}(7); \mathbb{Z}_2) &= \mathbb{Z}_2[u_4, u_6, u_7, v]. \end{aligned} \tag{3.54}$$

while $\gamma^* u_8 = 0$ and $\gamma^* q_2 = y$.

(ii) The morphism $\delta = (B\Delta^7)^*$ *has the following behaviour*

$$p_1 \mapsto 2q_1, \quad p_2 \mapsto -2y + q_1^2 \quad p_3 \mapsto q_3 - yq_1, \quad e \mapsto -y, \tag{3.55}$$

and for $j = 4, 6, 7$

$$w_j \mapsto u_j, \quad w_8 \mapsto v. \tag{3.56}$$

(iii) The kernel of δ *on integral cohomology is generated by* $4p_2 - p_1^2 + 8e$.

As G_2 is simply connected, the standard embedding $\mathsf{G}_2 \to \mathsf{SO}(7)$ lifts to an embedding $\rho : \mathsf{G}_2 \to \mathsf{Spin}(7)$. This induces a homotopy 7-sphere fibration

$$S^7 \to B\mathsf{G}_2 \to B\mathsf{Spin}(7). \tag{3.57}$$

The Thom-Gysin sequence implies that $H^k(B\mathsf{Spin}(7)) \to H^k(BG_2)$ is an isomorphism for $k \leq 7$. Therefore, characteristic classes will not distinguish between G_2-reductions on a given a spin 7-manifold.

3.4 Cohomology operations

Cohomology algebras exhibit additional structure beyond that of being just algebras. A cohomology operation is an encoding of some additional structure. See for example [95, Chapter 5 Section 9] for a more complete account of cohomology operations as well as the statements restated here. Our main aim here is to establish which classes will be characteristic elements of the intersection form as mentioned in Section 3.2.

Let G, G' be rings of coefficients, and $k, l \in \mathrm{N}_0$. A *cohomology operation of type* (k, G, l, G') is a natural transformation $\Theta : H^k(-; G) \to H^l(-; G')$. For example, the connecting morphism of a Bockstein sequence is a cohomology operation. We will consider only the Steenrod operators here. There exist analogous operators over ring of coefficients $\mathbb{Z}_p$, for prime p (See [6]).

Definition 3.4.1. The n^{th}-*Steenrod operator* (or *Steenrod square*) is a cohomology operation $\mathrm{Sq}^n : H^k(-; \mathbb{Z}_2) \to H^{k+n}(-; \mathbb{Z}_2)$. Together, they are completely characterized by the following conditions.

(i) Sq^n is natural (functorial) ie for $f : X \to Y$ continuous map $y \in H^k(Y; \mathbb{Z}_2)$, then $f^*(\mathrm{Sq}^n(y)) = \mathrm{Sq}^n(f^*(y))$.
(ii) Sq^0 is the identity.
(iii) For $x \in H^n(X; \mathbb{Z}_2)$, $\mathrm{Sq}^n(x) = x \smile x$.
(iv) For $n > \deg(x)$, $\mathrm{Sq}^n(x) = 0$.
(v) $\mathrm{Sq}^n(x \smile y) = \sum_{i+j=n} \mathrm{Sq}^i(x) \smile \mathrm{Sq}^j(y)$.

We define the *total Steenrod operator* to be $\mathrm{Sq} = \sum_{i=0} \mathrm{Sq}^i$.

Proposition 3.4.2. *We have the following basic properties of Steenrod operators.*

(i) Sq^1 *is the Bockstein homomorphism corresponding to the short exact sequence* $\mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2$.

(ii) *The Adem relations for* $0 < k < 2l$ *are*

$$\mathrm{Sq}^k \mathrm{Sq}^l = \sum_{0 \le j \le k/2} \binom{l-j-1}{k-2j} \mathrm{Sq}^{k+l-j} \mathrm{Sq}^j \tag{3.58}$$

In particular, for $B\mathsf{O}(n)$ we have the following *Wu formula* [71, Section 23.6]

$$Sq^i(w_j) = \sum_{k=0}^{i} \binom{j+k-i-1}{k} w_{i-k} w_{j+k}. \tag{3.59}$$

Definition 3.4.3. The *total Wu class* $v \in H(B\mathsf{O}; \mathbb{Z}_2)$ is completely characterized by $w = \mathrm{Sq}(v)$. The k^{th} *Wu class* $v_k \in H^k(B\mathsf{O}; \mathbb{Z}_2)$ is the component of v of degree k.

The definition of the Wu class is well defined in the sense that such a class exists and it is unique. The total Wu class has initial expansion

$$v = (w_1) + (w_2 + w_1^2) + (w_1 w_2) + (w_4 + w_3 w_1 + w_2^2 + w_1^4) + \dots \tag{3.60}$$

We have the following result for closed manifolds.

Proposition 3.4.4. *Let* X *be a closed* n*-manifold. For all* $x \in H^{n-k}(X; \mathbb{Z}_2)$, $\mathrm{Sq}^k(x) = v_k \smile x$. *In particular,* $v_k(X) = 0$ *for* $k > n/2$.

Definition 3.4.5. For a representation $\rho : H \to \mathsf{O}(n)$, the *degree* k *integral* ρ*-Wu structure* is the preimage via $H^\bullet(BH) \to H^\bullet(BH; \mathbb{Z}_2)$ of $B\rho^*(v_k) \in H^\bullet(BH; \mathbb{Z}_2)$. In the case that ρ is understood from the context and $n = 4k$, the *integral* H*-Wu structure* is the degree $2k$ integral ρ-Wu structure and is denoted by $\mathrm{Wu}(H) < H^{2k}(BH)$.

Corollary 3.4.6. *Let* $\rho : H \to \mathsf{O}(4k)$, *and suppose that* W *is a* $4k$*-manifold with boundary with* H*-structure. Then for* $w \in \mathrm{Wu}(H)$, $w(W) \in H^{2k}(W)$ *is a characteristic element of the intersection form of* W*. In particular,*

(i) *In the case of spin* 8*-manifolds, the first spin class* $q_1 \in \mathrm{Wu}(\mathsf{Spin}(8))$.

(ii) *In the case of almost complex* 8*-manifolds, then for any odd integers* a, b $ac_1^2 + bc_2 \in \mathrm{Wu}(\mathsf{U}(8))$.

3.5 Applications of the index theorem

The index theorem states that the analytic (or Fredholm) index of an elliptic operator on a compact manifold is equal to its topological index. The topological index is expressed as a rational linear combination of characteristic numbers. The analytic index is a priori an integer. Thus we have an integrality constraint on this rational linear combination. We are not particularly interested in the definition of the elliptic operator itself, and for the most part ignore it.

Some historical context: prior to the general index theorem (see [5]) special cases were known such as Gauss-Bonnet, Riemann-Roch, and the Hirzebruch signature theorem. In his concluding remarks of [101], Thom states that the signature is a *genus* on the rational oriented cobordism ring. That is, the signature induces a ring homomorphism $\Omega^\bullet_{\mathsf{SO}} \otimes \mathbb{Q} \to \mathbb{Q}$. A consequence of which is that the signature must be a rational linear combination of Pontrjagin classes. He computed the coefficients that appear in the 4-dimensional and 8-dimensional cases, the former previously conjectured by Wu. Hirzebruch [51] constructed the L-genus, which determines the rational coefficients for all dimensions. According to Sullivan [98], Hirzebruch produced this the afternoon of receiving Thom's paper.

Theorem 3.5.1. *Let X be a closed oriented $2k$-manifold with Euler class $e(X)$. Then*

$$\chi(X) = \langle e(X), [X] \rangle . \tag{3.61}$$

Recall that if X is an almost complex manifold then the Chern class $c_k(X) = e(X)$.

Theorem 3.5.2 (Novikov Additivity)**.** *Let W_i, $i = 0, 1$ be compact $4n$-manifolds with boundary and such that $M = \partial W_1 = -\partial W_0$. Then*

$$\sigma(W_0 \cup_M W_1) = \sigma(W_0) + \sigma(W_1) \tag{3.62}$$

(See [5, Proposition 7.2].)

Let $L \in \mathbb{Q}[[p_\bullet]]$ be the *L-genus.* It has initial expansion

$$\begin{aligned} L = 1 + \tfrac{1}{3}p_1 + \tfrac{1}{45}\left(7p_2 - p_1^2\right) + \tfrac{1}{945}\left(62p_3 - 13p_1p_2 + 2p_1^3\right) \\ + \tfrac{1}{14175}\left(381p_4 - 71p_1p_3 - 19p_2^2 + 22p_1^2p_2 - 3p_1^4\right) + \ldots \end{aligned} \tag{3.63}$$

See [51, Chapter II Theorem 8.2.2] for the following.

Theorem 3.5.3 (Hirzebruch Signature)**.** *Let X be a closed oriented manifold of dimension $4k$ with signature $\sigma(X)$. Then*

$$\sigma(X) = \langle L(X), [X] \rangle . \tag{3.64}$$

Let $\hat{A} \in \mathbb{Q}[[p_\bullet]]$ be the *$\hat{A}$-genus.* It has initial expansion

$$\begin{aligned} \hat{A} = 1 - \tfrac{1}{24}p_1 + \tfrac{1}{5760}(-4p_2 + 7p_1^2) + \tfrac{1}{967680}(-16p_3 + 44p_2p_1 - 31p_1^3) \\ + \tfrac{1}{464486400}(-192p_4 + 512p_3p_1 + 208p_2^2 - 904p_2p_1^2 + 381p_1^4) + \ldots \end{aligned} \tag{3.65}$$

See [68, Chapter IV Theorem 1.1].

Theorem 3.5.4 ($\hat{A}$-genus)**.** *Let X be a closed spin $4k$-manifold, and let D be the spin Dirac operator. Then*

$$\operatorname{Ind}(D) = \left\langle \hat{A}(X), [X] \right\rangle . \tag{3.66}$$

Moreover, if k is odd then $\operatorname{Ind}(D)$ *is even.*

Theorem 3.5.5. *Let X be an even dimensional closed manifold. Let $x \in H^2(X)$ be such that $x = w_2(X) \mod 2$. Then X has a spin^c structure with associated $\mathbb{C}$-line bundle L such that $c_1(L) = x$. Let D^+ be the associated spin^c Dirac operator. Then*

$$\operatorname{Ind}(D^+) = \left\langle \exp\left(\tfrac{x}{2}\right) \hat{A}(X), [X] \right\rangle . \tag{3.67}$$

(See [68, Chapter IV Theorem 1.3].) In the case that the spin^c structure is derived from an almost complex structure (see Section 2.5), then $c_1(L) = c_1(X)$. If $x = c_1(X)$, then $Td(X) = \exp\left(\frac{x}{2}\right)\hat{A}(X)$ is the *Todd genus*. The Todd genus has initial expansion

$$\begin{aligned} Td = 1 + \tfrac{1}{2}c_1 + \tfrac{1}{12}(c_2 + c_1^2) + \tfrac{1}{24}c_1c_2 \\ + \tfrac{1}{720}(-c_1^4 + 4c_2c_1^2 + 3c_2^2 + c_3c_1 - c_4) + \dots \end{aligned} \tag{3.68}$$

Theorem 3.5.6 (Twisted Dirac Operator)**.** *Let X be an even dimensional closed manifold. Let $x \in H^2(X)$ be such that $x = w_2(X) \mod 2$, and let E be a complex vector bundle on X. Then*

$$\left\langle \exp\left(\tfrac{x}{2}\right) ch(E)\hat{A}(X), [X] \right\rangle \tag{3.69}$$

is an integer, where $ch(E)$ is the total Chern character of E.

(See [51, Theorem 26.1.1].) Note that each of these can be considered to be special cases of the following generalization of Mayer [72] (see [51, Theorem 26.2.1]).

Theorem 3.5.7. *Let X be an even dimensional closed manifold. Let E be a $\mathsf{U}(l)$-bundle over X with total Chern character $ch(E)$. Let F be an $\mathsf{SO}(k)$-bundle over X with $k = 2s$ or $2s+1$. Let $x \in H^2(X)$ be such that $x = w_2(X) + w_2(F) \mod 2$. Suppose that y_i^2 are the formal Pontrjagin roots such that $p(F) = \prod_{i=1}^s (1 + y_i^2)$. Then*

$$\left\langle 2^s \exp\left(\tfrac{x}{2}\right) ch(E) \left(\textstyle\prod_{i=1}^s \cosh\left(\tfrac{y_i}{2}\right)\right) \hat{A}(X), [X] \right\rangle \tag{3.70}$$

is an integer.

Chapter 4

Bordism

In this chapter we review some aspects of obstruction theory, surgery theory, and bordism theory which are required in Chapter 5. Almost all the content can be found in the standard texts on the subject matter such as Stong [96] and Browder [15], but Propositions 4.3.4, 4.3.7 and 4.3.8 are new.

We have a loose dichotomy between problems involving structures on manifolds. Roughly speaking, this corresponds to whether one can sensibly perform surgery in the context under consideration. In Section 4.1, we discuss some results of obstruction theory. This does not require surgery techniques and is employed in Proposition 4.3.7 and 4.3.8.

The remaining sections discuss surgery and bordism theory which have been employed in classification problems of manifolds with incredible success. Some of the results have been included exclusively to fuel speculations discussed in Section 5.3.3.

4.1 Obstruction theory

In proving the existence of coboundaries with certain properties of the structure group, we require 'improving' an H-structure on some manifold X to a G-structure. Proposition 2.5.6 implies that this is equivalent to finding a section of the associated F-fibre bundle, where $F = H/G$. We employ a standard obstruction theory technique inducting extensions on skeleta.

Let (X, A) be a finite CW pair. Suppose that $E \to X$ is an F-fibre bundle over X. We will assume that the base space X is connected; that F is path-connected and $\pi_1(F)$ acts trivially on $\pi_n(F)$ for all n; and that $\pi_1(X)$ acts trivially on $\pi_n(F)$. Throughout this section we will assume that these conditions are always met.

Let $s \in \Gamma(E|_A)$ be a section of E over A. We attempt to extend s to X by induction on the skeleton. Suppose we have an extension $s_k \in \Gamma(E|_{X^k \cup A})$ of s to the k-skeleton X^k. For each $(k+1)$-cell $\Phi : (D^{k+1}, S^k) \to (X^{k+1}, X^k)$ not in A, we have $s_k \circ \Phi_{S^k} : S^k \to E$. As E trivializes over contractible space $\Phi(D^{k+1})$, we can fix a trivialization and treat $s_k \circ \Phi_{S^k} : S^k \to F$. This map depends on trivialization, but the homotopy class is independent of such a choice. Let $\omega_k : \pi_k(X^{k+1}, X^k) \to \pi_k(F)$ be the map sending the cell $[\Phi] \mapsto [s_k \circ \Phi_{S^k}]$.

Proposition 4.1.1 ([48, Section 3.3])**.** *In the notation above, ω_k is a $(k+1)$ cellular cochain that is a cocycle. The class $[\omega_k] \in H^{k+1}(X^{k+1}, X^k \cup A; \pi_k(F))$ vanishes if and only if the section s_{k-1} extends to a section s_{k+1}.*

Example 4.1.2 (Spin structure)*.* Let (X, A) be a finite CW pair with a principal $\mathsf{SO}(n)$ bundle $E \to X$ determining a map $(X, A) \to B\mathsf{SO}(n)$. Suppose $s : A \to B\mathsf{Spin}(n)$ is a lift via $B\mathsf{Spin}(n) \to B\mathsf{SO}(n)$. We have the fibration $K(\mathbb{Z}_2, 1) \to B\mathsf{Spin}(n) \to B\mathsf{SO}(n)$.

The characterizing property of the Eilenberg MacLane space $K(\mathbb{Z}_2, 1)$ is that it has a single nontrivial homotopy group $\pi_1(K(\mathbb{Z}_2, 1)) = \mathbb{Z}_2$. Thus the obstruction spaces $H^{k+1}(X^{k+1}, X^k \cup A; \pi_k(F))$ all vanish for $k \neq 1$. The first and only obstruction to extending s_0 to all X is then $\omega_2 = H^2(X^2, X^1 \cup A; \mathbb{Z}_2)$. In the case that $A = \varnothing$, then ω_2 corresponds to $w_2(E) \in H^2(X; \mathbb{Z}_2)$, the second Stiefel-Whitney class of E.

Proposition 4.1.3 ([48, Section 3.3])**.** *Let (X, A) be a finite CW pair. Suppose that $E \to X$ is an F-fibre bundle over X. We will assume that the base space X is connected; that F is path-connected and $\pi_1(F)$ acts trivially on $\pi_n(F)$ for all n; and that $\pi_1(X)$ acts trivially on $\pi_n(F)$.*

Suppose that $s \in \Gamma(E|_A)$ is a section of E over A. If F is $(k-1)$-connected, then the first nontrivial obstruction space is $H^{k+1}(X, A; \pi_k(F))$. Moreover, the obstruction class $c(E, s) \in H^{k+1}(X, A; \pi_k(F))$ of s is independent of choices.

In other words, the primary obstruction $c(E, s) \in H^{k+1}(X, A; \pi_k(F))$ is natural (ie functorial).

In the context of classifying spaces, we have the following setup. Suppose H is a topological group and, G is a subgroup of H. Let $F := H/G$, then we have the fibration $G \to H \to F$. Hence we have a fibration on classifying spaces $F \to BG \to BH$. Suppose G is the stabilizer of a vector of some representation $H \to \mathsf{SO}(k+1)$ such that H acts transitively on $F = S^k$. The Thom-Gysin sequence is

$$H^\bullet(BH) \to H^\bullet(BG) \to H^{\bullet - k}(BH) \xrightarrow{e\smile} H^{\bullet+1}(BH) \tag{4.1}$$

where e is the Euler class of the fibration.

Example 4.1.4 ($\mathsf{Spin}(7)$-structure)*.* Let (X, A) be a finite CW pair with a principal $\mathsf{Spin}(8)$ bundle $E \to X$ determining a map $(X, A) \to B\mathsf{Spin}(8)$. Suppose $s : A \to B\mathsf{Spin}(7)$ is a lift via $B\rho : B\mathsf{Spin}(7) \to B\mathsf{Spin}(8)$. We have the fibration $S^7 \to B\mathsf{Spin}(7) \to B\mathsf{Spin}(8)$. Thus the first nontrivial obstruction space is $H^8(X^8, X^7 \cup A; \mathbb{Z})$.

Suppose that A is trivial. For the standard inclusion $\rho : \mathsf{Spin}(7) \to \mathsf{Spin}(8)$, the obstruction class is the Euler class. If $\rho : \mathsf{Spin}(7) \to \mathsf{Spin}(8)$ is the positive spinor representation, then the obstruction class is the Euler class of the positive spinor bundle e_+, where $e_+ = \frac{1}{16}(4p_2 - p_1^2 + 8e)$, cf with Proposition 3.3.5.

Example 4.1.5 (G_2-structure)*.* Let (X, A) be a finite CW pair with a principal $\mathsf{Spin}(7)$ bundle $E \to X$ determining a map $(X, A) \to B\mathsf{Spin}(7)$. Suppose $s : A \to B\mathsf{G}_2$ is a lift via $B\rho : B\mathsf{G}_2 \to B\mathsf{Spin}(7)$. We have the fibration $S^7 \to B\mathsf{G}_2 \to B\mathsf{Spin}(7)$.

The first nontrivial obstruction space is $H^8(X^8, X^7 \cup A; \mathbb{Z})$. If X is a closed 7-manifold, then the extension of s to X is unobstructed.

We now consider the Euler class in the context of relative cohomology, as described by Šarafutdinov [91, Theorem 1.1]. Let (X, A) be a topological pair. Let $E \to X$ be a principal H bundle and via $\rho : H \to \mathsf{SO}(k+1)$ where H acts transitively on S^k. Let $E_V := \mathbb{R}^{k+1} \times_\rho E$ and $E_S := S^k \times_\rho E$ be the associated vector bundle and sphere bundle respectively.

Let $G := \mathrm{Stab}_H(v)$, the stabilizer of a unit vector $v \in S^k$. Suppose $s_A \in \Gamma(E_S|_A)$. We have a projection $p : (E_V, s_A(A)) \to (X, A)$, and embedding $j : (E_V, s_A(A)) \to (E_V, E_S)$. Let $\tau \in H^{k+1}(E_V, E_S)$ be the Thom (or fundamental) class of E_V. The Euler class relative to s is defined as $e(E_V, s) := ((p^*)^{-1} \circ j^*)(\tau)$. In the case that A is empty, then we have the absolute Euler class in the commonly understood sense and denoted $e(E_V)$.

The relative Euler class is the unique class satisfying the following axioms.

(i) Naturality: $f^*e(E_V, s) = e(f^*E_V, f^*s)$ for a morphism of CW-pairs $f : (Y, B) \to (X, A)$.

(ii) Multiplicative: $e(E_V \oplus E'_{V'}, s) = e(E_V, s)e(E'_{V'})$ for even dimensional oriented vector bundles E_V, and $E'_{V'}$, and section $s \in \Gamma(E_S|_A)$.

(iii) Normed: If $E_V = \mathcal{O}_{\mathbb{P}^1}(1)$ as a real vector bundle, and $X = S^2 = \mathbb{P}^1$, then $e(E_V) \in H^2(X)$ is the oriented generator.

Proposition 4.1.6. *With the notation and assumptions as above, the relative Euler class is the primary obstruction to extending $s_A \in \Gamma(E_S|_A)$ to X. That is*

$$c(E_S, s) = e(E_V, s) \tag{4.2}$$

For connected manifolds X and X' (either with or without boundary), if X and X' are oriented, then their connected sum $X\#X'$ is oriented. Moreover, if X and X' are spin manifolds, then $X\#X'$ is spin and is spin bordant to $X \sqcup X'$ (see for example [62, Lemma 2.1]). This is a special case of stable structures preserved by surgery as considered in Section 4.2. We have the following formula for spin 8-manifolds.

Proposition 4.1.7. *Let X, X' be closed spin 8-manifolds.*

$$e_+(X\#X') = e_+(X) + e_+(X') - 1. \tag{4.3}$$

Furthermore, if X, X' be are spin 8-manifolds with boundary

$$e_+(X\#X', s \sqcup s') = e_+(X, s) + e_+(X', s') - 1 \tag{4.4}$$

where s, s' are unit spinors of the boundaries of X and X' respectively.

Proof. Recall that $e(X\#X') = e(X) + e(X') - 2$, while $e_+ = \frac{1}{16}p_1^2 - \frac{1}{2}e - \frac{1}{4}p_2$. As the Pontrjagin classes are stable and behave additively under connect sums, the result follows. $\square$

Thus by taking connected sums, we can effectively kill off the relative Euler class to ensure the existence of, for example, a nowhere vanishing spinor.

4.2 Bordism and Surgery

The h-cobordism theorem encapsulates how bordism theory can be used in classification problems of manifolds. Surgery is the fundamental operation in differential topology, particularly in the context of normal B-bordisms. The theory of bordism and surgery are subtle and complicated theories, and we defer to standard texts on the subject for a comprehensive treatment.

Definition 4.2.1. $(W; M_0, M_1)$ is a *bordism* if W is a smooth compact oriented $(n+1)$-manifold, and ∂W is the disjoint union of two closed n-dimensional manifolds $-M_0$ and M_1.

Classification via bordism is typified by one of its initial incarnations in the celebrated h-cobordism theorem of Smale. Smale's result relies on the 'Whitney trick', but this only works in contexts where the dimension is sufficiently high. Thus the classification of manifolds is broadly partitioned by 'low' and 'high' dimensions. In the smooth context the Whitney trick fails in dimensions < 5 and in the topological context for dimensions < 4.

Definition 4.2.2. A bordism $(W; M_0, M_1)$ is an *h-bordism* if the inclusions $M_0, M_1 \to W$ are homotopy equivalences. An h-bordism $(W; M_0, M_1)$ is *trivial* if there exists a diffeomorphism

$$(W; M_0, M_1) \to M_0 \times (I; \{0\}, \{1\}). \tag{4.5}$$

Proposition 4.2.3 (h-Cobordism Theorem)**.** *Let $(W; M_0, M_1)$ be an h-bordism between simply connected n-manifolds M_1, M_0 with $n \geq 5$. Then $(W; M_0, M_1)$ is trivial.*

(See [93, Theorem 1.1].) There are analogous statements for topological and PL-manifolds. In some contexts we can also recover analogous results for manifolds with structure.

Definition 4.2.4. Let W be an oriented n-manifold, perhaps with boundary. Let $f : S^k \times D^{n-k} \to W$ be an orientation preserving embedding. The glueing

$$\mathrm{closure}(W \setminus f(S^k \times D^{n-k})) \cup_{S^k \times S^{n-k-1}} (D^{k+1} \times S^{n-k-1}) \tag{4.6}$$

can be canonically smoothed. A manifold W' diffeomorphic to (4.6) is *obtained from W by surgery via f*.

A surgery in effect exchanges a k-sphere with a $(n-k)$-sphere. There is an immediate connection between bordism and surgery.

Definition 4.2.5. Let M be a closed n-manifold and let $f : S^k \times D^{n-k} \to M$ be an orientation preserving embedding, as in Definition 4.2.4. The *trace* of the surgery is

$$W := (M \times I) \cup_f (D^{k+1} \times D^{n-k}) \tag{4.7}$$

where we consider $f : S^k \times D^{k+1} \to M \times \{1\}$.

The boundary of the trace is the disjoint union of $-M$ and the manifold M', where M' is obtained from M by surgery via f. In particular, $(W; M, M')$ is a bordism.

Proposition 4.2.6 ([74, Theorem 1])**.** *Two closed oriented manifolds M, M' are bordant if and only if there is a sequence $(M = M_0, \dots, M_k = M')$ such that M_{i+1} is obtained from M_i by surgery.*

Definition 4.2.7. Let $f_i : M_i \to X$, $i = 0, 1$ be two maps from n-manifolds M_i to a space X. A *bordism of maps* is a bordism $(W; M_0, M_1)$, together with a triple

$$(F; f_0, f_1) : (W; M_0, M_1) \to X \times (I; \{0\}, \{1\}). \tag{4.8}$$

f_0 and f_1 are *bordant* if such a bordism exists.

Two $\mathbb{R}$-vector bundles $E_0, E_1 \to X$ are *stably isomorphic* if there exists $n_0, n_1 \in \mathbb{N}_0$ such that $E_0 \oplus \mathbb{R}^{n_0}$ and $E_1 \oplus \mathbb{R}^{n_1}$ are isomorphic. A *stable vector bundle* is an isomorphism class of stably isomorphic vector bundles. We may refer to a stable vector bundle by a representative. Equivalently, it corresponds to a classifying map $f : X \to B\mathsf{O}$. Recall that the tangent bundle TM of an n-manifold M is determined up to homotopy by (and determines) a map $M \to B\mathsf{O}(n)$. An embedding $M \to \mathbb{R}^N$ determines a normal bundle $\nu(M)$. Any two normal bundles on M are stably isomorphic. The *stable normal bundle* on M is the stable bundle $\nu_M : M \to B\mathsf{O}$, and is characterized by the property that $TM \oplus \nu_M$ is stably trivial.

Let X be a space equipped with a stable vector bundle E. A *normal map* (M, f, ι) consists of a map $f : M \to X$ together with an isomorphism $\iota : f^*E \to \nu_M$ of stable bundles on M. Roughly speaking, typically we have a fibration $\gamma : X \to B\mathsf{O}$, and $E = \gamma^* E_{\mathsf{O}}$ is the pullback of the universal stable vector bundle. The map f is then a lift via γ. For example $X = B\mathsf{SO}$, $X = B\mathsf{Spin}$, $X = B\mathsf{Spin}^c$, $X = B\mathsf{U}$, are all standard cases in bordism theory. See [96, Chapter 2] for a proper account of such objects or see [66, Chapter 2] and the references therein.

Definition 4.2.8. Let $E \to X$ be a stable bundle over a space X. Let (W_i, f_i, ι_i), $i = 0, 1$ be normal maps $f_i : W_i \to X$. Then (W_0, f_0, ι_0) and (W_1, f_1, ι_1) are *normally bordant rel. boundary* provided that: $\partial W_0 = \partial W_1 (=: \partial W)$, $f_0|_{\partial W_0} = f_1|_{\partial W_1}$, $\iota_0|_{\partial W_0} = \iota_1|_{\partial W_1}$, and there exists a normal map (T, g, κ) to (X, E) such that $\partial T = W_0 \cup_{\partial W} W_1$, $g|_{\partial T} = f_0 \cup f_1$, and $\beta|_{\partial T} = \iota_0 \cup \iota_1$.

Normally bordant rel boundary implies normally bordant. The converse does not hold as, in general, it may not respect the boundary.

Proposition 4.2.9 ([66, Proposition 11.2])**.** *Let $E \to X$ be a stable bundle over a CW-complex X, assumed to have finite skeleta, and let W be an n-dimensional manifold perhaps with boundary ($n > 5$). Let (W, f, ι) be a normal map $f : W \to X$. Then (W, f, ι) is normally bordant rel. boundary to a normal map (W', f', ι') such that $f'_* : \pi_1(W') \xrightarrow{\cong} \pi_1(X)$, and*

$$f'_* : H_j(W'; \mathbb{Z}[\pi_1]) \to H_j(X; \mathbb{Z}[\pi_1]), \tag{4.9}$$

is an isomorphism for $j <$ $\mathrm{floor}(\frac{n}{2})$, and surjective for $j =$ $\mathrm{floor}(\frac{m}{2})$. In other words $f' : W' \to X$ is a $\mathrm{floor}(\frac{m}{2})$-equivalence.

Note that the homology groups in Proposition 4.2.9 are understood to have twisted coefficients. The moral of the Proposition is that given a normal bordism, surgery can kill off any class in $\ker(\pi_j(W) \to \pi_j(X))$. In the case that $\pi_j(M) \cong \pi_j(X)$, then W' is en route to becoming a normal h-bordism.

Surgery in the middle dimension is more delicate and complicated. There is an obstruction ϑ belonging to an obstruction space L. See [66, Section 12] for the rather involved explanation and definitions. In the case that $\pi_1(X) = 0$, and $n = 0 \mod 4$, then the obstruction space is $L^h_n(0) \cong \mathbb{Z}$ and $\vartheta(W, f, \iota)$ is the signature (see *loc. cit.*).

Theorem 4.2.10. *Let $E \to X$ be a stable bundle over a CW-complex X, assumed to have finite skeleta. Let $(W; M_0, M_1)$ be an n-dimensional bordism ($n > 5$) and that (W, f, ι) is a normal map $f : W \to X$. Suppose that $f|_{M_i}$ is a homotopy equivalence for $i = 0, 1$. Then W is normally bordant rel. boundary to an h-bordism if and only if the surgery obstruction*

$$\vartheta(W, f, \iota) \in L^h_n(\pi_1(W)) \tag{4.10}$$

This is (a version of) Wall's obstruction theorem. See [66, Theorem 14.6]. As mentioned, in simple cases the obstruction is the signature. In other contexts, the close correspondence with characteristic numbers remains.

The bordism group Ω^X_n consists of equivalence classes of normal maps (M, f, ι) where M is a closed n-manifold, $f : M \to X$, and two normal maps are equivalent if they are bordant. Stable bordism groups have been a staple object of study in bordism theory, with many of the groups having been computed.

In the context of $B\mathsf{O}$, $B\mathsf{SO}$ and $B\mathsf{Spin}$, there are immediate correspondences between a normal stable structure and a genuine tangential structure. For example, a normal stable spin structure on a manifold M corresponds to a spin structure in the sense of Definition 2.5.11. The case of $B\mathsf{Spin}^c$ is considered by Bunke [20, Section 3.3]. In this context, a normal Spin^c structure implies that there exists a tangential spin^c structure (see *loc. cit.*). In the context of $B\mathsf{U}$ or $B\mathsf{SU}$, then normal structures cannot be used to imply the existence of genuine tangential structures.

4.3 Bordism groups

We collect together some known results on bordism groups as well as several additional ones required in Chapter 5. For normal B-structures concerning the stable group G, the associated bordism group is denoted $\Omega^G_\bullet$. For a space X, $\Omega^G_\bullet(X)$ denotes the bordism group corresponding to the category in which each object W was additionally equipped with a map $W \to X$, defined up to homotopy, called a polarization and all morphisms respect polarizations. These are used to fix cohomology classes on $H^\bullet(M)$.

Proposition 4.3.1. *The oriented bordism groups Ω^{SO}_n for $n \leq 8$ are*

n	0	1	2	3	4	5	6	7	8
Ω^{SO}_n	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z} \oplus \mathbb{Z}$

See concluding remarks of Milnor Stasheff [78, Section 17].

Proposition 4.3.2. $\Omega_\bullet^{\mathsf{U}}$ *is a polynomial ring over* $\mathbb{Z}$ *with generators* $\{a_k : \deg(a_k) = 2k\}$.

See Novikov [85, Theorem 4].

Proposition 4.3.3. *The spin bordism groups* Ω_n^{Spin} *for* $n \leq 8$ *are*

n	0	1	2	3	4	5	6	7	8
Ω_n^{Spin}	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z} \oplus \mathbb{Z}$

See Milnor [77].

Our application requires that slightly stronger conditions are met by a coboundary. These can be subsumed by demonstrating that a coboundary exists which has a cohomology whose restriction map to the boundary is surjective in certain degrees. This is almost a consequence of the vanishing of the relevant bordism group of the form $\Omega_n^H(K)$, where K is an Eilenberg-MacLane space (See Section 2.4).

Proposition 4.3.4. *For any closed oriented* 7*-manifold* M *there exists an oriented* W *to* M *such that* $TH^4(W) = 0$ *and* $H^4(W) \to H^4(M)$ *is onto.*

Proof. By Proposition 4.3.1, $\Omega_7^{\mathsf{SO}} = 0$. Thus for an oriented 7-manifold M there exists an oriented coboundary W. By Proposition 4.2.9, we can perform surgery below the middle dimension. Thus we can assume that $\pi_k(W) \to \pi_k(B\mathsf{SO})$ is injective for $k < 4$. Recall that $\pi_k(B\mathsf{SO}) = \pi_{k-1}(SO)$ is $0, \mathbb{Z}_2, 0$ for $k = 1, 2, 3$ respectively. Either $\pi_2(W) = 0$, in which case W is 3-connected, or $\pi_2(W) = \mathbb{Z}_2$ and $W \to B\mathsf{SO}$ is a 3-equivalence.

By the Universal Coefficient Theorem $H_3(B\mathsf{SO}) \simeq FH^3(B\mathsf{SO}) \oplus TH^4(B\mathsf{SO})$, where $\simeq$ denotes a noncanonical equivalence. Note that $H_3(B\mathsf{SO}) = 0$. In the case that $W \to B\mathsf{SO}$ is a 3-equivalence so $H_3(W) = 0$.

In both cases $H_3(W) = 0$. Poincaré-Lefschetz duality implies $H_3(W) \cong H^5(W, M)$. Thus in the long exact sequence of cohomology associated to (W, M), $H^4(W) \to H^4(M)$ is onto. □

Proposition 4.3.5. *For any* $r, s \in \mathbb{N}_0$, $\Omega_7^{\mathsf{Spin}}(K(\mathbb{Z}^r, 2) \times K(\mathbb{Z}^s, 4))$ *is trivial.*

See [65, Theorem 6]. By standard arguments we can assume that $H_3(W) = 0$, and so for any polarized spin 7-manifold, there exists a polarized spin coboundary W such that $TH^4(W) = 0$ and $H^4(W) \to H^4(M)$ is onto.

We require results concerning bordism groups that are not of normal B-type. These do not seem to be available in the literature and so are proved here. These will be employed in Sections 5.2.5 and 5.3.1.

Proposition 4.3.6. *For any closed* 7*-manifold* M *with* $\mathsf{G}(2)$*-structure there exists a* $\mathsf{Spin}(7)$*-coboundary* W *to* M *such that* $TH^4(W) = 0$ *and* $H^4(W) \to H^4(M)$ *is onto.*

Proof. This is essentially the first part of the proof of Proposition 4.3.7. As M has a G_2-structure, it has a compatible spin structure. Proposition 4.3.3 implies that there exits a spin coboundary W', and by surgery we can assume 3-connected. In particular, $H^4(W')$ is free and $H^4(W') \to H^4(M)$ is onto. The obstruction of getting from a spin

8-manifold to an 8-manifold with $\mathsf{Spin}(7)$-structure is the relative Euler class of spinor bundle e_+. By Proposition 4.1.7, we can repeatedly connect sum with spin manifolds, for example, $S^4 \times S^4$ or T^8, to construct a $\mathsf{Spin}(7)$-coboundary W to M. The connect sum will not introduce torsion into the forth cohomology, or change the surjective nature of the restriction map. $\square$

Proposition 4.3.7. *For any closed* 7*-manifold* M *with* $\mathsf{SU}(3)$*-structure there exists an* $\mathsf{SU}(4)$*-coboundary* W *to* M *such that* $TH^4(W) = 0$ *and* $H^4(W) \to H^4(M)$ *is onto.*

Proof. An $\mathsf{SU}(3)$-structure on a 7-manifold determines a spin structure and a pair of orthonormal unit spinors. That is, the $\mathsf{SU}(3)$-structure is precisely the reduction of the spin structure defined as the stabilizer of the orthonormal pair of spinors.

By Proposition 4.3.3, the spin bordism group $\Omega^7_{\mathsf{Spin}} = 0$. Thus a spin coboundary of the spin manifold M exists. Moreover, by Proposition 4.2.9, together with Hurewicz theorem, we can choose that the coboundary W is 3-connected. In particular, $H_3(W) = 0$ implies that $H^4(W)$ is torsion free. By Poincaré-Lefschetz duality $H^5(W, M) \cong H_3(W) = 0$. The long exact sequence of cohomology given by the pair (W, M), then implies $H^4(W) \to H^4(M)$ is onto.

Let $s_i \in \Gamma(S_M)$ be the unit spinors determined by the $\mathsf{SU}(3)$-structure. It remains to extend the unit spinors $s_i \in \Gamma(S_M)$ to the interior of W. By Proposition 4.1.6, the primary obstruction to extending s_1 to skeleta of W is the relative Euler class $e(S_W, s_1) \in H^8(W, M; \pi_7(S^7)) \cong \mathbb{Z}$. Note that $e_+(S^4 \times S^4) = 2$, and $e_+(\mathbb{HP}^2) = 0$. By Proposition 4.1.7, taking successive connected sums of W with one of these, we may assume that $e_+(W, s_1) = 0$ and so assume that W admits a nowhere vanishing spinor field extending s_1. Note that W will remain 3-connected.

Now we consider extending s_2 such that it remains perpendicular to the extension of s_1. This is equivalent to extending the section of an S^6-fibre bundle. W is simply connected, so $H^7(W, M; \pi_6(S^6)) = 0$. Thus the primary obstruction of extending s_2 vanishes, and we have an extension of s_2 to W^7. The secondary obstruction space is $H^8(W, M; \pi_7(S^6)) \cong \mathbb{Z}_2$. We consider extending s_2 at the level of cells. There are precisely two possible homotopy classes: s_2 over ∂B^8_j corresponds to $0 \in \pi_7(S^6)$, and s_2 corresponds to $1 \in \pi_7(S^6)$. In the former case, we can extend s_2 over B^8_j, while in the latter we cannot.

If for each 8-cell we can extend s_2 from the boundary S^7 to the interior, then we are done. Assume there exists a cell $(B^8, S^7) \to (W^8, W^7)$ for which this is not possible. We now construct an 8-manifold with which we can replace each offending 8-cell and over the resulting manifold can extend s_2.

The manifold $\mathbb{HP}^2$ is spin and admits a $\mathsf{Spin}(7)$-structure but not an $\mathsf{SU}(4)$-structure [14, Theorem 5.7]. Moreover, puncturing at some $p \in \mathbb{HP}^2$ to obtain $\mathbb{HP}^2 \setminus \{p\}$ the $\mathsf{Spin}(7)$-structure does admit an $\mathsf{SU}(4)$-reduction. Fix an $\mathsf{SU}(4)$-structure $E \to \mathbb{HP}^2 \setminus \{p\}$. The restriction of E to an S^7 boundary of a neighbourhood of $p \in \mathbb{HP}^2$ is equivalent to that of a troublesome 8-cell.

The Euler characteristic $\chi(\mathbb{HP}^2) = 3$, while introducing three punctures results in $\chi(\mathbb{HP}^2 \setminus \{p, p', p''\}) = 0$. Thus on the triply punctured $\mathbb{HP}^2$ there exists a nowhere vanishing vector field assumed normal on the boundary. Fix a nowhere vanishing unit vector field and let $i : T(\mathbb{HP}^2 \setminus \{p, p', p''\}) \to T(\mathbb{HP}^2 \setminus \{p, p', p''\})$ be the orientation reversing bundle involution defined by reflecting along the vector field. We restrict E to $\mathbb{HP}^2 \setminus \{p, p', p''\}$. In a B^8 neighbourhood of p' and p'' we fix a trivialization of E. Let E' be the $\mathsf{SU}(4)$-structure induced via i. T^8 has the flat $\mathsf{SU}(4)$-structure, and we fix a B^8 subset. Thus we can glue together an $\mathsf{SU}(4)$-structure on $(-\mathbb{HP}^2)\#T^8\#T^8 \setminus \{p\}$ agreeing with E' on restriction. Let $X = (-\mathbb{HP}^2)\#T^8\#T^8$. Then $X \setminus \{p\}$ is an 8-manifold with boundary S^7 that has an $\mathsf{SU}(4)$-structure which on restriction to the boundary agrees with that of the offending 8-cells. Thus we replace each such 8-cell with $X \setminus \{p\}$.

Let W' be the resulting manifold after this process. We retain the property that $FH^4(W') \to H^4(M)$ is onto. □

Proposition 4.3.8. *For any closed 7-manifold M with $\mathsf{U}(3)$-structure there exists a $\mathsf{U}(4)$-coboundary W to M such that $TH^4(W) = 0$ and $H^4(W) \to H^4(M)$ is onto.*

Proof. The $\mathsf{U}(3)$-structure on M defines a spin^c structure on M. By Proposition 4.3.2, the stable unitary bordism group $\Omega^7_{\mathsf{U}} = 0$. Thus there exists a stable unitary coboundary W to M.

Note that a quintic in $Q \subset \mathbb{CP}^5$ has primitive c_1. Thus $Q \to B\mathsf{U}$ is a 3-equivalence. By taking a connect sum of W with Q we can assume that $\pi_k(W) \to \pi_k(B\mathsf{U})$ is a onto for $k < 4$. By Proposition 4.2.9, we may assume that this $W \to B\mathsf{U}$ is a 3-equivalence. In particular, $H_3(W) = 0$. $H_3(W) = 0$ implies that $H^4(W)$ is torsion free.

The unitary structure on W defines a $\mathsf{Spin}^c(8+N)$-structure on $TW \oplus \mathbb{R}^N$. Restricting to frames of TW, we recover a spin^c structure on TW. Moreover, W is a spin^c-coboundary to spin^c manifold M.

We improve the spin^c structure on W to a $\mathsf{U}(4)$-structure agreeing with the $\mathsf{U}(3)$-structure on the boundary. As

$$\mathsf{Spin}^c(8)/\mathsf{U}(4) \cong \mathsf{Spin}(8)/\mathsf{SU}(4) \cong S^7 \times S^6 \tag{4.11}$$

the proof proceeds identically to Proposition 4.3.7. □

In the proofs above, we construct the desired special unitary and unitary structures via spin and spin^c coboundaries respectively. This is because in the case of spin and spin^c the normal B-structure restricts to a genuine spin and spin^c structure respectively. The analogous logic does not hold for $\mathsf{U}(4)$ and $\mathsf{SU}(4)$.

Chapter 5

Invariants

This chapter draws together the main themes discussed in those preceding it. We describe a framework for defining invariants for manifolds with structure via coboundaries. The main principles here are not new and have been used many times previously as we shall see in the examples considered. We present a systematic approach applicable to many contexts.

In Section 5.2, we recast some previously defined boundary defect invariants in terms of the framework. Namely, Milnor's λ invariant, the Eells-Kuiper invariant and its generalization by Crowley and Nordström; the Kreck-Stolz invariants, the generalization by Hepworth and later by Crowley and Nordström; and the invariants for G_2-structures of Crowley and Nordström. In some cases, this presentation provides us with a transparent manner in which to further generalize the invariants. In Section 5.2.5, we present a new invariant for almost contact manifolds. In the interests of clarity and succinctness, we restrict attention to 7-manifolds. The framework allows for completely analogous treatment of other dimensions.

5.1 A framework for boundary defect invariants

We describe a framework for defining boundary defect invariants for closed manifolds with G-structure, where G is a boundary group (Definition 2.6.2).

5.1.1 Characteristic numbers

Let M be a closed n-manifold with G-structure, and let $\varphi_M : M \to BG$ be the classifying map. Let $\rho : G \to H$ be a Lie group homomorphism to H, a coboundary group to G (Definition 2.6.2). The concatenation $\widetilde{\varphi}_M := \varphi_M^* \circ B\rho^* : H^\bullet(BH) \to H^\bullet(M)$, defines the H-characteristic classes on M. We fix some submodule $\mathfrak{C} < H^\bullet(BH)$ from which we aim to derive characteristic numbers.

For a space X equipped with a map $\varphi_X : X \to H^\bullet(BH)$, let $\mathfrak{C}_X := \varphi_X^*(\mathfrak{C})$ denote the submodule of characteristic classes on X corresponding to $\mathfrak{C}$. We make a number of additional assumptions to extract the most utility from this process.

Definition 5.1.1. Let M be a closed n-manifold with G-structure, and H a coboundary group to G. Let $\mathfrak{C} < H^\bullet(BH)$ be a submodule of H-characteristic classes. Let $s : FH^\bullet(M) \to H^\bullet(M)$ be a section. Let W be an H-coboundary to M with $\alpha : H^\bullet(W) \to H^\bullet(M)$.

Then W is a *coboundary to* (M, s) *over* $\mathfrak{C}$ provided that there exists a $C < H^\bullet(W)$ such that

(i) $\mathrm{ps}(\mathrm{im}(F|_{\mathfrak{C}_M})) < \mathrm{im}(F \circ \alpha|_C)$,

(ii) The set $R(C, s)$ is nonempty.

(Recall the primitive supermodule ps from Definition 3.2.2.)

We assume for the rest of this section that W is a coboundary to (M, s) over $\mathfrak{C}$, and $C < H^\bullet(W)$ is as in Definition 5.1.1. The submodule C was accommodated in our treatment of products on compact oriented manifolds with boundary in Section 3.1. We inherit the notion we introduced there.

We wish to extend the notion of the shearing submodule to $\mathfrak{C}$. Let $q_m := FH^\bullet(M) \to FH^\bullet(M) \otimes \mathbb{Z}/m\mathbb{Z}$, and define

$$\mathfrak{C}_0 := \ker(F \circ \widetilde{\varphi}_M), \qquad \mathfrak{C}_m := \ker(q_m \circ F \circ \widetilde{\varphi}_M) \tag{5.1}$$

Then

$$\begin{aligned} \mathrm{Sh}(\widetilde{\varphi}_M)_0 &:= \left\{ \Phi \in \mathrm{Hom}([\mathcal{P}^{\geq 2}\mathfrak{C}]^{n+1}, \mathbb{Z}) \colon \Phi|_{[\mathcal{P}^{\geq 2}\mathfrak{C}_0]^{n+1}} = 0 \right\} \\ \mathrm{Sh}(\widetilde{\varphi}_M)_m &:= \left\{ \Phi \in \mathrm{Hom}([\mathcal{P}^{\geq 2}\mathfrak{C}]^{n+1}, \mathbb{Z}) \colon \Phi|_{[\mathcal{P}^{\geq 2}\mathfrak{C}_m]^{n+1}} = 0 \mod m \right\} \\ \mathrm{Sh}(\widetilde{\varphi}_M)^2_{2m} &:= \left\{ \Phi \in \mathrm{Hom}([\mathcal{P}^{\geq 2}\mathfrak{C}]^{n+1}, \mathbb{Z}) \colon \Phi|_{\widehat{\mathcal{P}}^2[\mathfrak{C}_{2m}]^{n'}} = 0 \mod 4m \right\} \end{aligned} \tag{5.2}$$

The shearing submodule associate to $\widetilde{\varphi}_M$ is

$$\mathrm{Sh}(\widetilde{\varphi}_M) := \mathrm{Sh}(\widetilde{\varphi}_M)_0 \cap \left(\bigcap_{m \in \mathbb{N}} \mathrm{Sh}(\widetilde{\varphi}_M)_m \right) \cap \left(\bigcap_{m \in \mathbb{N}} \mathrm{Sh}(\widetilde{\varphi}_M)^2_{2m} \right) \tag{5.3}$$

A cup product $\smile_\beta \in \mathrm{Hom}([\mathcal{P}^{\geq 2}C]^{n+1}, \mathbb{Q})$ pulls back to $(\varphi_W^*)^* \smile_\beta \in \mathrm{Hom}([\mathcal{P}^{\geq 2}\mathfrak{C}]^{n+1}, \mathbb{Q})$. The results of Section 3.1 claim that this product modulo $(\varphi_W^*)^*(\mathrm{Sh}(C))$ is independent of choice of $\beta \in R(C, s)$. Furthermore, the stipulation that W is a coboundary over $\mathfrak{C}$ implies that the pullback $(\varphi_W^*)^*(\mathrm{Sh}(C)) = \mathrm{Sh}(\widetilde{\varphi}_M)$.

