NON-DEGENERACY OF MULTI-BUBBLING SOLUTIONS FOR THE PRESCRIBED SCALAR CURVATURE EQUATIONS AND APPLICATIONS

YUXIA GUO, MONICA MUSSO, SHUANGJIE PENG AND SHUSEN YAN

Abstract. We consider the following prescribed scalar curvature equations in $\mathbb{R}^N$

$$-\Delta u = K(|y|)u^{2^*-1}, \quad u > 0 \ \text{ in } \mathbb{R}^N, \quad u \in D^{1,2}(\mathbb{R}^N), \tag{0.1}$$

where $K(r)$ is a positive function, $2^* = \frac{2N}{N-2}$. We first prove a non-degeneracy result for the positive multi-bubbling solutions constructed in [26] by using the local Pohozaev identities. Then we use this non-degeneracy result to glue together bubbles with different concentration rate to obtain new solutions.

1. Introduction

It is well known that by using the stereo-graphic projection, the prescribed scalar curvature problem on $\mathbb{S}^N$ can be changed to the following equation:

$$-\Delta u = K(y)u^{2^*-1}, \quad u > 0 \quad \text{in } \mathbb{R}^N, \ \ u \in D^{1,2}(\mathbb{R}^N). \tag{1.1}$$

Here $2^* = \frac{2N}{N-2}$ and $N \geq 3$. In the last three decades, enormous efforts have been devoted to the study of (1.1). We refer the readers to [1]–[11], [16]–[27] and the references therein.

If $K(y)$ is radial, infinitely many non-radial solutions are constructed in [26] for

$$-\Delta u = K(|y|)u^{2^*-1}, \quad u > 0 \ \text{ in } \mathbb{R}^N, \quad u \in D^{1,2}(\mathbb{R}^N), \tag{1.2}$$

under the following assumption on $K(r)$:

(K): There are $r_0 > 0$ and $c_0 > 0$, such that

$$K(r) = K(r_0) - c_0(r - r_0)^2 + O(|r - r_0|^3), \quad r \in (r_0 - \delta, r_0 + \delta). \tag{1.3}$$

Without loss of generality, we may assume that $K(r_0) = 1$. Let us briefly discuss the main results in [26].

It is well known (see [2, 25]) that all solutions to the following problem

$$-\Delta u = u^{2^*-1}, \quad u > 0 \ \text{ in } \mathbb{R}^N, \quad u \in D^{1,2}(\mathbb{R}^N) \tag{1.4}$$

are given by

$$U_{x,\mu}(y) = \frac{c_N \mu^{\frac{N-2}{2}}}{(1 + \mu^2|y - x|^2)^{\frac{N-2}{2}}}, \quad x \in \mathbb{R}^N, \ \mu > 0, \tag{1.5}$$

where c_N is a constant depending on N. Let k be an integer number and consider the vertices of a regular polygon with k edges in the (y_1, y_2)-plane given by

$$x_j = \Big(r\cos\frac{2(j-1)\pi}{k}, r\sin\frac{2(j-1)\pi}{k}, 0\Big), \quad j = 1, \cdots, k,$$

where 0 denotes the zero vector in $\mathbb{R}^{N-2}$ and $r \in [r_0-\delta, r_0+\delta]$. For any point $y \in \mathbb{R}^N$, we set $y=(y', y'')$, $y' \in \mathbb{R}^2$, $y'' \in \mathbb{R}^{N-2}$. Define

$$H_s = \Big\{ u : u \text{ is even in } y_h, h = 2, \cdots, N,$$
$$u(r\cos\theta, r\sin\theta, y'') = u(r\cos(\theta+\frac{2\pi j}{k}), r\sin(\theta+\frac{2\pi j}{k}), y'') \Big\}$$

and

$$W_{r,\mu}(y) = \sum_{j=1}^{k} U_{x_j,\mu}(y),$$

where $\mu > 0$ is large. For a function $u \in H_s \cap D^{1,2}(\mathbb{R}^N)$, we introduce the norm $\|u\|_*$ as follows:

$$\|u\|_* = \sup_{y\in\mathbb{R}^N} |u(y)| \Big(\sum_{j=1}^{k} \frac{\mu^{\frac{N-2}{2}}}{(1+\mu|y-x_j|)^{\frac{N-2}{2}+\tau}} \Big)^{-1},$$

where τ is any fixed number in $(\frac{N-4}{N-2}, 1+\theta)$, $\theta > 0$ is a small constant. The result obtained in [26] states the following:

Theorem A. *Suppose that $K(r)$ satisfies (K) and $N \geq 5$. Then there is an integer $k_0 > 0$, such that for any integer $k \geq k_0$, (1.2) has a solution u_k of the form*

$$u_k = W_{r_k,\mu_k}(y) + \omega_k,$$

where $\omega_k \in H_s \cap D^{1,2}(\mathbb{R}^N)$, and as $k \to +\infty$, $|r_k - r_0| = O\big(\frac{1}{\mu_k^{1+\sigma}}\big)$, $\mu_k \sim k^{\frac{N-2}{N-4}}$,

$$\|\omega_k\|_* = O\big(\frac{1}{\mu_k^{1+\sigma}}\big)$$

for some $\sigma > 0$.

The solutions predicted in Theorem A are obtained by gluing together a very large number of basic profiles (1.5) centered at the vertices of a regular polygon with a large number of edges, and scaled with a parameter μ that, as k is taken large, diverges to $+\infty$. The main term $W_{r_k,\mu_k}(y)$ of the solution u_k depends on y'' radially. To obtain a solution which depends on y'' radially, we can carry out the reduction procedure in the following space

$$D^{1,2}(\mathbb{R}^N) \cap \Big\{ u : u \text{ is even in } y_2;\ u(y', y'') = u(y', |y''|),$$
$$u(r\cos\theta, r\sin\theta, y'') = u(r\cos(\theta+\frac{2\pi j}{k}), r\sin(\theta+\frac{2\pi j}{k}), y'') \Big\},$$

to ensure that the error term ω_k also depends on y'' radially.

A direct consequence of the proof in [13], together with the estimates in section 2, is that the solution satisfying the conditions in Theorem A is unique. In particular, such solution must be radial in y''-variable.

Of course, we can also find a solution with n-bubbles, whose centers lie near the circle $|y| = r_0$ in the (y_3, y_4)-plane. The question we want to discuss in this paper is whether

these two solutions can be glued together to give rise to a new type of solutions. In other words, we are interested in finding a new solution to (1.1) whose shape is, at main order,

$$u \approx \sum_{j=1}^{k} U_{x_j,\mu} + \sum_{j=1}^{n} U_{p_j,\lambda}, \tag{1.6}$$

for k and n large integers, where

$$x_j = \Big(r\cos\frac{2(j-1)\pi}{k}, r\sin\frac{2(j-1)\pi}{k}, 0\Big), \quad j = 1, \cdots, k,$$

$$p_j = \Big(0, 0, t\cos\frac{2(j-1)\pi}{n}, t\sin\frac{2(j-1)\pi}{n}, 0, \cdots, 0\Big), \quad j = 1, \cdots, n,$$

and r and t are close to r_0. Equation (1.1) is the Euler-Lagrange equation associated to the energy functional

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{1}{2^*}\int_{\mathbb{R}^N} K(y)|u|^{2^*}.$$

Thus, roughly speaking, a function of the form (1.6) is an approximate solution to (1.1) provided that the radii r, t and the parameters μ and λ are such that

$$I'(\sum_{j=1}^{k} U_{x_j,\mu} + \sum_{j=1}^{n} U_{p_j,\lambda}) \sim 0.$$

Having in mind that $\mu, \lambda \to \infty$, and $r, t \sim r_0$, one easily gets that

$$\begin{aligned} &I\Big(\sum_{j=1}^{k} U_{x_j,\mu} + \sum_{j=1}^{n} U_{p_j,\lambda}\Big) \\ =&(k+n)A + k\Big(\frac{B_1}{\mu^2} + \frac{B_2}{\mu^2}(\mu r_0 - r)^2 - \frac{B_3 k^{N-2}}{\mu^{N-2}}\Big) + kO\Big(\frac{B_1}{\mu^{2+\sigma}} + \frac{B_2}{\mu^2}(\mu r_0 - r)^3\Big) \\ &+ n\Big(\frac{B_1}{\lambda^2} + \frac{B_2}{\lambda^2}(\lambda r_0 - t)^2 - \frac{B_3 n^{N-2}}{\lambda^{N-2}}\Big) + nO\Big(\frac{B_1}{\lambda^{2+\sigma}} + \frac{B_2}{\lambda^2}(\lambda r_0 - t)^3\Big), \end{aligned} \tag{1.7}$$

where $A = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla U_{0,1}|^2 - \frac{1}{2^*}\int_{\mathbb{R}^N} U_{0,1}^{2^*}$, B_1, B_2 and B_3 are some positive constants, and $\sigma > 0$ is a small constant. Observe now that, if $n >> k$, then the two terms in (1.7) are of different order, which makes it complicated to find a critical point for I. Therefore, it is very difficult to use a reduction argument to construct solutions of the form (1.6). In fact, this approach has been successfully used in [22] (see also [12] and [23]) to construct finite energy sign-changing solutions in the case $K(y) \equiv 1$, namely

$$-\Delta u = |u|^{2^*-2}u, \quad \text{in } \mathbb{R}^N, u \in D^{1,2}(\mathbb{R}^N).$$

In this paper, we propose an alternative approach and we consider the above problem from completely different point of view. Recall that our aim is to glue n-bubbles, whose

centers lie in the circle $|y| = r_0$ in the (y_3, y_4)-plane to the k-bubbling solution u_k described in Theorem A. The linear operator for such problem is

$$Q_n\xi = -\Delta\xi - (2^* - 1)K(y)\Big(u_k + \sum_{j=1}^{n} U_{p_j,\lambda}\Big)^{2^*-2}\xi.$$

Away from the points p_j, the operator Q_n can be approximated by the linearized operator around u_k, defined by:

$$L_k\xi = -\Delta\xi - (2^* - 1)K(y)u_k^{2^*-2}\xi. \tag{1.8}$$

The new approach we propose is to build the solution with k-bubbles in the (y_1, y_2)-plane and n-bubbles in the (y_3, y_4)-plane as a perturbation of the solution with the k-bubbles in the (y_1, y_2)-plane. In order to do so, an essential step is to understand the spectral properties of the liner operator L_k and study its invertibility in some suitable space.

The main result of this paper is the following.