The 'characteristic numbers' obtained via W clearly have a dependency on the choice of W. For example, a disjoint union with any closed oriented $(n+1)$-dimensional manifold with H-structure will also be a coboundary to M over $\mathfrak{C}$, and the resulting characteristic numbers will, in general, be different. For a closed oriented $(n+1)$ manifold X with H-structure described by classifying map $\varphi_X : X \to BH$ and cup product structure $\smile_X$, we get $(\varphi_X^*)^*(\smile_X) \in \mathrm{Hom}([\mathcal{P}^{\geq 2}\mathfrak{C}]^{n+1}, \mathbb{Z})$. Recall the notion of structure reversing map ι_H introduced in Section 2.6. We will assume that $\iota_H(M)$ has an H-coboundary over $\mathfrak{C}$.

Proposition 5.1.2. *In the notation above, let W_j, $j = 1, 2$, be coboundaries to (M, s) over $\mathfrak{C}$. Assume that $(\iota_H(M), (\iota_H^{-1})^* s)$ also admits a coboundary over $\mathfrak{C}$.*

Suppose that $\mathrm{Cl}(\mathfrak{C}) < \mathrm{Hom}([\mathcal{P}^{\geq 2}\mathfrak{C}]^{n+1}, \mathbb{Z})$ *is a submodule such that if* X *is a closed oriented manifold with* H*-structure then* $(\varphi_X^*)^*(\smile_X) \in \mathrm{Cl}(\mathfrak{C})$.

Then for $\beta_j \in R(C_j, s)$

$$(\varphi_{W_1}^*)^*(\smile_{\beta_1}) - (\varphi_{W_2}^*)^*(\smile_{\beta_2}) \in \mathrm{Cl}(\mathfrak{C}) + \mathrm{Sh}(\widetilde{\varphi}_M) \tag{5.4}$$

Proof. Let W' be an H coboundary to $(M', s') := (\iota_H(M), (\iota_H^{-1})^* s)$ over $\mathfrak{C}$, $\alpha' : H^\bullet(W') \to H^\bullet(M')$ and $\beta' \in R(C', s')$. We treat β' as a function on $FH^\bullet(M)$, via ι_H^*. We can construct two closed manifolds with H-structure $X_i = W_i \cup_M W'$.

As in the proofs of Section 3.1, it is sufficient to consider the product on $u \in [\mathcal{P}^{\geq 2}\mathfrak{C}]^{n+1}$ of monomial form. We will consider $u \in [\mathcal{P}^{\geq 2}\mathfrak{C}_m]^{n+1}$. Let $u = u_1 \cdot \ldots \cdot u_k$, $u_i \in \mathfrak{C}$. For a space Y with classifying map $\varphi_Y : Y \to BH$, let u_i^Y denote $\varphi_Y^* u_i \in H^\bullet(Y)$.

By assumption, there exists $\overline{u}_i^M \in FH^\bullet(M)$ such that $m\overline{u}_i^M = F(u_i^M)$. Let $\overline{u}_i^{W_j} := \beta_j(\overline{u}_i^M)$, and $\overline{u}_i^{W'} := -\beta'(\overline{u}_i^M)$. By insisting that β_j and β' are compatible with s, it follows that

$$(\overline{u}_i^{W_j}, \overline{u}_i^{W'}) \mapsto 0 \in H^\bullet(M) \tag{5.5}$$

Thus we can lift this class to $\overline{u}_i^{X_j}$. Then

$$u_i^{X_j} - m\overline{u}_i^{X_j} \mapsto (u_i^{W_j} - m\overline{u}_i^{W_j}, u_i^{W'} - m\overline{u}_i^{W'}) \tag{5.6}$$

Then

$$\begin{aligned} \smile_{X_j} (u^{X_j}) &= (u_1^{X_j} - m\overline{u}_1^{X_j}) \smile_{X_j} \ldots \smile_{X_j} (u_k^{X_j} - m\overline{u}_k^{X_j}) - m(\ldots) \\ &= \smile_{\beta_j} (u^{W_j}) + \smile_{\beta'} (u^{W'}) - m(\ldots) \end{aligned} \tag{5.7}$$

where $m(\ldots)$ is the collection of all the remaining products. Akin to the proofs of Section 3.1, we factor out m from each of these remaining terms.

$$\begin{aligned} ((\varphi_{W_1}^*)^*(\smile_{\beta_1}) - (\varphi_{W_2}^*)^*(\smile_{\beta_2}))(u) &= \smile_{\beta_1} (u^{W_1}) - \smile_{\beta_2} (u^{W_2}) \\ &= \smile_{X_1} (u^{X_1}) - \smile_{X_2} (u^{X_2}) - m(\ldots) \end{aligned} \tag{5.8}$$

u was an arbitrary monomial, and so

$$((\varphi_{W_1}^*)^*(\smile_{\beta_1}) - (\varphi_{W_2}^*)^*(\smile_{\beta_2})) \in \mathrm{Cl}(\mathfrak{C}) + \bigcap_m \mathrm{Sh}(\widetilde{\varphi}_M)_m \tag{5.9}$$

The cases of $\mathrm{Sh}(\widetilde{\varphi}_M)_0$, and $\mathrm{Sh}(\widetilde{\varphi}_M)^2_{2m}$ are logically analogous. The result follows. □

Corollary 5.1.3. *Let* W *be a coboundary of* (M, s) *over* $\mathfrak{C}$*, and* $\beta \in R(C, s)$*. Then*

$$[(\varphi_W^*)^*(\smile_\beta)] \in \mathrm{Hom}([\mathcal{P}^{\geq 2}\mathfrak{C}]^{n+1}, \mathbb{Q})/(\mathrm{Cl}(\mathfrak{C}) + \mathrm{Sh}(\widetilde{\varphi})) \tag{5.10}$$

is independent of choice of β *and of coboundary* W*, and depends only on* (M, s).

5.1.2 Maps that are additive under glueings

We aim to extend the map of (5.10). Let ϑ be a $\mathbb{Z}$-valued function on compact $(n+1)$-manifolds with boundary with H-structure. ϑ is *additive under glueing* if for any closed manifold $X = W_0 \cup W_1$ with H-structure formed by glueing the boundaries of $(W_0, \iota_H(M))$ and (W_1, M), then

$$\vartheta(X) = \vartheta(W_0) + \vartheta(W_1). \tag{5.11}$$

For example, the Euler characteristic χ is additive under glueing in even dimensions. (In odd dimensions this is trivially the case, since it uniformly vanishes.) This is particularly useful in cases of almost complex structures (cf Theorem 3.5.1). Novikov Additivity (Theorem 3.5.2) states that the signature is also additive under glueings. Both of these appear in our examples.

Suppose that ϑ is $\mathbb{Z}^l$-valued function on compact oriented $(n+1)$-manifolds with boundary with H-structure that is additive under glueings. As in (5.10), we wish to find a submodule of the codomain $\mathbb{Z}^l$ by which we can quotient to recover an invariant of M. For us, ϑ will not depend on a choice of $\beta \in R(C, s)$, and so the role of the shearing module is vacuous.

Let $\mathrm{Cl}(\vartheta) < \mathbb{Z}^l$ be the submodule generated by the image of closed manifolds under ϑ. Again, it is sufficient only that $\mathrm{Cl}(\vartheta)$ is a submodule containing the image of closed manifolds. Nonetheless, in our examples we demonstrate the sharpness of our $\mathrm{Cl}(\vartheta)$. Then if W is an H-coboundary to M $[\vartheta(W)] \in \mathbb{Z}^l/\mathrm{Cl}(\vartheta)$ is independent of choice W, and depends only on M.

We combine (5.10) and ϑ. In applications it may be convenient to specify a submodule $P < [\mathcal{P}^{\geq 2}\mathfrak{C}]^{n+1}$ on which to focus our attention and consider only the characteristic numbers in $P^\vee_{\mathbb{Q}} := P^\vee \otimes \mathbb{Q}$. We define $\Phi(W, \beta) := ((\varphi_W^*)^*(\smile_\beta), \vartheta(W)) \in (P^\vee_{\mathbb{Q}} \oplus \mathbb{Z}^l)$.

Corollary 5.1.4. *We adopt the notation of Corollary 5.1.3. Let W be a coboundary of (M, s) over $\mathfrak{C}$, and $\beta \in R(C, s)$. Let $P < [\mathcal{P}^{\geq 2}\mathfrak{C}]^{n+1}$, and ϑ be additive under glueings on compact $(n+1)$ manifolds with boundary with H-structure. Let $\mathrm{Cl}(P, \vartheta) < P^\vee_{\mathbb{Q}} \oplus \mathbb{Z}^l$ be the submodule generated by the image of closed manifolds. Then*

$$[((\varphi_W^*)^*(\smile_\beta), \vartheta(W))] \in (P^\vee_{\mathbb{Q}} \oplus \mathbb{Z}^l)/(\mathrm{Cl}(P, \vartheta) + \mathrm{Sh}(\widetilde{\varphi})) \tag{5.12}$$

is independent of choice of coboundary W, and depends only on (M, s).

5.1.3 A further refinement

In Section 3.1, we noted that in the absence of torsion on the boundary the characteristic numbers are integer valued. In the presence of torsion, we have some control over the denominators of the products. In the case that $n = (4k-1)$, whether M is free in the middle dimension or not, we may be able to achieve more precise invariants.

In Section 3.2, we discuss how a suitable coboundary determines the torsion linking form. Corollary 3.4.6 implies that certain characteristic classes, namely integral Wu classes (Definition 3.4.5), will always be characteristic elements of the intersection form or the coboundary.

Let M be a closed oriented $(4k-1)$-manifold with G-structure. Fix a section $s : FH^{2k}(M) \to H^{2k}(M)$. Let H be a coboundary group to G, and $c \in \mathrm{Wu}(H)$ be an integral Wu class for H-structured $4k$-manifolds. Assume that there exists an H-coboundary W to M such that $TH^{2k}(W) = 0$ and $H^{2k}(W) \to H^{2k}(M)$ is onto. Proposition 3.2.8 implies that we can compute the quadratic refinements of b_M associated to $c \in \mathrm{Wu}(H)$ via the intersection form of W.

Let $c \in \mathrm{Wu}(H)$ and $\beta \in R(C, s)$. For $x \in H^{2k}(W)$

$$2q_{c,s}(\alpha(x)) = -(x \smile_\beta x) - (c \smile_\beta x) \in \mathbb{Q}/2\mathbb{Z} \tag{5.13}$$

In addition, Proposition 3.2.6 implies there is a constraint between the signature of the intersection form, the integral Wu class, and the Arf invariant of the associated quadratic refinement.

Assume that ϑ includes the signature σ and $\mathrm{Wu}(H)$ is nonempty. The constraints above motivate the definition of the following object. We define the *q-refined module*

$$\begin{aligned} \mathrm{Qr}(P, \vartheta) := \{\eta \in P^\vee \oplus \mathbb{Z}^l \colon & \forall c \in \mathrm{Wu}(H), \quad \eta(c^{\cdot 2}) + \eta(\sigma) = 0 \mod 8, \\ & \forall x \in H^{2k}(BH), \quad x^{\cdot 2}, c \cdot x \in P, \\ & \eta(x^{\cdot 2}) + \eta(c \cdot x) = 0 \mod 2\}. \end{aligned} \tag{5.14}$$

For a suitable H-coboundary W to M, the image of $\Phi(W, \beta)$ must then lie in a $\mathrm{Qr}(P, \vartheta)$-coset in the additive group $P^\vee_{\mathbb{Q}} \oplus \mathbb{Z}^l$. Moreover, the coset is completely determined by the set of extended quadratic refinements

$$\mathbf{q}_{\mathrm{Wu}(H),s} := \{(H^{2k}(M), q_{c,s}, c(M), s) \colon c \in \mathrm{Wu}(H)\}. \tag{5.15}$$

In particular, the coset is not dependent on the choice of W. Let $\mathrm{Qr}(P, \vartheta)[\mathbf{q}_{\mathrm{Wu}(H),s}]$, denote the corresponding coset.

Let us introduce the shorthand to our cumbersome notation

$$\mathrm{Denom}(P, \vartheta, \widetilde{\varphi}) := ((\mathrm{Cl}(P, \vartheta) + \mathrm{Sh}(\widetilde{\varphi})) \cap \mathrm{Qr}(P, \vartheta)) \tag{5.16}$$

Following the logic of Corollary 3.2.9, we can work with coboundaries that do not have that $H^{2k}(W) \to H^{2k}(M)$ is surjective. It is sufficient that they are over $\mathfrak{C}$.

Corollary 5.1.5. *We adopt the notation of Corollary 5.1.4, and the above discussion. Suppose that $n = 4k-1$. Let W be a coboundary of M over $\mathfrak{C}$. Let s be a partial section of $H^\bullet(M) \to FH^\bullet(F)$ defined on $F \circ \widetilde{\varphi}(\mathfrak{C})$.*

Then for $\beta \in R(C, s)$

$$\overline{\Phi}(W, \beta) \in \mathrm{Qr}(P, \vartheta)[\mathbf{q}_{\mathrm{Wu}(H),s}]/\mathrm{Denom}(P, \vartheta, \mathfrak{C}_M) \tag{5.17}$$

is independent of choice of (W, β) and depends only on (M, s).

5.1.4 Additional remarks

Let us summarise how the previous chapters have prepared us for applying this result in examples. Section 2.6 defines the notion of boundary and coboundary so that classifying maps are homotopy invariant. Section 3.3 describes the characteristic classes in $H^\bullet(BH)$ from which we can pick a $\mathfrak{C}$. Moreover, Corollary 3.4.6 dictates which classes are integral Wu classes. Chapter 4, particularly the results of the concluding Section 4.3, assists in justifying the assumption that certain coboundaries exist. Section 3.5 helps in determining a submodule $\mathrm{Cl}(\mathfrak{C})$ that has the desired property. Together with some direct calculations (e.g those in Appendix A.1), we can show that a chosen $\mathrm{Cl}(\mathfrak{C})$ is sharp. Proposition 3.1.10, assists in computing the shearing submodule.

$\mathrm{Sh}(\widetilde{\varphi}_M)$ depends only $\mathfrak{C} \to FH^\bullet(M)$ as a morphism of $\mathbb{Z}$ modules. That is, for a morphism $\delta : \mathfrak{C} \to D$ to a finitely generated free $\mathbb{Z}$-module D, we can define the associated shearing module $\mathrm{Sh}(\delta)$. If $\gamma_M : D \to C$ is an isomorphism of $\mathbb{Z}$-modules such that $\widetilde{\varphi} = \gamma_M \circ \delta$ then $\mathrm{Sh}(\delta) = \mathrm{Sh}(\widetilde{\varphi}_M)$.

δ is a $\mathfrak{C}$-*model* provided that $\delta(\mathfrak{C}) < D$ is of full rank in D. Let $\delta' : \mathfrak{C} \to D'$ be a $\mathfrak{C}$-model. $\gamma : D \to D'$ is a $\mathfrak{C}$-*model isomorphism* if it is onto and $\delta' = \gamma \circ \delta$. Thus the shearing module depends on the $\mathfrak{C}$-model up to this equivalence.

We may consider the $\mathfrak{C}$-model, $\mathfrak{C}_M$, of M to be the $\mathfrak{C}$-*primary invariant* of M. The valid codomain of (5.17) depends on the $\mathfrak{C}$-primary invariant. Thus, we may consider (5.17) to be a *secondary invariant* of M.

If $\delta' : \mathfrak{C} \to D'$ factors through δ, then $\mathrm{Sh}(\delta') < \mathrm{Sh}(\delta)$. In particular, $\delta_0 : \mathfrak{C} \to \{0\}$ factors through any δ, and $\mathrm{Sh}(\delta_0) = 0$. We can define an invariant for the *trivial* $\mathfrak{C}$-*model* δ_0, and through which the invariant for all other $\mathfrak{C}$-models will factor. This allows us to define invariants on a set of $\mathfrak{C}$-models simultaneously.

To actually compute this invariant for a given M, we require a sufficiently explicit and amenable description of a suitable coboundary W on which we can compute characteristic numbers. This is not considered in this chapter.

In all our examples we consider only $P < \mathcal{P}^2\mathfrak{C}$. Where there is the potential for ambiguity we use '$\cdot$' to denote the product in $\mathcal{P}$ to distinguish it from a cup product at the level of the cohomology. For example, if $x, y \in \mathfrak{C}$ then $x^{\cdot 2}, x \cdot y \in \mathcal{P}^2\mathfrak{C}$. We write

$$(x \cdot y)(W, \beta) := \langle x(W) \smile_\beta y(W), [W, M] \rangle \,. \tag{5.18}$$

If it is known that x, y are trivial in the $\mathfrak{C}$-model, then there is no dependence on β. In this case we may write simply $(x \cdot y)(W)$.

The role of BH as the classifying space can be substituted for another classifying space with essentially no changes. For example, it is advantageous to introduce a polarization in some contexts. In which case, $BH \times K(\mathbb{Z}^r, k)$ is substituted for BH, and $BG \times K(\mathbb{Z}^r, k)$ for BG.

5.2 Examples of BDIs

5.2.1 Oriented structure

We begin by deriving Milnor's λ-invariant in the framework outlined in the previous section, and add a very minor generalization.

Let $H = \mathsf{SO}(8)$ with the standard representation be the coboundary group. Then $G = \mathsf{SO}(7)$ is the boundary group. Let $\mathfrak{C}$ be the submodule generated by $p_1 \in H^4(BH)$, and $P = \mathcal{P}^2\mathfrak{C}$.

Let $\vartheta = (\sigma)$ be the signature. Consider $\mathrm{Cl}(P, \vartheta)$. The signature theorem implies that for a closed oriented manifold X

$$\Phi(X) \in \langle (1,2), (0,7) \rangle \tag{5.19}$$

Evaluating Φ on X equal to $\mathbb{HP}^2$, $\mathbb{CP}^4$ and X_{Bott} gives $(4,1)$, $(25,1)$ and $(0,224)$ respectively (see A.1). Thus setting

$$\mathrm{Cl}(P, \vartheta) = \langle (1,2), (0,7) \rangle \tag{5.20}$$

is *sharp* in the sense it is the smallest submodule containing $\Phi(X)$ for any closed oriented 8-manifold X.

A $\mathfrak{C}$-model is essentially determined by the divisibility of the free part of $p_1(M)$. Let $m = \mathrm{gd}(F \circ \widetilde{\varphi}_M(p_1))$, ie the greatest divisor of $p_1(M)$ modulo torsion, and $\widehat{m} = \gcd(2, m)m$. Then $\mathrm{Cl}(P, \vartheta) + \mathrm{Sh}(\mathfrak{C}_M) = \langle (1,2), (0,7), (\widehat{m}, 0) \rangle$. There are no integral Wu classes, so we have no further refinement.

By Proposition 4.3.4, for a closed oriented 7-manifold M and for any section $s : FH^\bullet(M) \to H^\bullet(M)$, we have a coboundary W of (M, s) over $\mathfrak{C}$. The orientation reversal $\iota_H : M \to M'$ is the standard reversal of structure $M' = -M$.

Let $C < H^\bullet(W)$ be fixed as required. It follows then that for $\beta \in R(C, s)$

$$\overline{\Phi} : (W, \beta) \mapsto ((p_1 \cdot p_1)(W, \beta), \sigma(W)) \in (P^\vee_{\mathbb{Q}} \oplus \mathbb{Z}) / \langle (1,2), (0,7), (\widehat{m}, 0) \rangle \tag{5.21}$$

is well defined and dependent on (M, s) and independent of choice of coboundary (W, β). We can choose a *normalization* of $\overline{\Phi}$. Let

$$\lambda : (M, s) \mapsto (p_1 \cdot p_1)(W, \beta) - 2\sigma(W) \in \mathbb{Q}/\gcd(7, m)\mathbb{Z}. \tag{5.22}$$

Clearly (5.22), this will be of potential interest only if $7|m$. Milnor considered the case that $p_1(M) = 0$

$$\lambda(M) := 2p_1^2(W) - \sigma(W) \pmod 7. \tag{5.23}$$

Thus we have a very mild generalization to Milnor's result.

5.2.2 Spin structure

We now derive the generalized Eells-Kuiper invariant in the framework, restricting to the case dimension 7-case. The Eells-Kuiper invariant is an extension of Milnor's λ-invariant in the context of spin manifolds. The generalization is due to Crowley and Nordström.

Let $H = \mathsf{Spin}(8)$ be the coboundary group with boundary group $G = \mathsf{Spin}(7)$. Let $\mathfrak{C} < H^\bullet(BH)$ be the submodule generated by the first spin class $q_1 \in H^4(BH)$ (ie $q_1 = \frac{1}{2}p_1$) and $P = \mathcal{P}^2\mathfrak{C}$. By Proposition 4.3.5 for a spin 7-manifold M and for any section $s : FH^\bullet(M) \to H^\bullet(M)$, we have a coboundary W of (M, s) over $\mathfrak{C}$.

Let $C < H^\bullet(W)$ be fixed as required, and $\beta \in R(C, s)$ for (W, M). Let $\vartheta = (\sigma)$ be the signature. Then

$$\Phi : (W, \beta) \mapsto ((q_1 \cdot q_1)(W, \beta), \sigma(W)). \tag{5.24}$$

We consider the submodule $\mathrm{Cl}(P, \vartheta)$. Let X be a closed spin 8-manifold. Let L-genus and $\hat{A}$-genus in terms of spin classes $q_i \in H^\bullet(B\mathsf{Spin})$

$$\begin{aligned} L &= 1 + \tfrac{2}{3}q_1 + \tfrac{1}{15}q_1^2 + \tfrac{14}{45}q_2 + \dots \\ \hat{A} &= 1 - \tfrac{1}{12}q_1 + \tfrac{1}{240}q_1^2 - \tfrac{1}{720}q_2 + \dots \end{aligned} \tag{5.25}$$

By Theorem 3.5.4 in terms of

$$\left\langle (\tfrac{1}{240}q_1^2 - \tfrac{1}{720}q_2)(X), [X] \right\rangle = 0 \mod 1. \tag{5.26}$$

By the signature Theorem (Theorem 3.5.3)

$$\sigma(X) = \left\langle (\tfrac{1}{15}q_1^2 + \tfrac{14}{45}q_2)(X), [X] \right\rangle. \tag{5.27}$$

We eliminate the q_2 term

$$\left\langle q_1^2(X), [X] \right\rangle - \sigma(X) = 0 \mod 224. \tag{5.28}$$

Let $\mathrm{Cl}(P, \vartheta) = \langle (1,1), (0, 224) \rangle$. Note that if X is equal to $\mathbb{HP}^2$, and X_{Bott}, then $\Phi(X)$ is $(1,1)$, and $(0, 224)$ respectively. Thus $\mathrm{Cl}(P, \vartheta)$ is sharp.

q_1 is an integral Wu class by Corollary 3.4.6. Proposition 3.2.6 places a constraint on the image of Φ for coboundaries over $\mathfrak{C}$. Thus $\mathrm{Qr}(P, \vartheta) = \langle (1,1), (0,8) \rangle$. A $\mathfrak{C}$-model is essentially determined by the divisibility of the free part of $q_1(M)$. Let $m = \mathrm{gd}(F \circ \widetilde{\varphi}_M(q_1))$, ie the greatest divisor of $q_1(M)$ modulo torsion. As m is even $\widehat{m} = 2m$. Thus

$$\mathrm{Denom}(P, \vartheta, \mathfrak{C}_M) = \left\langle (1,1), (0, m') \right\rangle \tag{5.29}$$

where $m' = \mathrm{lcm}(8, 2\ \mathrm{gcd}(m, 224))$, equivalently $m' = 8\ \mathrm{gcd}(28,\ \mathrm{Num}(\frac{m}{4}))$.

It follows then that for a spin 7-manifold M with section s,

$$\overline{\Phi} : (M, s) \mapsto ((q_1 \cdot q_1)(W, \beta), \sigma(W)) \in (P^\vee_{\mathbb{Q}} \oplus \mathbb{Z}) / \left\langle (1,1), (0, m') \right\rangle \tag{5.30}$$

where W is a spin coboundary over $\mathfrak{C}_M$.

The $\mathrm{Qr}(P, \vartheta)$ coset containing $\overline{\Phi}(W, \beta)$ depends only on the Arf invariant $\mathrm{Arf}(q_{q_1})$ of the quadratic refinement q_{q_1} of (M, s) associated to the spin class q_1. Fix a $\mathfrak{C}$-model $\mathfrak{C}_M$ and an isomorphism class of an extended quadratic refinement (N, q_c, c_N, s_N). $\overline{\Phi}$ restricted to (M, s) with such $\mathfrak{C}$-model and quadratic refinement may take at most $\frac{m'}{8}$ distinct values.

We can choose a normalization of $\overline{\Phi}$. Let

$$\mu : (M, s) \mapsto \tfrac{1}{8}(\sigma(W) - (q_1 \cdot q_1)(W, \beta)) \in \mathbb{Q}/m'\mathbb{Z}. \tag{5.31}$$

In the case that $m = 0$, we recover the Eells-Kuiper invariant for 7-manifolds. One can carry out the above in an analogous fashion for closed spin manifolds of dimension $(4k-1)$ and is the context in which Eells Kuiper initially presented their invariant in [39]. This *generalized Eells-Kuiper* invariant was defined by Crowley and Nordström [32].

5.2.3 Polarized spin structure

We now derive a generalization of invariants presented by Hepworth [50]. Hepworth's invariants are extensions of Kreck-Stolz invariants, which in turn incorporate the Eells-Kuiper invariant.

Let $H = \mathsf{Spin}(8)$ be the coboundary group with boundary group $G = \mathsf{Spin}(7)$. In addition, we introduce a polarization on the forth cohomology. We take the classifying space of the coboundary to be $B\mathrm{Spin}(8) \times K(\mathbb{Z}^r, 2)$. Recall that $K(\mathbb{Z}^r, 2) \simeq (\mathbb{CP}^\infty)^r$. As remarked above, the general framework proceeds as in the case when the classifying space was that of a Lie group.

Again let q_1 be the first spin class. Let $x_i \in H^2(B\mathsf{Spin}(8) \times K(\mathbb{Z}^r, 2))$ correspond to the generator of i^{th} copy of $H^2(\mathbb{CP}^\infty)$ via the Kunneth formula, $1 \leq i \leq r$. Let $\mathfrak{C}$ be the submodule of $H^4(B\mathsf{Spin}(8) \times K(\mathbb{Z}^r, 2))$ generated by $\{q_1, x_{jk} \colon 1 \leq j \leq k \leq r\}$, where $x_{jk} = x_j \smile x_k$.

Let $P = \mathcal{P}^2\mathfrak{C}$. We can fix a basis $\{q_1 \cdot q_1, q_1 \cdot x_{jk}, x_{jk} \cdot x_{lm}\} \subset \mathcal{P}^2(\mathfrak{C})$, where the subscripts run over all possible permutations. Note that there are $r' = \binom{r+1}{2}$ terms of the form $q_1 \cdot x_{jk}$ and $\binom{r'+1}{2}$ terms of the form $x_{jk} \cdot x_{lm}$.

By Proposition 4.3.5, for a closed polarized spin 7-manifold M and for any section $s : FH^\bullet(M) \to H^\bullet(M)$, we have a coboundary W of (M, s) over $\mathfrak{C}$. Fix a section s, and compatible $\beta \in R(C, s)$. Let $\vartheta = (\sigma)$ be the signature. We define a map

$$\Phi : (W, \beta) \mapsto ((q_1 \cdot q_1, q_1 \cdot x_{jk}, x_{jk} \cdot x_{lm})(W, \beta), \sigma(W)) \in \mathbb{Q}^N \oplus \mathbb{Z} \tag{5.32}$$

where $N = 1 + r' + \binom{r'+1}{2}$. We proceed with general r until things become too convoluted for it to remain illustrative.

We consider the submodule $\mathrm{Cl}(P, \vartheta)$ which we do in parts. Let X be a closed polarized spin 8-manifold. For $a, b, c, d \in H^2(K(\mathbb{Z}, 2)^r)$

$$(a \smile b) \smile (c \smile d) = (a \smile c) \smile (b \smile d). \tag{5.33}$$

Let $Q < P$ be the rank $\binom{r'+1}{2}$ submodule spanned by $\{x_{jk} \cdot x_{lm}\}$. We define the symmetrization map $Q \to \mathcal{P}^4 H^2(K(\mathbb{Z}^r, 2))$, determined by $x_{jk} \cdot x_{lm} \mapsto x_j \cdot x_k \cdot x_l \cdot x_m$. Let $K = \ker(Q \to \mathcal{P}^4 H^2(K(\mathbb{Z}^r, 2))) < P$. Let

$$A := \ker(P^\vee \oplus \mathbb{Z} \to K^\vee). \tag{5.34}$$

The behaviour of A is a little different to other examples. We have not explicitly used a property of closed manifolds with structure such as an integrality constraint imposed by the index theorem. For a closed polarized spin 8-manifold X, we have a morphism $H^\bullet(BH) \to H^\bullet(X)$ of graded algebras. Thus for $k \in K$, we understand $\langle k(X), [X] \rangle \in \mathbb{Z}$.

However, on a manifold with boundary the product is not simply determined by $H^\bullet(W)$ but also by $H^\bullet(W, \partial W) \to H^\bullet(W)$. We consider this in more detail in Section 5.3.2.

We consider now the constraints corresponding to the twisted Dirac operator. That is Theorem 3.5.6 with $x = 0$, since X is spin. For E the trivial line bundle, $ch(E) = 1$, and we have the constraint of the previous example.

$$\left\langle q_1^2(X), [X] \right\rangle - \sigma(X) = 0 \mod 224 \tag{5.35}$$

Let

$$B_0 := \{(u, v_{jk}, w_{jk,lm}, s) \in P^\vee \oplus \mathbb{Z} \colon u + s = 0 \mod 224\}. \tag{5.36}$$

By (5.25) and eliminating q_2 via the signature theorem, we find that for a complex vector bundle $E \to X$

$$\left\langle ch_4(E) - \tfrac{1}{12} q_1(X) ch_2(E) + \tfrac{1}{224} \mathrm{rank}(E) q_1^2(X), [X] \right\rangle - \tfrac{1}{224}\sigma(X) = 0 \mod 1. \tag{5.37}$$

By taking the difference with a multiple of the trivial bundle, *ie* (5.35), we obtain the condition that

$$\left\langle ch_4(E) - \tfrac{1}{12} q_1(X) ch_2(E) \right\rangle = 0 \mod 1. \tag{5.38}$$

For any $a \in H^2(X)$ there exists a complex line bundle $E_a \to X$ such that $c_1(E) = a$. The Chern character for E_a is $ch(E_a) = 1 + a + \frac{1}{2}a^2 + \frac{1}{6}a^3 + \frac{1}{24}a^4$. For E_a (5.38) implies that

$$\left\langle q_1(X) a^2 - a^4, [X] \right\rangle = 0 \mod 24. \tag{5.39}$$

For a vector $a \in \mathbb{Z}^r$, define the submodule $B(a) < P^\vee \oplus \mathbb{Z}$ by

$$\begin{aligned} B(a) := \big\{ &(u, v_{jk}, w_{jk,lm}, s) \in P^\vee \oplus \mathbb{Z} \colon \\ &\textstyle\sum_{jk} a_{jk} v_{jk} - \sum_{jk,lm} a_{jk,lm} w_{jk,lm} = 0 \mod 24 \big\} \end{aligned} \tag{5.40}$$

where $a_{jk} = (2 - \delta_{jk}) a_j a_k$ and $a_{jk,lm} = (2 - \delta_{jk,lm}) a_{jk} a_{lm}$. (Here, δ_{jk} and $\delta_{jk,lm}$ are Kroneker deltas.) Let $B = B_0 \cap \bigcap_{a \in \mathbb{Z}^r} B(a)$.

Consider $A \cap B$. By a mild abuse of notation, we can consider the basis to be comprised of (u, v_{ij}, w_{jklm}, s), where the indices of w_{jklm} range over the unordered 4-tuples in r elements. In addition, we need only consider $B(a)$ for a in the finite set $\tilde{\mathbb{Z}}_{24}^r := \{a \in \mathbb{Z}^r \colon 0 \leq a_i < 24\}$. Better still, we need only consider the vectors in $\tilde{\mathbb{Z}}_{24}^r$ that are either primitive, or twice a primitive vector. This leads to the following constraints.

$$\begin{aligned} v_{ii} + w_{iiii} &= 0 \mod 24 \\ 2v_{ij} + 4w_{iiij} + 6w_{iijj} + 4w_{ijjj} &= 0 \mod 24 \quad i < j \\ 12 w_{iijj} &= 0 \mod 24 \quad i \leq j \\ 12 w_{iijk} + 12 w_{ijjk} + 12 w_{ijkk} &= 0 \mod 24 \quad i < j < k. \end{aligned} \tag{5.41}$$

At this point we can directly compare this presentation of $\mathrm{Cl}(P, \vartheta) = A \cap B$ to the invariants as defined by Hepworth [50]. In our vocabulary, [50] defines invariants valid on polarized spin 7-manifolds with trivial $\mathfrak{C}$-model. Equations 5.41 correspond to the invariants denoted $\sigma_i, \sigma_{ij}, \tau_{ij}$ and τ_{ijk}. The constraint $w_{ij,kl} - w_{ik,jl} = 0$ in the definition of A, corresponds to the invariant $m_{ij,kl}$. The properties of $m_{ij,kl}$ are discussed in Section 5.3.2.

Remark 5.2.1. The second Wu class of X vanishes as X is spin. For $a, b, c \in H^2(X;\mathbb{Z}_2)$, $\mathrm{Sq}^2(abc) = a^2bc + ab^2c + abc^2$. It follows that for $a, b, c \in H^2(X)$, $a^2bc + ab^2c + abc^2 = 0$ mod 2. Note that

$$\{(u, v_{jk}, w_{jk,lm}, s) \in L\colon w_{jj,kl} + w_{jk,kl} + w_{jk,ll} = 0 \mod 2\} \tag{5.42}$$

is a subset of B.

A brief thought is given to the q-refined submodule while still considering general r. As in the previous case, we have that q_1 is a characteristic element, Here, we have additional constraints coming from each of the x_{ij} as well. $\mathrm{Qr}(P, \vartheta) < P^\vee \oplus \mathbb{Z}$ is then an index $2^{3+r'}$ sublattice of full rank.

In the case that $r = 0$, we return to the invariant of Section 5.2.2. To proceed we restrict r since the general treatment is cumbersome, and we only use $r = 1$ in our applications. If $r = 1$, then $(u, v, w, s) \in L := P^\vee \oplus \mathbb{Z} \cong \mathbb{Z}^4$, $A = L$, $B = B_0 \cap B(1) \cap B(2)$. We find that $\mathrm{Cl}(P, \vartheta) = A \cap B$ has (row) echelon basis

$$(1,\ 0,\ 0,\ 1)\,, (0,\ 2,\ 2,\ 0)\,, (0,\ 0,\ 24,\ 0)\,, (0,\ 0,\ 0,\ 224)\,. \tag{5.43}$$

If $X = \mathbb{HP}^2$ and $x = 0 \in H^2(X)$, then $\Phi(X) = (1, 0, 0, 1)$. If $X = Q \subset \mathbb{P}^5$, a smooth quadric hypersurface, and $x \in H^2(X)$ is the class of a hyperplane section, then $\Phi(X) = (2, 2, 2, 2)$. If $X = (\mathbb{P}^1)^4$ and $x = \sum_i x_i \in H^2(X)$ where $x_i \in H^2(\mathbb{P}^1_i)$ is the oriented generator of the i^{th} $\mathbb{P}^1$ factor of X (via Kunneth theorem), then $\Phi(X) = (0, 0, 24, 0)$. If $X = X_{\text{Bott}}$ and $x = 0 \in H^2(X)$, then $\Phi(X) = (0, 0, 0, 224)$. Thus $\mathrm{Cl}(P, \vartheta)$ is sharp.

For $r = 1$, $\mathrm{Qr}(P, \vartheta)$ has echelon basis

$$(1,\ 0,\ 0,\ 1)\,, (0,\ 1,\ 1,\ 0)\,, (0,\ 0,\ 2,\ 0)\,, (0,\ 0,\ 0,\ 8)\,. \tag{5.44}$$

With respect to this basis, $\mathrm{Cl}(P, \vartheta)$ has basis

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 28 \end{pmatrix} \tag{5.45}$$

taking a convenient diagonal form. We can then read off that we can define three independent invariants that have domains modulo 2, 12, and 28 respectively. Following the lead of [50], we shall denote these τ, σ, and μ. Note that μ is as in (5.31). It is slightly unfortunate that the symbol σ is also used to denote the signature. However, it should be clear from the context which we mean.

We now consider the shearing submodule for a polarized spin 7-manifold M. For $r = 1$, the $\mathfrak{C}$-model has image with rank 0, 1, or 2. In the case that the $\mathfrak{C}$-model has image of rank 0, then $\mathfrak{C}_M$ it is the trivial $\mathfrak{C}$-model. Let $m, n \in \mathbb{N}_0$ be the divisibility of images $u_1 = q_1(M)$ and $u_2 = x^2(M)$ respectively. If $\mathrm{rank}(F \circ \widetilde{\varphi}_M(\mathfrak{C})) = 1$, then the $\mathfrak{C}$-model is one of the following cases:

(i) triple $(m, n, i) \in \mathbb{N}^2 \times \{0, 1\}$, such that $nu_1 = (-1)^i mu_2$.

(ii) $m = 0$, and $n > 0$.
(iii) $n = 0$, and $m > 0$.

If the image of $\mathfrak{C}$-model has rank 2 then $m, n \in \mathbb{N}$. We have freedom to choose the basis of $\mathrm{ps}(F \circ \widetilde{\varphi}_M(\mathfrak{C}))$. We may assume that $u_1 = (m, 0)$ and $u_2 = n \cdot (a, b)$ where $a, b \in \mathbb{Z}$ are coprime. Moreover, we may assume that $b > 0$ and $b > a \geq 0$. The correspondence between m, n, b, a and the resulting boundary defect invariant is algorithmic, but does not seem to have a succinct, pleasant formula. In any case, the invariant defined can be expressed in terms of τ, σ, and μ. We compute the invariants in several hundred examples (See Section 8.4).

Remark 5.2.2. Let us be explicit about what is new. Hepworth considers the case where the $\mathfrak{C}$-model is trivial. We have here a generalization valid for general $\mathfrak{C}$-model, in a fashion completely analogous to the generalized Eells-Kuiper invariant of the previous section.

Remark 5.2.3. Perhaps this provides an alternative answer to Hepworth's desire for a 'natural' choice of invariants [50, Question 1.4.1]. The 'unnatural' part reserved to a choice of normalization.

5.2.4 G_2-structure

We derive now the invariants of Crowley and Nordström [31] within the framework.

Let $H = \mathsf{Spin}(7)$ be the coboundary group with the spinor representation $\mathsf{Spin}(7) \to \mathsf{SO}(8)$. The boundary group is $G = \mathsf{G}_2$. By Proposition 4.3.6 for a closed 7-manifold M with G_2-structure and for any section $s : FH^\bullet(M) \to H^\bullet(M)$, we have a coboundary W of (M, s) over $\mathfrak{C}$. there exists a $\mathsf{Spin}(7)$-coboundary W such that $TH^4(W) = 0$ and $H^4(W) \to H^4(M)$ onto. Let $\mathfrak{C} < H^\bullet(BH)$ be the submodule generated by the first spin class $q_1 \in H^4(BH)$. Let $P = \mathcal{P}^2\mathfrak{C}$. Let $\vartheta = (\sigma, \chi)$ where χ is the Euler characteristic.

Consider $\mathrm{Cl}(P, \vartheta)$. As before, we have that for a closed spin 8-manifold X, $224\hat{A} = q_1^2(A) - \sigma(A)$. In addition, X has a $\mathsf{Spin}(7)$-structure if and only if $e_+(X) = 0$. By comparing characteristic classes, $e_+ = 24\hat{A} + \frac{1}{2}(\chi - 3\sigma)$. Thus $q_1^2 - \sigma = 0 \mod 224$ and $14\chi + 3q_1^2 - 45\sigma = 0$. If X is equal to $\mathbb{HP}^2$, then $\Phi(X) = (1, 1, 3)$. Note that $\chi(X_{Bott}) = 226$, and $\chi(S^4 \times S^4) = 4$. Let $X_k := X_{Bott} \#(S^4 \times S^4)^{\#k}$ be the connected sum of X_{Bott} and k copies of $S^4 \times S^4$. Then X_k is spin and $\Phi(X_k) = (0, 224, 226 + 2k)$. As $e_+(X_{297}) = 0$, X_{297} has a $\mathsf{Spin}(7)$-structure and $\Phi(X_{297}) = (0, 224, 720)$. $\mathrm{Cl}(P, \vartheta)$ has basis

$$(1,\ 1,\ 3)\,, (0,\ 224,\ 720)\,. \tag{5.46}$$

As in Section 5.2.2, $\mathrm{Qr}(P, \vartheta)$ has basis

$$(1, 1, 0), (0, 8, 0), (0, 0, 1). \tag{5.47}$$

With respect to this basis for $\mathrm{Qr}(P, \vartheta)$, $\mathrm{Cl}(P, \vartheta)$ has basis

$$(1,\ 0,\ 3)\,, (0,\ 28,\ 720)\,. \tag{5.48}$$

By considering the Smith normal form, we can define two invariants

$$\begin{aligned} g_1 &:= \tfrac{5}{8}\, q_1 \cdot q_1 - \tfrac{77}{8}\, \sigma + 3\, \chi \mod 4, \\ g_2 &:= \tfrac{3}{2}\, q_1 \cdot q_1 - \tfrac{45}{2}\, \sigma + 7\, \chi. \end{aligned} \tag{5.49}$$

Consider nontrivial $\mathfrak{C}$-model. As in previous cases this is completely determined by the divisibility m of $F(q_1(M))$. If $m > 0$, then $\mathrm{Denom}(P, \vartheta, \mathfrak{C}_M)$ has a basis (with respect to (5.47))

$$(1, 0, 3), (0, 28, 720), (0,\ \mathrm{Num}(\tfrac{m}{4}), 24 \cdot \mathrm{Num}(\tfrac{m}{4})). \tag{5.50}$$

By considering the Smith normal form, we find there are two invariants with codomain modulo $\gcd(4,\ \mathrm{Num}(\frac{m}{4}))$ and $48 \cdot \mathrm{Num}(\frac{m}{16})$ respectively. The precise linear combinations of g_1 and g_2 depends on m in an algorithmic yet convoluted manner.

[31] defines two boundary defect invariants for 7-manifolds with G_2-structure via a $\mathsf{Spin}(7)$-coboundary. For a closed 7-manifold M with G_2-structure, section $s : FH^4(M) \to H^4(M)$ with $\mathsf{Spin}(7)$-coboundary W such that it is over $\mathfrak{C}_M$, and for $\beta \in R(C, s)$

$$\begin{aligned} \nu'(M, s) &:= -3\sigma(W) + \chi(W) \mod 48, \\ \xi'(M, s) &:= \tfrac{3}{2}(q_1 \cdot q_1)(W, \beta) - \tfrac{45}{2}\sigma(W) + 7\chi(W) \in \mathbb{Q}/12 \cdot \mathrm{Num}(\tfrac{m}{4})\mathbb{Z}. \end{aligned} \tag{5.51}$$

By taking $\nu' = 12g_1 - 5g_2$. Their normalization has the advantage that only one invariant is dependent on the choice of s and the primary invariant m of M—convenient in applications. The cost is that these invariants are coupled ie the value of ν' is constrained by the value of ξ'.

Remark 5.2.4. [31, Definition 6.8] of ξ' includes a term g_W called the *Gauss refinement*. Strictly speaking, the Gauss refinement is only defined for 3-connected coboundaries, and is a key object of study in [32]. However, ξ' requires that W is a $\mathsf{Spin}(7)$-coboundary. The existence of the $\mathsf{Spin}(7)$-structure on W relies on being able to take arbitrary connect sums, as we did in Proposition 4.3.6, but this breaks 3-connectedness. For this reason, one is motivated to define the Gauss refinement for coboundaries W such that $H^4(W)$ is free and $H^4(W) \to H^4(M)$ is onto.

5.2.5 Almost contact structure

Within the previous examples, the framework has at most generalized known invariants to contexts where the $\mathfrak{C}$ is nontrivial. The invariants of this section are new (first appearing in [105]).

Let $H = \mathsf{U}(4)$ with the realification representation $H \to \mathsf{SO}(8)$. The boundary group is $G = \mathsf{U}(3)$. Let $G = \mathsf{U}(3)$ with representation $\mathsf{U}(3) \to \mathsf{SO}(7)$. Almost contact structures on 7-manifolds can be described by the triple (v, ω, J) as discussed in Examples 2.5.9. The orientation reversal ι_H 'reverses' the vector field v (see Section 2.6).

Let $\mathfrak{C} < H^\bullet(BH)$ be the submodule generated by c_1, c_1^2, c_2, c_3. Let $P < [\mathcal{P}^2\mathfrak{C}]^8$ be the $\mathbb{Z}$-module with basis $(c_1^2 \cdot c_1^2, c_1^2 \cdot c_2, c_2 \cdot c_2, c_1 \cdot c_3)$. Previously it has been apparent which products are considered in $\mathcal{P}^2$ from the context—here we use '$\cdot$' to denote the graded symmetric product.

By Proposition 4.3.8, for a closed almost contact 7-manifold M there exists a $\mathsf{U}(4)$-coboundary W such that $TH^4(W) = 0$ and $H^4(W) \to H^4(M)$ onto. We are deficient of the complete result: that W is over $\mathfrak{C}_M$ in degrees 2 and 6 as well, regardless of the choice of section s. However, in our applications we will assume or find that $c_1(M)$ and $c_3(M)$

rationally vanish, and in which case our bordism result is sufficient. The well definedness of the general invariants is subject to the existence of a suitable coboundary.

Let $\vartheta = (\sigma, \chi)$. As is the routine, fix a section s for M, let W be a suitable $\mathsf{U}(4)$-coboundary, and $\beta \in R(C, s)$. Then

$$\Phi(W, \beta) \in P_{\mathbb{Q}}^{\vee} \oplus \mathbb{Z}^2. \tag{5.52}$$

We consider now $\mathrm{Qr}(P, \vartheta)$. Recall that $c_i \mapsto w_{2i}$ via $H^{\bullet}(B\mathsf{U}) \to H^{\bullet}(B\mathsf{U}; \mathbb{Z}_2)$. As $v_4 = w_4 + w_2^2$, $(c_2 + c_1^2)$ is a characteristic element. Thus, $\mathrm{Qr}(P, \vartheta)$ has the row echelon basis matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.53}$$

We now consider the submodule $\mathrm{Cl}(P, \vartheta)$. For an almost complex 8-manifold X, $\chi(X) = \langle c_4(X), [X] \rangle$. For a complex vector bundle $E \to X$, the twisted Dirac operator implies that $\langle \mathrm{Td}(X)\mathrm{ch}(E), [X] \rangle$ is an integer. Let $E_k = \bigwedge^k TX$, so that for example $E_4 = \det(TX)$ and $E_1 = TX$. By computing the Chern character for each of these we will derive some constraints.

Let $c_i = c_i(X)$. Then

$$\begin{aligned} c(E_1) &:= 1 + c_1 + c_2 + c_3 + c_4 \\ c(E_2) &:= 1 + 3c_1 + 3c_1^2 + 2c_2 + c_1^3 + 4c_1c_2 + 2c_1^2c_2 + c_2^2 + c_1c_3 - 4c_4 \\ c(E_3) &:= 1 + 3c_1 + 3c_1^2 + c_2 + c_1^3 + 2c_1c_2 - c_3 + c_1^2c_2 - c_1c_3 + c_4 \\ c(E_4) &:= 1 + c_1 \end{aligned} \tag{5.54}$$

The Chern characters are then

$$\begin{aligned} \mathrm{ch}(E_1) &:= 4 + c_1 + \tfrac{1}{2}c_1^2 - c_2 + \tfrac{1}{6}c_1^3 - \tfrac{1}{2}c_1c_2 + \tfrac{1}{2}c_3 + \tfrac{1}{24}c_1^4 - \tfrac{1}{6}c_1^2c_2 + \tfrac{1}{12}c_2^2 + \tfrac{1}{6}c_1c_3 - \tfrac{1}{6}c_4 \\ \mathrm{ch}(E_2) &:= 6 + 3c_1 + \tfrac{3}{2}c_1^2 - 2c_2 + \tfrac{1}{2}c_1^3 - c_1c_2 + \tfrac{1}{8}c_1^4 - \tfrac{1}{3}c_1^2c_2 + \tfrac{1}{6}c_2^2 - \tfrac{1}{6}c_1c_3 + \tfrac{2}{3}c_4 \\ \mathrm{ch}(E_3) &:= 4 + 3c_1 + \tfrac{3}{2}c_1^2 - c_2 + \tfrac{1}{2}c_1^3 - \tfrac{1}{2}c_1c_2 - \tfrac{1}{2}c_3 + \tfrac{1}{8}c_1^4 - \tfrac{1}{6}c_1^2c_2 + \tfrac{1}{12}c_2^2 - \tfrac{1}{3}c_1c_3 - \tfrac{1}{6}c_4 \\ \mathrm{ch}(E_4) &:= 1 + c_1 + \tfrac{1}{2}c_1^2 + \tfrac{1}{6}c_1^3 + \tfrac{1}{24}c_1^4 \end{aligned} \tag{5.55}$$

For completeness let $E_0 = \mathbb{C}$ be the trivial bundle. Then we have the following integrality constraints.

$$\begin{array}{ll} E_0) & -\dfrac{1}{720}c_1^4+\dfrac{1}{180}c_1^2c_2+\dfrac{1}{240}c_2^2+\dfrac{1}{720}c_1c_3-\dfrac{1}{720}c_4=0 \mod 1 \\ E_1) & \dfrac{29}{180}c_1^4-\dfrac{71}{180}c_1^2c_2+\dfrac{1}{60}c_2^2+\dfrac{19}{45}c_1c_3-\dfrac{31}{180}c_4=0 \mod 1 \\ E_2) & \dfrac{59}{120}c_1^4-\dfrac{43}{60}c_1^2c_2+\dfrac{1}{40}c_2^2-\dfrac{19}{120}c_1c_3+\dfrac{79}{120}c_4=0 \mod 1 \\ E_3) & \dfrac{89}{180}c_1^4-\dfrac{41}{180}c_1^2c_2+\dfrac{1}{60}c_2^2-\dfrac{26}{45}c_1c_3-\dfrac{31}{180}c_4=0 \mod 1 \\ E_4) & \dfrac{119}{720}c_1^4+\dfrac{4}{45}c_1^2c_2+\dfrac{1}{240}c_2^2+\dfrac{1}{720}c_1c_3-\dfrac{1}{720}c_4=0 \mod 1 \end{array} \tag{5.56}$$

In addition, we have the signature theorem

$$45\sigma = -c_1^4+4\,c_1^2c_2+3\,c_2^2-14\,c_1c_3+14\,c_4 \tag{5.57}$$

The row echelon basis matrix satisfying these constraints is

$$\begin{pmatrix} 1 & 10 & 0 & 2 & 13 & 41 \\ 0 & 12 & 0 & 0 & 16 & 48 \\ 0 & 0 & 1 & 2 & 1 & 5 \\ 0 & 0 & 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 & 224 & 720 \end{pmatrix} \tag{5.58}$$

We take this to be basis for $\mathrm{Cl}(P,\vartheta)$.

Remark 5.2.5. It is sufficient to consider only three of the five complex bundles E_i, in order to compute the row basis matrix above. More precisely, $\{E_i, E_j, E_k\}$ determine the row basis matrix above if and only if (i,j,k) isis equal to one of the following $(0,1,2)$, $(0,1,4)$, $(0,2,3)$, $(0,3,4)$, $(1,2,4)$, $(2,3,4)$ (up to ordering). Considering less than three is insufficient.

In Appendix A.1.4 we compute the invariants of the complex 8-manifolds $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$, $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^2$, $\mathbb{P}^2\times\mathbb{P}^2$, and $\mathbb{P}^1\times\mathbb{P}^3$. Together with $\mathbb{P}^4$ and the almost complex manifold $S^2\times S^6$ we have the following row basis matrix of invariants

$$\begin{pmatrix} 384 & 192 & 96 & 64 & 0 & 16 \\ 432 & 204 & 96 & 60 & 0 & 12 \\ 486 & 216 & 99 & 54 & 1 & 9 \\ 512 & 224 & 96 & 56 & 0 & 8 \\ 625 & 250 & 100 & 50 & 1 & 5 \\ 0 & 0 & 0 & 4 & 0 & 4 \end{pmatrix} \tag{5.59}$$

The echelon form of this matrix is precisely (5.58), so our choice of $\mathrm{Cl}(P,\vartheta)$ is sharp. The row basis matrix of $\mathrm{Cl}(P,\vartheta)$ with respect to that of $\mathrm{Qr}(P,\vartheta)$ is

$$\begin{pmatrix} 1 & 5 & 0 & 2 & -1 & 41 \\ 0 & 6 & 0 & 0 & -1 & 48 \\ 0 & 0 & 1 & 2 & 0 & 5 \\ 0 & 0 & 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 & 28 & 720 \end{pmatrix} \tag{5.60}$$

We have the following Smith normal form

$$\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&4&0&0\\ 0&0&0&0&24&0 \end{pmatrix} = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&-20&0&0&1 \end{pmatrix} \begin{pmatrix} 1&5&0&2&-1&41\\ 0&6&0&0&-1&48\\ 0&0&1&2&0&5\\ 0&0&0&4&0&4\\ 0&0&0&0&28&720 \end{pmatrix} \begin{pmatrix} 1&-50&0&-2&10&-23\\ 0&-184&0&0&37&-86\\ 0&-45&1&-2&9&-21\\ 0&-15&0&1&3&-7\\ 0&-385&0&0&78&-180\\ 0&15&0&0&-3&7 \end{pmatrix} \tag{5.61}$$

Transforming back to the standard basis (and not that of $\mathrm{Qr}(P,\vartheta)$) motivates the following. Let

$$\begin{aligned} u_1 &:= 2\,c_1^2\cdot c_1^2 + 2\,c_2\cdot c_2 - c_1\cdot c_3 \mod 4\\ u_2 &:= \tfrac{1}{4}\,c_1^2\cdot c_1^2 - c_1^2\cdot c_2 - \tfrac{3}{4}\,c_2\cdot c_2 + 3\,c_1\cdot c_3 + \tfrac{39}{4}\,\sigma - 3\,\chi \mod 24\\ u_3 &:= -\tfrac{1}{2}\,c_1^2\cdot c_1^2 + 2\,c_1^2\cdot c_2 + \tfrac{3}{2}\,c_2\cdot c_2 - 7\,c_1\cdot c_3 - \tfrac{45}{2}\,\sigma + 7\,\chi \end{aligned} \tag{5.62}$$

These are invariants of almost contact manifolds with trivial $\mathfrak{C}$-model. We note that it may perhaps be more convenient to use, in place of u_2,

$$u_2' := 2\,u_2 + u_3 = c_1\cdot c_3 + 3\,\sigma - \chi \mod 48. \tag{5.63}$$

As with (5.51), this comes at the cost of coupling the invariants.

Consider now the shearing submodule. We have a particular interest in the cases where c_1 and c_3 are rationally trivial on M. This occurs for 2-connected 7-manifolds, or more generally for simply connected 7-manifolds where there is an $\mathsf{SU}(3)$ reduction of the $\mathsf{U}(3)$-structure. The additional benefit here for us is that Proposition 4.3.8 insures that a suitable coboundary exists in this case.

Assuming that c_1, c_3 are rationally trivial implies that the $\mathfrak{C}$-model is determined by the divisibility of $c_2(M)$ modulo torsion. Let m be the greatest divisor of $c_2(M)$ modulo torsion. As $c_1^2 + c_2$ is characteristic, and $c_1(M)$ is rationally trivial, so $c_2(M)$ must be divisible by 2 modulo torsion and $\widehat{m} = 2m$. Hence the expressions of the invariants u_1, u_2' remain valid.

For u_3 we must add this following observation. Until now, integer coefficients have been sufficient. As discussed in Section 3.1, the role of C can be exchanged for $C_{\mathbb{Q}}$ to allow for a mild extension. This is necessary here. Note that

$$-\tfrac{1}{2}\,c_1^2\cdot c_1^2 + 2\,c_1^2\cdot c_2 + \tfrac{3}{2}\,c_2\cdot c_2 = \tfrac{3}{2}(c_2 + \tfrac{2}{3}\,c_1^2)\cdot(c_2 + \tfrac{2}{3}\,c_1^2) - \tfrac{7}{6}c_1^2\cdot c_1^2. \tag{5.64}$$

As $c_1^2 \in \mathfrak{C}_0$, so $(c_2 + \frac{2}{3}c_1^2) \in \mathfrak{C}_{\mathbb{Q},m}$.

In summary, for a closed almost contact 7-manifold M such that c_1 and c_3 rationally vanish and m is the greatest divisor of $c_2(M)$ modulo torsion, then there exists a $\mathsf{U}(4)$-coboundary W such that is over $\mathfrak{C}$. Fix a section s, and $\beta \in R(C,s)$. Then the following are independent of choice of suitable coboundary W.

$$\begin{aligned} \nu(M,s) &:= -(c_1\cdot c_3)(W) - 3\sigma(W) + \chi(W) \ \in \mathbb{Q}/48\mathbb{Z},\\ \xi(M,s) &:= -(\tfrac{7}{6}c_1^2\cdot c_1^2 + 7c_1\cdot c_3)(W) + \tfrac{3}{2}(c_2 + \tfrac{2}{3}c_1^2)^{\cdot 2}(W,\beta)\\ &\qquad - \tfrac{45}{2}\sigma(W) + 7\chi(W) \in \mathbb{Q}/12\cdot\mathrm{Num}(\tfrac{m}{4})\mathbb{Z},\\ \varrho(M,s) &:= (2c_1^2\cdot c_1^2 - c_1\cdot c_3)(W) + 2(c_2\cdot c_2)(W,\beta) \ \in \mathbb{Q}/4\mathbb{Z}. \end{aligned} \tag{5.65}$$

The choice of normalization and notation is used to mirror the relationship with the invariants of Crowley and Nordström defined in the previous section. This is discussed in Section 5.3.1.

5.3 Remarks on BDI examples

We make a few extended remarks about some aspects of the examples of the previous section.

5.3.1 Relating G_2 and almost contact structures

A G_2-structure and an almost contact structure are *compatible* if they admit $\mathsf{SU}(3)$-reductions that are isomorphic when considered as reductions of the $\mathsf{SO}(7)$-structure (Definition 2.5.14). In the context where a manifold has multiple structures, we may reference the structures as arguments of the invariants for clarity.

Proposition 5.3.1. *On a 7-manifold M with G_2-structure φ, there exists a compatible $\mathsf{U}(3)$-structure (v, g, ω) with $c_1, c_3 = 0$. Moreover, the greatest divisors of $q_1(M)$ and $c_2(M)$ modulo torsion are equal and*

$$\nu'(M, \varphi) = \nu(M, v, g, \omega), \qquad \xi'(M, \varphi) = \xi(M, v, g, \omega). \tag{5.66}$$

In particular, this is independent of choice of compatible $\mathsf{U}(3)$-structure.

Proof. A spin 7-manifold admits an orthonormal pair of vector fields [103, Theorem 1.1]. Thus a G_2-structure admits a reduction to an $\mathsf{SU}(2)$-structure. Fix such a reduction and extend trivially to a $\mathsf{U}(3)$-structure. Such a $\mathsf{U}(3)$-structure clearly shares an $\mathsf{SU}(3)$-reduction with G_2, and $c_1, c_3 = 0$.

As $c_1(M)$ is torsion and $q_1 = \frac{1}{2}c_1^2 - c_2$, the greatest divisors of $q_1(M)$ and $c_2(M)$ modulo torsion are equal. By Proposition 4.3.7, there exists an $\mathsf{SU}(4)$-coboundary W to the shared $\mathsf{SU}(3)$-reduction such that $H^4(W)$ is free and $H^4(W) \to H^4(M)$ is onto. By considering the terms on such a coboundary, the equality of the invariants is immediate. $\square$

In light of this, we may refer to a $\mathsf{U}(4)$-coboundary of a manifold with G_2-structure.

Definition 5.3.2. Let M be a closed 7-manifold with a G_2-structure. A $\mathsf{U}(4)$-*coboundary* W to M is a coboundary such that the restriction of the $\mathsf{U}(4)$-structure to M is a compatible $\mathsf{U}(3)$-structure.

The following lemma is useful when checking compatible structures.

Lemma 5.3.3. *Let M be a 7-manifold with a G_2-structure φ and $\mathsf{U}(3)$-structure (v, ω, g) such that $c_1, c_3 = 0$. The structures are compatible provided that $\varphi \lrcorner v = \omega$, and $g_\varphi = g$.*

5.3.2 Formality as a BDI

In homotopy theory, determining the higher homotopy groups of even some of the most basic spaces is notoriously difficult. Rational homotopy theory neatly disposes of the unwieldy torsion part of the homotopy groups. The foundations for rational homotopy theory were established by Quillen [88] and Sullivan [97]. There are results of rational homotopy theory that have implications for classification problems in differential topology, and topological constraints of manifolds with special holonomy.

A principal motivation of Sullivan's original work on rational homotopy theory stems from the fact that the rational homotopy type (see Definition 5.3.4) of a simply connected manifold together with some data regarding characteristic classes and integral homology determine the diffeomorphism type up to finite ambiguity (see [97, Theorem 13.1]). Thus the rational homotopy types can give partial answers to classification questions in topology.

Definition 5.3.4. A continuous map $\varphi : X \to Y$ between simply connected topological spaces is a *rational homotopy equivalence* if φ_* is an isomorphism of rational homotopy groups. X and Y are *rational-homotopy equivalent* if there exists a zig-zag of rational homotopy equivalences between them. That is $X \leftarrow Z_0 \to Z_1 \leftarrow \cdots \to Y$, where each map is a rational homotopy equivalence. The *rational homotopy type* of X is the class of spaces rational homotopy equivalent to X

Rational homotopy theory is the study of rational homotopy types, and properties of maps and spaces invariant under rational homotopy equivalence.

Definition 5.3.5. A *commutative differential graded algebra* (CDGA) over $\mathbb{Q}$ is a cochain complex $(A^\bullet, d)$ endowed with two cochain maps: a graded commutative associative product and a *unit* $\eta : \mathbb{Q} \to (A^\bullet, d)$ A CDGA A is *r-connected* if $A^0 = \mathbb{Q}$ and $A^k = 0$ for $0 < k \leq r$. A morphism $\varphi : (A, d_A) \to (B, d_B)$ on CDGAs is a *quasi-isomorphism* if it induces an isomorphism of CDGAs at the level of cohomology. A 1-connected CDGA A is *formal* if there exist a zig-zag of quasi-isomorphisms of CDGAs between A and $H^\bullet(A)$.

For a simplicial set K (of finite type), one can define a $\mathbb{Q}$-CDGA $\Omega_{PL}(K)$ of piecewise-linear polynomial forms. The cohomology of $\Omega_{PL}(K)$ is isomorphic to that of standard cohomology with rational coefficients. We say K is *formal* if $\Omega_{PL}(K)$ is formal. In general, $\Omega_{PL}(K)$ is large and difficult to compute. There is an equivalence of categories of topological space up to rational homotopy equivalence, and $\mathbb{Q}$-CDGAs up to quasi-isomorphism. Thus, if a given manifold M is formal then its rational cohomology algebra $H^\bullet$ determines its rational homotopy type.

Here is a blitz of known results of formal spaces. Deligne et al [35] proved that Kähler manifolds are necessarily formal. Symmetric spaces are necessarily formal. Formality is preserved under fibre products and direct sums. Sufficiently connected spaces are necessarily formal. Miller [73] showed $(k-1)$-connected $(4k-2)$ manifolds are necessarily formal. (And this cannot be improved without further hypothesis, with examples by Dranishnikov [38].)

For $(4k-1)$-manifolds, the *critical dimension* is k. Cavalcanti [21] shows that a $(k-1)$-connected $(4k-1)$-manifold M is necessarily formal if the critical Betti number $b_k(M) \leq 1$. In addition, he shows that if M has a hard Lefschetz type property [1] then it is formal if $b_k(M) \leq 2$. Crowley and Nordström [27] improved the latter result: M, with a hard Lefschetz type property is formal if $b_k \leq 3$ (see Corollary 5.3.8.)

[1] X has a *hard Lefschetz type property*if there exists $\varphi \in H^{2k-1}(X)$ such that cup product $\varphi : H^k \xrightarrow{\simeq} H^{3k-1}(X)$

It is an interesting question whether there exists a nonformal manifold with special holonomy. A manifold with reducible holonomy can be written as a metric product, and so will be formal if and only if the irreducible components are formal. These two reductions lead us to considering the holonomy groups on 'Berger's list'. The families of holonomy groups $\mathsf{U}(n)$, $\mathsf{SU}(n)$, and $\mathsf{Sp}(n)$, correspond to Kähler, Calabi-Yau and Hyper-Kähler manifolds respectively. As all are Kähler, all are formal. Thus, if an example of a nonformal manifold with special holonomy is to exist then its holonomy group must be $\mathsf{Sp}(1) \cdot \mathsf{Sp}(n)$, G_2, or $\mathsf{Spin}(7)$.

We restrict our considerations to manifolds with special holonomy G_2, over the other two potential classes because there is a greater understanding of how to construct examples. A simply connected G_2-manifold has a hard Lefschetz property from the G_2-structure, so to be nonformal it must have $b_2 \geq 4$. Crowley and Nordström [27] define an invariant which characterizes formality on a certain class of manifolds including simply connected G_2-manifolds. It is defined both intrinsically and equivalently as a boundary defect invariant.

In Section 5.2.3 we noted that for $r \geq 2$ and trivial $\mathfrak{C}$-model case, the invariant defined there has a free part. Note that the spin structures play no part in the following invariant—it is only necessary that the manifolds are oriented.

Let $\mathfrak{C} = \langle x_{ij} \rangle \subset H^\bullet(B\mathsf{Spin}(8) \times K(\mathbb{Z}, 2)^r)$. Suppose we fix an $\mathfrak{C}$-model $\mathfrak{C}_0$, and let $E = [\ker(\mathfrak{C}_0)]^{(4)}$, the submodule of the kernel of the $\mathfrak{C}$-model of classes of degree 4. Let $P_E = \mathcal{P}^2 E$, which by design will ensure that the role of $\mathrm{Sh}(\mathfrak{C}_M)$ is redundant. Let $L_E = \mathrm{Hom}(P_E, \mathbb{Z})$. Let $\mathrm{Cl}_E = \mathrm{Ann}(K(E))$ where $K(E) := \ker(\mathcal{P}^2 E \to \mathcal{P}^4 H^2(K(\mathbb{Z}^r, 2)))$. Hence we can define an invariant, which we shall denote $\mathcal{B}$

$$\mathcal{B} : (W, \beta) \to (\varphi_W^*)^*(\smile_\beta) \in (L_E)_\mathbb{Q}/\mathrm{Cl}_E \tag{5.67}$$

Note that $(L_E)_\mathbb{Q}/\mathrm{Cl}_E \cong \mathrm{Hom}(K(E), \mathbb{Q})$. [27] refers to $\mathcal{B}$ as the *Bianchi Massey tensor*. We now describe explicitly how to compute the invariant $\mathcal{B}$. We begin with a digression on linear algebra.

Let V be a rank r free $\mathbb{Z}$-module or an r dimensional vector space. Assume we have (else fix) a basis $v_1, \ldots, v_r$ for V. Let $v_{i_1 \ldots i_{k-1}} \in P^k V$ denote $v_{i_1} v_{i_1} \ldots v_{i_{k-1}}$. The ordering of the subscripts is immaterial, but there is a unique representative where $i_1 \leq \cdots \leq i_{k-1}$, which we shall notationally prefer. Let $\mathcal{I}_r^k$ denote the set of $\binom{r+k-1}{k}$ such tuples. Then $\{v_s \colon s \in \mathcal{I}_r^k\}$ is a basis for $\mathcal{P}^k V$. Let $v_{ij,kl} \in \mathcal{P}^2\mathcal{P}^2 V$ denote $v_{ij} v_{kl}$. There is a unique representative where $i \leq j$, $k \leq l$, $i \leq k$ and if $i = k$ then $j \leq l$. Let $\mathcal{I}_r^{2,2}$ denote the set of all $\frac{1}{12}(r+2)(r+1)^2 r$ such tuples.

The symmetrization map $S : \mathcal{P}^2\mathcal{P}^2 V \to \mathcal{P}^4 V$ is determined by $v_{ij,kl} \mapsto v_{ijkl}$, for $((i,j),(k,l)) \in \mathcal{I}_r^{2,2}$. The kernel $K := \ker(S)$ is generated by the differences of pairs of basis vectors in the domain that share an image. Thus we have a straightforward way to describe a basis for K. For $(i,j,k,l) \in \mathcal{I}_r^4$ we have the following vectors form a basis for

K

$$\begin{aligned} v_{ij,kl}-v_{ik,jl}, v_{ij,kl}-v_{il,jk} \quad & \text{if} \quad i<j<k<l \\ v_{ii,kl}-v_{ik,il} \quad & \text{if} \quad i=j<k<l \\ v_{ij,jl}-v_{il,jj} \quad & \text{if} \quad i<j=k<l \\ v_{ij,kk}-v_{ik,jk} \quad & \text{if} \quad i<j<k=l \\ v_{ii,kk}-v_{ik,ik} \quad & \text{if} \quad i=j<k=l \end{aligned} \tag{5.68}$$

and that if any three of i, j, k, l are equal there is no contribution to the kernel. Note that $\dim(K) = 2\binom{r}{4} + 3\binom{r}{3} + \binom{r}{2}$, equivalently $\dim(K) = \frac{1}{12}(r+1)r^2(r-1)$.

More generally, for a subspace $E < \mathcal{P}^2 V$ let $K(E) := \mathcal{P}^2 E \cap K$. Suppose that $E < \mathcal{P}^2 V$ is of rank m with basis matrix C_{ipq}. That is, we have a basis $(u_i)_{i=1}^m$ of E where $u_i = \sum_{(p,q)\in\mathcal{I}_r^2} C_{ipq} v_{pq}$. We have a basis for $\mathcal{P}^2 E$ given by $(u_{ij})_{(i,j)\in\mathcal{I}_m^2}$, where

$$u_{ij} = \sum_{(p,q,r,s)\in\mathcal{I}_r^{2,2}} D_{ij;pq,rs} v_{pq,rs} \ , \quad D_{ij;pq,rs} = C_{ipq}C_{jrs} + (1-\delta_{pq,rs})C_{irs}C_{jpq} \tag{5.69}$$

Define the array L as follows. For $(i,j) \in \mathcal{I}_m^2$ and $(p,q,r,s) \in \mathcal{I}_r^4$

$$L_{ij;pqrs} := \begin{cases} D_{ij;pq,rs} + D_{ij;pr,qs} + D_{ij;ps,qr} & p<q<r<s \\ D_{ij;pp,rs} + D_{ij;pr,ps} & p=q<r<s \\ D_{ij;pq,qs} + D_{ij;ps,qq} & p<q=r<s \\ D_{ij;pq,rr} + D_{ij;pr,qr} & p<q<r=s \\ D_{ij;pp,rr} + D_{ij;pr,pr} & p=q<r=s \\ D_{ij;pp,ps} & p=q=r<s \\ D_{ij;pq,qq} & p<q=r=s \\ D_{ij;pp,pp} & p=q=r=s \end{cases} \tag{5.70}$$

Then $k \in K(E)$ with $k = \sum_{(i,j)\in\mathcal{I}_m^2} k_{ij} u_{ij}$ if and only if

$$\sum_{(i,j)\in\mathcal{I}_m^2} k_{ij} L_{ij;pqrs} = 0 \in \mathcal{P}^4 V \tag{5.71}$$

Thus for a subspace E, or its basis matrix C_{ipq}, we construct matrix $L_{ij;pqrs} : \mathcal{P}^2 E \to \mathcal{P}^4 V$. Then $K(E) := \ker(L_{ij;pqrs})$ understood as a matrix acting on the right.

We return to considering the invariant on 7-manifolds. Let $\alpha : V = \mathbb{Z}^r \to H^2(W)$ be the polarization. The invariant is determined by: the map $c \circ \mathcal{P}^2 V \to H^4(W)$ determined by the cup product c; $H^4(W,M) \to H^4(W)$; and pairing $H^4(W,M) \times H^4(W) \to \mathbb{Z}$. By fixing a basis of $H^4(W)$, we can take $H^4(W,M)$ to be in its dual basis, and let A be the matrix of $H^4(W,M) \to H^4(W)$ with respect to these bases. Let P be the matrix of $c \circ \mathcal{P}^2\alpha$.

Let $E = P^{-1}(\mathrm{im}(A)) < \mathcal{P}^2 V$. We have a bilinear form $B : E \times E \to \mathbb{Z}$ by $(u_i, u_j) \mapsto w_i.\tilde{w_j}$ where $w_i = Pu_i$ and $\tilde{w_j} \in H^4(W,M)$ is the lift of $Pu_j \in H_0^4(W)$. By the properties of the intersection form on $H_0^4(W)$ this is symmetric. Construct the linear functional $B' : \mathcal{P}^2 E \to \mathbb{Z}$ from B. Then the invariant $\mathcal{B}(W) = B'|_{K(E)}$.

Proposition 5.3.6. *For closed 1-connected 7-manifold M, M is formal if and only if $F : B(H^\bullet(M)) \to \mathbb{Q}$ is trivial.*

(See the more general result [27, Theorem 1.3].) A space X is said to be *intrinsically formal* if any space with cohomology algebra $H^\bullet(X)$ is rationally homotopy equivalent to X.

Proposition 5.3.7. *In the notation above, M is intrinsically formal if and only if $K(E)$ is empty which holds if and only if L has full rank* $\mathrm{Rank}(L) = \frac{1}{2}m(m+1)$ *where* $m = \dim(E)$.

(See the more general result [27, Corollary 1.13].)

Corollary 5.3.8. *Let M be a closed 1-connected 7-manifold. If $b_2(M) \leq 3$ and if there exists a $\varphi \in H^3(M)$ inducing an isomorphism $H^2(M) \xrightarrow{\sim} H^5(M)$ by cup product $x \mapsto \varphi \smile x$, then M is intrinsically formal.*

5.3.3 Classification

The main aims of the classification of manifolds can be summarized as follows. We wish to have a complete set of algebraic invariants such that:

(i) The invariants of a manifold are computable.
(ii) Two manifolds are isomorphic if and only if they have the same invariants.
(iii) There is a given list of non-isomorphic manifolds realizing every possible set of invariants.

Generally, a classification problem is broken down into incremental refinements. For example *when are two homotopic manifolds homeomorphic?* or *when are two homeomorphic manifolds diffeomorphic?* and so on.

The h-Cobordism Theorem (Proposition 4.2.3) demonstrates a profound link between bordism theory and the classification of manifolds. The existence of a diffeomorphism follows from the existence of an h-bordism, so the question becomes when and how can we find an h-bordism. Computing the h-bordism group is not really any easier than computing an isomorphism classification directly.

In contrast, less stringent types of bordisms are easier to construct. For example, suppose we have two closed manifolds that have a normal B-structure and are both null B-bordant. By taking the connected sum of their coboundaries, they are normal B-bordant. In our brief discussion of surgery in Section 4.2, we saw that in context of normal B-structures we can use surgery to try to obtain an h-bordism from a given one.

Thus we have a dichotomy of approaches to classification problems: those in which surgery techniques work ie high dimensional manifolds perhaps with normal B-structures; and those for which surgery techniques fail. We continue to consider the former.

Proposition 4.2.9 motivates us to consider normal B-structures, and B-bordisms for which the normal map $f : M \to B$ is a k-equivalence, for k as close to the middle dimension as we can manage. In this context, a B-bordism can then be improved almost to an h-bordism but potentially failing in the middle dimension.

As mentioned, surgery in the middle dimension is more difficult. The Wall obstruction theorem (Proposition 4.2.10) does not seem to be directly applicable in the contexts we have considered, since the condition that $f|_{M_i}$ is a homotopy equivalence is too strong.

Note that the obstruction is, in some contexts, the signature which is at least suggestive of a link between surgery obstructions and characteristic numbers. The converse, that characteristic numbers are an obstruction to finding an h-bordism, is clear. Consider the following example of Kreck.

Proposition 5.3.9. *Let M_i, $i = 0, 1$, be closed simply connected spin 7-manifolds. Suppose that $H^2(M_i) = \mathbb{Z}u_i$, $H^3(M_i) = 0$, $H^4(M_i)$ is a finite cyclic group generated by u_i^2, and that $|H^4(M_0)| = |H^4(M_1)|$. Then $M_0 \cong M_1$ if and only if there exists a bordism $(W; M_0, M_1)$ such that there exists $z \in H^2(W)$ such that under $H^2(W) \to H^2(M)$ $z \mapsto (-1)^i u_i$ and $\langle p_1^2(W), [W, \partial W]\rangle$, $\langle p_1(W)z^2, [W, \partial W]\rangle$, $\langle z^4, [W, \partial W]\rangle$, and $\sigma(W)$ all vanish.*

(See [67, Proposition 3.2] in which analogous statements are made for nonsmooth and non-spin cases.) The constraints on the bordism W amount to the relative characteristic numbers vanishing. We have the following improvement, from the perspective of application (See [67, Proposition 3.1]).

Proposition 5.3.10. *Let M_i be as in Proposition 5.3.9. Then $M_0 \cong M_1$ if and only if their Kreck-Stolz invariants agree.*

By Proposition 4.3.5, the existence of a bordism is immediate. Thus, in Proposition 5.3.10 all constraints on the bordism have been exchanged for constraints on the boundary. This has the added advantage in applications that we no longer need to consider pairs of manifolds when classifying, but can identify the class of a single manifold by computing the invariants.

For a second example, [32] completes the classification of Wall and Wilkens for 2-connected 7-manifolds. The diffeomorphism class of a 2-connected 7-manifold M is determined by the 4-tuple $(H^4(M), q_M^\circ, \mu, p_M)$ consisting of $H^4(M)$, a family of quadratic refinements of the torsion linking form q_M°, the generalized Eells-Kuiper invariant μ of M, and the spin class of M, p_M (See [32, Theorem 1.3]).

In the context of the Kreck-Stolz invariants, the $\mathfrak{C}$-model is trivial, and the notion of relative characteristic numbers is unambiguous. For the generalized Eells-Kuiper invariant, the $\mathfrak{C}$-model is determined by $\mathrm{gd}(q_1(M))$, the greatest divisor of the spin class. The relative characteristic numbers then include a mod $\widehat{m}$ term. Although not written in these terms, the results of [32] are tantamount to proving that the bordism can be improved to h-bordism if the characteristic numbers vanish.

In general, which relative characteristic numbers are well defined is determined by the $\mathfrak{C}$-model. We expect that one should be able to reformulate the recent results of Kreck [65] on the classification of simply connected 7-manifolds with torsion free second homology in terms of the BDIs described in Section 5.2.3.

In the case that G is a coboundary group but is not stabilized, for example $G = \mathsf{G}_2$ or $G = \mathsf{U}(3)$, we cannot use the same technique. Instead we should consider questions such as: given a fixed class of some stable structure, what is the classification of compatible G-structures. For example, [31] consider the question of G_2 structures over a given spin manifold.

Recall that a natural coboundary group is $\mathsf{Spin}(7)$, and that a $\mathsf{Spin}(7)$ structure on an 8-manifold is determined by a nowhere vanishing spinor field. In [31], the authors analyse spin bordisms together with a spinor field that is allowed to vanish transversally within the interior. Together with the results of the classification of 2-connected spin 7-manifolds they were able to prove Proposition 5.3.11 below.

For a closed connected spin 7-manifold M, let $\mathcal{G}_2$ denote the homotopy classes of G_2-structures on M. Then $\pi_0\mathcal{G}_2(M) \cong \mathbb{Z}$ ([31, Lemma 1.1]). Denote the quotient by spin diffeomorphism by $\pi_0\overline{\mathcal{G}}_2(M)$.

Proposition 5.3.11 ([31, Theorem 1.12 & 1.17])**.** *Let M be a 2-connected closed spin 7-manifold with torsion free cohomology such that $q_1(M) \neq 0$, and $m = \mathrm{gd}(q_1(M))$. Then*

$$|\pi_0\overline{\mathcal{G}}_2(M)| = 24 \cdot \mathrm{Num}\left(\frac{m}{2^4 \cdot 7}\right) \tag{5.72}$$

and (ν, ξ) is a complete invariant of $\pi_0\overline{\mathcal{G}}_2(M)$.

Chapter 6

The TCS construction

Ultimately, we wish to compute some of the invariants defined in Chapter 5 on G_2-manifolds. With this chapter we begin the second part of this thesis: constructing examples and carrying out computations of the invariants.

The Twisted Connected Sum construction was first presented by Kovalev [64], and later extended by Corti Haskins Nordström Pacini [25]. We review some aspects of the construction relevant to our needs, referring heavily to [25]. In Section 6.2, we extend the TCS construction to an almost complex coboundary in the sense of Definition 5.3.2. We use this later to carry out the computations of the invariants of Section 5.2.5 on TCS manifolds. Much of this section is taken verbatim from the preprint [105].

6.1 The TCS construction

The TCS involves glueing together two open manifolds with holonomy $\mathsf{SU}(3)$. Each of these manifolds is the product of a circle and an open Calabi-Yau threefold with a tame asymptotic end. We consider first these *Asymptotically Cylindrical Calabi-Yaus* (ACyl CYs), and the nature of the glueing in the TCS construction. We then discuss how to construct ACyl CYs from projective varieties.

6.1.1 TCS from Asymptotically cylindrical Calabi-Yaus

Let $(V_\infty, \omega_\infty, \Omega_\infty)$ be a $(2n)$-dimensional complete Calabi-Yau manifold (ie has holonomy $\mathsf{SU}(n)$). Suppose that $V_\infty \cong \mathbb{R}^+ \times X$ where X is a smooth closed manifold. We say V_∞ is a *Calabi-Yau (half)-cylinder* if $(\omega_\infty, \Omega_\infty)$ is $\mathbb{R}^+$-invariant and that the associated metric g_∞ is a product metric $dt^2 + g_X$. X is the *cross section* of V_∞.

Definition 6.1.1 ([25, Definition 3.3])**.** Let (V, ω, Ω) be a complete Calabi-Yau manifold. Suppose that there exists a compact set $K \subset V$, a Calabi-Yau cylinder V_∞, and a diffeomorphism $\Phi : V_\infty \to V \setminus K$ such that for all $k \in \mathbb{N}_0$, for some $\lambda > 0$, and as $t \to \infty$

$$\begin{aligned} \Phi^*\omega - \omega_\infty &= d\varrho, \quad \text{for } \varrho \text{ such that } |\nabla^k \varrho| = O(e^{-\lambda t}) \\ \Phi^*\Omega - \Omega_\infty &= d\varsigma, \quad \text{for } \varsigma \text{ such that } |\nabla^k \varsigma| = O(e^{-\lambda t}) \end{aligned} \tag{6.1}$$

where ∇, and $|\cdot|$ are defined in terms of g_∞ on V_∞. Then V is an *asymptotically cylindrical Calabi-Yau* (ACyl CY) manifold. V_∞ is the *asymptotic end* of V. The *cross section* of V is the cross section of V_∞.

The rate of convergence is important to the analysis justifying the ultimate existence of a torsion-free G_2-structure in the TCS. We care only that the $\mathsf{SU}(3)$-structure of the asymptotic end is an arbitrarily small perturbation from the cylindrical Calabi-Yau structure as we move along the neck. In particular, the torsion free structure is homotopic to an $\mathsf{SU}(3)$-structure that eventually agrees with that of the asymptotic end. For us the cross section is always $\Sigma \times S^1$, the product of a K3 and a circle. The asymptotic end has Calabi-Yau structure

$$\omega_\infty = dt \wedge d\alpha + \omega^I, \qquad \Omega_\infty = (d\alpha - idt) \wedge (\omega^J + i\omega^K) \tag{6.2}$$

for coordinates $(x, \alpha, t) \in \Sigma \times S^1 \times \mathbb{R}$, where $(\omega^I, \omega^J, \omega^K)$ is a hyper-Kähler triple on Σ. Recall that $\mathsf{SU}(2) = \mathsf{Sp}(1)$, so in dimension 4 Calabi-Yau geometry and hyper-Kähler geometry are synonymous. This hyper-Kähler K3 in the cross section of the asymptotic end is the *K3 at infinity* of V.

From an ACyl CY threefold (V, ω, Ω) we can construct a torsion free G_2-structure on $M := V \times S^1$ given by

$$\varphi := d\beta \wedge \omega + \mathrm{Re}(\Omega) \tag{6.3}$$

where β is the coordinate for the 'external' circle factor. Sensibly extending our definitions of ACyl CY to G_2, the asymptotically cylindrical G_2 end of M has G_2-structure

$$\varphi_\infty := d\beta \wedge dt \wedge d\alpha + d\beta \wedge \omega^I + d\alpha \wedge \omega^J + dt \wedge \omega^K \tag{6.4}$$

Again, φ is an arbitrarily small perturbation from φ_∞ as we move along the neck. We now turn our attention to defining a glueing on a pair of asymptotically cylindrical G_2 manifolds.

Definition 6.1.2. Let $(\omega^I_\pm, \omega^J_\pm, \omega^K_\pm)$ be hyper-Kähler triples on K3 surfaces $\Sigma_\pm$ respectively. A diffeomorphism $r : \Sigma_+ \to \Sigma_-$ is a *hyper-Kähler Rotation* (HKR) if $r^*\omega^I_- = \omega^J_+$, $r^*\omega^J_- = \omega^I_+$ and $r^*\omega^K_- = -\omega^K_+$.

Suppose that (M, φ) is an asymptotically cylindrical G_2-manifold obtained from ACyl CY (V, ω, Ω), and that Φ in Definition 6.1.1 has been specified. Adopting the notation of Definition 6.1.1, for fixed $T \gg 0$ we define forms (ω_T, Ω_T)

$$\omega_T := \omega - d(\eta_T(t)\varrho), \qquad \Omega_T := \Omega_\infty - d(\eta_T(t)\varsigma) \tag{6.5}$$

where $\eta_T : \mathbb{R} \to [0, 1]$ is a smooth cutoff function such that $\eta_T(t) = 0$ for $t < T - 1$, and $\eta_T(t) = 1$ for $t > T$. These forms are closed and interpolate between two torsion free $\mathsf{SU}(3)$-structures on the neck. Note that (ω_T, Ω_T) is not an $\mathsf{SU}(3)$-structure. Despite this, for sufficiently large T, the 3-form on M

$$\varphi_T := d\beta \wedge \omega_T + \mathrm{Re}(\Omega_T) \tag{6.6}$$

is a G_2-structure on M since the space of 3-forms defining G_2-structures is open in the space of 3-forms, and that φ_T is a small perturbation from φ. The torsion of φ_T is $O(e^{-\lambda T})$.

Suppose we have a pair of ACyl CY threefolds $V_\pm$ with cross sections $\Sigma_\pm \times S^1$, together with a HKR $r : \Sigma_+ \to \Sigma_-$ between K3s at infinity. We fix some coordinates $\Phi_\pm$ on the necks of $V_\pm$. Define a map

$$\begin{aligned} G_T : \Sigma_+ \times S^1 \times S^1 \times [T, T+1] &\to \Sigma_- \times S^1 \times S^1 \times [T, T+1] \\ (x, \alpha, \beta, T+t) &\mapsto (r(x), \beta, \alpha, T+1-t) \end{aligned} \tag{6.7}$$

Note that $G_T^* \varphi_\infty = \varphi_\infty$. Let $M(T)$ denote the truncation of M at neck length $T+1$.

Definition 6.1.3. Let $V_\pm$ be a pair of ACyl CY threefolds with cross section $\Sigma_\pm \times S^1$ respectively. Let $r : \Sigma_+ \to \Sigma_-$ be a HKR between the K3s at infinity. Then their *twisted connected sum via r $M := M_+(T) \cup_{G_T} M_-(T)$* is a 7-manifold defined by the glueing of $M_\pm(T)$ by the diffeomorphism G_T.

We endow M with the closed G_2-structure $\varphi_T := \varphi_{+,T} \cup_{G_T} \varphi_{-,T}$. We shall often refer to M as the TCS of $V_\pm$, assuming that r is clear from the context. Clearly, the twisted connected sum depends on T. However, we shall simply assume that T is sufficiently large such that the following Theorem holds. The resulting manifold (M, φ) is then closed with holonomy precisely G_2.

Theorem 6.1.4 ([25, Theorem 3.12])**.** *Let $V_\pm$ be pair of ACyl CY threefolds with a hyper-Kähler rotation $r : \Sigma_+ \to \Sigma_-$ between K3s at infinity. Let (M, φ_T) be their twisted connected sum. For sufficiently large T there exists a torsion free perturbation of φ_T within its cohomology class.*

6.1.2 ACyl CYs from algebraic geometry

ACyl CYs can be constructed from complex threefolds called building blocks. The glueing of ACyl CYs in the TCS construction can be recast as a *matching* of building blocks.

Definition 6.1.5. A *building block* is a nonsingular algebraic threefold Z together with a projective morphism $f : Z \to \mathbb{P}^1$ satisfying the following assumptions:
(i) the anticanonical class $-K_Z \in H^2(Z)$ is primitive.
(ii) $\Sigma := f^*(\infty)$ is a smooth K3 with $\Sigma \in |-K_Z|$.
Identify $H^2(\Sigma)$ with the K3 lattice L, ie choose a marking, and let $N := \mathrm{Im}(H^2(Z) \to H^2(\Sigma))$.
(iii) The inclusion $N \hookrightarrow L$ is primitive.
(iv) The group $H^3(Z)$ is torsion free.
We may refer to a building block by $f : Z \to \mathbb{P}^1$, (Z, Σ), or simply Z.

Definition 6.1.6. Let $Z_\pm$ be a pair of complex threefolds, and $\Sigma_\pm \in |-K_{Z_\pm}|$ be smooth anticanonical divisors. Let $k_\pm \in H^2(Z_\pm; \mathbb{R})$ be Kähler classes and $\Pi_\pm < H^2(\Sigma; \mathbb{R})$ be 2-planes of type $(2,0)+(0,2)$ treating $H^2(\Sigma; \mathbb{R}) < H^2(\Sigma; \mathbb{C})$. A diffeomorphism $r : \Sigma_+ \to \Sigma_-$ is a *matching* if $r^*(k_-|_{\Sigma_-}) \in \Pi_+$, $k_+|_{\Sigma_+} \in r^*(\Pi_-)$, and $\Pi_+ \cap r^*(\Pi_-)$ is nonempty.

We refer to r as a matching of $(Z_\pm, \Sigma_\pm, k_\pm)$ or a matching of $(Z_\pm, \Sigma_\pm)$ with respect to $k_\pm$. Where $k_\pm$ and/or $\Sigma_\pm$ exist but are not specified we refer to r simply as a matching of $(Z_\pm, \Sigma_\pm)$ or of $Z_\pm$. Note that our definition of a matching does not stipulate that $Z_\pm$ are building blocks. This allows for a cleaner presentation in subsequent sections. The following result allows us to reformulate finding a HKR between building blocks to finding a matching.

Proposition 6.1.7 ([25, Corollary 6.4]). *Let $(Z_\pm, \Sigma_\pm)$ be a pair of building blocks that have a matching $r : \Sigma_+ \to \Sigma_-$ with respect to $k_\pm$. Then $V_\pm := Z_\pm \setminus \Sigma_\pm$ admits a structure reduction to an ACyl CY structure (ie a torsion free $\mathsf{SU}(3)$-structure) such that r is a HKR on the K3s at infinity.*

We extend the definitions and notions of ACyl CYs to building blocks via their associated ACyl CYs. For example, we refer to the TCS of a pair of building blocks to mean the TCS of their associated ACyl CYs.

6.2 A coboundary

We now describe a construction of almost complex coboundaries to TCS manifolds. We start by noting that any manifold that is a sphere bundle has a coboundary given by the corresponding disc bundle. Thus, an asymptotically cylindrical G_2 $M = V \times S^1$ has a coboundary $V \times D^2$. We will use the complex structure of the building blocks to define almost complex structures on the coboundaries. The fiddly part is extending the glueing between building blocks to the coboundaries.

6.2.1 A compatible $\mathsf{U}(3)$-structure

We will describe a 'nearby' G_2-structure on a TCS manifold M which is homotopic to φ given by Theorem 6.1.4, and which aids our construction of a TCS $\mathsf{U}(4)$-coboundary in the sense of Definition 5.3.2.

Suppose that M is a TCS obtained from ACyl CYs $V_\pm$. We endow $V_\pm$ with new $\mathsf{SU}(3)$-structures $(\omega'_{\pm,T}, \Omega'_{\pm,T})$ such that

$$(\omega'_{\pm,T}, \Omega'_{\pm,T}) = \begin{cases} (\omega_\pm, \Omega_\pm) & \text{for } t < T-1 \\ (\omega_{\pm,\infty}, \Omega_{\pm,\infty}) & \text{for } t \geq T \end{cases} \tag{6.8}$$

We stipulate that $(\omega'_{\pm,T}, \Omega'_{\pm,T})$ define $\mathsf{SU}(3)$-structures, rather than remain closed in contrast to $(\omega_{\pm,T}, \Omega_{\pm,T})$ in (6.5). The space of $\mathsf{SU}(3)$-structures above $X = \Sigma \times S^1$ understood as reductions of frames of $TX \times \mathbb{R}$, is a (albeit infinite dimensional) manifold. In paticular, the space of $\mathsf{SU}(3)$-structures is locally path connected. This ensures the existence $(\omega'_{\pm,T}, \Omega'_{\pm,T})$ for sufficiently large T.

We endow $M_\pm = V_\pm \times S^1$ with the associated G_2-structures $\varphi'_{\pm,T}$. Let $\overline{V}_\pm$ be the compact manifold with boundary obtained by truncating $V_\pm$ at neck length T. Let $\overline{M}_\pm := \overline{V}_\pm \times S^1$. We introduce $M_0 := \Sigma_+ \times T^2 \times [0, \tau]$ to be a piece of the neck and endow it with the asymptotically cylindrical G_2-structure φ_∞. The parameter $\tau > 0$ will be chosen later and has no bearing on the homotopy class of the resulting G_2-structure. Note that M_0 has two boundary components both diffeomorphic to $\Sigma_+ \times T^2$.

We define glueings on the boundaries of $\overline{M}_\pm$ to M_0 by

$$\begin{aligned} G_\pm : \Sigma_\pm \times T^2 &\to \Sigma_+ \times T^2 \\ (x, \alpha, \beta) &\mapsto \begin{cases} (x, \alpha, \beta) & G_+ \\ (r^{-1}(x), \beta, \alpha) & G_- \end{cases} \end{aligned} \tag{6.9}$$

We note that in the next section we will construct a different hyper-Kähler structure on M_0, and with respect to which r will be an isomorphism of hyper-Kähler manifolds. We think of M as

$$M = \overline{M}_+ \cup_{G_+} M_0 \cup_{G_-} \overline{M}_- \tag{6.10}$$

The G_2-structure φ', formed from glueing together $\varphi'_{\pm,T}$ and φ_∞, is homotopic to the torsion free G_2-structure φ.

6.2.2 A TCS coboundary

Let M be the TCS of ACyl CYs $(V_\pm, \Sigma_\pm)$ with HKR $r : \Sigma_+ \to \Sigma_-$. The idea behind the construction of a $\mathsf{U}(4)$-coboundary is relatively straightforward—'rounding' the glueing of the coboundary used in [30] in order to handle a $\mathsf{U}(4)$-structure.

Let $W_\pm := \overline{V}_\pm \times D$, where $D \subset \mathbb{C}$ is the complex unit disc. We endow $W_\pm$ with the product $\mathsf{SU}(4)$-structure determined by the $\mathsf{SU}(3)$-structure $(\omega'_{\pm,T}, \Omega'_{\pm,T})$ on $\overline{V}_\pm$, and the $\mathsf{SU}(1)$-structure on D.

Note that $W_\pm$ are manifolds with corners: the boundary is the union of manifolds with boundary. We avoid engaging with the technicalities of the theory of manifolds with corners. We refer to the boundary components of $W_\pm$ as either *internal* or *external*. $\overline{V}_\pm \times S^1 \subset \partial W_\pm$ is the (external) boundary component identified with $\overline{M}_\pm$, while $\Sigma_\pm \times S^1 \times D \subset \partial W_\pm$ is the internal boundary. These two components meet along a common 6-dimensional submanifold $\Sigma_\pm \times T^2$. By construction the $\mathsf{U}(4)$-structure on $W_\pm$ restricts to a $\mathsf{U}(3)$-structure on $\overline{M}_\pm$ compatible with $\varphi'_\pm$.

We define a further manifold (with corners) $W_0 := \Sigma \times Q$, where Σ is a K3 and

$$Q := \{(z, w) \in \mathbb{C}^2 : \ |z|, |w| \leq 2, \ (2 - |z|)^2 + (2 - |w|)^2 \geq 1\} \tag{6.11}$$

The boundary of Q has three components

$$\begin{aligned} E_+ &:= \{(z, w) \in Q : \ |z| = 2\} \\ E_- &:= \{(z, w) \in Q : \ |w| = 2\} \\ Q_0 &:= \{(z, w) \in Q : \ (2 - |z|)^2 + (2 - |w|)^2 = 1\} \end{aligned} \tag{6.12}$$

We refer to $\Sigma \times E_\pm$ as the internal boundary components while $\Sigma \times Q_0$ is the (external) boundary of W_0. We will postpone a description of a $\mathsf{U}(4)$-structure on W_0. It is clear that as smooth manifolds with boundary $M_0 \cong \Sigma \times Q_0$.