Theorem 1.1. *Assume $N \geq 5$. Suppose that $K(y)$ satisfies (K) and*

$$\Delta K - \big(\Delta K + \frac{1}{2}(\Delta K)'\big)r \neq 0 \text{ at } r = r_0.$$

Let $\xi \in H_s \cap D^{1,2}(\mathbb{R}^N)$ be a solution of $L_k\xi = 0$. Then $\xi = 0$.

A direct consequence of Theorem 1.1 is the following result for the existence of new solutions.

Theorem 1.2. *Suppose that $K(r)$ satisfies the assumptions in Theorem 1.1 and $N \geq 7$. Let u_k be a solution in Theorem A and $k > 0$ is a large even number. Then there is an integer $n_0 > 0$, depending on k, such that for any even number $n \geq n_0$, (1.2) has a solution of the form (1.6) for some $t_n \to r_0$ and $\lambda_n \sim n^{\frac{N-2}{N-4}}$.*

Local uniqueness of single bubbling solutions for elliptic problems with critical growth was first studied in [14] by using a degree-counting method, while in [13, 15], the authors used the local Pohozaev identities to deal with the local uniqueness problem for multi-bubbling solutions. The use of the local Pohozaev identities not only simplifies the estimates, but it also makes it possible to study the local uniqueness of solutions with large numbers of bubbles. It is well known that the non-degeneracy of the solution and the uniqueness of such solution are two very closely related problems. In this paper, we shall show that the local Pohozaev identities also play an important role in the study of the non-degeneracy of the multi-bubbling solutions.

This paper is organized as follows. In section 2, we shall prove the main theorem by using the local Pohozaev identities. As an application of this main result, new solutions for (1.2) are constructed in section 3.

2. The non-degeneracy of the solutions

Let

$$-\Delta u = K(|y|)u^{2^*-1}, \tag{2.1}$$

and

$$-\Delta\xi = (2^*-1)K(|y|)u^{2^*-2}\xi. \tag{2.2}$$

Assume that Ω is a smooth domain in $\mathbb{R}^N$. We have the following identities.

Lemma 2.1. *It holds*

$$\begin{aligned} &-\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\frac{\partial\xi}{\partial y_i}-\int_{\partial\Omega}\frac{\partial\xi}{\partial\nu}\frac{\partial u}{\partial y_i}+\int_{\partial\Omega}\big\langle\nabla u,\nabla\xi\big\rangle\nu_i-\int_{\partial\Omega}K(|y|)u^{2^*-1}\xi\nu_i\\ =&-\int_\Omega u^{2^*-1}\xi\frac{\partial K(|y|)}{\partial y_i}, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &\int_\Omega u^{2^*-1}\xi\langle\nabla K(y),y-x_0\rangle\\ =&\int_{\partial\Omega}K(|y|)u^{2^*-1}\xi\langle\nu,y-x_0\rangle\\ &+\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\langle\nabla\xi,y-x_0\rangle+\int_{\partial\Omega}\frac{\partial\xi}{\partial\nu}\langle\nabla u,y-x_0\rangle-\int_{\partial\Omega}\big\langle\nabla u,\nabla\xi\big\rangle\langle\nu,y-x_0\rangle\\ &+\frac{N-2}{2}\int_{\partial\Omega}\xi\frac{\partial u}{\partial\nu}+\frac{N-2}{2}\int_{\partial\Omega}u\frac{\partial\xi}{\partial\nu}. \end{aligned} \tag{2.4}$$

Proof. Proof of (2.3). We have

$$\begin{aligned} &\int_\Omega\Big(-\Delta u\frac{\partial\xi}{\partial y_i}+(-\Delta\xi)\frac{\partial u}{\partial y_i}\Big)\\ =&\int_\Omega K(|y|)\Big(u^{2^*-1}\frac{\partial\xi}{\partial y_i}+(2^*-1)u^{2^*-2}\xi\frac{\partial u}{\partial y_i}\Big). \end{aligned} \tag{2.5}$$

It is easy to check that

$$\begin{aligned} &\int_\Omega K(|y|)\Big(u^{2^*-1}\frac{\partial\xi}{\partial y_i}+(2^*-1)u^{2^*-2}\xi\frac{\partial u}{\partial y_i}\Big)\\ =&\int_\Omega K(|y|)\frac{\partial(u^{2^*-1}\xi)}{\partial y_i}=-\int_\Omega u^{2^*-1}\xi\frac{\partial K(|y|)}{\partial y_i}+\int_{\partial\Omega}K(|y|)u^{2^*-1}\xi\nu_i. \end{aligned} \tag{2.6}$$

Moreover,

$$\begin{aligned} &\int_\Omega\Big(-\Delta u\frac{\partial\xi}{\partial y_i}+(-\Delta\xi)\frac{\partial u}{\partial y_i}\Big)\\ =&-\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\frac{\partial\xi}{\partial y_i}+\int_\Omega\frac{\partial u}{\partial y_j}\frac{\partial^2\xi}{\partial y_i\partial y_j}-\int_{\partial\Omega}\frac{\partial\xi}{\partial\nu}\frac{\partial u}{\partial y_i}+\int_\Omega\frac{\partial\xi}{\partial y_j}\frac{\partial^2u}{\partial y_i\partial y_j}\\ =&-\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\frac{\partial\xi}{\partial y_i}-\int_{\partial\Omega}\frac{\partial\xi}{\partial\nu}\frac{\partial u}{\partial y_i}+\int_\Omega\frac{\partial}{\partial y_i}\Big(\frac{\partial u}{\partial y_j}\frac{\partial\xi}{\partial y_j}\Big)\\ =&-\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\frac{\partial\xi}{\partial y_i}-\int_{\partial\Omega}\frac{\partial\xi}{\partial\nu}\frac{\partial u}{\partial y_i}+\int_{\partial\Omega}\big\langle\nabla u,\nabla\xi\big\rangle\nu_i. \end{aligned} \tag{2.7}$$

So we have proved (2.3).

Proof of (2.4). We have

$$\begin{aligned}&\int_\Omega \Big(-\Delta u\langle\nabla\xi, y-x_0\rangle + (-\Delta\xi)\langle\nabla u, y-x_0\rangle\Big)\\ =&\int_\Omega K(|y|)\Big(u^{2^*-1}\langle\nabla\xi, y-x_0\rangle + (2^*-1)u^{2^*-2}\xi\langle\nabla u, y-x_0\rangle\Big).\end{aligned} \tag{2.8}$$

It is easy to see that

$$\begin{aligned}&\int_\Omega K(|y|)\Big(u^{2^*-1}\langle\nabla\xi, y-x_0\rangle + (2^*-1)u^{2^*-2}\xi\langle\nabla u, y-x_0\rangle\Big)\\ =&\int_\Omega K(|y|)\langle\nabla(u^{2^*-1}\xi), y-x_0\rangle\\ =&\int_{\partial\Omega} K(|y|)u^{2^*-1}\xi\langle\nu, y-x_0\rangle - \int_\Omega u^{2^*-1}\xi\langle\nabla K(y), y-x_0\rangle - N\int_\Omega K(|y|)u^{2^*-1}\xi,\end{aligned} \tag{2.9}$$

where ν is the outward unit normal of $\partial\Omega$ at $y\in\partial\Omega$. Moreover,

$$\begin{aligned}&\int_\Omega \Big(-\Delta u\langle\nabla\xi, y-x_0\rangle + (-\Delta\xi)\langle\nabla u, y-x_0\rangle\Big)\\ =&-\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\langle\nabla\xi, y-x_0\rangle + \int_\Omega \frac{\partial u}{\partial y_j}\big\langle\nabla\frac{\partial\xi}{\partial y_j}, y-x_0\big\rangle + \int_\Omega\big\langle\nabla u,\nabla\xi\big\rangle\\ &-\int_{\partial\Omega}\frac{\partial\xi}{\partial\nu}\langle\nabla u, y-x_0\rangle + \int_\Omega \frac{\partial\xi}{\partial y_j}\big\langle\nabla\frac{\partial u}{\partial y_j}, y-x_0\big\rangle + \int_\Omega\big\langle\nabla u,\nabla\xi\big\rangle\\ =&-\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\langle\nabla\xi, y-x_0\rangle - \int_{\partial\Omega}\frac{\partial\xi}{\partial\nu}\langle\nabla u, y-x_0\rangle + \int_{\partial\Omega}\big\langle\nabla u,\nabla\xi\big\rangle\langle\nu, y-x_0\rangle\\ &+(2-N)\int_\Omega\big\langle\nabla u,\nabla\xi\big\rangle.\end{aligned} \tag{2.10}$$

We also have

$$\begin{aligned}2^*\int_\Omega K(|y|)u^{2^*-1}\xi =&\int_\Omega\big(-\xi\Delta u + u(-\Delta\xi)\big)\\ =&2\int_\Omega\big\langle\nabla u,\nabla\xi\big\rangle - \int_{\partial\Omega}\xi\frac{\partial u}{\partial\nu} - \int_{\partial\Omega}u\frac{\partial\xi}{\partial\nu},\end{aligned} \tag{2.11}$$

which gives

$$\int_\Omega\big\langle\nabla u,\nabla\xi\big\rangle = \frac{2^*}{2}\int_\Omega K(|y|)u^{2^*-1}\xi + \frac{1}{2}\int_{\partial\Omega}\xi\frac{\partial u}{\partial\nu} + \frac{1}{2}\int_{\partial\Omega}u\frac{\partial\xi}{\partial\nu}. \tag{2.12}$$

Thus, the result follows.

□

We will use the results of Lemma 2.1 to establish a fine estimate on the k-bubbling solution of (1.2) u_k obtained in Theorem A. We define the norms as

$$\|u\|_* = \sup_{y\in\mathbb{R}^N} |u(y)| \Big(\sum_{j=1}^k \frac{\mu_k^{\frac{N-2}{2}}}{(1+\mu_k|y-x_{k,j}|)^{\frac{N-2}{2}+\tau}}\Big)^{-1},$$

and

$$\|f\|_{**} = \sup_{y\in\mathbb{R}^N} |f(y)| \Big(\sum_{j=1}^k \frac{\mu_k^{\frac{N+2}{2}}}{(1+\mu_k|y-x_{k,j}|)^{\frac{N+2}{2}+\tau}}\Big)^{-1},$$

where $x_{k,j} = (r_k \cos\frac{2(j-1)\pi}{k}, r_k \sin\frac{2(j-1)\pi}{k}, 0)$. Here τ is any fixed number in $(\frac{N-4}{N-2}, 1+\theta)$, $\theta > 0$ is a small constant. With such choice, noting $\mu_k \sim k^{\frac{N-2}{N-4}}$, we find

$$\sum_{j=2}^k \frac{1}{(\mu_k|x_{k,j}-x_{k,1}|)^\tau} \le \frac{Ck^\tau}{\mu_k^\tau}\sum_{j=1}^k \frac{1}{j^\tau} \le \frac{C_1 k}{\mu_k^\tau} \le C'.$$

Let

$$\Omega_j = \Big\{y=(y',y'')\in\mathbb{R}^2\times\mathbb{R}^{N-2} : \Big\langle \frac{(y',0)}{|y'|}, \frac{x_{k,j}}{|x_{k,j}|}\Big\rangle \ge \cos\frac{\pi}{k}\Big\}.$$

We define the linear operator

$$L_k\xi = -\Delta\xi - (2^*-1)K(|y|)u_k^{2^*-2}\xi. \tag{2.13}$$

First, we prove the following lemma.