We define the following glueing maps between the internal boundaries of $W_\pm$ and W_0.

$$G_\pm : \Sigma_\pm \times S^1 \times D \to \Sigma \times E_\pm$$
$$(x, \alpha, w) \mapsto \begin{cases} (x, 2e^{i\alpha}, w) & \text{for } G_+ \\ (r^{-1}(x), w, 2e^{i\alpha}) & \text{for } G_- \end{cases} \tag{6.13}$$

This is an extension of the glueing maps of $\overline{M}_\pm$ to M_0 to the interior of the internal boundaries.

Proposition 6.2.1. *Let $W := W_+ \cup_{G_+} W_0 \cup_{G_-} W_-$ be the quotient space. Then W is a smooth manifold with boundary and $\partial W \cong M$.*

Proof. The glueing map G_+ is a diffeomorphism onto its image. A neighbourhood of $\Sigma \times E_+ \subset W$ has a natural parameterization of $\Sigma \times E_+ \times (-\varepsilon, \varepsilon)$, where $t \in (-\varepsilon, 0]$ belongs W_+, and otherwise W_0. The analogous holds for G_-. As the glueing is an extension of a glueing of M in (6.10), the boundary is smooth. That the boundary is diffeomorphic to M is clear from construction. □

Let M be the TCS of a pair of building blocks $Z_\pm$, and W the coboundary given in Proposition 6.2.1. We can extend the embedding of $\overline{V}_\pm \to W$, given by $\overline{V}_\pm \to V_\pm \times \{0\} \subset W$, to the associated building blocks such that $Z_\pm \to W_+ \cup_{G_+} W_0 \cup_{G_-} W_-$.

6.2.3 The $\mathsf{U}(4)$-structure on the coboundary

One would like to define a $\mathsf{U}(3)$-structure on M, use Lemma 5.3.3 to prove it is compatible with φ', and prove it extends to a $\mathsf{U}(4)$-structure on the interior of W. In practice, it seems easier to define a $\mathsf{U}(4)$-structure on W and check that the restriction to the boundary is compatible with φ'. The details are a little ugly.

Suppose that M is the TCS of some pair of building blocks $Z_\pm$, and that W is the coboundary as in Proposition 6.2.1. We have an embedding of $\overline{V}_\pm \to W$ by identifying $\overline{V}_\pm \cong \overline{V} \pm \times \{0\} \subset W_\pm \subset W$. The complement of $\overline{V}_\pm$ in $Z_\pm$ is holomorphic to $\Sigma_\pm \times D$ where $D \subset \mathbb{C}$ is the complex unit disk. We define an embedding $j_\pm : Z_\pm \to W$ extending $\overline{V}_\pm$ by $\Sigma_\pm \times D \to \Sigma \times Q$. The remainder of this section provides the proof of the following proposition.

Proposition 6.2.2. *Let W be as in Proposition 6.2.1. Then W admits a $\mathsf{U}(4)$-structure such that it is a $\mathsf{U}(4)$-coboundary to (M, φ').*

Moreover, if the TCS is obtained from building blocks $Z_\pm$, then the pullback of the $\mathsf{U}(4)$-structure via $j_\pm$ reduces to a $\mathsf{U}(3)$-structure that is homotopic to that induced by the complex structure on $Z_\pm$.

We define a $\mathsf{U}(4)$-structure on W by considering each of its components in turn, and checking that they agree across the glueings. $W_\pm$ are each equipped with a product $\mathsf{SU}(4)$-structure that on restricting to its external boundary agrees with the $\mathsf{SU}(3)$-structure on $\overline{M}_\pm$.

We define a $\mathsf{U}(4)$-structure on W_0 while considering the constraints introduced by the glueing of the internal boundaries of $W_\pm$ and the G_2-structure on M_0. The $\mathsf{U}(4)$-structure on W_0 will reduce to an $\mathsf{SU}(2) \times \mathsf{U}(2)$-structure and we consider Σ and Q in turn. W_0 can be viewed as a K3 fibration over base Q. The metric is the product metric $g_\Sigma + g_Q$ where g_Σ is the Ricci flat metric shared by $\Sigma_\pm$; and g_Q is the metric on Q that is yet to be determined. Let $(\omega^I, \omega^J, \omega^K)$ be the hyper-Kähler triple of Σ_+. We will endow each K3 fibre with a symplectic structure belonging to the S^2-family of forms this triple defines.

Let Q inherit the complex structure from $\mathbb{C}^2$, which will agree with the complex structure of the images of $G_\pm$ on $E_\pm$. Let Q have hermitian metric $h_1|dz|^2 + h_2|dw|^2$, where h_1, and h_2 are positive real functions on Q. The constraints that this agrees with the images of $G_\pm$ are that $h_1 = 1/4$, $h_2 = 1$ on E_+ and $h_1 = 1$, $h_2 = 1/4$ on E_-. We will give a precise description of the symplectic structure on K3 fibres and of h_i shortly. That such functions exist is clear and any such choices will endow on W_0 a $\mathsf{U}(4)$-structure.

As the $\mathsf{U}(4)$-structures of $W_\pm$ agree with that of W_0 across their respective internal boundaries, so the structures can be glued to form $\mathsf{U}(4)$-structures on W. As noted above, the $\mathsf{U}(4)$-structure on $W_\pm$ restricts to $\mathsf{U}(3)$-structures $(\partial_\beta, \omega'_{\pm,T}, \Omega'_{\pm,T})$ on $\overline{M}_\pm$ respectively.

It remains to specify h_i and the symplectic structure of the K3 fibres of W_0 such that the restriction to M_0 is compatible with the G_2-structure φ_∞. If (v, ω, g) is the $\mathsf{U}(3)$-structure, then this amounts to checking that $\varphi_\infty \lrcorner v = \omega$, and $g_{\varphi_\infty} = g$ by Lemma 5.3.3.

Fix coordinates on Q_0,

$$\begin{aligned} f : S^1 \times S^1 \times [0, \tfrac{\pi}{2}] &\to Q \\ (\alpha, \beta, \vartheta) &\mapsto \Big((2 - \sin\vartheta)e^{i\alpha}, (2 - \cos\vartheta)e^{i\beta}\Big) \end{aligned} \tag{6.14}$$

The derivative of f is

$$df = \begin{pmatrix} i(2 - \sin\vartheta)e^{i\alpha} & 0 & -\cos\vartheta e^{i\alpha} \\ 0 & i(2 - \cos\vartheta)e^{i\beta} & \sin\vartheta e^{i\beta} \end{pmatrix} \tag{6.15}$$

Thus, the pullback metric on Q_0 by f is

$$\begin{aligned} g_{Q_0} &:= f^* g_Q \\ &= (h_1 \cos^2\vartheta + h_2 \sin^2\vartheta)d\vartheta^2 + h_1(2 - \sin\vartheta)^2 d\alpha^2 + h_2(2 - \cos\vartheta)^2 d\beta^2 \end{aligned} \tag{6.16}$$

The outward normal of M_0 is solely in the Q component while the outward normal of Q_0 is

$$N = (h_2 \sin\vartheta e^{i\alpha}, h_1 \cos\vartheta e^{i\beta})^T \in TQ \tag{6.17}$$

Thus $J_Q(N)$ is a vector field on Q_0, where $J_Q : TQ \to TQ$ is the almost complex structure. Let v' be the preimage of $J_Q(N)$ under df. Then v' is given in $(\vartheta, \alpha, \beta)$ coordinates by

$$v' = \frac{h_2 \sin\vartheta}{2 - \sin\vartheta}\partial_\alpha + \frac{h_1 \cos\vartheta}{2 - \cos\vartheta}\partial_\beta \tag{6.18}$$

We set the distinguished unit vector field $v := v'/|v'|$ to be the normalized vector field.

Consider the symplectic form $\omega_Q = g_Q(J_Q\cdot,\cdot)$ on Q when pulled back to Q_0

$$\begin{aligned}\omega_{Q_0} &:= f^*\omega_Q \\ &= d\vartheta \wedge (-h_1 \cos\vartheta(2-\sin\vartheta)d\alpha + h_2 \sin\vartheta(2-\cos\vartheta)d\beta)\end{aligned} \tag{6.19}$$

Thus the $\mathsf{U}(3)$-structure on M_0 is $(v, g_\Sigma + g_{Q_0}, \omega_\Sigma + \omega_{Q_0})$, where ω_Σ is the symplectic form on the K3 fibres.

The G_2-form on M_0 is

$$\varphi_\infty = d\beta \wedge dt \wedge d\alpha + d\beta \wedge \omega^I + d\alpha \wedge \omega^J + dt \wedge \omega^K \tag{6.20}$$

where $t = t(\vartheta)$ is a reparameterization of ϑ. The metric associated to φ_∞ is $g_{\varphi_\infty} = g_{\Sigma_0} + dt^2 + d\alpha^2 + d\beta^2$. Thus, if this is to agree with g_∞, then on ∞

$$h_1 = (2-\sin\vartheta)^{-2}, \quad h_2 = (2-\cos\vartheta)^{-2}, \tag{6.21}$$

while

$$\frac{dt}{d\vartheta} = \left(\frac{\cos^2\vartheta}{(2-\sin\vartheta)^2} + \frac{\sin^2\vartheta}{(2-\cos\vartheta)^2}\right)^{\frac{1}{2}} \tag{6.22}$$

As the right-hand side is strictly positive, so $t(\vartheta)$ is strictly increasing.

In terms of parameter t, the symplectic form and distinguished vector of the $\mathsf{U}(3)$-structure on M_0 are

$$\begin{aligned}\omega &= \omega_\Sigma + \frac{d\vartheta}{dt}\left(\frac{\cos\vartheta}{2-\sin\vartheta}dt \wedge d\alpha + \frac{\sin\vartheta}{2-\cos\vartheta}d\beta \wedge dt\right), \\ v &= \frac{d\vartheta}{dt}\left(\frac{\sin\vartheta}{2-\cos\vartheta}\partial_\alpha + \frac{\cos\vartheta}{2-\sin\vartheta}\partial_\beta\right)\end{aligned} \tag{6.23}$$

Contracting φ_∞ with v gives

$$\varphi_\infty \lrcorner v = \frac{d\vartheta}{dt}\left(\frac{\sin\vartheta}{2-\cos\vartheta}(d\beta \wedge dt + \omega^J) + \frac{\cos\vartheta}{2-\sin\vartheta}(dt \wedge d\alpha + \omega^I)\right) \tag{6.24}$$

We now complete our description of the $\mathsf{U}(4)$-structure on W_0. Let $\rho := \tan^{-1}\left(\frac{2-|z|}{2-|w|}\right)$, and note that $\rho \circ f(\vartheta, \alpha, \beta) = \vartheta$.

$$\omega_\Sigma(\rho) := \left(\frac{\cos^2\rho}{(2-\sin\rho)^2} + \frac{\sin^2\rho}{(2-\cos\rho)^2}\right)^{-\frac{1}{2}}\left(\frac{\sin\rho}{2-\cos\rho}\omega^I + \frac{\cos\rho}{2-\sin\rho}\omega^J\right) \tag{6.25}$$

and extend the definitions of h_1 and h_2 to the interior of Q by

$$h_1 = (2-\sin\rho)^{-2}, \quad h_2 = (2-\cos\rho)^{-2} \tag{6.26}$$

Finally, the image of $Z_\pm$ restricted to W_0 is either $\Sigma \times \{|z| \leq 2\} \times \{0\}$ or $\Sigma \times \{0\} \times \{|w| \leq 2\}$. It is clear then that the pullback of the $\mathsf{U}(4)$-structure is homotopic to the $\mathsf{U}(3)$-structure on $Z_\pm$ given by its complex structure. This concludes our proof of Proposition 6.2.2.

6.3 Topology

In this section we shall describe the cohomology algebras of the TCS manifold M, and the coboundary W, as well as the long exact sequence given by (W, M). We also discuss the characteristic classes and numbers of these manifolds. Let us fix the following notation.

Definition 6.3.1. Let $L := H^2(\Sigma)$ be the abstract K3 lattice. Suppose we have a TCS from the building blocks $(Z_\pm, \Sigma_\pm)$. Let $K_\pm := \ker(H^2(Z_\pm) \to L)$, and the polarization lattices $N_\pm := \mathrm{im}(H^2(Z_\pm) \to L)$.

For a pair of primitive embeddings $(N_\pm \to L)$ (which may or may not have come from a TCS)

(i) $T_\pm := N_\pm^\perp \subset L$, the transcendental lattices;
(ii) $N_0 := N_+ \cap N_-$, the intersection lattice;
(iii) $P := N_+ + N_-$ the span of the images of $N_\pm \hookrightarrow L$;
(iv) $P_\pm := N_\pm \cap T_\mp$;
(v) $\Lambda_\pm := P_\mp^{\perp_P} \subset P$.

Note that our notation differs from [25] as here K is the kernel of $H^2(Z) \to H^2(\Sigma)$, and not of $H^2(V) \to H^2(\Sigma)$. By [25, Lemma 4.2], the Poincaré dual c_1 of $\Sigma \subset Z$ induces the we exact sequence

$$0 \to \mathbb{Z} \xrightarrow{c_1} H^2(Z) \to H^2(V) \to 0. \tag{6.27}$$

6.3.1 Cohomology of M

A detailed account of the topology of M is given in [25, Section 4]. The Van Kampen Theorem applied to $M = M_+ \cup M_-$ implies that $\pi_1(M) \cong \pi_1(V_+) \times \pi_1(V_-)$. The ACyl CYs associated to building blocks are simply connected. Thus the TCS of building blocks is simply connected.

A more detailed description of $H^\bullet(M)$ is found in [25, Theorem 4.8]. Applying Mayer-Vietoris to (M_+, M_-) we get that $H^1(M) = 0$ and

$$\begin{aligned} H^2(M) &\cong (K_+)_{/\langle \mathrm{PD}(\Sigma_+)\rangle} \oplus (K_-)_{/\langle \mathrm{PD}(\Sigma_-)\rangle} \oplus N_0, \\ H^3(M) &\cong L_{/P} \oplus P_- \oplus P_+ \oplus H^3(Z_+) \oplus H^3(Z_-) \oplus (K_+ \oplus K_-)_{/\langle \mathrm{PD}(\Sigma)\rangle}, \\ H^4(M) &\cong (T_+ \cap T_-) \oplus L_{/(N_- + T_+)} \oplus L_{/(N_+ + T_-)} \\ &\qquad \oplus H^3(Z_+) \oplus H^3(Z_-) \oplus (K_+^\vee \oplus_0 K_-^\vee) \end{aligned} \tag{6.28}$$

where $K_+^\vee \oplus_0 K_-^\vee$ denotes the codimensional 1 subspace of $(K_+ \oplus K_-)^\vee \cong K_+^\vee \oplus K_-^\vee$, annihilating the subspace $\langle \mathrm{PD}(\Sigma)\rangle < K_+ \oplus K_-$. Note that some of these splittings are not canonical so using $\cong$ is a slight abuse of notation. The torsion part $TH^\bullet(M)$ of $H^\bullet(M)$ has the following form

$$\begin{aligned} TH^3(M) &\cong TH^3(Z_+) \oplus TH^3(Z_-) \oplus \mathrm{Tor}(L_{/P}, \\ TH^4(M) &\cong TH^3(Z_+) \oplus TH^3(Z_-) \oplus \mathrm{Tor}(L_{/(N_+ + T_-)}) \oplus \mathrm{Tor}(L_{/(N_- + T_+)}). \end{aligned} \tag{6.29}$$

6.3.2 Cohomology of W

The long exact sequence in cohomology

$$\cdots \to H^k(W, M; R) \to H^k(W; R) \to H^k(M; R) \to H^{k+1}(W, M; R) \to \dots \tag{6.30}$$

relates the cohomologies of W and M and the relative cohomology of the pair (W, M).

Let $\widetilde{W}_\pm := W_\pm \cup_{G_\pm} W_0$. We apply Mayer-Vietoris to $W = \widetilde{W}_+ \cup_{W_0} \widetilde{W}_-$, noting that $\widetilde{W}_\pm \simeq Z_\pm$ and $W_0 \simeq \Sigma$. Thus

$$\cdots \to H^k(W) \to H^k(Z_+) \oplus H^k(Z_-) \to H^k(\Sigma) \to H^{k+1}(W) \to \dots \tag{6.31}$$

It follows that

$$\begin{aligned} &H^1(W) \cong 0, && H^2(W) \cong K_+ \oplus K_- \oplus N_0, \\ &H^3(W) \cong L/(N_+ + N_-) \oplus H^3(Z_+) \oplus H^3(Z_-), && H^4(W) \cong H^4(Z_+) \oplus_0 H^4(Z_-) \end{aligned} \tag{6.32}$$

where $H^4(Z_+) \oplus_0 H^4(Z_-)$ denotes the codimension 1 subspace of $H^4(Z_+) \oplus H^4(Z_-)$ of pairs that share an image in $H^4(\Sigma)$.

Recall that for CW complex pair (X, Y), with $(A, C), (B, D) \subset (X, Y)$, we have a relative Mayer-Vietoris sequence

$$\cdots \to H^k(A \cup B, C \cup D) \to H^k(A, C) \oplus H^k(B, D) \to H^k(A \cap B, C \cap D) \to \dots \tag{6.33}$$

Let $\widetilde{M}_\pm := M_\pm \cup M_0$. We apply the above sequence with $A = W$, $B = W$, $C = W_+ \cup \widetilde{M}_-$, and $D = W_- \cup \widetilde{M}_+$. Then $A \cap B = W$ and $C \cap D = M$. We have the following equivalences

$$\begin{aligned} (W, W_\mp \cup \widetilde{M}_\pm) &\simeq (Z_\pm \times D, Z_\pm \times S^1), \\ (W, W_+ \cup \widetilde{M}_-) \cup (W, W_- \cup \widetilde{M}_+) &\simeq (W, W_+ \cup W_- \cup M_0), \\ (W, W_+ \cup W_- \cup M_0) &\simeq (\Sigma \times B^4, \Sigma \times S^3).. \end{aligned} \tag{6.34}$$

Thus, the Mayer Vietoris sequence becomes

$$\cdots \to H^{k-4}(\Sigma) \to H^{k-2}(Z_+) \oplus H^{k-2}(Z_-) \to H^k(W, M) \to H^{k-3}(\Sigma) \to \dots \tag{6.35}$$

This is in a sense dual to the decomposition used to compute $H^\bullet(W)$. We find that the long exact sequence gives the following exact sequences:

$$\begin{aligned} &0 \to H^0(Z_+) \oplus H^0(Z_-) \to H^2(W, M) \to 0 \\ &0 \to H^3(W, M) \to H^0(\Sigma) \to H^2(Z_+) \oplus H^2(Z_-) \to H^4(W, M) \to 0 \end{aligned} \tag{6.36}$$

The map $H^0(\Sigma) \to H^2(Z_+) \oplus H^2(Z_-)$ is generated by $\mathrm{PD}_\Sigma(\Sigma) \mapsto (\mathrm{PD}_{Z_+}(\Sigma), \mathrm{PD}_{Z_-}(\Sigma))$, and, in particular, it is injective. Hence

$$\begin{aligned} &H^1(W, M) \cong 0, && H^2(W, M) \cong \langle \mathrm{PD}(Z_+), \mathrm{PD}(Z_-) \rangle, \\ &H^3(W, M) \cong 0, && H^4(W, M) \cong (H^2(Z_+) \oplus H^2(Z_-))/\langle \mathrm{PD}(\Sigma) \rangle. \end{aligned} \tag{6.37}$$

From the dualities of the cohomology of W and the relative cohomology of the pair (W, M), we have a complete description of their module structure.

6.3.3 Products and characteristic classes

Proposition 6.2.2 implies that the map $H^\bullet(\widetilde{W}_\pm) \to H^\bullet(Z_\pm)$ takes Chern classes $c_j(\widetilde{W}_\pm) \mapsto c_j(Z_\pm)$. Hence $H^\bullet(W) \to H^\bullet(Z_+) \oplus H^\bullet(Z_-)$ maps $c_j(W) \mapsto c_j(Z_+) \oplus c_j(Z_-)$. In order to compute the products, we consider the map $H^\bullet(W,M) \to H^\bullet(W)$.

In the Mayer-Vietoris sequence of $H^\bullet(W,M)$ we noted that we have homotopy equivalences (6.34) that allow us to express terms in $H^\bullet(W,M)$ in terms of $H^\bullet(\Sigma)$ and $H^\bullet(Z_\pm)$. The concatenation

$$H^\bullet(Z_\pm \times D, Z_\pm \times S^1) \to H^\bullet(W,M) \to H^\bullet(W) \to H^\bullet(Z_\pm) \tag{6.38}$$

agrees with the $H^\bullet(Z_\pm \times D, Z_\pm \times S^1) \to H^\bullet(Z_\pm)$ in the relative cohomology sequence. Thus it vanishes.

We now consider the cross terms

$$\gamma_\pm : H^\bullet(Z_\pm \times D, Z_\pm S^1) \to H^\bullet(W,M) \to H^\bullet(W) \to H^\bullet(Z_\mp). \tag{6.39}$$

The following two maps

$$(\Sigma \times D, \Sigma \times S^1) \to (Z_\pm \times D, Z_\pm \times S^1), \qquad (\Sigma \times D, \Sigma \times S^1) \to (Z_\mp, V_\mp) \tag{6.40}$$

are compatible with embeddings into W and are an inclusion and a homotopy equivalence respectively. Terms in the image of (6.39) must have support in $(\Sigma \times D, \Sigma \times S^1)$.

Identify of $H^{\bullet-2}(Z_\pm) \cong H^\bullet(Z_\pm \times D, Z_\pm \times S^1)$ and $H^{\bullet-2}(\Sigma) \cong\to H^\bullet(\Sigma \times D, \Sigma \times S^1)$. Then $H^{\bullet-2}(Z_\pm) \to H^{\bullet-2}(\Sigma)$ is the restriction map. A description of $H^\bullet(Z_\mp, V_\mp) \to H^\bullet(Z_\mp)$ is contained in [25, Lemma 4.2] which we summarize below.

Proposition 6.3.2. *Let (Z,Σ) be a building block and $V \simeq Z \setminus \Sigma$. Let $N = \mathrm{im}(H^2(Z) \to H^2(\Sigma))$, and $N^\vee$ be its dual lattice. We have the following exact sequences taken from the long exact sequence associated to (Z,V)*

$$0 \to \mathbb{Z} \xrightarrow{c_1(Z)} H^2(Z), \qquad 0 \to N^\vee \to H^4(Z). \tag{6.41}$$

Corollary 6.3.3. *The following maps describe some cross terms $H^\bullet(W,M) \to H^\bullet(W)$.*

$$\gamma^2_\pm : H^0(Z_\pm) \to H^2(Z_\mp), \qquad 1 \mapsto c_1(Z_\mp). \tag{6.42}$$

Let $\iota_\pm : H^\bullet(Z_\pm) \to H^\bullet(\Sigma)$ be the restriction maps. Then $\gamma^4_\pm = (\iota_-)^* \circ \widehat{\lambda}_L \circ \iota_+$ where $\widehat{\lambda}_L : L \to L^\vee$ is the adjoint map of the K3-lattice (L,λ).

Hence, in the identifications of (6.37), we have a lift of $c_1(W) \mapsto (1,1) \in H^2(W,M)$ and so $c_1^2(W) = [c_1(Z_+), c_1(Z_-)] \in H^4(W,M)$. The following products can then be calculated in terms of the product structure on $Z_\pm$. We find that

$$c_1(W)^4 = 0 \quad c_1(W)^2 c_2(W) = 48 \quad c_1(W)c_3(W) = \chi(Z_-) + \chi(Z_+). \tag{6.43}$$

The form of $H^4(W,M) \to H^4(W)$ is expressed in a completely isotropic basis. Thus the signature $\sigma(W) = 0$.

For computations it is advantageous to note that (assuming M is torsion free) letting $N'_\pm := \mathrm{im}(H^2(Z_\mp) \to H^2(Z_\pm))$, and $m_\pm := \mathrm{gd}(c_2(Z_\pm) \bmod (N'_\pm))$, then $m := \mathrm{gd}(c_2(M)) = \gcd(m_+, m_-)$. Recall that the spin class $q_1 = \frac{1}{2}c_1^2 - c_2$, so that on M where $c_1(M)$ is rationally trivial m is the greatest divisor of $q_1(M)$ modulo torsion. q_1 is an integral Wu class for the coboundary, so $2|m$. As M contains a K3 with trivial normal bundle, $c_2(M) \mapsto c_2(\Sigma) = \chi(\Sigma) = 24$. Thus $m|24$. The Euler characteristic $\chi(W) = \chi(Z_+) + \chi(Z_-) - 24$.

6.4 Configurations

We reformulate the condition of a matching of building blocks in terms of relatively rudimentary cohomological data called a *configuration* (Definition 6.4.1). This comes at a cost. We must deal with pairs of deformation families of building blocks instead of simply pairs of building blocks. In addition, we must introduce technical hypotheses which justify that when a configuration corresponds to a matching. This is considered in Section 7.7. In [25, Section 6] the authors consider orthogonal configurations, while the general case has been clarified in [30].

Definition 6.4.1. Suppose $r : \Sigma_+ \to \Sigma_-$ is a matching (Definition 6.1.6) between building blocks $(Z_\pm, \Sigma_\pm)$. After a choice of marking $H^2(\Sigma_+) \cong L$, we have a pair of primitive embeddings $N_\pm \to L$ of the polarization lattices which we call the *configuration* of r. It is well defined up to $\mathsf{O}(L)$.

A configuration is called *orthogonal* if the reflections of $L_\mathbb{R}$ in N_+ and N_- commute. If, in addition, $N_+ \cap N_-$ is trivial we call the configuration *perpendicular*. If a configuration is not orthogonal, it is said to be *skew*.

Motivated by the above, we fix the following notation. An *abstract configuration* $(N_\pm \to L)$ consists of a pair of primitive embeddings of nondegenerate lattices of signature $(1, r_\pm - 1)$ into the abstract K3 lattice L. We assume that their combined span $P := (N_+ + N_-) < L$ is of signature $(2, r_+ + r_- - 2)$. We adopt the notation of Definition 6.3.1.

In the presentation of [25], the authors define and use in an essential way the stack structure of deformation families of complex threefolds. To spare a digression on stacks or Kuranishi structures we refer the interested reader to that paper and references therein. Beyond employing the result [25, Theorem 6.7], we do not require an understanding of the structure of the deformation family, only that it is a set.

Definition 6.4.2. Let N be a lattice, and Y be a simply connected algebraic threefold. View $H^2(Y)$ as a lattice in which the quadratic form is derived by contracting the triple product with the anticanonical class. A map $i_Y : H^2(Y) \to N$ is said to be an *N-marking* if it is a surjective morphism of lattices.

We recall some terminology of lattice polarized K3s (see Section 7.7.1). The period domain of K3 surfaces is a space of oriented 2-planes in $L \otimes \mathbb{R}$. It can be identified with $\{\Pi \in \mathbb{P}(L \otimes \mathbb{C}) \colon \Pi^2 = 0, \Pi \wedge \overline{\Pi} > 0\}$, which inherits a complex structure. For a primitive sublattice $\Lambda < L$, the *period domain of Λ-polarized K3s* is $D_\Lambda := \{\Pi \in \mathbb{P}(\Lambda^\perp \otimes \mathbb{C}) \colon \Pi^2 = 0, \Pi \wedge \overline{\Pi} > 0\}$.

Definition 6.4.3. Let $N, \Lambda < L$ be primitive sublattices such that $N < \Lambda$. Let $C \subset N_\mathbb{R}$ be a nonempty open subcone of the positive cone in $N_\mathbb{R}$. A set $\mathcal{Y}$ of N-marked threefolds is (Λ, C)-*generic* if there exists a subset $U_\mathcal{Y} \subset D_\Lambda$ that is the complement of a countable union of complex analytic submanifolds of positive codimension with the property that: For any $\Pi \in U_\mathcal{Y}$, $k \in C$, there is $Y \in \mathcal{Y}$, a smooth anticanonical divisor $\Sigma \subset Y$, and a marking $h : H^2(\Sigma) \to L$ such that Π is the period of (Σ, h); the composition $H^2(Y) \to H^2(\Sigma) \to L$ agrees with the marking $i_Y : H^2(Y) \to N$; and $h^{-1}(k)$ is the image of the restriction to Σ of a Kähler class on Y.

Note that Definition 6.4.3 differs from [25, Definition 6.17]. There, $U_\mathcal{Y}$ is a complement of a locally finite union of complex analytic submanifolds. The weaker hypotheses above allow for genericity results needed for skew configurations to be feasibly proved.

Proposition 6.4.4. *Let $(N_\pm \to L)$ be an abstract configuration, and let $C_\pm \subset N_\pm \otimes \mathbb{R}$ be subcones of the positive cone, such that $P_\pm \cap C_\pm$ are nonempty. Let $\mathcal{Y}_\pm$ be sets of the threefolds with $N_\pm$-markings and assume that $\mathcal{Y}_\pm$ are $(\Lambda_\pm, C_\pm)$-generic.*

Then there exists generic subsets $\mathcal{K}_\pm \subset P_\pm \cap C_\pm$ such that: for all pairs $k_\pm \in \mathcal{K}_\pm$, there exists $Y_\pm \in \mathcal{Y}_\pm$ and smooth anticanonical divisors $\Sigma_\pm \to Y_\pm$, such that there exists a matching of $(Z_\pm, \Sigma_\pm, k_\pm)$ with configuration $(N_\pm \to L)$.

The proof now appears in [30, Proposition 5.8]. Strictly speaking the genericity result in [25, example No. 11] is not demonstrated in the sense of [25, Definition 6.17] since it is not sufficient to prove a statement for Λ-polarized K3 surfaces with Picard lattice $\mathrm{Pic}(\Sigma) \cong \Lambda$. However, genericity in the sense of Definition 6.4.3 is demonstrated, and the subsequent conclusions drawn from this example are still valid.

It is easier and mostly sufficient for us to work with $(N_\pm \to P)$ rather than $(N_\pm \to L)$. Note that $P \to L$ need not be primitive. There may exist overlattice refinements $P \hookrightarrow \tilde{P} < L$, where $|\tilde{P}/P|$ is finite and nonzero. This leads to TCS manifolds with torsion in H^3 of the TCS. For the most part we shall not consider this further. Much of the topology of the TCS is determined by $(N_\pm \to P)$ but clearly not all of it.

In the case of perpendicular or orthogonal matchings, P is the *pushout* of $N_\pm$. For a perpendicular matching $P \cong N_+ \oplus N_-$. An orthogonal matching is determined by specifying a shared isometric primitive sublattice $N_0 \to N_\pm$. Then $P \cong (N_+ \oplus N_-)/N_0$. To describe a skew matching we define a lattice P and some isometric embeddings $(N_\pm \to P)$.

6.5 Invariants for TCSs

Let W be the TCS coboundary to TCS manifold M. The spin class $q_1(M)$, equivalently the second Chern class $c_2(M)$, is contained in the image $I := \mathrm{im}(H^4(Z_+) \oplus_0 H^4(Z_-) \to H^4(M))$. The image of $H^4(W) \to H^4(M)$ contains I. Thus it contains all divisors of $c_2(M)$ (modulo torsion), and so is sufficient to compute the ξ-invariant.

Recall from Section 6.3.3 that for a TCS manifold M, the greatest divisor of the spin class $q_1(M)$ modulo torsion is an even divisor of 24. In the cases of torsion free cohomology, the generalized Eells-Kuiper invariant reduces to a constant $\mu \in \mathbb{Z}/\bar{m}\mathbb{Z}$, where $\bar{m} := \gcd\left(28, \mathrm{Num}\left(\frac{m}{4}\right)\right)$. Thus for TCS manifolds, $\mu(M)$ necessarily vanishes when $8 \nmid m$. In addition, the invariants μ, ν and ξ are subject to the relation

$$\frac{\xi - 7\nu}{12} = \mu \bmod \bar{m} \tag{6.44}$$

(see [31, Equation 38b]) from which we see that μ is completely determined by ν and ξ. We find similarly that if $6 \nmid m$ then ξ is determined by μ and ν.

Proposition 6.5.1. *Let (M, φ) be a TCS manifold and let W be the TCS coboundary. Then $\sigma(W) = 0$, $\chi(W) = \chi(Z_+) + \chi(Z_-) - 24$.*

$$\nu(\varphi) = 24 \pmod{48} \tag{6.45}$$

If in addition M has torsion free cohomology then on fixing any right inverse $\beta : H^4(M) \to H^4(M)$

$$\xi(\varphi) = \tfrac{3}{2}(c_2 \cdot c_2)(W, \beta) \text{ (mod 12 Num}(\tfrac{m}{4}). \tag{6.46}$$

Equation 6.45 agrees with [31, Theorem 1.7]. In particular simple configurations have trivial invariants.

Corollary 6.5.2. *If M is a 2-connected torsion free TCS manifold obtained via a perpendicular configuration then $\mu(M) = 0$ and $\xi(\varphi) = 0$.*

Chapter 7

Building blocks

The TCS construction requires as input pairs of building blocks (Definition 6.1.5). In this Chapter we explain a method of obtaining building blocks and computing relevant parts of their topological data.

There several ways to obtain building blocks. We focus on the 'ordinary' construction discussed in Section 7.1. This construction may take as input weak Fano threefolds. We discuss aspects of the theory of weak Fano threefolds which is itself an area of active research. The subject is involved, and we give but a cursory account. It is however necessary to get a reasonable grasp of the algebraic geometry underlying the theory since we use it calculate invariants and validate aspects of the TCS construction.

In Section 7.7, we consider the genericity problem that arises when attempting to construct a glueing from a configuration. In simple configurations this condition is often automatically satisfied by a one-size-fits-all result (Proposition 7.7.2). In more sophisticated configurations, this fails. The concept of genericity conditions is not new, but since simple configurations seem to have satisfied most users needs, these ideas have not been developed. As we require these more sophisticated configurations, it is necessary to consider the problem further. We propose a systematic approach to find *genericity conditions* for at least a subset of building blocks.

7.1 Ordinary building blocks

Proposition 7.1.1 ([26, Proposition 4.24]). *Let Y be a smooth closed projective threefold. Let $|\Sigma_0 : \Sigma_1| \subset |-K_Y|$ be a pencil with smooth base locus C, and let $\Sigma \in |\Sigma_0 : \Sigma_1|$ be a smooth divisor. Let $Z \to Y$ be the blowup along C. Then the proper transform of the pencil defines a K3 fibration $Z \to \mathbb{P}^1$, where the proper transform $\tilde{\Sigma}$ of Σ is isomorphic to Σ. Moreover, the image of the nef cone $\mathrm{Nef}(Z) \to H^2(\Sigma;\mathbb{R})$ contains the image of $\mathrm{Nef}(Y) \to H^2(\Sigma;\mathbb{R})$.*

Proposition 7.1.2 ([26, Proposition 5.7]). *Let Y be a weak Fano threefold, C the base locus of a generic pencil $|\Sigma_0 : \Sigma_1| \subset |-K_Y|$, and assume that C and $\Sigma \in |\Sigma_0 : \Sigma_1|$ are smooth. Let Z be the blowup of Y along C and $f : Z \to \mathbb{P}^1$ the fibration induced by the* *pencil. Then*

(i) The anticanonical class $-K_Z \in H^2(Z)$ *is primitive.*
(ii) The proper transform of the pencil $|\Sigma_0 : \Sigma_1|$ *is a fibration of K3s.*
(iii) The restriction maps $H^2(Z) \to H^2(\Sigma)$ *and* $H^2(Y) \to H^2(\Sigma)$ *have identical image.*
(iv) The kernel of $H^2(Z) \to H^2(\Sigma)$ *is generated by* $\mathrm{PD}(\tilde{\Sigma})$.
(v) $H^3(Z)$ *is torsion free if and only if* $H^3(Y)$ *is torsion free.*
(vi) $\pi_1(Z) = 0$.

If in addition we suppose that Y *is a semi Fano, then* $H^2(Y) \to H^2(\Sigma)$ *is a primitive embedding. Thus,* Z *is a building block.*

Construction 7.1.3. Suppose Y is a semi Fano threefold with basepoint free anticanonical system. The building block Z in Proposition 7.1.2 is the *ordinary building block associated to* Y. When the context makes this sufficiently clear, we will simply call it the building block of Y.

The topology of a building block obtained via Construction 7.1.3 can be computed from those of the associated semi Fano. See Section A.2.7.

7.2 Weak Fano threefolds

Mori and Mukai [80, 81] presented the classification of Fano threefolds using Mori theory. The classification of weak Fano threefolds of Picard rank 2 has been approached similarly. We recall the key notation and foundational results in Section A.3.1 and brief outline of Mori theory in Section A.3.3.

Definition 7.2.1. Let Y be a smooth 3 dimensional complex algebraic variety with anticanonical class $-K_Y$.

(i) If $-K_Y$ is ample, then Y is *Fano.*
(ii) If $-K_Y$ is big and nef (ie for all closed curves C on Y $-K_Y \cdot C \geq 0$, and $(-K_Y)^3 > 0$), then Y is *weak Fano.*
(iii) If $-K_Y$ is big and nef and semi-small (ie the associated morphism $\varphi_{-K_Y} : Y \to \mathbb{P}^N$ at worst contracts finitely many divisors to curves) then Y is *semi Fano.*

All Fanos are semi Fanos and all semi Fanos are weak Fanos. The number of deformation families of smooth weak Fano threefolds is finite ([26, Theorem 4.3]). The well-definedness of semi Fano follows from Proposition 7.2.2 (v).

We collect some basic facts about weak Fanos.

Proposition 7.2.2. *Let* Y *be a weak Fano threefold. Then:*

(i) $h^{0,i}(Y) = h^{i,0}(Y) = 0$ *for* $i > 0$*, and* $\mathrm{Pic}(Y) \cong H^2(Y)$.
(ii) $h^0(Y, -K_Y) = g + 2$ *where* $(-K_Y)^3 = 2g - 2$
(iii) $c_1(Y) \cdot c_2(Y) = 24$
(iv) A general anticanonical divisor $\Sigma \in |-K_Y|$ *is smooth.*
(v) $-K_Y$ *is semi-ample.*

(vi) *The anticanonical model X of Y is a Gorenstein Fano with at worst canonical singularities.*
(vii) *Y contains two smooth anticanonical divisors Σ_i that intersect transversally provided that its anticanonical model X has very ample $-K_X$.*
(viii) *If $|-K_Y|$ contains two members Σ_0, Σ_1 that intersect transversally, then curve $C = \Sigma_0 \cap \Sigma_1$ is a smooth canonically polarized curve (ie $K_C = -K_Y|_C$) of genus g.*
(ix) *Y is a semi Fano if the bilinear form $(c_1(Y), \cdot, \cdot)$ on $H^2(Y)$ is nondegenerate.*
(x) *$\pi_1(Y) = \{0\}$.*

Proof. for (iv) see [26, Theorem 4.7]; for (i), (iii), (ii) see [26, Corollary 4.3]; for (vi), (vii), (viii) see [26, Remark 4.10]. For (ix), we note that if Y is weak Fano and not semi Fano there exists a divisor $D \subset Y$ such that $|-K_Y|$ contracts D to a point. Equivalently $c_1(Y) \cdot \mathrm{PD}(D) = 0 \in H^4(Y)$; for (x) see [26, Theorem 5.7]; □

We call $g = Y(g)$ in the above Proposition 7.2.2 the *genus* of Y. It is equal to the genus of the canonically polarized curve C, and the genus of a smooth member $\Sigma \in |-K_Y|$. That is $g(C) = g(\Sigma) = g(Y)$.

7.3 The classification of Fano threefolds

The classification of Fano threefolds was solved in stages. Iskovskih [54, 53] building on the work of Fano classifies Fano threefolds with Picard rank 1. Using the work of Iskovskih and Šokurov [94],and Mori and Mukai [80, 81] completed the classification. In total there are 105 deformation families. The classification is given in [55] (see Theorem 7.1.1 and Chapter 12), but without the correction of [81].

We give a brief summary of some of the consequences of the classification. There are no Fano threefolds with Picard rank $\rho > 10$. The single class for each Picard rank $\rho \geq 6$, is of the form $\mathbb{P}^1 \times X_d$ where X_d is a del Pezzo surface. We tabulate the families by Picard rank.

ρ	1	2	3	4	5	6	7	8	9	10
#	17	36	31	12	3	1	1	1	1	1

Recall that the *Fano index* of threefold Y is $r := \gcd(c_1(Y))$. That is, there exists a class $H \in \mathrm{Pic}(Y)$ such that $-K_Y = rH$. H is the *fundamental class* or *fundamental divisor* of Y. The Fano threefold with the greatest Fano index is $\mathbb{P}^3$ with $r = 4$. The only family of Fano threefolds with Fano index $r = 3$ are quadric hypersurfaces $Q \subset \mathbb{P}^4$. (A result of Fujita states that the analogous holds for Fano varieties of dimension $n \geq 3$. See [53, Theorem 3.1.14].) These are both of Picard rank 1. There are 8 *del Pezzo threefolds* (or *Mukai threefolds*) characterized by $r = 2$. 5 del Pezzo threefolds have Picard rank 1. These are denoted by V_k, for $k = 1, \ldots, 5$, where $k = H^3$. The remaining three del Pezzo threefolds consist of of a smooth divisor $W \in |\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(1,1)|$, $V_7 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$, and

$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. All remaining Fano threefolds have Fano index 1. By abuse of notation we may refer to the deformation family of a Fano threefold by a representative.

Remark 7.3.1. The *degree* of a (weak) Fano threefold Y is defined to be $\deg(Y) := \left\langle -K_Y^3, [Y] \right\rangle$. Note that some authors define the degree to be $\left\langle H^3, [Y] \right\rangle$.

All but 2 families have basepoint free anticanonical systems: one has Picard rank 2 and degree 14; the other is $\mathbb{P}^1 \times X_1$, the sole family of Picard rank $\rho = 10$. (See [55, Page 216].)

For families with $\rho \geq 2$ all but 13 are *imprimitive*. A family of Fano threefolds $\mathcal{Y}$ is imprimitive if there is a family of Fano threefolds $\mathcal{Y}'$ such that for any member $Y \in \mathcal{Y}$, there exists $Y' \in \mathcal{Y}'$ with smooth curve $C \subset Y'$ such that $Y \to Y'$ is the blowup centred at C. We say a family is *primitive* if it cannot be described as above. As $\rho(Y') = \rho(Y) - 1$, all Picard rank 1 Fanos are vacuously primitive. The classification of Fano threefolds by Mori and Mukai is based first on the classification of primitive Fanos, and then considering what blowups are possible.

Of the primitive Fano threefolds with $\rho \geq 2$: two are products of projective space, three are projective bundles, three are divisors of Fano fourfolds, while the remaining five are double covers of other primitive Fano threefolds.

These descriptions allow us to compute the invariants of the Fano threefolds, construct building blocks and ultimately compute the topological invariants of G_2 manifolds produced via the twisted connected sum. Alternative descriptions such as those given [23] may prove more amenable for further study.

7.4 The classification of weak Fano threefolds with $\rho = 2$

A classification of weak Fano threefolds is far from complete. The classification of weak Fano threefolds of Picard rank 2 is almost complete. See authors [58, 100, 57, 2] and references therein.

We shall make no attempt to reproduce the classification here: it is too long and too complicated. Instead we summarise the broad structure of the classification, focusing on the aspects relevant to our needs.

All weak Fano threefolds of Picard rank 1 are necessarily Fano. For a Fano threefold X of Picard rank 2, the cone of curves is spanned by two extremal rays, both K_X negative. For a strictly weak Fano X (ie K_X is nef and big and not ample) of Picard rank 2, the cone of curves is spanned by a K_X negative, and a K_X trivial ray. Let $\varphi : X \to Y$ be the Mori-contraction on the K_X negative ray. The cone theorem does not apply to the second extremal ray, [1] since it is not K_X-negative. However, the basepoint free theorem implies that the anticanonical map (ie the *anticanonical model*) is a morphism. After Stein factorization, we get a morphism $\psi : X \to X'$, such that $\psi_* \mathcal{O}_X \cong \mathcal{O}_{X'}$. Following [58] we refer to ψ as the *anticanonical morphism*. Let F be the closed cone spanned by

[1] A note of caution: the definition of extremal ray varies between authors eg [63, Definition 1.15]) compared to [55, Section 1.2]

all $[C]$ such that $\psi(C)$ is a point. Then $\psi = \mathrm{cont}_F$, the contraction of F (see Definition A.3.19 and [58, Section 1].)

As ψ is big, $\dim(X') = 3$. Thus ψ is either *divisorial* (ie contracts a divisor) or it is *small* (ie contracts at most finitely many curves). In either case, φ must be of a type listed in Mori's Theorem A.3.20. The classification can be partitioned by considering each possible case of (φ, ψ) in turn.

In the case that ψ is divisorial, then X' is a Gorenstein Fano threefold of Picard rank 1. In the case that ψ is small, we have a flop (See [55, Definition 1.4.13]). A flop induces a symmetric relation between weak Fanos (X, X^+). Thus we can partition the classification further by considering types of pairs (φ, φ^+). See Figure 7.4.

$$\begin{tikzcd} & X \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ Y & & X' \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, dashed, "\chi"] \arrow[d, "\varphi"'] \arrow[dr, "\psi"'] & & X^+ \arrow[dl, "\psi^+"] \arrow[d, "\varphi^+"] \\ Y & X' & Y^+ \end{tikzcd}$$

Figure 7-1: Divisorial case (left) and small case (right) leading to a flop

Various authors considered different aspects of the classification and there is overlap between them. The classification programme of weak Fano threefolds of Picard rank 2 was initiated in [58]. Here they restrict to ψ divisorial, and work through each type of Mori contraction for φ.

They produce a list of restrictions that lead to a finite and exhaustible list of constructions. These are the *numerically feasible* but do not necessarily correspond to a *geometrically realizable* weak Fano threefold. In other words, all possible weak Fanos belong to a class listed, but some classes may be empty—they are not geometrically realized. Each numerically feasible case is considered to determine whether or not it can be geometrically realized. Some of these remain open (at the time of writing). This two stage approach of first producing a finite list of candidates and then attempting to prove or disprove the geometric realizability is a common feature of the other papers on this matter.

The results of [58] are tabulated at the end of their paper (Tables A2-5). They prove that 13, 12, 34, and 7 case of φ-type D, C, E1, and E2-5 respectively are geometrically realized. Precisely 2 numerically feasible cases were left open, corresponding to φ-type D (Table A2. row 7 and row 8). Both were later shown to be geometrically realizable by Fukuoka in [42, Table 1]. The weak Fanos that are not semi Fanos correspond precisely to Table A3. rows 1,9 and 12, and the set of 10 cases corresponding to Table A4. row 25.

[57] considers the case where ψ is small and at least one of the pair (φ, φ^+) is not birational. The results are tabulated (Appendix Tables 7.1-7). These correspond to (φ, φ^+)-type being DD, DC, DE2-5, DE1, CC, CE2-5, CE1 respectively. From the first four tables, containing del Pezzo fibrations, there exists 17, 5, 4, and 17 numerically feasible cases respectively. All open cases involved del Pezzo surfaces of degree 5 or 6. Those concerning del Pezzo fibrations of degree 5 were demonstrated to be geometrically realized by [100].

Those concerning del Pezzo fibrations of degree 6 were demonstrated to be geometrically realized by [42]. In Table 7.5 (CC-type) there are 2 cases both of which are geometrically realized. In Table 7.6 (CE2-5-type) there are 3 cases all of which are geometrically realized. In the final table, Table 7.7 (CE1-type), there are 13 numerically feasible cases, 7 of which are geometrically realized while the remaining 6 are open. In summary, there are 55 geometrically realized cases while a further 6 remain open. Note that the flop may map to the same class, or to a different one.

Remark 7.4.1. We note that there is a transcription error in the one of these remaining cases. Row 7 should consist of a blowup of Q which is how it appears in Proposition 7.13 (and not V_3).

[33] addresses the case where ψ is small and (φ, φ^+) are both birational. The results are tabulated in [33, Section 5].

Remark 7.4.2. The tables in the Arxiv version (v4) of the paper have been updated since the published version, with many previously open cases being resolved (see also [2]). Confusingly the rest of the text seems to have remained unchanged.

Of the E1E1-type, there are 89 cases, 77 of which are shown to be geometrically realized, while 12 remain open. (Note that the table also includes cases which fail to be geometrically realizable which accounts for there being 111 rows.) Of the E1E2, E1E3/4, E1E5 there are precisely 3, 4, 5 geometrically realized cases respectively, and no further open cases. Of the E2E2 cases, there exists 1 geometrically realized case, and one further that remains open. Of the E3/4E3/4 cases, there are precisely 2 geometrically realized cases and no further open cases. Finally of the E5E5-type, there is only 1 numerically feasible case, the geometric realizability of which remains open. In summary, there are 92 geometrically realized cases, and a further 14 cases remain open.

This concludes our summary of the current state of the classification of weak Fano threefolds of Picard rank 2. We note that other papers have covered aspects of the above from different perspectives. For example, Jahnke and Peternell classify *almost del Pezzo* threefolds in [56]. That is, the classification of smooth projective varieties of dimension n, with anticanonical class being nef and big, and divisible by $(n-1)$ in the Picard group. Takeuchi [100] considers cases where X has a del Pezzo fibration. Blanc and Lamy [12, 11] consider the cases where $Y = \mathbb{P}^3$ and $Y \subset \mathbb{P}^4$ as a cubic hypersurface. See these and citations therein for a more complete picture.

We summarize the classification in Tables 7.1, 7.2. The headers of the table reflect those found in the literature. The header ':)' is the number of geometrically realized cases. The header '?' is the number of numerically feasible cases left open by the referenced paper. The notation $\swarrow$ in the '?' column denotes that a paper has resolved some previously open cases. The header '+' is the number of previously open cases that were subsequently geometrically realized. No case left open by a cited paper has been demonstrated to not exist (outside the scope of the cited paper), so we have not included an 'x' column. The totals at the bottom are the total number of geometrically realized, and the total number of numerically feasible cases that remain open.

φ-type	:)	+	?	Reference
D	13		2	[58, A.2]
		2	$\swarrow$	[42, Table 1]
C	12			[58, A.3]
E1	25			[58, A.4]
E2-5	7			[58, A.5]
Total		59	0	

Table 7.1: ψ-divisorial case summary

φ-type	φ^+-type	:)	+	?	Reference
D	D	11		6	[57, A.1]
			2	$\swarrow$	[100, Theorem 2.13]
			4	$\swarrow$	[57, Table 2]
	C	3		2	[57, A.2]
			1	$\swarrow$	[100, Theorem 2.13]
			1	$\swarrow$	[42, Table 2]
	E2-5	4			[57, A.3]
	E1	11		6	[57, A.4]
			3	$\swarrow$	[100, Theorem 2.13]
			3	$\swarrow$	[42, Table 2]
C	C	2			[57, A.5]
	E2-5	3			[57, A.6]
	E1	7		6	[57, A.7]
E1	E1	77		12	[33, Section 5]
	E2	3			[33, Section 5]
	E3-4	4			[33, Section 5]
	E5	5			[33, Section 5]
E2	E2	1		1	[33, Section 5]
E3	E3	2			[33, Section 5]
E5	E5	0		1	[33, Section 5]
Total			147	20	

Table 7.2: ψ-small case summary

Proposition 7.4.3 ([57, Proposition 2.5]). *Let X be a weak Fano threefold (and not Fano). Then $-K_X$ is generated by global sections except when X is a deformation a Fano threefold of Picard rank 1, Fano index 1, and degree 2 (AGV.1.1), and arises as the complete intersection of a quadric cone, and a general sextic in $\mathbb{P}(1^4, 2, 3)$. In this exceptional case, the anticanonical morphism $X \to X'$ is small, $\rho = 2$, $(-K_X)^3 = 2$, $X \simeq X^+$, and φ is a del Pezzo contraction with fibre F, $K_F^2 = 1$.*

In subsequent sections we refer to weak Fano threefolds and/or their deformation families by the paper, table, and row in which they appear. This is described explicitly in Section B.1.

7.5 Deformation invariants

From the above classification, we might also hope to establish a description of each of the deformation families as we have in the case of Fano threefolds. However, the correspondence is not immediate. For example for each class CMv4.1.107.0, JPR05.3.11 and JPR05.3.12 φ is a blowup of V_5 in a curve of genus 0 and degree 6.

As mentioned, for the most part we wish to avoid introducing the notion of a Kuranishi family of a stack. For the majority of cases we need know nothing more about the structure of a deformation family except that it is a set. We do however wish to ascertain that certain properties are shared between members of a family. The cohomology algebra and Chern classes are invariant under deformations. The Kähler cone and the behaviour of the contractions (φ, ψ) may not be (see [86, Corollary 1.2]).

We are concerned with the deformation families of weak Fanos. That each ray class consists of members of a single deformation family is implicit in the descriptions presented. There are cases where a deformation family is represented more than once in the classification. For example, many flops are do not leave the deformation family. More substantially, some classes appear as special cases or boundary cases of others, and such is the case in the examples above.

We do not attempt to describe a complete set of deformation families. A deformation class has associated to it a set of topological invariants shared by all members. By computing these invariants for each class, we can produce a lower bound on the number of distinct deformation families.

Let Y be a simply connected threefold with $b_2 = 2$ and $c_1^3 > 0$. By the duality $H^2(Y) \cong H^4(Y)^\vee$, $\mathrm{Ann}(c_1^2) < H^2(Y)$ is a rank 1 subspace. Let $a \in H^2(Y)$ be primitive such that $ac_1(Y)^2 = 0$ and $a^3 > 0$; or if $a^3 = 0$ then $ac_2 > 0$; or if $a^3 = ac_2 = 0$ then fix a sign of a. We have the following deformation invariants

$$c_1^3, \quad a^2c_1, \quad a^3, \quad ac_2, \quad \gcd(c_1), \quad b_3, \quad TH^3(Y) \tag{7.1}$$

We have two additional invariants $\cot(c_1, a)$ and $\mathrm{cong}(c_1, a)$ which we define as follows. Let $\overline{c_1} := c_1/\gcd(c_1)$, and $\mathrm{m}(\overline{c_1}, a)$ be the matrix with rows $\overline{c_1}$ and a. Then $\cot(c_1, a)$ is the gcd of the maximal minors of $\mathrm{m}(\overline{c_1}, a)$, which in this case is the equal to the absolute determinant of $\mathrm{m}(\overline{c_1}, a)$. Then $\mathrm{cong}(c_1, a) = x \in \{1, \ldots, \cot(c_1, a)\}$ is such that it satisfies the congruence relation $(x\overline{c_1} - a) = 0 \mod \cot(c_1, a)$.

The properties of deformation families of weak Fanos required in the TCS construction include:

(i) $H^2(Y)$, and its triple product;

(ii) $b_3(Y)$ and $TH^3(Y)$;

(iii) whether the anticanonical class is basepoint free;

(iv) whether the anticanonical morphism is semi-small;

(v) the total Chern class;
(vi) the Nef cone.

Less information is necessary in the context of simple configurations such as perpendicular configurations. A more precise understanding is required when addressing genericity conditions. The Nef cone is not necessarily constant on a family, as discussed in section 7.7. In all our examples $TH^3(Y) = 0$, which is reassuring when it comes to computing our invariants via the TCS coboundary.

When Y is of type $E1$ or $E2$, ie the blowup of a Picard rank 1 Fano threefold along a smooth curve or a point, then we can establish all of the above. Either it is immediately read from the literature or can be computed straightforwardly from the formule in Sections A.2.7. We consider the $E1$ case below.

For Y of type $E3-5$, the base of the contraction is not smooth. Alternative methods are required to compute the topology. In cases that Y is of type C and D then it is often presented as a subvariety of a projective bundle over $\mathbb{P}^2$ or $\mathbb{P}^1$ respectively, such as in [100, 58, 57]. One can use standard methods to deduce the topology in these cases. We have not included these cases in any examples.

Remark 7.5.1. For a weak Fano threefold Y with anticanonical morphism that is divisorial and not semi-small, (ie Y a weak Fano and not a semi Fano), then one does not automatically get a building block of Construction 7.1.3. There are 13 such cases in the classification [58]: 10 of type E1 and 3 of type C. All have free anticanonical systems. Let Z be obtained from Y via Construction 7.1.3. One may still be able to use Z in the TCS method, but many results will not be automatic. For example $H^2(Z) \to H^2(\Sigma)$ must be shown to have primitive image, and the one-size-fits-all genericity result [25, Theorem 6.8] does not immediately apply.

The topology relevant to our needs of Fano threefolds of Picard rank 1 can either be read off directly from the cited sources or easily computed as follows. $H^2(Y)$ is generated by the hyperplane or *fundamental* class H. By definition, the (Fano) index r of Y is such that $c_1(Y) = rH$. As for any weak Fano threefold $c_1(Y) \cdot c_2(Y) = 24$, so $H \cdot c_2(Y) = 24/r$.

Most of the semi Fanos we consider are of type E1, while of the 88 Fano threefolds of Picard rank ≥ 2 all but 13 can be described as the blowup in some smooth curve. The topology can be computed straightforwardly via the results in Section A.2.7. We unpack this a little here.

Let $\pi : X \to Y$ be the blowup along a smooth curve C. Then $H^2(X)$ is spanned by $H^2(Y)$ and the exceptional class ζ. The product structure on even degrees is determined by the product structure on $H^2(Y)$ and

$$\zeta^3 = -\textstyle\int_C c_1(N_{C/Y}), \quad \pi^*(a) \cdot \zeta^2 = -\textstyle\int_C a, \quad \pi^*(b) \cdot \zeta = 0 \tag{7.2}$$

for $a \in H^2(Y)$, $b \in H^4(Y)$. The Chern classes are given by

$$c_1(X) = \pi^* c_1(Y) - \zeta, \quad c_2(X) = \pi^*(c_2(Y) + \mathrm{PD}(C)) - \pi^* c_1(Y) \cdot \zeta \tag{7.3}$$

Poincaré duality implies that $H^2(X) \cong H^4(X)^\vee$, and in computations it is helpful to express $H^4(X)$ in the dual basis of $H^2(X)$. In particular

$$\pi^*(a) \cdot c_2(X) = a \cdot c_2(Y) + \textstyle\int_C a, \quad \zeta \cdot c_2(X) = \textstyle\int_C c_1(Y). \tag{7.4}$$

As $H^3(X) \simeq H^3(Y) \oplus H^1(C)$, $b_3(X) = b_3(Y) + 2g(C)$ where $g(C)$ is the genus of C. In addition, $H^3(X)$ is torsion free cohomology if and only if $H^3(Y)$ is torsion free. Note that, all Fano threefolds are known to have torsion free cohomology.

Proposition 7.5.2. *Let $X \to Y$ be the blowup of a rank 1 Fano Y of Fano index r along a smooth curve $C \subset Y$ of degree d and genus g. Let H be the pullback of the fundamental class H' of Y and E be the exceptional class of the blowup. H and E form a basis of $H^2(X)$.*

The lattice on $\mathrm{Pic}(X)$ is determined by $H^2 = \deg(Y)/r^2$, $E.H = d$ and $E^2 = 2g - 2$. The ample cone is spanned by rays $G := kH - E$, where k is the least integer such that C is cut out by sections of $|kH'|$. If X is Fano then $1 \leq k < r$. If X is strictly weak Fano (weak Fano and not Fano) then $k = r$.

(See [30, Lemma 4.5] and the succeeding remarks.)

To compute the topology in the remaining cases requires applying appropriate results selected to suit the given description of a family. Section A.2 contains a collection of standard results used.

7.6 Building blocks with large b_2

7.6.1 Fanos with large b_2

There is only a single family of Fano threefolds with b_2 equal to each of $\{6, \ldots, 10\}$, and no Fano threefold has $b_2 \geq 11$. A member Y of the family with $b_2(Y) = 2 + k$, is the product $Y = \mathbb{P}^1 \times X_{9-k}$ where $X_{9-k} = \mathrm{Bl}_k\mathbb{P}^2$ is the blowup of k points in general position.

The del Pezzo surface $X = X_d$ has degree $c_1(X)^2 = d = 9 - k$. The cohomology of X is concentrated in even degrees. Let $y \in H^2(X)$ be the hyperplane class, and $z_i \in H^2(X)$ be the exceptional class of the i^{th} blowup. The cohomology of X is completely determined by:

$$y^2 = -\delta_{ij} z_i z_j, \quad y z_i = 0 \tag{7.5}$$

The Chern classes of X are then $c_1(X) = 3y - \sum_i z_i$, and $c_2(X) = \chi(X) = 3 + k$. The ample cone of a del Pezzo surface X_d can be obtained by computing the cone of curves in X_d. A curve is either exceptional or the proper transform of a curve on $\mathbb{P}^2$.

Let $Y := \mathbb{P}^1 \times X_d$. The cohomology of Y is given by the Kunneth Theorem (Proposition A.2.16). Let $x \in H^2(\mathbb{P}^1)$ be the oriented generator. The triple product on $H^2(Y)$ is determined by

$$xy^2 = 1, \quad xz_i^2 = -1, \tag{7.6}$$

and otherwise a triple product of generators vanish.

The total Chern class is

$$\begin{aligned} c(Y) &= c(\mathbb{P}^1)c(X_d) \\ &= 1 + 2x + 3y - \sum z_i + 6xy - 2\sum xz_i + (3+k)y^2 + 2(3+k)xy^2 \end{aligned} \tag{7.7}$$

Hence $c_1 = 2x + 3y - \sum z_i$, $c_2 = 6xy - 2\sum xz_i + (3+k)y^2$ and $\chi(Y) = 2(3+k)$. A quick sanity check confirms that $c_1(Y) \cdot c_2(Y) = 24$.

$$c_1^2(Y) = (9-k)y^2 + 12xy - 4\sum_i xz_i \tag{7.8}$$

The class $c_1^2(Y)$ is the Poincare dual to the canonical curve that is blown up to obtain the building block. The degree of Y is then $c_1^3(Y) = 2d + 3 \times 12 - 4(9-d) = 6d$.

The Picard lattice is obtained by contracting the triple products with $c_1(Y)$. The ample cone is the product of the cones on its factors.

For $d > 1$, the base locus of a generic anticanonical pencil is smooth. Hence we can appeal to Construction 7.1.3 to obtain the associated building block. For $d = 1$, $c_1(Y)$ is ample but not very ample.

For $d = 1$, the base locus of an anticanonical pencil is nonempty. The anticanonical linear system on X_1 corresponds to the proper transform of the pencil of cubic curves on $\mathbb{P}^2$ that pass through the 8 points of the blowup. All cubic curves in the pencil contain a 9^{th} point p_9. We cannot appeal to Construction 7.1.3, but can instead obtain a building block in via several blowups.

Let X_0 be the blowup of X_8 in p_9 and $Y_0 = \mathbb{P}^1 \times X_0$. Let $x, y, z_1, \ldots z_8$ be as above and let z_9 correspond to the additional blowup. The base locus of the anticanonical pencil of Y_0 is four distinct curves that each lie in a X_9 fibre above $\mathbb{P}^1$. Let Z be the blowup of each of these. Let w_i correspond to the Poincaré dual of the i^{th} exceptional divisor.

Then $w_1, \ldots, w_4, x, y, z_1, \ldots, z_9 \in H^2(Z)$ generate $H^2(Z)$. Denote the dual basis by $\hat{w}_1, \ldots, \hat{w}_4, \hat{x}, \hat{y}, \hat{z}_1, \ldots, \hat{z}_9 \in H^4(Z)$. The cup product on $H^2(Z)$ is determined by

$$\begin{array}{ccc} w_i x = 0 & w_i y = 3\hat{w}_i & w_i z_j = 4\hat{w}_i \\ x^2 = 0 & xy = \hat{y} & xz_j = -\hat{z}_j \\ y^2 = \hat{x} & yz_j = 0 & z_j z_k = -\delta_{jk}\hat{x} \end{array} \tag{7.9}$$

and

$$w_i w_j = \delta_{ij}\left(3\hat{y} - \sum_k \hat{z}_k\right). \tag{7.10}$$

Note that

$$c_1(Z) = \sum_i w_i + 2x + 3y - \sum_k z_k. \tag{7.11}$$

Z is a building block, provided we check that $H^2(Z) \to H^2(\Sigma)$ is a primitive embedding. This is observed with little difficulty. All terms $x, y, z_1, \ldots z_8$ factor through $H^2(Y) \to H^2(Z) \to H^2(\Sigma)$, while for each j, $w_j - x + 3y - \sum_k z_k \in \ker(H^2(Z) \to H^2(\Sigma))$. Thus N is equal to the image of $\mathrm{Pic}(Y) \to H^2(\Sigma)$ and so is primitive.

In $\mathbb{P}^1 \times X_0$, the form $-2x + 3y - \sum_1^9 z_j$ lies in the kernel of $H^2(\mathbb{P}^1 \times X_0) \to H^2(\Sigma)$. The obvious preimages of $x, y - z_1 \in H^2(\Sigma)$ are $x, y - z_1 \in H^2(\mathbb{P}^1 \times X_9)$. We can find constants $A, B > 0$ such that $Ax + B(y - z_1) + (-2x + 3y - \sum_1^9 z_j)$ is ample on $\mathbb{P}^1 \times X_9$, and its images lies in $N_0^{\perp}$.

7.6.2 Divisors of $\mathbb{P}^2 \times X_d$

Let $x \in H^2(\mathbb{P}^2)$ be the oriented generator, while $y, z_1, \ldots, z_k \in H^2(X_d)$ the cohomology of the del Pezzo X_{9-k}. The cohomology of $\mathbb{P}^2 \times X_d$ follows from the Kunneth formula (Proposition A.2.16):

$$H^2(\mathbb{P}^2 \times X_d) = \langle x, y, z_1, \ldots, z_k \rangle \tag{7.12}$$

which generates $H^\bullet(\mathbb{P}^2 \times X_d)$.

The line bundle $L = \mathcal{O}_{\mathbb{P}^2}(2) \otimes (-K_{X_d})$ is ample. Let $Y \in |L|$, a smooth divisor. The cohomology of Y below the middle dimension follows from the Lefschetz Hyperplane Theorem (Proposition A.2.9). In particular, Y is simply connected and

$$H^2(Y) = \langle x, y, z_1, \ldots, z_k \rangle . \tag{7.13}$$

This mild abuse of notation does not distinguish between the class of $\mathbb{P}^2 \times X_d$ and its restriction to Y. The triple product on $H^2(Y)$ is determined by

$$x^2 y = 3, \quad x^2 z_i = 1, \quad xy^2 = 2, \quad xz_i^2 = -2, \tag{7.14}$$

and otherwise a triple product of generators vanish.

The total Chern class of Y is

$$\begin{aligned} c(Y) &= [c(\mathbb{P}^2)c(X_d)c(L)^{-1}]|_Y \\ &= (1 + 3x + 3y - \textstyle\sum z_i + 3x^2 + 9xy - 3\sum xz_i + (3+k)y^2 + \\ &\quad\; 9x^2y - 3\textstyle\sum xz_i + 3(3+k)xy^2) \\ &\quad\; (1 - (2x + 3y - \textstyle\sum z_i) + (4x^2 + (9-k)y^2 + 12xy - 4\sum xz_i) - \\ &\quad\; (36x^2y + 6(9-k)xy^2 - 12\textstyle\sum xz_i)) \\ &= 1 + x + (x^2 + (3+k)y^2 + 6xy - 2\textstyle\sum xz_i) - 4(6-k)xy^2. \end{aligned} \tag{7.15}$$

Thus $c_1(Y) = x$, $c_2(Y) = x^2 + (3+k)y^2 + 6xy - 2\sum xz_i$ and $\chi(Y) = -8(6-k)$. Hence, $b_3 = 2 + 2b_2 - \chi = 6(9-k)$. By the Lefschetz Hyperplane Theorem $TH_2(Y) = \{0\}$. By the Universal Coefficient Theorem and $TH^3(Y) = \{0\}$. Thus $TH^\bullet(Y) = \{0\}$.

Note that $\mathbb{P}^2 \times X_d$ is the blowup of $\mathbb{P}^2 \times X_{d+1}$ in some surface $\mathbb{P}^2 \times \{pt\}$. Let $Y_d \subset \mathbb{P}^2 \times X_d$ as described above. Then Y_d is the blowup of Y_{d+1} along the curve $Y_{d+1} \cap \mathbb{P}^2 \times \{pt\}$.

An anticanonical divisor is the intersection of Y with $\mathbb{P}^1 \times X_d$. The image of $H^2(\mathbb{P}^1 \times X_d) \to H^2(\Sigma)$ is a primitive. The image of $H^2(Y) \to H^2(\Sigma)$ is identical, so too is primitive. The anticanonical class $-K_Y = x$ is basepoint free.

We can choose a pencil of K3 $|\Sigma_0 : \Sigma_1|$ of anticanonical divisors whose base locus is smooth curve $C \subset Y$ corresponding to the fibre x^2. We can take $C = \{pt\} \times C'$ where $C' \subset X_d$ is curve corresponding to the proper transform of a smooth cubic on $\mathbb{P}^2$ that contains the $(9-d)$ blown up to obtain X_d.

We check directly that we can construct a building block from Y. Let $\pi : Z \to Y$ be the blowup along C and let $w \in H^2(Z)$ correspond to the exceptional class. The proper transform of the pencil induces the projective morphism $Z \to \mathbb{P}^1$, a generic member of which is a smooth K3 by Proposition 7.1.1. $H^2(Z)$ is generated by $\pi^* H^2(Y)$ and w. The anticanonical class of Z is $\pi^*x - w$. In particular, it is primitive. It follows that Z is indeed a building block.

The ample cone of Y contains the image of the ample cone of $\mathbb{P}^2 \times X_d$, which is the product of the ample cones of the components. The induced polarizing lattice on anticanonical divisors is isomorphic to corresponding lattices in the case of $\mathbb{P}^1 \times X_d$. By Proposition 7.1.1, the image of the nef cone of Z on Σ contains the image of the nef cone of Y.

7.7 Genericity results

Let $\mathcal{Y}$ be a set of N-polarized smooth algebraic threefolds, such that for any $Y \in \mathcal{Y}$ a generic anticanonical divisor $\Sigma \in |-K_Y|$ is a smooth K3 surface. Suppose $N \hookrightarrow \Lambda$ is primitive lattice embedding, where $\Lambda \subset L_{K3}$ is a primitive sublattice of the abstract K3 lattice. In addition, let $C \subset N \otimes \mathbb{R}$ be a cone, such that for any $Y \in \mathcal{Y}$ the polarization $N \to \mathrm{Pic}(Y)$ maps C into the Kähler cone of Y.

The question: For a generic Λ-polarized K3 Σ, does there exist a member $Y \in \mathcal{Y}$ such that $\Sigma \to Y$ as an anticanonical divisor?

Definition 7.7.1. We say that $\mathcal{Y}$ is (Λ, C)-*generic* or simply Λ-generic if Λ answers the above to the affirmative. Conversely, we say that Λ is a $(\mathcal{Y}, C)$-*generic lattice*, or simply is $\mathcal{Y}$-*generic*, (for lack of a better word) if it is an overlattice $N \to \Lambda$ such that $\mathcal{Y}$ is (Λ, C)-generic. We denote the set of all $\mathcal{Y}$-generic lattices by $\mathbf{gen}(\mathcal{Y}, C)$.

An answer should take a form that is computationally useable. For a given set $\mathcal{Y}$ of N-marked threefolds, we want an exhaustive list of arithmetic conditions that, if met by an overlattice $(N \to \Lambda)$, then imply that Λ is $\mathcal{Y}$-generic.

The role of the cone C is a little subtle. Paoletti [86] describes how the Kähler cone on threefolds can jump, on subspaces of the deformation space of positive codimension.

The following is our first genericity result.

Proposition 7.7.2. *Let $\mathcal{Y}$ be a deformation family of semi Fano threefolds. Let $Y \in \mathcal{Y}$ be a generic member, and let $N = \mathrm{Pic}(Y)$ and $C = \mathrm{Amp}(Y) \subset N_{\mathbb{R}}$. Then $\mathcal{Y}$ is (N, C)-generic.*

See [26, Theorem 6.8] based on Beauville [8]. Recall in orthogonal configurations $\Lambda = N$. Thus we have sufficient genericity results for any orthogonal configuration. This has allowed for the mass production, particularly in the perpendicular cases (see [25, Section 8]).

The following section is predominantly written in the language of algebraic geometry. The notation and some fundamental results have been included in Section A.3.1, along with some aspects of the theory of algebraic curves in Section A.3.2. We consider an invertible sheaf, a divisor and a line bundle synonymous in most circumstances. Playing fast and loose with notation, we rarely distinguish between say a divisor D, and its corresponding class $[D]$. Obviously, a K3 surface will be considered a smooth simply connected complex surface with trivial canonical class. All our K3s are in fact algebraic (embeddable into $\mathbb{P}^N$).

7.7.1 K3 surfaces

We recall some basic theory of K3s and fix notation.

Proposition 7.7.3 ([7, Proposition 3.1-3])**.** *Let Σ be a K3. Then*

(i) $c_1(\Sigma) = 0$, $c_2(\Sigma) = 24$ *and* $\sigma(\Sigma) = -16$.

(ii) $H^\bullet(\Sigma)$ *is torsion free, and* $b_1(\Sigma) = 0$, $b_2(\Sigma) = 22$.

(iii) $H^2(\Sigma) \cong L$ *the even unimodular lattice of signature* $(3, 19)$. $L = U^{\oplus 3} \oplus (-E_8)^{\oplus 2}$, *where* U *is the rank* 2 *hyperbolic lattice* U *and* E_8 *is the* E_8*-lattice.*

(iv) $h^{p,q}(\Sigma) = 1$ *for* $(p, q) = \{(0, 0), (0, 2), (2, 0), (2, 2)\}$*;* $h^{1,1}(\Sigma) = 20$*; and* $h^{p,q} = 0$ *otherwise.*

Lemma 7.7.4 (Riemann-Roch for K3s)**.** *Let D be a divisor on a K3 surface Σ. Then*

$$\chi(\mathcal{O}_\Sigma(D)) = h^0(D) - h^1(D) + h^0(-D) = 2 + \frac{1}{2}D^2 \tag{7.16}$$

Corollary 7.7.5. *Let Σ be a K3 and $C \subset \Sigma$ an irreducible curve. $C^2 \geq -2$ with equality if and only if C is a smooth rational curve, called* (-2)-curves *or* nodal curves*. If $D \subset \Sigma$ is an effective divisor and $h^0(D) = 1$, then D is the sum of (-2)-curves.*

(See [87, Section 1].)

Corollary 7.7.6. *Let D be a divisor on a K3 Σ such that $D \neq 0$. Then*

(i) If $D^2 \geq -2$, then either $h^0(D) > 0$ and D effective; or $h^0(-D) > 0$ and $-D$ effective.

(ii) If $D \geq 0$, then either $h^0(D) \geq 2$; or $h^0(-D) \geq 2$.

Definition 7.7.7. For a K3 Σ a lattice isometry $h : H^2(\Sigma) \to L$ is a *marking* of Σ. A *marked K3* is a pair (Σ, h) consisting of a K3 Σ together with a marking $h : H^2(\Sigma) \to L$.

Let Λ be a nondegenerate even lattice of signature $(1, r-1)$. The set $V(\Lambda) := \{v \in \Lambda \otimes \mathbb{R} \colon v^2 > 0\}$ consists of two connected cones. Let $V(\Lambda)^+$ denote one of these connected cones. Let $\Delta(\Lambda) := \{D \in \Lambda \colon D^2 = -2\}$. Fix a partition $\Delta = \Delta^+ \sqcup \Delta^-$ such that $\Delta^- = \{-D \colon D \in \Delta^+\}$ and if $D = \sum n_i D_i$ is an effective sum (ie $n_i > 0$) for $D_i \in \Delta^+$ and $D^2 = -2$ then $D \in \Lambda^+$. Let $C^+(\Lambda) := \{v \in V(\Lambda) \cap \Lambda \colon \forall D \in \Delta^+, \ (v, D) > 0\}$.

Suppose that for some $H \in V(\Lambda)^+$, $\{D \in \Delta(\Lambda) \colon (H, D) = 0\} = \varnothing$, then H defines a partition on $\Delta(\Lambda)$ by $\Delta_H^+ := \{D \in \Delta \colon (H, D) > 0\}$.

Definition 7.7.8. A Λ*-polarized K3* (Σ, j) is a projective K3 Σ together with a primitive lattice embedding $j : \Lambda \to \mathrm{Pic}(\Sigma)$. In addition, we stipulate that the image of j meets the closure of the Kähler cone of Σ nontrivially.

Suppose that (Σ, j) is a Λ-polarized K3. We can fix the partition on $\Delta(\Lambda)$ by setting Δ^+ to consist precisely of $D \in \Delta(\Lambda)$ such that $j(D)$ is effective. Then $C^+(\Lambda)$ will consist of the ample classes of Σ in the image of j.

For an abstract lattice Λ, a Λ-polarized K3 can exist only if there exists a primitive embedding $i : \Lambda \to L$. The following result facilitates demonstrating the existence of i. For a nondegenerate lattice P, we denote by $m(P)$ the minimal number of generators of $P^\vee / P$.

Proposition 7.7.9 ([84, Theorem 1.12.4 and Corollary 1.12.3])**.** *Let P be an even nondegenerate lattice of signature (p_+, p_-), and let Q be an even unimodular lattice of indefinite signature (q_+, q_-). If $p_\pm \leq q_\pm$ and either*
(i) $2\ \mathrm{Rank}(P) \leq \mathrm{Rank}(Q)$*; or*
(ii) $\mathrm{Rank}(P) + m(P) < \mathrm{Rank}(Q)$*;*
Then there exists a primitive embedding $P \hookrightarrow Q$.

Corollary 7.7.10. *Let Λ be an even nondegenerate lattice of signature $(1, r-1)$. If $r \leq 10$ then there exists a primitive embedding $\Lambda \to L$. More generally, if $r + m(\Lambda) \leq 21$ then there exists a primitive embedding $\Lambda \to L$.*

An abstract lattice Λ satisfying the hypotheses of Corollary 7.7.10 is said to be a *polarizing lattice.* Note that Corollary 7.7.10 is sufficient but not strictly necessary.

Definition 7.7.11. The *period domain of K3 surfaces* is the complex manifold $\mathcal{D} := \{x \in \mathbb{P}(L \otimes \mathbb{C})\colon x^2 = 0, x \wedge \overline{x} > 0\}$. For a primitive sublattice $\Lambda < L$ of signature $(1, r-1)$, the *period domain of Λ-polarized K3 surfaces* is the complex submanifold $\mathcal{D}_\Lambda := \mathbb{P}(\Lambda^\perp \otimes \mathbb{C}) \cap \mathcal{D}$.

Note that $\dim_{\mathbb{C}}(\mathcal{D}) = 20$, and $\dim_{\mathbb{C}}(\mathcal{D}_\Lambda) = 20 - \mathrm{rank}(\Lambda)$.

Definition 7.7.12. The *period point* of a marked K3 (Σ, h) is $[h_{\mathbb{C}}(H^{2,0}(\Sigma))] \in \mathcal{D}$. Here $h_{\mathbb{C}} : H^2(\Sigma) \otimes \mathbb{C} = H^2(\Sigma, \mathbb{C}) \to L \otimes \mathbb{C}$ is the extension of h to an isomorphism of Hermitian forms, and $H^{2,0}(\Sigma) < H^2(\Sigma; \mathbb{C})$, so that $h_{\mathbb{C}}(H^{2,0}(\Sigma)) < L \otimes \mathbb{C}$ is a complex line.

It is clear that for a sublattice $\Lambda < L$, for a Λ-polarized K3 (Σ, j) with a compatible marking $h : H^2(\Sigma) \to L$ (ie with $h \circ j : \Lambda \to L$ the sublattice embedding), the period point of (Σ, h) is in $\mathcal{D}_\Lambda$.

Proposition 7.7.13 (K3 Torelli Theorem)**.** *A marked K3 (Σ, h) is uniquely determined by its period point. Conversely, for any $\Pi \in \mathcal{D}_\Lambda$ there exists a Λ-polarized K3 (Σ, j) together with a marking $h : H^2(\Sigma) \to L$ such that Π is the period point of (Σ, h).*

(See [92] or [37].)

Definition 7.7.14. The set $\mathcal{D}^\circ_\Lambda \subset \mathcal{D}_\Lambda$ is defined to be the periods $\Pi \in \mathcal{D}_\Lambda$ corresponding to marked K3s (Σ, h) where $\mathrm{Pic}(\Sigma) \cong \Lambda$.

Proposition 7.7.15. *$\mathcal{D}^\circ_\Lambda \subset \mathcal{D}_\Lambda$ is the complement of a countable union of subvarieties of positive codimension.*

By abuse of notation, we will not specify the lattice embedding $i : \Lambda \to L$ for $\mathcal{D}_\Lambda$. We will stipulate that Λ is a polarizing lattice and so i exists. Furthermore, we shall write $\Sigma \in \mathcal{D}^\circ_\Lambda$ to mean a polarized K3 (Σ, j) with lattice isomorphism $j : \Lambda \to \mathrm{Pic}(\Sigma)$, and such that there exists a marking $h : H^2(\Sigma) \to L$ such that $i = h \circ j$.

Definition 7.7.16. Let Σ be a K3 surface. For $D \in \mathrm{Pic}(\Sigma)$, such that $D^2 = -2$ we define a *Picard-Lefschetz reflection* to be the involution $r_D : \mathrm{Pic}(\Sigma) \to \mathrm{Pic}(\Sigma)$, $E \mapsto E + (E.D)D$. The group they generate is the *Picard-Lefschetz group.*

Lemma 7.7.17 ([7, VIII Proposition 3.9])**.** *The Picard-Lefschetz group of Σ acts on the positive cone $C^+ \subset \mathrm{Pic}(\Sigma) \otimes \mathbb{R}$. The Kähler cone is a fundamental domain of the group action.*

Thus on a K3 Σ with $\mathrm{Pic}(\Sigma) \cong \Lambda$ and a class $H \in \Lambda$ with $H^2 > 0$, we can choose a polarization $\Lambda \to \mathrm{Pic}(\Sigma)$ such that H is nef on Σ. We extend our notation such that $\Sigma \in \mathcal{D}^\circ_{(\Lambda,H)}$ implies that $\Sigma \in \mathcal{D}^\circ_\Lambda$ has polarization $j : \Lambda \to \mathrm{Pic}(\Sigma)$ such that $j(H)$ is nef. We say that the pair (Λ, H) of a polarizing lattice together with an element $H \in \Lambda$ such that $H^2 > 0$ is a *nef lattice*.

7.7.2 Projective models of K3s

Projective models of K3 surfaces are subtly related with special divisors on embedded curves (see Section A.3.2). The main reference is the classical paper of Saint-Donat [90]. We restate some foundational results from this paper.

Suppose H is a very ample divisor on Σ, determining a projective embedding $\varphi_H : \Sigma \to \mathbb{P}^g$. Then $g = \frac{1}{2}(H^2 - 1)$ is the *genus* of (Σ, H). By the Riemann-Roch theorem for K3s

$$h^0(\mathcal{I}_\Sigma \otimes \mathcal{O}_{\mathbb{P}^g}(k)) \geq \binom{g+k}{k} - \left(2 + (g-1)k^2\right) \tag{7.17}$$

Note that $H^0(\mathcal{I}_\Sigma \otimes \mathcal{O}_{\mathbb{P}^g}(k))$ corresponds to the degree k hypersurfaces of $\mathbb{P}^g$ that contain Σ.

For a linear system $|L|$ and $D \in |L|$, the *fixed component* of D is the greatest effective divisor F such that for all $D' \in |L|$, $D' - F \geq 0$, while $M = D - F$ is the *moveable component*.

Proposition 7.7.18. *Let D be an effective divisor on a K3 Σ. Then there exists an effective sum of nodal curves $F = \sum n_i \Gamma_i$ such that $M = D - F$ is effective, nef (potentially trivial), $M^2 \geq D^2$ and $H^0(M) = H^0(D)$. F and M are the fixed and moveable components of D.*

Proposition 7.7.19 ([90, Corollary 3.2])**.** *A complete linear system on a K3 has no basepoints outside its fixed component.*

In particular, if for a divisor D we can show that its fixed component is empty, then D is basepoint free.

Proposition 7.7.20 ([90, Proposition 2.6])**.** *Let Σ be a K3 with a line bundle H. Suppose that $|H| \neq \varnothing$ and that $|H|$ is basepoint free. If $H^2 > 0$, then a general member of $|H|$ is a smooth curve of genus $\frac{1}{2}H^2 + 1$. If $H^2 = 0$, then there exists an elliptic pencil E and $a \geq 1$ such that $H = aE$.*

Recall that an elliptic pencil is a linear system $E = |C_0 : C_1|$ where C_i are elliptic curves.

Proposition 7.7.21 ([90, Proposition 2.7]). *Let H be a nef line bundle on a K3 surface Σ. Then $|H|$ is not basepoint free if and only if there exist smooth curves E, D on Σ and an integer $k \geq 2$, such that*

$$H \sim kE + D \quad E^2 = 0 \quad E.D = 1 \quad D^2 = -2 \tag{7.18}$$

In this case we say that H is monogonal.

Proposition 7.7.22. *Let L be a basepoint free line bundle, with $L^2 > 0$ on a K3 Σ. The following are equivalent*

(i) *The morphism associated to L, φ_L is not birational.*
(ii) *There is a smooth hyperelliptic curve in $|L|$.*
(iii) *All smooth curves in $|L|$ are hyperelliptic.*
(iv) *Either $L^2 = 2$; or there exists a smooth elliptic curve E on Σ satisfying $E.L = 2$; or $L \sim 2B$ for a smooth curve B, with $B^2 = 2$.*

Proposition 7.7.23. *Let L be a basepoint free non hyperelliptic line bundle, with $L^2 \geq 8$ on a K3 Σ. Let $K = \ker(\mathrm{Sym}^\bullet H^0(L) \to \bigoplus_n H^0(nL))$. Then K is generated by quadrics and cubics. Moreover the following are equivalent.*

(i) *K is generated not solely by quadrics.*
(ii) *$|L|$ contains a smooth curve with a g^1_3 or a g^2_5. (see Section A.3.2 for the definition of g^r_d).*
(iii) *Smooth curves in $|L|$ all have a g^1_3 or all have a g^2_5.*
(iv) *There is a smooth elliptic curve E on Σ such that $E.L = 3$; or $L \sim 2B + D$ for a smooth curve B, with $B^2 = 2$ and D a smooth rational curve with $D^2 = -2$ and $B.D = 1$ (in particular $L^2 = 10$).*

Propositions 7.7.22 and 7.7.23 are results of [90], but following the formulation of [59] Theorem 1.1 and Theorem 1.3. Note that in the case of $L^2 = 4$ and $L^2 = 6$, all smooth curves in $|L|$ have genus 3 or 4, so necessarily have a g^1_3. A curve is hyperelliptic if and only if it has a g^1_2. These special divisors are detected by Clifford index (see Definition A.3.6). One consequence of these results is that if one smooth curve in the anticanonical system is Clifford special, then they all are. This was generalized by Green and Lazarsfeld (Theorem 7.7.27).

We next consider contributions by Mukai. Mukai extended the notion of Brill-Noether general from curves to polarized K3 surfaces.

Definition 7.7.24. A polarized K3 (Σ, H) of genus g is said to be *Brill-Noether general* if for all $A, B \in \Lambda \setminus \{0\}$ such that $A + B = H$ then $h^0(A)h^0(B) < h^0(H)$.

Theorem 7.7.25 ([83, Theorems 3.2, 4.7, 5.5]). *The projective models of Brill Noether general polarized K3s of small genus are as follows:*

g	Projective model
2	$\Sigma \to \mathbb{P}^2$ double covering with branched sextic.
3	$(4) \subset \mathbb{P}^3$ quartic hypersurface.
4	$(2,3) \subset \mathbb{P}^4$ complete intersection.
5	$(2^3) \subset \mathbb{P}^5$ complete intersection.
6	$(1^3, 2) \cap Gr(2,5) \subset \mathbb{P}^6$. $Gr(2,5) \subset \mathbb{P}^9$ is a 6 dimensional variety of degree 5.
7	$(1^8) \cap G^{10}_{12} \subset \mathbb{P}^7$. $G^{10}_{12} \subset \mathbb{P}^{15}$ is a 10 variety of degree 12.
8	$(1^6) \cap Gr(2,6) \subset \mathbb{P}^8$. $Gr(2,6) \subset \mathbb{P}^{14}$ is an 8 dimensional variety of degree 14.
9	$(1^4) \cap G^6_{16} \subset \mathbb{P}^9$. $G^6_{16} \subset \mathbb{P}^{13}$ is a 6 dimensional variety of degree 16.
10	$(1^3) \cap G^5_{18} \subset \mathbb{P}^{10}$, $G^5_{18} \subset \mathbb{P}^{13}$ is a 5 dimensional subvariety of degree 18.
12	$\Sigma_{12} = (1) \subset G^3_{12} \subset \mathbb{P}^{12}$.

For cases $g \geq 6$, the projective models occur as subspaces of certain homogeneous spaces G^n_d.

Corollary 7.7.26. *Let (Σ, H) be a smooth polarized K3 that is Brill Noether general, and of genus $6 \leq g \leq 10$ or $g = 12$. Then there exists a smooth Fano threefold $Y \subset \mathbb{P}^g$ of Picard rank 1, Fano index 1, and genus g, such that $\varphi_H : \Sigma \to Y$ embeds Σ as an anticanonical divisor.*

(See also [42, Proposition 2.6] and [2, Section 6].) The conditions of being Brill Noether general does not immediately translate to an arithmetic constraint on a polarizing lattice. In addition, a class being Brill Noether general is not sufficient to ensure a lattice is generic in all cases *eg* Proposition 7.7.36. The following result of Green and Lazarsfeld [45] is a, if not the, key ingredient to understanding the projective embeddings of lattice polarized K3s. Recall that a divisor A on a curve C *computes* the Clifford index if $\mathrm{Cliff}(C) = \mathsf{deg}(A) - 2(h^0(A) - 1)$ (Definition A.3.6).

Theorem 7.7.27 ([45])**.** *Let Σ be a projective K3, and $C \subset \Sigma$ be a smooth irreducible curve of genus $g \geq 2$. Then $\mathrm{Cliff}(C') = \mathrm{Cliff}(C)$ for every smooth curve $C' \in |C|$. Furthermore, if $\mathrm{Cliff}(C)$ is strictly less than the generic value of $\mathrm{floor}(\frac{1}{2}(g-1))$, then there exists a line bundle L on C such that for any smooth curve $C' \in |C|$, $L|_{C'}$ computes the Clifford index of C'.*

Definition 7.7.28. Let L be a basepoint free line bundle on a K3 Σ. The *Clifford index of* (Σ, L) is $\mathrm{Cliff}_L(\Sigma) := \mathrm{Cliff}(C)$ for a smooth curve $C \in |L|$. (Σ, L) is of *non-general Clifford index* if $\mathrm{Cliff}_L(\Sigma) < \mathrm{floor}(\frac{g-1}{2})$.

Let C be a smooth curve of genus g, and let r, d be integers. Let the subset $V^r_d(C) \subset \mathrm{Pic}^d(C)$ be the set of line bundles A such that $h^0(A) = r+1$, $\deg(A) = d$ and both A and

$A^\vee \otimes \omega_C$ are generated by global sections. V^r_d is an open subset of Brill Noether loci of C (see Definition A.3.8). The following can be found in [69, Section 1].

Proposition 7.7.29. *Let (Σ, H) be a polarized K3. Let $C \in |H|$ be a smooth canonical curve, and let $A \in V^r_d(C)$. Then there exists a vector bundle F on Σ of rank* $\mathrm{rank}(F) = (r+1)$*, such that $H^0(F) = H^2(F^\vee) = 0$, $H^1(F) = H^1(F^\vee) = 0$, $h^0(F^\vee) = h^0(A) + h^1(A)$, $c_1(F) = -[H]$, and $c_2(F) = d$. Moreover $F^\vee$ is globally generated by its sections.*

7.7.3 Application to lattices

We abstract the notions that apply to algebraic K3 surfaces to abstract lattices.

Lemma 7.7.30. *Let Λ be a nondegenerate lattice of signature $(1, r-1)$. If $H \in \Lambda$ and $H^2 > 0$, then the set*

$$\Lambda_H(d, s) := \{D \in \Lambda \colon D.H = d,\ D^2 = s\} \tag{7.19}$$

is finite.

Computing $\Lambda_H(d, s)$ is algorithmic, and feasible in all cases of interest to us.

Proposition 7.7.31. *Let (Λ, H) be a nef lattice. If $\Lambda_H(\frac{1}{2}(H^2 - 2), -2) = \varnothing$ then H is basepoint free on $\Sigma \in \mathcal{D}^\circ_{(\Lambda,H)}$.*

Proof. By assumption $H \in \mathrm{Pic}(\Sigma)$ is nef. By Proposition 7.7.21 H is basepoint free if and only if it is not monogonal. If H is monogonal then there exists a class $D \in \mathrm{Pic}(\Sigma)$, such that $H.D = \frac{1}{2}(H^2 - 2)$ and $D^2 = -2$. By assumption no such class exists and the result follows. $\square$

We shall say that a pair (Λ, H) satisfying the hypotheses of Proposition 7.7.31 is a *basepoint free lattice.* For $\Sigma \in \mathcal{D}^\circ_{(\Lambda,H)}$, H defines a morphism $\varphi_H : \Sigma \to \mathbb{P}^g$, where $g = \frac{1}{2}(H^2 + 2)$.

Proposition 7.7.32. *Let (Λ, H) be a basepoint free lattice. If $\Lambda_H(0, -2) = \varnothing$ then H is ample on $\Sigma \in \mathcal{D}^\circ_{(\Lambda,H)}$.*

Proof. A class in $\Lambda_H(0, -2) \subset \mathrm{Pic}(\Sigma)$ are represented by hyperelliptic canonical curves. The result is thus a reformulation of Proposition 7.7.22. $\square$

We shall say that a pair (Λ, H) satisfying the hypotheses of Proposition 7.7.32 is an *ample lattice.* The morphism defined by φ_H does not contract any (-2)-curves.

Proposition 7.7.33. *Let (Λ, H) be an ample lattice with $H^2 \geq 4$. If $\Lambda_H(2, 0) = \varnothing$, and $\nexists D \in \Lambda$ such that $2D = H$ and $D^2 = 2$, then for $\Sigma \in \mathcal{D}^\circ_{(\Lambda,H)}$, $\varphi_H : \Sigma \to \mathbb{P}^g$ is an embedding.*

Proof. By assumption $H \in \mathrm{Pic}(\Sigma)$ is ample. The additional hypotheses imply that H is not hyperelliptic by Proposition 7.7.22, and also that φ_H is birational. $\square$

We shall say that a pair (Λ, H) satisfying the hypotheses of Proposition 7.7.33 is a *very ample lattice*. We consolidate these results to the following.

Corollary 7.7.34. *Let Λ be a polarizing lattice, and $H \in \Lambda$ be primitive with $H^2 \geq 4$. If $\Lambda_H(\frac{1}{2}H^2 - 1, -2) = \varnothing$, $\Lambda_H(0, -2) = \varnothing$, and $\Lambda_H(2, 0) = \varnothing$, then for $\Sigma \in \mathcal{D}^\circ_{(\Lambda,H)}$, $\varphi_H : \Sigma \to \mathbb{P}^g$ is an embedding.*

7.7.4 Genericity results for $\rho = 1$ and $r \geq 2$

We fix the following notation. We may refer to a deformation family by a representative. We are not explicit about what exactly is meant by a deformation family. We will assume that all members are smooth. There are 2 deformation families of Fano threefolds with Picard rank $\rho = 1$ and Fano index $r > 2$. Namely $\mathbb{P}^3$ with $r = 4$ and quadric hypersurfaces $Q \subset \mathbb{P}^4$ with $r = 3$. There are 5 families of Picard rank 1 del Pezzo threefolds (Fano threefolds with $r = 2$), denoted V_d where $1 \leq d \leq 5$ is the degree of the fundamental class $H := \frac{1}{r}(-K_Y)$ of any member Y of the family.

The set of $\mathcal{Y}$-generic lattices will be denoted by $\mathbf{gen}(\mathcal{Y})$, since the cone $C \subset N$ is essentially unambiguous.

Proposition 7.7.35. *Let Λ be a polarizing lattice with very ample class $H \in \Lambda$ such that $H^2 = 4$. Then $\Lambda \in \mathbf{gen}(\mathbb{P}^3)$.*

Proof. Immediate. □

Proposition 7.7.36. *Let Λ be a polarizing lattice with very ample class $H \in \Lambda$ such that $H^2 = 6$. If $\Lambda_H(3, 0) = \varnothing$ then $\Lambda \in \mathbf{gen}(Q)$.*

Proof. We have a nondegenerate embedding $\varphi_H : \Sigma \to \mathbb{P}^4$. By (7.17), the family of quadric hypersurfaces containing Σ is non empty, but perhaps 0-dimensional. Let Q be a quadric containing Σ. If Q were reducible then $Q = P + P'$ where P, P' are hyperplanes, but this contradicts that φ_H is nondegenerate. Thus Q is irreducible.

Σ is a smooth Cartier divisor in Q. By Lemma A.3.4, then Q has at worst isolated singularities and these are away from Σ.

Suppose Q is singular, then Q is a cone with a vertex away from Σ. We can fix coordinates such that $Q = \{x_0x_1 - x_2x_3\} \subset \mathbb{P}^4$. Consider then $D = \Sigma \cap \{x_0\}$ which consists of cubic curves $C_i = \Sigma \cap \{x_0 = 0, x_i = 0\}$ for $i = 2, 3$. As D corresponds to the hyperplane class so $D = C_2 + C_3$, $C_2^2 = C_3^2 = 0$ and $C_2 \cdot C_3 = 3$. Then $C_i \in \Lambda_H(3, 0)$, contradicting $\Lambda_H(3, 0) = \varnothing$.