Lemma 2.2. *There exists a constant $C>0$ such that*

$$|u_k(y)| \le C\sum_{j=1}^k \frac{\mu_k^{\frac{N-2}{2}}}{(1+\mu_k|y-x_{k,j}|)^{N-2}}, \quad \textit{for all} \quad y\in\mathbb{R}^N.$$

Proof. Let $\tilde{u}_k(y) = \mu_k^{-\frac{N-2}{2}} u_k(\mu_k^{-1}y)$. Then

$$-\Delta\tilde{u}_k = K(\mu_k^{-1}y)\tilde{u}_k^{2^*-1}.$$

We have

$$\tilde{u}_k(y) = \int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}} K(\mu_k^{-1}z)\tilde{u}_k^{2^*-1}(z)\,dz.$$

By Theorem A, we find

$$\begin{aligned}
|\tilde{u}_k(y)| \leq & C\int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}} u_k^{2^*-1}\,dz \\
\leq & C\int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}} \Big(\sum_{j=1}^{k} \frac{1}{(1+|z-\tilde{x}_{k,j}|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*-1} \\
\leq & C\int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}} \sum_{j=1}^{k} \frac{1}{(1+|z-\tilde{x}_{k,j}|)^{\frac{N+2}{2}+\tau_1+\frac{(N+2)(\tau-\tau_1)}{N-2}}} \Big(\sum_{j=1}^{k} \frac{1}{|\tilde{x}_{k,1}-\tilde{x}_{k,j}|^{\tau_1}}\Big)^{\frac{4}{N-2}} \\
\leq & C\sum_{j=1}^{k} \frac{1}{(1+|y-\tilde{x}_{k,j}|)^{\frac{N-2}{2}+\tau_1+\frac{(N+2)(\tau-\tau_1)}{N-2}}},
\end{aligned}$$

where $\tilde{x}_{k,j} = \mu_k x_{k,j}$, and $\tau_1 \in (\frac{N-4}{N-2}, \tau)$. Noting that

$$\frac{N-2}{2} + \tau_1 + \frac{(N+2)(\tau-\tau_1)}{N-2} = \frac{N-2}{2} + \tau + \frac{4(\tau-\tau_1)}{N-2} > \frac{N-2}{2} + \tau,$$

we can continue this process to prove the result.

□

We now prove Theorem 1.1, arguing by contradiction. Suppose that there are $k_m \to +\infty$, satisfying $\|\xi_m\|_* = 1$, and

$$L_{k_m}\xi_m = 0. \tag{2.14}$$

Let

$$\tilde{\xi}_m(y) = \mu_{k_m}^{-\frac{N-2}{2}} \xi_m(\mu_{k_m}^{-1} y + x_{k_m,1}). \tag{2.15}$$

Lemma 2.3. *It holds*

$$\tilde{\xi}_m \to b_0\psi_0 + b_1\psi_1, \tag{2.16}$$

uniformly in $C^1(B_R(0))$ for any $R > 0$, where b_0 and b_1 are some constants,

$$\psi_0 = \frac{\partial U_{0,\mu}}{\partial \mu}\Big|_{\mu=1}, \quad \psi_i = \frac{\partial U_{0,1}}{\partial y_i}, \quad i = 1, ..., N.$$

Proof. In view of $|\tilde{\xi}_m| \leq C$, we may assume that $\tilde{\xi}_m \to \xi$ in $C_{loc}(\mathbb{R}^N)$. Then ξ satisfies

$$-\Delta\xi = (2^*-1)U^{2^*-2}\xi, \quad \text{in } \mathbb{R}^N, \tag{2.17}$$

which gives

$$\xi = \sum_{i=0}^{N} b_i\psi_i. \tag{2.18}$$

Since ξ_m is even in y_i, $i = 2, \cdots, N$, it holds $b_i = 0$, $i = 2, \cdots, N$.

□

We decompose

$$\xi_m(y)=b_{0,m}\mu_{k_m}\sum_{j=1}^{k}\frac{\partial U_{x_{k_m,j},\mu_{k_m}}}{\partial\mu_{k_m}}+b_{1,m}\mu_{k_m}^{-1}\sum_{j=1}^{k}\frac{\partial U_{x_{k_m,j},\mu_{k_m}}}{\partial r}+\xi_m^*,$$

where ξ_m^* satisfies

$$\int_{\mathbb{R}^N}U_{x_{k_m,j},\mu_{k_m}}^{2^*-1}\frac{\partial U_{x_{k_m,j},\mu_{k_m}}}{\partial\mu_{k_m}}\xi_m^*=\int_{\mathbb{R}^N}U_{x_{k_m,j},\mu_{k_m}}^{2^*-1}\frac{\partial U_{x_{k_m,j},\mu_{k_m}}}{\partial r}\xi_m^*=0.$$

It follows from Lemma 2.3 that $b_{0,m}$ and $b_{1,m}$ are bounded.

Lemma 2.4. *It holds*

$$\|\xi_m^*\|_*\le C\mu_{k_m}^{-1-\sigma},$$

where $\sigma>0$ is a small constant.

Proof. It is easy to see that

$$\begin{aligned}L_{k_m}\xi_m^*:=&-\Delta\xi_m^*-(2^*-1)K(|y|)u_{k_m}^{2^*-2}\xi_m^*\\=&-(2^*-1)\big(K(|y|)-1\big)u_{k_m}^{2^*-2}\sum_{j=1}^{k_m}\Big(b_{0,m}\mu_{k_m}\frac{\partial U_{x_{k_m,j},\mu_{k_m}}}{\partial\mu_{k_m}}+b_{1,n}\mu_{k_m}^{-1}\frac{\partial U_{x_{k_m,j},\mu_{k_m}}}{\partial r}\Big)\\&-(2^*-1)\sum_{j=1}^{k}\big(u_{k_m}^{2^*-2}-U_{x_{k_m,j},\mu_{k_m}}^{2^*-2}\big)\Big(b_{0,m}\mu_{k_m}\frac{\partial U_{x_{k_m,j},\mu_{k_m}}}{\partial\mu_{k_m}}+b_{1,n}\mu_{k_m}^{-1}\frac{\partial U_{x_{k_m,j},\mu_{k_m}}}{\partial r}\Big).\end{aligned}$$

Similar to the proof of Lemma 2.5 in [26], we can prove

$$\begin{aligned}&\Big\|\big(K(|y|)-1\big)u_{k_m}^{2^*-2}\sum_{j=1}^{k_m}\Big(b_{0,m}\mu_{k_m}\frac{\partial U_{x_{k_m,j},\mu_{k_m}}}{\partial\mu_{k_m}}+b_{1,n}\mu_{k_m}^{-1}\frac{\partial U_{x_{k_m,j},\mu_{k_m}}}{\partial r}\Big)\Big\|_{**}\\\le&\Big\|\big(K(|y|)-1\big)\big(\sum_{j=1}^{k_m}U_{x_{k_m,j},\mu_{k_m}}\big)^{2^*-1}\Big\|_{**}\le C\mu_k^{-1-\sigma}.\end{aligned}$$

Without loss of generality, we assume $y\in\Omega_1$. Now we have

$$\begin{aligned}&\Big|\Big(\big(\sum_{i=1}^{k_m}U_{x_{k_m,i},\mu_{k_m}}\big)^{2^*-2}-U_{x_{k_m,1},\mu_{k_m}}^{2^*-2}\Big)\Big(b_{0,m}\mu_{k_m}\frac{\partial U_{x_{k_m,1},\mu_{k_m}}}{\partial\mu_{k_m}}+b_{1,n}\mu_{k_m}^{-1}\frac{\partial U_{x_{k_m,1},\mu_{k_m}}}{\partial r}\Big)\Big|\\\le&CU_{x_{k_m,1},\mu_{k_m}}^{2^*-2}\sum_{j=2}^{k_m}U_{x_{k_m,j},\mu_{k_m}}\\\le&C\frac{\mu_{k_m}^{\frac{N+2}{2}}}{(1+\mu_{k_m}|y-x_{k_m,1}|)^{\frac{N+2}{2}+\tau}}\sum_{j=2}^{k_m}\frac{1}{(\mu_{k_m}|x_{k_m,j}-x_{k_m,1}|)^{\frac{N+2}{2}-\tau}}\\\le&\frac{C\mu_{k_m}^{\frac{N+2}{2}}}{(1+\mu_{k_m}|y-x_{k_m,1}|)^{\frac{N+2}{2}+\tau}}\mu_k^{-1-\sigma},\end{aligned}$$

and

$$\begin{aligned}
&\Big|\sum_{j=2}^{k}\Big((\sum_{i=1}^{k_m}U_{x_{k_m,i},\mu_{k_m}})^{2^*-2}-U^{2^*-2}_{x_{k_m,j},\mu_{k_m}}\Big)\Big(b_{0,m}\mu_{k_m}\frac{\partial U_{x_{k_m,j},\mu_{k_m}}}{\partial\mu_{k_m}}+b_{1,n}\mu_{k_m}^{-1}\frac{\partial U_{x_{k_m,j},\mu_{k_m}}}{\partial r}\Big)\Big|\\
\le&\sum_{j=2}^{k}\Big(U^{2^*-2}_{x_{k_m,1},\mu_{k_m}}+(\sum_{i=2}^{k_m}U_{x_{k_m,i},\mu_{k_m}})^{2^*-2}\Big)U_{x_{k_m,j},\mu_{k_m}}\\
\le&\sum_{j=2}^{k}\Big(U^{2^*-2}_{x_{k_m,1},\mu_{k_m}}+\frac{C\mu_{k_m}^2}{(1+\mu_{k_m}|y-x_{k_m,1}|)^{4-\frac{4\tau}{N-2}}}\Big)U_{x_{k_m,j},\mu_{k_m}}\\
\le&\frac{C\mu_{k_m}^{\frac{N+2}{2}}}{(1+\mu_{k_m}|y-x_{k_m,1}|)^{\frac{N+2}{2}+\tau}}\mu_k^{-1-\sigma}.
\end{aligned}$$