Therefore Q is smooth. □

Proposition 7.7.37. *Let Λ be a polarizing lattice and let $H \in \Lambda$ be very ample and such that $H^2 = 10$. If $\Lambda_H(3, 0)$ and $\Lambda_H(5, 2)$ are empty, then Λ embeds into a possibly singular threefold that appears on the boundary of V_5.*

Our proof stops short of the what is desired—it is yet to demonstrate that a threefold in which the K3 embeds is smooth.

Proof of Proposition 7.7.37. A generic canonical curve of (Σ, H) is a smooth curve C of genus 6. As H is very ample, H is not hyperelliptic, so C does not carry a g^1_2. If C carries either a g^1_3 or a g^2_5 then the Clifford index $\mathrm{Cliff}(C) \leq 1$. By Theorem 7.7.27, a line bundle describing these special divisors exists on Σ. However, this would contradict $\Lambda_H(3,0) = \varnothing$ and $\Lambda_H(5,2) = \varnothing$.

By Proposition A.3.9, C carries a complete g^1_4. Let A be a line bundle on C corresponding to the g^1_4. The residual of g^1_4 is g^2_6, and hence $h^1(A) = 3$. As there are no g^1_3 or g^2_5, both A and $A^\vee \otimes \omega_C$ are free. By Proposition 7.7.29 there exists a vector bundle F on Σ with certain properties. Let V be the dual to the vector bundle F. Then V is of rank 2 and

$$h^0(V) = 5 \quad c_1(V) = [H] \quad c_2(V) = 4 \tag{7.20}$$

and V is globally generated by its sections. Thus we have an induced morphism

$$\alpha : \Sigma \to \mathrm{Gr}(2,5) \tag{7.21}$$

The Plucker embedding $pl : \mathrm{Gr}(2,5) \to \mathbb{P}^9$ is a degree 5 embedding. By construction, there is a linear embedding $\iota : \mathbb{P}^6 \to \mathbb{P}^9$, such that $\iota \circ \varphi_H = pl \circ \alpha : \Sigma \to \mathbb{P}^9$. Note that $\iota(\mathbb{P}^6)$ is *span* of $pl \circ \alpha(\Sigma) \subset \mathbb{P}^9$, and so is the unique 6-plane in $\mathbb{P}^9$ containing $pl \circ \alpha(\Sigma)$. □

Conjecture 7.7.38. *In the notation of Proposition 7.7.37, if in addition $\Lambda_H(4,0)$ is empty, then $\Lambda \in \mathbf{gen}(V_5)$.*

In the notation of the proof of Proposition 7.7.37, let $Y = \iota(\mathbb{P}^6) \cap pl(\mathrm{Gr}(2,5))$. We wish to show that Y is a smooth threefold. It remains to check that $\dim(Y) = 3$, Y is irreducible, and Y is smooth. This can be done by considering Schubert cells of the Grassmannian. We do not pursue this further.

Proposition 7.7.39. *Let Λ be a polarizing lattice and let $H \in \Lambda$ be very ample and such that $H^2 = 8$. If $\Lambda_H(3,0) = \varnothing$ then $\Lambda \in \mathbf{gen}(V_4)$.*

Proof. We have a smooth nondegenerate embedding $\varphi_H : \Sigma \to \mathbb{P}^5$. By (7.17), the linear system L of quadric hypersurfaces containing Σ has dimension at least 3. By Theorem 7.7.23, Σ is cut out by quadrics containing it if and only if a smooth anticanonical C has no g^1_3. If C had a g^1_3 then by Theorem 7.7.27 it would appear as a class in the $\mathrm{Pic}(\Sigma)$, contradicting the assumption $\Lambda_H(3,0) = \varnothing$.

Let N be a net of quadrics containing Σ. As φ_H is nondegenerate, any quadric in N is irreducible, ie it cannot be the union of two hyperplanes. If Σ is the complete intersection of a net of quadrics, then by Lemmas A.3.2 and A.3.4, two generic members $Q_0, Q_1 \in N$ intersect along a smooth threefold $Y = Q_0 \cap Q_1$. In which case $Y \in V_4$.

It remains to consider if Σ is not a complete intersection. In which case, any two distinct $Q_i \in N$ intersect on a reducible threefold, and at least one component of which is contained by all other quadrics of N. Let $Z = Q_0 \cap Q_1 = \sum Z_i$ where Z_i are each irreducible. As $\Sigma \subset Z$ assume that $\Sigma \subset Z_1$.

If $\deg(Z_1) = 1$ then Z_1 is a hyperplane, contradicting φ_H nondegenerate. If $\deg(Z_1) = 2$ then there is an embedding $Z_1 \to \mathbb{P}^4$, again contradicting φ_H nondegenerate. If

$\deg(Z_1) = 4$ then $Z = Z_1$ and we shall return to the case of the complete intersection. It remains to consider $Z = Z_1 + Z_2$ where Z_1 is a threefold of degree 3 in $\mathbb{P}^5$.

Z_1 is nondegenerate (does not lie in any hyperplane). By Enriques (See [90, Corollary 1.12]) Z_1 is either a cone over Hirzebruch surface $F_1 \subset \mathbb{P}^4$, or the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$.

If Z_1 is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$, then $\Lambda_H(3, 0) \neq \varnothing$ (*cf* Proposition 7.7.48).

Suppose that Z_1 is a cone over $F_1 \subset \mathbb{P}^4$. F_1 is isomorphic to $\mathbb{P}^2$ blown up in a point. Let $F, G \in \mathrm{Pic}(F_1)$ correspond to the proper transforms of a lines on $\mathbb{P}^2$ with only the latter passing through the blowup point. Then $h^0(F) = 2$ and $h^0(G) = 3$. The linear system of $G + F$ determines a map $F_1 \to \mathbb{P}^4$. As the vertex cannot be contained in Σ by Lemma A.3.4, we have a projection $p : \Sigma \to F_1$ from the vertex of Z_1. Then $H^2 = d(F + G)^2$ where d is the degree of morphism, but $(F + G)^2 = 3$ which is impossible. □

Remark 7.7.40. Note that if a smooth anticanonical curve C does carry a g^1_3, then it cannot be the complete intersection of quadrics. In fact, by a classical theorem of Enriques-Petri, the homogeneous ideal of a canonical curve with a g^1_3 is generated by both quadrics and cubics. *cf* with Proposition 7.7.48

Proposition 7.7.41. *Let Λ be a polarizing lattice and let $H \in \Lambda$ be very ample and such that $H^2 = 6$. Then $\Lambda \in$ **gen**(V_3).*

Proof. We have a smooth nondegenerate embedding $\varphi_H : \Sigma \to \mathbb{P}^4$. By (7.17), there exists at least a 5 dimensional family of cubic hypersurfaces containing Σ. By Proposition 7.7.36, Σ is contained in a quadric Q that may or may not have a singular point. There is a precisely 4-dimensional subspace consisting of reducible cubics of the form $Q + \Pi$, where Π is a hyperplane. Let W be an irreducible cubic hypersurface. Then $\Sigma = W \cap (Q + \Pi) \cap (Q + \Pi')$. Thus $\mathrm{Bs}(H^0(\mathcal{I}_\Sigma \otimes \mathcal{O}_{\mathbb{P}^4}(3))) = \Sigma$. By Bertini and Lemma A.3.4, so Σ is contained in a smooth cubic hypersurface $W \in V_3$. □

Proposition 7.7.42. *Let Λ be a polarizing lattice and let $H \in \Lambda$ be ample and such that $H^2 = 4$. Then $\Lambda \in$ **gen**(V_2).*

Proof. We have a smooth nondegenerate embedding $\varphi_H : \Sigma \to \mathbb{P}^3$. Let $Y \to \mathbb{P}^3$ be the double cover branched over $\varphi_H(\Sigma)$. Then the preimage of $\varphi_H(\Sigma)$ in Y is isomorphic to Σ and $Y \in V_2$. □

Proposition 7.7.43. *Let Λ be a polarizing lattice and let $H \in \Lambda$ be very ample and such that $H^2 = 2$. Then $\Lambda \in$ **gen**(V_1).*

Proof. We have a double cover $\varphi_H : \Sigma \to \mathbb{P}^2$ branched over a smooth sextic curve C. Suppose that C is the zero locus of $f(x_0, x_1, x_2)$. Then Σ is isomorphic to the zero locus of $x_3^2 + f(x_0, x_1, x_2)$ in weighted projective space $\mathbb{P}(1^3, 3)$. Let Y be a degree 6 hypersurface in the weighted projective space $\mathbb{P}(1^3, 2, 3)$. Then Σ is isomorphic to a quadric section of Y, and so $\Lambda \in$ **gen**(V_1). □

7.7.5 Genericity results for $\rho = 1$ and $r = 1$

For the cases of $\rho = 1$, $r = 1$, and $g \in \{6, \ldots, 12\} \setminus \{11\}$ Corollary 7.7.26 describes genericity type conditions but in terms of being Brill Noether general. We require a formulation purely in terms of lattice arithmetic. The following result reformulates the question in terms of Clifford generality.

Proposition 7.7.44 ([59, Theorem 10.5])**.** *Let (Σ, H) be a polarized K3 of genus $g \in \{2, \ldots, 9\} \setminus \{8\}$. Then (Σ, H) is Brill Noether general if and only if it is Clifford general.*

If $g = 8$ (respectively $g = 10$), then (Σ, H) is Clifford general but not Brill Noether general if and only if there is an effective divisor D satisfying $D^2 = 2$ and $D.H = 7$ (respectively $D.H = 8$), and there is no divisor E such that $2E = H$.

By Theorem 7.7.27, failure to be Brill Noether general will be exhibited by the existence of certain classes of the polarizing lattice. It remains only to list the square and degree of classes that will depress the Clifford index in these cases. We denote the Fano families with Picard rank 1, Fano rank 1, and genus g by A_g, for $g = 2, 3, 4, 5$.

Lemma 7.7.45. *Let $X \subset \mathbb{P}^g$ be a smooth hypersurface of degree k. Then there exists a smooth hypersurface $X' \subset \mathbb{P}^{g+1}$ of degree k and hyperplane $H \cong \mathbb{P}^g \subset \mathbb{P}^{g+1}$ such that $X = X' \cap H$*

Proof. This is a special case of Lemma A.3.3. Directly, suppose $f(x_0, \ldots, x_g)$ is the homogeneous polynomial of degree k describing X. Let $F(x_0, \ldots, x_{g+1}) = x_{g+1}^k - f(x_0, \ldots, x_g)$ describe X' and $H = \{x_{g+1} = 0\}$. By design $X = X' \cap H$. Note that $\partial_{g+1} F = k x_{g+1}^{k-1}$, so X' is smooth away from $X \cap H$. By Lemma A.3.4 or otherwise, X' is smooth. □

Proposition 7.7.46. *Let Λ be a polarizing lattice and $H \in \Lambda$ be very ample.*
(i) Suppose that $H^2 = 8$. If $\Lambda_H(3, 0) = \varnothing$ then $\Lambda \in \mathbf{gen}(A_5)$.
(ii) Suppose that $H^2 = 6$. If $\Lambda_H(3, 0) = \varnothing$ then $\Lambda \in \mathbf{gen}(A_4)$.
(iii) Suppose that $H^2 = 4$ then $\Lambda \in \mathbf{gen}(A_3)$.

Proof. Suppose $H^2 = 8$. We have a smooth nondegenerate embedding into $\varphi_H : \Sigma \to \mathbb{P}^5$. By Proposition 7.7.39, Σ is the complete intersection of 3 quadrics in $\mathbb{P}^5$. Moreover, we may assume each of these is smooth. By Lemma A.3.3 we can extend each of these to a smooth quadric in $\mathbb{P}^6$. Moreover, we can extend in such a way that they form a smooth complete intersection. The intersection is a threefold $Y \in A_5$.

For the case that $H^2 = 6$, the proof is as above using Proposition 7.7.41. For the case that $H^2 = 4$, the proof is as above using Proposition 7.7.42. □

Proposition 7.7.47. *Let Λ be a polarizing lattice and $H \in \Lambda$ be such that $H^2 = 2$ and assume that $\nexists E, D \in \Lambda$, $k \geq 2$ such that $H = kE + D$, $E^2 = 0$, $E.D = 0$, $D^2 = -2$. Then $\Lambda \in \mathbf{gen}(A_2)$.*

Proof. Let $\Sigma \in \mathcal{D}^{\circ}_{\Lambda}$. By Proposition 7.7.21, $H \in \mathrm{Pic}(\Sigma)$ is basepoint free. Hence $\varphi_H : \Sigma \to \mathbb{P}^2$ is a smooth morphism. φ_H is a double cover ramified in a smooth curve C of degree 6.

Embed $\mathbb{P}^2 \to \mathbb{P}^3$ as a hyperplane By Lemma A.3.3 we can extend C to a smooth degree 6 surface S on $\mathbb{P}^3$. Let $Y \to \mathbb{P}^3$ be the double cover ramified along S. Then $Y \in A_2$. □

7.7.6 A Primitive case

There are 13 primitive Fano threefolds. Five are explicitly described as double covers of other Fano threefolds. Two are the products $\mathbb{P}^1 \times \mathbb{P}^2$, and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Four are divisors in $\mathbb{P}^2 \times \mathbb{P}^2$ (note that the $\mathbb{P}^1 \times \mathbb{P}^2$ is one such divisor). The three others (AGV.2.35, AGV.3.2, AGV.3.31) have different, more involved presentations.

Proposition 7.7.48. *Let $Y = \mathbb{P}^1 \times \mathbb{P}^2$, and $\mathcal{Y}$ the associated deformation family. Let N be the rank 2 lattice with basis G, H determined by $G^2 = 0$, $G.H = 3$ and $H^2 = 2$. Let Λ be a polarizing lattice that is a primitive overlattice of N, and let $A = G + H$. If all:*
(i) $\Lambda_A(0, -2) = \varnothing$ and $\Lambda_A(2, 0) = \varnothing$
(ii) $\nexists D \in \Lambda$ such that $0 < A.D < 3$ and $D^2 = -2$, and $G.D < 0$.
(iii) $\nexists D \in \Lambda$ such that $0 < A.D < 5$ and $D^2 = -2$, and $H.D < 0$.
then $\Lambda \in \mathbf{gen}(\mathcal{Y})$.

Proof. We will show that $\varphi_A : \Sigma \to \mathbb{P}^5$ is the anticanonical divisor of a Segre embedding $\mathbb{P}^1 \times \mathbb{P}^2$.

$A^2 = 8$, A is primitive, and by assumption $\Lambda_A(0, -2) = \varnothing$ and $\Lambda_A(2, 0) = \varnothing$, Hence by Corollary 7.7.34 (Λ, A) is very ample, and so $\varphi_A : \Sigma \to \mathbb{P}^5$ is an embedding.

Both G, H have squares ≥ 0 and their products with A are each positive. Hence by Corollary 7.7.6 G and H are effective. Note that H is now the residual of G on (Σ, A).

Suppose that G is not nef. Then there would exist an effective (-2)-curve Γ such that $G.\Gamma < 0$ and $G - \Gamma$ is effective. In particular, $0 < A.(G - \Gamma) = 3 - A.\Gamma$. However, this contradicts the assumption $\nexists D \in \Lambda$ such that $0 < A.D < 3$ and $D^2 = -2$, and $G.D < 0$. Thus G is nef. Similarly the assumption that $\nexists D \in \Lambda$ such that $0 < A.D < 5$ and $D^2 = -2$, and $H.D < 0$ implies that H is nef.

Then for a canonical curve $C \in |A|$, $G|_C$ is a complete g^1_3. By Proposition 7.7.23, φ_A is not the complete intersection of 3 quadrics. It is then either contained in the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ of a cone over the Hirzebruch surface $F_1 \subset \mathbb{P}^4$ embedded by $|b + 2f|$. (cf Proposition 7.7.39). We prove that it is the former.

As G is nontrivial, effective, nef and $G^2 = 0$, there exists a free elliptic pencil $|E|$ such that $G = aE$, by Proposition 7.7.20. As G is primitive, $a = 1$, and so $h^0(G) = 2$. As H is nef and big, $h^0(H) = \frac{1}{2}H^2 + 2 = 3$, by Proposition 7.7.20.

We have the following diagram.

$$\begin{tikzcd}
 & \mathbb{P}^1 \times \mathbb{P}^2 \arrow[d, "|\mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{O}_{\mathbb{P}^2}(1)|"] \\
\Sigma \arrow[ur, "(\backslash|G\backslash|{,}\backslash|H\backslash|)"] \arrow[r, "|G+H|"'] & \mathbb{P}^5
\end{tikzcd}$$

We claim that this diagram commutes or, equivalently, that $H^0(G) \otimes H^0(H) \to H^0(A)$ is an isomorphism.

It suffices to show that the kernel is trivial since both domain and codomain have dimension 6. Let $s_0, s_1 \in H^0(G)$ be a basis, and let $S_i = s_i^{-1}(0)$ be two fibres of the free pencil. Suppose that $s_0 \otimes t_0 + s_1 \otimes t_1$ belongs to the kernel. Firstly, t_i must both be nonzero. If $t_0 = 0$, then $\forall x \in \Sigma \setminus S_1$, $t_1(x) = 0$ which is open dense and so $t_1 = 0$ contradicting $r \neq 0$. Analogous contradiction is obtained by taking $t_1 = 0$.

On S_0, s_1 is nonzero, so $t_1|_{S_0} = 0$. Thus $S_0 \subset t_1^{-1}(0)$. Hence $H - G$ is also effective. As H is nef, we arrive then at the contradiction $H.(H - G) = -1$. □

7.7.7 Genericity result for imprimitive threefolds

A smooth Fano threefold X with Picard rank $\rho > 1$ is *primitive* if it cannot be described as a blowup $X \to Y$ along a smooth curve $C \subset Y$ on a Fano threefold Y. Otherwise Y is *imprimitive*.

Suppose that C is a smooth curve on a Fano threefold Y, and that $X \to Y$ is the blowup of Y along C. Irreducible curves on X are either fibres of the blowup or the proper transform of a curve on Y. If F is a fibre of the blowup, then $-K_X.F = 1$. Suppose that A is a smooth, degree k curve on Y, and that it is an l-secant curve to C. Then $-K_X.\tilde{A} = -K_Y.A - l$, where $\tilde{A}$ is the proper transform of A. By assumption that Y is Fano, $-K_Y.A > 0$. Thus X may fail to be Fano or weak Fano if there exists a curve A with a sufficiently low $k : l$ ratio.

Of the imprimitive cases, [80] describes curves central to the blowup either as the scheme theoretic intersection of divisors, or by the genus and degree of the curve or both. It is presented as a complete determination of each of the deformation families. For example, in the case of AGV.2.9, the deformation family is presented as the blowup of a smooth genus 5 degree 7 curve that is cut out by cubic surfaces. That is, there exists $V < H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$ such that $C = \mathrm{Bs}(|V|)$. There exist cases of genus 5 degree 7 curves that fail to be cut out by cubic surfaces, eg if C admits a 4-secant line l then l is contained in every cubic surface containing C.

In the classification of weak Fanos of Picard rank 2, the picture is less complete. For a class to be deemed geometrically realized, the class is simply shown to be nonempty. The generic nature of a given class is less readily available in the literature. This is noted in [12] where the authors treat the cases of the blowup of $\mathbb{P}^3$ along a smooth curve. [12, Theorem 1.1] (see Proposition 7.7.50) characterizes the generic nature of the curves in each case in terms of secant curves. For those classes which obtain a Fano, this gives an alternative characterization of generality to those of [80].

We wish to encode these ‘bad’ high secant curves as properties of the polarizing lattice. Our approach in addressing this is suboptimal, in the sense that it will exclude cases that do not lead to ‘bad curves’. However, it is relatively straightforward and doesn't require a case by case consideration.

Suppose that $C \subset \Sigma$ is a smooth curve, and that $\Sigma \subset |-K_Y|$ on a Fano threefold Y. Then $|-K_Y|$ cuts out $C \subset Y$ if and only if $-K_Y|_\Sigma$ cuts out C on Σ, which, by definition, is if and only if $-K_Y|_\Sigma - C$ is basepoint free. This final condition is then an arithmetic constraint on the polarizing lattice of the K3 via Proposition 7.7.21.

Proposition 7.7.49. *Let $\mathcal{Y}$ be a deformation family of N-polarized threefolds, let Λ be a polarizing lattice and $H \in \Lambda$ be very ample, and let $V \subset N \otimes \mathbb{R}$ be a cone such that $\mathcal{Y}$ is*

(Λ, V)-generic.

Suppose that $\Sigma \in \mathcal{D}^{\circ}_{(\Lambda,H)}$, $\Sigma \in |-K_Y|$ for some $Y \in \mathcal{Y}$, $C \in \Lambda$ corresponds to a smooth curve on Σ and that $-K_Y|_\Sigma - C$ is basepoint free. Let $\pi : X \to Y$ be the blowup along C with exceptional class E.

Then X is weak Fano, and the cone generated V' by $-K_X$ and $\pi^* V$ is contained in the nef cone of X. Moreover, let $\mathcal{X}$ denote the set of all X constructed as above for a fixed class $C \in \Lambda$. Then $\mathcal{X}$ is (Λ, V')-generic.

The ample cones for each of the rank 2 Fano cases are described in [30]. However, Proposition 7.7.49 does not guarantee that the threefold we embed the K3 into is a genuine Fano. In particular, we cannot guarantee that a configuration necessarily leads to matching unless the proposed ample class belongs to the interior of a more restrictive cone. Note that for weak Fanos with Picard rank 2, the ample cone is immediate from the construction ie the edges of the cone are the anticanonical class and pullback of the fundamental class.

The following result of Blanc and Lamy [12] gives a precise description of the deformation family for the weak Fanos that are obtained as a blowup of $\mathbb{P}^3$ along a smooth curve. In addition, it presents an alternative characterization of each deformation family of Fano threefolds obtained in this manner, namely in terms of high secant curves.

Proposition 7.7.50. *Let $Y = \mathbb{P}^3$. Let $\mathcal{H}_{g,d}$ denote the Hilbert scheme of smooth irreducible genus g, degree d curves on Y.*

$$\begin{aligned}
A_0 &:= \{(6,5),(10,6),(8,8),(12,9)\} \\
A_1 &:= \{(0,1),(0,2),(0,3),(0,4),(1,3),(1,4),(2,5),(4,6)\} \\
A_2 &:= \{(1,5),(3,6),(5,7),(10,9)\} \\
A_3 &:= \{(0,5),(0,6),(1,6),(2,6),(3,4),(3,7),(4,7),(6,7), \\
&\qquad (6,8),(7,8),(9,8),(9,9),(12,10),(19,12)\} \\
A_4 &:= \{(0,7),(1,7),(2,7),(2,8),(3,8), \\
&\qquad (4,8),(5,8),(6,9),(7,9),(8,9),(10,10),(11,10),(14,11)\}
\end{aligned} \tag{7.22}$$

Let $C \in \mathcal{H}_{g,d}$ for some pair $(g,d) \in A_i$. Let $X \to Y$ be the blowup along C. Then

(i) X is Fano if and only if either $(g,d) \in A_1$ or $(g,d) \in A_2$ and there is no 4-secant line to C.

(ii) X is weak Fano if and only if one of the following holds

- *(i) $(g,d) \in A_1 \cup \{(3,6)\}$;*
- *(ii) $(g,d) \in A_2 \setminus \{(3,6)\}$ and there is no 5-secant line to C;*
- *(iii) $(g,d) \in A_3$, there is no 5-secant line to C, and C is contained in a smooth quartic;*
- *(iv) $(g,d) \in A_4$, there is no 5-secant line, 9-secant conic, nor 13-secant twisted cubic to C, and C is contained in a smooth quartic.*

Conversely, if $X \to Y$ is the blowup along a smooth curve C and X is Fano or weak Fano, then it appears as one of the cases above.

(See [12, Theorem 1.1])

Remark 7.7.51. It seems there is a typo in the statement of [12, Theorem 1.1]. In the paper, one condition on being weak Fano reads $(g, d) \in A_2 \setminus \{(3, 6)\}$ *and there is no* 4-*secant line to* C. Comparing the statement with [12, Proposition 4.2] suggests that this should read as stated in Proposition 7.7.50.

The inclusion A_0 is to complete a partition of all pairs that lie in region determined by application of more elementary considerations. See *loc. cit.*.

If a smooth curve C on a quartic surface Σ in $\mathbb{P}^3$ has a $4k+1$-secant degree k curve l, then l is contained in Σ. In particular, there exists a class in $\mathrm{Pic}(\Sigma)$ corresponding to l. Then $-K_Y|_\Sigma - C$ would be negative on l and in particular fails to be basepoint free. [11] presents the analogous result for cubic hypersurfaces in $\mathbb{P}^4$.

More generally, high secant varieties to a curve correspond to special divisors on the curve (see Propositions A.3.10 and A.3.11). These divisors depress the Clifford index to a nongeneric value. For a smooth canonical curve with nongeneric Clifford index in a K3, this is 'picked up' in $\mathrm{Pic}(\Sigma)$ by Theorem 7.7.27. We have yet to make this precise.

Collecting together the results from the section we can formulate genericity conditions for many of the Fano threefolds, and the weak Fano threefolds with φ-type E1.

Chapter 8

Examples

We will compute some of the boundary defect invariants defined in Chapter 5 on G_2-manifolds obtained via the TCS method. Let us summarise the components of this procedure.

In Sections 7.3 and 7.4 we discussed the classification of Fano threefolds and of weak Fano threefolds of Picard rank 2. In Section 7.5 we described topological data that can be computed for threefolds with Picard rank 2 via the results in Section A.2. We restrict ourselves to Fano threefolds and semi Fanos of φ-type $E1$ or $E2$. The deformation invariants of these cases are tabulated in Section B.2. Section B.1 has an explanation of the naming convention.

Building blocks can be obtained from semi Fanos via Construction 7.1.3. We compute the topological data relevant to the TCS construction in each case. In particular, we calculate the polarizing lattice $N_\pm$ and vectors describing the image of the ample cone or at least a subcone of it.

From the Picard lattices and cone corresponding to ample vectors, we construct configurations $P = N_+ + N_-$ as discussed in Section 6.4. From P we obtain the two overlattices $\Lambda_\pm$, each of signature vector $(1, r_\pm - 1, 0)$, where $r_\pm$ depend on the properties of P. We then check the genericity conditions with the results of Section 7.7. This guarantees that the configuration corresponds to a genuine G_2-manifold by Proposition 6.4.4. Section 6.3 discussed the topology of the TCS in terms of the configuration and the topological data for the building blocks.

In Section 8.1, we describe an example of a skew matching with empty intersection to illustrate how this procedure works. In Section 8.2, we describe examples of TCS manifolds which are diffeomorphic and with G_2-structures distinguished by the ξ invariant. In Section 8.3, we describe an example of an orthogonal matching with N_0 of rank 1. In Section 8.4, we describe the computations of the $\mathfrak{C}$-model and the polarized spin invariants defined in Section 5.2.3. In Section 8.5, we consider constructions of TCS manifolds with large b_2 and compute the BMT.

8.1 A skew matching

We shall give an example in detail of a TCS manifold obtained via a skew matching of two Picard rank 2 semi Fanos. This example appears in Table 8.1.

Let $Y_+ = \mathbb{P}^1 \times \mathbb{P}^2$. Y_+ is clearly Fano and has label AGV.2.34 [1], row 189 of the table of Section B.2. The Picard lattice of Y_+ with respect to the pullback of the hyperplane class via projections onto its components is

$$N_+ = \begin{pmatrix} 0 & 3 \\ 3 & 2 \end{pmatrix} \tag{8.1}$$

The anticanonical class with respect to this basis is $(2, 3)$. The nef cone is spanned by $(1, 0)$ and $(0, 1)$. Y_+ has a free anticanonical linear system and torsion free cohomology, and so by Construction 7.1.3 we have an associated building block Z_+.

Let Y_- be the blowup of $\mathbb{P}^3$ along a genus 2 degree 8 curve C. Y_- is weak Fano with label CMv4.1.49.0 and appears row 24 of the table of Section B.2. By Proposition 7.7.50, Y_- is a weak Fano provided that C has no 5-secant line, no 9-secant conic, and no 13-secant twisted cubic. The Picard lattice of Y_- with respect to the (pullback of the) hyperplane class, and the exceptional class is

$$N_- = \begin{pmatrix} 4 & 8 \\ 8 & 2 \end{pmatrix} \tag{8.2}$$

The anticanonical class with respect to this basis is $(4, -1)$. The nef cone is spanned by $(4, -1)$ and $(1, 0)$. Y_- has a free anticanonical linear system and torsion free cohomology, and so by Construction 7.1.3 we have an associated building block Z_-.

We can construct a configuration of Z_+ and Z_- with the following lattice.

$$P = \begin{pmatrix} 0 & 3 & -2 & -10 \\ 3 & 2 & 1 & 5 \\ -2 & 1 & 4 & 8 \\ -10 & 5 & 8 & 2 \end{pmatrix} \tag{8.3}$$

The vector $(8, -1)$ is ample on Y_- and under inclusion into P is orthogonal to the image of N_+. Likewise the vector $(4, 1)$ is ample on Y_+ and under inclusion into P is orthogonal to the image of N_-. The signature vector of P is $(2, 2, 0)$ as required. We can compute the lattices $\Lambda_\pm$.

$$\Lambda_+ = \begin{pmatrix} 0 & 3 & 22 \\ 3 & 2 & -11 \\ 22 & -11 & -308 \end{pmatrix}, \quad \Lambda_- = \begin{pmatrix} 4 & 8 & -20 \\ 8 & 2 & -100 \\ -20 & -100 & -180 \end{pmatrix} \tag{8.4}$$

Both are nondegenerate overlattices of the corresponding Picard lattices and each have the expected signature vector $(1, 2, 0)$.

[1] See Section B.1 for the naming convention

For Λ_- we check that $H = (1,0,0)$ is very ample by Corollary 7.7.34 verifying that the necessary sets are empty. $4H - C = (4,-1,0)$ is nef and basepoint free by Proposition 7.7.31. Let V be the cone spanned by $(1,0,0)$ and $(4,-1,0)$ and in which $(8,-1,0)$ belongs to the interior. By Proposition 7.7.49, the deformation family of Y_- is (Λ_-, V)-generic. Likewise, for Λ_+ we use Proposition 7.7.48 to prove that Y_+ is Λ_+-generic.

By Proposition 6.4.4, we have a matching between the building blocks $Z_\pm$, and ultimately there exists a corresponding G_2-manifold. The topology can then be computed via the formula of Section 6.3. Note that in particular, the intersection $N_\pm$ in P is trivial and so the resulting 7-manifold is 2-connected.

8.2 Disconnecting the G_2-moduli space

We use the ξ invariant defined in Section 5.2.5 to prove that the moduli space of G_2-metrics on some spin manifolds is disconnected. The examples here are discussed in the preprint [105].

The spin diffeomorphism class of a 2-connected spin 7-manifold with torsion free cohomology, is determined by the triple $(b_3, \mathrm{gd}(q_1(M)), \mu(M))$ consisting of the third Betti number, the greatest divisor of the spin class, and the generalized Eells-Kuiper invariant. (See Section 5.3.3 or [32, Theorem 1.3].)

For a closed, 2-connected spin 7-manifold M with torsion free cohomology, M and $-M$ will have the same (b_3, m) invariants, where m is the greatest divisor of the spin class. Moreover, in our context where $\bar{m} := \gcd\left(28, \mathrm{Num}\left(\frac{m}{4}\right)\right)$ equals 1 or 2, μ agrees on M and $-M$ (see Section 6.5). Hence, there exists an orientation reversing diffeomorphism $r : M \to M$. A G_2-structure φ is diffeomorphic to $-r^*(\varphi)$, yet $\xi(\varphi) = -(\xi(-r^*\varphi))$. We are interested in G_2-manifolds with torsion free cohomology, for which (b_3, m) agree and ξ differs and not simply by a sign change.

We consider skew configurations with empty intersection involving the Picard rank 2 threefolds tabulated in Section B.2. Before considering genericity conditions there are in excess of $40,000$ configurations (see Section 8.6).

Table 8.1 contains the data in the construction of the two pairs of TCS manifolds for which ξ distinguishes the homotopy class of their G_2-structures. The columns consist of the invariants b_3, m, and ξ followed by the data of a TCS with such invariants. The invariants are not stated if they are shared with the previous line. The data corresponding to a TCS include the quadratic form P of the span of the polarizations, the Fano or semi Fano from which the building block is obtained, the ample vectors $A_\pm$ that are perpendicular to $N_\mp$ in P, and the overlattices $\Lambda_\pm$.

Table 8.1: Disconnected G_2-moduli

b_3	m	ξ	P	$Y_\pm$	$A_\pm$	$\Lambda_\pm$

71	6	0	$\begin{pmatrix} 18 & 0 \\ 0 & 18 \end{pmatrix}$	AGV.1.9	(1)	(18)
				AGV.1.9	(1)	(18)
		12	$\begin{pmatrix} 4 & 11 & 1 & 4 \\ 11 & 26 & 5 & 20 \\ 1 & 5 & 6 & 6 \\ 4 & 20 & 6 & -2 \end{pmatrix}$	CMv4.1.52.0	$(6 \quad -1)$	$\begin{pmatrix} 4 & 11 & -23 \\ 11 & 26 & -115 \\ -23 & -115 & -552 \end{pmatrix}$
				CMv4.1.109.0	$(4 \quad -1)$	$\begin{pmatrix} 6 & 6 & -16 \\ 6 & -2 & -64 \\ -16 & -64 & -272 \end{pmatrix}$
85	24	36	$\begin{pmatrix} 0 & 3 & -2 & -10 \\ 3 & 2 & 1 & 5 \\ -2 & 1 & 4 & 8 \\ -10 & 5 & 8 & 2 \end{pmatrix}$	AGV.2.34	$(4 \quad 1)$	$\begin{pmatrix} 0 & 3 & 22 \\ 3 & 2 & -11 \\ 22 & -11 & -308 \end{pmatrix}$
				CMv4.1.49.0	$(5 \quad -1)$	$\begin{pmatrix} 4 & 8 & -20 \\ 8 & 2 & -100 \\ -20 & -100 & -180 \end{pmatrix}$
		60	$\begin{pmatrix} 4 & 11 & 1 & 5 \\ 11 & 26 & 5 & 25 \\ 1 & 5 & 4 & 11 \\ 5 & 25 & 11 & 26 \end{pmatrix}$	CMv4.1.52.0	$(6 \quad -1)$	$\begin{pmatrix} 4 & 11 & -16 \\ 11 & 26 & -80 \\ -16 & -80 & -272 \end{pmatrix}$
				CMv4.1.52.0	$(5 \quad -1)$	$\begin{pmatrix} 4 & 11 & -16 \\ 11 & 26 & -80 \\ -16 & -80 & -272 \end{pmatrix}$

8.3 An orthogonal matching

We give a description of a TCS manifold with $b_2 = 1$. In particular, where the contribution to $H^2(M)$ is from the N_0 component (in the notation of Section 6.3), rather than $K_\pm/\mathrm{PD}(\Sigma_\pm)$.

Let V_4 be the complete intersection of two quadrics in $\mathbb{P}^5$. V_4 is a Fano threefold belonging to [55, Chapter 12, Table 1, Row 14]. Thus in our notation, we reference the family by AGV.1.14.

Let $Y \to V_4$ be the blowup along genus 2 degree 8 curve. Then Y is a semi Fano belonging to CMv4.1.34.0 (Table B.1, Row 28). The topology of Y is determined in a procedural fashion by the results of Section A.2.7. Let $H, E \in H^2(Y)$ be the basis corresponding to the pullback of the hyperplane class, and the exceptional class of the blowup respectively. Then

$$H^3 = 4 \quad H^2E = 0 \quad HE^2 = -8 \quad E^3 = -18 \tag{8.5}$$

The first Chern class $c_1 = (2, -1)$. Then $a = (-7, 4)$ (see Section 7.5 for the definition of a). Fix the dual basis on $H^4(Y) \cong H^2(Y)^\vee$. By the formula of Lemma A.2.21, $c_2(Y) = (20, 16)$.

The Picard lattice of Y is of the form

$$\begin{pmatrix} 8 & 8 \\ 8 & 2 \end{pmatrix} \tag{8.6}$$

The ample cone is spanned by $H = (1,0)$ and $c_1(Y) = (2,-1)$ as Y is weak Fano and not Fano. In particular, the class $v := (5,-2)$ is ample. The class $u = (3,-2)$ is orthogonal to v. The square of u is -16. The class $w = (-1,1)$ is complementary to u. In the basis u, w the Picard lattice is

$$\begin{pmatrix} -16 & 12 \\ 12 & -6 \end{pmatrix} \tag{8.7}$$

The lattice

$$Q = \begin{pmatrix} -16 & 12 & 12 \\ 12 & -6 & -9 \\ 12 & -9 & -6 \end{pmatrix} \tag{8.8}$$

has signature vector $(2,1,0)$. We can embed two copies of the Picard lattice of Y into Q via the matrices (acting on the right)

$$E_+ = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & 0 \end{pmatrix} \quad , \quad E_- = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 3 \end{pmatrix}, \tag{8.9}$$

Note that both map u to $(1,0,0)$.

$$E_+ \; Q \; E_-^T = \begin{pmatrix} -4 & -10 \\ -10 & -25 \end{pmatrix} \tag{8.10}$$

and v lies in the kernel of this map, whether it acts from the left or right. Let Z be the building block associated to Y via Construction 7.1.3. Thus we have an orthogonal matching of Z with itself.

The topology of Z is computed using the same formulas. Z is the blowup of a curve C that is the complete intersection of two anticanonical divisors on Y. Thus $\deg(N_{C/Y}) = 2\deg(Y)$ and the class of the curve in $H^4(Y)$ is $c_1^2(Y) = (8,14)$. Fix the basis of $H^2(Z)$ to be the pullback of the basis of $H^2(Y)$ extended by $c_1(Z)$. The exceptional class of the blowup with respect to this basis is then $(2,-1,-1)$. $\pi^*D.c_2(Z) = D.(c_2(Y) + c_1^2(Y))$ for in $D \in H^2(Y)$ and $c_1(Z)c_2(Z) = 24$. Thus $c_2(Z) = (28,30,24)$. We require also $(\pi^*u)^2 \in H^4(Z)$. This is determined by $u^2 \in H^4(Y)$ and $(\pi^*u)^2 c_1(Z) = u^2 c_1(Y) = -16$. Thus $(\pi^*u)^2 = (4,24,-16)$.

Let M be the TCS with this configuration, and W its TCS coboundary. Using the results of Section 6.3 we can compute that $\pi_1(M) = 0$, $H^2(M) \cong \mathbb{Z}$ and $b_3(M) = 46$. Let $Z_\pm$ be two copies of Z. We have an injective map $H^k(W) \to H^k(Z_+) \oplus H^k(Z_-)$ for $k = 2,4$. Let $\tilde{x} \in H^2(W)$ correspond to the vector of N_0. Then $\tilde{x} \mapsto x \in H^2(M)$, where x is a generator. Recall $q_1(W) = -c_2(W)$. We have the following.

$$\begin{aligned} c_1(W) &\mapsto (0,0,1) \oplus (0,0,1) & &\in H^2(Z_+) \oplus H^2(Z_-) \\ \tilde{x} &\mapsto (3,-2,0) \oplus (3,-2,0) & &\in H^2(Z_+) \oplus H^2(Z_-) \\ q_1(W) &\mapsto -((28,30,24) \oplus (28,30,24)) & &\in H^4(Z_+) \oplus H^4(Z_-) \\ \tilde{x}^2 &\mapsto (4,24,-16) \oplus (4,24,-16) & &\in H^4(Z_+) \oplus H^4(Z_-) \end{aligned} \tag{8.11}$$

We fix a map $H^4(W,M) \to H^2(Z_+) \oplus H^2(Z_-)$, that is a lift $(H^2(Z_+) \oplus H^2(Z_-))/\langle \mathrm{PD}(\Sigma) \rangle$. The intersection form can the then be described by the matrix $H^2(Z_+) \oplus H^2(Z_-) \to H^4(Z_+) \oplus H^4(Z_-)$,

$$\left(\begin{array}{ccc|ccc} 0 & 0 & 0 & -4 & -10 & 0 \\ 0 & 0 & 0 & -10 & -25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -4 & -10 & 0 & 0 & 0 & 0 \\ -10 & -25 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{8.12}$$

The image $H_0^4(W)$ of $H^4(W,M) \to H^4(W)$ corresponds to the kernel of $H^4(W) \to H^4(M)$. The map $H^4(W) \to H^2(M)$ is onto, so we identify $H^4(M)$ with quotient space $H^4(W)/H_0^4(W)$. In terms of matrices this is as follows. We extend the basis matrix of the rank 2 image of this map to a full basis

$$\left(\begin{array}{cccccc} 2 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 5 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 \end{array}\right) \tag{8.13}$$

With respect to this basis, $q_1(W)$ and $\tilde{x}^2$ have the following span.

$$\left(\begin{array}{cc|cccc} 26 & 26 & -24 & 80 & -24 & 80 \\ 16 & 16 & -16 & 28 & -16 & 28 \end{array}\right) \tag{8.14}$$

By projecting on to the latter 4 coordinates allows us to determine our $\mathfrak{C}$ model. Note that $(-24, 80, -24, 80) = 8(-3, 10, -3, 10) = 8e_1$ where $\gcd(e_1) = 1$. Let $e_2 = (-2, 7, -2, 7)$. Then $(-16, 28, -16, 28) = 4(14e_1 + 19e_2)$. Thus our $\mathfrak{C}$-model is $(8, 4, 19, 14)$. By projecting on to the first 2 coordinates we are in the image of $H^4(W,M) \to H^4(W)$. Observe that by applying the dual change of basis to $H^4(W,M)$ the intersection form has a diagonal matrix with entries $(1, 1, 0, \ldots, 0)$. Thus $q_1^2 = 2(26^2) = 8 \mod 16$, $q_1\tilde{x}^2 = 0 \mod 4$ and $\tilde{x}^4 = 0 \mod 8$.

We compute $L^{sh}(\mathfrak{C}_M)$ for $(8, 4, 19, 14)$. The torsion of the cokernel of the $\mathfrak{C}$-model is then 152-torsion. Recall that $\mu = \frac{1}{8}(q_1^2 - \sigma)$ (here σ is the signature), $\sigma = \frac{1}{2}(q_1\tilde{x}^2 - \tilde{x}^4)$ and $\tau = \tilde{x}^4$. We have well-defined invariants

$$\mu \mod 2, \quad \sigma \mod 2, \quad \tau \mod 2 \tag{8.15}$$

Thus

$$\mu = 1 \mod 2, \quad \sigma = 0 \mod 2, \quad \tau = 0 \mod 2 \tag{8.16}$$

8.4 TCS manifolds with $b_2 = 1$

We construct matchings of the form described in Section 8.3 from the rank 2 semi Fanos tabulated in Section B.2. We also compute orthogonal matchings involving the rank 3 Fano threefolds, where the topological data can be derive with reasonable ease. Our search for configurations was not exhaustive.

The TCS manifolds are tabulated in Section B.3. The invariants defined are described as linear combinations of μ, σ and τ as defined in Section 5.2.3. These are then computed in each case.

8.5 BMT Examples

Corollary 5.3.8 implies that to have any chance of finding a nonformal TCS manifold M then $b_2(M) \geq 4$. Moreover, the cup product on the coboundary $H^2(W) \times H^2(W) \to H^4(W)$ must be nontrivial. Heuristically, the greater the rank of this map, the greater the chance of finding a nonformal example. Thus we consider configurations with large N_0. In particular, the building blocks must have large b_2.

In Section 7.6, we discussed building blocks obtained from Fano threefolds of the form $\mathbb{P}^1 \times X_d$, and those obtained from sections of $\mathbb{P}^2 \times X_d$. They have identical polarizing lattices and ample cones, but different cup products. Thus, an arrangement of polarizing lattices corresponding to a configuration will allow us to obtain three different TCS manifolds (one for each unordered pair).

Let $N_\pm$ be polarizing lattices corresponding to some $\mathbb{P}^1 \times X_d$. If we consider orthogonal configurations we can determine $N_0 \to N_\pm$, by identifying its orthogonal complements $N_0^\perp \subset N_\pm$. As we aspire for N_0 to be of large rank, we look for $N_0^\perp$ to be small.

Let $N_0^\perp = \langle x, y - z_1 \rangle$ in both $N_\pm$. The orthogonal push-out $P = N_- + N_+$ is a lattice with signature vector $(2, 11 - d, 0)$. (By Proposition 7.7.9 there exists a primitive embedding $P \hookrightarrow L$.) Thus it is a valid configuration, and leads to TCS manifolds.

When the threefold Y from which the building block is obtained has $b_2 < 6$ then the configuration described has $\mathrm{rank}(N_0) \leq 3$. The TCS is then intrinsically formal by Corollary 5.3.8. The configurations described have trivial BMT with domains of rank $3, 21, 60, 110, 220$, for $b_2(Y) = 6, 7, 8, 9, 10$ respectively, for each of the three possible pairs.

We have also considered configurations with the building block obtained from $\mathbb{P}^1 \times X_1$ described in Section 7.6.1. In addition, we introduced small 'perturbations' of the configuration, to find other configurations with large N_0. In all examples considered, the BMT was trivial.

8.6 Computing on scale

Our ultimate goal is to use our invariants to find topologically peculiar phenomena amongst manifolds with holonomy G_2. To this end, we wish to compute the invariants on a large pool of examples. This scale motivates the employment of computers. The collection of results in Chapters 6 and 7 were transposed into a series of programs written in `sage` [89]. The program is able to find configurations and compute invariants from topological data of threefolds. The parameters and input data are motivated by what it is we are looking for. We describe some aspects of the procedure here.

The input for TCS are (mostly) building blocks derived from semi Fano threefolds. We begin with a data set covering all the relevant data of all the semi Fano threefolds we care to include. For example, Section B.2 displays some of the data for the 191 semi Fano threefolds of Picard rank 2 that we used. From this we compute the data of the associated building blocks.

Perpendicular matchings require the least understanding of the geometry of the input threefold. The only obstruction to two semi Fano threefold admitting a perpendicular matching is if the direct sum of the Picard lattices cannot be embedded in to the K3 lattice (cf Proposition 7.7.9). If an embedding exists the configuration is essentially unique. For example, if we restrict to ordinary building blocks derived from the tabulated 191 semi Fanos then this obstruction is not encountered. Thus we would obtain $\frac{1}{2} \cdot 191 \cdot 190$ perpendicular configurations.

Non-perpendicular configurations require some knowledge of the whereabouts of the ample cone of building blocks. More precisely, a configuration requires that the image of a member of the ample cone of one building block is orthogonal to the image of the Picard lattice of the other building block. Part of the data of each of the building blocks in our data set is a subcone of the ample cone.

Let us consider a skew configuration case with $N_0 = 0$. Suppose the pair (N, a) consists of a symmetric matrix N representing a nondegenerate quadratic form of signature vector $(1, s)$, together with a vector a which is primitive and has positive square. Denote the inner product with respect to a symmetric matrix N by $(\cdot,\cdot)_N$. Suppose we have two such pairs $(N_\pm, a_\pm)$. We wish to find all lattices W together with embeddings $N_\pm \to W$ such that $N_+ \cap N_- = \{0\}$, and $a_\pm$ is orthogonal to $N_\mp$ in W.

Let T be an integer matrix that transforms N into block diagonal form such that $T_{j1} = [a_j]$ and

$$T^t N T = (a^2) \oplus B \tag{8.17}$$

Where B is a symmetric matrix that represents a quadratic form of signature $(0, s)$. Rationally T is a change of basis, but in general $\mathrm{Det}(T) \neq \pm 1$.

Working with a rational basis, we can assume that W has the following form.

$$W = \begin{pmatrix} a_+^2 & & & \\ & B_+ & & D \\ & & a_-^2 & \\ & D^T & & B_- \end{pmatrix} \tag{8.18}$$

where $D \in \mathbb{Z}^{s_+ \times s_-}$ is to be determined. The constraint on the signature of W implies that for all $(s_+ + s_-)$ length vectors $(u_+, u_-) \neq 0$, that

$$(u_+, u_+)_{B_+} + (u_-, u_-)_{B_-} + 2\sum_{j,k} u_{+,j} D_{jk} u_{-,k} < 0 \tag{8.19}$$

The coefficients of D must then lie in some bounded region. For example, it is necessarily for each D_{jk}, $|D_{jk}| < \frac{1}{2}(|(B_+)_{jj}| + |(B_-)_{kk}|)$, although clearly this is insufficient in general. As $D_{jk} \in \mathbb{Z}$, this leads to a finite and often feasibly computable list of possible candidates for D. The condition that W corresponds to an integral lattice of $N_{\pm}$ gives the additional constraint that

$$\sum_{j,k>1}[T_+^{-t}]_{ij}D_{j-1,k-1}[T_-^{-1}]_{kl} \in \mathbb{Z} \tag{8.20}$$

We exclude any candidate D failing this condition from our list. All remaining candidate D's correspond to a solution W. Solving D is essentially finding integral points inside an ellipsoid. There exists good standard libraries for problems of this type.