So we have proved

$$\begin{aligned}
&\Big\|\sum_{j=1}^{k}\Big((\sum_{i=1}^{k_m}U_{x_{k_m,i},\mu_{k_m}})^{2^*-2}-U^{2^*-2}_{x_{k_m,j},\mu_{k_m}}\Big)\Big(b_{0,m}\mu_{k_m}\frac{\partial U_{x_{k_m,j},\mu_{k_m}}}{\partial\mu_{k_m}}+b_{1,n}\mu_{k_m}^{-1}\frac{\partial U_{x_{k_m,j},\mu_{k_m}}}{\partial r}\Big)\Big\|_{**}\\
\le&\mu_k^{-1-\sigma}.
\end{aligned}$$

It is also easy to prove

$$\begin{aligned}
&\Big\|\sum_{j=1}^{k_m}\Big(u_{k_m}^{2^*-2}-(\sum_{j=1}^{k_m}U_{x_{k_m,j},\mu_{k_m}})^{2^*-2}\Big)\Big(b_{0,m}\mu_{k_m}\frac{\partial U_{x_{k_m,j},\mu_{k_m}}}{\partial\mu_{k_m}}+b_{1,n}\mu_{k_m}^{-1}\frac{\partial U_{x_{k_m,j},\mu_{k_m}}}{\partial r}\Big)\Big\|_{**}\\
\le&\Big\|\Big(\sum_{j=1}^{k_m}U_{x_{k_m,j},\mu_{k_m}}\Big)^{2^*-2}\omega_{k_m}\Big\|_{**}\le C\|\omega_{k_m}\|_*\le C\mu_k^{-1-\sigma}.
\end{aligned}$$

Moreover, from

$$\int_{\mathbb{R}^N}U^{2^*-1}_{x_{k_m,j},\mu_{k_m}}\frac{\partial U_{x_{k_m,j},\mu_{k_m}}}{\partial\mu_{k_m}}\xi_m^*=\int_{\mathbb{R}^N}U^{2^*-1}_{x_{k_m,j},\mu_{k_m}}\frac{\partial U_{x_{k_m,j},\mu_{k_m}}}{\partial r}\xi_m^*=0,$$

and Lemma 2.2, we can prove that there exists $\rho>0$, such that

$$\|L_{k_m}\xi_m^*\|_{**}\ge\rho\|\xi_m^*\|_*.$$

Thus, the result follows.

□

Lemma 2.5. *If* $\Delta K-\big(\Delta K+\frac{1}{2}(\Delta K)'\big)r\neq 0$ *at* $r=r_0$, *then*

$$\tilde{\xi}_m\to 0 \tag{2.19}$$

uniformly in $C^1(B_R(0))$ *for any* $R>0$

Proof. **Step 1.** Recall that

$$\Omega_j = \Big\{ y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \Big\langle \frac{(y', 0)}{|y'|}, \frac{x_{k,j}}{|x_{k,j}|} \Big\rangle \geq \cos \frac{\pi}{k} \Big\}.$$

To prove $b_{1,m} \to 0$, we apply the identities in Lemma 2.1 in the domain Ω_1:

$$\begin{aligned} &- \int_{\partial\Omega_1} \frac{\partial u_{k_m}}{\partial \nu} \frac{\partial \xi_m}{\partial y_1} - \int_{\partial\Omega_1} \frac{\partial \xi_m}{\partial \nu} \frac{\partial u_{k_m}}{\partial y_1} + \int_{\partial\Omega_1} \big\langle \nabla u_{k_m}, \nabla \xi_m \big\rangle \nu_1 - \int_{\partial\Omega_1} K(|y|) u_{k_m}^{2^*-1} \xi_m \nu_1 \\ =& - \int_{\Omega_1} u_{k_n}^{2^*-1} \xi_m \frac{\partial K(|y|)}{\partial y_1}. \end{aligned} \tag{2.20}$$

Now we estimate the left hand side of (2.20). By the symmetry, $\frac{\partial u_{k_m}}{\partial \nu} = 0$ and $\frac{\partial \xi_m}{\partial \nu} = 0$ on $\partial\Omega_1$. So

$$\begin{aligned} &- \int_{\partial\Omega_1} \frac{\partial u_{k_m}}{\partial \nu} \frac{\partial \xi_m}{\partial y_i} - \int_{\partial\Omega_1} \frac{\partial \xi_m}{\partial \nu} \frac{\partial u_{k_m}}{\partial y_i} + \int_{\partial\Omega_1} \big\langle \nabla u_{k_m}, \nabla \xi_m \big\rangle \nu_1 \\ &- \int_{\partial\Omega_1} K(|y|) u_{k_m}^{2^*-1} \xi_m \nu_1 \\ =& \int_{\partial\Omega_1} \big\langle \nabla u_{k_m}, \nabla \xi_m \big\rangle \nu_1 - \int_{\partial\Omega_1} K(|y|) u_{k_m}^{2^*-1} \xi_m \nu_1 \\ =& - \sin \frac{\pi}{k_m} \Big(\int_{\partial\Omega_1} \big\langle \nabla u_{k_m}, \nabla \xi_m \big\rangle - \int_{\partial\Omega_1} K(|y|) u_{k_m}^{2^*-1} \xi_m \Big). \end{aligned} \tag{2.21}$$

Combining (2.21) and (2.20), we obtain

$$\begin{aligned} &\sin \frac{\pi}{k_m} \Big(\int_{\partial\Omega_1} \big\langle \nabla u_{k_m}, \nabla \xi_m \big\rangle - \int_{\partial\Omega_1} K(|y|) u_{k_m}^{2^*-1} \xi_m \Big) \\ =& \int_{\Omega_1} u_{k_n}^{2^*-1} \xi_m \frac{\partial K(|y|)}{\partial y_1}. \end{aligned} \tag{2.22}$$

To estimate the left hand side in (2.22), we use (2.4) in Ω_1. Using the symmetry, we obtain

$$\begin{aligned} &\int_{\Omega_1} u_{k_m}^{2^*-1} \xi_m \langle \nabla K(y), y - x_{k_m,1} \rangle \\ =& \int_{\partial\Omega_1} K(|y|) u_{k_m}^{2^*-1} \xi_m \langle \nu, y - x_{k_m,1} \rangle - \int_{\partial\Omega_1} \big\langle \nabla u_{k_m}, \nabla \xi_m \big\rangle \langle \nu, y - x_{k_m,1} \rangle. \end{aligned} \tag{2.23}$$

On $\partial\Omega_1$, it holds $\langle \nu, y \rangle = 0$,

$$\langle \nu, x_{k_m,1} \rangle = - \sin \frac{\pi}{k_m}.$$

Thus, (2.23) becomes

$$\begin{aligned}&\int_{\Omega_1} u_{k_m}^{2^*-1}\xi_m\langle\nabla K(y), y-x_{k_m,1}\rangle\\ =&\sin\frac{\pi}{k_m}\Big(\int_{\partial\Omega_1} K(|y|)u_{k_m}^{2^*-1}\xi_m-\int_{\partial\Omega_1}\big\langle\nabla u_{k_m},\nabla\xi_m\big\rangle\Big).\end{aligned}\tag{2.24}$$

Combining (2.22) and (2.24), we obtain

$$\int_{\Omega_1} u_{k_n}^{2^*-1}\xi_m\frac{\partial K(|y|)}{\partial y_1}=-\int_{\Omega_1} u_{k_m}^{2^*-1}\xi_m\langle\nabla K(y), y-x_{k_m,1}\rangle.\tag{2.25}$$

Since $\nabla K(x_{k_m,1})=O(||x_{k_m,1}|-r_0|)=O(\mu_{k_m}^{-1-\sigma})$, and

$$\begin{aligned}\int_{\Omega_1} u_{k_m}^{2^*-1}\xi_m=&\int_{(\Omega_1)_{x_{k_m,1},\mu_{k_m}}}\Big(\mu_{k_m}^{-\frac{N-2}{2}}u_{k_m}(\mu_{k_m}^{-1}y+x_{k_m,1})\Big)^{2^*-1}\tilde{\xi}_m\\ =&\int_{\mathbb{R}^N} U^{2^*-1}\Big(b_{0,m}\psi_0+b_{1,m}\psi_1+\mu_{k_m}^{-\frac{N-2}{2}}\xi_m^*(\mu_k^{-1}y+x_{k_m,1})\Big)+O(\mu_{k_m}^{-2})\\ =&O(\mu_{k_m}^{-1-\sigma}),\end{aligned}\tag{2.26}$$

where $(\Omega_1)_{x_{k_m,1},\mu_{k_m}}=\big\{y\,:\,\mu_{k_m}^{-1}y+x_{k_m,1}\in\Omega_1\big\}$, we find

$$\begin{aligned}&\int_{\Omega_1} u_{k_m}^{2^*-1}\xi_m\frac{\partial K(y)}{\partial y_1}\\ =&\int_{\Omega_1} u_{k_m}^{2^*-1}\xi_m\Big(\frac{\partial K(y)}{\partial y_1}-\frac{\partial K(x_{k_m,1})}{\partial y_1}\Big)+O(\mu_{k_m}^{-2-2\sigma})\\ =&\int_{\Omega_1} u_{k_m}^{2^*-1}\xi_m\Big(\Big\langle\nabla\frac{\partial K(x_{k_m,1})}{\partial y_1}, y-x_{k_m,1}\Big\rangle+\frac{1}{2}\Big\langle\nabla^2\frac{\partial K(x_{k_m,1})}{\partial y_1}(y-x_{k_m,1}), y-x_{k_m,1}\Big\rangle\\ &\qquad+O\big(|y-x_{k_m,1}|^3\big)\Big)+O(\mu_{k_m}^{-2-2\sigma})\\ =&\int_{\mathbb{R}^N} U^{2^*-1}\big(b_{0,m}\psi_0+b_{1,m}\psi_1\big)\Big(\Big\langle\nabla\frac{\partial K(x_{k_m,1})}{\partial y_1},\frac{y}{\mu_{k_m}}\Big\rangle+\frac{1}{2}\Big\langle\nabla^2\frac{\partial K(x_{k_m,1})}{\partial y_1}\frac{y}{\mu_{k_m}},\frac{y}{\mu_{k_m}}\Big\rangle\Big)\\ &+O(\mu_{k_m}^{-2-\sigma})\\ =&\frac{K''(x_{k_m,1})b_{m,1}}{\mu_{k_m}}\int_{\mathbb{R}^N} U^{2^*-1}\psi_1 y_1+\frac{\frac{\partial\Delta K(x_{k_m,1})}{\partial y_1}b_{m,0}}{2N\mu_{k_m}^2}\int_{\mathbb{R}^N} U^{2^*-1}\psi_0|y|^2+O(\mu_{k_m}^{-2-\sigma}).\end{aligned}\tag{2.27}$$

On the other hand, we have

$$
\begin{aligned}
&\int_{\Omega_1} u_{k_m}^{2^*-1}\xi_m\langle\nabla K(y), y-x_{k_m,1}\rangle\\
=&\int_{\Omega_1} u_{k_m}^{2^*-1}\xi_m\langle\nabla K(y)-\nabla K(x_{k_m,1}), y-x_{k_m,1}\rangle + O(\mu_{k_m}^{-2-\sigma})\\
=&\int_{\Omega_1} u_{k_m}^{2^*-1}\xi_m\langle\nabla^2 K(x_{k_m,1})(y-x_{k_m,1}), y-x_{k_m,1}\rangle + O(\mu_{k_m}^{-2-\sigma})\\
=&\int_{\mathbb{R}^N} U^{2^*-1}\big(b_{0,m}\psi_0+b_{1,m}\psi_1\big)\big\langle\nabla^2 K(x_{k_m,1})\mu_{k_m}^{-1}y, \mu_{k_m}^{-1}y\big\rangle + O(\mu_{k_m}^{-2-\sigma})\\
=&\frac{b_{0,m}\Delta K(x_{k_m,1})}{N\mu_{k_m}^2}\int_{\mathbb{R}^N} U^{2^*-1}\psi_0|y|^2 + O(\mu_{k_m}^{-2-\sigma}).
\end{aligned}
\tag{2.28}
$$

Therefore, (2.27) and (2.28) give

$$
b_{m,1} = -\frac{\displaystyle\int_{\mathbb{R}^N} U^{2^*-1}\psi_0|y|^2}{\mu_{k_m}K''(x_{k_m,1})\displaystyle\int_{\mathbb{R}^N} U^{2^*-1}\psi_1 y_1}\Big(\frac{K''(x_{k_m,1})}{N}+\frac{\frac{\partial\Delta K(x_{k_m,1})}{\partial y_1}}{2N}\Big)b_{m,0}+O\Big(\frac{1}{\mu_{k_m}^{1+\sigma}}\Big). \tag{2.29}
$$

Step 2. Next, we use (2.4) to obtain

$$
\int_{\mathbb{R}^N} u_{k_m}^{2^*-1}\xi_m\langle\nabla K(y), y\rangle = 0, \tag{2.30}
$$

which gives

$$
\int_{\Omega_1} u_{k_m}^{2^*-1}\xi_m\langle\nabla K(y), y\rangle = 0. \tag{2.31}
$$

On the other hand, proceeding as in the proof of (2.26), we find

$$
\begin{aligned}
&\int_{\Omega_1} u_{k_m}^{2^*-1}\xi_m\langle\nabla K(x_{k_m,1}), y\rangle\\
=&\int_{\Omega_1} u_{k_m}^{2^*-1}\xi_m\langle\nabla K(x_{k_m,1}), y-x_{\mu_{k_m},1}\rangle + \int_{\Omega_1} u_{k_m}^{2^*-1}\xi_m\langle\nabla K(x_{k_m,1}), x_{k_m,1}\rangle\\
=&O(\mu_{k_m}^{-2-\sigma}).
\end{aligned}
$$

So from (2.29),

$$
\begin{aligned}
&\int_{\Omega_1} u_{k_m}^{2^*-1}\xi_m\langle\nabla K(y),y\rangle\\
=&\int_{\Omega_1} u_{k_m}^{2^*-1}\xi_m\langle\nabla K(y)-\nabla K(x_{k_m,1}),y\rangle+O(\mu_{k_m}^{-2-\sigma})\\
=&\int_{\Omega_1} u_{k_m}^{2^*-1}\xi_m\langle\nabla^2 K(x_{k_m,1})(y-x_{k_m,1}),y\rangle+O(\mu_{k_m}^{-2-\sigma})\\
=&\int_{\mathbb{R}^N} U^{2^*-1}\big(b_{0,m}\psi_0+b_{1,m}\psi_1\big)\big\langle\nabla^2 K(x_{k_m,1})\mu_{k_m}^{-1}y,\mu_{k_m}^{-1}y+x_{\mu_{k_m},1}\big\rangle+O(\mu_{k_m}^{-2-\sigma})\\
=&\frac{b_{0,m}\Delta K(x_{k_m,1})}{N\mu_{k_m}^2}\int_{\mathbb{R}^N}U^{2^*-1}\psi_0|y|^2+b_{1,m}\mu_{k_m}^{-1}\int_{\mathbb{R}^N}U^{2^*-1}\psi_1\big\langle\nabla^2K(x_{k_m,1})y,x_{k_m,1}\big\rangle\\
&+O(\mu_{k_m}^{-2-\sigma})\\
=&\frac{b_{0,m}\Delta K(x_{k_m,1})}{N\mu_{k_m}^2}\int_{\mathbb{R}^N}U^{2^*-1}\psi_0|y|^2+b_{1,m}\mu_k^{-1}K''(x_{k_m,1})|x_{k_m,1}|\int_{\mathbb{R}^N}U^{2^*-1}\psi_1y_1\\
&+O(\mu_{k_m}^{-2-\sigma})\\
=&\frac{b_{0,m}\Delta K(x_{k_m,1})}{N\mu_{k_m}^2}\int_{\mathbb{R}^N}U^{2^*-1}\psi_0|y|^2\\
&-\Big(\frac{K''(x_{k_m,1})}{N}+\frac{\frac{\partial\Delta K(x_{k_m,1})}{\partial y_1}}{2N}\Big)\int_{\mathbb{R}^N}U^{2^*-1}\psi_0|y|^2\frac{|x_{k_m,1}|b_{0,m}}{\mu_{k_m}^2}\\
&+O(\mu_{k_m}^{-2-\sigma}).
\end{aligned}
\tag{2.32}
$$

Since $\Delta K-\big(\Delta K+\frac{1}{2}(\Delta K)'\big)r\neq 0$ at $r=r_0$, we find

$$b_{0,m}=O(\mu_{k_m}^{-1})=o(1).$$

So $b_{1,m}=o(1)$.

□

Proof of Theorem 1.1. We have

$$\xi_m(y)=(2^*-1)\int_{\mathbb{R}^N}\frac{1}{|z-y|^{N-2}}K(|z|)u_{k_m}^{2^*-2}(z)\xi_m(z)\,dz. \tag{2.33}$$

Now we estimate

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^N}\frac{1}{|z-y|^{N-2}}u_{k_m}^{2^*-2}(z)\xi_m(z)\,dz\Big|\\
\le& C\|\xi_m\|_*\int_{\mathbb{R}^N}\frac{1}{|z-y|^{N-2}}u_{k_m}^{2^*-2}(z)\sum_{j=1}^{k_m}\frac{\mu_{k_m}^{\frac{N-2}{2}}}{(1+\mu_{k_m}|z-x_{k_m,j}|)^{\frac{N-2}{2}+\theta}}\\
\le& C\|\xi_m\|_*\sum_{j=1}^{k_m}\frac{\mu_{k_m}^{\frac{N-2}{2}}}{(1+\mu_{k_m}|y-x_{k_m,j}|)^{\frac{N-2}{2}+\tau+\theta}},
\end{aligned}$$

for some $\theta>0$. So we obtain

$$\frac{|\xi_m(y)|}{\sum_{j=1}^{k_m}\frac{\mu_{k_m}^{\frac{N-2}{2}}}{(1+\mu_{k_m}|y-x_{k_m,j}|)^{\frac{N-2}{2}+\tau}}}\le C\|\xi_k\|_*\frac{\sum_{j=1}^{k_m}\frac{\mu_{k_m}^{\frac{N-2}{2}}}{(1+\mu_{k_m}|y-x_{k_m,j}|)^{\frac{N-2}{2}+\tau+\theta}}}{\sum_{j=1}^{k_m}\frac{\mu_{k_m}^{\frac{N-2}{2}}}{(1+\mu_{k_m}|y-x_{k_m,j}|)^{\frac{N-2}{2}+\tau}}}.$$

Since $\xi_m\to 0$ in $B_{R\mu_{k_m}^{-1}}(x_{k_m,j})$ and $\|\xi_m\|_*=1$, we know that

$$\frac{|\xi_m(y)|}{\sum_{j=1}^{k_m}\frac{\mu_{k_m}^{\frac{N-2}{2}}}{(1+\mu_{k_m}|y-x_{k_m,j}|)^{\frac{N-2}{2}+\tau}}}$$

attains its maximum in $\mathbb{R}^N\setminus\cup_{j=1}^{k_m}B_{R\mu_{k_m}^{-1}}(x_{k_m,j})$. Thus

$$\|\xi_m\|_*\le o(1)\|\xi_m\|_*.$$

So $\|\xi_m\|_*\to 0$ as $m\to+\infty$. This is a contradiction to $\|\xi_m\|_*=1$.

□

3. Proof of the main result

Let u_k be the k-bubbling solutions in Theorem A, where $k>0$ is a large even integer. Since k is even, u_k is even in each y_j, $j=1,\cdots,N$. Moreover, u_k is radial in $y''=(y_3,\cdots,y_N)$.

Let $n\ge k$ be a large even integer. Set

$$p_j=\Big(0,0,t\cos\frac{2(j-1)\pi}{n},t\sin\frac{2(j-1)\pi}{n},0\Big),\quad j=1,\cdots,n,$$

where t is close to r_0.

Define

$$\begin{aligned}
X_s=\Big\{u:&u\in H_s,u\text{ is even in }y_h,h=1,\cdots,N,\\
&u(y_1,y_2,t\cos\theta,t\sin\theta,y^*)=u(y_1,y_2,t\cos(\theta+\frac{2\pi j}{n}),t\sin(\theta+\frac{2\pi j}{n}),y^*)\Big\}.
\end{aligned}$$

Here $y^*=(y_5,\cdots,y_N)$.