In [30], the authors consider skew matchings of building blocks derived from Fano threefolds of Picard rank 2. (Excluding the Fano threefold with anticanonical linear system that is not free.) The above is much simplified in this case since B is of rank 1. They compute that no configurations exist between building blocks if the ratio of the square of a to the absolute discriminant Δ of N is greater than $\frac{8}{5}$. Of the semi Fanos we consider, the largest ratio of Δ/a^2 is achieved by CMv4.1.31.0 (Row 2 of B.1). More specifically we have that

$$N = \begin{pmatrix} 10 & 9 \\ 9 & -2 \end{pmatrix}, \qquad a = (3, -1) \quad , \quad \frac{\Delta}{a^2} = \tfrac{101}{34}. \tag{8.21}$$

Hence if we consider skew matchings with the semi Fanos listed, we consider only vectors a such that $a^2/\Delta < 3$. This is an exhaustive list. With the tabulated semi Fanos using this bound we found $> 40,000$ candidate skew configurations, before considering genericity conditions.

For skew configurations involving building blocks from semi Fanos with $b_2 > 2$, finding an upper bound on a^2 is much more involved. We are far from being able to achieve an exhaustive set of matchings from semi Fanos in many respects: we do not yet have a complete classification of all weak Fano threefolds with Picard rank 2, let alone higher rank; we have not computed all the data for the ones that are classified; and we have, at most, a suboptimal subcone of the ample cones for Fano threefolds with $b_2 > 2$. With all these things in place, the amount of computations would likely become formidable.

In [30], the authors also consider matchings in which N_0 is nontrivial. As the Picard lattices of the semi Fanos considered have rank 2, the configurations are necessarily orthogonal. They find exactly 19 configurations of this type. Again, the Picard rank 2 case is much simpler than the general case, but the idea generalizes.

For each building block, fix an ample vector a. Consider a negative definite $N_0 < N$ orthogonal to a. Definite quadratic forms of small rank are classified in terms of feasibly computable invariants. Thus we can begin to form lists of isomorphism classes of N_0's, and attempt to find configurations whenever we have a pair of isomorphic N_0's. This includes of course self matchings. In this case, the configuration lattice W (8.18) has an additional diagonal term N_0. The only off diagonal block is still D and solving D is done as above.

In each case, one then considers the corresponding genericity results for the lattices $\Lambda_{\pm} < W$. The genericity results of Section 7.7, are framed in the existence or nonexistence of classes of certain properties. This again amounts to finding integral points in, or perhaps on, an ellipsoid. If $\Lambda_{\pm}$ meet the required genericity results, then we proceed to calculate the relevant invariants of the corresponding TCS.

Apart from a massive bookkeeping exercise there are some other problems to be considered. There are various points where we could over count. For example, if D drops rank then one will get the same solution multiple times. Also, if the semi Fano has symmetry such as $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, then we will find solutions which are essentially identical. In cases of using Fanos of the type $\mathbb{P}^1 \times X_d$, and other building blocks of large b_2 described, there are large symmetry groups in the Picard lattice. Trying to quotient by these symmetry groups, while attempting an exhaustive search for configurations becomes a bit of headache as well as a computational challenge on a standard computer. The problem of finding integral points in ellipsoid also becomes computationally problematic in high dimensional vector spaces. Smart choices of initial bases can speed things up in borderline cases, while poor choices of initial bases will see the system fall over.

With more and improved input data one should find many more examples.

Appendix A

Constructions and topology

A.1 Some 8-manifolds

We compute the invariants of certain 8-manifolds.

A.1.1 $\mathbb{CP}^4$

Let $X = \mathbb{CP}^4$, and let $a \in H^2(X)$ be the oriented generator. The cohomology ring $H^\bullet(X) = \langle a \rangle / (a^5)$, and $\langle a^4, [X] \rangle = 1$. The total Chern class is $c(X) = 1 + 5a + 10a^2 + 10a^3 + 5a^4$. Thus Chern numbers are

$$c_{(1111)}(X) = 625 \quad c_{(112)}(X) = 250 \quad c_{(22)}(X) = 100 \quad c_{(13)}(X) = 50 \quad c_{(4)}(X) = 5 \tag{A.1}$$

The Euler characteristic $\chi(X) = 5$, and the signature $\sigma(X) = 1$. As c_1 is not divisible by 2, $\mathbb{CP}^4$ is not spin. As $p_1 = 2c_2 - c_1^2$, and $p_2 = 2c_4 - 2c_1c_3 + c_2^2$,

$$p_{(11)}(X) = 25 \quad p_{(2)}(X) = 10 \tag{A.2}$$

A.1.2 Quadric hypersurface

Let $Q \to \mathbb{P}^5$ be a smooth quadric. The cohomology below the middle dimension is determined by the Lefschetz hyperplane theorem. Let $x \in H^\bullet(Q)$ be the hyperplane class, then $\langle x^4, [X] \rangle = 2$. The total Chern class is given by

$$\begin{aligned} c(Q) &= \pi^*(c(\mathbb{P}^5)/(1+2H)) \\ &= (1 + 6x + 15x^2 + 20x^3 + 15x^4)(1 - 2x + 4x^2 - 8x^3 + 16x^4) \\ &= 1 + 4x + 7x^2 + 6x^3 + 3x^4 \end{aligned} \tag{A.3}$$

The Chern numbers are

$$c_{(1111)}(X) = 256 \quad c_{(112)}(X) = 224 \quad c_{(22)}(X) = 98 \quad c_{(13)}(X) = 48 \quad c_{(4)}(X) = 6 \tag{A.4}$$

The Euler characteristic $\chi(X) = 6$. As $c_1(X)$ is divisible by 2, Q is spin, and $q_1(Q) = \frac{1}{2}c_1^2(Q) - c_2(Q) = x^2$, As $b_4(Q) = \chi(Q) - (2 + b_2(Q)) = 2$ and $q_1^2(Q) - \sigma(Q) = 2 - \sigma(Q) = 0$ mod 8, $\sigma(Q) = 2$.

A.1.3 Bott towers

Bott towers are complex manifolds obtained via an iterative construction. Let $X_0 = *$ be a point. We obtain X_{j+1} from X_j as follows. Let $L_j \to X_j$ be a complex vector bundle with Chern class $a_j := c_1(L_j) \in H^2(X_j)$. Define $X_{j+1} = \mathbb{P}(\underline{\mathbb{C}} \oplus L_j)$, where $\underline{C} \to X_j$ is the trivial complex line bundle. Note that $L_0 = \mathbb{C}$, and so $X_1 = \mathbb{CP}^1$.

Let $y_j \in H^2(X_j)$ correspond to the antitautological bundle of X_j. The cohomology of X_j is generated by y_j as an $H^\bullet(X_{j-1})$ algebra, subject to the single relation that $y_j^2 + a_j y_j = 0$. We do not distinguish in notation between a class in $H^\bullet(X_{j-1})$ and its pullback to $H^\bullet(X_j)$. Using this we can iteratively construct the cohomology ring as a $\mathbb{Z}$ algebra. We note that $H^\bullet(X_1) = \mathbb{Z} \oplus y_1\mathbb{Z}$, and $a_1 = 0$.

The Betti numbers of X_j are $b_{2k}(X_j) = \binom{j}{k}$, and 0 otherwise. We can choose a basis $y^\alpha := \prod_{k=1}^j y_k^{\alpha_k}$ where $\alpha \in \mathbb{Z}_2^j$. The oriented generator of $H^{2j}(X_j)$ is then $y_1 \dots y_j$. The total Chern class is

$$c(X_j) = \prod_{k=0}^{j} (1 + 2y_k + a_k) \tag{A.5}$$

By specifying the coefficients $a_j = \sum_{k=1}^{j-1} a_{jk} y_k$.

For complex fourfolds we have the following Chern classes (index shifted)

$$\begin{aligned}
c_1 &= 2y_3 + (a_{32} + 2)y_2 + (a_{21} + a_{31} + 2)y_1 + (a_{10} + a_{20} + a_{30} + 2)y_0 \\
c_2 &= 4y_3y_2 + (2a_{21} + 4)y_3y_1 + (-a_{21}a_{32} + 2a_{31} + 2a_{32} + 4)y_2y_1 + (2a_{10} + 2a_{20} + 4)y_3y_0 \\
&\quad + (a_{10}a_{32} - a_{20}a_{32} + 2a_{10} + 2a_{30} + 2a_{32} + 4)y_2y_0 \\
&\quad + (-a_{10}a_{21}a_{31} - a_{10}a_{21} + a_{21}a_{30} - a_{10}a_{31} + a_{20}a_{31} + 2a_{20} + 2a_{21} + 2a_{30} + 2a_{31} + 4)y_1y_0 \\
c_3 &= 8y_3y_2y_1 + (4a_{10} + 8)y_3y_2y_0 + (-2a_{10}a_{21} + 4a_{20} + 4a_{21} + 8)y_3y_1y_0 + \\
&\quad (a_{10}a_{21}a_{32} - 2a_{10}a_{31} - 2a_{20}a_{32} - 2a_{21}a_{32} + 4a_{30} + 4a_{31} + 4a_{32} + 8)y_2y_1y_0 \\
c_4 &= 16y_3y_2y_1y_0
\end{aligned} \tag{A.6}$$

One can then express the Chern numbers in terms of coefficients, and so too, the Pontrjagin classes.

The Pontrjagin classes

$$\begin{aligned}
p_1 &= (a_{21}a_{32}^2 - 2a_{31}a_{32})y_2y_1 + (a_{20}a_{32}^2 - 2a_{30}a_{32})y_2y_0 \\
&\quad + (a_{10}a_{21}^2 + a_{10}a_{31}^2 - 2a_{20}a_{21} - 2a_{30}a_{31})y_1y_0 \\
p_2 &= 0
\end{aligned} \tag{A.7}$$

The Chern numbers are

$$\begin{aligned}
c_{(4)} &= 16 \\
c_{(13)} &= 64 \\
c_{(22)} &= 96 \\
c_{(112)} &= -2a_{10}a_{21}^2a_{32}^2 + 4a_{10}a_{21}a_{31}a_{32} + 2a_{10}a_{21}a_{32}^2 + 4a_{20}a_{21}a_{32}^2 - 4a_{10}a_{21}^2 \\
&\quad - 4a_{10}a_{31}^2 - 4a_{21}a_{30}a_{32} - 4a_{10}a_{31}a_{32} - 4a_{20}a_{31}a_{32} - 4a_{20}a_{32}^2 - 4a_{21}a_{32}^2 \\
&\quad + 8a_{20}a_{21} + 8a_{30}a_{31} + 8a_{30}a_{32} + 8a_{31}a_{32} + 192 \\
c_{(1111)} &= -8a_{10}a_{21}^2a_{32}^2 + 16a_{10}a_{21}a_{31}a_{32} + 8a_{10}a_{21}a_{32}^2 + 16a_{20}a_{21}a_{32}^2 - 16a_{10}a_{21}^2 \\
&\quad - 16a_{10}a_{31}^2 - 16a_{21}a_{30}a_{32} - 16a_{10}a_{31}a_{32} - 16a_{20}a_{31}a_{32} - 16a_{20}a_{32}^2 \\
&\quad - 16a_{21}a_{32}^2 + 32a_{20}a_{21} + 32a_{30}a_{31} + 32a_{30}a_{32} + 32a_{31}a_{32} + 384
\end{aligned} \tag{A.8}$$

The Pontrjagin numbers are

$$p_{(2)} = 0 \quad p_{(11)} = 0 \tag{A.9}$$

Remark A.1.1. A quick check confirms that this holds in the trivial case $a_i = 0$. That is, when the Bott tower is $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

A.1.4 Products of projective space

As remarked $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, is a Bott tower.

$$c_{(1111)} = 384 \quad c_{(112)} = 192 \quad c_{(22)} = 96 \quad c_{(13)} = 64 \quad c_{(4)} = 16 \tag{A.10}$$

The Even Betti numbers are $b_{2j} = \binom{4}{j}$ while the odd Betti numbers are 0. Thus the Euler Characteristic $\chi = 16$. The signature $\sigma = 0$.

Consider $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$. The total Chern class

$$c(X) = (1 + 2a)(1 + 2b)(1 + 3c + 3c^2) \tag{A.11}$$

and $\langle abc^2, [X] \rangle = 1$.

$$\begin{aligned}
c_1 &= 2a + 2b + 3c & c_2 &= 4ab + 6ac + 6bc + 3c^2 \\
c_3 &= 12abc + 6ac^2 + 6bc^2 & c_4 &= 12abc^2
\end{aligned} \tag{A.12}$$

$$c_{(1111)} = 432 \quad c_{(112)} = 204 \quad c_{(22)} = 96 \quad c_{(13)} = 60 \quad c_{(4)} = 12 \tag{A.13}$$

The Betti numbers are $b_1 = 0$, $b_2 = 3$, $b_3 = 0$, and $b_4 = 4$. Thus the Euler Characteristic $\chi = 12$. The signature $\sigma = 0$.

Consider $X = \mathbb{P}^2 \times \mathbb{P}^2$. The total Chern class

$$c(X) = (1 + 3a + 3a^2)(1 + 3b + 3b^2) \tag{A.14}$$

and $\langle a^2b^2, [X]\rangle = 1$.

$$\begin{aligned} c_1 &= 3a+3b & c_2 &= 3a^2+9ab+3b^2 \\ c_3 &= 9a^2b+9ab^2 & c_4 &= 9a^2b^2 \end{aligned} \tag{A.15}$$

$$c_{(1111)} = 486 \quad c_{(112)} = 216 \quad c_{(22)} = 99 \quad c_{(13)} = 54 \quad c_{(4)} = 9 \tag{A.16}$$

The Betti numbers are $b_1 = 0$, $b_2 = 2$, $b_3 = 0$, and $b_4 = 3$. Thus the Euler Characteristic $\chi = 9$. The signature $\sigma = 1$.

Consider $X = \mathbb{P}^1 \times \mathbb{P}^3$. The total Chern class

$$c(X) = (1+2a)(1+4b+6b^2+4b^3) \tag{A.17}$$

and $\langle ab^3, [X]\rangle = 1$.

$$\begin{aligned} c_1 &= 2a+4b & c_2 &= 8ab+6b^2 \\ c_3 &= 12ab^2+4b^3 & c_4 &= 8ab^3 \end{aligned} \tag{A.18}$$

$$c_{(1111)} = 512 \quad c_{(112)} = 224 \quad c_{(22)} = 96 \quad c_{(13)} = 56 \quad c_{(4)} = 8 \tag{A.19}$$

The Betti numbers are $b_1 = 0$, $b_2 = 2$, $b_3 = 0$, and $b_4 = 2$. Thus the Euler Characteristic $\chi = 8$. The signature $\sigma = 0$.

A.1.5 $\mathbb{HP}^2$

Let $X = \mathbb{HP}^2$, and let $a \in H^4(X)$ be the oriented generator. Then $\langle a^2, [X]\rangle = 1$. The total Pontrjagin class is $p(X) = 1 + 2a + 7a^2$ (See [99, Corollary 2.3]). The Pontrjagin numbers are

$$p_{(11)}(X) = 4 \quad p_{(2)}(X) = 7 \tag{A.20}$$

The Euler characteristic $\chi(X) = 3$, and the signature $\sigma(X) = 1$.

A.1.6 X_{Bott}

The E_8 lattice is a rank 8 free $\mathbb{Z}$-module, with positive definite even unimodular form. In fact, this characterizes the E_8 lattice.

Proposition A.1.2. *Let Q be an even quadratic form. For $k > 1$ there exists a $(4k)$-manifold W with boundary such that*
(i) W is $(2k-1)$-connected, and ∂W is $(2k-2)$-connected.
(ii) The intersection form of W is isomorphic to Q.
Moreover, ∂W is a homotopy sphere if and only if Q is unimodular.

See Browder [15, Theorem V.2.1 & V.2.7]. Let W_{E_8} be an 8-manifold with boundary described by Proposition A.1.2, where Q is the E_8-lattice. Then the boundary ∂W_{E_8} is a homotopy sphere. As the group of smooth structures on the homotopy 7-sphere is isomorphic to $\mathbb{Z}_{28}$, the boundary connect sum of 28 copies of W_{E_8} has boundary diffeomorphic to S^7. We let X_{Bott} be the closed 8-manifold formed by glueing an 8-ball, along the S^7-boundary. Thus X_{Bott} is 3-connected, with intersection form $(E_8)^{\oplus 28}$. It has vanishing spin class.

A.2 Constructions in complex geometry

This section collects together some standard results that facilitate the computations of topological data of complex manifolds obtained via a variety of constructions. Much of this can be found in standard texts on the matter, such as [46]. Some results appear in the introductory sections of [3] and [34].

We recall the following standard definitions. Let X be a topological space and $E \to X$ a vector bundle. $\chi(E)$ denoted the Euler characteristic of E, and when X is differentiable $\chi(X) := \chi(TX)$. For complex E, $c_k(E)$ denotes the k^{th} Chern class. $c(E) = 1 + \sum_{k\geq 1} c_k$ is the *total Chern class* (or *complete Chern class*), $c_t(E) = 1 + \sum_{k\geq 1} c_k(E)t^k$ is the *Chern polynomial*, $ch(E) = \mathrm{rk}(E) + \sum_{k\geq 1} ch_k(E)$ is the complete Chern character, where $ch_k(E)$ is the k^{th} *Chern character*.

Lemma A.2.1. *Let X be a complex variety of dimension n. Then $\langle c_n(X), [X]\rangle = \chi(X)$.*

Lemma A.2.2. *Let $E_i \to X$ be complex vector bundles over a manifold X. If $E = E_1 \oplus E_2$, then $c(E) = c(E_1)c(E_2)$ and $ch(E) = ch(E_1) + ch(E_2)$. If $E = E_1 \otimes E_2$, then $ch(E) = ch(E_1)ch(E_2)$.*

The Chern character for a vector bundle E relate to the Chern classes via

$$ch(E) = \sum_k i, \quad ch_k(E) = \sum_j \exp(x_j), \qquad c(E) = \sum_k, \quad c_k(E) = \prod_j (1 + x_j). \tag{A.21}$$

$ch_0(E)$ is the rank of the vector bundle. The lowest order Chern characters are

$$\begin{aligned} ch_1 &= c_1, & ch_3 &= \tfrac{1}{6}\left(c_1^3 - 3c_1c_2 + 3c_3\right), \\ ch_2 &= \tfrac{1}{2}\left(c_1^2 - 2c_2\right), & ch_4 &= \tfrac{1}{24}\left(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4\right). \end{aligned} \tag{A.22}$$

It may also be helpful to use the relations defined iteratively

$$\begin{aligned} c_1 &= ch_1, & c_3 &= \tfrac{1}{3}\left(c_2p_1 - c_1p_2 + p_3\right), \\ c_2 &= \tfrac{1}{2}\left(c_1p_1 - p_2\right), & c_4 &= \tfrac{1}{4}\left(c_3p_1 - c_2p_2 + c_1p_3 - p_4\right) \end{aligned} \tag{A.23}$$

where $p_k = k! ch_k$.

Example A.2.3. $E = T_{\mathbb{P}^n}$, then

$$c(E) = (1 + x)^{n+1}, \qquad ch(E) = (n + 1)e^x - 1. \tag{A.24}$$

Both follow from considering the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \longrightarrow T_{\mathbb{P}^n} \longrightarrow 0$$

Proposition A.2.4 (Splitting Principal)**.** *Let $E \to X$ be a complex vector bundle of rank k over a paracompact space X. There exists a space $Y :=$ Fl(E) (ie the* associated flag bundle *of E) and a map $p : Y \to X$ such that $p^*H^\bullet(X) \to H^\bullet(Y)$ is injective, and the pullback $p^*E \to Y$ splits as a direct sum of line $p^*E = L_1 \oplus \cdots \oplus L_k$.*

A.2.1 Homogeneous spaces

Definition A.2.5. A *homogeneous space* X is a space with a transitive action by a Lie group G. As a space it is diffeomorphic to some coset space G/H.

Homogeneous spaces form the atoms of many constructions. Some prototypical examples include:

(i) (complex) Projective space $\mathbb{P}^n \cong \mathsf{U}(n+1)/(\mathsf{U}(n) \times \mathsf{U}(1))$.

(ii) (complex) Grassmanians $\mathrm{Gr}(n,k) \cong \mathsf{U}(n)/(\mathsf{U}(n-k) \times \mathsf{U}(k))$.

(iii) spheres $S^n = \mathsf{SO}(n+1)/\mathsf{SO}(n)$.

Lemma A.2.6. *Let $h = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$. Then the total Chern Class*

$$c(\mathbb{P}^n) = (1+h)^{n+1}. \tag{A.25}$$

Equivalently $c_k(\mathbb{P}^n) = \binom{n+1}{k} h^k$. The Chern Character

$$ch(\mathbb{P}^n) = n + \sum_{k\geq 1} \frac{n+1}{k!} h^k. \tag{A.26}$$

Definition A.2.7. Let V be a complex vector bundle of rank n, let $k \in \mathbb{N}$ be such that $k < n$. The *Grassmanian* $\mathrm{Gr}(k,V)$ is the space of all k dimensional subspaces of V. It is naturally a smooth complex manifold of dimension $k(n-k)$. We may denote the Grassmanian by $\mathrm{Gr}(n,k)$ when $V = \mathbb{C}^n$.

Different authors have used different conventions in the definition of Grassmanians. Here k denotes the dimension of the subspace as defined by [46, Chapter 1, Section 5] and [82, Chapter 8], whereas k denotes the codimension of the subspace in [13, Section 23]. Although the two are of course isomorphic, there is a discrepancy in the behaviour of the *tautological* bundle.

The *tautological bundle* P of $\mathrm{Gr}(k,V)$ is the bundle that over each point in $\mathrm{Gr}(k,V)$ is the $(n-k)$ plane it represents. The *product bundle* is simply $\mathrm{Gr}(k,V) \times V$. The *quotient bundle* Q is defined as the cokernel of the inclusion $T \to \mathrm{Gr}(k,V) \times V$, ie we have an exact sequence

$$0 \to P \to \mathrm{Gr}(k,V) \times V \to Q \to 0 \tag{A.27}$$

which is called the *tautological sequence.*

Proposition A.2.8. *Let P be the tautological bundle over $\mathrm{Gr}(n,k)$ with total Chern class $1 + x_1 + \cdots + x_k$. Let Q be the quotient bundle over $\mathrm{Gr}(n,k)$ with total Chern class $1 + p_1 + \cdots + p_{n-k}$. Define the formal power series*

$$\textstyle\sum_{j\geq 0} p_j(x_1,\ldots,x_k)t^j = (1 + \sum_1^r x_i t^i) \tag{A.28}$$

Then

$$H^\bullet(\mathrm{Gr}(n,k)) = \langle x_i \colon 1 \leq i \leq k \rangle / (p_{n-k+1},\ldots,p_n) \tag{A.29}$$

In particular, the cohomology of $\mathrm{Gr}(n,k)$ is torsion free.

(See [82, Equation (8.12)] and surrounding discussion.)

A.2.2 Subvarieties

Proposition A.2.9 (Lefschetz Hyperplane Theorem)**.** *Let $X \subset \mathbb{P}^N$ be a projective variety such that $\dim_{\mathbb{C}}(X) = n+1$, and Y a hyperplane section of X such that $X \setminus Y$ is smooth. Then*

(i) $\iota_* : \pi_k(Y) \to \pi_k(X)$ *is an isomorphism for $k < n$ and surjective for $k = n$.*
(ii) $\iota_* : H_k(Y) \to H_k(X)$ *is an isomorphism for $k < n$ and surjective for $k = n$.*
(iii) $\iota^* : H^k(X) \to H^k(X)$ *is an isomorphism for $k < n$ and injective for $k = n$.*

Proposition A.2.10 ([26, Proposition 3.10])**.** *Let Y be a nonsingular projective n-fold. Suppose $f :\to \mathbb{P}^N$ is a semi-small morphism and let $X \in |f^*\mathcal{O}_{\mathbb{P}^N}(1)|$. Then the restriction $H^m(Y) \to H^m(X)$ is an isomorphic for $m < n-1$, and primitive injective for $m = n-1$.*

This is a corollary of Goresky MacPherson's generalization of the Lefschetz Hyperplane theorem [43, pg 151, Theorem 1.1].

Lemma A.2.11. *Let D be an ample divisor of X with associated line bundle L. Then*

$$c(D) = (c(X)/(1 + c_1(L))|_D \tag{A.30}$$

Let X be a projective variety with $\dim_{\mathbb{C}}(X) = n+1$ and $L \to X$ an ample line bundle. Let $Y \in |L|$. Then $H^n(Y) \simeq T_{n-1} \oplus \mathbb{Z}^{b_n}$ (where $\simeq$ denotes noncanonical isomorphism), where $T_{n-1} = TH_{n-1}(Y)$ by the Universal Coefficient Theorem. We can calculate $TH_{n-1}(Y)$ via the Lefschetz Hyperplane Theorem. We can calculate the top Chern class of Y by Lemmas A.2.11, and A.2.1. An ample class on X restricts to an ample class on Y.

Lemma A.2.12. *Let Y be a smooth projective variety with vector bundle $E \to Y$ of rank r. Let s be a smooth section of E and $X = s^{-1}(0)$, its zero locus. Assume that X is smooth and of dimension* $\mathrm{Y} - \mathrm{rk}(E)$*. Then*

$$ch_k(X) = (ch_k(Y) - ch_k(E))|_X \tag{A.31}$$

In the particular case that X is a smooth complete intersection of divisors D_i, then

$$ch_k(X) = \left(ch_k(Y) - \frac{1}{k!}\sum D_i^k\right)|_X \tag{A.32}$$

A.2.3 Ramified covers

Proposition A.2.13. *Let $f : X \to \mathbb{P}^n$ be a finite mapping of degree d. Then*

$$f^* : H^k(\mathbb{P}^n, \mathbb{C}) \to H^k(X, \mathbb{C}) \tag{A.33}$$

is an isomorphism for $k \leq n + 1 - d$.

(Lazarsfeld A Barth-Type Theorem for Branched Coverings of Projective Space Theorem 1)

Lemma A.2.14. *Let X, Y be smooth projective varieties. Let $f : X^n \to Y^n$ be a finite map of degree d branched over divisor $S \subset Y$ and ramified in $R \subset X$. Suppose that R and S are smooth. Then*

$$\chi(X) = d\chi(Y) - (d-1)\chi(S). \tag{A.34}$$

Proof. By the assumption that R is smooth, $\chi(X \setminus R) = \chi(X) - \chi(R)$. The analogous holds for Y and S. $f : R \to S$ is an isomorphism. In particular, $\chi(R) = \chi(S)$. $f : (X \setminus R) \to (Y \setminus S)$ is an unramified d cover. Thus $\chi(X \setminus R) = d\chi(Y \setminus S)$. □

Proposition A.2.15. *Let X, Y, R, S be as above. Suppose $f : X \to Y$ is a double cover. Then $c_1(X) = f^*(c_1(Y) - \frac{1}{2}S)$, and $c_2(X) = f^*(c_2(Y) + \frac{1}{2}S^2 - \frac{1}{2}c_1(Y) \cdot S)$. Equivalently $ch_2(X) = f^*(ch_2(Y) - \frac{3}{8}S^2)$.*

Proof. Follows from the exact sequence

$$0 \to f^*\Omega_Y \to \Omega_X \to \mathcal{O}(-R)|_R \to 0 \tag{A.35}$$

□

A.2.4 Products

Proposition A.2.16 (Kunneth Theorem)**.** *Let X and Y be topological spaces and K a field*

$$H^r(X \times Y; K) \cong \bigoplus_{p+q=r} H^p(X; K) \otimes H^q(Y; K) \tag{A.36}$$

If either X or Y are torsion free then the result also holds with $K = \mathbb{Z}$.

Lemma A.2.17. *The ample cone on a product variety is the product cone of the ample cones of its components.*

A.2.5 Projective bundles

Proposition A.2.18. *Let $\mathcal{E} \to X$ be a complex vector bundle or rank r over smooth manifold X. Let $\xi = c_1(\mathcal{O}_\pi(1))$ denote the class of the antitautological bundle over $\mathbb{P}(\mathcal{E})$. Then $H^\bullet(\mathbb{P}(\mathcal{E}$ is generated as an $H^\bullet(X)$-algebra by ξ, subject to the single relation:*

$$\xi^r + c_1(\mathcal{E})\xi^{r+1} + \cdots + c_r(\mathcal{E}) \tag{A.37}$$

Let $E \to X$ be a complex vector bundle over a smooth algebraic variety X. Let $\pi : \mathbb{P}(E) \to X$, $T_{\mathbb{P}(E)/X} := \ker(\pi_* : T_{\mathbb{P}(E)} \to \pi^*T_X)$, and $\mathcal{O}_{\mathbb{P}(E)}(-1)$ be the tautological bundle. We have two short exact sequences of sheaves

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(E)} \longrightarrow \pi^*E \otimes \mathcal{O}_{\mathbb{P}(E)}(1) \longrightarrow T_{\mathbb{P}(E)/X} \longrightarrow 0$$

$$0 \longrightarrow T_{\mathbb{P}(E)/X} \longrightarrow T_{\mathbb{P}(E)} \longrightarrow \pi^*T_X \longrightarrow 0$$

Thus the Chern characters $ch(T_{\mathbb{P}(E)/X}) = \pi^*ch(E)e^{\xi} - 1$, while $ch(T_{\mathbb{P}(E)}) = ch(T_{\mathbb{P}(E)/X}) + \pi^*ch(T_X)$, where $\xi = c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$.

Lemma A.2.19. *Let $\mathcal{E} \to X$ be a complex vector bundle of rank r over smooth manifold X. Let $\pi : \mathbb{P}(\mathcal{E}) \to X$, and $\xi = c_1(\mathcal{O}_\pi(1))$. Then*

$$\begin{aligned} c_1(\mathbb{P}(\mathcal{E})) &= \pi^*(c_1(X) + c_1(\mathcal{E})) + r\xi \\ ch_2(\mathbb{P}(\mathcal{E})) &= \pi^*(ch_2(X) + ch_2(\mathcal{E})) - \pi^*c_1(\mathcal{E}) \cdot \xi + \frac{r}{2}\xi^2 \end{aligned} \tag{A.38}$$

Equivalently

$$c_2(\mathbb{P}(\mathcal{E})) = \pi^*(c_2(X) + c_1(X)c_1(\mathcal{E}) + c_2(\mathcal{E})) + \pi^*(rc_1(X) + (r-1)c_1(\mathcal{E})) \cdot \xi + \tfrac{1}{2}r(r-1)\xi^2 \tag{A.39}$$

In particular, when $r = 2$

$$c_2(\mathbb{P}(\mathcal{E})) = \pi^*(c_2(X) + c_1(X)c_1(\mathcal{E})) + 2\pi^*c_1(X) \cdot \xi \tag{A.40}$$

A.2.6 Projective bundles over curves

In this more specific setting we can say slightly more. Let C be a smooth curve. Let $E \to C$ be a rank r vector bundle.

All vector bundles over curves split as the direct sum of line bundles $E = L_1 \oplus \cdots \oplus L_n$. Let $a_i = \int_C c_i(L)$ and write $L_i = \mathcal{O}(a_i)$.

For any line bundle $L \to C$, $\mathbb{P}(E) \cong \mathbb{P}(E \otimes L)$. Thus without loss of generality, we assume that for $X : \mathbb{P}(E)$, $E = \bigoplus_1^n \mathcal{O}(a_i)$, $0 = a_1 \leq a_2 \leq a_n$. Let ξ be the antitautological class and f be the class of a fibre $\mathbb{P}(E) \to C$. The nef cone $\mathrm{Nef}(X) = \langle \xi, f \rangle$.

A.2.7 Blowups

The following results can be found in, say [46, page 605] or [55, page 39].

Lemma A.2.20. *Let X' be a smooth n manifold and $Y \subset X$ a smooth submanifold. Let $X \to X'$ be the blowup centered along a smooth Y, and E the exceptional class of the blowup. As additive groups*

$$H^\bullet(X) \cong \pi^*H^\bullet(X') \oplus H^\bullet(E)/\pi^\bullet(Y) \tag{A.41}$$

Moreover, $c_1(X) = \pi^*c_1(X') - (n-k-1)E$.

Lemma A.2.21. *Let $X \to X'$ be the blowup of the algebraic threefold X' centered on smooth curve $C \to X'$. Let E be the exceptional class and $\vartheta_C \in H^4(X')$ the class of the curve C. The product structure (for even degrees) is given by $H^*(X')$ and*

$$E^3 = -\deg(N_{C/X'}) \quad E^2\pi^*D = -\vartheta_C D \quad E\pi^*(F) = 0 \tag{A.42}$$

for $D \in H^2(X')$, $F \in H^4(X)$. Moreover $c_2(X) = \pi^*(c_2(X') + \vartheta_C) - \pi^*c_1(X') \cdot E$.

Note that in the notation of the above

$$\deg(N_{C/X'}) = 2g(C) - 2 - K_{X'} \cdot C \tag{A.43}$$

Lemma A.2.22. *Let $X \to X'$ be the blowup of a threefold X' at a smooth point. Then $c_2(X) = \pi^*(X')$.*

A.3 Aspects of algebraic geometry

A.3.1 Notation and foundational results

Let X be a variety which, unless stated otherwise, is assumed to be complete irreducible normal projective over $\mathbb{C}$. Let n be the dimension of X. Let $Z_k(X)$ denote the *k-cycles* on X, ie the free abelian group generated by all irreducible reduced subvarieties of dimension k on X. Thus $Z_1(X)$ is the free abelian group generated by curves; $Z_{n-1}(X)$ is the group of *Weil divisors* on X. Let $\mathrm{Div}(X)$ denote the set of Cartier divisors on X. Recall that there is a natural injection $\mathrm{Div}(X) \to Z_{n-1}(X)$.

The *Picard group* $\mathrm{Pic}(X)$ of X is the group of Cartier divisors up to linear equivalence. There is a natural one-to-one correspondence with invertible sheaves on X, which in turn is in one-to-one correspondence with line bundles on X.

For a Cartier divisor $D \in \mathrm{Div}(X)$ and a smooth curve $C \subset X$, their *intersection* $D.C := \deg(\mathcal{O}_X(D)|_C)$. For an invertible sheaf $L \in \mathrm{Pic}(X)$, $L.C := \deg(L|_C)$. Two 1-cycles $\alpha, \beta \in Z_1(X)$ are *numerically equivalent* if for all $L \in \mathrm{Pic}(X)$, $L.\alpha = L.\beta$. By duality, we define *numerical equivalence* on $\mathrm{Pic}(X)$, and denote both equivalence relations by $\equiv$. Set $N_1(X) := (Z_1(X)/\equiv) \otimes \mathbb{R}$ and $N^1(X) := (\mathrm{Pic}(X)/\equiv) \otimes \mathbb{R}$. The pairing $\mathrm{Pic}(X) \times Z_1(X) \to \mathbb{Z}$ induces a perfect pairing $N^1(X) \times N_1(X) \to \mathbb{Z}$.

Let $NE(X) \subset N_1(X)$ be the least convex cone containing all effective 1-cycles. The closure $\overline{NE}(X)$ of $NE(X)$ in the standard $\mathbb{R}$-topology is the *Mori cone* or *cone of curves* of X. Let $NS(X)$ denote the *Néron Severi* group defined as the quotient of $\mathrm{Pic}(X)$ by the connected component of the identity $\mathrm{Pic}^0(X)$. The *Picard rank* or *Picard number* of X is $\rho(X) := \mathrm{rank}(NS(X)) = \dim_{\mathbb{R}}(N_1(X))$. This is finite, by the Néron Severi Theorem.

Let $L \in \mathrm{Pic}(X)$ be a line bundle on projective variety X. L is *very ample* if $H^0(X, L)$ determines an embedding of X into some $\mathbb{P}^N$. L is *ample* if there exists an $m > 0$ such that $L^{\otimes m}$ is very ample. L is *semi-ample* if there exists an $m > 0$ such that the linear system $|L^{\otimes m}|$ is basepoint free. L is *nef* if for every curve $C \subset Y$, $\deg(L|_C) \geq 0$. L is *big* if the there exists an $m > 0$ such that $H^0(X, L^{\otimes m})$ determines a map that is birational onto its image. The first Chern class $c_1 : \mathrm{Pic}(L) \to H^2(X; \mathbb{Z})$.

Let $V < H^0(X, L)$ be a nonzero subspace. $|V| := \mathbb{P}(V)$ is a *linear series* or *linear system*. In particular, $|L| := \mathbb{P}(H^0(X, L))$ is a *complete linear system*. The *evaluation morphism* $\mathrm{ev}_V : V \otimes \mathcal{O}_X \to L$. The *base ideal* of $|V|$ is the image of $V \otimes L^\vee \to \mathcal{O}_X$ determined be ev_V. The *base locus* $Bs(|V|)$ is the closed subset determined by the base ideal. $|V|$ is *free* or *basepoint free* if the base locus is empty. A divisor, or line bundle is *free* or *basepoint free* if the associated complete linear system is free. In this case, we may also use the completely synonymous terms *generated by global sections* or simply *globally generated.*

Let $C \subset Y$ be a subvariety of Y and let $L \to Y$ be a basepoint free line bundle on Y. C is *scheme theoretically cut out* by L, or simply *cut out*, if $\mathcal{I}_C \otimes L$ is globally generated. Equivalently, there exists a linear system $V < H^0(Y, L)$ such that $C = \mathrm{Bs}(|V|)$ as a scheme.

An effective divisor D on X determines a short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_D(D) \to 0 \tag{A.44}$$

As D is a (Cartier) divisor $\mathcal{O}_X(D)$ is locally free. Thus this is the short exact sequence of the ideal sheaf twisted by $\mathcal{O}_X(D)$. When D is smooth $\mathcal{O}_D(D)$ is (isomorphic to) the normal bundle of $D \subset X$.

Let $Y \subset X$ be a subvariety. We have short exact sequence of sheaves

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0 \tag{A.45}$$

For any invertible sheaf $\mathcal{F}$ on X we can tensor the sequence and remain exact

$$0 \to \mathcal{I}_Y \otimes \mathcal{F} \to \mathcal{F} \to \mathcal{O}_Y \otimes \mathcal{F} \to 0 \tag{A.46}$$

where $\mathcal{O}_Y \otimes \mathcal{F}$ is equivalent to $\mathcal{F}|_Y$. The global sections of $\mathcal{F}$ vanishing on Y is then $H^0(\mathcal{I}_Y \otimes \mathcal{F})$. By the long exact sequence in cohomology we get

$$h^0(\mathcal{I}_Y \otimes \mathcal{F}) \geq h^0(\mathcal{F}) - h^0(\mathcal{F}|_Y) \tag{A.47}$$

Lemma A.3.1. *Let X be a nonsingular projective variety; let L be an invertible sheaf over X. Let $V < H^0(L)$ be a subspace, ie a linear system. A generic divisor $D \in |V|$ is smooth away from the base locus* $\mathrm{Bs}(V)$.

This result is known as Bertini's Theorem. See [47, Corollary III.10.9, and Remark III.10.9.2]. We have the following results of a similar nature.

Lemma A.3.2 ([52, Lemma 2.3])**.** *Let L be a linear system of dimension n on X. Assume that the base locus B is smooth of codimension $n+1$, (ie B is the complete intersection of basis divisors of L), then a general member of L is smooth.*

Lemma A.3.3 ([52, Lemma 2.4])**.** *Let W be a smooth divisor on X and L a line bundle. Let $D \subset W$ be a smooth member of the linear system $|L_W|$. Assume that $H^1(X, L(-W)) = 0$ and the linear system $|L(-W)|$ is basepoint free. Then D has a smooth extension ie there exists a divisor $\tilde{D} \subset |L|$ on X such that $D = \tilde{D} \cap W$.*

This can be extended to multiple divisors.

Lemma A.3.4. *Let D be a Cartier divisor on a normal variety X. If D is smooth, then* $\mathrm{Sing}(X) \cap D = \varnothing$*, and X has only isolated singularities.*

(See say [22])

A.3.2 Algebraic curves

The following terms and definitions can be found in standard texts, such as [4] and [47]. Unless otherwise stated, an *algebraic curve* will be assumed to be reduced and complete. If C is a smooth curve, we will assume that it is also irreducible. Recall that a projective curve is *canonical* if it is embedded via its canonical linear system. A smooth curve of genus $g > 1$ is *hyperelliptic* if there exists a finite morphism $C \to \mathbb{P}^1$ of degree 2.

Let D be a divisor on C, and $V < |D|$ a linear system. We say that V is a g^r_d if $\deg(D) = d$ and $\dim(V) = r + 1$. Recall Riemann-Roch for smooth curves: for a divisor D on a curve C of genus g

$$h^0(D) - h^0(K_C - D) = \deg(D) - g + 1 \tag{A.48}$$

Thus if $\deg(D) < 0$, then $h^0(D) = 0$; and if $\deg(D) > 2g-2$, then $h^0(D) = \deg(D)-g+1$. The divisor $K_C - D$ is the *residual* of D. By Serre duality $h^0(K_C - D) = h^1(D)$. An effective divisor D is *special* if $h^0(K_C-D) > 0$. A special divisor is *exceptional* if $h^0(D) > \max(0, \deg(D) - g)$.

Definition A.3.5. Let C be a smooth curve. The *gonality* of C is the minimum degree of a morphism $C \to \mathbb{P}^1$.

Definition A.3.6. Let C be a smooth curve of genus $g \geq 2$. For $g \geq 4$, the *Clifford index* of C is

$$\mathrm{Cliff}(C) := \min\{\deg(A) - 2(h^0(A) - 1) \colon \} \tag{A.49}$$

considered over all line bundles A on C such that $h^0(A) \geq 2$ and $h^1(A) \geq 2$. For $2 \leq g < 4$, we have ad hoc definitions. If $g = 2$ or if $g = 3$ and C is hyperelliptic, then $\mathrm{Cliff}(C) := 0$. If $g = 3$ and C is nonhyperelliptic, then $\mathrm{Cliff}(C) := 1$.

A divisor A on C such that $h^0(A) \geq 2$ and $h^1(A) \geq 2$ *computes the Clifford index* of C if

$$\mathrm{Cliff}(C) = \deg(A) - 2(h^0(A) - 1) \tag{A.50}$$

The gonality and Clifford index both provide some measure of the behaviour of linear systems on curves, and in doing provide insight into its geometry. This is manifest in the classical result of Clifford.

Proposition A.3.7 (Clifford's Theorem)**.** *Let C be a smooth curve of genus $g \geq 2$. Then* $\mathrm{Cliff}(C) \geq 0$ *with equality if and only if C is hyperelliptic.* $\mathrm{Cliff}(C) = 1$ *if and only if C trigonal or a plane quintic.*

The Picard variety $\mathrm{Pic}(C)$ of C consists of the complete linear systems on C. $\mathrm{Pic}^d(C) \subset \mathrm{Pic}(C)$ is the subvariety of complete linear systems of degree d.

Definition A.3.8. The *Brill-Noether loci of C* are defined as the subvarieties

$$W^r_d(C) := \{|D| \in \mathrm{Pic}^d(C) \colon \dim(|D|) > r\} \tag{A.51}$$

There is a natural filtration $W^0_d(C) \supset W^1_d(C) \supset \ldots$ of $\mathrm{Pic}^d(C)$. Elements of $W^r_d \backslash W^{r+1}_d$ are precisely the complete g^r_d's on C. We have the *residuation morphism* $W^r_d(C) \to W^{g-d+r-1}_{2g-2-d}(C)$ by $|D| \mapsto |K_C - D|$. See [4, Chapter V].

Let g, d, r be integers with $d \geq 1$, $g, r \geq 0$. The *Brill Noether number* is

$$\rho(g, d, r) := g - (r+1)(g - d + r) \tag{A.52}$$

The Brill Noether number of a divisor D is $\rho(D) = \rho(g, \deg(D), h^0(D) - 1)$. Note that $\rho(D) = g - h^0(D)h^0(K_C - D) = g - h^0(D)h^1(D)$.

Proposition A.3.9 (Existence Theorem). *Let C be a smooth curve of genus g. Let d, r be integers with $d \geq 1$, $r \geq 0$. If $\rho = \rho(g, d, r) \geq 0$ then $W^r_d(C)$ is nonempty. Moreover if $r \geq d - g$ then the dimension of (each component of) $W^r_d(C)$ is at least ρ.*

(See [4, Chapter V, (1.1)])

By Proposition A.3.9, a general smooth curve C of genus g possesses a line bundle L with $h^0(L) \geq h^0$ and $h^1(L) \geq h^1$ if and only if $h^0h^1 < g$. Thus, for a curve C of genus $g \geq 2$ $\mathrm{Cliff}(C) \leq \mathrm{floor}\left(\frac{g-1}{2}\right)$. A curve is *Clifford general* if $\mathrm{Cliff}(C) = \mathrm{floor}\left(\frac{g-1}{2}\right)$, and is *Clifford exceptional* otherwise.

Clifford special divisors on a canonical curve on a projective K3 appear in the K3 lattice by Theorem 7.7.27. There is a close relationship between special divisors on curves, and high secant varieties appearing in their projective embeddings.

Proposition A.3.10 ([24, Theorem A]). *Let $C \to \mathbb{P}^k$ be a smooth projective curve of $d \geq 4k - 7$ where $(k \geq 2)$. Assuming that C is linearly normal (ie $\mathcal{O}_{\mathbb{P}^k}(1) \to \mathcal{O}_C \otimes \mathcal{O}(1)$ is onto) then C has a $(2k-3)$-secant $(k-2)$-plane.*

Recall that an s-secant variety V of a projective curve C meets C in s points (with multiplicity 1). A general canonical curve of genus $g \geq 4$ has a $(2g-2)$-secant $(g-2)$-plane. Curves that fail to have a $(2g-2)$-secant $(g-2)$-plane have an infinite number of $(2g-3)$-secant $(g-2)$-planes. These curves are Clifford exceptional.

Proposition A.3.11 ([24, Theorem B]). *Let C be a smooth canonical curve of genus $g \geq 2$. If C is a k-gonal curve $(k \geq 3)$ of genus g, any g^r_d on C of degree d, such that $k - 3 \geq d \geq 2g - 2 - (k-3)$, satisfies $2r \leq d - (k-3)$.*

A.3.3 Mori Theory

Our treatment is very terse, and is really only fixing the notation. [55] [70] and [63] all give a clear exposition in their introductory sections of most if not all of the following.

Lemma A.3.12. *Let $L \in \mathrm{Pic}(X)$ be a nef line bundle on projective variety X. Then L is big if and only if $\int_X c_1(L)^{\dim(X)} > 0$.*

Proposition A.3.13 (Nakai-Moishezon-Kleiman Criterion). *Let L be a line bundle on a projective scheme X. Then L is ample if and only if for any irreducible subvariety of positive dimension $V \subset X$, $\int_V D^{\dim(V)} > 0$.*

(See [70, Theorem 1.4.9].)

In particular, if D and D' are numerically equivalent, then D is ample if and only if D' is ample. The *ample cone* $\mathrm{Amp}(X) \subset N^1(X)$ is the convex cone generated by all ample classes on X. The *nef cone* $\mathrm{Nef}(X) \subset N^1(X)$ is the convex cone generated by all nef classes on X.

Proposition A.3.14. *Let X be a projective variety. Then $\mathrm{Nef}(X) = \overline{\mathrm{Amp}}(X)$, the topological closure; $\mathrm{Amp}(X) = \mathrm{int}(\mathrm{Amp}(X))$, the (topological) interior.*

Furthermore, with respect to the pairing, $\mathrm{Nef}(X)$ and $\overline{NE}(X)$ are dual.

(See [70, Theorem 1.4.23, Proposition 1.4.28])

Definition A.3.15. A half line R in $\overline{NE}(X)$ is an *extremal ray* provided that:
(i) For an effective 1-cycle C such that $R = \mathbb{R}_+[C]$, $(-K_X \cdot C) > 0$; and
(ii) For $z_1, z_2 \in NE(X)$, if $z_1 + z_2 \in NE(X)$ then $z_1, z_2 \in NE(X)$.
A rational curve l on X is *extremal* if $\mathbb{R}_+[l]$ is an extremal ray, and $(-K_X \cdot l) \leq \dim(X)+1$.

For a divisor D on X, we denote by $\overline{NE}(X)_{D\geq 0}$, the subset of classes α such that $D.\alpha \geq 0$.