Let

$$D_j = \Big\{ y = (y', y_3, y_4, y^*) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^{N-4} : \Big\langle \frac{(0,0,y_3,y_4,0,\cdots,0)}{|(y_3,y_4)|}, \frac{p_j}{|p_j|} \Big\rangle \geq \cos\frac{\pi}{n} \Big\}.$$

Note that both u_k and $\sum_{j=1}^n U_{p_j,\lambda}$ belong to X_s, while u_k and $\sum_{j=1}^n U_{p_j,\lambda}$ are separated from each other. We aim to construct a solution for (1.2) of the form

$$u = u_k + \sum_{j=1}^n U_{p_j,\lambda} + \xi,$$

where $\xi \in X_s$ is a small perturbed term.

We define the linear operator

$$Q_n\xi = -\Delta\xi - (2^*-1)K(|y|)\Big(u_k + \sum_{j=1}^n U_{p_j,\lambda}\Big)^{2^*-2}\xi, \quad \xi \in X_s. \tag{3.1}$$

We can regard $Q_n\xi$ as a function in X_s, such that

$$\big\langle Q_n\xi, \phi\big\rangle = \int_{\mathbb{R}^N}\Big(\nabla\xi\nabla\phi - (2^*-1)K(|y|)\big(u_k + \sum_{j=1}^n U_{p_j,\lambda}\big)^{2^*-2}\xi\phi\Big), \quad \xi,\, \phi \in X_s. \tag{3.2}$$

Let

$$Z_{j,1} = \frac{\partial U_{p_j,\lambda}}{\partial r}, \;\; j = 1,\cdots,k, \quad Z_{j,2} = \frac{\partial U_{p_j,\lambda}}{\partial \lambda}$$

Let $h_n \in X_s$. Consider

$$\begin{cases} Q_n\xi_n = h_n + \sum\limits_{i=1}^2 a_{n,i}\sum\limits_{j=1}^n Z_{j,i}, \\ \xi_n \in X_s, \\ \int_{\mathbb{R}^N} U_{p_j,\lambda}^{2^*-2} Z_{j,i}\xi_n = 0, \;\; i = 1,2,\; j = 1,\cdots,n, \end{cases} \tag{3.3}$$

for some constants $a_{n,i}$, depending on ξ_n.

Lemma 3.1. *Assume that ξ_n solve* (3.3). *If $\|h_n\|_{D^{1,2}} \to 0$, then $\|\xi_n\|_{D^{1,2}} \to 0$.*

Proof. For simplicity, we will use $\|u\|$ to denote $\|u\|_{D^{1,2}}$.

We argue by contradiction. Suppose that there are $p_{n,j}$, λ_n, h_n and ξ_n, satisfying (3.3), $\lambda_n \to +\infty$, $\|h_n\| \to 0$ and $\|\xi_n\| \geq c > 0$. We may assume $\|\xi_n\|^2 = n$. Then $\|h_n\|^2 = o(n)$.

First, we estimate $a_{n,i}$. We have

$$\sum_{i=1}^2 a_{n,i}\sum_{j=1}^n \int_{\mathbb{R}^N} U_{p_{n,j},\lambda}^{2^*-2} Z_{j,i} Z_{1,l} = \big\langle Q_n\xi_n - h_n, Z_{1,l}\big\rangle.$$

It holds

$$\begin{aligned}\langle h_n, Z_{1,l}\rangle =& \int_{D_1} \nabla h_n \sum_{j=1}^n \nabla Z_{1,l}(z+p_{n,j}) \\ =& O\Big(\Big(\int_{D_1} |\nabla h_n|^2\Big)^{\frac12} \int_{D_1} \Big(\Big|\sum_{j=1}^n \nabla Z_{1,l}(z+p_{n,j})\Big|^2\Big)^{\frac12}\Big) = o(\lambda_n^{\tau(l)}),\end{aligned}$$

where $\tau(l)=1$ if $l=1$, while $\tau(l)=-1$ if $l=2$.

On the other hand,

$$\int_{\mathbb{R}^N} Q_n \xi_n Z_{1,l} = (2^*-1)\int_{\mathbb{R}^N} \Big(U_{p_{n,1},\lambda_n}^{2^*-2} - K(|y|)\Big(u_k + \sum_{j=1}^n U_{p_{n,j},\lambda_n}\Big)^{2^*-2}\Big)\xi_n Z_{1,l}.$$

In view of

$$|u_k(y)| \le \frac{C}{(1+|y|)^{N-2}}, \tag{3.4}$$

we can prove

$$\begin{aligned}\int_{\mathbb{R}^N} u_k^{2^*-2}|\xi_n Z_{1,l}| =& O\Big(\int_{D_1} |\xi_n| u_k^{2^*-2} \sum_{j=1}^n |Z_{1,l}(z+p_{n,j})|\Big) \\ =& O\Big(\lambda_n^{\tau(l)} \int_{D_1} |\xi_n| u_k^{2^*-2} \frac{\lambda_n^{\frac{N-2}{2}}}{(1+\lambda_n|y-p_{n,1}|)^{N-2-\tau}}\Big) = o\big(\lambda_n^{\tau(l)}\big).\end{aligned}$$

As a result,

$$\begin{aligned}&\int_{\mathbb{R}^N} Q_n \xi_n Z_{1,l} \\ =&(2^*-1)\int_{\mathbb{R}^N} \Big(U_{p_{n,1},\lambda_n}^{2^*-2} - K(|y|)\big(\sum_{j=1}^n U_{p_{n,j},\lambda_n}\big)^{2^*-2}\Big)\xi_n Z_{1,l} + o(\lambda_n^{\tau(l)}) \\ =&(2^*-1)\int_{D_1} \Big(U_{p_{n,1},\lambda_n}^{2^*-2} - K(|y|)\big(\sum_{j=1}^n U_{p_{n,j},\lambda_n}\big)^{2^*-2}\Big)\xi_n \sum_{j=1}^n Z_{1,l}(z+p_{n,j}) + o(\lambda_n^{\tau(l)}) \\ =&o(\lambda_n^{\tau(l)}).\end{aligned}$$

Moreover, it holds

$$\sum_{j=1}^n \int_{\mathbb{R}^N} U_{p_{n,j},\lambda_n}^{2^*-2} Z_{j,i} Z_{1,l} = \lambda_n^{2\tau(l)}\big(A+o(1)\big)\delta_{il}$$

for some constant $A\neq 0$. So we have proved $a_{n,i} = o(\lambda_n^{-\tau(i)})$.

From $\int_{\mathbb{R}^N} U_{p_{n,j},\lambda_n}^{2^*-2} Z_j \xi_n = 0$, it is easy to prove that

$$\lambda_n^{-\frac{N-2}{2}} \xi_n(\lambda_n^{-1}y + p_{n,j}) \to 0, \quad \text{in } H^1_{loc}(\mathbb{R}^N).$$

Moreover, since $\frac{1}{\sqrt{n}}\xi_n$ is bounded in $D^{1,2}(\mathbb{R}^N)$, we can assume that

$$\frac{1}{\sqrt{n}}\xi_n \rightharpoonup \xi, \quad \text{weakly in } \ D^{1,2}(\mathbb{R}^N),$$

and

$$\frac{1}{\sqrt{n}}\xi_n \to \xi, \quad \text{strongly in } \ L^2_{loc}(\mathbb{R}^N).$$

Thus, ξ satisfies

$$-\Delta\xi - (2^*-1)K(|y|)u_k^{2^*-2}\xi = 0, \quad \text{in } \mathbb{R}^N.$$

By Theorem 1.1, $\xi = 0$. Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^N}\big(u_k + \sum_{j=1}^n U_{p_{n,j},\lambda_n}\big)^{2^*-2}\xi_n^2 \\
=&\int_{\mathbb{R}^N\setminus(B_R(0)\cup(\cup_{j=1}^n B_{R\lambda_n^{-1}}(p_{n,j}))}\big(u_k + \sum_{j=1}^n U_{p_{n,j},\lambda_n}\big)^{2^*-2}\xi_n^2 + o(n) \\
=&n\int_{D_1\setminus(B_R(0)\cup(\cup_{j=1}^n B_{R\lambda_n^{-1}}(p_{n,j}))}\big(u_k + \sum_{j=1}^n U_{p_{n,j},\lambda_n}\big)^{2^*-2}\xi_n^2 + o(n).
\end{aligned}$$

For $y \in D_1$, it holds

$$\sum_{j=1}^n U_{p_{n,j},\lambda_n}(y) = U_{p_{n,1},\lambda_n}(y) + \lambda_n^{\frac{N-2}{2}} O\Big(\frac{1}{(1+\lambda_n|y-p_{n,1}|)^{N-2-\tau}}\Big).$$

Therefore, using (3.4), we obtain

$$\begin{aligned}
&\int_{D_1\setminus(B_R(0)\cup(\cup_{j=1}^n B_{R\lambda_n^{-1}}(p_{n,j})))}\big(u_k + \sum_{j=1}^n U_{p_{n,j},\lambda_n}\big)^{2^*-2}\xi_n^2 \\
\le& C\int_{D_1\setminus(B_R(0)\cup(\cup_{j=1}^n B_{R\lambda_n^{-1}}(p_{n,j})))}\big(u_k^{2^*-2} + U_{p_{n,1},\lambda_n}^{2^*-2}\big)\xi_n^2 \\
&+ C\int_{D_1\setminus(B_R(0)\cup(\cup_{j=1}^n B_{R\lambda_n^{-1}}(p_{n,j})))}\Big(\frac{\lambda_n^{\frac{N-2}{2}}}{(1+\lambda_n|y-p_{n,1}|)^{N-2-\tau}}\Big)^{2^*-2}\xi_n^2 \\
=&o(1).
\end{aligned}$$

We also have

$$\big\langle h_n, \xi_n\big\rangle = o(n),$$

and

$$\Big\langle \sum_{j=1}^n Z_{j,i}, \xi_n\Big\rangle = O(n\lambda_n^{\tau(i)}).$$

So we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla\xi_n|^2\\ =&O\Big(\int_{\mathbb{R}^N}\Big(u_k+\sum_{j=1}^n U_{p_{n,j},\lambda_n}\Big)^{2^*-2}\xi_n^2\Big)+\big\langle h_n,\xi_n\big\rangle+\int_{\mathbb{R}^N}\sum_{i=1}^2 a_{n,i}\Big\langle\sum_{j=1}^n Z_{j,i},\xi_n\Big\rangle\\ =&o(n).\end{aligned}$$

This is a contradiction.

□

From now on, we assume that $N\geq 7$. We want to construct a solution u for (1.2) with

$$u=u_k+\sum_{j=1}^n U_{p_j,\lambda}+\omega,$$

where $\omega\in X_s$ is a small perturbed term, satisfying

$$\int_{\mathbb{R}^N}U_{p_j,\lambda}^{2^*-2}Z_{j,l}\omega=0,\quad j=1,\cdots,n,\ l=1,2.$$

Then ω satisfies

$$Q_n\xi_n=l_n+R(\xi_n),\tag{3.5}$$

where

$$l_n=K(|y|)\Big(u_k+\sum_{j=1}^n U_{p_j,\lambda}\Big)^{2^*-1}-K(|y|)u_k^{2^*-1}-\sum_{j=1}^n U_{p_j,\lambda}^{2^*-1},\tag{3.6}$$

and

$$\begin{aligned}R_n(\xi)=&K(|y|)\Big(u_k+\sum_{j=1}^n U_{p_j,\lambda}+\xi\Big)^{2^*-1}-K(|y|)\Big(u_k+\sum_{j=1}^n U_{p_j,\lambda}\Big)^{2^*-1}\\ &-(2^*-1)K(|y|)\Big(u_k+\sum_{j=1}^n U_{p_j,\lambda}\Big)^{2^*-2}\xi.\end{aligned}\tag{3.7}$$

We have the following estimate for $\|l_n\|$.

Lemma 3.2. *There is a small $\sigma>0$, such that*

$$\|l_n\|\leq\frac{C\sqrt{n}}{\lambda^{1+\sigma}}.$$

Proof. Write

$$\begin{aligned} l_n =&K(|y|)\Big(\big(u_k+\sum_{j=1}^n U_{p_j,\lambda}\big)^{2^*-1}-u_k^{2^*-1}-\big(\sum_{j=1}^n U_{p_j,\lambda}\big)^{2^*-1}\Big)\\ &+\big(\sum_{j=1}^n U_{p_j,\lambda}\big)^{2^*-1}-\sum_{j=1}^n U_{p_j,\lambda}^{2^*-1}\\ &+\big(K(|y|)-1\big)\big(\sum_{j=1}^n U_{p_j,\lambda}\big)^{2^*-1}\\ :=&J_1+J_2+J_3. \end{aligned}$$

For $y\in D_1\cap B_{\lambda^{-\frac12}}(p_1)$, we have

$$u_k\le CU_{p_1,\lambda}.$$

So for any $\theta,\ \bar\theta\in(0,1)$,

$$\begin{aligned} |J_1|\le&C\big(\sum_{j=1}^n U_{p_j,\lambda}\big)^{2^*-2}u_k+u_k^{2^*-1}\\ \le&CU_{p_1,\lambda}^{2^*-2}u_k+u_k^{2^*-1}+C\big(\sum_{j=2}^n U_{p_j,\lambda}\big)^{2^*-2}u_k\\ \le&CU_{p_1,\lambda}^{2^*-2+\theta}+C\big(\sum_{j=2}^n U_{p_j,\lambda}\big)^{2^*-2}U_{p_1,\lambda}^{\bar\theta}. \end{aligned}$$

Take $\theta=\frac{N-6}{2(N-2)}+\theta_1$, where $\theta_1>0$ is small. Then $\frac{2N(2^*-2+\theta)(N-2)}{N+2}>N$. Thus

$$\begin{aligned} \int_{D_1\cap B_{\lambda^{-\frac12}}(p_1)}|U_{p_1,\lambda}^{2^*-2+\theta}\phi|\le&\Big(\int_{D_1\cap B_{\lambda^{-\frac12}}(p_1)}U_{p_1,\lambda}^{\frac{2^*(2^*-2+\theta)}{2^*-1}}\Big)^{1-\frac{1}{2^*}}\Big(\int_{D_1}|\phi|^{2^*}\Big)^{\frac{1}{2^*}}\\ \le&\frac{C}{\lambda^{\frac{(1-\theta)(N-2)}{2}}}\Big(\int_{D_1}|\phi|^{2^*}\Big)^{\frac{1}{2^*}}\le\frac{Cn^{-\frac12+\frac1N}}{\lambda^{\frac{(1-\theta)(N-2)}{2}}}\|\phi\|\\ =&\frac{Cn^{-\frac12+\frac1N}}{\lambda^{\frac{N-2}{4}+1-\frac{(N-2)\theta_1}{2}}}\|\phi\|\le\frac{Cn^{-\frac12}}{\lambda^{1+\sigma}}\|\phi\|, \end{aligned}$$

if $\theta_1>0$ is small.

For $y\in D_1$, we have

$$\begin{aligned} \sum_{j=2}^n U_{p_j,\lambda}\le&\frac{C\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-p_1|)^{(N-2)(1-\tau_1)}}\sum_{j=2}^n\frac{1}{(\lambda|p_j-p_1|)^{(N-2)\tau_1}}\\ \le&\frac{C\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-p_1|)^{(N-2)(1-\tau_1)}}, \end{aligned}$$

where $\tau_1 = \frac{N-4}{(N-2)^2}$. Hence

$$\big(\sum_{j=2}^{n} U_{p_j,\lambda}\big)^{2^*-2} U^{\bar\theta}_{p_1,\lambda} \le C\Big(\frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-p_1|)^{(N-2)(1-\tau_1)}}\Big)^{2^*-2+\bar\theta}.$$

Take $\bar\theta = \frac{1}{N-2}\big(\frac{N+2}{2(1-\tau_1)} - 4 + \theta_1\big)$, where $\theta_1 > 0$ is small. Then $\frac{2N(2^*-2+\bar\theta)(N-2)(1-\tau_1)}{N+2} > N$. Thus

$$\begin{aligned}
&\int_{D_1\cap B_{\lambda^{-\frac12}}(p_1)} |\phi|\big(\sum_{j=2}^{n} U_{p_j,\lambda}\big)^{2^*-2} U^{\bar\theta}_{p_1,\lambda} \\
\le& C\Big(\int_{D_1\cap B_{\lambda^{-\frac12}}(p_1)} \Big(\frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-p_1|)^{(N-2)(1-\tau_1)}}\Big)^{\frac{2^*(2^*-2+\bar\theta)}{2^*-1}}\Big)^{1-\frac{1}{2^*}} \Big(\int_{D_1} |\phi|^{2^*}\Big)^{\frac{1}{2^*}} \\
\le& \frac{C}{\lambda^{\frac{(1-\bar\theta)(N-2)}{2}}}\Big(\int_{D_1} |\phi|^{2^*}\Big)^{\frac{1}{2^*}} = \frac{C}{\lambda^{\frac{N+2}{2}(1-\frac{1}{2(1-\tau_1)})-\frac{\theta_1}{2}}}\Big(\int_{D_1} |\phi|^{2^*}\Big)^{\frac{1}{2^*}} \\
\le& \frac{Cn^{-\frac12}}{\lambda^{1+\sigma}}\|\phi\|
\end{aligned}$$

if $\theta_1 > 0$ is small. So we have proved

$$\int_{\cup_{j=1}^n (D_j\cap B_{\lambda^{-\frac12}}(p_j))} |J_1\phi| = n\int_{D_1\cap B_{\lambda^{-\frac12}}(p_1)} |J_1\phi| \le \frac{Cn^{\frac12}}{\lambda^{1+\sigma}}\|\phi\|.$$

For $y \in \mathbb{R}^N \setminus \cup_{j=1}^n \big(D_j \cap B_{\lambda^{-\frac12}}(p_j)\big)$, we have

$$\sum_{j=1}^{n} U_{p_j,\lambda} \le C u_k.$$

Thus

$$\begin{aligned}
|J_1| \le& C u_k^{2^*-2} \sum_{j=1}^{n} U_{p_j,\lambda} + \big(\sum_{j=1}^{n} U_{p_j,\lambda}\big)^{2^*-1} \\
\le& C u_k^{2^*-2} \sum_{j=1}^{n} U_{p_j,\lambda} \le \frac{C}{(1+|y|)^4}\sum_{j=1}^{n} U_{p_j,\lambda}.
\end{aligned}$$

For $N \ge 7$,

$$\begin{aligned}
\int_{\mathbb{R}^N\setminus\cup_{j=1}^n \left(D_j\cap B_{\lambda^{-\frac12}}(p_j)\right)} |J_1||\phi| \le& C\int_{\mathbb{R}^N} \sum_{j=1}^{n} U_{p_j,\lambda}|\phi| \\
\le& \frac{Cn}{\lambda^2}\|\phi\| \le \frac{Cn^{-\frac12}}{\lambda^{1+\sigma}}\|\phi\|.
\end{aligned}$$

For $y \in D_1$, we have

$$\sum_{j=2}^{n} U_{p_j,\lambda}^{2^*-1} \le C U_{p_1,\lambda}^{2^*-2} \sum_{j=2}^{n} U_{p_j,\lambda}$$

and

$$\begin{aligned}\sum_{j=2}^{n} U_{p_j,\lambda} \le & \frac{C\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-p_1|)^{(N-2)\theta}} \sum_{j=2}^{n} \frac{1}{(\lambda|p_j-p_1|)^{(N-2)(1-\theta)}} \\ \le & \frac{C\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-p_1|)^{(N-2)\theta}} \Big(\frac{n^{N-2}}{\lambda^{N-2}}\Big)^{1-\theta}.\end{aligned}$$

As a result,

$$|J_2| \le \frac{C\lambda^{\frac{N+2}{2}}}{(1+\lambda|y-p_1|)^{4+(N-2)\theta}} \Big(\frac{n^{N-2}}{\lambda^{N-2}}\Big)^{1-\theta}, \quad y \in D_1.$$

Therefore, for any $\theta > 0$ satisfying $\theta > \frac{N-6}{2(N-2)}$, it holds

$$\begin{aligned}\int_{\mathbb{R}^N} |J_2\phi| = & n \int_{D_1} |J_2\phi| \\ \le & n \Big(\int_{D_1} |J_2|^{\frac{2^*}{2^*-1}}\Big)^{1-\frac{1}{2^*}} \Big(\int_{D_1} |\phi|^{2^*}\Big)^{\frac{1}{2^*}} \\ \le & Cn \Big(\frac{n^{N-2}}{\lambda^{N-2}}\Big)^{1-\theta} \Big(\int_{D_1} |\phi|^{2^*}\Big)^{\frac{1}{2^*}} \le Cn^{1-\frac{1}{2^*}} \Big(\frac{n^{N-2}}{\lambda^{N-2}}\Big)^{1-\theta} \|\phi\|.\end{aligned}$$

Note that $\lambda \sim n^{\frac{N-2}{N-4}}$. Take $\theta = \frac{N-6}{2(N-2)} + \theta_1$, where $\theta_1 > 0$ is small. Then $1-\theta = \frac{1}{2} + \frac{2}{N-2} - \theta_1$. This gives

$$\begin{aligned}n^{1-\frac{1}{2^*}} \Big(\frac{n^{N-2}}{\lambda^{N-2}}\Big)^{1-\theta} \le & C n^{1-\frac{1}{2^*}} \frac{1}{\lambda^{2(1-\theta)}} \\ \le & \frac{C\sqrt{n} n^{\frac{1}{N}}}{\lambda^{1+\frac{4}{N-2}-2\theta_1}} \le \frac{C\sqrt{n}}{\lambda^{1+\sigma}}.\end{aligned}$$

It is also easy to check

$$\begin{aligned}\int_{\mathbb{R}^N} |J_3\phi| = & n \int_{D_1} |J_3\phi| \le \frac{Cn}{\lambda^2} \Big(\int_{D_1} |\phi|^{2^*}\Big)^{\frac{1}{2^*}} \\ \le & \frac{Cn^{1-\frac{1}{2^*}}}{\lambda^2} \|\phi\| \le \frac{C\sqrt{n}}{\lambda^{1+\sigma}} \|\phi\|.\end{aligned}$$

□

We also need the following lemma.