Theorem A.3.16 (Cone Theorem)**.** *Let X be a nonsingular projective variety. Then*
(i) There are countably many rational curves $C_i \subset X$ such that $0 < (C_i.K_X) \leq \dim(X+1)$ and

$$\overline{NE}(X) = \overline{NE}(X)_{K_X \geq 0} + \textstyle\sum_i \mathbb{R}_{\geq 0}[C_i]. \tag{A.53}$$

(ii) For any $\varepsilon > 0$ and ample divisor H

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X+\varepsilon H)\geq 0} + \textstyle\sum_{i=1}^N \mathbb{R}_{\geq 0}[C_i]. \tag{A.54}$$

(See [63, Theorem 1.24])

Corollary A.3.17. *Let X be a Fano threefold. There are finitely many extremal rational curves $l_1, \cdots l_r$ on X such that $NE(X) = \sum_1^r \mathbb{R}_+[l_i]$. In particular, the cone is polyhedral and closed.*

Definition A.3.18. Let V be a vector space over $\mathbb{R}$ or $\mathbb{Q}$. A subset $N \subset V$ is a *cone* if $0 \in N$ and it is closed under multiplication by positive scalars.

A subcone $M \subset N$ is *extremal* or is an *extremal face* if $u, v \in N$ such that $u + v \in M$ imply $u, v \in M$. If M is one dimensional, it is called an *extremal ray.*

Definition A.3.19. Let X be a normal projective variety and $F \subset \overline{NE}(X)$ an extremal face. A morphism $\mathrm{cont}_F : X \to Z$ is a *contraction* of F if the following conditions hold:
(i) $\mathrm{cont}_F(C) =$ point for irreducible curve C if and only if $[C] \in F$.
(ii) $(\mathrm{cont}_F)_* \mathcal{O}_X = \mathcal{O}_Z$.

The existence of a contraction is not guaranteed. However, if it exists then it is unique.

Theorem A.3.20 (Mori contractions)**.** *Let X be a nonsingular projective threefold over $\mathbb{C}$, and let R by an K_X-negative extremal face of $\overline{NE}(X)$. Then* cont_R *exist and is one of the following types:*

E : *(Exceptional)* $\dim Y = 3$, cont_R *is birational and there are 5 types of local behaviour near the contracted surface.*

- *E1 :* cont_R *is the blowup of a smooth curve in a smooth threefold Y. E a ruled surface.*
- *E2 :* cont_R *is the blowup of a smooth point on a smooth threefold Y. $E \simeq \mathbb{P}^2$ and $\mathcal{O}_E(E) \simeq \mathcal{O}_{\mathbb{P}^2}(-1)$*
- *E3 :* cont_R *is the blowup of an ordinary double point on Y ie a point where Y is locally analytically given by $x^2+y^2+z^2+w^2=0$. $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{O}_E(E) \simeq \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1)$.*
- *E4 :* cont_R *is the blowup of a point on Y where Y is locally analytically given by $x^2+y^2+z^2+w^3=0$. E is a quadric cone in $\mathbb{P}^3$ and $\mathcal{O}_E(E) \simeq \mathcal{O}_E \otimes \mathcal{O}_{\mathbb{P}^3}(-1)$.*
- *E5 :* cont_R *contracts a smooth $E \cong \mathbb{P}^2$ with normal bundle $\mathcal{O}_E(E) \cong \mathcal{O}(-2)$ to a point of multiplicity 4 on Y which is locally analytically the quotient of $\mathbb{C}^3$ by the involution $(x,y,z) \mapsto (-x,-y,-z)$.*

C : *(Conic bundle)* $\dim(Y) = 2$ *and* cont_R *is a fibration whose (general) fibres are smooth plane conics. $Y \cong \mathbb{P}^2$.*

D : *(del Pezzo fibration)* $\dim(Y) = 1$ *and general fibres of* cont_R *are smooth del Pezzo surfaces. $Y \cong \mathbb{P}^1$.*

F : *(Fano variety)* $\dim(Y) = 0$, *$-K_X$ is ample and hence X is a Fano variety.*

(See [55, Theorem 1.4.3].) The contractions of K_X extremal faces are called *Mori contractions.* In cases E1-E5, D, and the following sequence is exact

$$0 \to \mathrm{Pic}(Y) \xrightarrow{\mathrm{cont}_R} \mathrm{Pic}(X) \xrightarrow{\cdot l} \mathbb{Z} \to 0. \tag{A.55}$$

In the case of C, this sequence is exact provided that either the fibration has a singular fibre or that Y is rational.

Corollary A.3.21. *Let $f : X \to Y$ as in Theorem A.3.20. Then $h^i(X, \mathcal{O}(X)) = h^i(Y, \mathcal{O}(Y))$ for all $i > 0$.*

Appendix B

Tables

B.1 Naming convention of threefolds

Given the amount of different (families of) threefolds at play, we will fix the following naming convention. There are significant drawbacks to various ways of doing this. We opt for labels that have little geometric inspiration but are somewhat logical and directly reference a source of their actual definition. Even then this is a bit of headache, and I apologize now before going any further. We shall cite only four sources, and each has its own caveats.

For Fano threefolds we shall follow (for the most part) the ordering of [55, Chapter 12]. A label for a Fano is given as

$$\text{AGV}.b_2.\text{row}$$

For $b_2 = 1$, rows are ordered ascending firstly in Fano rank and secondly in degree. Thus the first 10 are Fano rank $r = 1$, the next 5 are del Pezzo threefolds ($r = 2$). The quadric Q is then denoted by AGV.1.16, and $\mathbb{P}^3$ by AGV.1.17. For $b_2 \geq 2$, rows are ordered as they appear in [80] ascending in degree, although there are many cases where this does not distinguish multiple classes. For $b_2 \leq 5$, b_2 is the same as the table number. Note that for $b_2 = 4$, there is a correction [81]. This case has degree 26, and so we denote it by AGV.4.2, and all subsquent rows with $b_2 = 4$ are incremented by 1 as to their appearence in the references. For $b_2 > 5$, these are appended to the bottom of table 5.

For weak Fanos with divisorial anticanonical morphism we cite Jahnke, Peternell, Radloff's paper [58]. A label for a divisorial weak Fano is given as

$$\text{JPR05.table.row}$$

The tables are found in appendix A. If 'table' is 3, then the threefold has φ-type E1; if 'table' is 4, then the threefold has φ-type E2. The 'table' indices are one less than the corresponding subsections of the corresponding paper.

For weak Fanos with small anticanonical morphism we cite Jahnke, Peternell, Radloff's paper [57] or Cutrone Marshburn [33]. A label for these weak Fanos is given as

$$\text{paper.table.row.flop}$$

The paper is either JPR07 or CMv4. The names were selected as follows. JPR07 is dated by its preprint, and to distinguish it from JPR05. CMv4 refers to the fact that the arxiv version v4 has tables that are more complete than those of the published version. The table numbers follow the subsection number in which the table appears. The rows are as they appear in the relevant paper. Each row is a flop and corresponds to two threefolds. Thus the ‘flop’ is 0 for a threefold that appears on the left; the ‘flop’ is one for a threefold that appears on the right.

B.2 Threefolds with $b_2 = 2$

We tabulate the deformation invariants of threefolds that are semi Fano, including Fano. For those that are strictly semi Fano we have only included cases where the extremal ray contraction φ is of type $E1$ or $E2$. The naming convention is explained in Section B.1. The column headers are discussed in Section 7.5. We have listed unique values: if two classes have the same invariants, then we include only one.

All cases are torsion free so, in particular, $TH^3(Y)$ is omitted from the table. Note that for simply connected almost complex 6-manifolds Y:

(i) b_3 is even;

(ii) for $x, y \in H^2(Y)$, $xy(x + y + c_2) = 0 \mod 2$;

(iii) for $x \in H^2(Y)$, $x(c_1^2 + c_2) = x^3 \mod 3$;

(See [104, Theorem 3].) A quick check ascertains that in all cases these conditions are met.

Table B.1: Topological data of some Fano and weak Fano threefolds

#	c_1^3	a^2c_1	a^3	ac_2	r	b_3	cot	x	φ	Ref
1	2	−250	5532	−246	1	0	2	1	E1	CMv4.1.8.0
2	2	−202	4044	−222	1	0	2	1	E1	CMv4.1.31.0
3	2	−170	3092	−202	1	4	2	1	E1	CMv4.1.7.0
4	2	−154	2596	−194	1	4	2	1	E1	CMv4.1.20.0
5	2	−138	2284	−182	1	4	2	1	E1	CMv4.1.44.0
6	2	−130	2084	−178	1	4	2	1	E1	CMv4.1.30.0
7	2	−122	1876	−170	1	8	2	1	E1	CMv4.1.38.0
8	2	−106	1516	−158	1	10	2	1	E1	CMv4.1.5.0
9	2	−98	1316	−154	1	8	2	1	E1	CMv4.1.13.0
10	2	−90	1116	−150	1	8	2	1	E1	CMv4.1.19.0
11	2	−82	1044	−138	1	12	2	1	E1	CMv4.1.50.0
12	2	−74	884	−130	1	16	2	1	E1	CMv4.1.46.0
13	2	−74	892	−134	1	10	2	1	E1	CMv4.1.29.0
14	2	−66	732	−126	1	12	2	1	E1	CMv4.1.37.0
15	2	−58	572	−118	1	16	2	1	E1	CMv4.1.42.0

#	c_1^3	a^2c_1	a^3	ac_2	r	b_3	cot	x	φ	Ref
16	2	−58	612	−114	1	20	2	1	E1	CMv4.1.3.0
17	2	−50	460	−110	1	16	2	1	E1	CMv4.1.11.0
18	2	−50	490	−110	1	2	1	1	E1	CMv4.1.15.0
19	2	−40	355	−98	1	4	1	1	E1	CMv4.1.35.0
20	2	−34	268	−86	1	28	2	1	E1	CMv4.1.52.0
21	2	−34	276	−90	1	6	1	1	E1	CMv4.1.6.0
22	2	−32	248	−88	1	6	1	1	E1	CMv4.1.14.0
23	2	−28	192	−84	1	6	1	1	E1	CMv4.1.23.0
24	2	−28	209	−82	1	4	1	1	E1	CMv4.1.49.0
25	2	−26	156	−78	1	28	2	1	E1	CMv4.1.48.0
26	2	−26	186	−78	1	10	1	1	E1	CMv4.1.45.0
27	2	−26	188	−70	1	40	2	1	E1	CMv4.1.1.0
28	2	−24	164	−76	1	8	1	1	E1	CMv4.1.34.0
29	2	−22	141	−72	1	12	1	1	E1	CMv4.1.40.0
30	2	−20	124	−68	1	14	1	1	E1	CMv4.1.4.0
31	2	−18	52	−74	1	20	2	1	E1	CMv4.1.41.0
32	2	−18	102	−66	1	12	1	1	E1	CMv4.1.12.0
33	2	−16	80	−64	1	10	1	1	E1	CMv4.1.18.0
34	2	−14	73	−56	1	20	1	1	E1	CMv4.1.51.0
35	2	−12	56	−52	1	22	1	1	E1	JPR05.3.13
36	2	−12	57	−54	1	14	1	1	E1	CMv4.1.33.0
37	2	−10	40	−50	1	16	1	1	E1	JPR05.3.7
38	2	−10	44	−46	1	28	1	1	E1	JPR05.3.1
39	2	−8	28	−44	1	22	1	1	E1	JPR05.3.2
40	4	−324	5256	−180	1	2	4	3	E1	CMv4.1.65.1
41	4	−324	5688	−180	1	2	4	1	E1	CMv4.1.65.0
42	4	−228	3224	−148	1	6	4	1	E1	CMv4.1.70.1
43	4	−228	3368	−148	1	6	4	3	E1	CMv4.1.70.0
44	4	−196	2296	−140	1	6	4	1	E1	CMv4.1.64.1
45	4	−196	2632	−140	1	6	4	3	E1	CMv4.1.64.0
46	4	−132	1416	−108	1	14	4	3	E1	CMv4.1.74.1
47	4	−132	1464	−108	1	14	4	1	E1	CMv4.1.74.0
48	4	−108	1072	−104	1	0	2	1	E1	CMv4.1.57.0
49	4	−100	680	−100	1	12	4	3	E1	CMv4.1.63.1
50	4	−100	856	−92	1	18	4	1	E1	JPR07.4.17.1
51	4	−100	920	−100	1	12	4	1	E1	CMv4.1.63.0
52	4	−76	656	−88	1	0	2	1	E1	CMv4.1.71.0
53	4	−68	568	−68	1	28	4	1	E1	JPR07.7.11.1

#	c_1^3	a^2c_1	a^3	ac_2	r	b_3	cot	x	φ	Ref
54	4	−60	440	−76	1	6	2	1	E1	CMv4.1.68.0
55	4	−44	272	−64	1	10	2	1	E1	CMv4.1.55.0
56	4	−36	168	−60	1	8	2	1	E1	JPR07.4.15.1
57	4	−20	80	−40	1	22	2	1	E1	CMv4.1.76.0
58	4	−18	72	−42	1	4	1	1	E1	CMv4.1.56.0
59	4	−12	41	−34	1	6	1	1	E1	CMv4.1.75.0
60	4	−10	30	−30	1	12	1	1	E1	JPR05.3.14
61	4	−8	20	−28	1	10	1	1	E1	JPR05.3.8
62	4	−6	9	−24	1	14	1	1	E1	CMv4.1.69.0
63	4	−6	14	−22	1	20	1	1	E1	JPR05.3.3
64	4	−6	15	−24	1	14	1	1	E2	CMv4.2.1.1
65	4	−4	8	20	1	44	4	3		AGV.2.1
66	6	−366	4980	−138	1	4	6	5	E1	CMv4.1.81.1
67	6	−366	5172	−138	1	4	6	1	E1	CMv4.1.81.0
68	6	−222	2364	−102	1	10	6	1	E1	JPR07.7.9.1
69	6	−150	1308	−78	1	16	6	5	E1	JPR07.4.14.1
70	6	−96	552	−72	1	2	3	2	E1	CMv4.1.86.1
71	6	−96	744	−72	1	2	3	1	E1	CMv4.1.86.0
72	6	−62	356	−58	1	0	2	1	E1	CMv4.1.79.0
73	6	−60	285	−54	1	8	3	1	E1	CMv4.1.89.1
74	6	−60	363	−54	1	8	3	2	E1	CMv4.1.89.0
75	6	−42	174	−42	1	14	3	2	E1	CMv4.1.87.0
76	6	−42	201	−48	1	10	3	2	E1	CMv4.2.2.0
77	6	−42	204	−42	1	14	3	1	E1	CMv4.1.87.1
78	6	−42	231	−48	1	10	3	1	E2	CMv4.2.2.1
79	6	−38	188	−46	1	0	2	1	E1	CMv4.1.90.0
80	6	−30	124	−38	1	8	2	1	E1	CMv4.1.88.0
81	6	−22	68	−34	1	8	2	1	E1	CMv4.1.84.0
82	6	−14	41	−28	1	0	1	1	E1	CMv4.1.83.0
83	6	−14	44	−22	1	20	2	1	E1	CMv4.1.77.0
84	6	−8	16	−20	1	6	1	1	E1	JPR05.3.4
85	6	−6	6	−18	1	6	1	1	E1	JPR07.4.13.1
86	6	−6	12	6	1	40	3	2		AGV.2.2
87	6	−2	1	−8	1	24	1	1	E1	JPR05.3.20
88	6	−2	2	−10	1	20	1	1	E1	JPR05.3.5
89	8	−392	1904	−112	1	2	8	1	E1	CMv4.1.98.1
90	8	−392	5264	−112	1	2	8	7	E1	CMv4.1.98.0
91	8	−328	2736	−96	1	6	8	5	E1	CMv4.1.94.0

#	c_1^3	a^2c_1	a^3	ac_2	r	b_3	cot	x	φ	Ref
92	8	−328	3408	−96	1	6	8	3	E1	CMv4.1.94.1
93	8	−200	2096	−64	1	14	8	5	E1	JPR07.4.12.1
94	8	−136	624	−48	1	18	8	1	E1	CMv4.4.3.0
95	8	−104	592	−56	1	4	4	1	E1	CMv4.1.97.1
96	8	−104	688	−56	1	4	4	3	E1	CMv4.1.97.0
97	8	−72	336	−48	1	6	4	3	E1	JPR07.4.11.1
98	8	−32	112	−32	1	2	2	1	E1	CMv4.1.96.0
99	8	−16	40	−20	1	10	2	1	E1	CMv4.1.92.0
100	8	−16	48	−24	1	6	2	1	E2	CMv4.5.1.0
101	8	−10	18	−18	1	0	1	1	E1	CMv4.1.93.0
102	8	−8	16	16	1	22	4	3		AGV.2.3
103	8	−4	5	−10	1	10	1	1	E1	JPR05.3.21
104	8	−4	6	−12	2	20	1	1	E2	JPR05.4.6
105	8	−2	0	−6	1	16	1	1	E1	JPR05.3.15
106	8	−2	1	−8	1	12	1	1	E1	JPR05.3.6
107	10	−690	8060	−118	1	0	10	3	E1	CMv4.1.101.1
108	10	−690	9940	−122	1	0	10	7	E1	CMv4.1.101.0
109	10	−410	4220	−94	1	4	10	9	E1	CMv4.1.103.1
110	10	−410	4780	−86	1	4	10	1	E1	CMv4.1.103.0
111	10	−290	3340	−62	1	10	10	7	E1	JPR07.7.5.1
112	10	−210	380	−46	1	12	10	1	E1	CMv4.4.5.0
113	10	−210	1380	−66	1	10	10	1	E1	CMv4.4.4.0
114	10	−90	460	−38	1	6	5	3	E1	JPR07.4.9.1
115	10	−90	460	−38	1	20	10	3	E1	JPR07.4.10.1
116	10	−90	540	18	1	20	10	7		AGV.2.4
117	10	−90	585	−48	1	4	5	3	E2	JPR07.3.2.1
118	10	−60	205	−26	1	12	5	1	E1	CMv4.3.4.0
119	10	−18	36	−18	1	4	2	1	E1	CMv4.1.100.0
120	10	−4	4	−8	1	6	1	1	E1	JPR05.3.16
121	10	−2	0	−6	1	8	1	1	E1	JPR05.3.9
122	12	−588	5208	−84	1	2	12	11	E1	CMv4.1.105.1
123	12	−588	6888	−84	1	2	12	1	E1	CMv4.1.105.0
124	12	−300	840	−60	1	6	12	1	E1	CMv4.4.6.0
125	12	−84	192	−24	1	8	6	1	E1	CMv4.3.5.0
126	12	−44	136	−20	1	6	4	1	E1	CMv4.1.106.0
127	12	−44	184	−20	1	6	4	3	E1	CMv4.1.106.1
128	12	−30	78	−18	1	4	3	2	E1	JPR07.7.4.1
129	12	−20	32	−16	1	0	2	1	E1	CMv4.1.104.0

#	c_1^3	a^2c_1	a^3	ac_2	r	b_3	cot	x	φ	Ref
130	12	-12	3	-6	1	14	3	1	E1	JPR07.4.8.1
131	12	-12	24	12	1	12	4	3		AGV.2.5
132	12	-4	0	0	1	18	2	1		AGV.2.6
133	12	-2	1	4	1	6	1	1	E1	JPR05.3.10
134	14	-742	6636	-78	1	0	14	9	E1	JPR07.7.3.1
135	14	-518	3948	-54	1	4	14	3	E1	CMv4.1.108.0
136	14	-518	7028	-58	1	4	14	11	E1	CMv4.1.108.1
137	14	-406	1204	46	1	4	14	13	E1	CMv4.4.7.0
138	14	-350	1988	-34	1	8	14	5	E1	JPR07.4.6.1
139	14	-154	903	-36	1	0	7	2	E1	CMv4.2.3.0
140	14	-154	1155	-48	1	0	7	5	E2	CMv4.2.3.1
141	14	-112	364	-32	1	4	7	1	E1	CMv4.3.6.0
142	14	-56	119	-22	1	10	7	2	E1	JPR07.4.7.1
143	14	-56	224	8	1	10	7	5		AGV.2.7
144	14	-28	126	-12	1	18	7	3		AGV.2.8
145	14	-4	3	-6	1	0	1	1	E1	CMv4.1.107.0
146	16	-400	96	-24	1	6	16	5	E1	JPR07.4.5.1
147	16	-272	224	8	1	10	16	9		AGV.2.9
148	16	-144	96	-24	1	2	8	1	E1	CMv4.3.7.0
149	16	-144	672	-24	1	2	8	3	E1	JPR07.4.4.1
150	16	-48	128	-16	1	0	4	1	E1	CMv4.1.109.0
151	16	-48	128	-16	1	0	4	3	E1	CMv4.1.109.1
152	16	-16	32	8	1	6	4	3		AGV.2.10
153	16	-12	24	-12	2	10	2	1	E2	JPR05.4.7
154	18	-450	1332	-42	1	4	18	5	E1	JPR07.4.2.1
155	18	-234	1260	-30	1	10	18	7		AGV.2.11
156	18	-126	369	-12	1	4	9	4	E1	JPR07.7.2.1
157	18	-90	180	-18	1	0	6	1	E1	CMv4.1.110.0
158	18	-90	468	-18	1	0	6	5	E1	CMv4.1.110.1
159	18	-18	18	-6	1	2	3	1	E1	JPR07.4.3.1
160	18	-2	4	-2	1	12	2	1	E1	JPR05.3.22
161	20	-30	20	2	1	4	5	3		AGV.2.13
162	20	-20	40	4	1	2	4	3		AGV.2.14
163	20	-4	0	0	1	6	2	1	E1	JPR05.3.23
164	22	-814	4708	-26	1	0	22	9	E1	JPR07.7.1.1
165	22	-198	143	-16	1	0	11	3	E1	JPR07.4.1.1
166	22	-110	88	-14	1	4	11	4		AGV.2.16
167	22	-66	429	-12	1	8	11	5		AGV.2.15

#	c_1^3	a^2c_1	a^3	ac_2	r	b_3	cot	x	φ	Ref
168	22	−6	4	−2	1	0	2	1	E1	JPR05.3.24
169	24	−600	1680	0	1	2	24	13		AGV.2.17
170	24	−24	60	−12	2	4	3	2	E2	JPR07.3.1.1
171	24	−6	6	−6	1	4	3	1		AGV.2.18
172	26	−754	1404	18	1	0	26	17		AGV.2.20
173	26	−442	2444	−22	1	4	26	11		AGV.2.19
174	26	−2	4	−2	1	4	2	1	E1	JPR05.3.17
175	28	−4	0	0	1	0	2	1		AGV.2.21
176	30	−120	1020	−12	1	2	15	7		AGV.2.23
177	30	−70	180	−18	1	0	10	3		AGV.2.24
178	30	−20	5	2	1	0	5	3		AGV.2.22
179	32	−40	120	−12	2	0	4	3	E2	JPR07.6.1.1
180	32	−32	64	−8	1	2	8	3		AGV.2.25
181	34	−714	4012	−14	1	0	34	15		AGV.2.26
182	38	−646	3420	−30	1	0	38	13		AGV.2.27
183	40	−360	9840	−48	1	2	40	13		AGV.2.28
184	40	−10	10	−2	1	0	5	2		AGV.2.29
185	46	−598	828	−6	1	0	46	17		AGV.2.31
186	46	−138	713	−16	1	0	23	7		AGV.2.30
187	48	−4	0	0	2	0	2	1		AGV.2.32
188	54	−54	108	−6	1	0	18	5		AGV.2.33
189	54	−54	108	−6	1	0	18	11		AGV.2.34
190	56	−28	63	−6	2	0	7	3		AGV.2.35
191	62	−310	7812	−42	1	0	62	37		AGV.2.36

B.3 TCS with $b_2 = 1$

The following table displays the invariants of Section 5.2.3 on TCS manifolds with $b_2 = 1$ obtained via orthogonal matchings of Picard rank 2 threefolds tabulated in Section B.2 as well as Fanos with Picard rank 3. An example of such a matching is given in Section 8.3, and the general approach is described in Section 8.4.

The entries are ordered first by $\mathfrak{C}$-model and then by b_3. All manifolds have torsion free cohomology. The $\mathfrak{C}$-model is presented as a 4-tuple. Below this are the invariants defined for this case, described as linear combinations of μ, σ and τ as discussed in Section

5.2.3. For example

$$\begin{aligned} &(12,2,1,0)\\ \sigma+\mu\ \ &(2)\\ \tau\ \ &(2) \end{aligned} \tag{B.1}$$

means that for the $\mathfrak{C}$-model, we have two invariants both defined modulo 2. If none are listed then the invariants are defined only modulo 1 and so are vacuous.

To the right of the $\mathfrak{C}$-model and description of the invariants defined we list the TCS manifolds with this $\mathfrak{C}$-model. A superscript asterisk denotes that the TCS is identical to the one immediately preceding it. A superscript exclamation mark denotes that the TCS is distinguished from the one immediately preceding it only by the evaluation of the boundary defect invariants. There is one case of this and it occurs for the $\mathfrak{C}$-model $(8,4,1,0)$. The column $\overline{\Phi}$ contains the evaluation of the secondary invariants in the order that they appear on the left.

$\mathfrak{C}$	Examples #	b_3	$\overline{\Phi}$	Y_+	Y_-
(2, 1, 1, 0)	1	32		CMv4.1.8.0	CMv4.1.15.0
	2	44		CMv4.1.44.0	CMv4.1.45.0
	3	46		CMv4.1.30.0	CMv4.1.107.0
	4	50		CMv4.1.11.0	CMv4.1.49.0
	* 5	50		CMv4.1.37.0	CMv4.1.34.0
	* 6	50		CMv4.1.89.0	CMv4.1.101.0
	7	56		CMv4.1.35.0	AGV.3.6
	* 8	56		CMv4.1.70.1	CMv4.1.87.1
	* 9	56		CMv4.1.70.1	CMv4.3.5.0
	10	58		CMv4.1.46.0	CMv4.1.40.0
	* 11	58		CMv4.1.74.0	CMv4.1.89.0
	12	60		CMv4.1.40.0	AGV.3.5
	* 13	60		CMv4.1.75.0	AGV.3.6
	14	62		CMv4.1.75.0	AGV.3.10
	* 15	62		CMv4.1.88.0	AGV.3.4
	16	64		CMv4.1.11.0	AGV.3.5
	* 17	64		CMv4.1.33.0	AGV.2.14
	* 18	64		CMv4.1.37.0	AGV.2.7
	* 19	64		CMv4.1.40.0	AGV.3.3
	* 20	64		JPR05.3.21	AGV.3.5
	21	70		AGV.2.7	AGV.3.5
	* 22	70		CMv4.1.97.0	AGV.3.15
	* 23	70		JPR07.7.11.1	CMv4.1.97.0
	24	74		AGV.2.7	AGV.3.3
	* 25	74		CMv4.1.76.0	AGV.3.4
	26	76		CMv4.1.40.0	AGV.3.17
	* 27	76		CMv4.1.76.0	AGV.3.3
	* 28	76		CMv4.1.89.0	AGV.3.18
	29	80		CMv4.1.37.0	AGV.2.29
	30	82		CMv4.1.76.0	AGV.3.13
	31	84		CMv4.1.3.0	AGV.3.18
	32	86		AGV.2.29	AGV.3.5
	* 33	86		AGV.2.7	AGV.3.17
	34	88		CMv4.1.76.0	AGV.3.17
	35	90		AGV.2.29	AGV.3.3
	36	102		AGV.2.29	AGV.3.17

$\mathfrak{C}$	#	b_3	$\overline{\Phi}$	Y_+	Y_-
(2, 1, 2, 1)	1	48		CMv4.1.45.0	CMv4.1.75.0
	2	54		CMv4.1.75.0	JPR07.4.4.1
	3	60		CMv4.5.1.0	AGV.3.5
	4	62		CMv4.1.87.0	CMv4.2.2.0
	5	64		CMv4.1.75.0	AGV.3.1
	* 6	64		JPR05.3.8	AGV.3.3
	7	66		JPR05.3.1	CMv4.1.75.0
	8	68		CMv4.1.76.0	CMv4.2.2.0
	* 9	68		CMv4.2.2.0	AGV.2.12
	10	70		CMv4.2.2.0	AGV.2.21
	* 11	70		CMv4.5.1.0	AGV.3.11
	12	80		CMv4.1.76.0	AGV.2.21
	* 13	80		CMv4.1.76.0	AGV.3.1
	14	82		AGV.2.14	AGV.2.25
	* 15	82		CMv4.1.76.0	AGV.2.6
	16	84		CMv4.1.75.0	AGV.3.27
(2, 1, 3, 1)	1	40		CMv4.1.19.0	CMv4.1.57.0
	2	54		CMv4.1.57.0	AGV.2.5
	* 3	54		CMv4.1.89.0	CMv4.1.88.0
	4	62		CMv4.1.51.0	CMv4.1.88.0
	5	66		CMv4.1.88.0	AGV.3.10
	6	70		CMv4.1.89.0	AGV.2.24
	7	78		CMv4.1.51.0	AGV.2.24
	8	82		AGV.2.24	AGV.3.10
	9	86		CMv4.1.88.0	AGV.2.2
(2, 1, 4, 1)	1	70	0	CMv4.1.70.1	AGV.2.25
$\sigma + \mu$ (2)	2	78	0	CMv4.1.76.0	AGV.3.7
	3	100	0	CMv4.1.76.0	AGV.3.27
(2, 1, 4, 3)	1	40	1	CMv4.1.86.0	CMv4.1.83.0
$\tau + \sigma + \mu$ (2)	2	42	0	CMv4.1.75.0	CMv4.1.83.0
	3	48	0	JPR07.4.15.1	CMv4.1.75.0
	4	54	0	JPR05.3.7	CMv4.1.75.0
	5	60	0	JPR05.3.8	AGV.3.5
	6	62	0	CMv4.1.75.0	AGV.2.16
	7	66	0	CMv4.1.75.0	AGV.2.3
	8	76	0	JPR05.3.8	AGV.3.17
(2, 1, 6, 1)	1	84		AGV.2.5	AGV.2.25
(2, 1, 8, 1)	1	66	0	CMv4.1.30.0	AGV.2.25
$3\sigma + \mu$ (4)					
(2, 1, 8, 5)	1	66	0	CMv4.1.76.0	JPR07.4.11.1
$2\tau + 3\sigma + \mu$ (4)	2	100	0	CMv4.1.76.0	AGV.2.32

$\mathfrak{C}$	#	b_3	$\overline{\Phi}$	Y_+	Y_-
(2, 1, 9, 4)	1	68		CMv4.1.4.0	AGV.3.7
	2	86		AGV.2.25	AGV.3.7
(2, 1, 10, 7)	1	72		CMv4.1.76.0	CMv4.1.87.0
	2	78		CMv4.1.76.0	AGV.2.12
(2, 1, 15, 4)	1	36		CMv4.1.65.0	CMv4.1.57.0
(2, 1, 16, 5) $2\tau + 3\sigma + \mu$ (4)	1	74	0	CMv4.1.37.0	AGV.2.25
(2, 1, 21, 4)	1	52		CMv4.1.88.0	CMv4.1.97.0
(2, 1, 36, 19) $\tau + \sigma + \mu$ (2)	1	68	0	CMv4.2.2.1	AGV.3.7
(2, 1, 40, 37) $2\tau + 3\sigma + \mu$ (4)	1	78	0	CMv4.1.76.0	CMv4.1.76.0
(2, 2, 1, 0) τ (2)	1	52	0	CMv4.1.29.0	CMv4.1.40.0
	* 2	52	0	JPR05.3.14	CMv4.1.101.0
	3	56	0	CMv4.1.42.0	CMv4.1.45.0
	4	58	0	CMv4.1.101.0	JPR07.7.2.1
	* 5	58	0	CMv4.1.51.0	CMv4.1.101.0
	6	60	0	CMv4.1.74.0	JPR05.3.14
	7	62	0	CMv4.1.101.0	AGV.3.10
	8	64	0	CMv4.1.45.0	AGV.2.20
	* 9	64	0	CMv4.4.3.0	CMv4.1.97.0
	10	66	0	CMv4.1.51.0	CMv4.1.74.0
	* 11	66	0	CMv4.1.74.0	AGV.3.4
	* 12	66	0	CMv4.1.74.0	JPR07.7.2.1
	13	68	0	CMv4.1.87.0	AGV.3.4
	* 14	68	0	CMv4.1.97.0	AGV.2.19
	15	70	0	CMv4.1.74.0	AGV.3.10
	* 16	70	0	CMv4.1.87.0	AGV.3.3
	17	76	0	AGV.2.14	AGV.2.18
	* 18	76	0	CMv4.1.18.0	AGV.2.27
	* 19	76	0	CMv4.1.87.0	AGV.3.13
	20	80	0	CMv4.1.42.0	AGV.3.18
	21	82	0	AGV.2.12	AGV.3.13
	* 22	82	0	CMv4.1.87.0	AGV.3.17
	23	84	0	AGV.2.21	AGV.3.13
	24	88	0	AGV.2.20	AGV.3.18
	25	90	0	AGV.3.10	AGV.3.21
	26	94	0	CMv4.1.89.1	AGV.2.33
	* 27	94	0	JPR05.3.1	AGV.3.21
(2, 2, 2, 1) τ (2)	1	88	0	AGV.2.12	AGV.3.17
	2	90	0	AGV.2.21	AGV.3.17
(2, 3, 1, 0)	1	48		CMv4.1.7.0	CMv4.1.33.0

$\mathfrak{C}$	#	b_3	$\overline{\Phi}$	Y_+	Y_-
(2, 4, 1, 0)	1	56	0	CMv4.1.30.0	CMv4.1.54.0
τ (2)	* 2	56	0	CMv4.1.37.0	CMv4.1.4.0
	* 3	56	0	CMv4.1.50.0	CMv4.1.4.0
	4	62	0	CMv4.1.30.0	AGV.2.22
	5	70	0	AGV.2.10	AGV.3.4
	6	72	0	CMv4.1.54.0	AGV.2.14
	7	74	0	AGV.2.12	AGV.3.4
	8	76	0	AGV.2.12	AGV.3.3
	* 9	76	0	AGV.2.21	AGV.3.4
	10	78	0	AGV.2.21	AGV.3.3
	* 11	78	0	JPR05.3.14	AGV.3.18
(2, 4, 3, 1) τ (2)	1	66	0	CMv4.1.4.0	AGV.2.5
(2, 4, 5, 4) τ (2)	1	52	0	CMv4.1.4.0	CMv4.1.70.1
(2, 4, 13, 8) τ (2)	1	40	0	CMv4.1.30.0	CMv4.1.23.0
(2, 5, 1, 0)	1	68		CMv4.1.11.0	AGV.3.3
	2	74		CMv4.1.68.0	AGV.3.21
	3	80		CMv4.1.11.0	AGV.3.17
(2, 6, 1, 0)	1	54	0	CMv4.1.74.0	CMv4.1.96.0
τ (2)	2	56	0	CMv4.1.74.0	CMv4.1.97.0
	3	64	0	CMv4.1.12.0	AGV.3.6
	4	68	0	JPR05.3.2	CMv4.1.74.0
	5	72	0	CMv4.1.74.0	AGV.2.18
	6	76	0	CMv4.1.51.0	AGV.3.12
	7	90	0	CMv4.1.74.0	AGV.2.2
	8	96	0	JPR05.3.16	AGV.2.33
	9	106	0	AGV.2.33	AGV.3.10
(2, 8, 1, 0)	1	58	0	CMv4.1.86.0	AGV.3.6
τ (2)	2	60	0	CMv4.1.14.0	AGV.3.10
	* 3	60	0	CMv4.1.86.0	AGV.3.10
(2, 10, 1, 0)	1	70	0	CMv4.1.35.0	AGV.3.21
τ (2)	* 2	70	0	CMv4.1.89.1	AGV.2.4
	3	82	0	JPR05.3.7	AGV.3.21
	4	90	0	AGV.2.16	AGV.3.21
(2, 11, 1, 0)	1	84		CMv4.2.3.1	AGV.3.25
(2, 16, 1, 0) τ (2)	1	64	0	CMv4.1.18.0	AGV.2.9
(2, 18, 1, 0)	1	60	0	CMv4.1.65.0	AGV.3.12
τ (2)	2	82	0	AGV.2.4	AGV.3.10
(2, 20, 1, 0) τ (2)	1	56	0	CMv4.1.23.0	AGV.2.14

$\mathfrak{C}$	#	b_3	$\overline{\Phi}$	Y_+	Y_-
(2, 20, 5, 3)	1	78	0	AGV.2.14	AGV.2.22
τ (2)					
(2, 30, 1, 0)	1	72	0	AGV.2.4	JPR05.3.16
τ (2)					
(4, 1, 1, 0)	1	54		CMv4.1.49.0	AGV.2.14
	2	70		CMv4.3.4.0	AGV.2.14
(4, 1, 2, 1)	1	38	1	CMv4.1.30.0	CMv4.1.49.0
$\tau + \mu$ (2)					
(4, 1, 3, 1)	1	56		CMv4.1.7.0	JPR05.3.13
(4, 1, 3, 2)	1	64		JPR05.3.2	CMv4.1.88.0
(4, 1, 6, 5)	1	72	1	JPR05.3.14	AGV.2.24
$\tau + \mu$ (2)					
(4, 1, 12, 11)	1	78	1	JPR07.7.2.1	AGV.2.24
$\tau + \mu$ (2)					
(4, 1, 16, 15)	1	52	0	CMv4.1.70.1	CMv4.2.2.1
$\tau + \mu$ (2)					
(4, 2, 1, 0)	1	60	0	CMv4.1.87.0	JPR07.4.11.1
τ (2)	2	66	0	JPR07.4.11.1	AGV.2.12
	3	68	0	JPR07.4.11.1	AGV.2.21
	4	72	0	CMv4.1.87.0	AGV.3.7
	5	74	0	CMv4.1.87.0	AGV.2.21
	* 6	74	0	CMv4.1.87.0	AGV.3.1
	7	76	0	CMv4.1.87.0	AGV.2.6
	8	78	0	AGV.2.12	AGV.3.7
	9	80	0	AGV.2.21	AGV.3.7
	10	94	0	CMv4.1.87.0	AGV.2.32
	* 11	94	0	CMv4.1.87.0	AGV.3.27
	12	100	0	AGV.2.12	AGV.2.32
	* 13	100	0	AGV.2.12	AGV.3.27
	14	102	0	AGV.2.21	AGV.2.32
	* 15	102	0	AGV.2.21	AGV.3.27
(4, 2, 5, 2)	1	72	0	CMv4.1.87.0	AGV.2.12
τ (2)					
(4, 3, 1, 0)	1	56		CMv4.1.7.0	AGV.2.13

$\mathfrak{C}$	#	b_3	$\overline{\Phi}$	Y_+	Y_-
(4, 4, 1, 0)	1	50	0,0	CMv4.1.30.0	CMv4.2.1.0
σ (2)	2	66	0,0	CMv4.2.1.0	AGV.2.14
τ (2)	* 3	66	0,0	CMv4.2.1.1	AGV.2.14
	* 4	66	0,0	CMv4.2.1.1	AGV.2.14
	5	76	0,0	AGV.2.14	AGV.3.1
	6	80	0,0	AGV.2.12	AGV.2.21
	* 7	80	0,0	AGV.2.12	AGV.3.1
	8	82	0,0	AGV.2.21	AGV.3.1
	* 9	82	0,0	AGV.2.6	AGV.2.12
	* 10	82	0,0	CMv4.1.86.0	AGV.3.27
	11	84	0,0	AGV.2.6	AGV.2.21
	12	96	0,0	AGV.2.14	AGV.3.27
(4, 4, 5, 1)	1	78	0,0	AGV.2.12	AGV.2.12
σ (2)					
τ (2)					
(4, 4, 7, 6)	1	82	0,0	AGV.2.21	AGV.2.21
σ (2)					
τ (2)					
(4, 4, 8, 7)	1	52	0,0	CMv4.1.86.0	JPR07.4.4.1
σ (2)					
τ (2)					
(4, 4, 13, 3)	1	50	0,0	CMv4.1.30.0	CMv4.2.1.1
σ (2)					
τ (2)					
(4, 5, 1, 0)	1	72		AGV.2.13	AGV.2.14
	* 2	72		JPR05.3.13	AGV.2.14
(4, 8, 1, 0)	1	62	0,0	CMv4.1.86.0	AGV.3.1
σ (2)	2	64	0,0	JPR05.3.1	CMv4.1.86.0
τ (2)					
(6, 1, 1, 0)	1	60		CMv4.1.3.0	CMv4.1.45.0
	2	64		CMv4.2.2.0	AGV.3.4
	3	66		CMv4.2.2.0	AGV.3.3
	4	68		JPR05.3.21	AGV.3.3
	5	70		CMv4.2.3.1	AGV.3.13
	6	72		CMv4.2.2.0	AGV.3.13
	7	78		AGV.2.24	AGV.3.4
	* 8	78		CMv4.2.2.0	AGV.3.17
	9	80		JPR05.3.21	AGV.3.17
(6, 1, 2, 1)	1	70		CMv4.2.2.0	AGV.3.1
	2	72		CMv4.2.2.0	AGV.2.6

$\mathfrak{C}$	#	b_3	$\overline{\Phi}$	Y_+	Y_-
(6, 1, 4, 1)	1	56	0	CMv4.2.2.0	JPR07.4.11.1
$\tau + \sigma + \mu$ (2)	2	68	0	CMv4.2.2.0	AGV.3.7
	3	90	0	CMv4.2.2.0	AGV.2.32
	* 4	90	0	CMv4.2.2.0	AGV.3.27
(6, 1, 16, 13)	1	58	1	CMv4.2.2.0	CMv4.2.2.0
$\tau + \sigma + \mu$ (2)					
(6, 1, 26, 3)	1	66		CMv4.2.3.1	AGV.2.12
(6, 1, 52, 29)	1	66	1	CMv4.1.76.0	CMv4.2.3.1
$\tau + \sigma + \mu$ (2)					
(6, 2, 1, 0)	1	46	0	CMv4.1.96.0	CMv4.1.101.0
τ (2)	2	48	0	CMv4.1.97.0	CMv4.1.101.0
	3	58	0	CMv4.1.101.0	AGV.3.4
	4	60	0	JPR05.3.2	CMv4.1.101.0
	5	62	0	JPR07.4.11.1	AGV.3.4
	6	64	0	CMv4.1.101.0	AGV.2.18
	* 7	64	0	JPR07.4.11.1	AGV.3.3
	8	70	0	JPR07.4.11.1	AGV.3.13
	9	74	0	AGV.3.4	AGV.3.7
	10	76	0	AGV.3.3	AGV.3.7
	* 11	76	0	JPR07.4.11.1	AGV.3.17
	12	78	0	AGV.3.4	AGV.3.13
	13	80	0	AGV.3.3	AGV.3.13
	14	82	0	AGV.2.2	CMv4.1.101.0
	* 15	82	0	AGV.3.7	AGV.3.13
	16	84	0	AGV.3.1	AGV.3.13
	* 17	84	0	AGV.3.4	AGV.3.17
	18	86	0	AGV.2.6	AGV.3.13
	* 19	86	0	AGV.3.13	AGV.3.13
	* 20	86	0	AGV.3.3	AGV.3.17
	21	88	0	AGV.3.7	AGV.3.17
	22	92	0	AGV.3.13	AGV.3.17
	23	96	0	AGV.2.32	AGV.3.4
	* 24	96	0	AGV.3.4	AGV.3.27
	25	98	0	AGV.2.32	AGV.3.3
	* 26	98	0	AGV.3.3	AGV.3.27
	27	104	0	AGV.2.32	AGV.3.13
	* 28	104	0	AGV.3.13	AGV.3.27
	29	110	0	AGV.2.32	AGV.3.17
	* 30	110	0	AGV.3.17	AGV.3.27
(6, 2, 2, 1)	1	90	0	AGV.3.1	AGV.3.17
τ (2)	2	92	0	AGV.2.6	AGV.3.17
	3	98	0	AGV.3.17	AGV.3.17

$\mathfrak{C}$	#	b_3	$\overline{\Phi}$	Y_+	Y_-
(6, 2, 3, 1)	1	68	0	CMv4.1.52.0	CMv4.1.18.0
τ (2)					
(6, 3, 1, 0)	1	44	2	CMv4.1.71.0	CMv4.1.88.0
σ (3)	2	68	0	CMv4.1.97.0	AGV.2.24
	3	102	0	AGV.2.2	AGV.2.24
	4	108	0	AGV.2.18	AGV.2.34
(6, 4, 1, 0)	1	70	0	AGV.3.4	AGV.3.4
τ (2)	2	72	0	AGV.3.3	AGV.3.4
	3	74	0	AGV.3.3	AGV.3.3
	4	76	0	AGV.3.1	AGV.3.4
	5	78	0	AGV.2.6	AGV.3.4
	* 6	78	0	AGV.3.1	AGV.3.3
	7	80	0	AGV.2.6	AGV.3.3
(6, 6, 1, 0)	1	102	0,0	CMv4.1.51.0	AGV.2.34
2σ (3)					
τ (2)					
(6, 8, 1, 0)	1	56	0	CMv4.1.14.0	AGV.3.4
τ (2)					
(6, 9, 50, 9)	1	58	0	CMv4.1.65.0	AGV.3.7
σ (3)					
(8, 1, 1, 0)	1	54	1	CMv4.1.30.0	CMv4.3.4.0
μ (2)					
(8, 1, 3, 1)	1	50	0	CMv4.1.88.0	CMv4.1.96.0
μ (2)	2	56	1	JPR05.3.14	CMv4.1.88.0
	3	62	1	CMv4.1.88.0	JPR07.7.2.1
	4	68	1	CMv4.1.88.0	AGV.2.18
(8, 2, 1, 0)	1	46	1,0	CMv4.1.45.0	CMv4.1.86.0
μ (2)					
τ (2)					
(8, 2, 5, 4)	1	66	1,0	CMv4.1.87.0	CMv4.1.87.0
μ (2)					
τ (2)					
(8, 4, 1, 0)	1	52	1,0,0	JPR05.3.7	CMv4.1.86.0
μ (2)	2	58	0,0,0	CMv4.1.34.0	AGV.2.10
σ (2)	3	60	0,0,0	CMv4.1.30.0	AGV.3.1
τ (2)	! 4	60	1,0,0	CMv4.1.86.0	AGV.2.16
(8, 4, 2, 1)	1	80	0,0,0	CMv4.1.30.0	AGV.3.27
σ (2)					
$3\sigma + \mu$ (4)					
τ (2)					
(8, 4, 19, 14)	1	46	1,0,0	CMv4.1.34.0	CMv4.1.34.0
μ (2)					
σ (2)					
τ (2)					

$\mathfrak{C}$	#	b_3	$\overline{\Phi}$	Y_+	Y_-
(8, 8, 1, 0)	1	46	0,0,0	JPR07.4.15.1	CMv4.1.86.0
μ (2)	2	64	0,2,0	CMv4.1.86.0	AGV.2.3
σ (4)					
τ (2)					
(8, 16, 2, 1)	1	70	0,0,0	AGV.2.10	AGV.2.10
μ (2)					
σ (4)					
τ (2)					
(12, 2, 1, 0)	1	68	0	JPR07.4.11.1	AGV.3.1
τ (2)	2	70	0	JPR07.4.11.1	AGV.2.6
	3	80	0	AGV.3.1	AGV.3.7
	4	82	0	AGV.2.6	AGV.3.7
	5	102	0	AGV.2.32	AGV.3.1
	* 6	102	0	AGV.3.1	AGV.3.27
	7	104	0	AGV.2.6	AGV.2.32
	* 8	104	0	AGV.2.6	AGV.3.27
(12, 2, 2, 1)	1	66	0,0	JPR07.4.11.1	AGV.3.7
$\sigma+\mu$ (2)	2	78	0,0	AGV.3.7	AGV.3.7
τ (2)	3	88	0,0	JPR07.4.11.1	AGV.3.27
	4	100	0,0	AGV.2.32	AGV.3.7
	* 5	100	0,0	AGV.3.7	AGV.3.27
	6	122	0,0	AGV.2.32	AGV.3.27
	* 7	122	0,0	AGV.3.27	AGV.3.27
(12, 2, 4, 3)	1	54	0,0	JPR07.4.11.1	JPR07.4.11.1
$3\sigma+\mu$ (4)	2	88	0,0	JPR07.4.11.1	AGV.2.32
τ (2)	3	122	0,0	AGV.2.32	AGV.2.32
(12, 3, 1, 0)	1	80	0	JPR05.3.2	AGV.2.24
σ (3)					
(12, 3, 2, 1)	1	84	0,1	AGV.2.18	AGV.2.24
2σ (3)					
$\tau+\mu$ (2)					
(12, 3, 4, 3)	1	66	0,0	CMv4.1.96.0	AGV.2.24
2σ (3)					
$\tau+\mu$ (2)					
(12, 4, 1, 0)	1	82	0,0	AGV.3.1	AGV.3.1
σ (2)	2	84	0,0	AGV.2.6	AGV.3.1
τ (2)	3	86	0,0	AGV.2.6	AGV.2.6
(12, 16, 1, 0)	1	64	0,0	CMv4.5.1.0	AGV.3.3
σ (2)					
τ (2)					
(12, 16, 3, 2)	1	76	0,0	CMv4.5.1.0	AGV.3.17
σ (2)					
τ (2)					

$\mathfrak{C}$	#	b_3	$\overline{\Phi}$	Y_+	Y_-

Table B.2: Invariants of simply connected spin 7-manifolds with $b_2 = 1$.

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