Lemma 3.3.

$$\|R_n(\xi)\| \le C\|\xi\|^{\min(2^*-1,2)}.$$

Moreover,

$$\|R_n(\xi_1) - R_n(\xi_2)\| \le C\Big(\|\xi_1\|^{\min(2^*-2,1)} + \|\xi_2\|^{\min(2^*-2,1)}\Big)\|\xi_1 - \xi_2\|.$$

Proof. The proof of this lemma is standard and can be found in [26]. Thus we omit it. □

We consider the following problem:

$$\begin{cases} Q_n\xi_n = l_n + R_n(\xi_n) + \sum\limits_{i=1}^{2} a_{n,i} \sum\limits_{j=1}^{n} Z_{j,i}, \\ \xi_n \in X_s, \\ \int_{\mathbb{R}^N} U_{p_j,\lambda}^{p-1} Z_{j,l}\xi_n = 0, \quad j = 1, \cdots, n,\ l = 1, 2. \end{cases} \tag{3.8}$$

Using Lemmas 3.1, 3.2 and 3.3, we can prove the following proposition in a standard way.

Proposition 3.4. *There is an integer $n_0 > 0$, such that for each $n \ge n_0$ and $(t, \lambda) \in (r_0 - \delta,\ r_0 + \delta) \times [\lambda_0 n \ln n, \lambda_1 n \ln n]$, (3.8) has a solution ξ_n for some constants $a_{n,i}$. Moreover, ξ_n is a C^1 map from $(r_0 - \delta,\ r_0 + \delta) \times [\lambda_0 n \ln n, \lambda_1 n \ln n]$ to X_s, and*

$$\|\xi_n\| \le \frac{C\sqrt{n}}{\lambda^{1+\sigma}}$$

for some $\sigma > 0$.

Define

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |Du|^2 - \frac{1}{2^*}\int_{\mathbb{R}^N} K(|y|)|u|^{2^*}.$$

Let

$$F(t, \lambda) = I\Big(u_k + \sum_{j=1}^{n} U_{p_j,\lambda} + \xi_n\Big).$$

To obtain a solution of the form $u_k + \sum_{j=1}^{n} U_{p_j,\lambda} + \xi_n$, we just need to find a critical point for $F(t, \lambda)$ in $[r_0 - \delta, r_0 + \delta] \times [\lambda_0 n^{\frac{N-2}{N-4}}, \lambda_1 n^{\frac{N-2}{N-4}}]$.

Proof of Theorem 1.2. We have

$$F(t, \lambda) = I\Big(u_k + \sum_{j=1}^{n} U_{p_j,\lambda}\Big) + nO\big(\frac{1}{\lambda^{2+\sigma}}\big). \tag{3.9}$$

On the other hand,

$$\begin{aligned} I\Big(u_k + \sum_{j=1}^{n} U_{p_j,\lambda}\Big) =& I\Big(\sum_{j=1}^{n} U_{p_j,\lambda}\Big) + I(u_k) + \frac{1}{2}\sum_{j=1}^{n}\int_{\mathbb{R}^N} K(|y|)u_k^{2^*-1}U_{p_j,\lambda} \\ &- \frac{1}{2^*}\int_{\mathbb{R}^N} K(|y|)\Big(\big(u_k + \sum_{j=1}^{n} U_{p_j,\lambda}\big)^{2^*} - \big(\sum_{j=1}^{n} U_{p_j,\lambda}\big)^{2^*} - u_k^{2^*}\Big). \end{aligned} \tag{3.10}$$

It is easy to check

$$\int_{\mathbb{R}^N} K(|y|)u_k^{2^*-1}U_{p_j,\lambda} = O\Big(\frac{1}{\lambda^{\frac{N-2}{2}}}\Big).$$

For $y \in \mathbb{R}^N \setminus \cup_{j=1}^n \big(D_j \cap B_{\lambda^{-\frac{1}{2}}}(p_j)\big)$, we have

$$\begin{aligned}
&\Big|\big(u_k + \sum_{j=1}^n U_{p_j,\lambda}\big)^{2^*} - u_k^{2^*} - \big(\sum_{j=1}^n U_{p_j,\lambda}\big)^{2^*}\Big| \\
\leq & C u_k^{2^*-1}\sum_{j=1}^n U_{p_j,\lambda} + C\big(\sum_{j=1}^n U_{p_j,\lambda}\big)^{2^*} \leq C' u_k^{2^*-1}\sum_{j=1}^n U_{p_j,\lambda}.
\end{aligned}$$

As a result,

$$\begin{aligned}
&\int_{\mathbb{R}^N\setminus\cup_{j=1}^n \left(D_j\cap B_{\lambda^{-\frac{1}{2}}}(p_j)\right)} K(|y|)\Big|\big(u_k + \sum_{j=1}^n U_{p_j,\lambda}\big)^{2^*} - u_k^{2^*} - \big(\sum_{j=1}^n U_{p_j,\lambda}\big)^{2^*}\Big| \\
\leq & C\int_{\mathbb{R}^N} u_k^{2^*-1}\sum_{j=1}^n U_{p_j,\lambda} = O\Big(\frac{n}{\lambda^{\frac{N-2}{2}}}\Big).
\end{aligned}$$

We also have

$$\begin{aligned}
&\int_{\cup_{j=1}^n \left(D_j\cap B_{\lambda^{-\frac{1}{2}}}(p_j)\right)} K(|y|)\Big(\big(u_k + \sum_{j=1}^n U_{p_j,\lambda}\big)^{2^*} - u_k^{2^*} - \big(\sum_{j=1}^n U_{p_j,\lambda}\big)^{2^*}\Big) \\
= & n\int_{D_1\cap B_{\lambda^{-\frac{1}{2}}}(p_1)} K(|y|)\Big(\big(u_k + \sum_{j=1}^n U_{p_j,\lambda}\big)^{2^*} - u_k^{2^*} - \big(\sum_{j=1}^n U_{p_j,\lambda}\big)^{2^*}\Big).
\end{aligned}$$

It holds

$$\int_{D_1\cap B_{\lambda^{-\frac{1}{2}}}(p_1)} K(|y|)u_k^{2^*} = O\Big(\frac{1}{\lambda^{\frac{N}{2}}}\Big),$$

and

$$\begin{aligned}
&\int_{D_1\cap B_{\lambda^{-\frac{1}{2}}}(p_1)} K(|y|)\Big|\big(u_k + \sum_{j=1}^n U_{p_j,\lambda}\big)^{2^*} - \big(\sum_{j=1}^n U_{p_j,\lambda}\big)^{2^*}\Big| \\
\leq & C\int_{D_1\cap B_{\lambda^{-\frac{1}{2}}}(p_1)} \big(\sum_{j=1}^n U_{p_j,\lambda}\big)^{2^*-1} \\
\leq & C\int_{D_1\cap B_{\lambda^{-\frac{1}{2}}}(p_1)} \Big(U_{p_1,\lambda}^{2^*-1} + \frac{\lambda^{\frac{N+2}{2}}}{(1+\lambda|y-p_1|)^{(2^*-1)(N-2)(1-\tau_1)}}\Big) \\
\leq & \frac{C}{\lambda^{\frac{N-2}{2}}},
\end{aligned}$$

where $\tau_1 = \frac{N-4}{(N-2)^2}$.

So we have proved

$$I\Big(u_k+\sum_{j=1}^n U_{p_j,\lambda}\Big)=I\Big(\sum_{j=1}^n U_{p_j,\lambda}\Big)+I(u_k)+O\big(\frac{n}{\lambda^{\frac{N-2}{2}}}\big). \tag{3.11}$$

Combining (3.9) and (3.11), and proceeding as in [26], we obtain

$$\begin{aligned} F(t,\lambda)=&I\Big(\sum_{j=1}^n U_{p_j,\lambda}\Big)+I(u_k)+nO\big(\frac{1}{\lambda^{2+\sigma}}\big)\\ =&I(u_k)+nA+n\Big(\frac{B_1}{\lambda^2}+\frac{B_2}{\lambda^2}(\lambda r_0-t)^2-\frac{B_3n^{N-2}}{\lambda^{N-2}}\Big)\\ &+nO\Big(\frac{B_1}{\lambda^{2+\sigma}}+\frac{B_2}{\lambda^2}(\lambda r_0-t)^3\Big), \end{aligned} \tag{3.12}$$

where $A=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla U_{0,1}|^2-\frac{1}{2^*}\int_{\mathbb{R}^N}U_{0,1}^{2^*}$, B_1, B_2 and B_3 are some positive constants, and $\sigma>0$ is a small constant.

Now to find a critical point for $F(t,\lambda)$, we just need to proceed exactly as in [26]. □

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Department of Mathematics, Tsinghua University, Beijing, 100084, P.R.China
E-mail address: yguo@mail.tsinghua.edu.cn

Department of Mathematical Sciences University of Bath, Bath BA2 7AY, United Kingdom
E-mail address: m.musso@bath.ac.uk

School of Mathematics and Statistics, Central China Normal University, Wuhan, P.R. China
E-mail address: sjpeng@mail.ccnu.edu.cn

School of Mathematics and Statistics, Central China Normal University, Wuhan, P.R. China
E-mail address: syan@mail.ccnu.edu.cn